Generalized Metrics

A Thesis Submitted to the
College of Graduate Studies and Research
in Partial Fulfillment of the Requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon

By
Samer Assaf

Permission to Use

In presenting this thesis in partial fulfilment of the requirements for a Postgraduate degree from the University of Saskatchewan, I agree that the Libraries of this University may make it freely available for inspection. I further agree that permission for copying of this thesis in any manner, in whole or in part, for scholarly purposes may be granted by the professor or professors who supervised my thesis work or, in their absence, by the Head of the Department or the Dean of the College in which my thesis work was done. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to the University of Saskatchewan in any scholarly use which may be made of any material in my thesis.

Requests for permission to copy or to make other use of material in this thesis in whole or part should be addressed to:

Head of the Department of Mathematics and Statistics
Room 142 McLean Hall
106 Wiggins Road
University of Saskatchewan
Saskatoon, Saskatchewan
Canada
S7N 5E6

Abstract

A distance on a set is a comparative function. The smaller the distance between two elements of that set, the closer, or more similar, those elements are. Fréchet axiomatized the notion of distance into what is today known as a metric. In this thesis we study several generalizations of Fréchet's axioms. These include partial metric, strong partial metric, partial $n-\mathfrak{M}$etric and strong partial $n-\mathfrak{M}$etric. Those generalizations allow for negative distances, non-zero distances between a point and itself and even the comparison of $n-$tuples. We then present the scoring of a DNA sequence, a comparative function that is not a metric but can be modeled as a strong partial metric.

Using the generalized metrics mentioned above we create topological spaces and investigate convergence, limits and continuity in them. As an application, we discuss contractiveness in the language of our generalized metrics and present Banach-like fixed, common fixed and coincidence point theorems.

Acknowledgements

The Ph.D. program was by far my most challenging endeavor. Luckily, I did not go through it alone. I would like to take a moment to thank the silent heroes who helped me make this possible:

Professor emeritus Edward Tymchatyn: Prof. Tymchatyn was my supervisor during my Ph.D. program. He had an open door policy, always made sure to make time for me and broadened my horizons by exposing me to topological conferences in which I presented my findings. No question was too small for him to answer and, with an extensive knowledge in topology, he was an inexhaustible source of knowledge to be envied for. His attention to minute details, mathematical rigor and perfectionist approach provided lessons I hope to keep with me on my journey.

Doctor Koushik Pal: Dr. Pal is my co-author for the first two papers submitted. As an undergraduate student, I was used to being presented with a question and finding a way to prove it. At the graduate level however, things are not as straightforward. It is with him that I learnt that as mathematical investigators, we may actually change the question. This is a simple lesson but a very important one. It is with Dr. Pal that my journey as a researcher started.

Associate Professor Christopher Dutchyn: Prof. Dutchyn introduced me to the scoring of DNA strands mentioned in Section 2.2. His valuable insight on DNA scoring scheme guided me as I investigated the subject. He also took time from his busy schedule to read and discuss my first results in partial metrics. Additionally, he was my go-to reference in latex coding. The organized bibliography and referencing system are all thanks to him.

Assistant Professor Katarzyna Kuhlmann: Prof. Kuhlmann was the one who introduced me to the notion of generalized metrics. One day, while she was walking down the hall, I saw a partial metric paper in her hand and asked her about it. She took the time to introduce it, explain it and point me in the right direction for further research. She also took time to discuss dislocated metrics and present some examples. It is thanks to her positive attitude and willingness to help that ignited my interest in the topic which I ended up researching.

Professor Franz-Viktor Kuhlmann: Prof. Kuhlmann organized the fixed point theorem seminars in our department. During those seminars, I was exposed to various techniques which sparked the main questions for my first paper. Additionally, he took the time to read the first paper and suggested valuable amendments.

Assistant Professor Walid Abou Salem: There is more to the graduate program than the mathematics part and Prof. Abou Salem was a refreshing reference. A true teacher at heart, he understands the difficulties we experience as students and always provided valuable advice. He was a tremendous support on this journey.

Mathematics Department Office Administrators: Simply stated, they are amazing. With an uplifting smile and the knowledge to navigate policies in the most efficient way, they made every problem, no matter how big, seem easily manageable. A detailed explanation of procedures, email reminders and ingenious solutions were always happily provided in such a clear way that I had no problem staying on top of what was required of me.

Mathematics Department Systems Manager Richard Kondra: From downloading a program I needed, to updating Bakoma packages, to adjusting screen problems, to setting lecture conferences and the list goes on and on. Mr. Kondra was our IT savior and thankfully, a very patient one.

Last and most certainly not least, my parents: To my mother who spent countless hours nurturing the young mathematician in me and my father who became an expatriate to support us, a deep and heartfelt thanks. Both are hugely responsible for the man I am today. I hope they know that I would not even dream of being here without their encouragement, patience and support.

This thesis is dedicated to every graduate student who feels overwhelmed. It is normal to doubt your capabilities when you are dealing with professors who spent their entire career researching that field. Be patient: You do have what it takes, you just need experience. Persevere and enjoy the journey.

"We fail to realize that mastery is not about perfection. It's a process, a journey. The master stays on the path day after day, year after year. The master is willing to try and fail and try again, for as long as he/she lives."—George Leonard

Contents

Chapter 1

Introduction

1.1 Why do we generalize a metric?

Given a set X, a metric is a function $d : X \times X \to \mathbb{R}$ as defined in Section 2.1. The point of a metric is for it to be a comparative function. For any two given elements x and y of X, the value (also called the distance) $d(x, y)$ represents how close x and y are to each other. Here, we use "close" in a loose sense as its meaning may vary depending on what information we need when comparing the elements of X. For example, when we say the distance between Paris and Saskatoon is 6944.32 km, the information we are looking for is the amount of space that needs to be traveled to go from one to the other in a direct straight flight. In Computer Science, when we say the error between the actual value of π and 3.14 is of the order of 10^{-2}, the information we are conveying is how good of an estimation is 3.14 to the real value of π?

Unfortunately some comparative functions, although very meaningful, do not adhere to the strict axioms of a metric presented in Section 2.1. One example is the scoring presented in Section 2.3. A scoring is a function used to compare the similarity between two DNA strands. We generalize the known metric in two steps depicted in the figure below:

Metric.

$\Downarrow$

Partial metric and Strong partial metric.

(Section 2.2 and Section 2.3)

$\Downarrow$

Partial $n - \mathfrak{M}$etric and Strong Partial $n - \mathfrak{M}$etric.

(Section 2.5 and Section 2.6)

First we generalize the metric into a partial metric by allowing the self distance (the distance between a point and itself) to take real values that may be different than zero. Then we generalize a partial metric that acts on pairs into a‘ partial $n - \mathfrak{M}$etric that acts on $n-$tuples. The Partial $n - \mathfrak{M}$etric is the most general case presented. As an application, we use these generalized metrics to form fixed point theorems.

1.2 Why do we study fixed point theory?

Fixed point theory is a branch of topology that studies the conditions under which a function from a set to itself has a fixed point.

Advancements in fixed point theory enrich many scientific fields such as biology, chemistry, computer science, economics and game theory. Below we give one example from each field which illustrates the use of fixed point theory in that science.

In biology, fixed point theory helps with studying how cancer cells replicate. Statistical data is collected, modeled, and fixed point theorems are used to form an educated guess of how those cells will progress in the future.

In chemistry, the branch that uses computer programs and simulations is called computational chemistry. This branch of chemistry investigates the charge density and electronic charge per volume of atoms, ions and molecules. The bigger the molecule studied, the more complex the problem becomes. Some of the problems are too complex to be solved analytically, hence, fixed point theory is used to develop an iterative process that will converge to a solution.

In computer science, fixed point theory is used to check whether a program will stabilize or run indefinitely.

In economics and game theory, fixed point theory is used to prove that a certain simulation has at least one equilibrium point.

1.3 How are fixed point theorems classified?

There are two approaches to the study of fixed point theory. The first is an existential approach where, under some criteria, we ensure that a function has a fixed point. Unfortunately, the theory itself does not provide a way of constructing the fixed point. One example of such an approach is the Brouwer's Fixed Point Theorem which ensures the existence of a fixed point for a continuous function over a compact convex set in a Euclidian space.

The second approach is an iterative one where, under some criteria, we are able to construct a sequence whose terms converge to a limit which is a fixed point of the function in question. The main example is the Banach Fixed Point Theorem which asserts that continuous functions on a complete metric space have a fixed point should those functions be contractive.

1.4 What are we presenting in this thesis?

In this thesis, we restrict ourselves to iterative approaches to fixed point theorems. Our major concern is that this method relies heavily on a metric to allow us to see if successive iterations are getting "closer" fast enough or not. In some cases, such as in comparing DNA strands, the use of a metric is too computationally taxing. One would like to use a comparison score which may not satisfy all the axioms of a metric. Generalized metrics are obtained by amending metric axioms to allow meaningful comparison scores to be used.

We present four improvements to the metrics and fixed point theorems at hand:

1) A metric assigns a number to each pair of points; the smaller the number, the closer the points are together. Gähler [16, 17] presented the following question: Can we do a similar thing to assess a triplet? Will our new "metric" be stable enough to describe a T_o topology? Many mathematicians have attempted to define such a "metric"; some with more success than others. In this thesis we define and generalize n-metrics by assigning a number to each n-tuple and generate a T_o topology from that generalized metric.

2) Edelstein [10, 11, 12] generalized the Banach Fixed Point Theorem to allow the continuous function of a metric space to itself to be contractive only on the orbit of a certain point $x_o \in X$. We adapt Edelstein's idea and use it with the new generalized metrics discussed above. Our functions may not even be continuous but are contractive on an orbit.

3) Matthews [7, 25] extended Banach's Fixed Point Theorem to "metrics" where self distances are not necessarily zero. Those "metrics" are also known as partial metrics. His theorems, however, only applied where the limit has a self distance (in this paper called a central distance) of zero. O'Neill [29] extended the definitions of partial metrics to allow negative values. We extend Matthews' theorems to partial metrics in the sense of O'Neill. We also extend those theorems to allow the fixed point to have a self distance (the distance between that fixed point and itself) to be any real number rather than be restricted to the value zero.

4) Markov [24] began investigating common fixed point theorems for commuting maps. His Theorem was later generalized by Kakutani [20] to give us the Markov-Kakutani fixed-point theorem. Eilenberg and Montgomery [13] were among the first to investigate coincidence points for non-self maps, where the domain set is ordered. We generalize the techniques used in fixed point theorems to common and coincidence point theorems. Our approach allows us to present common and coincidence point theorems where neither commuting functions nor ordered spaces are required.

1.5 How do we organize our work?

We organize our work into six chapters:

In Chapter 2, we concentrate on generalizing the original notion of metric and partial metric. We define strong partial metrics, partial $n-\mathfrak{M}$etrics, and strong partial $n-\mathfrak{M}$etrics. We also present other generalized metrics already found in the literature. We give examples of each and derive some inequalities needed for the chapters to come.

In Chapter 3, we define open balls for each of the generalized metrics defined in Chapter 2. We show that these balls form a basis for a topology that is at least T_o and, in some generalized metric cases, even T_1. The T_0 property is important because we want our topologies to be able to distinguish points.

In Chapter 4, we define Cauchy sequences for each of our generalized metrics. We investigate the properties of the limits of Cauchy sequences in each case. We then move on to define a stronger version of the topological limit called the special limit, study its properties and define a complete space. We also introduce the notion of Cauchy pairs. Those are pairs of sequences whose terms eventually get arbitrarily "close" to one another.

In Chapter 5, we present a variety of contractive conditions on functions, or pairs of functions, sufficient for the use of iterative methods to obtain Cauchy sequences.

In Chapter 6, we investigate some criteria of functions, such as a relaxed version of continuity, and study their effect on the limits of the sequences found in Chapter 5. Those properties will make sure the iterations lead to a fixed point, common fixed point, or a coincidence point as shown in Chapter 7.

Chapter 7 is dedicated to our main theorems. The proofs will be short and direct due to the extensive scaffolding created in the previous chapters. Our main theorems consist of fixed point, common fixed point and coincidence point theorems in each of the generalized metric spaces. The fixed point theorems use a technique similar to Edelstein's by taking a function that is contracting on an orbit and provide additional constraints needed for the function to have a fixed point. The common fixed point theorems have a similar contractive approach. Fisher [14], Yeh [33] and many others developed common fixed point theorems that rely on the two functions commuting ($f(g(x)) = g(f(x))$), or a weaker form. We adapt our contractive approach to generate common fixed point theorems that do not require the two functions to commute. Finally, we formulate a coincidence point theorem where the only requirement on the domain is for it to be a complete generalized metric space.

1.6 Notations.

Let $x_1, x_2, ..., x_n$ and a be elements of a set X.

$\mathbb{N}$:	Set of natural numbers.
$\mathbb{R}$:	Set of real numbers.
$\mathbb{R}^{\geq 0}$:	Set of non-negative real numbers.
$\mathcal{P}(X)$:	Power set (set of all subsets) of X.
$\mathbb{R}^{>0}$:	Set of positive real numbers.
$\langle a \rangle^n$:	The $n-$tuple $(a, a, a, ..., a)$.
$\langle x_i \rangle_{i=1}^n$:	The $n-$tuple $(x_1, x_2, x_3, ..., x_n)$.
$(\langle x_i \rangle_{i=1}^k, \langle y_j \rangle_{j=1}^{n-k})$:	The $n-$tuple $(x_1, x_2, ..., x_k, y_1, y_2, ..., y_{n-k})$.

It is very important not to confuse

$$\langle x_i \rangle_{i=1}^{n} = (x_1, x_2, x_3, ..., x_n)$$

with

$$\langle x_i \rangle^{n} = (x_i, x_i, ..., x_i, x_i).$$

Chapter 2

Metrics and their generalizations

Distances have been used in the world since times immemorial. Fréchet [15] was the first to axiomatize a distance, he called it écart. Hausdorff [19] coined the term metric space and used metrics to define a topology. M. Deza and E. Deza in their book *Encyclopedia of Distances* [8] provided us with an unparalleled reference for a variety of metrics that appear in the literature.

In this chapter we present the various generalized metrics used in this thesis. Some of these generalizations are already found in the literature. We will state the motivation for each and provide some examples. In order not to make this chapter too cumbersome, we give proofs only in non-trivial cases.

2.1 Metric

Definition 2.1.1. *A* ***metric*** *d on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following axioms: For all $x, y, z \in X$,*

(m-lbnd): $0 \leq d(x, y)$. *(non-negative axiom or lower boundary axiom)*

(m-sym): $d(x, y) = d(y, x)$. *(symmetry axiom)*

(m-sep): $d(x, y) = 0 \iff x = y$. *(separation axiom)*

(m-inq): $d(x, y) \leq d(x, z) + d(z, y)$. *(triangular inequality axiom)*

We remark that in his paper Fréchet [15] assumed but did not actually state (m-sym). The reader who would like more information on metrics and metric topologies may consult [26].

2.2 Partial Metric

In **1992**, Matthews [25] considered finite sequences as partially computed versions of infinite sequences. He noticed that depending on the computational approach, from the computer's perspective, two computed sequences of the same original infinite sequence need not be identical. Therefore, a distance between a sequence and itself need not be zero. Motivated by this observation, Matthews generalized metrics into what he called partial metrics where a distance between a point and itself need no longer be zero. This change led him to modify the initial metric axioms (see Definition 2.1.1). His partial metrics however, allowed only non-negative values.

In **1996**, O'Neill [29] generalized Matthews' partial metric to allow negative values. When we mention a partial metric from here on, we will be referring to O'Neill's partial metric.

Definition 2.2.1. *A **partial metric** p on a set X is a function $p : X \times X \to \mathbb{R}$ satisfying the following axioms:*
For all $x, y, z \in X$,
(p-lbnd): $p(x, x) \leq p(x, y)$.
(p-sym): $p(x, y) = p(y, x)$.
(p-sep): $p(x, x) = p(x, y) = p(y, y) \iff x = y$.
(p-inq): $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

It is easy to see that every metric is a partial metric with self distance (i.e. distance of a point to itself) equal to zero. Every partial metric defines a metric as follows:

Lemma 2.2.1. *(O'Neill [29]: Lemma 2.7) Let p be a partial metric defined on a set X. Define $d : X \times X \to \mathbb{R}$ as follows:*
For all $x, y \in X$,

$$d(x, y) = 2p(x, y) - p(x, x) - p(y, y).$$

Then, d is a metric on X.

Proof: This Lemma is a special case of Theorem 2.5.5.

We will show in Chapter 3 that the topological structure on a set induced by a partial metric is coarser than that induced by the metric described in Lemma 2.2.1.

We give some examples of simple partial metrics.

Example 2.2.2. ***(Basic Partial Metric):***
Consider the set $X = \{x, y\}$. Let $p : X \times X \to \mathbb{R}$ be defined by:

$$p(x, x) = 0, p(x, y) = 1 \text{ and } p(y, y) = 1.$$

Then p is a partial metric on X.

Example 2.2.3. ***(Maximum Partial Metric [25]):***
Consider $X \subseteq \mathbb{R}$. Let $p : X \times X \to \mathbb{R}$ be defined by setting for all $x, y \in X$,

$$p(x, y) = \max\{x, y\}.$$

Then p is a partial metric on X.

Proof: Let $x, y, z \in X$.

Without loss of generality, we may assume that $x \leq y$. Thus,

$$(\star) : p(x, x) = x,\ p(y, y) = y, \text{ and } p(x, y) = y.$$

Proof of (p-lbnd): From ($\star$) we deduce that

$$p(x,x) = x \leq y = p(x,y) \text{ and } p(y,y) = y = p(x,y).$$

Proof of (p-sym): From ($\star$) we have $p(x,y) = y$ and $p(y,x) = y$, so, $p(x,y) = p(y,x)$.

Proof of (p-sep):

($\Leftarrow$) Trivial.

($\Rightarrow$): Suppose

$$p(x,x) = p(x,y) = p(y,y).$$

Then from ($\star$) it is trivial to see that $x = y$.

Proof of (p-inq): Knowing that $x \leq y$, to prove (p-inq) we need to consider three cases corresponding to the three possible orderings of $\{x,y,z\}$:

Case 1: Suppose that $z \leq x \leq y$. Then $p(x,z) + p(y,z) - p(z,z) = x + y - z = y + (x - z) \geq y = p(x,y)$.

Case 2: Suppose that $x \leq z \leq y$. Then $p(x,z) + p(y,z) - p(z,z) = z + y - z = y = p(x,y)$.

Case 3: Suppose that $x \leq y \leq z$. Then $p(x,z) + p(y,z) - p(z,z) = z + z - z = z \geq y = p(x,y)$.

Therefore, in all cases,

$$p(x,y) \leq p(x,z) + p(z,y) - p(z,z). \qquad \square$$

Example 2.2.4. ***(Augmented Real Line):***

Consider the set $X = \mathbb{R} \cup \{a\}$ where $a \notin \mathbb{R}$. Let $p : X \times X \to \mathbb{R}$ be defined by:

For all $x, y \in \mathbb{R}$,

$$p(a,a) = 0, p(a,x) = |x| \text{ and } p(x,y) = |x - y| - 1.$$

Then p is a partial metric on X.

Proof: Let $x, y, z \in \mathbb{R}$.

Proof of (p-lbnd): Three cases arise.

Case 1: $p(a,a) = 0$ and $p(a,x) = |x|$, hence, $p(a,a) \leq p(a,x)$.

Case 2: $p(x,x) = -1$ and $p(a,x) = |x|$, hence, $p(x,x) \leq p(a,x)$.

Case 3: $p(x,x) = -1$ and $p(x,y) = |x - y| - 1$, hence, $p(x,x) \leq p(x,y)$.

Proof of (p-sym): Trivial.

Proof of (p-sep):

($\Leftarrow$) Trivial.

($\Rightarrow$): Let $x, y \in \mathbb{R}$.

Suppose that

$$p(x, x) = p(x, y) = p(y, y).$$

Then, $-1 = |x - y| - 1$ giving us that $x = y$.

Now suppose that

$$p(a, a) = p(a, x) = p(x, x).$$

Then, $p(a, a) = p(x, x)$ which is a contradiction since $0 \neq -1$.

Proof of (p-inq): The proof of (p-inq) is given by considering four possible claims. Let $x, y, z \in X$.

<u>Claim 1</u>: $p(a, a) \leq p(a, x) + p(a, x) - p(x, x)$.

Proof of claim 1:

$$p(a, x) + p(a, x) - p(x, x) = |x| + |x| - (-1) = 2|x| + 1 \geq 0 = p(a, a).$$

<u>Claim 2</u>: $p(x, x) \leq p(a, x) + p(a, x) - p(a, a)$.

Proof of claim 2:

$$p(a, x) + p(a, x) - p(a, a) = 2|x| \geq -1 = p(x, x).$$

<u>Claim 3</u>: $p(x, y) \leq p(a, x) + p(a, y) - p(a, a)$.

Proof of claim 3:

$$p(a, x) + p(a, y) - p(a, a) = |x| + |y| \geq |x - y| \geq |x - y| - 1 = p(x, y).$$

<u>Claim 4</u>: $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Proof of claim 4: We have

$$p(x, z) + p(z, y) - p(z, z) = |x - z| - 1 + |y - z| - 1 - (-1)$$

$= |x - z| + |y - z| - 1 \geq |(x - z) - (y - z)| - 1$ (by the triangular inequality for the absolute value in $\mathbb{R}$)

$= |x - y| - 1 = p(x, y)$. $\square$

2.3 Strong Partial Metric

DNA strands, proteins and words are all examples of finite sequences generated from a finite alphabet $\mathcal{A}$. A generic question is: Given two finite sequences $x = \langle x_i' \rangle_{i=1}^{s}$ and $y = \langle y_j' \rangle_{j=1}^{t}$, how similar are these two sequences?

In the case of DNA for example, the alphabet $\mathcal{A} = \{C, G, A, T\}$. While studying mutation from a sequence $x = \langle x_i' \rangle_{i=1}^{s}$ to a sequence $y = \langle y_j' \rangle_{j=1}^{t}$, it becomes important to come up with a measure that can effectively compare partial DNA strands. One such measure is the following commonly used *scoring scheme* [32] :

We first align two given words by inserting gaps so that their lengths match. Formally we adjoin the symbol $-$, called a gap to form a new alphabet $\mathcal{A}^\star = \mathcal{A} \cup \{-\}$ where $- \notin \mathcal{A}$. Obviously more than one alignment of words is possible: We consider one such alignment $\mathcal{L}$ where we represent x in the alignment by $\langle x_i \rangle_{i=1}^n$ and y by $\langle y_i \rangle_{i=1}^n$. Fixing $\alpha', \beta', \gamma' \in \mathbb{R}$ we then do a letter-by-letter comparison while assigning a score to each of four distinct possibilities. Namely, a score of zero is assigned if both letters are $-$. The score γ' is assigned if only one of the letters is in $\mathcal{A}$. The score α' is assigned if the two letters match (are the same) and are in $\mathcal{A}$. The score β' is assigned if the two letters mismatch (are distinct) and are both in $\mathcal{A}$. Then these scores are summed up to assign a total score $h'_{\mathcal{L}}(x, y)$. Finally

$$s'(x, y) = \max\{h'_{\mathcal{L}}(x, y) | \mathcal{L} \text{ is an alignment}\}$$

denotes the highest possible score of all alignments of x and y. Then $s'(x, y)$ is used as a measure of similarity or dissimilarity of the two words. As an example, assume

$$\alpha' = +1, \beta' = -1 \text{ and } \gamma' = -2.$$

Therefore, the total score of the pair (CGATC, CAGA) for the particular alignment

$-$	C	G	A	$-$	T	C
$-$	C	$-$	A	G	A	$-$
0	$+1$	-2	$+1$	-2	-1	-2

is

$$0 + 1 - 2 + 1 - 2 - 1 - 2 = -5.$$

It is not hard to show that the best possible score for this pair of words is -2 arising from the alignment (CGATC and C—AGA).

In our conventional view of metrics, we need a scoring function s in which closeness is indicated by smaller numbers in the usual ordering of $\mathbb{R}$. For that reason in [3], we investigate the function $s(x, y) = -s'(x, y)$ and from it derive the axioms for a strong partial metric. And hence in the example above, the score given by the best alignment becomes $+2$:

C	G	A	T	C
C	$-$	A	G	A
-1	$+2$	-1	$+1$	$+1$

Remark 2.3.1. *As mentioned above, the difference between the scoring we are investigating and the one presented in [32] is that we are taking the negative of their score i.e.*

$$s(x, y) = -s'(x, y).$$

Definition 2.3.2. *A* ***strong partial metric*** *s on a set X is a function $s : X \times X \to \mathbb{R}$ satisfying the following axioms:*
For all $x, y, z \in X$,
(s-lbnd): $s(x,x) < s(x,y)$, for $x \neq y$.
(s-sym): $s(x,y) = s(y,x)$.
(s-inq): $s(x,y) \leq s(x,z) + s(z,y) - s(z,z)$.

Notice that a separation axiom (s-sep) is hidden in (s-lbnd) as:

$$s(x,y) \leq s(x,x) \iff x = y.$$

Clearly a strong partial metric is a partial metric. Therefore, a strong partial metric s defines a metric d as defined by Lemma 2.2.1.

Example 2.3.3. ***(Shifted Metric):***
Let $d : X \times X \to \mathbb{R}$ be a metric defined on a set X. For a real number r, let $s_r : X \times X \to \mathbb{R}$ be defined as

$$s_r(x,y) = d(x,y) + r.$$

Then s_r is a strong partial metric on X.

Proof: Let $x, y, z \in X$.
Proof of (s-lbnd): From (m-lbnd) and (m-sep) we get if $x \neq y$ then $d(x,x) = 0 < d(x,y)$. Hence,

$$d(x,x) + r < d(x,y) + r$$

giving us that

$$s_r(x,x) < s_r(x,y).$$

Proof of (s-sym): From (m-sym) we get

$$s_r(x,y) = d(x,y) + r = d(y,x) + r = s_r(y,x).$$

Proof of (s-inq): From (m-inq) we know that

$$d(x,y) \leq d(x,z) + d(z,y).$$

Adding r to both sides we get that

$$d(x,y) + r \leq d(x,z) + d(z,y) + r.$$

From (m-sep), we know that $d(x,x) = 0$. With the above we deduce that

$$d(x,y) + r \leq d(x,z) + r + d(y,z) + r + d(x,x) - r.$$

Finally we use the definition of s_r to obtain

$$s_r(x,y) \leq s_r(x,z) + s_r(z,y) - s_r(z,z). \qquad \square$$

Example 2.3.4. ***(Positive Real Line):***

Consider X to be the set of all positive real numbers. Let $s : X \times X \to \mathbb{R}$ be the function defined by setting :

$$s(x,x) = x \text{ and } s(x,y) = x + y \text{ for } x \neq y.$$

Then s is a strong partial metric on X.

Proof: Let $x, y, z \in X$.

The proof of (s-lbnd) and (s-sym) are quite straight forward.

Proof of (s-inq): There are four cases.

Case 1: If $x = y = z$ then

$$s(x,y) = x = x + x - x = s(x,z) + s(y,z) - s(z,z).$$

Case 2: If $x = y$ and $y \neq z$ then

$$s(x,y) = x \leq x + x + z = x + z + y + z - z = s(x,z) + s(y,z) - s(z,z).$$

Case 3: If $x \neq y$ and $x = z$ then

$$s(x,y) = x + y = x + y + z - z = s(x,z) + s(y,z) - s(z,z).$$

Case 4: If $x \neq y, y \neq z$, and $z \neq x$ then

$$s(x,y) = x + y \leq x + z + y = x + z + y + z - z \leq s(x,z) + s(y,z) - s(z,z). \qquad \square$$

As in Bio-informatics [32], we try to find out how similar (similarity as measured by the partial metric s) are the two finite sequences $x = \langle x'_i \rangle_{i=1}^{s}$ and $y = \langle y'_j \rangle_{j=1}^{t}$. For the convenience of the reader, we restate the definition of our scoring function s.

Definition 2.3.5. *Consider X to be the set of finite sequences generated by a finite alphabet $\mathcal{A}$. First we augment the alphabet $\mathcal{A}$ by adding a gap element $-$ i.e. $\mathcal{A}^\star = \mathcal{A} \cup \{-\}$ where $- \notin \mathcal{A}$. Then for $\alpha, \beta, \gamma \in \mathbb{R}$, we define a **scoring function** $s : X \times X \to \mathbb{R}$ by following the blueprint presented in the introduction of Section 2.3. For each $x, y \in X$, we use "deletions" and "insertions" (called InDels or gaps) to align the two sequences so that their lengths match i.e. we represent x by $\langle x_i \rangle_{i=1}^{n}$ and y by $\langle y_i \rangle_{i=1}^{n}$. We denote this alignment $\mathcal{L}$. We then compare the i^{th} terms and assign a score $h_{(\mathcal{L},i)}(x,y)$ in the manner below:*

Letter-by-letter comparison	*Terminology*	$h_{(\mathcal{L},i)}(x,y)$
$x_i \in \mathcal{A}$ *and* $y_i = -$	*Deletion*	γ
$x_i = -$ *and* $y_i \in \mathcal{A}$	*Insert*	γ
$x_i = y_i \in \mathcal{A}$	*Match*	α
$x_i, y_i \in \mathcal{A}$ *but* $x_i \neq y_i$	*Mismatch*	β
$x_i = y_i = -$	*Relay*	0

$$h_{\mathcal{L}}(x,y) = s(x,y).$$

A Relay will have no effect in comparing these two sequences. That is why it is given a score of zero. The best score $s(x,y)$ is attained when we get the alignment with the smallest possible score i.e.

$$s(x,y) = \min\{h_{\mathcal{L}}(x,y) | \mathcal{L} \text{ is an alignment }\}.$$

We call $s(x,y)$ the ***score*** *of the pair (x,y).*

We should note that more than one possible alignment may give us the best score, but the score itself is unique.

Remark 2.3.6. *In [32], the "best score" $s'(x,y)$ is given by taking the score of the alignment $\mathcal{L}$ that gives us the biggest value of $h'_{\mathcal{L}}(x,y)$. Thus in [32], $s'(x,y) = -s(x,y)$ and the relative letter-by-letter scores are taken as $\alpha' = -\alpha, \beta' = -\beta$ and $\gamma' = -\gamma$.*

Notation 2.3.7. *From this point forward, we will denote $h_{(\mathcal{L},i)}(x,y)$ and $h_{\mathcal{L}}(x,y)$ by $h_i(x,y)$ and $h(x,y)$ respectively when it is clear to which alignment $\mathcal{L}$ we are referring.*

To ensure s is a strong partial metric we need to require that a Match is strictly our most favorable occurrence. A Mismatch is at least as good as two InDels and a Relay is strictly better than an InDel.

Lemma 2.3.1. *Consider X the set of finite sequences generated from the finite alphabet $\mathcal{A}$. Let $s : X \times X \to \mathbb{R}$ be the scoring function in Definition 2.3.5. If $\alpha < \min\{\beta, \gamma, 0\}$, $\beta \leq 2\gamma$ and γ is strictly positive then s is a strong partial metric.*

Proof: Let $x, y, z \in X$.

Proof of (s-lbnd): We compare the sequences after optimally aligning them. I.e. $s(x,y) = h(x,y)$ where we represent x and y by $\langle x_i \rangle_{i=1}^n$ and $\langle y_i \rangle_{i=1}^n$ respectively. For all $i \in \{1, ..., n\}$

(x_i, y_i)	$h_i(x,x)$	$s_i(x,y)$	Comparison
$x_i = y_i \in \mathcal{A}$	α	α	$\alpha \leq \alpha$
$x_i \in \mathcal{A}$ and $y_i = -$	α	γ	$\alpha < \gamma$
$x_i = -$ and $y_i \in \mathcal{A}$	0	γ	$0 < \gamma$
$x_i, y_i \in \mathcal{A}$ but $x_i \neq y_i$	α	β	$\alpha < \beta$

Thus, for all $i \in \{1, ..., n\}$ $h_i(x,x) \leq s_i(x,y)$. Now since $x \neq y$ then there exists an i_o such that $s_i(x,y) \neq \alpha$. Since a relay has a score of 0 then for these particular representations of x and y we have

$$s(x,x) = h(x,x) = \sum_{i=1}^{n} h_i(x,x) < \sum_{i=1}^{n} s_i(x,y) = s(x,y).$$

Hence, $s(x,x) < s(x,y)$ for $x \neq y$.

Proof of (s-inq): To compare three sequences x, y and z, we are going to optimally align x with z and y

with z by adding the necessary InDels and Relays so that they all have the same length.

Before giving the formal proof of (s-inq) we give a simple example illustrating this step. Consider $x = CGT$, $y = AGAGT$ and $z = CAGC$. Now for some scoring scheme, assume the optimal alignment of x to z and y to z is given below

$x:$	C	$-$	G	$-$	T
$z:$	C	A	G	C	$-$

and

$y:$	$-$	A	G	A	G	T
$z:$	C	A	G	$-$	$-$	C

By using Relays we can amend the above alignments such that the representations of x, y and z have the same length. Also note that a relay has a score of zero, hence, $s(x, z)$ and $s(y, z)$ may be computed using the representations of x, y and z in the box below. Here a gap in either representation of z above appears as a gap in the representation of z below.

$x:$	C	$-$	G	$-$	$-$	$-$	T
$y:$	$-$	A	G	A	G	T	$-$
$z:$	C	A	G	$-$	$-$	C	$-$

As the example above demonstrates, we may consider x, y and z having the same representations length $\langle x_i \rangle_{i=1}^n$, $\langle y_i \rangle_{i=1}^n$and $\langle z_i \rangle_{i=1}^n$, respectively and optimally aligned to z. Then

$$s(z, z) = h(z, z)$$

$$s(x, z) = h(x, z) = \sum_{i=1}^{n} h_i(x, z)$$

and

$$s(y, z) = h(y, z) = \sum_{i=1}^{n} h_i(x, y).$$

This leaves us with x and y not necessarily optimally aligned but $s(x, y) \leq h(x, y)$.

