All Dressed Up and Ready to Go: a Tale of Two Quarks

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Abstract

The existence of unexpected states (states not predicted by the conventional quark model) in heavy Quarkonia, specifically Charmonium and Bottomonium like states, is of great interest to modern particle physics. States like $X(3872), Y_b(10890), Z_b^{\pm}(10610)$, and $Z_b^{\pm}(10650)$ have proven difficult to reconcile with the conventional quark model. However, analysis of diquark constituent masses has pointed towards tetraquark configurations being responsible for many of these exotic states.

Thus far, the diquark correlations required for a tetraquark configuration of X(3872) have been primarily examined through the use of diquark correlation functions where the Schwinger string is introduced to extract gauge invariant information from said diquark correlator. Here, research is presented on the use of the dressed field formalism in the context of diquark correlation functions. Results for doubly light, light-heavy, and doubly heavy diquark systems have been obtained, and all results have shown this dressed field method to be an effective means of extracting gauge invariant results from diquark correlation functions.

Acknowledgements

First and foremost, I absolutely must thank Tom Steele with every fibre of my being — building a student up from nothing is a huge (and undoubtedly often frustrating) task. The raw amount of time offered up, just so that I could develop as a physicist, is incredible on its own; and that's just the beginning. From sending me to conferences just so I could learn a little more to discussing strange physics concepts to going out with our entire research group, you gave much with very little concern for the cost. Thank you for being such an amazing guide for the first leg of my research journey.

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To those whom I love.

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List of Abbreviations

BSM	Beyond Standard Model
MS	Minimal Subtraction
$\overline{\text{MS}}$	Modified-Minimal Subtraction
OPE	Operator Product Expansion
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
SM	Standard Model

Chapter 1

Introduction

1.1 Motivation for Research

It was the best of times, it was the worst of times; a time of both new discovery and a suspicious lack of answers; it was 2003 and the Belle collaboration announced the discovery of a new, mysterious particle, $X(3872)$ [17]. The particle's decay patterns clearly indicated that it was hadronic, its J^{PC} value of 1^{++} was normal, and it was right in the middle of the charmonium spectrum. But the data produced by Belle and later confirmed by Babar [7], D0 [3], CDF [5], and LHCb [1] clearly demonstrated that $X(3872)$ was not a member of the charmonium spectrum predicted by the conventional quark model. As its identity and make-up were a mystery, it was given the distinction of being labelled with an X — indicating its lack of a proper particle family.

Perhaps unsurprisingly, $X(3872)$ was only the first of many so-called "XYZ" states that have been discovered over the last decade or so by various collaborations all over the globe. None of these new states could be explained using the conventional quark model that had been used for the last 40 years: some new interpretation was needed. Tetraquark and pentaquark (exotic) states have been proposed by many [24, 25, 48, 50, 64] to be the possible structure of these XYZ states. These predictions have been rooted in the early results from constituent

models which show $X(3872)$ and others to be promising four or five quark state candidates [6, 19, 46, 49, 65, 66]. These developments make a quantum chromodynamics (QCD) approach to four-quark states appealing: if the mass predictions from QCD point towards an exotic explanation, a prediction and classification framework can be built for these newly discovered particles.

That said, mass predictions for four-quark configurations aren't necessarily straightforward. It has been suggested that there are different ways that the quarks within such a state can bind together: two popular structures that have been proposed for these four-quark states are a diquark-antidiquark cluster and a mesonic molecular configuration. Diquarks are a hypothetical structure where two quarks are strongly correlated within a hadron [40] — in order to complete any QCD analysis on a tetraquark style structure, the diquark correlation function must first be calculated.

Herein lies the challenge this thesis looks to tackle: the diquark correlator, when calculated using standard means, is not gauge invariant. This means that any physical prediction made using the diquark result can be shifted by an arbitrary amount. This isn't good: we want to extract gauge invariant information from said correlator.

Traditionally, extraction of gauge invariant information has been accomplished through the application of the Schwinger string [23]. Unfortunately, calculations using this method tend to be incredibly difficult to extend to massive quarks. So, we look to an alternative option: the dressed field formalism designed primarily by Lavelle, McMullan, and Bagan [10–13, 37]. This dressed field technique is simple in its essence: a gauge dependent field is acted upon by a specific operator, making the field locally gauge independent. Lavelle *et al* demonstrated the validity of this method for QCD in various papers [10]; however the field

dressing had never been applied in a diquark context.