We now move to prove that

$$h(x, y) \leq h(x, z) + h(y, z) - h(z, z) = s(x, z) + s(y, z) - s(z, z).$$

We do this term by term, thus ten cases arise:

Case 1	$x_i = y_i, z_i \in \mathcal{A}$ with z_i distinct from x_i
Case 2	$x_i = y_i \in \mathcal{A}$, and $z_i = -$
Case 3	$x_i, y_i, z_i \in \mathcal{A}$ and all three are distinct
Case 4	$x_i = z_i, y_i \in \mathcal{A}$ with y_i distinct from x_i
Case 5	x_i and y_i are distinct elements of $\mathcal{A}$, and $z_i = -$
Case 6	$x_i = -$, and $y_i = z_i \in \mathcal{A}$
Case 7	y_i and z_i are distinct elements of $\mathcal{A}$, and $x_i = -$
Case 8	$x_i = z_i = -$, and $y_i \in \mathcal{A}$
Case 9	$x_i = y_i = -$, and $z_i \in \mathcal{A}$
Case 10	$x_i = y_i = z_i = -$

We remind the reader that $\alpha < \min\{\beta, \gamma, 0\}$, $\beta \leq 2\gamma$ and $\gamma > 0$

Case 1: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \beta + \beta - \alpha$ and $h_i(x,y) = \alpha$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\alpha < \beta$.

Case 2: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \gamma + \gamma - 0$ and $h_i(x,y) = \alpha$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\gamma > 0$ and, therefore, $\alpha < 0 < 2\gamma$.

Case 3: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \beta + \beta - \alpha$ and $h_i(x,y) = \beta$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\alpha < \beta$.

Case 4: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \alpha + \beta - \alpha$ and $h_i(x,y) = \beta$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$.

Case 5: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \gamma + \gamma - 0$ and $h_i(x,y) = \beta$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\beta \leq 2\gamma$.

Case 6: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \gamma + \alpha - \alpha$ and $h_i(x,y) = \gamma$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$.

Case 7: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \gamma + \beta - \alpha$ and $h_i(x,y) = \gamma$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\alpha < \beta$.

Case 8: $h_i(x,z) + h_i(y,z) - h_i(z,z) = 0 + \gamma - 0$ and $h_i(x,y) = \gamma$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$.

Case 9: $h_i(x,z) + h_i(y,z) - h_i(z,z) = \gamma + \gamma - \alpha$ and $h_i(x,y) = 0$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\gamma > 0$ and, therefore, $\alpha < 0 < 2\gamma$.

Case 10: $h_i(x,z) + h_i(y,z) - h_i(z,z) = 0 + 0 - 0$ and $h_i(x,y) = 0$.
Hence, $h_i(x,y) \leq h_i(x,z) + h_i(y,z) - h_i(z,z)$ since $\alpha < \beta$.

In fact, Case 10 need not arise when comparing three sequences. It is a useful tool however when using multiple alignment schemes discussed in Section 2.6. □

Remark 2.3.8. *Lemma 2.3.1 remains valid even if α and β are functions with* $\max\{\alpha\} < \min\{\beta, \gamma, 0\}$, $\max\{\beta\} \leq 2\gamma$, $\gamma > 0$, *and Case 3 of the triangular inequality holds for distinct $x_i, y_i, z_i \in \mathcal{A}$.*

Example 2.3.9. ***(BLOSUM62):***
One example of the scoring scheme in Lemma 2.3.1is BLOSUM62 [32].

We also note that in the alignment schemes, available in the bio-informatics literature, the letter-by-letter scores are restricted to:

$$\alpha < 0, \beta \geq 0 \text{ and } \gamma \geq 0.$$

Hence, $\beta \leq 2\gamma$ and Case 3 are the remaining requirements to check for s to be a strong partial metric in those schemes.

2.4 $G-$metric and $n - \mathfrak{M}$etric

Many mathematicians attempted to generalize a distance, a functions that assigns values to pairs, to a function that assigns values to triplets or even to $n-$tuples. In **1963**, Gähler [16, 17] attempted a generalization to triplets by modeling his axioms to mimic the area of the triangle whose vertices are three given points. While he called the function a $2-$metric, we will refer to it as Gähler's $2-$metric to avoid any confusion that may arise from later definitions.

Definition 2.4.1. *A Gähler's $2-$**metric** σ on a set X is a function $\sigma : X \times X \times X \to \mathbb{R}$ satisfying the following axioms:*
For all $x, y, z, a \in X$,
(2-lbnd): $0 = \sigma(x,x,y) \leq \sigma(x,y,z)$.
(2-sym): $\sigma(x,y,z) = \sigma(\Pi\{x,y,z\})$, *where Π denotes a permutation on $\{x,y,z\}$.*
(2-sep): If $x \neq y$ then there is at least one element $z \in X$ such that $\sigma(x,y,z) \neq 0$.
(2-inq): $\sigma(x,y,z) \leq \sigma(x,y,a) + \sigma(x,a,z) + \sigma(a,y,z)$.

In **1988**, Ha, Cho and White [18] showed that Gähler's $2-$metric is not a generalization of a distance by giving an example of a metric space which is not a Gähler's $2-$metric space and and an example of a Gähler's $2-$metric space which is not a metric space. To remedy this, in **1992**, Dhage [9] defined a $D-$metric by modeling his axioms on the perimeter of the triangle whose vertices are three given points.

Definition 2.4.2. *A $\underline{D-\textit{metric}}$ D on a set X is a function $D : X \times X \times X \to \mathbb{R}$ satisfying the following axioms:*

For all $x, y, z, a \in X$,

(D-lbnd): $0 \leq D(x, y, z)$.

(D-sym): $D(x, y, z) = D(\Pi\{x, y, z\})$, *where* Π *denotes a permutation on* $\{x, y, z\}$.

(D-sep): $D(x, y, z) = 0 \iff x = y = z$

(D-inq): $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$.

In **2004**, Mustafa and Sims [27] showed that most of Dhage's claims about the structure of a $D-$metric space were incorrect. In **2006**, Mustapha and Sims [28] modified Dhage's axioms by changing (D-inq) and introducing an additional boundary axiom. They called their function a G-metric.

Definition 2.4.3. *A $\underline{G-\textit{metric}}$ G on a set X is a function $G : X \times X \times X \to \mathbb{R}$ satisfying the following axioms:*

For all $x, y, z, a \in X$,

(G-lbnd): $0 < G(x, x, y) \leq G(x, y, z)$, *for* $x \neq y$ *and* $y \neq z$,

(G-sym): $G(x, y, z) = G(\Pi\{x, y, z\})$, *where* Π *denotes a permutation on* $\{x, y, z\}$.

(G-sep): $G(x, y, z) = 0 \iff x = y = z$,

(G-inq): $G(x, y, z) \leq G(x, y, a) + G(a, a, z)$.

In **2012**, Khan [23] extended the Mustafa-Sims $G-$metric above into what he called a $K-$metric which is a function $K : X^n \to \mathbb{R}^{\geq 0}$ for $n \geq 2$. The notation used is presented in Section 1.6.

Definition 2.4.4. *For $n \geq 2$, a $\underline{K-\textit{metric}}$ K on a set X is a function $K : X^n \to \mathbb{R}$ satisfying the following axioms:*

For all $(\langle x_i \rangle_{i=1}^n, a) \in X^{n+1}$,

(K-lbnd): $0 < K(\langle x_1 \rangle^{n-1}, x_2) \leq K(\langle x_i \rangle_{i=1}^n)$ *for all distinct elements* $x_1, x_2, ..., x_n$.

(K-sym): $K(\langle x_i \rangle_{i=1}^n) = K(\langle x_{\pi(i)} \rangle_{i=1}^n)$, *where* π *is a permutation on* $\{1, ..., n\}$.

(K-sep): $K(\langle x_i \rangle_{i=1}^n) = 0 \iff x_1 = x_2 = x_3 = ... = x_n$,

(K-inq): $K(\langle x_i \rangle_{i=1}^n) \leq K(\langle x_i \rangle_{i=1}^{n-1}, a) + K(\langle a \rangle^{n-1}, x_n)$.

The $K-$metric is modeled on the perimeter of an $n-$simplex (i.e. the sum of the length of the sides of the n-simplex). For $n = 2$, the $K-$metric axioms are just the usual metric axioms. For $n = 3$, Khan's definition coincides with Mustafa and Sim's definition of a $G-$metric. We found that the axioms Khan proposed were unnecessarily restrictive. That is why in [2], we proposed the $n - \mathfrak{M}$etric. The reader should note the use of the scripted $\mathfrak{M}$ to differentiate it from other generalizations found in the literature.

Definition 2.4.5. *For $n \geq 2$, an* $\underline{n-\mathfrak{Metric}}$ *M on a set X is a function $M : X^n \to \mathbb{R}$ satisfying the following axioms:*
For all $(\langle x_i \rangle_{i=1}^n, a) \in X^{n+1}$,
(n-lbnd): $0 \leq M(\langle x_1 \rangle^{n-1}, x_2)$.
(n-sym): $M(\langle x_i \rangle_{i=1}^n) = M(\langle x_{\pi(i)} \rangle_{i=1}^n)$, *where π is a permutation on* $\{1, ..., n\}$.
(n-sep): $M(\langle x_1 \rangle^{n-1}, x_2) = 0 \iff x_1 = x_2$,
(n-inq): $M(\langle x_i \rangle_{i=1}^n) \leq M(\langle x_i \rangle_{i=1}^{n-1}, a) + M(\langle a \rangle^{n-1}, x_n)$.

Remark 2.4.6. *The $2-\mathfrak{M}$etric, not to be confused with Gähler's $2-$metric, is simply a metric. As mentioned below a $K-$metric (see Definition 2.4.4) is a special case of an $n-\mathfrak{M}$etric. Hence, a $G-$metric (see Definition 2.4.3) is a special case of a $3-\mathfrak{M}$etric.*

For $n \geq 3$, our $n-\mathfrak{M}$etric axioms relax Khan's $K-$metric axioms in three ways. First we allow negative values (see Example 2.4.9). The second difference lies in (n-lbnd) which is a major weakening of (K-lbnd). Third, our (n-sep) is weaker than (K-sep).

Mustafa and Sims [28] proved several properties of $G-$metrics. Not all of those properties hold for our $3-\mathfrak{M}$etric due to our weakened (n-lbnd) condition mentioned above. One property, however,

$$G(a, a, b) \leq 2G(b, b, a)$$

still holds for a $3-\mathfrak{M}$etric. We were able to generalize it into a tool for term replacement in the $n-\mathfrak{M}$etric case in Theorem 2.4.1.

Theorem 2.4.1. ***(Term Replacement):***
Let M be an $n-\mathfrak{M}$etric on a set X. For all $\langle x_i \rangle_{i=1}^n, \langle y_i \rangle_{i=1}^n \in X^n$ and for $t \in \{1, ..., n\}$,

$$M(\langle x_i \rangle_{i=1}^n) \leq M(\langle y_j \rangle_{j=1}^t, \langle x_i \rangle_{i=t+1}^n) + \sum_{j=1}^{t} M(\langle y_j \rangle^{n-1}, x_j).$$

Proof: Let $\langle x_i \rangle_{i=1}^n, \langle y_i \rangle_{i=1}^n \in X^n$. For $t = 1$, the result follows by (n-inq) and (n-sym). Let $t \in \{2, ..., n-1\}$ and assume that the inequality holds for $t-1$. Then

$$M(\langle x_i \rangle_{i=1}^n) \leq M(\langle y_j \rangle_{i=1}^{t-1}, \langle x_i \rangle_{i=t}^n) + \sum_{j=1}^{t-1} M(\langle y_j \rangle^{n-1}, x_j)$$

by (n-sym)

$$= M(\langle y_j \rangle_{i=1}^{t-1}, \langle x_i \rangle_{i=t+1}^n, x_t) + \sum_{j=1}^{t-1} M(\langle y_j \rangle^{n-1}, x_j)$$

by (n-inq)

$$\leq M(\langle y_j \rangle_{i=1}^{t-1}, \langle x_i \rangle_{i=t+1}^n, y_t) + M(\langle y_t \rangle^{n-1}, x_t) + \sum_{j=1}^{t-1} M(\langle y_j \rangle^{n-1}, x_j)$$

by (n-sym)

$$M(\langle y_j \rangle_{i=1}^{t}, \langle x_i \rangle_{i=t+1}^n) + \sum_{j=1}^{t} M(\langle y_j \rangle^{n-1}, x_j). \qquad \square$$

Remark 2.4.7. *The theorem above gives rise to important tools used in Section 3.3. We present them in the corollaries below.*

Corollary 2.4.2. *Let M be an $n - \mathfrak{M}etric$ on a set X. For all $\langle x_i\rangle_{i=1}^n, \langle y_i\rangle_{i=1}^n \in X^n$*

$$M(\langle x_i\rangle_{i=1}^n) \leq M(\langle y_i\rangle_{i=1}^n) + \sum_{j=1}^{n} M(\langle y_j\rangle^{n-1}, x_j).$$

Proof: This is the case of Theorem 2.4.1 when $t = n$. $\square$

Corollary 2.4.3. *Let M be an $n - \mathfrak{M}etric$ on a set X. Then for $a, b \in X$ for $t \in \{1, \ldots\ldots, n\}$,*

$$M(\langle a\rangle^t, \langle b\rangle^{n-t}) \leq tM(\langle b\rangle^{n-1}, a).$$

Proof: By Corollary 2.4.2 where $\langle x_i\rangle_{i=1}^n = (\langle a\rangle_1^t, \langle b\rangle_{t+1}^n)$ and $\langle y_j\rangle_{j=1}^n = \langle b\rangle^n$ we get

$$M(\langle a\rangle^t, \langle b\rangle_{t+1}^n) \leq M(\langle b\rangle^n) + \sum_{j=1}^{n} M(\langle y_j\rangle^{n-1}, x_j)$$

by (n-sep)

$$= 0 + \sum_{j=1}^{t} M(\langle y_j\rangle^{n-1}, x_j) + \sum_{j=t+1}^{n} M(\langle y_j\rangle^{n-1}, x_j)$$

$$= \sum_{j=1}^{t} M(\langle b\rangle^{n-1}, a) + \sum_{j=t+1}^{n} M(\langle b\rangle^n)$$

by (n-sep) again

$$= \sum_{j=1}^{t} M(\langle b\rangle^{n-1}, a) = tM(\langle b\rangle^{n-1}, a). \qquad \square$$

Corollary 2.4.4. *Let M be an $n - \mathfrak{M}etric$ on a set X. Then for $a, b \in X$,*

$$M(\langle a\rangle^{n-1}, b) \leq (n-1)M(\langle b\rangle^{n-1}, a).$$

Proof: This is the case of Corollary 2.4.3 when $t = n - 1$. $\square$

We now show that each $n - \mathfrak{M}$etric on a set X naturally induces a metric on X.

Theorem 2.4.5. ***(Metric from an $n - \mathfrak{M}etric$):***
Let M be an $n - \mathfrak{M}etric$ on a set X. For $x, y \in X$ let

$$d(x, y) = M(y, \langle x\rangle^{n-1}) + M(x, \langle y\rangle^{n-1}).$$

Then d is a metric on the set X.

Proof: Let $x, y, z \in X$.

Proof of (n-lbnd): From (n-lbnd), we know that $M(x, \langle y\rangle^{n-1}) \geq 0$ and $M(y, \langle x\rangle^{n-1}) \geq 0$. Hence, $d(x, y) = M(x, \langle y\rangle^{n-1}) + M(y, \langle x\rangle^{n-1}) \geq 0$.

Proof of (n-sym): Symmetry of d follows from the symmetry of addition of real numbers.

Proof of (n-sep):

($\Rightarrow$) From the definition of d, if $d(x, y) = 0$ then

$$M(x, \langle y \rangle^{n-1}) + M(y, \langle x \rangle^{n-1}) = 0.$$

By (n-lbnd)

$$M(x, \langle y \rangle^{n-1}) = 0 = M(y, \langle x \rangle^{n-1}).$$

By (K-sep), $x = y$.

($\Leftarrow$) If $x = y$ then $M(x, \langle y \rangle^{n-1}) = M(y, \langle x \rangle^{n-1}) = M(\langle x \rangle^{n}) = 0$ by (n-sep).

Hence,

$$d(x, y) = M(y, \langle x \rangle_1^{n-1}) + M(x, \langle y \rangle_1^{n-1}) = 0.$$

Proof of (n-inq): From (n-sym) and (n-inq) we get

$$M(x, \langle y \rangle^{n-1}) = M(\langle y \rangle^{n-1}, x)$$

$$\leq M(\langle y \rangle^{n-1}, z) + M(\langle z \rangle^{n-1}, x)$$

$$= M(x, \langle z \rangle^{n-1}) + M(z, \langle y \rangle^{n-1}).$$

Similarly

$$M(y, \langle x \rangle^{n-1}) \leq M(y, \langle z \rangle^{n-1}) + M(z, \langle x \rangle^{n-1}).$$

Hence,

$$d(x, y) = M(x, \langle y \rangle^{n-1}) + M(y, \langle x \rangle^{n-1})$$

$$\leq M(x, \langle z \rangle^{n-1}) + M(z, \langle y \rangle^{n-1}) + M(z, \langle x \rangle^{n-1}) + M(y, \langle z \rangle^{n-1})$$

$$= M(x, \langle z \rangle^{n-1}) + M(z, \langle x \rangle^{n-1}) + M(z, \langle y \rangle^{n-1}) + M(y, \langle z \rangle^{n-1})$$

$$= d(x, z) + d(z, y). \qquad \square$$

We shall show in Section 3.3 that the topology on X induced by the $n-\mathfrak{M}$etric coincides with the topology on X induced by the metric presented in Theorem 2.4.5.

Example 2.4.8. ***(Unit $n - \mathfrak{M}$etric):***

Consider $X = \mathbb{R}$. Let $M : X^n \to \mathbb{R}$ be defined as follows:

For all $\langle x_i \rangle_{i=1}^n \in \mathbb{R}^n, M(\langle x_i \rangle_{i=1}^n) = \begin{cases} 0 & \text{if } x_1 = x_2 = ... = x_n. \\ 1 & \text{otherwise.} \end{cases}$

Then M is an $n - \mathfrak{M}$etric.

Example 2.4.9. *(5 − $\mathfrak{M}$etric with Negative values):*

Consider $X = \{a, b\}$. Let $M : X^5 \to \mathbb{R}$ be defined as follows:

For all $\langle x_i\rangle_{i=1}^5 \in X^5$, $M(\langle x_i\rangle_{i=1}^5) = M(\langle x_{\pi(i)}\rangle_{i=1}^5)$. (Where π is a permutation on $\{1, 2, 3, 4, 5\}$)

Furthermore,

$M(a, a, a, a, a) = 0, M(b, b, b, b, b) = 0,$

$M(a, a, a, a, b) = 3,$ $M(a, b, b, b, b) = 4,$

$M(a, a, a, b, b) = -1,$ *and* $M(a, a, b, b, b) = 2.$

Then M is a 5 − $\mathfrak{M}$etric on the set X.

Proof: (n-lbnd), (n-sym), and (n-sep) are direct results from the definition of M.

Proof of (n-inq): There are ten cases:

$M(a, a, a, a, a) \le M(a, a, a, a, b) + M(b, b, b, b, a)$, since $(0 \le 3 + 4)$.

$M(a, a, a, a, b) \le M(a, a, a, b, b) + M(b, b, b, b, a)$, since $(3 \le -1 + 4)$.

$M(a, a, a, a, b) \le M(a, a, a, a, a) + M(a, a, a, a, b)$, since $(3 \le 0 + 3)$.

$M(a, a, a, b, b) \le M(a, a, b, b, b) + M(b, b, b, b, a)$, since $(-1 \le 2 + 4)$.

$M(a, a, a, b, b) \le M(a, a, a, a, b) + M(a, a, a, a, b)$, since $(-1 \le 3 + 3)$.

$M(a, a, b, b, b) \le M(a, b, b, b, b) + M(b, b, b, b, a)$, since $(2 \le 4 + 4)$.

$M(a, a, b, b, b) \le M(a, a, a, b, b) + M(a, a, a, a, b)$, since $(2 \le -1 + 3)$.

$M(a, b, b, b, b) \le M(b, b, b, b, b) + M(b, b, b, b, a)$, since $(4 \le 0 + 4)$.

$M(a, b, b, b, b) \le M(a, a, b, b, b) + M(a, a, a, a, b)$, since $(4 \le 2 + 3)$.

$M(b, b, b, b, b) \le M(a, b, b, b, b) + M(a, a, a, a, b)$, since $(0 \le 4 + 3)$. □

Theorem 2.4.6. *(n − $\mathfrak{M}$etric from a metric):*

Every metric d on a set X naturally defines an n−metric M on X as follows:

For all $\langle x_i\rangle_{i=1}^n \in X^n$, let

$$M(\langle x_i\rangle_{i=1}^n) = \sum_{t=2}^{n}\sum_{i=1}^{t-1} d(x_i, x_t).$$

Proof: Let $(\langle x_j\rangle_{j=1}^n, a) \in X^{n+1}$.

Proof of (n-lbnd): Since d is a metric on X, for $x_1 \neq x_2$

$$\begin{aligned} M(\langle x_1\rangle^{n-1}, x_2) &= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} d(x_1, x_1) + \sum_{i=1}^{n-1} d(x_1, x_2) \\ &= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} 0 + \sum_{i=1}^{n-1} d(x_1, x_2) \\ &= (n-1)d(x_1, x_2) > 0. \end{aligned}$$

Proof of (n-sym): Follows from (n-sym).

Proof of (n-sep):
(⇐) If $x_1 = x_2$, then $M(\langle x_1\rangle^{n-1}, x_2) = \sum\limits_{t=2}^{n}\sum\limits_{i=1}^{t-1} d(x_1, x_1) = 0.$
(⇒) Now if $M(\langle x_1\rangle^{n-1}, x_2) = 0$, then

$$\sum_{t=2}^{n-1}\sum_{i=1}^{t-1} d(x_1, x_1) + \sum_{i=1}^{n-1} d(x_1, x_2) = 0,$$

and, hence,

$$(n-1)d(x_1, x_2) = 0$$

i.e.

$$d(x_1, x_2) = 0.$$

By (n-sep) we get that $x_1 = x_2$.

Proof of (n-inq): Using the definition of $M(\langle x_i\rangle_{i=1}^{n})$ and (n-inq) on $\sum\limits_{i=1}^{n-1} d(x_i, x_n)$ we get

$$M(\langle x_i\rangle_{i=1}^{n}) = \sum_{t=2}^{n}\sum_{i=1}^{t-1} d(x_i, x_t)$$

$$= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} d(x_i, x_t) + \sum_{i=1}^{n-1} d(x_i, x_n)$$

and, hence, by (d-inq)

$$\leq \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} d(x_i, x_t) + \sum_{i=1}^{n-1} (d(x_i, a) + d(a, x_n))$$

$$= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} d(x_i, x_t) + \sum_{i=1}^{n-1} d(x_i, a) + \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} d(a, a) + \sum_{i=1}^{n-1} d(a, x_n)$$

$$= M(\langle x_i\rangle_{i=1}^{n-1}, a) + M(\langle a\rangle^{n-1}, x_n). \qquad \square$$

2.5 G_p-Metric and Partial $n-\mathfrak{M}$etric

In **2011**, Zand and Nezhad [34] defined a function which they called a G_p-metric. A G_p- metric acts on triplets and is in fact a combination of the idea of partial metrics and $G-$metrics. A partial metric acts on pairs and has the property where the distance of a point to itself need not be zero. The $G-$metric on the other hand, acts on triplets but the distance of a triplet made up of the same point remains zero ($G(x, x, x) = 0$). As a generalization of both, the G_p-metric acts on triplets and the distance of a triplet made up of the same point need not be zero.

Definition 2.5.1. *A $\underline{G_p-\textbf{metric}}$ G_p on a set X is a function $G_p : X^3 \to \mathbb{R}$ satisfying the following axioms: For all $x, y, z, a \in X$,*
(G_p-lbnd): $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$.

(G_p-sym): $G_p(x,y,z) = G_p(\Pi\{x,y,z\})$, *where* Π *denotes a permutation on* $\{x,y,z\}$.
(G_p-sep): $G_p(x,y,z) = G_p(x,x,x) = G_p(y,y,y) = G_p(z,z,z) \iff x = y = z$.
(G_p-inq): $G_p(x,y,z) \leq G_p(x,y,a) + G_p(a,a,z) - G_p(a,a,a)$.

(G_p-lbnd) restricts G_p to have exclusively nonnegative values. (G_p-lbnd) combined with (G_p-sym) forces $G_p(x,x,y) \leq G_p(y,y,x)$ by taking $z = y$ and, hence, $G_p(x,x,y) = G_p(y,y,x)$. In the previous section, and while investigating the $G-$metric, we found (G-lbnd) to be too restrictive. We proposed the $n - \mathfrak{M}$etric such that (n-lbnd) of the $3 - \mathfrak{M}$etric was a weakening of (G-lbnd). We now proceed to generalize the $n - \mathfrak{M}$etric into what we call a partial $n - \mathfrak{M}$etric.

Definition 2.5.2. *A* ***partial*** $n - \mathfrak{M}$***etric*** *P on a set X is a function $P : X^n \to \mathbb{R}$ satisfying the following axioms:*
For all $(\langle x_i\rangle_{i=1}^n, a) \in X^{n+1}$,
(P_n-lbnd): $P(\langle x_1\rangle^n) \leq P(\langle x_1\rangle^{n-1}, x_2)$.
(P_n-sym): $P(\langle x_i\rangle_{i=1}^n) = P(\langle x_{\pi(i)}\rangle_{i=1}^n)$, *where* π *is a permutation on* $\{1, ..., n\}$.
(P_n-sep): $P(\langle x_1\rangle^{n-1}, x_2) = P(\langle x_1\rangle^n)$ *and* $P(\langle x_2\rangle^{n-1}, x_1) = P(\langle x_2\rangle^n) \iff x_1 = x_2$.
(P_n-inq): $P(\langle x_i\rangle_{i=1}^n) \leq P(\langle x_i\rangle_{i=1}^{n-1}, a) + P(\langle a\rangle^{n-1}, x_n) - P(\langle a\rangle^n)$.

Remark 2.5.3. *A G_p-metric is a special case of a partial $3 - \mathfrak{M}$etric. A partial $2 - \mathfrak{M}$etric is the same as a partial metric.*

The cornerstone of the above generalization is to allow $P(\langle x_1\rangle^n)$ to have non-zero values. The axioms were amended to flow with this adjustment. Note that in our proposed partial $3 - \mathfrak{M}$etric, our (P_n-lbnd) is weaker than the (G_p-lbnd), allowing P to have negative values. This weakening also allows $P(x,x,y)$ and $P(y,y,x)$ to be related only by the triangular inequality. i.e. $P(x,x,y)$ and $P(y,y,x)$ need not be equal.

We start by generalizing Theorem 2.4.1 to the partial $n - \mathfrak{M}$etric case.

Theorem 2.5.1. ***(Term Replacement):***
Let P be a partial $n - \mathfrak{M}$etric on a set X. For all $\langle x_i\rangle_{i=1}^n, \langle y_i\rangle_{i=1}^n \in X^n$ and for $t \in \{1, ..., n\}$,

$$P(\langle x_i\rangle_{i=1}^n) \leq P(\langle y_j\rangle_{j=1}^t, \langle x_i\rangle_{i=t+1}^n) + \sum_{j=1}^t [P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)].$$

Proof: Let $\langle x_i\rangle_{i=1}^n, \langle y_i\rangle_{i=1}^n \in X^n$. For $t = 1$, the result follows from (P_n-inq) and (P_n-sym).

Let $t \in \{2, ..., n-1\}$ and assume the inequality holds for $t - 1$. Then

$$P(\langle x_i\rangle_{i=1}^n) \leq P(\langle y_j\rangle_{i=1}^{t-1}, \langle x_i\rangle_{i=t}^n) + \sum_{j=1}^{t-1} [P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)]$$

by (P_n-sym)

$$= P(\langle y_j\rangle_{i=1}^{t-1}, \langle x_i\rangle_{i=t+1}^n, x_t) + \sum_{j=1}^{t-1} [P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)]$$

by (P_n-inq)

$$\leq P(\langle y_j\rangle_{i=1}^{t-1}, \langle x_i\rangle_{i=t+1}^{n}, y_t) + P(\langle y_t\rangle^{n-1}, x_t) - P(\langle y_t\rangle^n) + \sum_{j=1}^{t-1}[P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)]$$

by (P_n-sym)

$$= P(\langle y_j\rangle_{i=1}^{t}, \langle x_i\rangle_{i=t+1}^{n}) + \sum_{j=1}^{t}[P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)]. \qquad \square$$

Remark 2.5.4. *As does Theorem 2.4.1, Theorem 2.5.1 gives rise to useful replacement tools which are presented in the three corollaries below.*

Corollary 2.5.2. *Let P be a partial $n-\mathfrak{M}$etric on a set X. For all $\langle x_i\rangle_{i=1}^n, \langle y_i\rangle_{i=1}^n \in X^n$*

$$P(\langle x_i\rangle_{i=1}^n) \leq P(\langle y_i\rangle_{i=1}^n) + \sum_{j=1}^{n}[P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)].$$

Proof: This is the case of Theorem 2.5.1 when $t = n$. $\square$

Corollary 2.5.3. *Let P be a partial $n-\mathfrak{M}$etric on a set X. Then for $a, b \in X$ and $t \in \{1, ..., n\}$,*

$$P(\langle a\rangle^t, \langle b\rangle^{n-t}) \leq tP(\langle b\rangle^{n-1}, a) - (t-1)P(\langle b\rangle^n).$$

Proof: The proof follows from Corollary 2.5.2 where $\langle x_i\rangle_{i=1}^n = (\langle a\rangle^t, \langle b\rangle^{n-t})$ and $\langle y_j\rangle_{j=1}^n = \langle b\rangle^n$ we get

$$P(\langle a\rangle^t, \langle b\rangle^{n-t}) \leq P(\langle b\rangle^n) + \sum_{j=1}^{n}[P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)]$$

$$= P(\langle b\rangle^n) + \sum_{j=1}^{t}[P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)] + \sum_{j=t+1}^{n}[P(\langle y_j\rangle^{n-1}, x_j) - P(\langle y_j\rangle^n)]$$

$$= P(\langle b\rangle^n) + \sum_{j=1}^{t}[P(\langle b\rangle^{n-1}, a) - P(\langle b\rangle^n)] + \sum_{j=t+1}^{n}[P(\langle b\rangle^{n-1}, b) - P(\langle b\rangle^n)]$$

$$= P(\langle b\rangle^n) + t[P(\langle b\rangle^{n-1}, a) - P(\langle b\rangle^n)]$$

$$= tP(\langle b\rangle^{n-1}, a) - (t-1)P(\langle b\rangle^n). \qquad \square$$

The above corollary hints to as why we were able to choose such a simple (P_n-sep) axiom. A very useful corollary is when $t = n-1$.

Corollary 2.5.4. *Let P be a partial $n-\mathfrak{M}$etric on a set X. Then for $a, b \in X$,*

$$P(\langle a\rangle^{n-1}, b) \leq (n-1)P(\langle b\rangle^{n-1}, a) - (n-2)P(\langle b\rangle^n).$$

Proof: This is the case of Corollary 2.4.3 when $t = n-1$. $\square$

Similarly to the case of an $n-\mathfrak{M}$etric, every partial $n-\mathfrak{M}$etric on a set X induces a metric on X. Thus Theorem 2.4.5 is a special case of Theorem 2.5.5 where for every $x \in X$, $P(\langle x\rangle^n) = 0$.

Theorem 2.5.5. ***(Metric from Partial** $n-\mathfrak{Metric}$**):***
Let P be a partial $n-\mathfrak{Metric}$ on a set X. For $x, y \in X$ let

$$d(x,y) = P(y, \langle x\rangle^{n-1}) - P(\langle x\rangle^n) + P(x, \langle y\rangle^{n-1}) - P(\langle y\rangle^n).$$

Then d is a metric on the set X.

Proof: Let $x, y, z \in X$.

Proof of (m-bnd): From (P_n-lbnd) we have

$$P(\langle x\rangle^n) \leq P(y, \langle x\rangle^{n-1}) \text{ and } P(\langle y\rangle^n) \leq P(x, \langle y\rangle^{n-1}).$$

Therefore,

$$d(x,y) = P(y, \langle x\rangle^{n-1}) - P(\langle x\rangle^n) + P(x, \langle y\rangle^{n-1}) - P(\langle y\rangle^n) \geq 0.$$

Proof of (n-sym): Symmetry of d follows from the symmetry of addition of real numbers.

Proof of (n-sep):
($\Leftarrow$) From the definition of d we get

$$d(x,x) = P(\langle x\rangle^n) - P(\langle x\rangle^n) + P(\langle x\rangle^n) - P(\langle x\rangle^n) = 0.$$

($\Rightarrow$) If $d(x,y) = 0$, then

$$P(y, \langle x\rangle^{n-1}) - P(\langle x\rangle_1^n) + P(x, \langle y\rangle_1^{n-1}) - P(\langle y\rangle_1^n) = 0.$$

From (P_n-lbnd) we know that

$$P(y, \langle x\rangle^{n-1}) - P(\langle x\rangle_1^n) \geq 0 \text{ and } P(x, \langle y\rangle^{n-1}) - P(\langle y\rangle^n) \geq 0.$$

Therefore,

$$P(y, \langle x\rangle^{n-1}) - P(< x >^n) = 0 \text{ and } P(x, < y >^{n-1}) - P(< y >^n) = 0,$$

which means that

$$P(y, < x >^{n-1}) = P(< x >^n) \text{ and } P(x, < y >^{n-1}) = P(< y >^n).$$

From (P_n-sep) we deduce that $x = y$.

Proof of (n-inq): From (P_n-inq) and (P_n-sym) we get

$$d(x,y) = P(y, \langle x\rangle^{n-1}) - P(\langle x\rangle^n) + P(x, \langle y\rangle^{n-1}) - P(\langle y\rangle^n)$$

$$= P(\langle x\rangle^{n-1}, y) - P(\langle x\rangle^n) + P(\langle y\rangle^{n-1}, x) - P(\langle y\rangle^n)$$

$$\leq P(\langle x\rangle^{n-1}, z) + P(\langle z\rangle^{n-1}, y) - P(\langle z\rangle^n) - P(\langle x\rangle^n) + P(\langle y\rangle^{n-1}, z) + P(\langle z\rangle^{n-1}, x) - P(\langle z\rangle^n) - P(\langle y\rangle^n)$$

$$= P(x, \langle z\rangle^{n-1}) - P(\langle z\rangle^n) + P(z, \langle x\rangle^{n-1}) - P(\langle x\rangle^n) + P(y, \langle z\rangle^{n-1}) - P(\langle z\rangle^n) + P(z, \langle y\rangle^{n-1}) - P(\langle y\rangle^n)$$

$$= d(x, z) + d(z, y). \qquad \square$$

Example 2.5.5. ***($\{-1,1\}-$Discrete Partial $n-\mathfrak{M}$etric):***
Consider X to be any arbitrary set. Let $P : X^n \to \mathbb{R}$ be defined by:

For all $\langle x_i\rangle_{i=1}^n \in X^n, P(\langle x_i\rangle_{i=1}^n) = \begin{cases} -1 & \text{if } x_1 = x_2 = \ldots = x_n. \\ 1 & \text{otherwise.} \end{cases}$

Then P is a partial $n-\mathfrak{M}$etric on the set X.

Example 2.5.6. ***(Maximum Partial $n-\mathfrak{M}$etric):***
Consider the set X to be a subset of $\mathbb{R}$. For all $\langle x_i\rangle_{i=1}^n \in X$ let

$$P(\langle x_i\rangle_{i=1}^n) = \max\{x_i\}_{i=1}^n.$$

Then P is a partial $n-\mathfrak{M}$etric on the set X.

Proof: Let $(\langle x_i\rangle_{i=1}^n, a) \in X^{n+1}$.

Proof of (P_n-bnd): $a \leq \max\{a, x_1\}$ and, hence, $P(\langle a\rangle^n) \leq P(\langle a\rangle^{n-1}, x_1)$.

Proof of (P_n-sym): The maximum of a finite set does not change under the permutation of the set.

Proof of (P_n-sep):
($\Leftarrow$)is trivial.

($\Rightarrow$) If

$$P(\langle x_1\rangle^{n-1}, x_2) = P(\langle x_1\rangle^n) \text{ and } P(\langle x_2\rangle^{n-1}, x_1) = P(\langle x_2\rangle^n),$$

then

$$\max\{x_1, x_2\} = x_1 \text{ and } \max\{x_1, x_2\} = x_2.$$

Therefore, $x_1 = x_2$.

Proof of (P_n-inq): Without loss of generality and due to (P_n-sym), we may assume that $x_1 \leq x_2 \leq \ldots. \leq x_n$.
Hence, $P(\langle x_i\rangle_{i=1}^n) = x_n$. Three cases arise:
<u>Case 1</u>: Suppose $a \leq x_{n-1} \leq x_n$. Then

$$P(\langle x_i\rangle_{i=1}^{n-1}, a) + P(\langle a\rangle^{n-1}, x_n) - P(\langle a\rangle^n) = x_{n-1} + x_n - a \geq x_n = P(\langle x_i\rangle_{i=1}^n).$$

Case 2: Suppose $x_{n-1} \leq a \leq x_n$. Then

$$P(\langle x_i\rangle_{i=1}^{n-1}, a) + P(\langle a\rangle^{n-1}, x_n) - P(\langle a\rangle^n) = a + x_n - a = x_n = P(\langle x_i\rangle_{i=1}^{n}).$$

Case 3: Suppose $x_{n-1} \leq x_n \leq a$. Then

$$P(\langle x_i\rangle_{i=1}^{n-1}, a) + P(x_n, \langle a\rangle^{n-1}) - P(\langle a\rangle^n) = a + a - a = a \geq x_n = P(\langle x_i\rangle_{i=1}^{n}).$$

Therefore, for all $(\langle x_i\rangle_{i=1}^{n}, a) \in X^{n+1}$,

$$P(\langle x_i\rangle_{i=1}^{n}) \leq P(\langle x_i\rangle_{i=1}^{n-1}, a) + P(\langle a\rangle^{n-1}, x_n) - P(\langle a\rangle^n). \qquad \square$$

Theorem 2.5.6. ***(Partial** $n-\mathfrak{M}$**etric from a partial metric):***
Every partial metric p on a set X naturally defines a partial $n-$metric P on X as follows: For all $\langle x_i\rangle_{i=1}^{n} \in X^n$,

$$P(\langle x_i\rangle_{i=1}^{n}) = \sum_{t=2}^{n}\sum_{i=1}^{t-1} p(x_i, x_t).$$

Proof: Let $(\langle x_i\rangle_{i=1}^{n}, a) \in X^{n+1}$.

Proof of (P_n-lbnd): From the definition of P we get

$$P(\langle x_1\rangle^n) = \sum_{t=2}^{n}\sum_{i=1}^{t-1} p(x_1, x_1) = \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_1, x_1) + \sum_{i=1}^{n-1} p(x_1, x_1).$$

Now using (P_n-lbnd) we get

$$P(\langle x_1\rangle^n) \leq \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_1, x_1) + \sum_{i=1}^{n-1} p(x_1, x_2) = P(\langle x_1\rangle^{n-1}, x_2).$$

Proof of (P_n-sym): (P_n-sym) follows from the above and (p-sym)

Proof of (P-ineq): Using the definition of $P(\langle x_i\rangle_{i=1}^{n})$ we get

$$P(\langle x_i\rangle_{i=1}^{n}) = \sum_{t=2}^{n}\sum_{i=1}^{t-1} p(x_i, x_t) = \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_i, x_t) + \sum_{i=1}^{n-1} p(x_i, x_n)$$

by (p-inq)

$$\leq \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_i, x_t) + \sum_{i=1}^{n-1}(p(x_i, a) + p(a, x_n) - p(a, a))$$

$$= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_i, x_t) + \sum_{i=1}^{n-1} p(x_i, a) + \underline{\underline{0}} + \sum_{i=1}^{n-1} p(a, x_n) - \sum_{i=1}^{n-1} p(a, a)$$

$$= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_i, x_t) + \sum_{i=1}^{n-1} p(x_i, a) + \underline{\underline{\sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(a, a) - \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(a, a)}} + \sum_{i=1}^{n-1} p(a, x_n) - \sum_{i=1}^{n-1} p(a, a)$$

$$= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_i, x_t) + \sum_{i=1}^{n-1} p(x_i, a) + \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(a, a) + \sum_{i=1}^{n-1} p(a, x_n) - \underline{\underline{\underline{\sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(a, a) - \sum_{i=1}^{n-1} p(a, a)}}}$$

$$= \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(x_i, x_t) + \sum_{i=1}^{n-1} p(x_i, a) + \sum_{t=2}^{n-1}\sum_{i=1}^{t-1} p(a, a) + \sum_{i=1}^{n-1} p(a, x_n) - \underline{\underline{\underline{\sum_{t=2}^{n}\sum_{i=1}^{t-1} p(a, a)}}}$$

$$= P(\langle x_i \rangle^{n-1}, a) + P(\langle a \rangle^{n-1}, x_n) - P(\langle a \rangle^n). \qquad \square$$

2.6 Strong Partial $n - \mathfrak{M}$etric

In Section 2.3 we presented the strong partial metric, a generalized metric aimed at simulating scoring schemes set up to compare two finite sequences.

In bio-informatics [32], we have scoring schemes which allow us to align and compare multiple DNA strands at the same time. We call those types of schemes multiple sequence alignment schemes. That is why in **2015** [2], we introduced a stronger version of the partial $n-\mathfrak{M}$etric by combining (P_n-lbnd) and (P_n-sep) into a stronger axiom. The result is a generalized metric capable of emulating a multiple sequence alignment schemes for a set of finite sequences generated from a finite alphabet. We called it a strong partial $n-\mathfrak{M}$etric.

Definition 2.6.1. *A **strong partial** $n - \mathfrak{M}$**etric** S on a set X is a function $S : X^n \to \mathbb{R}$ satisfying the following axioms:*
For all $(\langle x_i \rangle_{i=1}^n, a) \in X^{n+1}$,
(S_n-lbnd): $S(\langle x_1 \rangle^n) < S(\langle x_1 \rangle^{n-1}, x_2)$.
(S_n-sym): $S(\langle x_i \rangle_{i=1}^n) = S(\langle x_{\pi(i)} \rangle_{i=1}^n)$, *where π is a permutation on* $\{1, ..., n\}$.
(S_n-inq): $S(\langle x_i \rangle_{i=1}^n) \leq S(\langle x_i \rangle_{i=1}^{n-1}, a) + S(\langle a \rangle^{n-1}, x_n) - S(\langle a \rangle^n)$.

Remark 2.6.2. *A strong partial $2 - \mathfrak{M}$etric is a strong partial metric.*

Notice that an (S_n-sep) is hidden in (S_n-lbnd) as

$$S(\langle x_1 \rangle^{n-1}, x_2) = S(\langle x_1 \rangle^n) \iff x_1 = x_2.$$

Clearly, a strong partial $n-\mathfrak{M}$etric S on a set X is a partial $n-\mathfrak{M}$etric on X. Hence, as in Theorem 2.5.5, S induces a metric d on X defined as follows:
For all $x, y \in X$

$$d(x, y) = S(y, \langle x \rangle^{n-1}) - S(\langle x \rangle^n) + S(x, \langle y \rangle^{n-1}) - S(\langle y \rangle^n).$$

Example 2.6.3. ***(Shifted** $n - \mathfrak{M}$**etric):***
Let $M : X^n \to \mathbb{R}$ be an $n - \mathfrak{M}$etric defined on a set X. For any $r \in \mathbb{R}$, let $S_r : X^n \to \mathbb{R}$ be defined by:

$$S_r(\langle x_i \rangle_{i=1}^n) = M(\langle x_i \rangle_{i=1}^n) + r.$$

Then S_r is a strong partial $n - \mathfrak{M}$etric on the set X.

Proof: Let $(\langle x_i \rangle_{i=1}^n, a) \in X^{n+1}$.

Proof of (S_n-lbnd): For $a \neq x_1$,

$$S_r(\langle a \rangle^n) = M(\langle a \rangle^n) + r = r.$$

From (n-lbnd) and (n-sep) and since $a \neq x_1$ we get

$$M(\langle a \rangle^{n-1}, x_1) > 0.$$

Hence,

$$S_r(\langle a \rangle^n) = r < M(\langle a \rangle^{n-1}, x_1) + r = S_r(\langle a \rangle^{n-1}, x_1).$$

Proof of (S_n-sym): Follows directly from (n-sym).

Proof of (S_n-inq): From (n-inq) we know that

$$M(\langle x_i \rangle_{i=1}^n) \leq M(\langle x_i \rangle_{i=1}^{n-1}, a) + M(\langle a \rangle^{n-1}, x_n)$$
$$= M(\langle x_i \rangle_{i=1}^{n-1}, a) + M(\langle a \rangle^{n-1}, x_n) - M(\langle a \rangle^n).$$

By (n-sep), $M(\langle a \rangle^n) = 0$. Therefore,

$$S_r(\langle x_i \rangle_{i=1}^n) = M(\langle x_i \rangle_{i=1}^n) + r$$
$$= M(\langle x_i \rangle_{i=1}^n) + 2r - r - M(\langle a \rangle^n)$$
$$\leq M(\langle x_i \rangle_{i=1}^{n-1}, a) + M(\langle a \rangle^{n-1}, x_n) + 2r - r - M(\langle a \rangle^n)$$
$$= [M(\langle x_i \rangle_{i=1}^{n-1}, a) + r] + [M(\langle a \rangle_1^{n-1}, x_n) + r] - [M(\langle a \rangle_{i=1}^n) + r]$$
$$= S_r(\langle x_i \rangle_{i=1}^{n-1}, a) + S_r(\langle a \rangle^{n-1}, x_n) - S_r(\langle a \rangle^n). \qquad \square$$

Theorem 2.6.1. ***(Strong partial*** $n - \mathfrak{Metric}$ ***from a strong partial metric):*** *Every strong partial metric s on a set X naturally defines a strong partial $n-$metric S on X as follows: For all $\langle x_i \rangle_{i=1}^n \in X^n$,*

$$S(\langle x_i \rangle_{i=1}^n) = \sum_{t=2}^{n} \sum_{i=1}^{t-1} s(x_i, x_t).$$

Proof: For the proof of (S_n-sym) and (S_n-inq), please refer to Example 2.5.5.

Proof of (S_n-lbnd): Let a and x_1 be two distinct elements of X. Using the definition of $S(\langle a \rangle^n)$ we get

$$S(\langle a \rangle^n) = \sum_{t=2}^{n} \sum_{i=1}^{t-1} s(a, a) = \sum_{t=2}^{n-1} \sum_{i=1}^{t-1} s(a, a) + \sum_{i=1}^{n-1} s(a, a).$$

By (s-lbnd) $s(a, a) < s(a, x_1)$ and, hence,

$$S(\langle a \rangle^n) < \sum_{t=2}^{n-1} \sum_{i=1}^{t-1} s(a, a) + \sum_{i=1}^{n-1} s(a, x_1) = S(\langle a \rangle^{n-1}, x_1). \qquad \square$$

The above example is a general form of a multiple sequence alignment scheme [32], since each pairwise alignment scheme is a strong partial metric. (see Example 2.3.9.)

Chapter 3

Topology

In **1914**, Hausdorff [19] used set theory to generalize a Euclidean space while retaining concepts such as continuity, convergence and connectedness. He used the metric defined by Fréchet [15] (see Definition 2.1.1) to generate a topological space called a metric space. The definitions and lemmas in this subsection and in Section 3.1, with the exception of Definition 3.1.4, can be found in [26] and every book on undergraduate topology.

Definition 3.0.1. *A **topology** $\mathcal{T}$ on a set X is a subset of $\mathcal{P}(X)$ satisfying the following axioms:*
(R1): $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
(R2): If $\{O_i\}_{i\in\mathcal{I}} \subseteq \mathcal{T}$ then $\bigcup\limits_{i\in\mathcal{I}} O_i \in \mathcal{T}$. I.e. an arbitrary union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
(R3): If $\{O_i\}_{i=1}^{n} \subseteq \mathcal{T}$ and n is a positive integer, then $\bigcap\limits_{i=1}^{n} O_i \in \mathcal{T}$. I.e. a finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

*We call the pair $(X, \mathcal{T})$ a **topological space**.*

Definition 3.0.2. *The **open sets** of a topological space $(X, \mathcal{T})$ are the elements of $\mathcal{T}$. While the **closed sets** are the complement of the elements of $\mathcal{T}$. I.e.*

$$O \in X \text{ is called an } \underline{\textbf{open set}} \text{ of } X \iff O \in \mathcal{T}.$$
$$F \in X \text{ is called a } \underline{\textbf{closed set}} \text{ of } X \iff X - F \in \mathcal{T}.$$

Definition 3.0.3. *Let $(X, \mathcal{T})$ be a topological space. For each x in X, every open set containing x is called an **open neighborhood** of x.*

Lemma 3.0.1. *Let $(X, \mathcal{T})$ be a topological space and the set $U \subseteq X$. U is an open set if and only if for each $x \in U$ there exists an open neighborhood V_x of x such that $V_x \subseteq U$.*

When considering a topological space one usually does not need to deal with all open sets. Lemma 3.0.1 makes it sufficient to deal with a sub-collection of open sets called a basis. A key feature of a basis of a topology is that it can be used to generate all open sets [31].

Definition 3.0.4. *A* ***basis*** $\mathcal{B}$ *on a set* X *is a subset of* $\mathcal{P}(X)$ *satisfying the following axioms:*

(B1): For all $x \in X$*, there exists* $B \in \mathcal{B}$ *such that* $x \in B$*.*

(B2): For all $B_1, B_2 \in \mathcal{B}$ *and for each element* $x \in B_1 \cap B_2$*, there exists* $B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subseteq B_1 \cap B_2$*.*

Given a basis $\mathcal{B}$ on a set X, let $\mathcal{T}_{\mathcal{B}} = \{\bigcup \mathcal{C} \mid \mathcal{C} \in \mathcal{B}\} \bigcup \{\emptyset\}$. Then $\mathcal{T}_{\mathcal{B}}$ is a topology on X [26]. We say that $\mathcal{T}_{\mathcal{B}}$ is the topology generated by $\mathcal{B}$.

In the case of metric spaces the following well-known technique is used [25] to prove that a certain subset of $\mathcal{P}(X)$ is a basis.

Lemma 3.0.2. *Given a set* X *and a function* $b : \mathbb{R}^{>0} \times X \to \mathcal{P}(X)$ *satisfying the conditions:*
(b_1)*: For all* $x \in X$ *and for each* $\alpha \in \mathbb{R}^{>0}$*,* $x \in b(\alpha, x)$*.*
(b_2)*: For all* $x \in X$ *and for all positive real numbers* $\alpha \leq \beta$*,* $b(\alpha, x) \subseteq b(\beta, x)$*.*
(b_3)*: For all* $x, y \in X$ *and* $\alpha \in \mathbb{R}^{>0}$ *such that* $y \in b(\alpha, x)$*, there exists a positive real number* ϵ *such that* $b(\epsilon, y) \subseteq b(\alpha, x)$*.*
Then $b(\mathbb{R}^{>0} \times X)$ *is a basis on* X*.*

Definition 3.0.5. *Let* $(X, \mathcal{T})$ *be a topological space and a set* $A \subseteq X$*. The* ***closure*** *of* A*, denoted* $\bar{A}$ *or* $Cl(A)$*, is the intersection of all closed sets containing* A*.*

Lemma 3.0.3. *Let* $(X, \mathcal{T})$ *be a topological space and a set* $A \subseteq X$*.* $Cl(A)$ *is the smallest closed set containing* A*.*

Lemma 3.0.4. *Let* $(X, \mathcal{T})$ *be a topological space and a set* $A \subseteq X$*.* $x \in Cl(A)$ *if and only if every open neighborhood of* x *intersects* A*.*

To prove Banach-type fixed point theorems on a topological space, we need to be able to topologically distinguish distinct points.

Definition 3.0.6. *Let* $(X, \mathcal{T})$ *be a topological space. We say that* $(X, \mathcal{T})$ *is* $\underline{\mathbf{T_0}}$ *if for every two distinct elements* x *and* y *of* X*, there exists an open set* U *in* $\mathcal{T}$ *such that either*

$$[x \in U \text{ and } y \notin U] \ \underline{\textbf{\textit{or}}}\ [x \notin U \text{ and } y \in U].$$

Definition 3.0.7. *Let* $(X, \mathcal{T})$ *be a topological space. We say that* $(X, \mathcal{T})$ *is* $\underline{\mathbf{T_1}}$ *if for every two distinct elements* x *and* y *of* X*, there exist two open sets* U *and* V *in* $\mathcal{T}$ *such that*

$$[x \in U \text{ and } y \notin U] \ \underline{\textbf{\textit{and}}}\ [x \notin V \text{ and } y \in V].$$

Definition 3.0.8. *Let* $(X, \mathcal{T})$ *be a topological space. We say that* $(X, \mathcal{T})$ *is* $\underline{\mathbf{T_2}}$ *(or* ***Hausdorff****) if for every two distinct elements* x *and* y *of* X*, there exists two open sets* U *and* V *in* $\mathcal{T}$ *such that*

$$x \in U,\ y \in V \text{and } U \cap V = \emptyset.$$

Clearly if a topological space is T_2 then it is T_1 and if a topological space is T_1 then it is T_0 [26].

Definition 3.0.9. *Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is* ***first countable*** *if for every $x \in X$, there exists a countable collection $\mathcal{B}_x = \{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{T}$ such that for each open set U containing x there is an index $i \in \mathbb{N}$ with $x \in B_i \subseteq U$. We call $\mathcal{B}_x$ a countable local basis at x.*

Finally we introduce the notion of "partial ordering" on topologies of on a particular domain.

Definition 3.0.10. *Let $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ be two topological spaces.*
We say $\mathcal{T}$ is ***coarser*** *than $\mathcal{T}'$ or $\mathcal{T}'$ is* ***finer*** *than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. Alternatively, we may denote it as $(X, \mathcal{T})$ is coarser than $(X, \mathcal{T}')$ or $(X, \mathcal{T}')$ is finer than $(X, \mathcal{T})$.*

In the sections to follow, we will use the generalized metrics defined in Chapter 2 to define topologies, thus, giving us the structure needed for the study of convergence.

3.1 Metric Space

Definition 3.1.1. *Let d be a metric on a set X. For each element $x \in X$ and positive real number ϵ, the* ***d-open ball*** *around x of radius ϵ is*

$$B_\epsilon^d(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

Lemma 3.1.1. *Let d be a metric on a set X. The collection of all d-open balls on X, $\mathcal{B}^d = \{B_\epsilon^d(x)\}_{x \in X}^{\epsilon \in \mathbb{R}^{>0}}$ forms a basis on X.*

Definition 3.1.2. *Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is a* ***metric space*** *if there exists a metric d on X such that*

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^d}.$$

(See Definition 3.0.4.)

Notation 3.1.3. *Let d be a metric on a set X. We denote the topological space $(X, \mathcal{T}_{\mathcal{B}^d})$ by (X, d).*

Lemma 3.1.2. *Every metric space $(X, \mathcal{T})$ is first countable.*

Lemma 3.1.3. *Every metric space (X, d) is T_2.*

Definition 3.1.4. *Let (X, d) be a metric space. For each element $x \in X$ and positive real number ϵ, the* ***d-gilded ball*** *around x of radius ϵ is*

$$\tilde{B}_\epsilon^d(x) = \{y \in X \mid d(x, y) \leq \epsilon\}.$$

Lemma 3.1.4. *If (X, d) is a metric space then every d-gilded ball is closed in (X, d).*

Remark 3.1.5. *In the literature, a d-gilded ball is referred to as a d-closed ball. When discussing a partial metric p, a p-gilded ball need not be a closed set in the relevant topology. Hence, we chose to change the usual name to avoid ambiguity.*

3.2 Partial Metric and Strong Partial Metric Space

Although other possible definitions were given for an open ball relative to a partial metric [25], we felt that O'Neill's definition [29] is the most natural generalization of a d-open ball into the partial metric case.

Definition 3.2.1. *Let p be a partial metric on a set X. For each element $x \in X$ and positive real number ϵ, the* ***p-open ball*** *around x of radius ϵ is*

$$B^p_\epsilon(x) = \{y \in X \mid p(x,y) - p(x,x) < \epsilon\}.$$

Matthews ([7]: Definition 13) noted that

$$B^{p\star}_\epsilon(x) = \{y \in X \mid p(x,y) - p(y,y) < \epsilon\}$$

is another possible definition which may give us a totally different topology. He also presented us with Lemma 3.2.1 which is in a fact a special case of our Lemma 3.4.1.

Lemma 3.2.1. *Let p be a partial metric on a set X. The collection of all p-open balls on X, $\mathcal{B}^p = \{B^p_\epsilon(x)\}^{\epsilon \in \mathbb{R}^{>0}}_{x \in X}$ forms a basis on X.*

Proof: Use Lemma 3.0.2.

Definition 3.2.2. *Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is a* ***(strong) partial metric space*** *if there exists a (strong) partial metric p on X such that*

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^p}.$$

(See Definition 3.0.4.)

Notation 3.2.3. *Let p be a partial metric on a set X. We denote the topological space $(X, \mathcal{T}_{\mathcal{B}^p})$ by (X, p).*

Lemma 3.2.2. *Every partial metric space $(X, \mathcal{T})$ is first countable.*

Proof: Trivial.

Lemma 3.2.3. *(Matthews [7]) Every partial metric space $(X, \mathcal{T})$ is T_0.*

Proof: Let $(X, \mathcal{T})$ be a partial metric space. By Definition 3.2.2, let p be a partial metric on X such that

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^p}.$$

Consider two distinct elements x and y in X. From (p-lbnd) and (p-sep) we get

$$p(x,y) > p(x,x) \text{ or } p(x,y) > p(y,y).$$

Then

$$\epsilon_x = p(x,y) - p(x,x) > 0 \text{ or } \epsilon_y = p(x,y) - p(y,y) > 0.$$

Without any loss of generality we may assume that $\epsilon_x > 0$. Hence, $B^p_{\epsilon_x}$ exists. Additionally

$$x \in B^p_{\epsilon_x}(x) \text{ and } y \notin B^p_{\epsilon_x}(x) \qquad \square$$

Note that a partial metric space need not be T_1.

Example 3.2.4. ***(A partial metric space that is not T_1):***
Let p be a partial metric on $X = \mathbb{R} \cup \{a\}$ where $a \notin \mathbb{R}$ as defined in Example 2.2.4 by

$$p(a,a) = 0, p(a,x) = |x| \textit{ and } p(x,y) = |x-y| - 1.$$

Then (X,p) is not T_1.

Proof: We know that $a \neq 0$ and for every $\epsilon \in \mathbb{R}^{>0}$,

$$p(a,0) - p(a,a) = 0 - 0 = 0 < \epsilon.$$

Hence, $0 \in B^p_\epsilon(a)$. $\square$

Lemma 3.2.4. *Every strong partial metric space (X,s) is T_1.*

Proof: Let $(X,\mathcal{T})$ be a strong partial metric space. By Definition 3.2.2, let s be a strong partial metric on X such that

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^s}.$$

Consider two distinct elements x and y in X. From (s-lbnd) we get

$$s(x,y) > s(x,x) \text{ and } s(x,y) > s(y,y).$$

Let

$$\epsilon_x = s(x,y) - s(x,x) > 0 \text{ and } \epsilon_y = s(x,y) - s(y,y) > 0.$$

Then,

$$x \in B^s_{\epsilon_x}(x) \ , \ y \notin B^s_{\epsilon_x}(x) \ , \ y \in B^s_{\epsilon_y}(y) \ , \text{ and } x \notin B^s_{\epsilon_y}(y). \qquad \square$$

Note that a strong partial metric space need not be T_2.

Example 3.2.5. ***(A strong partial metric space that is not T_2):***
Let s be the strong partial metric on $X = (0,+\infty)$ defined in Example 2.3.4 by

$$s(x,y) = \begin{cases} x & \textit{if } x = y. \\ x + y & \textit{if } x \neq y. \end{cases}$$

Then (X,s) is not T_2.

Proof: For each element $x \in X$ and positive real number ϵ,

$$B_\epsilon^s(x) = \{y \in X \mid s(x,y) - s(x,x) < \epsilon\} = \{x\} \cup \{y \in X \mid y < \epsilon\}.$$

Consider two distinct elements x and y in X and two positive real numbers ϵ and δ.
Let $z < \min\{\epsilon, \delta\}$ then $z \in B_\epsilon^s(x) \cap B_\delta^s(y)$. □

Definition 3.2.6. *Let (X,p) be a partial metric space. For each element $x \in X$ and positive real number ϵ, the **p-gilded ball** around x of radius ϵ is*

$$\tilde{B}_\epsilon^p(x) = \{y \in X \mid p(x,y) - p(x,x) \leq \epsilon\}.$$

The next example shows that if p is a partial metric on a set X, a p-gilded ball need not be closed in (X,p).

Example 3.2.7. ***(p-gilded but not closed ball):***
Let s be the strong partial metric on $X = (0, +\infty)$ defined in Example 2.3.4 by

$$s(x,y) = \begin{cases} x & \text{if } x = y. \\ x + y & \text{if } x \neq y. \end{cases}$$

Then, for each $x \in X$ and positive real number ϵ, $\tilde{B}_\epsilon^s(x)$ is not closed in (X,s).

Proof: Let $x \in X$ and ϵ a positive real number. Then

$$\tilde{B}_\epsilon^s(x) = \{y \in X \mid y \leq \epsilon\} \cup \{x\}.$$

For each $y \in X$ and every positive real number δ,

$$\emptyset \neq (0, \min\{\epsilon, \delta\}) \subseteq B_\delta^s(y) \cap \tilde{B}_\epsilon^s(x).$$

Hence, y is in the closure of $\tilde{B}_\epsilon^s(x)$. I.e. the closure of $\tilde{B}_\epsilon^s(x)$ is X and, therefore, $\tilde{B}_\epsilon^s(x)$ is not closed in (X,s). □

3.3 $G-$metric and $n - \mathfrak{M}$etric Space

Mustapha and Sims [28] defined a topology on a set X having a $G-$metric (see Definition 2.4.1). Later Khan [23] used the $K-$metric, a stronger version of our $n-\mathfrak{M}$etric (see Definition 2.4.2), on a set X to define a topology on X. He called this topological space a $K-$metric space [2].

In this section we will develop the $n - \mathfrak{M}$etric space, which is in fact a generalization of the $G-$metric space and the $K-$metric space. We will also prove that an $n-\mathfrak{M}$etric space is simply a metric space. Hence, we will be dropping $n - \mathfrak{M}$etrics as of Chapter 4.

Definition 3.3.1. *Let M be an $n-\mathfrak{M}$etric on a set X. For each element $x \in X$ and positive real number ϵ, the **M-open ball** around x of radius ϵ is*

$$B_\epsilon^M(x) = \{y \in X \mid M(\langle x\rangle^{n-1}, y) < \epsilon\}.$$

Note that this is not the only possible definition. For example

$$B_\epsilon^{M\star}(x) = \{y \in X \mid M(\langle y\rangle^{n-1}, x)) < \epsilon\}$$

is another possible definition but which will still generate the same metric topology.

Lemma 3.3.1. *Let M be an $n-\mathfrak{M}$etric on a set X. The collection of all M-open balls on X, $\mathcal{B}^M = \{B_\epsilon^M(x)\}_{x\in X}^{\epsilon\in\mathbb{R}^{>0}}$ forms a basis on X.*

Proof: Consider the function $b : \mathbb{R}^{>0} \times X \to \mathcal{P}(X)$ given by $b(\epsilon, x) = B_\epsilon^M(x)$. We now check that b satisfies the conditions of Lemma 3.0.2.
(b_1): For every $x \in X$, by (M-sep) we know that $M(\langle x\rangle^{n-1}, x) = 0$. Hence, for each positive real number ϵ, $x \in B_\epsilon^M(x)$.
(b_2): Clearly, if $0 < \delta \leq \epsilon$ then $B_\delta^M(x) \subseteq B_\epsilon^M(x)$ for each $x \in X$ by the transitivity of the order on $\mathbb{R}$. Hence, $b(\delta, x) \subseteq b(\epsilon, x)$.
(b_3): For every $x \in X$ and $y \in B_\epsilon^M(x)$, we know that $M(\langle x\rangle^{n-1}, y) < \epsilon$. Let $\delta = \epsilon - M(\langle x\rangle^{n-1}, y)$. For each element $z \in B_\delta^M(y)$ (i.e $M(\langle y\rangle^{n-1}, z) < \delta$), by (M-inq) we get

$$M(\langle x\rangle^{n-1}, z) \leq M(\langle x\rangle^{n-1}, y) + M(\langle y\rangle^{n-1}, z) < M(\langle x\rangle^{n-1}, y) + \delta = \epsilon$$

and, hence,

$$M(\langle x\rangle^{n-1}, z) < \epsilon.$$

Thus, $z \in B_\epsilon^M(x)$ and $B_\delta^M(y) \subseteq B_\epsilon^M(x)$. $\square$

Definition 3.3.2. *Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is an **$n-\mathfrak{M}$etric space** if there exists an $n-\mathfrak{M}$etric M on X such that*

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^M}.$$

Notation 3.3.3. *Let M be an $n-\mathfrak{M}$etric on a set X. We denote the topological space $(X, \mathcal{T}_{\mathcal{B}^M})$ by (X, M).*

Theorem 3.3.2. *Every $n-\mathfrak{M}$etric space $(X, \mathcal{T})$ is a metric space.*

Proof: Let $(X, \mathcal{T})$ be an $n-\mathfrak{M}$etric space. By Definition 3.4.2, let M be an $n-\mathfrak{M}$etric on X such that

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^M}.$$

In Theorem 2.4.5 we proved that the function

$$d(x, y) = M(\langle x\rangle^{n-1}, y) + M(\langle y\rangle^{n-1}, x)$$

is a metric on X.

First we prove that (X,d) is finer than (X,M). For each element $x \in X$ and $y \in B^d_\epsilon(x)$ we know that

$$d(x,y) < \epsilon.$$

I.e.

$$M(\langle x \rangle^{n-1}, y) + M(x, \langle y \rangle^{n-1}) < \epsilon.$$

From (M-lbnd) we know that

$$M(x, \langle y \rangle^{n-1}) \geq 0$$

and, hence,

$$M(y, \langle x \rangle^{n-1}) \leq M(\langle x \rangle^{n-1}, y) + M(x, \langle y \rangle^{n-1}) < \epsilon.$$

Thus, $y \in B^M_\epsilon(x)$ and, as a result, $B^d_\epsilon(x) \subseteq B^M_\epsilon(x)$.

Similarly we prove that (X,M) is finer than (X,d). For each element $x \in X$ and $y \in B^M_\epsilon(x)$ we know that

$$M(\langle x \rangle^{n-1}, y) < \epsilon.$$

Let $\delta = \frac{\epsilon}{n}$. For each element $y \in B^M_\delta(x)$ we know that

$$M(\langle x \rangle^{n-1}, y) < \delta.$$

From Corollary 2.4.4 we get

$$M(x, \langle y \rangle^{n-1}) \leq (n-1)M(\langle x \rangle^{n-1}, y).$$

Hence,

$$\begin{aligned} d(x,y) &\leq M(\langle x \rangle^{n-1}, y) + (n-1)M(\langle x \rangle^{n-1}, y) \\ &= nM(\langle x \rangle^{n-1}, y) < n\delta = \epsilon. \end{aligned}$$

Thus, $y \in B^d_\epsilon(x)$ and $B^M_\delta(x) \subseteq B^d_\epsilon(x)$. □

3.4 Partial $n - \mathfrak{M}$etric and Strong Partial $n - \mathfrak{M}$etric Space

As mentioned in Chapter 2, a partial $n-\mathfrak{M}$etric is a generalization of a partial metric and an $n-\mathfrak{M}$etric. Unlike in the chapters to come, no special techniques are needed other than the ones presented in Section 3.2 and Section 3.3. Hence, we will present our statements without proofs.