This thesis does exactly that: the diquark correlation function is dressed in accordance to the formalism within the doubly-light, light-heavy, and doubly-heavy quark scenarios. It is demonstrated that not only is a gauge invariant result obtained via this technique, but the calculations themselves are feasible. This both verifies the theoretical framework laid by Lavelle *et al* and lays the foundation for future prediction of four-quark masses.

1.2 The Quark Model

1.2.1 Quarks and Flavour

Over the course of the 30s, 40s, and 50s, new particle after new particle was discovered as new experimental innovations began to come to the fore. Growing from the basic vision of a sub-atomic world that consisted primarily of protons, neutrons, and electrons, a new image was born: complete with pions, muons, and a host of other particles. Classification into the simple groups of Fermions and Bosons proved to be not enough so additional classifications based on particle behaviour were introduced. For the purposes of this analysis, the important subatomic particles are the composites: hadrons. The hadron group can be divided further based on spin. Mesons are hadrons with integer spin, whereas as baryons are hadrons with half-integer spin. Ever since the introduction of the quark model by Zweig and Gell-Mann (independently) in 1964 [29, 69], the composite components of hadrons have been called "quarks".

Current theory, coupled with experiment, suggests that there exist six flavours of quarks:

up, down, strange, charm, top, and bottom. The up, down, and strange quarks are referred to as light quarks, while the remaining three flavours make up the heavy quarks. These various flavours of quarks come together to make the composite hadrons: for example, two up quarks and a down quark form a proton. Originally only three compositions were theorized: triple quark combinations, triple antiquark combinations, and quark antiquark combinations. Since each individual quark behaves as a fermion, the triple combinations are baryons whereas the diquarks are mesons.

Further classification of particles into multiplets based off their J^P helps to reveal the underlying symmetries present within the particle zoo. As per usual, $J = L+S$ and quantifies total angular momentum (L is the relative orbital angular momentum and S is the spin), and $P = (+, -)$ parity (additionally, for J^{PC} values $C = (+, -)$ represents charge conjugation). Mass differences within light quark multiplets are typically minor — if ignored they allow for a particle's identity to be determined simply through examination of its constituent quarks. If transformations between quark combinations are examined, a flavour symmetry reflecting that of the SU(3) flavour group is found. This holds for both baryons and mesons, and is reflected in the structure of the various multiplets of both classes: meson multiplets form nonet structures, whereas baryon multiplets form octet and decuplet structures. Two examples of multiplets can be seen in figures 1 and 2 [58].

From a group theory point of view we have a $3 \otimes \bar{3}$ flavour representation for mesons and a $3 \otimes 3 \otimes 3$ for baryons. These can be reduced to a singlet plus octet configuration ($3 \otimes \bar{3} = 1 \oplus 8$) and a singlet, double octet, and decuplet configuration ($3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$), respectively.

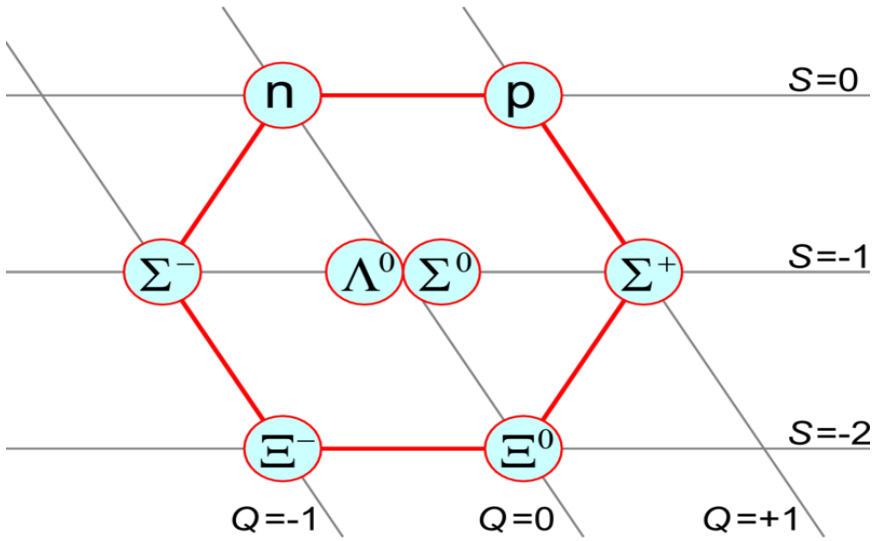

Figure 1.1: Baryon octet structure for $J^P = \frac{1}{2}^+$. Here, Q represents particle charge and S represents particle strangeness [58].