Definition 3.4.1. *Let P be a partial $n - \mathfrak{M}$etric on a set X. For each element $x \in X$ and positive real number ϵ, the* ***P-open ball*** *around x of radius ϵ is*

$$B^P_\epsilon(x) = \{y \in X \mid P(\langle x \rangle^{n-1}, y) - P(\langle x \rangle^n) < \epsilon\}.$$

Note that this is not the only possible definition. For example

$$B_\epsilon^{P\star}(x) = \{y \in X \mid P(\langle y\rangle^{n-1}, x) - P(\langle y\rangle^n) < \epsilon\}$$

is another possible definition which may give us a totally different topology.

Lemma 3.4.1. *Let P be a partial $n - \mathfrak{M}$etric on a set X. The collection of all P-open balls on X, $\mathcal{B}^P = \{B_\epsilon^P(x)\}_{x\in X}^{\epsilon\in\mathbb{R}^{>0}}$ forms a basis on X.*

Definition 3.4.2. *Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is a **(strong) partial $n - \mathfrak{M}$etric space** if there exists a (strong) partial $n - \mathfrak{M}$etric P on X such that*

$$\mathcal{T} = \mathcal{T}_{\mathcal{B}^P}.$$

Notation 3.4.3. *Let P be a partial $n - \mathfrak{M}$etric on a set X. We denote the topological space $(X, \mathcal{T}_{\mathcal{B}^P})$ by (X, P).*

Lemma 3.4.2. *Every partial $n - \mathfrak{M}$etric space $(X, \mathcal{T})$ is first countable.*

Lemma 3.4.3. *Every partial $n - \mathfrak{M}$etric space $(X, \mathcal{T})$ is T_0.*

Note that a partial $n - \mathfrak{M}$etric space need not be T_1.

Example 3.4.4. ***(A partial $n - \mathfrak{M}$etric space that is not T_1):***

For $n = 2$, a partial $n - \mathfrak{M}$etric space is simply a partial metric space. In Example 3.2.4 we gave an example of a partial metric space that is not T_1.

Lemma 3.4.4. *Every strong partial $n - \mathfrak{M}$etric space $(X, \mathcal{T})$ is T_1.*

Note that a strong partial $n - \mathfrak{M}$etric space need not be T_2.

Example 3.4.5. ***(A strong partial $n - \mathfrak{M}$etric space that is not T_2):***

For $n = 2$, a strong partial $n - \mathfrak{M}$etric space is a strong partial metric space. In Example 3.2.5 we gave an example of a strong partial metric space that is not T_2.

Definition 3.4.6. *Let (X, P) be a partial $n - \mathfrak{M}$etric space. For each element $x \in X$ and positive real number ϵ, the **P-gilded ball** around x of radius ϵ is*

$$\tilde{B}_\epsilon^P(x) = \{y \in X \mid P(\langle x\rangle^{n-1}, y) - P(\langle x\rangle^n) \leq \epsilon\}.$$

Example 3.4.7. ***(P-gilded but not closed ball):***

For $n = 2$, a strong partial $n - \mathfrak{M}$etric space is a strong partial metric space. In Example 3.2.7 we gave an example of an s-gilded ball that is not closed.

Chapter 4

Sequences and Limits

Definition 4.0.1. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence in a topological space* $(X, \mathcal{T})$*. We say that* a *is a* ***limit*** *of* $\{x_i\}_{i\in\mathbb{N}}$ *if and only if for every open neighborhood* U *of* a*, there exists a natural number* N *such that for all* $i > N$*,* $x_i \in U$*.*

Chapter 4 deals with limits, Cauchy sequences and Cauchy pairs. As mentioned in Chapter 2, metrics are a special case of partial metrics which in turn are a special case of partial $n - \mathfrak{M}$etrics. That is why, in Section 4.1 we will be presenting the results without proofs. In Section 4.2, we supply the proofs because they are considerably simpler than their counterparts in Section 4.3. Additionally for many users, we expect the level of generality found in Section 4.2 to suffice.

4.1 Metric Space

The proofs for this section are readily available in [26].

Definition 4.1.1. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence in a metric space* (X, d)*. We say that the sequence* $\{x_i\}_{i\in\mathbb{N}}$ *is* ***Cauchy*** *if and only if for each positive real number* ϵ *there exists a natural number* N *such that for all* $i \geq j > N$*,*

$$d(x_i, x_j) < \epsilon.$$

Merging Definition 4.0.1 and Definition 4.1.1, we can now identify a limit by analyzing distances.

Lemma 4.1.1. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence in a metric space* (X, d)*. A point* a *in* X *is a limit of* $\{x_i\}_{i\in\mathbb{N}}$ *if and only if for each positive real number* ϵ *there exists a natural number* N *such that for all* $i \geq j > N$*,*

$$d(a, x_i) < \epsilon.$$

Lemma 4.1.2. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a Cauchy sequence in a metric space* (X, d)*. If* $\{x_i\}_{i\in\mathbb{N}}$ *has a limit in* X *then that limit is unique.*

Definition 4.1.2. *Let* (X, d) *be a metric space. We say that* (X, d) *is* ***complete*** *if and only if every Cauchy sequence in* X *has a limit in* X*.*

Suppose we now have two different sequences in a topological space. The study of their respective terms and whether or not they get, in some sense, closer to each other proves to be quite useful in common fixed point theorems.

Definition 4.1.3. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be sequences in a metric space (X,d). We say that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a **Cauchy pair** if and only if for each positive real number ϵ there exists a natural number N such that for all $i,j>N$,*

$$d(x_i,y_j)<\epsilon.$$

Theorem 4.1.3. ***(Cauchy pair term comparison):***
Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences in a metric space (X,d). The statements below are equivalent.
(a) $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a Cauchy pair.
(b) For every positive real number ϵ there exists a natural number N such that for all $i\geq j>N$,

$$d(x_i,y_j)<\epsilon.$$

(c) For every positive real number ϵ there exists a natural number N such that for all $i\geq j>N$,

$$d(x_j,y_i)<\epsilon.$$

When we introduce a Cauchy pair, not only are we considering a pair of sequences whose corresponding terms get eventually arbitrarily close, but also the sequences are Cauchy.

Lemma 4.1.4. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be a Cauchy pair in a metric space (X,d). Then $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Additionally, if a is a limit of $\{x_i\}_{i\in\mathbb{N}}$ then a is also a limit of $\{y_i\}_{i\in\mathbb{N}}$.*

4.2 Partial Metric Space

Matthews [25] generalized the notion of a Cauchy sequence to partial metrics. Even though Matthews' work was restricted to non-negative values in $\mathbb{R}$, we will use his generalization of Cauchy sequences when considering O'Neill's definition [29] of partial metrics.

Definition 4.2.1. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial metric space (X,p). We say that the sequence $\{x_i\}_{i\in\mathbb{N}}$ is **Cauchy** if and only if there exists a real number r such that for each positive real number ϵ there exists a natural number N where for all $i\geq j>N$,*

$$-\epsilon<p(x_i,x_j)-r<\epsilon.$$

*We call r the **central distance** of $\{x_i\}_{i\in\mathbb{N}}$.*

Notice that from Definition 4.2.1, $p(x_i,x_i)$ tends to r. Hence, in the metric case the central distance r of a Cauchy sequence is 0 and coincides with Definition 4.1.1.

From Definition 4.0.1 we get the lemma below.

Lemma 4.2.1. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial metric space (X,p). A point a in X is a limit of $\{x_i\}_{i\in\mathbb{N}}$ if and only if for each positive real number ϵ there exists a natural number N such that for all $i > N$,*

$$p(a,x_i) - p(a,a) < \epsilon.$$

Proof: Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial metric space (X,p).

($\Rightarrow$) Let a be a limit of $\{x_i\}_{i\in\mathbb{N}}$. From Definition 3.2.2 we know that for each positive real number ϵ, $B^p_\epsilon(a)$ is an open neighborhood of a. Since a is a limit of $\{x_i\}_{i\in\mathbb{N}}$ then there exists a natural number N such that for all $i > N$,

$$x_i \in B^p_\epsilon(a).$$

Therefore, from Definition 3.2.1

$$p(a,x_i) - p(a,a) < \epsilon.$$

($\Leftarrow$) Assume that for each positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$p(a,x_i) - p(a,a) < \epsilon.$$

By Definition 3.2.1

$$x_i \in B^p_\epsilon(a).$$

Consider U an open neighborhood of a. From Definition 3.2.2 we know that there exists a positive real number ϵ such that

$$B^p_\epsilon(a) \subseteq U.$$

Therefore, there exists a natural number N such that for all $i > N$,

$$x_i \in B^p_\epsilon(a) \subseteq U. \qquad \square$$

Lemma 4.2.2. *Let (X,p) be a partial metric space. Consider a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with central distance r. If the point a in X is a limit of $\{x_i\}_{i\in\mathbb{N}}$, then*

$$r \leq p(a,a).$$

Proof: Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in X with central distance r. Then, from Definition 4.2.1, for every positive real number ϵ there exists a natural number N_1 such that for all $i \geq j > N_1$,

$$-\frac{\epsilon}{3} < p(x_i,x_j) - r < \frac{\epsilon}{3}$$

therefore,

$$r < p(x_i,x_j) + \frac{\epsilon}{3}.$$

Since a is a limit of $\{x_i\}_{i\in\mathbb{N}}$, from Lemma 4.2.1, there exists a natural number N_2 such that for all $i > N_2$,

$$p(a,x_i) - p(a,a) < \frac{\epsilon}{3}$$

therefore,

$$p(a, x_i) < p(a, a) + \frac{\epsilon}{3}.$$

Now taking $N = \max\{N_1, N_2\}$ we get that for all $i \geq j > N$

$$r < p(x_i, x_j) + \frac{\epsilon}{3}$$

by (p-inq)

$$\leq p(x_i, a) + p(a, x_j) - p(a, a) + \frac{\epsilon}{3}$$

by (p-sym)

$$\begin{aligned} & p(a, x_i) + p(a, x_j) - p(a, a) + \frac{\epsilon}{3} \\ < & (p(a, a) + \frac{\epsilon}{3}) + (p(a, a) + \frac{\epsilon}{3}) - p(a, a) + \frac{\epsilon}{3} \\ = & p(a, a) + \epsilon. \end{aligned}$$

Hence, for every positive real number ϵ

$$r < p(a, a) + \epsilon$$

and, therefore,

$$r \leq p(a, a). \qquad \square$$

In a metric space, a limit to a Cauchy sequence is unique. In a partial metric space though, that need not be the case.

Example 4.2.2. ***(Multiple Limits):***
Let p be a partial metric on $X = \mathbb{R} \cup \{a\}$ where $a \notin \mathbb{R}$ as defined in Example 2.2.4 by:
For all $x, y \in \mathbb{R}$,

$$p(a, a) = 0, p(a, x) = |x| \text{ and } p(x, y) = |x - y| - 1.$$

Then the sequence $\{\frac{1}{2^n}\}_{n \in \mathbb{N}}$ in X is Cauchy with a central distance $r = -1$. Moreover, 0 and a are both limits of $\{\frac{1}{2^n}\}_{n \in \mathbb{N}}$.

Proof: Assuming that $i \geq j$, then

$$\begin{aligned} p(\frac{1}{2^i}, \frac{1}{2^j}) &= |\frac{1}{2^i} - \frac{1}{2^j}| - 1 \\ &= \frac{1}{2^j} - \frac{1}{2^i} - 1. \end{aligned}$$

Therefore,

$$-1 - \frac{1}{2^i} < p(\frac{1}{2^i}, \frac{1}{2^j}) < \frac{1}{2^j} - 1.$$

Hence, for every positive real number ϵ there exists a natural number N such that $\frac{1}{2^N} < \epsilon$. Thus, for all $i \geq j > N$,

$$-1 - \epsilon < -1 - \frac{1}{2^N} < -1 - \frac{1}{2^i} < p(\frac{1}{2^i}, \frac{1}{2^j}) < \frac{1}{2^j} - 1 < \frac{1}{2^N} - 1 < \epsilon - 1.$$

From Definition 4.2.1, $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence with a central distance $r = -1$.

Additionally, for all natural numbers i,

$$p(0, \frac{1}{2^i}) - p(0,0) = |0 - \frac{1}{2^i}| - 1 - (-1) = \frac{1}{2^i}$$

and

$$p(a, \frac{1}{2^i}) - p(a,a) = |\frac{1}{2^i}| - 0 = \frac{1}{2^i}.$$

Hence, for every positive real number ϵ there exists a natural number N such that for all $i > N$,

$$p(0, \frac{1}{2^i}) - p(0,0) = \frac{1}{2^i} < \frac{1}{2^N} < \epsilon$$

and

$$p(a, \frac{1}{2^i}) - p(a,a) = \frac{1}{2^i} < \frac{1}{2^N} < \epsilon.$$

Therefore, by Lemma 4.2.1, 0 and a are both limits of $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$. $\square$

In the above example both 0 and a are limits of $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$. However, $p(a,a) = 0 \neq \lim_{i,j\to+\infty} p(\frac{1}{2^i}, \frac{1}{2^j})$ while $p(0,0) = -1 = \lim_{i,j\to+\infty} p(\frac{1}{2^i}, \frac{1}{2^j})$. Hence, in some sense, 0 has more significance to the sequence $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ than a does. We will call limits like this: special limits.

Definition 4.2.3. *Let (X,p) be a partial metric space. Consider a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with a central distance r. A point a in X is called a* ***special limit*** *of $\{x_i\}_{i\in\mathbb{N}}$ if and only if a is a limit of $\{x_i\}_{i\in\mathbb{N}}$ and $r = p(a,a)$.*

In a partial metric space, the special limit is analogous to the limit in a metric space since, if it exists, it is unique.

Lemma 4.2.3. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in a partial metric space (X,p). If $\{x_i\}_{i\in\mathbb{N}}$ has a special limit in X then that special limit is unique.*

Proof: Consider the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with a central distance r. If the points a and b of X are both special limits of $\{x_i\}_{i\in\mathbb{N}}$ then, by Definition 4.2.3,

$$p(a,a) = r = p(b,b).$$

Furthermore, for every positive real number ϵ there exists a natural number N_1 such that for all $i > N_1$,

$$r - \frac{\epsilon}{3} < p(x_i, x_j) < r + \frac{\epsilon}{3}$$

i.e.

$$-p(x_i, x_i) < -r + \frac{\epsilon}{3}.$$

By Definition 4.2.3, both a and b are limits of $\{x_i\}_{i\in\mathbb{N}}$. Then, by Lemma 4.2.1, there exists a natural number N_2 such that for all $i > N_2$,

$$p(a, x_i) - r = p(a, x_i) - p(a,a) < \frac{\epsilon}{3}$$

i.e.

$$p(a, x_i) < r + \frac{\epsilon}{3}$$

and there exists a natural number N_3 such that for all $i > N_3$,

$$p(b, x_i) - r = p(b, x_i) - p(b, b) < \frac{\epsilon}{3}$$

i.e.

$$p(b, x_i) < r + \frac{\epsilon}{3}.$$

Hence, using (p-lbnd) we get for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$,

$$p(a, a) \leq p(a, b)$$

by (p-inq)

$$\leq p(a, x_i) + p(x_i, b) - p(x_i, x_i)$$

by (p-sym)

$$= p(a, x_i) + p(b, x_i) - p(x_i, x_i)$$

$$< r + \frac{\epsilon}{3} + r + \frac{\epsilon}{3} - r + \frac{\epsilon}{3}$$

$$= r + \epsilon = p(a, a) + \epsilon.$$

Therefore,

$$p(a, a) \leq p(a, b) < p(a, a) + \epsilon$$

i.e.

$$p(a, a) = p(a, b).$$

Similarly $p(b, b) = p(b, a)$ and, hence, by (p-sep) $a = b$. □

An additional property of a special limit is that it preserves a notion of sequential continuity.

Lemma 4.2.4. *Let (X, p) be a partial metric space. Let a be the special limit of the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with a central distance r. For every y in X and positive real number ϵ, there exists a natural number N such that for all $i > N$,*

$$-\epsilon < p(y, x_i) - p(y, a) < \epsilon.$$

I.e.

$$\lim_{i\to+\infty} p(y, x_i) = p(y, a).$$

Proof: Let a be the special limit of the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with a central distance r. From Definition 4.2.3 we have

$$p(a, a) = r.$$

For every y in X and positive real number ϵ, there exists a natural number N_1 such that for all $i > N_1$,

$$p(a, x_i) - r = p(a, x_i) - p(a, a) < \frac{\epsilon}{2}$$

i.e.

$$p(a, x_i) < r + \frac{\epsilon}{2}$$

additionally, there exists a natural number N_2 such that for all $i > N_2$,

$$r - \frac{\epsilon}{2} < p(x_i, x_j) < r + \frac{\epsilon}{2}.$$

In particular for $i = j$,

$$-p(x_i, x_i) < -r + \frac{\epsilon}{2}.$$

Hence, using (p-inq) we get that for every positive real number ϵ, there exists a natural number $N = \max\{N_1, N_2\}$, such that for all $i > N$,

$$p(y, x_i) - p(y, a) \leq p(y, a) + p(a, x_i) - p(a, a) - p(y, a)$$

$$= p(a, x_i) - p(a, a) < \frac{\epsilon}{2} < \epsilon.$$

Additionally, by (p-inq)

$$p(y, a) \leq p(y, x_i) + p(x_i, a) - p(x_i, x_i)$$

by (p-sym)

$$= p(y, x_i) + p(a, x_i) - p(x_i, x_i)$$

$$< p(y, x_i) + r + \frac{\epsilon}{2} - r + \frac{\epsilon}{2}$$

$$= p(y, x_i) + \epsilon.$$

Therefore,

$$-\epsilon < p(y, x_i) - p(y, a). \qquad \square$$

Definition 4.2.4. *Let (X, p) be a partial metric space. We say that (X, p) is* ***complete*** *if and only if every Cauchy sequence has a special limit in X.*

We conclude this section by extending the definition of a Cauchy pair to the partial metric case.

Definition 4.2.5. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences in a partial metric space (X, p). We say that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a* ***Cauchy pair*** *if and only if there exists a real number r such that for every positive real number ϵ there exists a natural number N where for all $i, j > N$,*

$$r - \epsilon < \min\{p(x_i, x_i), p(y_j, y_j)\} \leq p(x_i, y_j) < r + \epsilon.$$

We call r the ***central distance*** *of the Cauchy pair $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$.*

Theorem 4.2.5. ***(Cauchy pair term comparison):***

Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences in a partial metric space (X,p). The statements below are equivalent.

(a) $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a Cauchy pair with central distance r.

(b) There exists a real number r where for every positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$r - \epsilon < \min\{p(x_i,x_i), p(y_j,y_j)\} \leq p(x_i,y_j) < r + \epsilon.$$

(c) There exists a real number r where for every positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$r - \epsilon < \min\{p(x_j,x_j), p(y_i,y_i)\} \leq p(x_j,y_i) < r + \epsilon.$$

Proof: It is clear that (a) is true if and only if (b) and (c) are true.

(b)$\Rightarrow$ (c): For every positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$r - \frac{\epsilon}{5} < p(x_i,x_i)\ ,\ r - \frac{\epsilon}{5} < p(y_j,y_j) \text{ and } p(x_i,y_j) < r + \frac{\epsilon}{5}$$

i.e.

$$-p(x_i,x_i) < -r + \frac{\epsilon}{5}\ ,\ -p(y_j,y_j) < -r + \frac{\epsilon}{5} \text{ and } p(x_i,y_j) < r + \frac{\epsilon}{5}. \qquad (\nabla)$$

Hence, for every positive real number ϵ for all $i \geq j > N$, from (∇) and using (p-lbnd)

$$r - \epsilon < r - \frac{\epsilon}{5} < r - \epsilon < \min\{p(x_j,x_j), p(y_i,y_i)\} \leq p(x_j,x_j) \leq p(x_j,y_i)$$

using (p-inq) twice we get

$$\leq p(x_j,y_j) + p(y_j,y_i) - p(y_j,y_j)$$

$$\leq p(x_j,y_j) + p(y_j,x_i) + p(x_i,y_i) - p(x_i,x_i) - p(y_j,y_j)$$

by (p-sym)

$$= p(x_j,y_j) + p(x_i,y_j) + p(x_i,y_i) - p(x_i,x_i) - p(y_j,y_j)$$

by (∇)

$$< r + \frac{\epsilon}{5} + r + \frac{\epsilon}{5} + r + \frac{\epsilon}{5} - r + \frac{\epsilon}{5} - r + \frac{\epsilon}{5} = r + \epsilon.$$

Therefore, for every positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$r - \epsilon < \min\{p(x_j,x_j), p(y_i,y_i)\} \leq p(x_j,y_i) < r + \epsilon.$$

Similarly we can prove that (c)$\Rightarrow$ (b). $\square$

Lemma 4.2.6. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be a Cauchy pair with a central distance r in a partial metric space (X,p). Then $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are both Cauchy sequences with central distance r. If a is a (special) limit of $\{x_i\}_{i\in\mathbb{N}}$ then a is also a (special) limit of $\{y_i\}_{i\in\mathbb{N}}$.*

Proof: Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be a Cauchy pair in X with a central distance r. Then for every positive real number ϵ there exists a natural number N_1 such that for all $i, j > N_1$,

$$r - \frac{\epsilon}{3} < p(x_i, x_i) \leq p(x_i, y_j) < r + \frac{\epsilon}{3}$$

and

$$r - \frac{\epsilon}{3} < p(y_j, y_j) \leq p(x_i, y_j) < r + \frac{\epsilon}{3}.$$

In particular, since the above is true, then for all $i, j > N_1$,

$$p(x_j, y_j) < r + \frac{\epsilon}{3}$$

and

$$-p(y_j, y_j) < -r + \frac{\epsilon}{3}.$$

Hence, by (p-lbnd)

$$r - \epsilon < r - \frac{\epsilon}{3} < p(x_i, x_i) \leq p(x_i, x_j)$$

by (p-inq)

$$\leq p(x_i, y_j) + p(y_j, x_j) - p(y_j, y_j)$$

by (p-sym)

$$= p(x_i, y_j) + p(x_j, y_j) - p(y_j, y_j)$$
$$< r + \frac{\epsilon}{3} + r + \frac{\epsilon}{3} - r + \frac{\epsilon}{3} = r + \epsilon.$$

Therefore, $\{x_i\}_{i\in\mathbb{N}}$ (and similarly $\{y_i\}_{i\in\mathbb{N}}$) is a Cauchy sequence with central distance r. Therefore for every positive real number ϵ there exists a natural number N_2, such that for all $i, j > N_2$,

$$r - \frac{\epsilon}{3} < p(x_i, x_j) < r + \frac{\epsilon}{3}$$

in particular, for $i = j$,

$$-p(x_i, x_i) < -r + \frac{\epsilon}{3}.$$

Now assume that a is a limit of $\{x_i\}_{i\in\mathbb{N}}$. By Lemma 4.2.1, for every positive real number ϵ there exists a natural number N_3 such that for all $i > N_3$,

$$p(a, x_i) - p(a, a) < \frac{\epsilon}{3}$$

i.e.

$$p(a, x_i) < p(a, a) + \frac{\epsilon}{3}.$$

Therefore, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i, j > N$, by (p-inq)

$$p(a, y_j) - p(a, a) \leq p(a, x_i) + p(x_i, y_j) - p(x_i, x_i) - p(a, a)$$

$$< p(a,a) + \frac{\epsilon}{3} + r + \frac{\epsilon}{3} - r + \frac{\epsilon}{3} - p(a,a) = \epsilon.$$

The special limit case follows from the fact that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ have the same central distance r as shown above. $\square$

4.3 Partial $n - \mathfrak{M}$etric Space

In [2], and to retain the feel of Definition 4.2.1, we generalized a Cauchy sequence to the partial $n-\mathfrak{M}$etric space in the manner below.

Definition 4.3.1. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial $n-\mathfrak{M}$etric space (X,P). We say that the sequence $\{x_i\}_{i\in\mathbb{N}}$ is **Cauchy** if and only if there exists a real number r where for each positive number ϵ there exists a natural number N such that for all $i_1, i_2, ..., i_n > N$,*

$$-\epsilon < P(\langle x_{i_t}\rangle_{t=1}^{n}) - r < \epsilon.$$

*We call r the **central distance** of $\{x_i\}_{i\in\mathbb{N}}$.*

Although the above definition is a natural generalization from the partial metric case, it may seem to the reader that it is a condition that is difficult to check in practice. Theorem 4.3.1 makes it much simpler to check if a sequence in a partial $n-\mathfrak{M}$etric space is Cauchy.

Theorem 4.3.1. ***(Cauchy Sequence Two Term Comparison):***
Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial $n-\mathfrak{M}$etric space (X,P). Then the statements below are equivalent.
(a) $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with a central distance r.
(b) There exists a positive real number r where for every positive real number ϵ there exists a natural number N such that for all $i,j > N$,

$$-\epsilon < P(\langle x_i\rangle^{n-1}, x_j) - r < \epsilon.$$

(c) There exists a positive real number r where for every positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$-\epsilon < P(\langle x_i\rangle^{n-1}, x_j) - r < \epsilon.$$

(d) There exists a positive real number r where for every positive real number ϵ there exists a natural number N such that for all $i \geq j > N$,

$$-\epsilon < P(\langle x_j\rangle^{n-1}, x_i) - r < \epsilon.$$

Proof: (a)$\Rightarrow$ (b) and (b)$\Rightarrow$ (c) are trivial.
(c) $\Rightarrow$ (d): For every positive real number ϵ, let $\epsilon' = \frac{\epsilon}{2n-3} \leq \epsilon$. Then, there exists a natural number N such that for all $i \geq j > N$,

$$-\epsilon' < P(\langle x_i\rangle^{n-1}, x_j) - r < \epsilon'.$$

In particular, for $i = j$

$$-\epsilon' < P(\langle x_j \rangle^n) - r$$

and

$$-P(\langle x_i \rangle^n) < -r + \epsilon'.$$

By (P_n-lbnd) for all $i \geq j > N$,

$$-\epsilon \leq -\epsilon' < P(\langle x_j \rangle^n) - r \leq P(\langle x_j \rangle^{n-1}, x_i) - r$$

by Corollary 2.5.4,

$$\leq (n-1)P(\langle x_i \rangle^{n-1}, x_j) - (n-2)P(\langle x_i \rangle^n) - r$$

$$= (n-1)P(\langle x_i \rangle^{n-1}, x_j) + (n-2)[-P(\langle x_i \rangle^n)] - r$$

$$< (n-1)(r + \epsilon') + (n-2)(-r + \epsilon') - r$$

$$= (2n-3)\epsilon' = \epsilon.$$

Hence,

$$-\epsilon < P(\langle x_j \rangle^{n-1}, x_i) - r < \epsilon.$$

(d)$\Rightarrow$ (a): For every positive real number ϵ, let $\epsilon' = \frac{\epsilon}{2n-3} \leq \epsilon$. Then, there exists a natural number N such that for all $i \geq j > N$,

$$-\epsilon' < P(\langle x_j \rangle^{n-1}, x_i) - r < \epsilon'.$$

For all $i_1, i_2, ..., i_n > N$, we will prove that

$$-\epsilon < P(\langle x_{i_t} \rangle_{t=1}^n) - r < \epsilon'.$$

Without loss of generality by (P_n-sym), assume that $i_n \geq i_{n-1} \geq ..... \geq i_2 \geq i_1 > N$. Hence, from (d), for all $t \geq k$ in $\{1, ..., n\}$ we get

$$(\star) \begin{cases} P(\langle x_{i_k} \rangle^{n-1}, x_{i_t}) < r + \epsilon' \\ P(\langle x_{i_k} \rangle^n) < r + \epsilon' \\ -\epsilon' < P(\langle x_{i_t} \rangle^n) - r \\ -P(\langle x_{i_k} \rangle^n) < -r + \epsilon' \end{cases}$$

In particular, for $t = n$ in $(\star)$,

$$-\epsilon' < P(\langle x_{i_n} \rangle^n) - r$$

by (P_n-lbnd)

$$\leq P(\langle x_{i_n} \rangle^{n-1}, x_{i_1}) - r$$

by (P_n-sym)

$$= P(\langle x_{i_n} \rangle^{n-2}, x_{i_1}, x_{i_n}) - r$$

by Theorem 2.5.1,

$$\leq P(\langle x_{i_k}\rangle_{k=2}^{n-1}, x_{i_1}, x_{i_n}) + \sum_{k=2}^{n-1}[P(\langle x_{i_k}\rangle^{n-1}, x_{i_n}) - P(\langle x_{i_k}\rangle^{n})] - r$$

by (P_n-sym)

$$= P(\langle x_{i_k}\rangle_{k=1}^{n}) + \sum_{k=2}^{n-1}[P(\langle x_{i_k}\rangle^{n-1}, x_{i_n}) - P(\langle x_{i_k}\rangle^{n})] - r$$

by ($\star$) and for $t = n$,

$$< P(\langle x_{i_k}\rangle_{k=1}^{n}) + \sum_{k=2}^{n-1}[(r+\epsilon') + (-r+\epsilon')] - r$$

$$= P(\langle x_{i_k}\rangle_{k=1}^{n}) + \sum_{k=2}^{n-1} 2\epsilon' - r$$

$$= P(\langle x_{i_k}\rangle_{k=1}^{n}) + (n-2)(2\epsilon') - r$$

$$= P(\langle x_{i_k}\rangle_{k=1}^{n}) - r + (2n-4)\epsilon'$$

and, hence,

$$-\epsilon' < P(\langle x_{i_k}\rangle_{k=1}^{n}) - r + (2n-4)\epsilon'$$

therefore,

$$P(\langle x_{i_k}\rangle_{k=1}^{n}) - r > -(2n-3)\epsilon' = -\epsilon.$$

On the other hand by Corollary 2.5.2,

$$P(\langle x_{i_t}\rangle_{t=1}^{n}) - r \leq P(\langle x_{i_1}\rangle^{n}) + \sum_{t=2}^{n-1}[P(\langle x_{i_1}\rangle^{n-1}, x_{i_t}) - P(\langle x_{i_1}\rangle^{n})] - r$$

by ($\star$) and for $k = 1$,

$$< r + \epsilon' + \sum_{t=2}^{n-1}[(r+\epsilon') + (-r+\epsilon')] - r$$

$$= \epsilon' + \sum_{t=2}^{n-1} 2\epsilon' = \epsilon' + (n-2)(2\epsilon')$$

$$= (2n-3)\epsilon' = \epsilon.$$

Therefore

$$-\epsilon < P(\langle x_{i_t}\rangle_{t=1}^{n}) - r < \epsilon. \qquad \square$$

Lemma 4.3.2. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial $n-\mathfrak{M}$etric space (X, P). A point a in X is a limit of $\{x_i\}_{i\in\mathbb{N}}$ if and only if for every positive real number ϵ there exists a natural number N such that for all $i > N$,*

$$P(\langle a\rangle^{n-1}, x_i) - P(\langle a\rangle^{n}) < \epsilon.$$

Proof: Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a partial $n-\mathfrak{M}$etric space (X,P).

($\Rightarrow$) Let a be a limit of $\{x_i\}_{i\in\mathbb{N}}$ then for every positive real number ϵ, $B^P_\epsilon(a)$ is an open neighborhood of a. Hence, there exists a positive natural number N such that for all $i>N$,

$$x_i \in B^P_\epsilon(a).$$

Therefore, from Definition 3.3.1

$$P(\langle a\rangle^{n-1},x_i)-P(\langle a\rangle^n)<\epsilon.$$

($\Leftarrow$) Assume that for each positive real number ϵ there exists a natural number N such that for all $i>N$,

$$P(\langle a\rangle^{n-1},x_i)-P(\langle a\rangle^n)<\epsilon$$

and, hence, from Definition 3.3.1

$$x_i \in B^P_\epsilon(a).$$

For every open neighborhood U of a, from Definition 3.3.2 we know that their exists a positive real number ϵ such that

$$B^P_\epsilon(a)\subseteq U.$$

Therefore, there exists a natural number N such that for all $i>N$,

$$x_i \in B^P_\epsilon(a)\subseteq U. \qquad \square$$

The next theorem shows that partial $n-\mathfrak{M}$etrics possess a kind of upper semi-continuity property.

Theorem 4.3.3. ***(Upper Semi Continuity):***
Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in a partial $n-\mathfrak{M}$etric space (X,P) with limit a in X. Then for every $0\le q\le n-1$, $\{b_k\}_{k=1}^q\subseteq X$ and positive real number ϵ, there exists a natural number N such that for all $i_1,i_2,...,i_{n-q}>N$,

$$P(\langle x_{i_t}\rangle_{t=1}^{n-q},\langle b_k\rangle_{k=1}^q)<P(\langle a\rangle^{n-q},\langle b_k\rangle_{k=1}^q)+\epsilon.$$

Proof: Let $0\le q\le n-1$, $\{b_k\}_{k=1}^q\subseteq X$ and positive real number ϵ. Let $\epsilon'=\frac{\epsilon}{n-q}$. Then, by Lemma 4.3.2, there exists a natural number N such that for all $i_1,i_2,...,i_{n-q}>N$,

$$P(\langle a\rangle^{n-1},x_{i_t})-P(\langle a\rangle^n)<\epsilon'.$$

Hence, without loss of generality by (P_n-sym) we may assume that $i_{n-q}\ge i_{n-q-1}\ge...\ge i_2\ge i_1>N$. Then by Theorem 2.5.1 ,

$$\begin{aligned}P(\langle x_{i_t}\rangle_{t=1}^{n-q},\langle b_k\rangle_{k=1}^q) &< P(\langle a\rangle^{n-q},\langle b_k\rangle_{k=1}^q)+\sum_{t=1}^{n-q}[P(\langle a\rangle^{n-1},x_{i_t})-P(\langle a\rangle^n)]\\ &< P(\langle a\rangle^{n-q},\langle b_k\rangle_{k=1}^q)+\sum_{t=1}^{n-q}\epsilon'\\ &= P(\langle a\rangle^{n-q},\langle b_k\rangle_{k=1}^q)+(n-q)\epsilon' = P(\langle a\rangle^{n-q},\langle b_k\rangle_{k=1}^q)+\epsilon. \qquad \square\end{aligned}$$

Corollary 4.3.4. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in a partial $n-\mathfrak{M}$etric space (X,P) with a limit a in X. Then for every positive real number ϵ there exists a natural number N such that for all $i_1, i_2, ..., i_n > N$ and all $0 \leq q \leq n$ the statements below hold true.*
(a) $P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle a\rangle^q) < P(\langle a\rangle^n) + \epsilon$.
(b) $P(\langle x_{i_t}\rangle_{t=1}^{n}) < P(\langle a\rangle^n) + \epsilon$.
(c) $P(\langle x_{i_t}\rangle_{t=1}^{n-1}, a) < P(\langle a\rangle^n) + \epsilon$.

The above Corollary is trivial to prove using Theorem 4.3.3 while varying $1 \leq q \leq n-1$ and taking for all k, $b_k = a$. The case where $q = n$ is trivial since

$$P(\langle a\rangle^n) < P(\langle a\rangle^n) + \epsilon$$

for any positive real number ϵ.

As in Example 4.2.2, a limit of a Cauchy sequence need not be unique.

Definition 4.3.2. *Let (X,p) be a partial $n-\mathfrak{M}$etric space. Consider a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with a central distance r. A point a in X is called a* ***special limit*** *of $\{x_i\}_{i\in\mathbb{N}}$ if and only if a is a limit of $\{x_i\}_{i\in\mathbb{N}}$ and $r = P(\langle a\rangle^n)$.*

As in the case of a partial metric space a special limit is unique.

Theorem 4.3.5. ***(Uniqueness of Special Limits):***
Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in a partial $n-\mathfrak{M}$etric space (X,P). If $\{x_i\}_{i\in\mathbb{N}}$ has a special limit in X then that special limit is unique.

Proof: Consider the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with a central distance r. If a and b are both special limits of $\{x_i\}_{i\in\mathbb{N}}$, then by Definition 4.3.2

$$P(\langle a\rangle^n) = r = P(\langle b\rangle^n).$$

Furthermore, from Definition 4.3.1, for every positive real number ϵ there exists a natural number N_1 such that for all $i > N_1$,

$$-\frac{\epsilon}{3} < P(\langle x_i\rangle^n) - r,$$

i.e.

$$-P(\langle x_i\rangle^n) < -r + \frac{\epsilon}{3}.$$

The special limit a of $\{x_i\}_{i\in\mathbb{N}}$ is also a limit of $\{x_i\}_{i\in\mathbb{N}}$. Hence, by Corollary 4.3.4 for every positive real number ϵ there exists a natural number N_2 such that for all $i > N_2$,

$$P(\langle a\rangle^{n-1}, x_i) - P(\langle a\rangle^n) < \frac{\epsilon}{3}$$

i.e.

$$P(\langle a\rangle^{n-1}, x_i) < r + \frac{\epsilon}{3}.$$

The special limit b of $\{x_i\}_{i\in\mathbb{N}}$ is also a limit of $\{x_i\}_{i\in\mathbb{N}}$. Hence, by Corollary 4.3.7 there exists a natural number N_3 such that for all $i > N_3$,

$$P(\langle x_i\rangle^{n-1}, b) < P(\langle b\rangle^n) + \frac{\epsilon}{3},$$

i.e.

$$P(\langle x_i\rangle^{n-1}, b) < r + \frac{\epsilon}{3}.$$

Hence, using (P_n-lbnd) we get that for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$,

$$P(\langle a\rangle^n) \leq P(\langle a\rangle^{n-1}, b)$$

by (P_n-inq)

$$\begin{aligned}
&\leq P(\langle a\rangle^{n-1}, x_i) + P(\langle x_i\rangle^{n-1}, b) - P(\langle x_i\rangle^n) \\
&< r + \frac{\epsilon}{3} + r + \frac{\epsilon}{3} - r + \frac{\epsilon}{3} \\
&= r + \epsilon = P(\langle a\rangle^n) + \epsilon
\end{aligned}$$

therefore,

$$P(\langle a\rangle^n) = P(\langle a\rangle^{n-1}, b).$$

Similarly $P(\langle b\rangle^n) = P(\langle b\rangle^{n-1}, a)$ and, hence, by (P_n-sep)

$$a = b. \qquad \square$$

Theorem 4.3.6. ***(Continuity of Special Limits):***
Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in a partial $n-\mathfrak{M}$etric space (X, P) with special limit a in X. Then for every $0 \leq q \leq n-1$, $\{b_k\}_{k=1}^q \subseteq X$ and positive real number ϵ, there exists a natural number N such that for all $i_1, i_2, ..., i_{n-q} > N$,

$$P(\langle a\rangle^{n-q}, \langle b_k\rangle_{k=1}^q) - \epsilon < P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q) < P(\langle a\rangle^{n-q}, \langle b_k\rangle_{k=1}^q) + \epsilon.$$

Proof: Let r be the central distance of $\{x_i\}_{i\in\mathbb{N}}$. By Definition 4.3.2

$$P(\langle a\rangle^n) = r.$$

Let $0 \leq q \leq n-1$, $\{b_k\}_{k=1}^q \subseteq X$, and positive real number ϵ. Let $\epsilon' = \frac{\epsilon}{2(n-q)}$. By Definition 4.3.2, a is also a limit of $\{x_i\}_{i\in\mathbb{N}}$ and, hence, there exists a natural number N_1 such that for all $i_t > N_1$,

$$P(\langle x_{i_t}\rangle^{n-1}, a) < P(\langle a\rangle^n) + \epsilon',$$

i.e.

$$P(\langle x_{i_t}\rangle^{n-1}, a) < r + \epsilon'.$$

By Definition 4.3.1, there exists a natural number N_2 such that for all $i_t > N_2$,

$$-\epsilon' < P(\langle x_{i_t}\rangle^n) - r,$$

i.e.

$$-P(\langle x_{i_t}\rangle^n) < -r + \epsilon'.$$

Hence, using Theorem 2.5.1 there exists a natural number $N = \max\{N_1, N_2\}$ such that for all $i_1, i_2, ..., i_{n-q} > N$,

$$\begin{aligned}
&P(\langle a\rangle^{n-q}, \langle b_k\rangle_{k=1}^q) - \epsilon \\
&\leq P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q) + \sum_{t=1}^{n-q}[P(\langle x_{i_t}\rangle^{n-1}, a) - P(\langle x_{i_t}\rangle^n)] - \epsilon \\
&< P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q) + \sum_{t=1}^{n-q}[r + \epsilon' - r + \epsilon'] - \epsilon \\
&= P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q) + \sum_{t=1}^{n-q} 2\epsilon' - \epsilon \\
&= P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q) + (n-q)(2\epsilon') - \epsilon \\
&= P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q) + \epsilon - \epsilon = P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle b_k\rangle_{k=1}^q).
\end{aligned}$$

The right side of the inequality was already proved in Theorem 4.3.3. □

Corollary 4.3.7. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in a partial $n-\mathfrak{M}$etric space (X, P) with special limit a in X. Then for every positive real number ϵ there exists a natural number N such that for all $i_1, i_2, ..., i_n > N$ and $0 \leq q \leq n$ the statements below hold true.*
(a) $-\epsilon < P(\langle x_{i_t}\rangle_{t=1}^{n-q}, \langle a\rangle^q) - P(\langle a\rangle^n) < \epsilon$.
(b) $-\epsilon < P(\langle x_{i_t}\rangle_{t=1}^{n}) - P(\langle a\rangle^n) < \epsilon$.
(c) $-\epsilon < P(\langle x_{i_t}\rangle_{t=1}^{n-1}, a) - P(\langle a\rangle^n) < \epsilon$.

The above Corollary is trivial to prove using Theorem 4.3.6 while varying $1 \leq q \leq n-1$ and taking for all k, $b_k = a$. The case where $q = n$ is trivial.

Definition 4.3.3. *Let (X, P) be a partial $n-\mathfrak{M}$etric space. We say that (X, P) is **complete** if and only if every Cauchy sequence has a special limit in X.*

Definition 4.3.4. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences in a partial $n-\mathfrak{M}$etric space (X, P). We say that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a **Cauchy pair** if and only if there exists a real number r such that for every positive real number ϵ there exists a natural number N such that for all $i, j > N$,*

$$r - \epsilon < \min\{P(\langle x_i\rangle^n), P(\langle y_i\rangle^n)\} \leq P(\langle x_i\rangle^{n-1}, y_j) < r + \epsilon.$$

*We call r the **central distance** of the Cauchy pair $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$.*

Theorem 4.3.8. ***(Cauchy pair term comparison):***

Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences in a partial $n-\mathfrak{M}$etric space (X,P). The statements below are equivalent.

(a) $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a Cauchy pair with central distance r.

(b) There exists a real number r where for every positive real number ϵ there exists a natural number N such that for all $i,j>N$,

$$r-\epsilon<\min\{P(\langle x_i\rangle^n),P(\langle y_i\rangle^n)\}\le P(\langle y_j\rangle^{n-1},x_i)<r+\epsilon.$$

(c) There exists a real number r where for every positive real number ϵ there exists a natural number N such that for all $i\ge j>N$,

$$r-\epsilon<\min\{P(\langle x_i\rangle^n),P(\langle y_i\rangle^n)\}\le P(\langle x_i\rangle^{n-1},y_j)<r+\epsilon.$$

(d) There exists a real number r where for every positive real number ϵ there exists a natural number N such that for all $i\ge j>N$,

$$r-\epsilon<\min\{P(\langle x_i\rangle^n),P(\langle y_i\rangle^n)\}\le P(\langle x_j\rangle^{n-1},y_i)<r+\epsilon.$$

Proof: It is clear that (a) is true if and only if (c) and (d) are true. Hence, it will be enough to prove that (a) is equivalent to (b) and (c) is equivalent to (d).

(a)$\Rightarrow$(b): Assume that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ form a Cauchy pair with central distance r. For every positive real number ϵ, let $\epsilon'=\frac{\epsilon}{2n-3}\le\epsilon$. Hence, there exists a natural number N such that for all $i,j>N$,

$$r-\epsilon'<\min\{P(\langle x_i\rangle^n),P(\langle y_i\rangle^n)\}\le P(\langle x_i\rangle^{n-1},y_j)<r+\epsilon'$$

in particular,

$$-P(\langle x_i\rangle^n)<-r+\epsilon'.$$

Therefore,

$$r-\epsilon\le r-\epsilon'\le\min\{P(\langle x_i\rangle^n),P(\langle y_i\rangle^n)\}$$

from (P_n-lbnd)

$$\le P(\langle y_i\rangle^n)\le P(\langle y_j\rangle^{n-1},x_i)$$

by Corollary 2.5.4

$$\le(n-1)P(\langle x_i\rangle^{n-1},y_j)-(n-2)P(\langle x_i\rangle^n)$$

$$<(n-1)(r+\epsilon')+(n-2)(-r+\epsilon')=r+(2n-3)\epsilon'=r+\epsilon.$$

The proof that (b)$\Rightarrow$ (b) is similar.

(c)$\Rightarrow$ (d): For every positive real number ϵ, let $\epsilon'=\frac{\epsilon}{2n+1}\le\epsilon$. Hence, there exists a natural number N such that for all $i\ge j>N$,

$$r-\epsilon'<\min\{P(\langle x_i\rangle^n),P(\langle y_i\rangle^n)\}\le P(\langle x_i\rangle^{n-1},y_j)<r+\epsilon'$$

in particular,

$$-P(\langle x_i\rangle^n) < -r+\epsilon' \ , \ -P(\langle y_j\rangle^n) < -r+\epsilon',$$

$$P(\langle x_i\rangle^{n-1}, y_i) < r+\epsilon' \text{ and } P(\langle x_j\rangle^{n-1}, y_j) < r+\epsilon'.$$

Therefore,

$$r-\epsilon \leq r-\epsilon' \leq \min\{P(\langle x_i\rangle^n), P(\langle y_i\rangle^n)\}$$

from (P_n-lbnd)

$$\leq P(\langle x_j\rangle^n) \leq P(\langle x_j\rangle^{n-1}, y_i)$$

by using (P_n-inq) twice we get

$$\leq P(\langle x_j\rangle^{n-1}, y_j) + P(\langle y_j\rangle^{n-1}, y_i) - P(\langle y_j\rangle^n)$$

$$\leq P(\langle x_j\rangle^{n-1}, y_j) + P(\langle y_j\rangle^{n-1}, x_i) + P(\langle x_i\rangle^{n-1}, y_i) - P(\langle x_i\rangle^n) - P(\langle y_j\rangle^n)$$

by Corollary 2.5.4

$$\leq P(\langle x_j\rangle^{n-1}, y_j) + (n-1)P(\langle x_i\rangle^{n-1}, y_j) - (n-2)P(\langle x_i\rangle^n) + P(\langle x_i\rangle^{n-1}, y_i) - P(\langle x_i\rangle^n) - P(\langle y_j\rangle^n)$$

$$< (r+\epsilon') + (n-1)(r+\epsilon') + (n-2)(-r+\epsilon') + (r+\epsilon') + (-r+\epsilon') + (-r+\epsilon')$$

$$= r + (2n+1)\epsilon' = r+\epsilon.$$

Similarly we can prove that (d)⇒(c). $\square$

Lemma 4.3.9. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be a Cauchy pair with a central distance r in a partial $n-\mathfrak{M}$etric space (X,P). Then $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are both Cauchy sequences with central distance r. If a is a (special) limit of $\{x_i\}_{i\in\mathbb{N}}$ then a is a (special) limit of $\{y_i\}_{i\in\mathbb{N}}$.*

Proof: Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be a Cauchy pair in X with a central distance r. Then for every positive real number ϵ there exists a natural number N_1 such that for all $i,j > N_1$,

$$r-\frac{\epsilon}{3} < \min\{P(\langle x_i\rangle^n), P(\langle y_i\rangle^n)\} \leq P(\langle x_i\rangle^{n-1}, y_j) < r+\frac{\epsilon}{3}$$

and

$$r-\frac{\epsilon}{3} < \min\{P(\langle x_i\rangle^n), P(\langle y_i\rangle^n)\} \leq P(\langle y_i\rangle^{n-1}, x_j) < r+\frac{\epsilon}{3}.$$

Hence,

$$-P(\langle x_i\rangle^n) < -r+\frac{\epsilon}{3}.$$

By (P_n-lbnd)

$$r-\epsilon < r-\frac{\epsilon}{3} < P(\langle y_i\rangle^n) \leq P(\langle y_i\rangle^{n-1}, y_j)$$

by (P_n-inq)

$$\leq P(\langle y_i\rangle^{n-1}, x_j) + P(\langle x_i\rangle^{n-1}, y_j) - P(\langle x_i\rangle^n)$$

$$< r + \frac{\epsilon}{3} + r + \frac{\epsilon}{3} - r + \frac{\epsilon}{3} = r + \epsilon.$$

Hence, by Theorem 4.3.1, $\{y_i\}_{i\in\mathbb{N}}$ (and similarly $\{x_i\}_{i\in\mathbb{N}}$) is a Cauchy sequence with central distance r.

Now assume that a is a limit of $\{x_i\}_{i\in\mathbb{N}}$. By Lemma 4.3.2, for every positive real number ϵ, there exists a natural number N_2 such that for all $i > N_2$,

$$P(\langle a\rangle^{n-1}, x_i) - P(\langle a\rangle^n) < \frac{\epsilon}{3}.$$

Therefore, for every positive number ϵ, there exists a natural number $N = \max\{N_1, N_2\}$ such that for all $i > N$,

$$P(\langle a\rangle^{n-1}, y_i) - P(\langle a\rangle^n)$$

by (P_n-inq)

$$\leq P(\langle a\rangle^{n-1}, x_i) + P(\langle x_i\rangle^{n-1}, y_i) - P(\langle x_i\rangle^n) - P(\langle a\rangle^n)$$

$$= P(\langle a\rangle^{n-1}, x_i) - P(\langle a\rangle^n) + P(\langle x_i\rangle^{n-1}, y_i) - P(\langle x_i\rangle^n)$$

$$< \frac{\epsilon}{3} + r + \frac{\epsilon}{3} - r + \frac{\epsilon}{3} = \epsilon.$$

The special limit case follows from the fact that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ have the same central distance r as shown above. $\square$

Chapter 5

Cauchy Functions

It all started in **1922** with Banach [5]. Given a metric space (X, d) and a function $f : X \to X$, Banach gave contracting criteria on f allowing him to generate from that function a Cauchy sequence. He then proved that the limit of this sequence is a fixed point. His fixed point theorem was generalized in many ways, but all generalizations had the same flow.

Step 1: Give criteria for the function to generate a Cauchy sequence.

Step 2: Make sure the limit of that Cauchy sequence exists.

Step 3: Give criteria on that function that leads to sequential continuity on the limit of the Cauchy sequence. This ensures the existence of a fixed point is found.

The most notable generalization was given in **1962** by Edelstein [10], who restricted his attention to continuous functions contractive on an orbit. In **1977**, Alber and Guerre-Delabriere [1] generalized contraction to what they called weak φ-contraction. Their work was restricted to maps on Hilbert spaces. In **2001**, Rhoades [30] showed that the results in [1] still hold in any Banach space.

In Chapter 5 we will investigate some contractive criteria on a function that enables it to generate a Cauchy sequence in its domain.

Definition 5.0.1. *Let $(X, \mathcal{T})$ be a topological space with x_o in X. Let $f : X \to X$ be a function on X. Denote $f^0(x_o) = x_0$, $f^1(x_o) = f(x_o)$ and inductively $f^{n+1}(x_o) = f(f^n(x_o))$. The* ***orbit of f at x_o*** *is the sequence $\{f^i(x_o)\}_{i\in\mathbb{N}}$.*

We will state the definitions below on a partial $n - \mathfrak{M}$etric space knowing that it includes all other cases discussed in this thesis.

Definition 5.0.2. *Let (X, P) be a partial $n - \mathfrak{M}$etric space with x_o in X and suppose $f : X \to X$ is a function on X. We say that f is a* ***Cauchy function*** *at x_o if and only if $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is a Cauchy sequence. We say that f is a* ***Cauchy function*** *if and only if for every x in X, $\{f^i(x)\}_{i\in\mathbb{N}}$ is a Cauchy sequence.*

Definition 5.0.3. *Let (X, P) be a partial $n - \mathfrak{M}$etric space with x_o and y_o in X and suppose $f : X \to X$ and $g : X \to X$ are two functions on X. We say that f and g form a* ***Cauchy pair over*** *(x_o, y_o) if and only if $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. I.e. there exists a real number r such that*

$$\lim_{i,j\to+\infty} P(\langle x_i\rangle^{n-1}, y_j) = r.$$

5.1 Metric Space

The material in this section will be presented more completely in the more general case in Section 5.2 as partial metric spaces.

Definition 5.1.1. *(Edelstein [10]) Let (X, d) be a metric space with x_o in X. Let $f : X \to X$ be a function on X and $0 < c < 1$ be a real number. We say that f is an* ***orbital*** c_0-***contraction at*** x_o *(or f is* ***orbitally*** c_0-***contractive at*** x_o*) if and only if for all natural numbers i,*

$$d(f^{i+1}(x_o), f^i(x_0) \leq cd(f^i(x_o), f^{i-1}(x_o)).$$

We say that f is an ***orbital*** c_0-***contraction*** *(or f is* ***orbitally*** c_0-***contractive****) if and only if for every x in X, f is an orbital contraction at x.*

In the partial metric case, the central distance r of the Cauchy sequence obtained need not be 0. That is why the notation **orbital** c_0-**contraction** was presented to allow the use of the term **orbital** c_r-**contraction** in the more general case.

Lemma 5.1.1. *(Edelstein [10]) Let (X, d) be a metric space with x_o in X. Let $f : X \to X$ be a function on X. If f is an orbital contraction at x_o then f is a Cauchy function at x_o.*

We build upon the work of Rhoades [30] and Edelstein [10, 11, 12] to present the definition below.

Definition 5.1.2. *Let (X, d) be a metric space with x_o in X. Let $f : X \to X$ be a function on X and $\varphi : [0, +\infty) \subset \mathbb{R} \to [0, +\infty)$ be a non-decreasing function such that*

$$\varphi(t) = 0 \text{ if and only if } t = 0.$$

We say that f is an ***orbital*** φ_0-***contraction at*** x_o *(or f is* ***orbitally*** φ_0-***contractive at*** x_o*) if and only if for all i and j,*

$$d(f^{i+1}(x_o), f^{j+1}(x_o)) \leq d(f^i(x_o), f^j(x_o)) - \varphi(d(f^i(x_o), f^j(x_o)).$$

We say that f is an ***orbital*** φ_0-***contraction*** *(or f is* ***orbitally*** φ_0-***contractive****) if and only if for every x in X, f is an orbital φ_0-contraction at x.*

Remark 5.1.3. *The reader should note that any orbital c_0-contraction is an orbital φ_0-contraction by taking*

$$\varphi(t) = (1 - c)t.$$

Lemma 5.1.2. *Let (X, d) be a metric space with x_o in X. Let $f : X \to X$ be a function on X. If f is an orbital φ_0-contraction at x_o then f is a Cauchy function at x_o.*

While searching the literature for common fixed point theorems, those we found required the two functions to commute and the space was required to have certain conditions in addition to being complete [14]. We wanted to present a common fixed point theorem that relies on a contraction criteria. In **2009**, Zhanga and Song [35] presented us with just that, however, their contraction is defined over the whole space rather than an orbit.

Definition 5.1.4. *Let (X,d) be a metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions on X and $0 < c < 1$ be a real number. We say that f and g are* ***$f-$pairwise c_0-contractive over** (x_o, y_o)* *if and only if for all natural numbers i,*

$$\begin{cases} d(f^{i+1}(x_0), g^i(y_o)) \leq cd(f^i(x_o), g^{i-1}(y_o)) \\ d(f^i(x_0), g^i(y_o)) \leq cd(f^{i-1}(x_o), g^{i-1}(y_o)) \end{cases}$$

In fact Definition 5.1.4 is more general and much easier to check than the following possible alternative definition,

$$d(f^{i+1}(x_0), g^{j+1}(y_o)) \leq cd(f^i(x_o), g^j(y_o)).$$

Theorem 5.1.3. ***(Cauchy $f-$Pairwise c_0-contractive):***
Let (X,d) be a metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions on X. If f and g are $f-$pairwise c_0-contractive over (x_o, y_o) then f and g form a Cauchy pair over (x_o, y_o).

Definition 5.1.5. *Let (X,l) and (Y,d) be two metric spaces. Let $f : X \to Y$ and $g : X \to Y$ two functions on X. Let $A \geq 0$ and $0 < c < 1$ be two real numbers. We say that f and g are* ***mutually c_0-contractive*** *if and only if for each x in X we can find an element z in X such that*

$$d(f(z), g(z)) \leq cd(f(x), g(x))$$

and

$$l(x,z) \leq Ad(f(x), g(x)).$$

Theorem 5.1.4 is a special case of Theorem 5.2.4. We have chosen to include the proof because, as far as we have been able to determine, the result is new and the proof increases in complexity in the partial metric case.

Theorem 5.1.4. ***(Cauchy Mutually c_0-Contractive):***
Let (X,l) and (Y,d) be two metric spaces. Let $f : X \to Y$ and $g : X \to Y$ be two functions on X. If f and g are mutually c_0-contractive then there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that $\{f(x_i)\}_{n\in\mathbb{N}}$ and $\{g(x_i)\}_{n\in\mathbb{N}}$ form a Cauchy pair in Y.

Proof: Since f and g are mutually contractive then there exist two real numbers $0 < c < 1$ and $A \geq 0$ such that for each x in X, there exists a z in X such that

$$d(f(z), g(z)) \leq cd(f(x), g(x))$$

and

$$l(x,z) \leq Ad(f(x), g(x)).$$

Let us take an arbitrary element x_1 in X then there exists an element x_2 in X such that

$$d(f(x_2), g(x_2)) \leq cd(f(x_1), g(x_1)) \text{ and } l(x_1, x_2) \leq Ad(f(x_1), g(x_1)).$$

There exists an element x_3 in X such that

$$d(f(x_3), g(x_3)) \leq cd(f(x_2), g(x_2)) \text{ and } l(x_2, x_3) \leq Ad(f(x_2), g(x_2)).$$

We continue the above process to generate the sequence $\{x_i\}_{i\in\mathbb{N}}$ such that for all i,

$$d(f(x_{i+1}), g(x_{i+1})) \leq cd(f(x_i), g(x_i)) \text{ and } l(x_i, x_{i+1}) \leq Ad(f(x_i), g(x_i)). \qquad (\ddot{\diamond})$$

We now prove that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with two steps.

Step 1: For all i let $t_i = d(f(x_i), g(x_i))$. Then from $(\ddot{\diamond})$

$$t_{i+1} = d(f(x_{i+1}), g(x_{i+1})) \leq cd(f(x_i), g(x_i)) \leq c^2 d(f(x_{i-1}), g(x_{i-1}))$$

and, hence, by induction

$$t_{i+1} \leq c^i d(f(x_1), g(x_1)) = c^i t_1. \qquad (\check{\diamond})$$

Since $0 < c < 1$ then for every positive real number ϵ there exists a natural number N' such that $c^{N'-1}t_1 < \epsilon$ and, hence, for all $i > N'$,

$$d(f(x_i), g(x_i)) = t_i \leq c^{i-1}t_1 < c^{N'-1}t_1 < \epsilon.$$

Step 2: For all $j > i$ and by repeatedly using (d-inq) we get,

$$l(x_i, x_j) \leq l(x_i, x_{i+1}) + l(x_{i+1}, x_{i+2}) + ...l(x_{j-1}, x_j)$$

from Definition 5.1.5

$$\leq Ad(f(x_i), g(x_i)) + Ad(f(x_{i+1}), g(x_{i+1})) + ... + Ad(f(x_{j-1}), g(x_{j-1}))$$

$$= At_i + At_{i+1} + ... + At_{j-1}$$

by $(\check{\diamond})$

$$\leq Ac^{i-1}t_1 + Ac^i t_1 + ... + Ac^{j-2}t_1 = At_1 \sum_{k=i-1}^{j-1} c^k = At_1 c^{i-1} \sum_{k=i-1}^{j-i-1} c^k.$$

Hence, knowing that $At_1 \geq 0$ we get

$$l(x_i, x_j) \leq At_1 c^{i-1} \sum_{k=0}^{j-i-1} c^k \leq At_1 \sum_{k=i-1}^{+\infty} c^k$$

by the geometric series formula

$$= At_1 c^{i-1} \sum_{k=0}^{+\infty} c^k = \frac{At_1}{1-c} c^{i-1}.$$

Therefore, for every positive number ϵ there exists a natural number N where $\frac{At_1}{1-c}c^{N-1} < \epsilon$ and, hence, for all $j \geq i > N$,

$$l(x_j, x_i) \leq \frac{At_1}{1-c}c^{i-1} < \frac{At_1}{1-c}c^{N-1} < \epsilon. \qquad \square$$

The corollary below is straight forward by taking $Y = X$.

Corollary 5.1.5. *Let (X, d) be a metric space. Let $f : X \to X$ and $g : X \to X$ be two functions on X. If f and g are mutually contractive then there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that $\{f(x_i)\}_{n\in\mathbb{N}}$ and $\{g(x_i)\}_{n\in\mathbb{N}}$ form a Cauchy pair.*

5.2 Partial Metric Space

The results in this section are an extension of the work of Matthews et al. [7, 25] and Karapinar et al. [21, 22]. Although their theorems were quite elegant, we felt that by requiring their Cauchy sequences to have a central distance $r = 0$, the partial metric spaces were not used to their full potential. In [3], we give contracting criteria on a function f so that the central distance of the Cauchy sequence generated is not restricted to be 0.

Definition 5.2.1. *Let (X, p) be a partial metric space with x_o in X. Let $f : X \to X$ be a function on X. Let r and $0 < c < 1$ be two real numbers. We say that f is an* ***orbital c_r-contraction at x_o*** *(or f is* ***orbitally c_r-contractive at x_o****) if and only if for all natural numbers i,*

$$r \leq p(f^i(x_o), f^i(x_0))$$

and

$$p(f^{i+2}(x_o), f^{i+1}(x_0)) \leq r + c^{i+1}|p(f(x_o), x_o)|.$$

We say that f is an ***orbital c_r-contraction*** *(or f is* ***orbitally c_r-contractive****) if and only if for every x in X, f is a orbital $r-$contraction at x.*

Lemma 5.2.1. *Let (X, p) be a partial metric space with x_o in X. Let $f : X \to X$ be a function on X. If f is an orbital c_r-contraction at x_o then f is a Cauchy function at x_o.*

Proof: Let $x_o \in X$ and suppose $f : X \to X$ is an orbital c_r-contraction at x_o. Denote $x_i = f^i(x_o)$. We now move to prove that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Let $M = |p(x_1, x_o)| = |p(f(x_o), x_o)|$. Then from Definition 5.2.1 and by (p-lbnd) we get for all i,

$$r \leq p(x_i, x_i) \leq p(x_{i+1}, x_i) \leq r + c^i M \qquad (\triangle)$$

and, hence,

$$-p(x_i, x_i) \leq -r.$$

For all $j \geq i$, and by using (p-inq)

$$p(x_j, x_i) \leq p(x_j, x_{i+1}) + p(x_{i+1}, x_i) - p(x_{i+1}, x_{i+1})$$

by ($\triangle$)

$$\leq p(x_j, x_{i+1}) + r + c^i M - r = p(x_j, x_{i+1}) + c^i M$$

by (p-inq)

$$\leq p(x_j, x_{i+2}) + p(x_{i+2}, x_{i+1}) - p(x_{i+1}, x_{i+1}) + c^i M$$

by ($\triangle$)

$$\leq p(x_j, x_{i+2}) + r + c^{i+1} M - r + c^i M$$

$$= p(x_j, x_{i+2}) + c^i M + c^{i+1} M$$

by repeating this process

$$\leq p(x_j, x_{j-1}) + \sum_{t=i}^{j-2} c^t M$$

by ($\triangle$)

$$\leq r + c^{j-1} M + \sum_{t=i}^{j-2} c^t M$$

$$= r + \sum_{t=i}^{j-1} c^t M = r + c^i M \sum_{t=0}^{j-i-1} c^t.$$

We know that $M \geq 0$ and, hence, from the geometric series formula

$$p(x_j, x_i) \leq r + c^i M \sum_{t=0}^{j-i-1} c^t$$

$$\leq r + c^i M \sum_{t=0}^{+\infty} c^t = r + c^i \frac{M}{1-c}.$$

Since $0 < c < 1$, then for every positive real number ϵ there exists a natural number N such that

$$c^N \frac{M}{1-c} < \epsilon$$

and hence, by (p-lbnd) and ($\triangle$) for all $j \geq i > N$,

$$r - \epsilon < r \leq p(x_i, x_i) \leq p(x_j, x_i)$$

$$\leq r + c^i \frac{M}{1-c} < r + c^N \frac{M}{1-c} < r + \epsilon. \qquad \square$$

As we mentioned before, Karapinar et al. [21] generalized what was defined as a $\varphi-$weak contraction on a metric space [30] to what they called a weak $\varphi-$contraction on a partial metric space. Their proposed generalization contracted over the whole space and the central distance of the obtained Cauchy sequence was forced to be zero. In [3], we relax the constraints on the function f requiring it only to be contracting on an orbit. We also allow the central distance of the obtained Cauchy sequence to be any arbitrary real number r.

Definition 5.2.2. *Let (X, p) be a partial metric space with x_o in X. Let $f : X \to X$ be a function on X. Let r be a real number and $\varphi : [r, +\infty) \subset \mathbb{R} \to [0, +\infty)$ be a non-decreasing function such that*

$$\varphi(t) = 0 \text{ if and only if } t = r.$$

We say that f is an ***orbital φ_r-Contraction at x_o*** *(or f is* ***orbitally φ_r-contractive at x_o****) if and only if for all i and j,*

$$r \leq p(f^i(x_o), f^i(x_o))$$

and

$$p(f^{i+1}(x_o), f^{j+1}(x_o)) \leq p(f^i(x_o), f^j(x_o)) - \varphi(p(f^i(x_o), f^j(x_o))).$$

We say that f is an ***orbital φ_r-contraction*** *(or f is* ***orbitally φ_r-contractive*** *) if and only if for every x in X, f is an orbital φ_r-contraction at x.*

Lemma 5.2.2. *Let (X, p) be a partial metric space with x_o in X. Let $f : X \to X$ be a function on X. If f is an orbital φ_r-contraction at x_o then f is a Cauchy function at x_o.*

Proof: Let $x_o \in X$ and suppose $f : X \to X$ is an orbital φ_r-contraction at x_o. Denote $x_i = f^i(x_o)$.

Step 1: Let $t_i = p(x_{i+1}, x_i)$. In this step we will show that in the topological space $\mathbb{R}$ endowed with the standard topology, $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence that converges to r.
From (p-lbnd) and since $\varphi(t_i) \geq 0$,

$$r \leq p(x_{i+1}, x_{i+1}) \leq p(x_{i+2}, x_{i+1}) = t_{i+1}$$

and

$$t_{i+1} = p(x_{i+2}, x_{i+1}) \leq p(x_{i+1}, x_i) - \varphi(p(x_{i+1}, x_i))$$
$$= t_i - \varphi(t_i) \leq t_i.$$

Hence, for all i,

$$r \leq t_{i+1} \leq t_i$$

i.e. $\{t_i\}_{i\in\mathbb{N}}$ is a non-increasing sequence in $\mathbb{R}$ bounded below by r and, therefore, $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ with the standard topology is a complete metric space, $\{t_i\}_{i\in\mathbb{N}}$ has a limit L such that for all i,

$$t_i \geq L \geq r$$

and, since φ is a non-decreasing function,

$$\varphi(t_i) \geq \varphi(L) \geq \varphi(r) = 0$$

i.e.

$$-\varphi(t_i) \leq -\varphi(L) \leq 0.$$

Hence, by Definition 5.2.2

$$r \le t_{i+1} \le t_i - \varphi(t_i) \le t_i - \varphi(L)$$

$$\le t_{i-1} - \varphi(t_{i-1}) - \varphi(L) \le t_{i-1} - 2\varphi(L)$$

by induction

$$t_{i+1} \le t_1 - i\varphi(L).$$

Assume that $L > r$ then by Definition 5.2.2 $\varphi(L) > 0$. By taking $i > \frac{t_1 - r}{\varphi(L)}$ we get

$$t_{i+1} \le t_1 - i\varphi(L) < t_1 - \frac{t_1 - r}{\varphi(L)}\varphi(L) = r$$

a contradiction since $t_i \ge r$. Therefore, $L = r$.