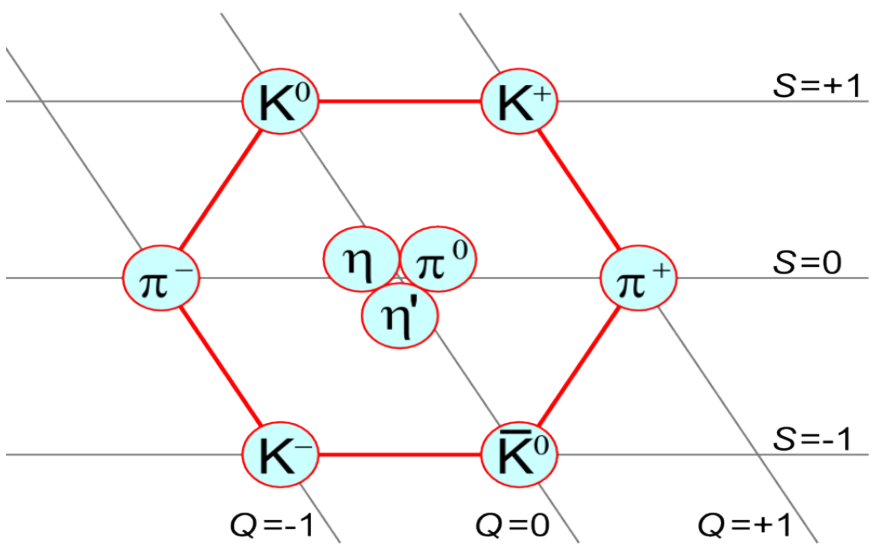

Figure 1.2: Meson nonet with $J^{PC} = 1^{--}$. Here, Q represents particle charge and S represents particle strangeness [58].

1.2.2 Colour Charge

Early on in the development of quark theory, it became apparent that flavour alone was not enough to describe the characteristics of quarks. The quintessential example of this is that of the Δ^{++} particle (made up of three up quarks). Like the Δ^{+}, Δ^{0}, and Δ^{-}, the Δ^{++} is in the ground state ($L = 0$) and holds a similar mass to its sister particles. This implies that the spatial wave function is symmetric. Also, since the spin of the Δ^{++} is $\frac{3}{2}$, all the component up quarks must have the same spin — thus the spin wave function is also symmetric. The flavour wave function is obviously symmetric as well. Combining all these wave functions leads to a result of a symmetric total wave function; this violates spin-statistics as the quarks composing Δ^{++} are fermions. Colour charge was introduced to deal with this inconsistency.

The mechanics of colour charge are, similar to flavour, governed by the mathematics of SU(3). There are six types of colour charge: green, red, blue, anti-green, anti-red, and anti-blue. Any quark can take on any one of the three colours and any anti-quark can take on any of the anti-colours. Additionally there is a limiting condition (referred to as colour confinement) on colour charge — the overall charge of a composite particle must be colour neutral. There are two primary ways of this occurring: for baryons, three quarks of different colours (or anti-colours) combine to make a colour neutral composition, while for mesons a coloured quark can combine with an anti-quark of the anti-colour. Colour can be transferred through the strong interaction via gluons and has no physical significance outside of providing an additional symmetry factor within the total wave function. For example, a red down quark behaves identically to a green down quark.

Ultimately, $SU(3)$ colour symmetry provides the foundation for the gauge theory of quark

interactions. Calling back to the principles introduced in the group theory basics of flavour symmetry, we see similar representation relations present for colour. Again, mesons have a representation of $3 \otimes \bar{3} = 1 \oplus 8$ whereas baryons have $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ for an irreducible representation. The singlet present in these relations implies that a colour neutral hadron can be formed.

Within the confines of this report, one of the central concepts of colour plays a key role in equation results: colour is conserved through strong interactions. As such, the incoming colour of, say, a diquark current would be identical to that of an outgoing diquark current after a loop interaction.

1.3 The Standard Model

The Standard Model (SM) of particle physics is a theory developed through the 1960s and 70s that describes the fundamental interactions of the electromagnetic, strong, and weak forces on the quantum scale. The roots of the SM go back to the 1961 work of Sheldon Glashow, where his paper "*Partial-symmetries of weak interactions*" demonstrated the possibility of connecting the electromagnetic and weak forces into the so-called electroweak force [30]. Other core components (including the primary subjects of the prior section) of the theory arrived as the decade progressed: the light quarks (up, down, and strange) were proposed in 1964 [29, 69], a fourth quark, the "charm", was theorized in 1965 [8, 15], and the concept of colour charge was introduced that same year [32]. These developments allowed for this primordial SM to make some rather successful classifications of the pre-existing particles and some accurate predictions — but the theory was still lacking on both the experimental and