Step 2: We now show that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with central distance r by supposing that it is not (a contrapositive approach). From Definition 5.2.2 we know that for all i and j,

$$r \le p(x_i, x_i) \le p(x_i, x_j)$$

in particular, for $i = j$,

$$-p(x_i, x_i) \le -r.$$

Hence, if $\{x_i\}_{i\in\mathbb{N}}$ is not a Cauchy sequence with central distance r then there exists a positive real number δ such that for every natural number N, there exists $i, j > N$ where

$$p(x_i, x_j) \ge r + \delta > r$$

and from step 1, by choosing N big enough

$$r \le p(x_i, x_i) \le p(x_i, x_{i+1}) < r + \delta.$$

Then there exist $j_1 > m_1 > N$ such that

$$p(x_{m_1}, x_{j_1}) \ge r + \delta > r.$$

Let n_1 be the smallest number with $n_1 > m_1$ and

$$p(x_{m_1}, x_{n_1}) \ge r + \delta.$$

Note

$$p(x_{m_1}, x_{n_1 - 1}) < r + \delta.$$

There exist $j_2 > m_2 > n_1$ such that

$$p(x_{m_2}, x_{j_2}) \ge r + \delta > r.$$

Let n_2 be the smallest number with $n_2 > m_2$ and

$$p(x_{m_2}, x_{n_2}) \ge r + \delta.$$

Then

$$p(x_{m_2}, x_{n_2-1}) < r + \delta.$$

Continuing this process, we build two increasing sequences in $\mathbb{N}$, $\{m_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ such that for all k,

$$p(x_{m_k}, x_{n_k-1}) < r + \delta \leq p(x_{m_k}, x_{n_k}).$$

For all k, denote $s_k = p(x_{m_k}, x_{n_k})$. By (p-inq)

$$s_k = p(x_{m_k}, x_{n_k}) \leq p(x_{m_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) - p(x_{n_k-1}, x_{n_k-1})$$

by (p-sym) and Step 1

$$= p(x_{m_k}, x_{n_k-1}) + p(x_{n_k}, x_{n_k-1}) - p(x_{n_k-1}, x_{n_k-1})$$

$$\leq p(x_{m_k}, x_{n_k-1}) + t_{n_k-1} - r$$

and, hence,

$$s_k \leq p(x_{m_k}, x_{n_k-1}) + t_{n_k-1} - r. \qquad (\triangle')$$

Additionally, since $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence tending to r, for every positive real number ϵ there exists a natural number N such that for all $n_k - 1 > N$,

$$r \leq t_{n_k-1} < r + \epsilon.$$

Since $\{m_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ are increasing sequences, there exists a natural number N' such that for all $k > N'$, $n_k - 1 > N$. Therefore for all $k > N'$ and since $r + \delta \leq s_k$,

$$0 \leq s_k - (r + \delta)$$

from $(\triangle')$

$$\leq p(x_{m_k}, x_{n_k-1}) + t_{n_k-1} - r - (r + \delta)$$

$$< (r + \delta) + (r + \epsilon) - r - (r + \delta) = \epsilon.$$

Therefore, $\{s_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence with $r + \delta$ as a limit. On the other hand by applying (p-inq) twice we get

$$s_k = p(x_{m_k}, x_{n_k}) \leq p(x_{m_k}, x_{n_k+1}) + p(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1})$$

$$\leq p(x_{m_k}, x_{m_k+1}) + p(x_{m_k+1}, x_{n_k+1}) - p(x_{m_k+1}, x_{m_k+1}) + p(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1})$$

from (p-sym) and Step 1

$$= p(x_{m_k+1}, x_{m_k}) + p(x_{m_k+1}, x_{n_k+1}) - p(x_{m_k+1}, x_{m_k+1}) + p(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1})$$

$$= t_{m_k} + p(x_{m_k+1}, x_{n_k+1}) - p(x_{m_k+1}, x_{m_k+1}) + t_{n_k} - p(x_{n_k+1}, x_{n_k+1})$$

$$\leq t_{m_k} + p(x_{m_k+1}, x_{n_k+1}) - r + t_{n_k} - r$$

from Definition 5.2.2

$$\leq t_{m_k} + p(x_{m_k}, x_{n_k}) - \varphi(p(x_{m_k}, x_{n_k})) - r + t_{n_k} - r$$

and, hence,

$$s_k \leq t_{m_k} + s_k - \varphi(s_k) + t_{n_k} - 2r$$

i.e.

$$\varphi(s_k) \leq t_{m_k} + t_{n_k} - 2r.$$

Since $r < r + \delta \leq s_k$ and from Definition 5.2.2 we get

$$0 < \varphi(r + \delta) \leq \varphi(s_k) \leq t_{m_k} + t_{n_k} - 2r.$$

Since $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence that tends to r then for every positive real number ϵ, there exists a natural number N such that for all $m_k, n_k > N$,

$$0 < \varphi(r + \delta) \leq t_{m_k} + t_{n_k} - 2r < (r + \frac{\epsilon}{2}) + (r + \frac{\epsilon}{2}) - 2r = \epsilon$$

and, hence,

$$0 < \varphi(r + \delta) \leq 0$$

a clear contradiction. Therefore, the assumption considered at the beginning of Step 2 is incorrect proving that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with central distance r. □

We now move to generalizing pairwise contractive functions.

Definition 5.2.3. *Let (X, p) be a partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions on X. Let r and $0 < c < 1$ be two real numbers. We say that f and g are* ***$f-$pairwise c_r-contractive over (x_o, y_o)*** *if and only if for all natural numbers i,*

$$\begin{cases} r \leq \min\{p(f^i(x_o), f^i(x_o)), p(g^i(y_o), g^i(y_o))\} \\ p(f^{i+1}(x_0), g^i(y_o)) \leq r + c^i M \\ p(f^i(x_0), g^i(y_o)) \leq r + c^i M \end{cases}$$

where $M = \max\{|p(f(x_o), y_o)|, |p(x_o, y_o)|\}$.

Theorem 5.2.3. ***(Cauchy $f-$Pairwise c_r-Contractive):***
Let (X, p) be a partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions on X. If f and g are $f-$pairwise c_r-contractive over (x_o, y_o) then f and g form a Cauchy pair over (x_o, y_o).

Proof: For all i, let $x_i = f^i(x_o)$ and $y_i = g^i(y_o)$. Let $M = \max\{|p(f(x_o), y_o)|, |p(x_o, y_o)|\}$.

First we show that for all $i \geq j$,

$$r \leq \min\{p(x_i, x_i), p(y_j, y_j)\} \leq p(x_i, y_j) \leq r + c^j \frac{2M}{1 - c}.$$

From Definition 5.2.3 we know that

$$\begin{cases} r \le \min\{p(x_i, x_i), p(y_i, y_i)\} \\ p(x_{i+1}, y_i) \le r + c^i M \\ p(x_i, y_i) \le r + c^i M \end{cases} \qquad (\tilde{\triangle})$$

In particular, for all i,

$$-p(x_i, x_i) \le -r \text{ and } -p(y_i, y_i) \le -r.$$

By Theorem 4.2.5 it suffices to bound $p(x_i, y_j)$ for $i \ge j$. Let us first investigate what happens for the specific values of $i = 6$ and $j = 3$. By repeatedly using (p-inq) we get

$$p(x_6, y_3) \le p(x_6, y_5) - p(y_5, y_5) + p(y_5, x_5) - p(x_5, x_5) + p(x_5, y_4) - p(y_4, y_4) + p(y_4, x_4) - p(x_4, x_4) + p(x_4, y_3)$$

by (p-sym)

$$= p(x_6, y_5) - p(y_5, y_5) + p(x_5, y_5) - p(x_5, x_5) + p(x_5, y_4) - p(y_4, y_4) + p(x_4, y_4) - p(x_4, x_4) + p(x_4, y_3)$$

by $(\tilde{\triangle})$

$$\le (r + c^5 M) - r + (r + c^5 M) - r + (r + c^4 M) - r + (r + c^4 M) - r + r + c^3 M$$

since $c^j M \ge 0$

$$\le r + 2[c^5 M + c^4 M + c^3 M].$$

We now move to derive an upper bound for $p(x_i, y_j)$ where $i \ge j$.
Case 1: If $i = j$, by $(\tilde{\triangle})$ and since $0 < c < 1$ and $c^j M \ge 0$,

$$r \le \min\{p(x_j, x_j), p(y_j, y_j)\} \le p(x_j, y_j) \le r + c^j M \le r + c^j \frac{2M}{1-c}.$$

Case 2: If $i > j$, by $(\tilde{\triangle})$ and (p-lbnd) we get

$$r \le p(x_i, x_i) \le p(x_i, y_j)$$

by repeatedly using (p-inq)

$$\le \sum_{t=j+1}^{i-1} [p(x_{t+1}, y_t) - p(y_t, y_t) + p(y_t, x_t) - p(x_t, x_t)] + p(x_{j+1}, y_j)$$

by (p-sym)

$$\le \sum_{t=j+1}^{i-1} [p(x_{t+1}, y_t) - p(y_t, y_t) + p(x_t, y_t) - p(x_t, x_t)] + p(x_{j+1}, y_j)$$

by $(\tilde{\triangle})$

$$\le \sum_{t=j+1}^{i-1} [(r + c^t M - r + (r + c^t M) - r] + r + c^j M = 2 \sum_{t=j+1}^{i-1} [c^t M] + r + c^j M$$

since $c^t M \geq 0$

$$\leq r + 2\sum_{t=j}^{i-1} c^t M = r + 2c^j \sum_{t=0}^{i-j-1} c^t M \leq r + 2c^j \sum_{t=0}^{+\infty} c^t M$$

finally, using the geometric series formula

$$= r + c^j \frac{2M}{1-c}.$$

Hence, for all $i \geq j$,

$$r \leq \min\{p(x_i, x_i), p(y_j, y_j)\} \leq p(x_i, y_j) \leq r + c^j \frac{2M}{1-c}.$$

For all positive real numbers ϵ there exists a natural number N such that $c^N \frac{2M}{1-c} < \epsilon$. Hence, for all $i \geq j > N$,

$$r - \epsilon < r \leq \min\{p(x_i, x_i), p(y_j, y_j)\} \leq p(x_i, y_j) \leq r + c^j \frac{2M}{1-c} < r + \frac{M}{1-c} c^N < r + \epsilon.$$

Therefore, by Theorem 4.2.5, f and g are Cauchy pairs over (x_o, y_o). □

Definition 5.2.4. *Let (X, p) and (Y, h) be two partial metric spaces and suppose $f : X \to Y$ and $g : X \to Y$ are two functions on X. Let r, $A \geq 0$ and $0 < c < 1$ be three real numbers. We say that f and g are* ***$f-$mutually c_r-contractive*** *if and only for each x in X we can find an element z of X such that*

$$h(f(z), g(z)) - h(f(z), f(z)) \leq c[h(f(x), g(x)) - h(f(x), f(x))]$$

and

$$r \leq p(z, z) \leq p(x, z) \leq r + A[h(f(x), g(x)) - h(f(x), f(x))].$$

In the above definition, putting a heavier emphasis on one function makes it much easier to apply the theorem below on a pair of functions when one is much more complex than the other. The above contraction is enough to generate a Cauchy sequence in X as shown in Theorem 5.2.4. In the case of (Y, h) being a strong partial metric space, $f-$mutually c_r-contraction is used to obtain a coincidence point. However, when (Y, h) is a partial metric space, we need a stronger version.

Definition 5.2.5. *Let (X, p) and (Y, h) be two partial metric spaces and suppose $f : X \to Y$ and $g : X \to Y$ are two functions on X. Let r, $A \geq 0$ and $0 < c < 1$ be three real numbers. We say that f and g are* ***$(f, g)-$mutually c_r-contractive*** *if and only for each x in X we can find an element z of X such that*

$$h(f(z), g(z)) - h(f(z), f(z)) \leq c[h(f(x), g(x)) - h(f(x), f(x))]$$

$$h(f(z), g(z)) - h(g(z), g(z)) \leq c[h(f(x), g(x)) - h(g(x), g(x))]$$

$$r \leq p(z, z) \leq p(x, z) \leq r + A[h(f(x), g(x)) - h(f(x), f(x))]$$

and

$$r \leq p(z, z) \leq p(x, z) \leq r + A[h(f(x), g(x)) - h(g(x), g(x))].$$

It is clear that if two functions f and g are $(f,g)-$mutually c_r-contractive then they are $f-$mutually c_r-contractive and $g-$mutually c_r-contractive.

Theorem 5.2.4. ***(Cauchy $f-$Mutually c_r-Contractive):***
Let (X,p) and (Y,h) be two partial metric spaces. Let $f : X \to Y$ and $g : X \to Y$ be two functions on X. If f and g are $f-$mutually c_r-contractive then there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with central distance r such that for all natural numbers i, $r \le p(x_i, x_i)$. Additionally for every positive real number ϵ there exists a natural number N such that for all $i > N$,

$$h(f(x_i), g(x_i)) - h(f(x_i), f(x_i)) < \epsilon.$$

Proof: Since f and g are $f-$mutually c_r-contractive then there exist two real numbers $0 < c < 1$ and $A \ge 0$ such that for each x in X, there exists a z in X such that

$$h(f(z), g(z)) - h(f(z), f(z)) \le c[h(f(x), g(x)) - h(f(x), f(x))]$$

and

$$r \le p(z,z) \le p(x,z) \le r + A[h(f(x), g(x)) - h(f(x), f(x))].$$

Let us take an arbitrary element x_1 in X then there exists an element x_2 in X such that

$$h(f(x_2), g(x_2)) - h(f(x_2), f(x_2)) \le c[h(f(x_1), g(x_1)) - h(f(x_1), f(x_1))]$$

and

$$r \le p(x_2, x_2) \le p(x_1, x_2) \le r + A[h(f(x_1), g(x_1)) - h(f(x_1), f(x_1))].$$

There exists an element x_3 in X such that

$$h(f(x_3), g(x_3)) - h(f(x_3), f(x_3)) \le c[h(f(x_2), g(x_2)) - h(f(x_2), f(x_2))]$$

and

$$r \le p(x_3, x_3) \le p(x_2, x_3) \le r + A[h(f(x_2), g(x_2)) - h(f(x_2), f(x_2))].$$

We continue the above process to generate the sequence $\{x_i\}_{i\in\mathbb{N}}$ such that for all i,

$$h(f(x_{i+1}), g(x_{i+1})) - h(f(x_{i+1}), f(x_{i+1})) \le c[h(f(x_i), g(x_i)) - h(f(x_i), f(x_i))]$$

and

$$r \le p(x_{i+1}, x_{i+1}) \le p(x_i, x_{i+1}) \le r + A[h(f(x_i), g(x_i)) - h(f(x_i), f(x_i))] \qquad (\ddot{\triangle})$$

in particular,

$$r \le p(x_i, x_i)$$

i.e.

$$-p(x_i, x_i) \le r.$$

We now prove that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with central distance r.

Step 1: For all i, let $t_i = h(f(x_i), g(x_i)) - h(f(x_i), f(x_i))$. Then from $(\ddot{\triangle})$

$$t_{i+1} = h(f(x_{i+1}), g(x_{i+1})) - h(f(x_{i+1}), f(x_{i+1}))$$

$$\leq c[h(f(x_i), g(x_i)) - h(f(x_i), f(x_i))] \leq c^2[h(f(x_{i-1}), g(x_{i-1})) - h(f(x_{i-1}), f(x_{i-1}))]$$

and, hence, by induction

$$t_{i+1} \leq c^i[h(f(x_1), g(x_1)) - h(f(x_1), g(x_1))] = c^i t_1. \qquad (\check{\triangle})$$

Therefore, and since $0 < c < 1$ for every positive real number ϵ there exists natural number N' such that for all $i > N'$,

$$h(f(x_i), g(x_i)) - h(f(x_i), f(x_i)) < \epsilon.$$

Step 2: For all $j > i$, by $(\ddot{\triangle})$ and repeatedly using (p-inq) we get,

$$r \leq p(x_i, x_i) \leq p(x_i, x_j)$$

$$\leq [p(x_i, x_{i+1}) - p(x_{i+1}, x_{i+1})] + [p(x_{i+1}, x_{i+2}) - p(x_{i+2}, x_{i+2})] + ... + [p(x_{j-2}, x_{j-1}) - p(x_{j-1}, x_{j-1})] + p(x_{j-1}, x_j)$$

by $(\ddot{\triangle})$

$$\leq [r + At_i - r] + [r + At_{i+1} - r] + ... + [r + At_{j-2} - r] + r + At_{j-1} = r + \sum_{k=i}^{j-1} At_k$$

by $(\check{\triangle})$

$$\leq r + \sum_{k=i}^{j-1} Ac^{k-1}t_1 = r + At_1c^{i-1} \sum_{k=0}^{j-i-1} c^k$$

knowing that $At_1 \geq 0$ and by the geometric series formula

$$\leq r + At_1 \sum_{k=0}^{+\infty} c^k$$

by the geometric series formula

$$= r + At_1c^{i-1} \sum_{k=0}^{+\infty} c^k = r + \frac{At_1}{c(1-c)}c^i.$$

Therefore, for every positive number ϵ there exists a natural number N where $\frac{At_1}{c(1-c)}c^N < \epsilon$ and, hence, for all $j \geq i > N$,

$$r - \epsilon < r \leq p(x_j, x_i) \leq r + \frac{At_1}{c(1-c)}c^i < r + \frac{At_1}{c(1-c)}c^N < r + \epsilon. \qquad \square$$

The corollary below is straight forward by taking $Y = X$.

Corollary 5.2.5. *Let (X, p) be a partial metric space. Let $f : X \to X$ and $g : X \to X$ be two functions on X. If f and g are $f-$mutually c_r-contractive then there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with central distance r such that for all natural numbers i, $r \leq p(x_i, x_i)$. Additionally for every positive real number ϵ there exists a natural number N such that for all $i > N$,*

$$p(f(x_i), g(x_i)) - p(f(x_i), f(x_i)) < \epsilon.$$

5.3 Partial $n - \mathfrak{M}$etric Space

The inspiration for this section mainly came from Ayadi et al. [4]. As explained in Chapter 2, we have relaxed the axioms of a G_p-metric (see [34]) to obtain the partial $3 - \mathfrak{M}$etric. We also have a much less restrictive condition on the contracting functions and the central distance of a Cauchy sequence. To this end, we will use Theorem 4.3.1 to check whether a sequence is Cauchy or not by comparing the elements pairwise rather than comparing $n-$tuples.

Computations with partial $n - \mathfrak{M}$etrics when $n > 2$ require more attention than their partial metric counterparts because of the following: Let P be a partial $n - \mathfrak{M}$etric on X. If $n = 2$ then P is a partial metric. Hence, for any two elements a and b in a set X we have by (p-sym)

$$P(\langle a\rangle^{2-1}, b) = P(\langle b\rangle^{2-1}, a).$$

In the more general case with $n > 2$, we have by Corollary 2.5.4,

$$P(\langle a\rangle^{n-1}, b) \leq (n-1)P(\langle b\rangle^{n-1}, a) - (n-2)P(\langle b\rangle^{n}).$$

The proofs of Section 5.2 generalize to the proofs of Section 5.3 by adding steps and considering a different ϵ to compensate for the that fact. Otherwise, the proofs are very similar to the proofs in Section 5.2 and, hence, may be skipped if the reader so desires.

Definition 5.3.1. *Let (X, P) be a partial $n-\mathfrak{M}$etric space with x_o in X and suppose $f : X \to X$ is a function on X. Let r and $0 < c < 1$ be two real numbers. We say that f is an **orbital c_r-contraction at x_o** (or f is **orbitally c_r-contractive at x_o**) if and only if for all natural numbers i,*

$$r \leq P(\langle f^{i+1}(x_o)\rangle^n)$$

and

$$P(\langle f^i(x_o)\rangle^{n-1}, f^{i+1}(x_0)) \leq r + c^i|P(\langle x_o\rangle^{n-1}, f(x_o))|.$$

*We say that f is an **orbital c_r-contraction** (or f is **orbitally c_r-contractive**) if and only if for every x in X, f is an orbital c_r-contraction at x.*

Lemma 5.3.1. *Let (X, P) be a partial $n - \mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a function on X. If f is an orbital c_r-contraction at x_o then f is a Cauchy function at x_o.*

Proof: Let $x_o \in X$ and suppose $f : X \to X$ is an orbital c_r-contraction at x_o. Denote $x_i = f^i(x_o)$. We now move to prove that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Let

$$M = |P(\langle x_o\rangle^{n-1}, x_1)| = |P(\langle x_o\rangle^{n-1}, f(x_o))|.$$

Then, from Definition 5.3.1 and by (P_n-lbnd) we get for all i,

$$r \leq P(\langle x_i\rangle^n) \leq P(\langle x_i\rangle^{n-1}, x_{i+1}) \leq r + c^iM \qquad (\otimes)$$

and, hence,

$$-P(\langle x_i \rangle^n) \leq -r.$$

For all $j \geq i$, and by using (P_n-inq)

$$P(\langle x_i \rangle^{n-1}, x_j) \leq P(\langle x_i \rangle^{n-1}, x_{i+1}) + P(\langle x_{i+1} \rangle^{n-1}, x_j) - P(\langle x_{i+1} \rangle^n)$$

by ($\otimes$)

$$\leq r + c^i M + P(\langle x_{i+1} \rangle^{n-1}, x_j) - r = P(\langle x_{i+1} \rangle^{n-1}, x_j) + c^i M$$

by (P_n-inq)

$$\leq P(\langle x_{i+1} \rangle^{n-1}, x_{i+2}) + P(\langle x_{i+2} \rangle^{n-1}, x_j) - P(\langle x_{i+2} \rangle^n) + c^i M$$

by ($\otimes$)

$$\leq r + c^{i+1} M + P(\langle x_{i+2} \rangle^{n-1}, x_j) - r + c^i M$$

$$= P(\langle x_{i+2} \rangle^{n-1}, x_j) + c^i M + c^{i+1} M$$

by repeating this process

$$\leq P(\langle x_{j-1} \rangle^{n-1}, x_j) + \sum_{k=i}^{j-2} c^k M$$

by ($\otimes$)

$$\leq r + c^{j-1} + \sum_{k=i}^{j-2} c^k M = r + M \sum_{k=i}^{j-1} c^k = r + Mc^i \sum_{k=0}^{j-i-1} c^k.$$

We know that $M \geq 0$and, hence, from the geometric series formula

$$P(\langle x_i \rangle^{n-1}, x_j) \leq r + Mc^i \sum_{k=0}^{j-i-1} c^k$$

$$\leq r + Mc^i \sum_{k=0}^{+\infty} c^k = r + c^i \frac{M}{1-c}.$$

Since $0 < c < 1$, then for every positive real number ϵ there exists a natural number N such that

$$c^N \frac{M}{1-c} < \epsilon$$

and, hence, for all $j \geq i > N$,

$$r - \epsilon < r \leq P(\langle x_i \rangle^{n-1}, x_j) \leq r + c^i \frac{M}{1-c} < r + c^N \frac{M}{1-c} < r + \epsilon.$$

Therefore, by Theorem 4.3.1, $\{x_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence with central distance r. $\square$

Bilgili et al. [6] generalized the idea of a weak contraction into a G_p-metric space. We build on his work and generalize it to a partial $n - \mathfrak{M}$etric case.

Definition 5.3.2. *Let (X,P) be a partial $n-\mathfrak{M}$etric space with x_o in X and suppose $f: X \to X$ is a function on X. Let r be a real number and $\varphi : [r,+\infty) \subset \mathbb{R} \to [0,+\infty)$ be a non-decreasing function such that*

$$\varphi(t) = 0 \text{ if and only if } t = r.$$

We say that f is an <u>***orbital*** φ_r***−contraction at*** x_o</u> *(or f is* <u>***orbitally*** φ_r***−contractive at*** x_o</u>*) if and only if for all i and j,*

$$r \leq P(\langle f^i(x_o)\rangle^n)$$

and

$$P(\langle f^{i+1}(x_o)\rangle^{n-1}, f^{j+1}(x_o)) \leq P(\langle f^i(x_o)\rangle^{n-1}, f^j(x_o)) - \varphi(P(\langle f^i(x_o)\rangle^{n-1}, f^j(x_o))).$$

We say that f is an <u>***orbital*** φ_r***−contraction***</u> *(or f is* <u>***orbitally*** φ_r***−contractive***</u>*) if and only if for every x in X, f is an orbital φ_r-contraction at x.*

Lemma 5.3.2. *Let (X,P) be a partial $n-\mathfrak{M}$etric space with x_o in X. Let $f: X \to X$ be a function on X. If f is an orbital φ_r-contraction at x_o then f is a Cauchy function at x_o.*

Proof: Let $x_o \in X$ and suppose $f: X \to X$ is an orbital φ_r-contraction at x_o. Denote $x_i = f^i(x_o)$.

<u>Step 1:</u> Let $t_i = P(\langle x_i\rangle^{n-1}, x_{i+1})$. In this step we will show that in the topological space $\mathbb{R}$ endowed with the standard topology, $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence that converges to r.
From (P_n-lbnd) and Definition 5.3.2,

$$r \leq P(\langle x_{i+1}\rangle^n) \leq P(\langle x_{i+1}\rangle^{n-1}, x_{i+2}) = t_{i+1}$$

$$\leq P(\langle x_i\rangle^{n-1}, x_{i+1}) - \varphi(P(\langle x_i\rangle^{n-1}, x_{i+1}))$$

since $\varphi(t_i) \geq 0$

$$= t_i - \varphi(t_i) \leq t_i.$$

Hence, for all n,

$$r \leq t_{i+1} \leq t_i$$

i.e. $\{t_i\}_{i\in\mathbb{N}}$ is a non-increasing sequence in $\mathbb{R}$ bounded below by r and, therefore, $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ with the standard topology is a complete metric space, $\{t_i\}_{i\in\mathbb{N}}$ has a limit L such that for all i,

$$\varphi(t_i) \geq \varphi(L) \geq \varphi(r) = 0$$

i.e.

$$-\varphi(t_i) \leq -\varphi(L) \leq 0.$$

Hence, by Definition 5.3.2

$$r \leq t_{i+1} \leq t_i - \varphi(t_i) \leq t_i - \varphi(L)$$

$$\leq t_{i-1} - \varphi(t_{i-1}) - \varphi(L) \leq t_{i-1} - 2\varphi(L)$$

by induction

$$t_{i+1} \leq t_1 - i\varphi(L).$$

Assume that $L > r$ then by Definition 5.3.2 $\varphi(L) > 0$. By taking $i > \frac{t_1 - r}{\varphi(L)}$ we get

$$t_{i+1} \leq t_1 - i\varphi(L) < t_1 - \frac{t_1 - r}{\varphi(L)}\varphi(L) = r$$

a contradiction since $t_i \geq r$. Therefore, $L = r$.

Step 2: We now show that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with central distance r by supposing that it is not (a contrapositive approach). To do that we refer the reader back to Theorem 4.3.1 (c). From Definition 5.3.2 we know that for all i and j,

$$r \leq P(\langle x_i\rangle^n) \leq P(\langle x_i\rangle^{n-1}, x_{i+1})$$

in particular, for $i = j$

$$-P(\langle x_i\rangle^n) \leq -r.$$

Hence, if $\{x_i\}_{i\in\mathbb{N}}$ is not a Cauchy sequence with central distance r then by Theorem 4.3.1 (d) there exists a positive real number δ such that for every natural number N, there exists $j \geq i > N$ where

$$P(\langle x_i\rangle^{n-1}, x_j) \geq r + \delta > r$$

and from step 1, by choosing N big enough

$$r \leq P(\langle x_i\rangle^n) \leq P(\langle x_i\rangle^{n-1}, x_{i+1}) < r + \delta.$$

Then there exist $j_1 > m_1 > N$ such that

$$P(\langle x_{m_1}\rangle^{n-1}, x_{j_1})) \geq r + \delta > r.$$

Let n_1 be the smallest number with $n_1 > m_1$ and

$$P(\langle x_{m_1}\rangle^{n-1}, x_{n_1}) \geq r + \delta.$$

Note

$$P(\langle x_{m_1}\rangle^{n-1}, x_{n_1-1}) < r + \delta.$$

There exist $j_2 > m_2 > n_1$ such that

$$P(\langle x_{m_2}\rangle^{n-1}, x_{j_2}) \geq r + \delta \geq r + \delta > r.$$

Let n_2 be the smallest number with $n_2 > m_2$ and

$$P(\langle x_{m_2}\rangle^{n-1}, x_{n_2}) \geq r + \delta \geq r + \delta.$$

Then

$$P(\langle x_{m_2}\rangle^{n-1}, x_{n_2-1}) < r + \delta.$$

Continuing this process, we build two increasing sequences in $\mathbb{N}$, $\{m_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ such that for all k,

$$P(\langle x_{m_k}\rangle^{n-1}, x_{n_k-1}) < r+\delta \leq P(\langle x_{m_k}\rangle^{n-1}, x_{n_k}).$$

For all k, denote $s_k = P(\langle x_{m_k}\rangle^{n-1}, x_{n_k})$. By ($P_n$-inq)

$$s_k = P(\langle x_{m_k}\rangle^{n-1}, x_{n_k}) \leq P(\langle x_{m_k}\rangle^{n-1}, x_{n_k-1}) + P(\langle x_{n_k-1}\rangle^{n-1}, x_{n_k}) - P(\langle x_{n_k-1}\rangle^{n})$$

by Step 1

$$s_k \leq P(\langle x_{m_k}\rangle^{n-1}, x_{n_k-1}) + t_{n_k-1} - r \qquad (\tilde{\otimes})$$

Additionally, since $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence tending to r, for every positive real number ϵ there exists a natural number N such that for all $n_k - 1 > N$,

$$r \leq t_{n_k-1} < r+\epsilon.$$

Since $\{m_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ are increasing sequences, there exists a natural number N' such that for all $k > N'$, $n_k - 1 > N$. Therefore for all $k > N'$ and since $r+\delta \leq s_k$,

$$0 \leq s_k - (r+\delta)$$

from ($\tilde{\otimes}$)

$$\leq P(\langle x_{m_k}\rangle^{n-1}, x_{n_k-1}) + t_{n_k-1} - r - (r+\delta)$$

$$< (r+\delta) + (r+\epsilon) - r - (r+\delta) = \epsilon.$$

Therefore, $\{s_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence with $r+\delta$ as a limit. On the other hand by applying (P_n-inq) we get

$$s_k = P(\langle x_{m_k}\rangle^{n-1}, x_{n_k}) \leq P(\langle x_{m_k}\rangle^{n-1}, x_{m_k+1}) + P(\langle x_{m_k+1}\rangle^{n-1}, x_{n_k}) - P(\langle x_{m_k+1}\rangle^{n})$$

by Step 1

$$\leq t_{m_k} + P(\langle x_{m_k+1}\rangle^{n-1}, x_{n_k}) - r$$

by (P_n-inq)

$$\leq t_{m_k} + P(\langle x_{m_k+1}\rangle^{n-1}, x_{n_k+1}) + P(\langle x_{n_k+1}\rangle^{n-1}, x_{n_k}) - P(\langle x_{n_k+1}\rangle^{n}) - r$$

by Corollary 2.5.4

$$\leq t_{m_k} + P(\langle x_{m_k+1}\rangle^{n-1}, x_{n_k+1}) + (n-1)P(\langle x_{n_k}\rangle^{n-1}, x_{n_k+1}) - (n-2)P(\langle x_{n_k}\rangle^{n}) - P(\langle x_{n_k+1}\rangle^{n}) - r$$

by Step 1

$$\leq t_{m_k} + P(\langle x_{m_k+1}\rangle^{n-1}, x_{n_k+1}) + (n-1)t_{n_k} - (n-2)r - r - r$$

by Definition 5.3.2

$$\leq t_{m_k} + P(\langle x_{m_k}\rangle^{n-1}, x_{n_k}) - \varphi(P(\langle x_{m_k}\rangle^{n-1}, x_{n_k})) + (n-1)t_{n_k} - (n-2)r - r - r$$

and, hence,

$$s_k \leq t_{m_k} + s_k - \varphi(s_k) + (n-1)t_{n_k} - nr$$

therefore,

$$\varphi(s_k) \leq t_{m_k} + (n-1)t_{n_k} - nr.$$

Since $r < r + \delta \leq s_k$ and from Definition 5.3.2 we get

$$0 < \varphi(r+\delta) \leq \varphi(s_k) \leq t_{m_k} + (n-1)t_{n_k} - nr$$

Additionally, since $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence tending to r, for every positive real number ϵ there exists a natural number N such that for all $n_k > m_k > N$,

$$r \leq t_{n_k} \leq t_{m_k} < r + \frac{\epsilon}{n}$$

therefore, for every positive real number ϵ there exists a natural number N such that for all $n_k > m_k > N$,

$$0 < \varphi(r+\delta) < r + \frac{\epsilon}{n} + (n-1)(r + \frac{\epsilon}{n}) - nr = \epsilon$$

and, hence,

$$0 < \varphi(r+\delta) \leq 0$$

a clear contradiction. Therefore, the assumption considered at the beginning of Step 2 is incorrect proving that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence with central distance r. $\square$

Definition 5.3.3. *Let (X,P) be a partial $n-\mathfrak{M}etric$ space with x_o and y_o in X and suppose $f : X \to X$ and $g : X \to X$ are two functions on X. Let r and $0 < c < 1$ be two real numbers. We say that f and g are* <u>***$f-$pairwise c_r-contractive over** (x_o, y_o)*</u> *if and only if for all natural numbers i,*

$$\begin{cases} r \leq \min\{P(\langle f^i(x_o)\rangle^n), P(\langle g^i(x_o)\rangle^n\} \\ P(\langle f^{i+1}(x_o)\rangle^{n-1}, g^i(y_o)) \leq r + c^i M \\ P(\langle f^i(x_o)\rangle^{n-1}, g^i(y_o)) \leq r + c^i M \end{cases}$$

where $M = \max\{|P(\langle f(x_o)\rangle^{n-1}, y_o)|, |P(\langle x_o\rangle^{n-1}, y_o)|\}$.

Theorem 5.3.3. ***(Cauchy $f-$Pairwise c_r-Contractive):***
Let (X,P) be a partial $n-\mathfrak{M}etric$ space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions on X. If f and g are $f-$pairwise c_r-contractive over (x_o, y_o) then f and g form a Cauchy pair over (x_o, y_o).

Proof: For all i, let $x_i = f^i(x_o)$ and $y_i = g^i(y_o)$. Let

$$M = \max\{|P(\langle f(x_o)\rangle^{n-1}, y_o)|, |P(\langle x_o\rangle^{n-1}, y_o)|\}.$$

First we prove that for all $i \geq j$,

$$r \leq \min\{P(\langle x_i \rangle^n), P(\langle y_i \rangle^n)\} \leq P(\langle x_i \rangle^{n-1}, y_j) \leq r + c^j \frac{nM}{1-c}.$$

From Definition 5.3.3 we know that

$$\begin{cases} r \leq \min\{P(\langle x_i \rangle^n), P(\langle y_i \rangle^n)\} \\ P(\langle x_{i+1} \rangle^{n-1}, y_i) \leq r + c^i M \\ P(\langle x_i \rangle^{n-1}, y_i) \leq r + c^i M \end{cases} \qquad (\ddot{\otimes})$$

In particular, for all i,

$$-P(\langle x_i \rangle^n) \leq -r \text{ and } -P(\langle y_i \rangle^n) \leq -r.$$

By Theorem 4.3.8 it suffices to bound $P(\langle x_i \rangle^{n-1}, y_j)$ for $i \geq j$. Let us first investigate what happens for the specific values of $i = 6$ and $j = 3$. By repeatedly using (P_n-inq) we get

$$P(\langle x_6 \rangle^{n-1}, y_3)$$

$$\leq P(\langle x_6 \rangle^{n-1}, y_5) - P(\langle y_5 \rangle^n) + P(\langle y_5 \rangle^{n-1}, x_5) - P(\langle x_5 \rangle^n) + P(\langle x_5 \rangle^{n-1}, y_4) - P(\langle y_4 \rangle^n)$$

$$+P(\langle y_4 \rangle^{n-1}, x_4) - P(\langle x_4 \rangle^n) + P(\langle x_4 \rangle^{n-1}, y_3)$$

by Corollary 2.5.4

$$\leq P(\langle x_6 \rangle^{n-1}, y_5) - P(\langle y_5 \rangle^n) + [(n-1)P(\langle x_5 \rangle^{n-1}, y_5) - (n-2)P(\langle x_5 \rangle^n)] - P(\langle x_5 \rangle^n) + P(\langle x_5 \rangle^{n-1}, y_4) - P(\langle y_4 \rangle^n)$$

$$+[(n-1)P(\langle x_4 \rangle^{n-1}, y_4) - (n-2)P(\langle x_4 \rangle^n)] - P(\langle x_4 \rangle^n) + P(\langle x_4 \rangle^{n-1}, y_3)$$

by ($\ddot{\otimes}$)

$$\leq (r+c^5M)-r+(n-1)(r+c^5M)+(n-2)(-r)-r+(r+c^4M)-r+(n-1)(r+c^4M)+(n-2)(-r)-r+(r+c^3M)$$

$$= r + n[c^5M + c^4M] + c^3M$$

since $c^jM \geq 0$

$$\leq r + n[c^5M + c^4M + c^3M].$$

We now move to derive an upper bound for $P(\langle x_i \rangle^{n-1}, y_j)$ where $i \geq j$.
Case 1: If $i = j$, by ($\ddot{\otimes}$) and since $0 < c < 1$ and $c^jM \geq 0$,

$$r \leq \min\{P(\langle x_j \rangle^n), P(\langle y_j \rangle^n)\} \leq P(\langle x_j \rangle^{n-1}, y_j) \leq r + c^jM \leq r + c^j \frac{nM}{1-c}.$$

Case 2: If $i > j$, by ($\ddot{\otimes}$) and (P_n-lbnd) we get

$$r \leq \min\{P(\langle x_i \rangle^n), P(\langle y_j \rangle^n)\} \leq P(\langle x_i \rangle^{n-1}, y_j)$$

by repeatedly using (P_n-inq)

$$\leq \sum_{t=j+1}^{i-1} [P(\langle x_{t+1}\rangle^{n-1}, y_t) - P(\langle y_t\rangle^n) + P(\langle y_t\rangle^{n-1}, x_t) - P(\langle x_t\rangle^n)] + P(\langle x_{j+1}\rangle^{n-1}, y_j)$$

by Corollary 2.5.4

$$\leq \sum_{t=j+1}^{i-1} [P(\langle x_{t+1}\rangle^{n-1}, y_t) - P(\langle y_t\rangle^n) + (n-1)P(\langle y_t\rangle^{n-1}, x_t) - (n-2)P(\langle x_t\rangle^n) - P(\langle x_t\rangle^n)] + P(\langle x_{j+1}\rangle^{n-1}, y_j)$$

by ($\ddot{\otimes}$)

$$\leq \sum_{t=j+1}^{i-1} [(r + c^t M) - r + (n-1)(r + c^t M) + (n-2)(-r) - r] + (r + c^j M)$$

$$= \sum_{t=j+1}^{i-1} [nc^t M] + (r + c^j M)$$

since $c^j M \geq 0$

$$\leq r + nM \sum_{t=j+1}^{i-1} [c^t] + nMc^j = r + nM \sum_{t=j}^{i-1} c^t = r + nMc^j \sum_{t=0}^{i-j-1} c^t \leq r + nMc^j \sum_{t=0}^{+\infty} c^t$$

finally, using the geometric series formula

$$= r + c^j \frac{nM}{1-c}.$$

Hence, for all $i \geq j$,

$$r \leq \min\{P(\langle x_i\rangle^n), P(\langle y_i\rangle^n)\} \leq P(\langle x_i\rangle^{n-1}, y_j) \leq r + c^j \frac{nM}{1-c}.$$

For all positive real number ϵ there exists a natural number N such that $c^N \frac{nM}{1-c} < \epsilon$. Hence, for all $i \geq j > N$,

$$r - \epsilon < r \leq \min\{P(\langle x_i\rangle^n), P(\langle y_i\rangle^n)\} \leq P(\langle x_i\rangle^{n-1}, y_j) \leq r + c^j \frac{nM}{1-c} < r + c^N \frac{nM}{1-c} < r + \epsilon.$$

Therefore, by Theorem 4.3.8, f and g are Cauchy pairs over (x_o, y_o). $\square$

Definition 5.3.4. *Let (X, P) and (Y, H) be two partial $n - \mathfrak{M}$etric spaces and suppose $f : X \to Y$ and $g : X \to Y$ are two functions on X. Let r, $A \geq 0$ and $0 < c < 1$ be three real numbers. We say that f and g are* ***f−mutually c_r−contractive*** *if and only if for each x in X we can find an element z in X such that*

$$H(\langle f(z)\rangle^{n-1}, g(z)) - H(\langle f(z)\rangle^n) \leq c[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle f(x)\rangle^n)]$$

and

$$r \leq P(\langle z\rangle^n) \leq P(\langle z\rangle^{n-1}, x) \leq r + A[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle f(x)\rangle^n)].$$

As in the case of partial metrics, the above definition is used in coincidence point theorems of strong partial $n - \mathfrak{M}$etric. In the partial $n - \mathfrak{M}$etric case, a stronger contraction is needed.

Definition 5.3.5. *Let (X, P) and (Y, H) be two partial $n - \mathfrak{M}$etric spaces and suppose $f : X \to Y$ and $g : X \to Y$ are two functions on X. Let r, $A \geq 0$ and $0 < c < 1$ be three real numbers. We say that f and g are* $\underline{(f,g)-\textbf{\textit{mutually}}\ c_r-\textbf{\textit{contractive}}}$ *if and only if for each x in X we can find an element z in X such that*

$$H(\langle f(z)\rangle^{n-1}, g(z)) - H(\langle f(z)\rangle^n) \leq c[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle f(x)\rangle^n)]$$

$$H(\langle f(z)\rangle^{n-1}, g(z)) - H(\langle g(z)\rangle^n) \leq c[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle g(x)\rangle^n)]$$

$$r \leq P(\langle z\rangle^n) \leq P(\langle z\rangle^{n-1}, x) \leq r + A[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle f(x)\rangle^n)]$$

and

$$r \leq P(\langle z\rangle^n) \leq P(\langle z\rangle^{n-1}, x) \leq r + A[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle g(x)\rangle^n)].$$

Theorem 5.3.4. ***(Cauchy $f-$Mutually c_r-Contractive):***
Let (X, P) and (Y, H) be two partial $n - \mathfrak{M}$etric spaces. Let $f : X \to Y$ and $g : X \to Y$ be two functions on X. If f and g are $f-$mutually c_r-contractive then there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with central distance r such that for all natural numbers i, $r \leq P(\langle x_i\rangle^n)$. Additionally for every positive real number ϵ there exists a natural number N such that for all $i > N$,

$$H(\langle f(x_i)\rangle^{n-1}, g(x_i)) - H(\langle f(x_i)\rangle^n) < \epsilon.$$

Proof: Since f and g are $f-$mutually c_r-contractive then there exists two real numbers $0 < c < 1$ and $A \geq 0$ such that for each x in X, there exists a z in X where

$$H(\langle f(z)\rangle^{n-1}, g(z)) - H(\langle f(z)\rangle^n) \leq c[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle f(x)\rangle^n)]$$

and

$$r \leq P(\langle z\rangle^n) \leq P(\langle z\rangle^{n-1}, x) \leq r + A[H(\langle f(x)\rangle^{n-1}, g(x)) - H(\langle f(x)\rangle^n)].$$

Let us take an arbitrary element x_1 in X then there exists an element x_2 in X such that

$$H(\langle f(x_2)\rangle^{n-1}, g(x_2)) - H(\langle f(x_2)\rangle^n) \leq c[H(\langle f(x_1)\rangle^{n-1}, g(x_1)) - H(\langle f(x_1)\rangle^n)]$$

and

$$r \leq P(\langle x_2\rangle^n) \leq P(\langle x_2\rangle^{n-1}, x_1) \leq r + A[H(\langle f(x_1)\rangle^{n-1}, g(x_1)) - H(\langle f(x_1)\rangle^n)].$$

There exists an element x_3 in X such that

$$H(\langle f(x_3)\rangle^{n-1}, g(x_3)) - H(\langle f(x_3)\rangle^n) \leq c[H(\langle f(x_2)\rangle^{n-1}, g(x_2)) - H(\langle f(x_2)\rangle^n)]$$

and

$$r \leq P(\langle x_3\rangle^n) \leq P(\langle x_3\rangle^{n-1}, x_2) \leq r + A[H(\langle f(x_2)\rangle^{n-1}, g(x_2)) - H(\langle f(x_2)\rangle^n)].$$

We continue the above process to generate a sequence $\{x_i\}_{i\in\mathbb{N}}$ such that for all i,

$$H(\langle f(x_{i+1})\rangle^{n-1}, g(x_{i+1})) - H(\langle f(x_{i+1})\rangle^n) \leq c[H(\langle f(x_i)\rangle^{n-1}, g(x_i)) - H(\langle f(x_i)\rangle^n)]$$

and

$$r \leq P(\langle x_{i+1}\rangle^n) \leq P(\langle x_{i+1}\rangle^{n-1}, x_n) \leq r + A[H(\langle f(x_i)\rangle^{n-1}, g(x_i)) - H(\langle f(x_i)\rangle^n)] \qquad (\check{\otimes})$$

in particular, it is easy to see that for all i,

$$r \leq P(\langle x_i\rangle^n)$$

i.e.

$$-P(\langle x_i\rangle^n) \leq -r.$$

<u>Step 1:</u> For all i let $t_i = H(\langle f(x_i)\rangle^{n-1}, g(x_i)) - H(\langle f(x_i)\rangle^n)$. Then, from $(\check{\otimes})$

$$t_{i+1} = H(\langle f(x_{i+1})\rangle^{n-1}, g(x_{i+1})) - H(\langle f(x_{i+1})\rangle^n)$$

$$\leq c[H(\langle f(x_i)\rangle^{n-1}, g(x_i)) - H(\langle f(x_i)\rangle^n)]$$

and, hence, by induction

$$t_{i+1} \leq c^i H(\langle f(x_1)\rangle^{n-1}, g(x_1)) - H(\langle f(x_1)\rangle^n). \qquad (\hat{\otimes})$$

Therefore, and since $0 < c < 1$ for every positive real number ϵ there exists natural number N' such that for all $i > N'$,

$$H(\langle f(x_i)\rangle^{n-1}, g(x_i)) - H(\langle f(x_i)\rangle^n) < \epsilon.$$

<u>Step 2:</u> For all $i > j$, by $(\check{\otimes})$ and repeatedly using (P_n-inq) we get,

$$r \leq P(\langle x_i\rangle^n) \leq P(\langle x_i\rangle^{n-1}, x_j)$$

$$\leq P(\langle x_i\rangle^{n-1}, x_{i-1}) - P(\langle x_{i-1}\rangle^n) + P(\langle x_{i-1}\rangle^{n-1}, x_{i-2}) - P(\langle x_{i-2}\rangle^n)$$

$$+ ... + P(\langle x_{j+2}\rangle^{n-1}, x_{j+1}) - P(\langle x_{j+1}\rangle^n) + P(\langle x_{j+1}\rangle^{n-1}, x_j)$$

$$\leq \sum_{k=j+1}^{i-1} [P(\langle x_{k+1}\rangle^{n-1}, x_k) - P(\langle x_k\rangle^n)] + P(\langle x_{j+1}\rangle^{n-1}, y_j)$$

$$\leq \sum_{k=j+1}^{i-1} [r + A[H(\langle f(x_k)\rangle^{n-1}, g(x_k)) - H(\langle f(x_k)\rangle^n)] - r] + r + A[H(\langle f(x_j)\rangle^{n-1}, g(x_j)) - H(\langle f(x_j)\rangle^n)]$$

$$= \sum_{k=j+1}^{i-1} [At_k] + r + A[t_j] = r + A\sum_{k=j}^{i-1} t_k$$

by $(\hat{\otimes})$

$$\leq r + A\sum_{k=j}^{i-1} c^{k-1} t_1 = r + c^{j-1} A t_1 \sum_{k=0}^{i-j-1} c^k$$

since $Ac^j t_k \geq 0$ and by the geometric series formula

$$\leq r + c^{j-1} A t_1 \sum_{k=0}^{+\infty} c^k = r + c^{j-1} \frac{At_1}{1-c}.$$

Therefore, for every positive real number ϵ there exists a natural number N where $c^{N-1}\frac{At_1}{1-c} < \epsilon$ and, hence, for all $i \geq j > N$,

$$r - \epsilon < r \leq P(\langle x_i \rangle^n) \leq P(\langle x_i \rangle^{n-1}, x_j) \leq r + c^{j-1}\frac{At_1}{1-c} < r + c^{N-1}\frac{At_1}{1-c} < r + \epsilon. \qquad \square$$

The corollary below is straightforward by taking $X = Y$.

Corollary 5.3.5. *Let (X, P) be a partial $n-\mathfrak{M}$etric space. Let $f : X \to X$ and $g : X \to X$ be two functions on X. If f and g are $f-$mutually c_r-contractive then there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with central distance r such that for all natural numbers i, $r \leq P(\langle x_i \rangle^n)$. Additionally for every positive real number ϵ there exists a natural number N such that for all $i > N$,*

$$P(\langle f(x_i) \rangle^{n-1}, g(x_i)) - P(\langle f(x_i) \rangle^n) < \epsilon.$$

Chapter 6

Continuity and Non-Expansiveness

In Chapter 5, we established some criteria on functions that are sufficient to generate Cauchy sequences and Cauchy pairs. Given that the limits (or special limits) of these sequences exist, we will need extra criteria on the functions for them to have a fixed point, common fixed point or coincidence point.

Definition 6.0.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Let $f : X \to Y$ be a function on X. We say that f is* ***continuous*** *if and only if for every set U open in Y, $f^{-1}(U)$ is open in X.*

Definition 6.0.2. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Let $f : X \to Y$ be a function on X. We say that f is* ***sequentially continuous*** *if and only if for every sequence $\{x_i\}_{i\in\mathbb{N}}$ in X having a limit a in X, $f(a)$ is a limit of $\{f(x_i)\}_{i\in\mathbb{N}}$.*

Theorem 6.0.1. ***(Continuity vs. Sequential Continuity):***
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Let $f : X \to Y$ be a function on X. If $(X, \mathcal{T}_X)$ is first countable, then the the two statements below are equivalent.
(a) f is continuous.
(b) f is sequentially continuous.

The proof of Theorem 6.0.1 is found in [26]: Theorem 21.3. We mention this theorem since all topologies discussed in this thesis are first countable. In most cases we only need sequential continuity on the orbit, even less, we only need sequential continuity for the special limit rather than for all limits on that orbit.

Definition 6.0.3. *Let $(X, \mathcal{T}_X)$ be a topological space with x_o in X. Let $f : X \to X$ be a function on X. We say that f is* ***orbitally continuous at*** *x_o if and only if a is a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ implies that $f(a)$ is a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$.*

The definition of a special limit is not a topological definition, but rather a definition deduced from our generalized metrics. However, we will define weakly orbitally continuous functions now to avoid repeating the definition in each section. We will state the definition on a partial $n-\mathfrak{M}$etric space knowing that it includes all other cases discussed in this thesis.

Definition 6.0.4. *Let (X, P) a partial $n-\mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a function on X. We say that f is* ***weakly orbitally continuous at*** *x_o if and only if a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ implies that $f(a)$ is a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$.*

Notice that the difference between orbitally continuous and weakly orbitally continuous is that in the latter we can only guarantee that $f(a)$ is a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ if a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$. Theorem 6.0.1 shows that continuity and sequential continuity become equivalent notions in first countable spaces. Hence, in first countable spaces,

Continuous $\iff$ Sequentially Continuous $\Rightarrow$ Orbitally Continuous at $x_o \Rightarrow$ Weakly Orbitally Continuous at x_o.

The two notions left for us to define are non-expansiveness and consistency. We will present the definition in the partial $n - \mathfrak{M}$etric case as it is our most general case.

Definition 6.0.5. *Let (X, P) a partial $n - \mathfrak{M}$etric space. Let $f : X \to X$ be a function on X. We say that f is **non-expansive** if and only if for every two elements x and y in X,*

$$P(\langle f(x)\rangle^{n-1}, f(y)) \leq P(\langle x\rangle^{n-1}, y).$$

Definition 6.0.6. *Let (X, P) and (Y, H) two partial $n - \mathfrak{M}$etric spaces. Let $f : X \to Y$ be a function on X. We say that f is **consistent** if and only if for every two elements x and z in X,*

$$P(\langle x\rangle^n) \leq P(\langle z\rangle^n)$$

implies

$$H(\langle f(x)\rangle^n) \leq H(\langle f(z)\rangle^n).$$

6.1 Metric Space

In the case of a metric space, the definition of non-expansive functions given in Definition 6.0.5 is written as

$$d(f(x), f(y)) \leq d(x, y).$$

The Lemmas presented in this section are folklore. Hence, we will state them while providing a minimal proof when needed.

Lemma 6.1.1. *Let (X, d) be a metric space. If $f : X \to X$ is a non-expansive function on X then f is continuous.*

Lemma 6.1.2. *Let (X, d) be a metric space with x_o in X. Let $f : X \to X$ be a non-expansive function on X. If a is a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$, then $f(a) = a$.* □

Proof: A non-expansive function on a metric space is continuous and, hence, by Theorem 6.0.1 sequentially continuous. Therefore, by Definition 6.0.2 $f(a) = a$. □

In a metric space the definitions of orbitally continuous and weakly orbitally continuous coincide since the definitions of special limits and limits coincide.

Lemma 6.1.3. *Let (X, d) be a metric space with x_o in X . Let $f : X \to X$ be a weakly orbitally continuous function at x_o. If a is a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$, then $f(a) = a$.*

Proof: In a metric space the limit is unique and, hence, by Definition 6.0.2 $f(a) = a$. □

6.2 Partial Metric Space

As in Section 5.1 the results in this section are special cases of results in Section 6.3. We include the proofs because they are much simpler than those of the more general results. In the case of a partial metric space, the definition of non-expansive functions given in Definition 6.0.5 is written as

$$p(f(x), f(y)) \leq p(x, y).$$

Remark 6.2.1. *In a metric space, a non-expansive function is continuous and, hence, weakly orbitally continuous. Additionally in the metric case, as pointed out in Section 6.1, the notions of orbital continuity and weak orbital continuity coincide. On the other hand, in a partial metric space, a non-expansive function need not be continuous or even weakly orbitally continuous. Moreover, a weakly orbitally continuous function need not be orbitally continuous. We show these important differences using the three examples below.*

Example 6.2.2. ***(Non-Expansiveness vs Continuity):***
Let p be a partial metric on $X = \mathbb{R} \cup \{a\}$ where $a \notin \mathbb{R}$ as defined in Example 2.2.4 by:
For all $x, y \in \mathbb{R}$,

$$p(a, a) = 0, p(a, x) = |x| \text{ and } p(x, y) = |x - y| - 1.$$

Let

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R}. \\ 1 & \text{if } x = a. \end{cases}$$

The function f is non-expansive over $\mathbb{R}$ since

$$p(f(a), f(a)) = p(1, 1) = -1 \leq 0 = p(a, a).$$

Additionally, for each $x \in \mathbb{R}$,

$$p(f(a), f(x)) = p(1, x) = |x - 1| - 1 \leq |x| = p(a, x).$$

On the other hand, a is a limit of $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ but $f(a) = 1$ is not. Hence, f is not sequentially continuous and by Theorem 6.0.1 is not continuous.

Example 6.2.3. ***(Non-Expansiveness vs. Weak Orbital Continuity):***
Let $p : X \times X \to \mathbb{R}$ be a partial metric on $X = [-1, 1]$ as defined in Example 2.2.3 by:
For all $x, y \in \mathbb{R}$,

$$p(x, y) = \max\{x, y\}.$$

Let

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 0. \\ -1 & \text{if } x = 0. \end{cases}$$

For each $x \in [-1, 0) \cup (0, 1]$

$$p(f(0), f(x)) = p(-1, \frac{x}{2}) = \max\{-1, \frac{x}{2}\} \leq \max\{0, x\} = p(0, x).$$

Hence, f is non-expansive. Showing that 0 is a special limit of $\{f^n(1)\}_{n\in\mathbb{N}} = \{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ is left as an exercise to the reader. On the other hand, $f(0) = -1$ is not a limit of $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ since for each $n \in \mathbb{N}$ and $\epsilon < 1$,

$$p(-1, \frac{1}{2^n}) - p(-1, -1) = \frac{1}{2^n} + 1 > 1 > \epsilon.$$

Example 6.2.4. ***(Weak vs Usual Orbital Continuity):***
Let p be a partial metric on $X = \mathbb{R} \cup \{a\}$ where $a \notin \mathbb{R}$ as defined in Example 2.2.4 by

$$p(x, y) = \begin{cases} 0 & \text{if } x = y = a. \\ |y| & \text{if } x = a \text{ and } y \in \mathbb{R}. \\ |x - y| - 1 & \text{if } \{x, y\} \subseteq \mathbb{R}. \end{cases}$$

Let

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \mathbb{R} - \{0\}. \\ a & \text{if } x = 0. \\ 5 & \text{if } x = a. \end{cases}$$

Then, the sequence $\{f^n(1)\}_{n\in\mathbb{N}} = \{\frac{1}{2^n}\}_{n\in\mathbb{N}}$. As shown in Example 2.2.4, 0 is a special limit and a is a limit of the sequence $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$. Moreover, $f(0) = a$ is a limit (not a special limit though) of $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ whereas $f(a) = 5$ is not. Hence, f is weakly orbitally continuous (but not orbitally continuous) at $x_o = 1$.

Lemma 6.2.1. *Let (X, p) a partial metric space with x_o in X. Let $f : X \to X$ be a non-expansive function on X. If a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ then $p(a, f(a)) = p(a, a)$.*

Proof: Since a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ (see Definition 4.2.3) this sequence is a Cauchy sequence with central distance $r = p(a, a)$. From (p-lbnd) we know that

$$p(a, a) \leq p(a, f(a))$$

by (p-inq)

$$\leq p(f(a), f^{i+1}(x_o)) + p(f^{i+1}(x_o), a) - p(f^{i+1}(x_o), f^{i+1}(x_o))$$

by (p-sym)

$$= p(f(a), f^{i+1}(x_o)) + p(a, f^{i+1}(x_o)) - p(f^{i+1}(x_o), f^{i+1}(x_o))$$

since f is non-expansive

$$\leq p(a, f^i(x_o)) + p(a, f^{i+1}(x_o)) - p(f^{i+1}(x_o), f^{i+1}(x_o)).$$

For every positive real number ϵ by Definition 4.2.1 there exists a natural number N_1 such that for all $i > N_1$

$$-p(f^{i+1}(x_o), f^{i+1}(x_o)) < -r + \frac{\epsilon}{3} = -p(a, a) + \frac{\epsilon}{3}$$

and, since a special limit is a limit, by Lemma 4.2.1 there exists a natural number N_2 such that for all $i > N_2$,

$$p(a, f^i(x_o)) < p(a, a) + \frac{\epsilon}{3}.$$

Therefore, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2\}$ such that for all $i > N$,

$$\begin{aligned} p(a, a) \leq p(a, f(a)) &\leq p(a, f^i(x_o)) + p(a, f^{i+1}(x_o)) - p(f^{i+1}(x_o), f^{i+1}(x_o)) \\ &< p(a, a) + \frac{\epsilon}{3} + p(a, a) + \frac{\epsilon}{3} - p(a, a) + \frac{\epsilon}{3} = p(a, a) + \epsilon. \end{aligned}$$

Hence, $p(a, a) = p(a, f(a))$. $\square$

Lemma 6.2.2. *Let (X, p) a partial metric space with $x_o \in X$. Let $f : X \to X$ be a weakly orbitally continuous function at x_o. If a is a special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$ then $p(a, f(a)) = p(f(a), f(a))$.*

Proof: From Definition 4.2.3, since a is a special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$ then that sequence is a Cauchy sequence with central distance $r = p(a, a)$. From (p-lbnd) we know that

$$p(f(a), f(a)) \leq p(a, f(a))$$

by (p-inq)

$$\leq p(f(a), f^{i+1}(x_o)) + p(f^{i+1}(x_o), a) - p(f^{i+1}(x_o), f^{i+1}(x_o))$$

by (p-sym)

$$= p(f(a), f^{i+1}(x_o)) + p(a, f^{i+1}(x_o)) - p(f^{i+1}(x_o), f^{i+1}(x_o)).$$

For every positive real number ϵ by Definition 4.2.1 there exists a natural number N_1 such that for all $i > N_1$

$$-p(f^{i+1}(x_o), f^{i+1}(x_o)) < -r + \frac{\epsilon}{3} = -p(a, a) + \frac{\epsilon}{3}.$$

Since a is a limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$ by Lemma 4.2.1 there exists a natural number N_2 such that for all $i > N_2$,

$$p(a, f^i(x_o) < p(a, a) + \frac{\epsilon}{3}.$$

Furthermore, f is weakly orbitally continuous at x_o and a is a special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$ hence, $f(a)$ is a limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$. Therefore, from Lemma 4.2.1 there exists a natural number N_3 such that for all $i > N_3$,

$$p(f(a), f^i(x_o)) < p(f(a), f(a)) + \frac{\epsilon}{3}.$$

Hence, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$,

$$p(f(a), f(a)) \leq p(a, f(a) \leq p(f(a), f^{i+1}(x_o)) + p(f^{i+1}(x_o), a) - p(f^{i+1}(x_o), f^{i+1}(x_o))$$

$$< p(f(a), f(a)) + \frac{\epsilon}{3} + p(a, a) + \frac{\epsilon}{3} - p(a, a) = p(f(a), f(a)) + \epsilon.$$

Therefore, $p(f(a), f(a)) = p(a, f(a))$. □

6.3 Partial $n - \mathfrak{M}$etric space

As in Section 5.3, the proofs in Section 6.3 are quite similar to those in Section 6.2 aside form the need to change ϵ to fit our needs. We still present the proofs for completeness.

Lemma 6.3.1. *Let (X, P) a partial $n - \mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a non-expansive function on X. If a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ then*

$$P(\langle a\rangle^{n-1}, f(a)) = P(\langle a\rangle^n)$$

and

$$P(\langle f(a)\rangle^{n-1}, a) \leq P(\langle a\rangle^n).$$

Proof: From Definition 4.3.2, since a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ then this sequence is a Cauchy sequence with central distance $r = P(\langle a\rangle^n)$. From ($P_n$-lbnd) we know that

$$P(\langle a\rangle^n) \leq P(\langle a\rangle^{n-1}, f(a))$$

by (P_n-inq)

$$\leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + P(\langle f^{i+1}(x_o)\rangle^{n-1}, f(a)) - P(\langle f^{i+1}(x_o)\rangle^n)$$

since f is non-expansive

$$\leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + P(\langle f^i(x_o)\rangle^{n-1}, a) - P(\langle f^{i+1}(x_o)\rangle^n)$$

and by Corollary 2.5.4

$$\leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + (n-1)P(\langle a\rangle^{n-1}, f^i(x_o)) - (n-2)P(\langle a\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n).$$

By Definition 4.3.1, for every positive real number ϵ there exists a natural number N_1 such that for all $i > N_1$,

$$-P(\langle f^i(x_o)\rangle^n) < -r + \frac{\epsilon}{n+1} = -P(\langle a\rangle^n) + \frac{\epsilon}{n+1}.$$

Since a special limit is a limit, by Lemma 4.3.2 there exists a natural number N_2 such that for all $i > N_2$,

$$P(\langle a\rangle^{n-1}, f^i(x_o)) < P(\langle a\rangle^n) + \frac{\epsilon}{n+1}.$$

Therefore, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$,

$$P(\langle a\rangle^n) \leq P(\langle a\rangle^{n-1}, f(a))$$
$$\leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + (n-1)P(\langle a\rangle^{n-1}, f^i(x_o)) - (n-2)P(\langle a\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n)$$
$$< P(\langle a\rangle^n) + \frac{\epsilon}{n+1} + (n-1)(P(\langle a\rangle^n) + \frac{\epsilon}{n+1}) + (n-2)P(\langle a\rangle^n) - P(\langle a\rangle^n) + \frac{\epsilon}{n+1}$$
$$= P(\langle a\rangle^n) + (n+1)\frac{\epsilon}{n+1} = P(\langle a\rangle^n) + \epsilon.$$

Hence, $P(\langle a\rangle^n) = P(\langle a\rangle^{n-1}, f(a))$.

Similarly by (P_n-inq)

$$P(\langle f(a)\rangle^{n-1}, a) \leq P(\langle f(a)\rangle^{n-1}, f^{i+1}(x_o)) + P(\langle f^{i+1}(x_o)\rangle^{n-1}, a) - P(\langle f^{i+1}(x_o)\rangle^n)$$

since f is non-expansive

$$\leq P(\langle a\rangle^{n-1}, f^i(x_o)) + P(\langle f^{i+1}(x_o)\rangle^{n-1}, a) - P(\langle f^{i+1}(x_o)\rangle^n)$$

by Corollary 2.5.4

$$\leq P(\langle a\rangle^{n-1}, f^i(x_o)) + (n-1)P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) - (n-2)P(\langle a\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n).$$

Hence, for every positive real number ϵ there exists a natural number N such that for all $i > N$,

$$P(\langle f(a)\rangle^{n-1}, a)$$
$$\leq P(\langle a\rangle^{n-1}, f^i(x_o)) + (n-1)P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) - (n-2)P(\langle a\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n)$$
$$< P(\langle a\rangle^n) + \frac{\epsilon}{n+1} + (n-1)(P(\langle a\rangle^n) + \frac{\epsilon}{n+1}) - (n-2)P(\langle a\rangle^n) - P(\langle a\rangle^n) + \frac{\epsilon}{n+1}$$
$$= P(\langle a\rangle^n) + (n+1)\frac{\epsilon}{n+1} = P(\langle a\rangle^n) + \epsilon.$$

Therefore, $P(\langle f(a)\rangle^{n-1}, a) \leq P(\langle a\rangle^n)$. $\square$

Lemma 6.3.2. *Let (X, P) a partial $n-\mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be weakly orbitally continuous function at x_o. If a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ then*

$$P(\langle f(a)\rangle^{n-1}, a) = P(\langle f(a)\rangle^n)$$

and

$$P(\langle a\rangle^{n-1}, f(a)) \leq P(\langle f(a)\rangle^n).$$

Proof: From Definition 4.3.2, since a is a special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ then that sequence is a Cauchy sequence with central distance $r = P(\langle a\rangle^n)$. From ($P_n$-lbnd) we know that

$$P(\langle f(a)\rangle^n) \leq P(\langle f(a)\rangle^{n-1}, a)$$

by (P_n-inq)

$$\leq P(\langle f(a)\rangle^{n-1}, f^{i+1}(x_o)) + P(\langle f^{i+1}(x_o)\rangle^{n-1}, a) - P(\langle f^{i+1}(x_o)\rangle^n)$$

by Corollary 2.5.4

$$\leq P(\langle f(a)\rangle^{n-1}, f^{i+1}(x_o)) + (n-1)P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) - (n-2)P(\langle a\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n).$$

By Definition 4.3.1, for every positive real number ϵ there exists a natural number N_1 such that for all $i > N_1$,

$$-P(\langle f^i(x_o)\rangle^n) < -r + \frac{\epsilon}{2n-1} = -P(\langle a\rangle^n) + \frac{\epsilon}{n+1}.$$

Since a special limit is a limit, by Lemma 4.3.2 there exists a natural number N_2 such that for all $i > N_2$,

$$P(\langle a\rangle^{n-1}, f^i(x_o)) < P(\langle a\rangle^n) + \frac{\epsilon}{n+1}.$$

Since f is weakly orbitally continuous at x_o then $f(a)$ is also a limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$ then there exists a natural number N_3 such that for all $i > N_3$,

$$P(\langle f(a)\rangle^{n-1}, f^i(x_o)) < P(\langle f(a)\rangle^n) + \frac{\epsilon}{n+1}.$$

Therefore, $N = \max\{N_1, N_2, N_3\}$ is a natural number such that for all $i > N$,

$$P(\langle f(a)\rangle^n) \leq P(\langle f(a)\rangle^{n-1}, a)$$

$$\begin{aligned}
&\leq P(\langle f(a)\rangle^{n-1}, f^{i+1}(x_o)) + (n-1)P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) - (n-2)P(\langle a\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n) \\
&< P(\langle f(a)\rangle^n) + \frac{\epsilon}{n+1} + (n-1)(P(\langle a\rangle^n) + \frac{\epsilon}{n+1}) + (n-2)P(\langle a\rangle^n) - P(\langle a\rangle^n) + \frac{\epsilon}{n+1} \\
&= P(\langle f(a)\rangle^n) + (n+1)\frac{\epsilon}{n+1} = P(\langle f(a)\rangle^n) + \epsilon.
\end{aligned}$$

Hence, $P(\langle f(a)\rangle^n) = P(\langle f(a)\rangle^{n-1}, a)$.
Similarly by (P_n-inq)

$$P(\langle a\rangle^{n-1}, f(a)) \leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + P(\langle f^{i+1}(x_o)\rangle^{n-1}, f(a)) - P(\langle f^{i+1}(x_o)\rangle^n)$$

by Corollary 2.5.4

$$\leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + (n-1)P(\langle f(a)\rangle^{n-1}, f^{i+1}(x_o)) - (n-2)P(\langle f(a)\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n).$$

Hence, for every positive real number ϵ there exists a natural number N such that for all $i > N$,

$$P(\langle a\rangle^{n-1}, f(a))$$

$$\begin{aligned}
&\leq P(\langle a\rangle^{n-1}, f^{i+1}(x_o)) + (n-1)P(\langle f(a)\rangle^{n-1}, f^{i+1}(x_o)) - (n-2)P(\langle f(a)\rangle^n) - P(\langle f^{i+1}(x_o)\rangle^n) \\
&< P(\langle a\rangle^n) + \frac{\epsilon}{n+1} + (n-1)(P(\langle f(a)\rangle^n) + \frac{\epsilon}{n+1}) - (n-2)P(\langle f(a)\rangle^n) - P(\langle a\rangle^n) + \frac{\epsilon}{n+1} \\
&= P(\langle f(a)\rangle^n) + (n+1)\frac{\epsilon}{n+1} = P(\langle f(a)\rangle^n) + \epsilon.
\end{aligned}$$

Therefore, $P(\langle a\rangle^{n-1}, f(a)) \leq P(\langle f(a)\rangle^n)$. $\square$

Chapter 7

Applications to Fixed point and Coincidence Point Theory.

We have reached the end of the rainbow to find our pot of gold. In this chapter we state fixed, common fixed and coincidence point theorems whose sole constraint on the generalized metric spaces is that they be complete. This thesis was intentionally written in a way that minimizes the proofs in this section. The theorems and lemmas in previous chapters are building blocks for the theorems ahead. As previously stated, a (strong) partial $n-\mathfrak{M}$etric is a generalization of a (strong) partial metric. Any special technique needed for the (strong) partial $n-\mathfrak{M}$etric case has already been presented in previous chapters. That is why we will be omitting the proofs of Section 7.4 and Section 7.5 to spare the reader any redundancy. We start with some basic definitions.

Definition 7.0.1. *Let X be a non-empty set. Let $f : X \to X$ be a function on X. We say that x in X is a* ***fixed point of f*** *if and only if $f(x) = x$.*

Definition 7.0.2. *Let X be a non-empty set. Let $f : X \to X$ and $g : X \to X$ be two functions on X. We say that x in X is a* ***common fixed point of f and g*** *if and only if $f(x) = x = g(x)$.*

Definition 7.0.3. *Let X and Y be two non-empty sets. Let $f : X \to Y$ and $g : X \to Y$ be two functions on X. We say that x in X is a* ***coincidence point of f and g*** *if and only if $f(x) = g(x)$.*

7.1 Metric Space

Depending on the type of the contractive function used, Theorem 7.1.1 and Theorem 7.1.2 can be attributed to either Edelstein [10, 11, 12] or Alber and Guerre-Delabriere [1].

Theorem 7.1.1. ***(Fixed point and Non-expansive):***
Let (X, d) be a complete metric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is non-expansive then f has a fixed point in X.

Proof: Since f is Cauchy at x_o then by Definition 5.0.2, $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space then by Definition 4.1.2 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ has a limit a in X. Finally f is non-expansive, then by Lemma 6.1.1 $f(a) = a$ and, hence, by Definition 7.0.3 a is a fixed point of f. $\square$

In fact, Theorem 7.1.1 can be considered a corollary of Theorem 7.1.2 since any non-expansive function in a metric space is continuous.

Theorem 7.1.2. ***(Fixed point and Weak orbital continuity):***
Let (X, d) be a complete metric space with x_o in X. Let $f : X \to X$ a Cauchy function at x_o. If f is weakly orbitally continuous at x_o then f has a fixed point in X.

Proof: Since f is Cauchy at x_o then by Definition 5.0.2, $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space then by Definition 4.1.2 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ has a limit a in X. Finally f is weakly orbitally continuous at x_o, then by Lemma 6.1.2 $f(a) = a$ and, hence, by Definition 7.0.3 a is a fixed point of f. □

Remark 7.1.1. *From Lemma 5.1.1 and Lemma 5.1.2, if f is orbitally c_0-contractive or orbitally φ_0-contractive at x_o then f is Cauchy at x_o.*

Theorem 7.1.3. ***(Common fixed point and Non-expansive):***
Let (X, d) be a complete metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are non-expansive then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.0.3 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X, d) is a complete metric space then by Definition 4.1.2 and Lemma 4.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same limit a in X. Finally f and g are both non-expansive, then by Lemma 6.1.1 $f(a) = a = g(a)$ and, hence, by Definition 7.0.2 a is a common fixed point of f and g. □

Theorem 7.1.4. ***(Common fixed point and Weak orbital continuity):***
Let (X, d) be a complete metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are weakly orbitally continuous at x_o and y_o respectively then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.0.3 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X, d) is a complete metric space then by Definition 4.1.2 and Lemma 4.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same limit a in X. Finally f and g are weakly orbitally continuous at x_o and y_o respectively, then by Lemma 6.1.2 $f(a) = a = g(a)$ and, hence, by Definition 7.0.2 a is a common fixed point of f and g. □

Theorem 7.1.5. ***(Common fixed point and Mixed criteria):***
Let (X, d) be a complete metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f is non-expansive and g is weakly orbitally continuous at y_o then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.0.3 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X, d) is a complete metric space then by Definition 4.1.2 and Lemma 4.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same limit a in X. Finally f is non-expansive then by Lemma 6.1.1 $f(a) = a$ and g is weakly orbitally continuous at y_o then by Lemma 6.1.2 $a = g(a)$ and, hence, by Definition 7.0.2 a is a common fixed point of f and g. $\square$

Remark 7.1.2. *From Theorem 5.1.3, if f and g are $f-$pairwise c_0-contractive (similarly $g-$pairwise c_0-contractive) over (x_o, y_o) then f and g form a Cauchy pair over (x_o, y_o).*

Theorem 7.1.6. ***(Coincidence Point Theorem):***
Let (X, l) be a complete metric space and let (Y, d) be a metric space. Let $f : X \to Y$ and $g : X \to Y$ be two sequentially continuous functions on X. If f and g are mutually c_0-contractive then f and g have a coincidence point.

Proof: Since f and g are mutually c_0-contractive then by Theorem 5.1.4 there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X where $\{f(x_i)\}_{i\in\mathbb{N}}$ and $\{g(x_i)\}_{i\in\mathbb{N}}$ form a Cauchy pair in Y. Hence, by Definition 4.1.3 for every positive real number ϵ there exists a natural number N_1 such that for all $i > N_1$,

$$d(f(x_i), g(x_i)) < \frac{\epsilon}{3}.$$

Since (X, l) is complete then $\{x_i\}_{i\in\mathbb{N}}$ has a limit a in X. Since f and g are sequentially continuous then by Definition 6.0.2 $f(a)$ and $g(a)$ are limits of $\{f(x_i)\}_{i\in\mathbb{N}}$ and $\{g(x_i)\}_{i\in\mathbb{N}}$ respectively.
$f(a)$ is the limit of $\{f(x_i)\}_{i\in\mathbb{N}}$ hence, by Lemma 4.1.1 there exists a natural number N_2 such that for all $i > N_2$,

$$d(f(a), f(x_i)) < \frac{\epsilon}{3}$$

$g(a)$ is the limit of $\{g(x_i)\}_{i\in\mathbb{N}}$ hence, by Lemma 4.1.1 there exists a natural number N_3 such that for all $i > N_3$,

$$d(f(a), f(x_i)) < \frac{\epsilon}{3}.$$

Hence, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$, by (m-lbnd)

$$0 \le d(f(a), g(a))$$

using (m-inq) twice we get

$$\le d(f(a), f(x_i)) + d(f(x_i), g(x_i)) + d(g(x_i), g(a))$$

by (m-sym)

$$d(f(a), f(x_i)) + d(f(x_i), g(x_i)) + d(g(a), g(x_i))$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Therefore, $d(f(a), g(a)) = 0$ and by (d-sep) $f(a) = g(a)$ and, hence, by Definition 7.0.3 a is a coincidence point of f and g. $\square$

7.2 Partial Metric Space

Theorem 7.2.1. ***(Fixed point and Partial metrics[3]):***
Let (X, p) be a complete partial metric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is non-expansive and weakly orbitally continuous at x_o then f has a fixed point.

Proof: Since f is Cauchy at x_o then by Definition 5.0.2, $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Since (X, p) is a complete partial metric space then by Definition 4.2.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ has a special limit a in X.
Since f is non-expansive then by Lemma 6.2.1

$$p(a, f(a)) = p(a, a).$$

Since f is weakly orbitally continuous at x_o then by Lemma 6.2.2

$$p(a, f(a)) = p(f(a), f(a)).$$

Hence, by (p-sep) $f(a) = a$. Therefore, by Definition 7.0.1 a is a fixed point of f. □

Corollary 7.2.2. *Let (X, p) be a complete partial metric space with x_o in X. Let $f : X \to X$ be a non-expansive weakly orbitally continuous function at x_o. If either one of the below criteria holds true:*
a) f is orbitally c_r-contractive at x_o.
b) f is orbitally φ_r-contractive at x_o.
then f has a fixed point.

Proof: From Lemma 5.2.2 and Theorem 5.2.3, if f is orbitally c_r-contractive or orbitally φ_r-contractive at x_o then f is Cauchy at x_o.

Theorem 7.2.3. ***(Common fixed point and Partial metrics):***
Let (X, p) be a complete partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are non-expansive and f and g are weakly orbitally continuous at x_o and y_o respectively. then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.0.3 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X, p) is a complete partial metric space then by Definition 4.2.4 and Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same special limit a in X.
Since f and g are both non-expansive then by Lemma 6.2.1

$$p(a, a) = p(a, f(a)) \text{ and } p(a, a) = p(a, g(a)).$$

Since f and g are weakly orbitally continuous on x_o and y_o respectively then by Lemma 6.2.2

$$p(f(a), f(a)) = p(a, f(a)) \text{ and } p(g(a), g(a)) = p(a, g(a)).$$

Hence, by (p-sep) $f(a) = a = g(a)$. Therefore, by Definition 7.0.2 a is a common fixed point of f and g. $\square$

Corollary 7.2.4. *Let (X,p) be a complete partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two non-expansive functions with f and g weakly orbitally continuous at x_o and y_o respectively. If either one of the below criteria holds true:*
a) f and g are $f-$pairwise c_r-contractive over (x_o, y_o).
b) f and g are $g-$pairwise c_r-contractive over (x_o, y_o).
then f and g have a common fixed point.

Proof: From Theorem 5.2.3, if f and g are $f-$pairwise c_r-contractive (similarly $g-$pairwise c_r-contractive) over (x_o, y_o) then f and g form a Cauchy pair over (x_o, y_o).

Theorem 7.2.5. ***(Coincidence Point Theorem):***
Let (X,p) be a complete partial metric space and let (Y,h) be a partial metric space. Let $f : X \to Y$ and $g : X \to Y$ be two sequentially continuous and consistent functions on X. If f and g are $(f,g)-$mutually c_r-contractive then f and g have a coincidence point.

Proof: Since f and g are $(f,g)-$mutually c_r-contractive then they are $f-$mutually c_r-contractive . Hence, by Theorem 5.2.4 there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X. Let r be the central distance of $\{x_i\}_{i\in\mathbb{N}}$ then again by Theorem 5.2.4 for all natural numbers i,

$$r \leq p(x_i, x_i).$$

Since (X,p) is complete partial metric space then $\{x_i\}_{i\in\mathbb{N}}$ has a special limit a in X. From Definition 4.2.3 for all natural numbers i,

$$r = p(a,a) \leq p(x_i, x_i).$$

g is consistent then

$$h(g(a), g(a)) \leq h(g(x_i), g(x_i))$$

and, hence,

$$-h(g(x_i), g(x_i)) \leq -h(g(a), g(a)).$$

For every positive real number ϵ we know that:
Since g is sequentially continuous, by Definition 6.0.2 $g(a)$ is a limit of $\{g(x_i)\}_{i\in\mathbb{N}}$. Therefore, there exists a natural number N_1 such that for all $i > N_1$,

$$h(g(a), g(x_i)) < h(g(a), g(a)) + \frac{\epsilon}{3}.$$

Similarly, f is sequentially continuous hence, by Definition 6.0.2 $f(a)$ is limit of $\{f(x_i)\}_{i\in\mathbb{N}}$. Therefore, there exists a natural number N_2 such that for all $i > N_2$,

$$h(f(a), f(x_i)) < h(f(a), f(a)) + \frac{\epsilon}{3}.$$

By Theorem 5.2.4, there exists a natural number N_3 such that for all $i > N_3$,

$$h(f(x_i), g(x_i)) - h(f(x_i), f(x_i)) < \frac{\epsilon}{3}.$$

Hence, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$,by (p-lbnd)

$$h(f(a), f(a)) \leq h(f(a), g(a))$$

using (p-inq) twice we get for all i

$$\leq h(f(a), f(x_i)) - h(f(x_i), f(x_i)) + h(f(x_i), g(x_i)) - h(g(x_i), g(x_i)) + h(g(x_i), g(a))$$

by (p-sym)

$$\begin{aligned} &= h(f(a), f(x_i)) + h(f(x_i), g(x_i)) - h(f(x_i), f(x_i)) - h(g(x_i), g(x_i)) + h(g(a), g(x_i)) \\ &< h(f(a), f(a)) + \frac{\epsilon}{3} + \frac{\epsilon}{3} - h(g(a), g(a)) + h(g(a), g(a)) + \frac{\epsilon}{3} \\ &= h(f(a), f(a)) + \epsilon. \end{aligned}$$

Therefore, $h(f(a), f(a)) = h(f(a), g(a))$. Repeating the above process with f and g being $g-$mutually c_r-contractive and f being consistent we get $h(g(a), g(a)) = h(f(a), g(a))$ and, hence, by (p-sep) $f(a) = g(a)$. Therefore, by Definition 7.0.3 a is a coincidence point of f and g. □

7.3 Strong Partial Metric Space

We remind our reader that (s-lbnd) is a stronger version of (p-sep). Hence, if (X, s) is a strong partial metric space, for any two element x and z in X, it is enough to have $s(x, z) \leq s(x, x)$ to deduce that $x = z$. Therefore, we are able to relax the requirements on the functions studied to assert the existence of the fixed point, common fixed point or coincidence point in question.

Theorem 7.3.1. ***(Fixed point and Non-expansive [3]):***
Let (X, s) be a complete strong partial metric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is non-expansive then f has a fixed point.

Proof: Since f is Cauchy at x_o then by Definition 5.0.2, $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Since (X, s) is a complete strong partial metric space then by Definition 4.2.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ has a special limit a in X. Since f is non-expansive then by Lemma 6.2.1

$$s(a, f(a)) = s(a, a)$$

and, hence by (s-lbnd) $f(a) = a$. Therefore, by Definition 7.0.1 a is a fixed point of f. □

Theorem 7.3.2. ***(Fixed point and Weak orbital continuity [3]):***
Let (X, s) be a complete strong partial metric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is weakly orbitally continuous at x_o then f has a fixed point.

Proof: Since f is Cauchy at x_o then by Definition 5.0.2, $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Since (X,p) is a complete strong partial metric space then by Definition 4.2.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ has a special limit a in X. Since f is weakly orbitally continuous at x_o then by Lemma 6.2.2

$$s(a, f(a)) = s(f(a), f(a))$$

and, hence, by (s-lbnd) $f(a) = a$. Therefore, by Definition 7.0.1 a is a fixed point of f. □

Corollary 7.3.3. *Let (X,s) be a complete strong partial metric space with x_o in X. Let $f : X \to X$ be a non-expansive function or a weakly orbitally continuous function at x_o. If either one of the below criteria holds true:*
a) f is orbitally c_r-contractive at x_o.
b) f is orbitally φ_r-contractive at x_o.
then f has a fixed point.

Proof: From Lemma 5.2.2 and Lemma 5.2.1, if f is orbitally c_r-contractive or orbitally φ_r-contractive at x_o then f is Cauchy at x_o.

Theorem 7.3.4. ***(Common fixed point and Non-expansive):***
Let (X,s) be a complete strong partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are non-expansive then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.0.3 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X,s) is a complete strong partial metric space then by Definition 4.2.4 and Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same special limit a in X. Since f and g are both non-expansive then by Lemma 6.2.2

$$p(a,a) = p(a, f(a)) \text{ and } p(a,a) = p(a, g(a))$$

and, hence, by (s-lbnd) $f(a) = a = g(a)$. Therefore, by Definition 7.0.2 a is a common fixed point of f and g. □

Theorem 7.3.5. ***(Common fixed point and Weak orbital continuity):***
Let (X,s) be a complete strong partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are weakly orbitally continuous on x_o and y_o respectively then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.0.3 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X,s) is a complete strong partial metric space then by Definition 4.2.4 and Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and

$\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same special limit a in X. Since f and g are weakly orbitally continuous on x_o and y_o respectively then

$$p(f(a), f(a)) = p(a, f(a)) \text{ and } p(g(a), g(a)) = p(a, g(a))$$

and, hence, by (s-lbnd) $f(a) = a = g(a)$. Therefore, by Definition 7.0.2 a is a common fixed point of f and g. $\square$

Theorem 7.3.6. ***(Common fixed point and Mixed criteria):***
Let (X, s) be a complete strong partial metric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f is non-expansive and g is weakly orbitally continuous on y_o then f and g have a common fixed point.

Proof: Since f and g form a Cauchy pair at (x_o, y_o) then by Definition 5.1.4 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ form a Cauchy pair. By Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ are both Cauchy sequences. Since (X, p) is a complete partial metric space then by Definition 4.2.4 and Lemma 4.2.6 $\{f^i(x_o)\}_{i\in\mathbb{N}}$ and $\{g^i(y_o)\}_{i\in\mathbb{N}}$ both have the same special limit a in X.
Since f is non-expansive then by Lemma 6.2.1

$$p(a, a) = p(a, f(a))$$

and, hence by (s-lbnd)

$$f(a) = a.$$

Since g is weakly orbitally continuous on y_o then by Lemma 6.2.2

$$p(g(a), g(a)) = p(a, g(a))$$

and, hence, by (s-lbnd) $f(a) = a = g(a)$. Therefore, by Definition 7.0.2 a is a common fixed point of f and g. $\square$

Corollary 7.3.7. *Let (X, s) be a complete strong partial metric space with x_o and y_o in X. Let $f : X \to X$ be a non-expansive function or a weakly orbitally continuous function at x_o. Similarly, let $g : X \to X$ be a non-expansive function or a weakly orbitally continuous function at y_o. If either one of the below criteria holds true:*
a) f and g are $f-$pairwise c_r-contractive over (x_o, y_o).
b) f and g are $g-$pairwise c_r-contractive over (x_o, y_o).
then f and g have a common fixed point.

Proof: From Theorem 5.2.3 , if f and g are $f-$pairwise c_r-contractive (similarly $g-$pairwise c_r-contractive) over (x_o, y_o) then f and g form a Cauchy pair over (x_o, y_o).

Theorem 7.3.8. ***(Coincidence Point Theorem):***

Let (X, p) be a complete partial metric space and let (Y, s) be a strong partial metric space. Let $f : X \to Y$ and $g : X \to Y$ two sequentially continuous functions on X. If f and g are $f-$mutually c_r-contractive and g is consistent then f and g have a coincidence point.

Proof: Since f and g are $f-$mutually c_r-contractive then by Theorem 5.2.4 there exists a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ in X. Let r be the central distance of $\{x_i\}_{i\in\mathbb{N}}$ then again by Theorem 5.2.4 for all natural numbers i,

$$r \leq p(x_i, x_i).$$

Since (X, p) is complete partial metric space then $\{x_i\}_{i\in\mathbb{N}}$ has a special limit a in X. From Definition 4.2.3 for all natural numbers i,

$$r = p(a, a) \leq p(x_i, x_i).$$

g is consistent then

$$s(g(a), g(a)) \leq s(g(x_i), g(x_i))$$

and, hence,

$$-s(g(x_i), g(x_i)) \leq -s(g(a), g(a)).$$

For every positive real number ϵ we know that:
since g is sequentially continuous, by Definition 6.0.2 $g(a)$ is limit of $\{g(x_i)\}_{i\in\mathbb{N}}$. Therefore, there exists a natural number N_1 such that for all $i > N_1$,

$$s(g(a), g(x_i)) < s(g(a), g(a)) + \frac{\epsilon}{3}.$$

Similarly, f is sequentially continuous hence, by Definition 6.0.2 $f(a)$ is limit of $\{f(x_i)\}_{i\in\mathbb{N}}$. Therefore, there exists a natural number N_2 such that for all $i > N_2$,

$$s(f(a), f(x_i)) < s(f(a), f(a)) + \frac{\epsilon}{3}.$$

By Theorem 5.2.4, there exists a natural number N_3 such that for all $i > N_3$,

$$s(f(x_i), g(x_i)) - s(f(x_i), f(x_i)) < \frac{\epsilon}{3}.$$

Hence, for every positive real number ϵ there exists a natural number $N = \max\{N_1, N_2, N_3\}$ such that for all $i > N$, using (s-inq) twice

$$h(f(a), g(a)) \leq s(f(a), f(x_i)) - s(f(x_i), f(x_i)) + s(f(x_i), g(x_i)) - s(g(x_i), g(x_i)) + s(g(x_i), g(a))$$

by (s-sym)

$$= s(f(a), f(x_i)) + s(f(x_i), g(x_i)) - s(f(x_i), f(x_i)) - s(g(x_i), g(x_i)) + s(g(a), g(x_i))$$

$$< h(f(a), f(a)) + \frac{\epsilon}{3} + \frac{\epsilon}{3} - h(g(a), g(a)) + h(g(a), g(a)) + \frac{\epsilon}{3}$$

$$= s(f(a), f(a)) + \epsilon.$$

Therefore, $s(f(a), g(a)) \leq s(f(a), f(a))$. Hence, by (s-lbnd) $f(a) = g(a)$. Therefore, by Definition 7.0.3 a is a coincidence point of f and g. $\square$

7.4 Partial $n - \mathfrak{M}$etric Space

Theorem 7.4.1. ***(Fixed point and Partial*** $n - \mathfrak{M}$***etrics[2]):***
Let (X, P) be a complete partial $n - \mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is non-expansive and weakly orbitally continuous at x_o then f has a fixed point.

Corollary 7.4.2. *Let (X, P) be a complete partial $n - \mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a non-expansive weakly orbitally continuous function at x_o. If either one of the below criteria holds true:*
a) f is orbitally c_r-contractive at x_o.
b) f is orbitally φ_r-contractive at x_o.
then f has a fixed point.

Theorem 7.4.3. ***(Common fixed point and Partial*** $n - \mathfrak{M}$***etrics):***
Let (X, P) be a complete partial $n - \mathfrak{M}$etric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are non-expansive and f and g are weakly orbitally continuous on x_o and y_o respectively then f and g have a common fixed point.

Corollary 7.4.4. *Let (X, P) be a complete partial $n-\mathfrak{M}$etric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two non-expansive functions with f and g weakly orbitally continuous at x_o and y_o respectively. If either one of the below criteria holds true:*
a) f and g are $f-$pairwise c_r-contractive over (x_o, y_o).
b) f and g are $g-$pairwise c_r-contractive over (x_o, y_o).
then f and g have a common fixed point.

Theorem 7.4.5. ***(Coincidence Point Theorem):***
Let (X, P) be a complete partial $n-\mathfrak{M}$etric space and let (Y, H) be a partial $n-\mathfrak{M}$etric space. Let $f : X \to Y$ and $g : X \to Y$ be two sequentially continuous and consistent functions on X. If f and g are $(f, g)-$mutually c_r-contractive then f and g have a coincidence point.

7.5 Strong Partial $n - \mathfrak{M}$etric Space

In this section again, we remind our reader that (S_n-lbnd) is a stronger version of (P_n-sep). Hence, if (X, S) is a strong partial $n - \mathfrak{M}$etric space, for any two element x and z in X, it is enough to have $S(\langle x\rangle^{n-1}, z) \leq S(\langle x\rangle^n)$ to deduce that $x = z$. Therefore, we are able to relax the requirements on the functions studied in Section 7.4 to assert the existence of the fixed point, common fixed point or coincidence point in question.

Theorem 7.5.1. ***(Fixed point and Non-expansive [2]):***
Let (X, S) be a complete strong partial $n-\mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is non-expansive then f has a fixed point.

Theorem 7.5.2. ***(Fixed point and Weak orbital continuity [2]):***
Let (X, S) be a complete strong partial $n-\mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a Cauchy function at x_o. If f is weakly orbitally continuous at x_o then f has a fixed point.

Corollary 7.5.3. *Let (X, S) be a complete strong partial $n-\mathfrak{M}$etric space with x_o in X. Let $f : X \to X$ be a non-expansive function or a weakly orbitally continuous function at x_o. If either one of the below criteria holds true:*
a) f is orbitally c_r-contractive at x_o.
b) f is orbitally φ_r-contractive at x_o.
then f has a fixed point.

Theorem 7.5.4. ***(Common fixed point and Non-expansive):***
Let (X, S) be a complete strong partial $n-\mathfrak{M}$etric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be a Cauchy pair over (x_o, y_o). If f and g are non-expansive then f and g have a common fixed point.

Theorem 7.5.5. ***(Common fixed point and Weak orbital continuity):***
Let (X, S) be a complete strong partial $n-\mathfrak{M}$etric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f and g are weakly orbitally continuous on x_o and y_o respectively then f and g have a common fixed point.

Theorem 7.5.6. ***(Common fixed point and Mixed criteria):***
Let (X, S) be a complete strong partial $n-\mathfrak{M}$etric space with x_o and y_o in X. Let $f : X \to X$ and $g : X \to X$ be two functions that form a Cauchy pair over (x_o, y_o). If f is non-expansive and g is weakly orbitally continuous on y_o then f and g have a common fixed point.

Corollary 7.5.7. *Let (X, S) be a complete strong partial $n - \mathfrak{M}$etric space with x_o and y_o in X. Let $f : X \to X$ be a non-expansive function or a weakly orbitally continuous function at x_o. Similarly, let $g : X \to X$ be a non-expansive function or a weakly orbitally continuous function at y_o. If either one of the below criteria holds true:*
a) f and g are $f-$pairwise c_r-contractive over (x_o, y_o).
b) f and g are $g-$pairwise c_r-contractive over (x_o, y_o).
then f and g have a common fixed point.

Theorem 7.5.8. ***(Coincidence Point Theorem):***

Let (X, P) be a complete partial $n - \mathfrak{M}$etric space and let (Y, S) be a strong partial $n - \mathfrak{M}$etric space. Let $f : X \to Y$ and $g : X \to Y$ be two sequentially continuous functions on X. If f and g are $f-$mutually c_r-contractive and g is consistent then f and g have a coincidence point.

References

[1] YA. I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in hilbert spaces*, Operator Theory Advances and Applications **98** (1997).

[2] Samer Assaf and Koushik Pal, *Partial $n-\mathfrak{M}$etric spaces and fixed point theorem*, Submitted (2015), http://arxiv.org/abs/1502.05320.

[3] ———, *Partial metric spaces with negative distances and fixed point theorems*, Topology Proceedings (Accepted).

[4] Hassen Ayadi, Erdal Karapinar, and Peyman Salimi, *Some fixed point results in G_p-metric spaces*, Journal of Applied Mathematics (2012), ID 891713.

[5] Stefan Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équation intégrales*, Fundamenta Mathematicae **3** (1922), 133–181.

[6] Nurcan Bilgili, Erdal Karapinar, and Peyman Salimi, *Fixed point theorems for generalized contractions on G_p-metric spaces*, Journal of Inequalities and Applications **39** (2013).

[7] Michael Bukatin, Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh, *Partial metric spaces*, The American Mathematical Monthly **116** (2009), no. 8, 708–719.

[8] Michel Marie Deza and Elena Deza, *Encyclopedia of distances*, Springer, 2009.

[9] Bapurao C. Dhage, *Generalised metric space and mappings with fixed point*, Bulletin of Calcutta Mathematical Society **84** (1992), no. 4, 329–336.

[10] Michael Edelstein, *An extension of Banach's contraction principle*, Proceedings of the American Mathematical Society **12** (1961), no. 1, 7–10.

[11] ———, *On fixed and periodic points under contractive mappings*, Journal of the London Mathematical Society **37** (1962), 74–79.

[12] ———, *On nonexpansive mappings*, Proceedings of the American Mathematical Society **15** (1964), no. 5, 689–695.

[13] Samuel Eilenberg and Deane Montgomery, *Fixed point theorems for multi-valued transformations*, American Journal of Mathematics **68** (1946), no. 2, 214–222.

[14] Brian Fisher, *A common fixed point theorem for commuting mappings*, Mathematics Seminar Notes **7** (1979), no. 297-300.

[15] Maurice Fréchet, *Sur quelques points de calcul fonctionnel*, Ph.D. thesis, Faculté des sciences de Paris, 1906.

[16] Von Siegfried Gähler, *$2-$metriche raume undhire topologische strukture*, Mathematische Nachrichten **26** (1963), 115–148.

[17] ———, *Zur geometric $2-$metriche raume*, Revue Roumaine de Mathématiques Pures et Aplliquée **11** (1966), 664–669.

[18] Ki Sik Ha, Yeol Je Cho, and Albert White, *Strictly convex and* $2-$*convex* $2-$*normed spaces*, Mathematica Japonica **33** (1988), 375–384.

[19] Felix Hausdorff, *Grundzüge der mengenlehre*, Chelsea, 1914.

[20] Shizuo Kakutani, *Two fixed point theorems concerning bicompact convex sets.*, Proc. Imp. Akad. **14** (1938), 242–245.

[21] Erdal Karapinar, *Weak* $\phi-$*contraction on partial metric spaces and existance of fixed points in partially ordered sets*, Mathematica Aeterna **1** (2011), no. 4, 237–244.

[22] Erdal Karapinar and Salvador Romaguera, *Nonunique fixed point theorems in partial metric spaces*, University of Niš, Serbia **27** (2013), no. 7, 1305–1314.

[23] Kamran Alam Khan, *On the possibility of* $n-$*topological spaces*, International Journal of Mathematical Archive **3** (2012), no. 7, 2520–2523.

[24] Andrey Markov, *Quelques théorèmes sur les ensembles abéliens*, Doklady Akademii Nauk **10** (1936), 311–314.

[25] Stephen G. Matthews, *Partial metric topology*, Annals of the New York Academy of Sciences, vol. 728, 1994, pp. 183–197.

[26] James R. Munkres, *Topology*, second ed., Pearson Education, 2000.

[27] Zead Mustafa and Brailey Sims, *Some remarks concerning* $D-$*metric spaces*, Proceedings of the International Conference on Fixed Point Theory and Applications (2003), 189–198.

[28] ______, *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis **7** (2006), no. 2, 289–297.

[29] S. J. O'Neill, *Partial metrics, valuations, and domain theory*, Annals of the New York Academy of Sciences **806** (1996), 304–315.

[30] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis **47** (2001), 2683–2693.

[31] Horst Schubert, *Topologie*, B. G. Teubner Verlagsgesellschaft, 1964.

[32] Paul M. Selzer, Andreas Rohwer, and Richard J. Marhöfer, *Applied bioinformatics*, Springer, 2008.

[33] Cheh Chih Yeh, *On common fixed point of continuous mappings.*, Indian Journal of Pure and Applied Mathematics **10** (1979), no. 4, 415–420.

[34] M. R. Zand and A. Dehghan Nezhad, *A generalization of partial metric spaces*, Journal of Contemporary Applied Mathematics **1** (2011), no. 1.

[35] Qingnian Zhang and Yisheng Song, *Fixed point theory for generalized* $\varphi-$*weak contractions*, Applied Mathematics Letters **22** (2009), no. 75-78.