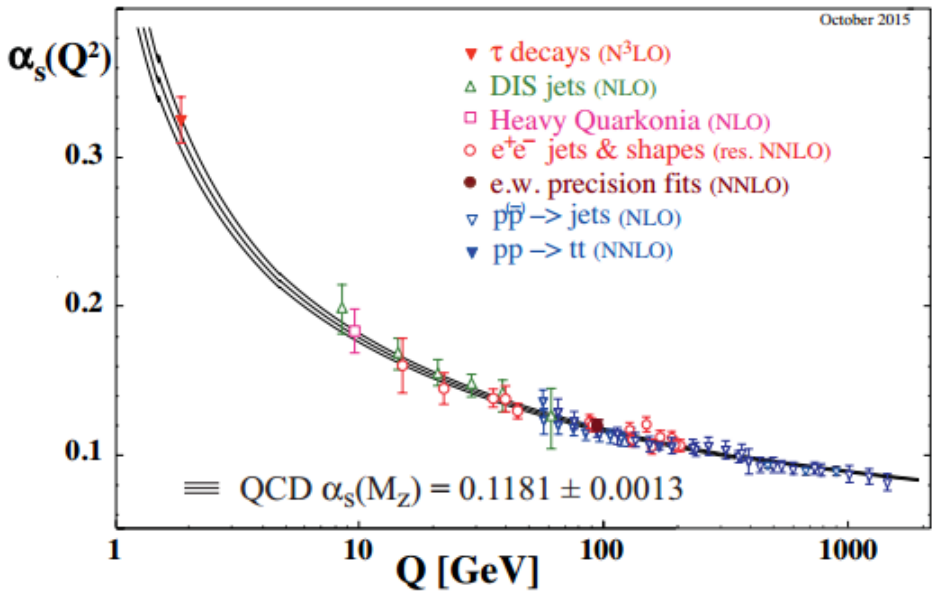

Figure 1.3: Summary of measurements of α_s as a function of the energy scale Q. The respective degree of QCD perturbation theory used in the extraction of α_s is indicated in brackets (NLO: next-to-leading order; $NNLO$: next-to-next-to leading order; res. $NNLO$: NNLO matched with resummed next-to-leading logs; N^3LO: next-to-$NNLO$). [54]

theoretical front: quarks, W and Z bosons, and tau neutrino had not been observed, while the model still lacked explanations for quark based interactions and the mass of gauge bosons.

Steven Weinberg and Abdus Salam provided the first steps towards resolving the latter of these issues when, in 1967, they introduced the Higgs mechanism (previously developed in [27, 34, 36]) into the SM as a means of generating mass for Gauge bosons [67]. More explicitly, this consists of an addition to the SM, the Higgs field, which permeates all space and causes spontaneous symmetry breaking; this symmetry breaking triggers the Higgs mechanism gifting the bosons it interacts with mass. Slightly later, in 1973, the theoretical explanation for quark interactions would be crafted: quantum chromodynamics (QCD). On a fundamental level QCD is a gauge theory of $SU(3)$ colour symmetry: in a similar fashion to its electromagnetism predecessor, quantum electrodynamics (QED), QCD explains how

Figure 1.4: Particle content of the Standard Model, minus the Higgs Boson. [35]

quarks interact via the strong force carrier, the gluon. Around this time David Politzer, David Gross, and Frank Wilczek discovered a crucial property of the strong force: bonds between quarks became asymptotically weaker as energy levels increased and distances scales decreased [33, 56]. This result is referred to as asymptotic freedom and can be seen in figure 1.3. With these additions to the primitive SM of the mid-60s, in 1974 the core of the modern SM was complete.

Experimental confirmation of the SM lagged behind the development of the theory sector for years and, in some cases, decades. The first true experimental evidence for quarks came in the late 1960s when during an electron-proton scattering experiment it appeared that the electrons were bouncing off of small, weakly-bound hard cores within the protons. A few years later indications of electrically neutral weak interactions were discovered. However, neither of these results provided hard evidence for the SM — many, many pieces were still

missing. Fortunately, these began to fall into place about a decade later with the discovery of the W^+, W^- and Z bosons using experiments developed by Simon van der Meer and Carlo Rubbia [20, 21]. The later discoveries of the top quark in 1995 [2, 4], the tau neutrino in 2000 [42], and Higgs boson in 2012 [16] provided successively stronger confirmations of the SM, with the Higgs discovery providing the crucial bit of evidence for the Higgs field and thus Higgs mechanism which are central to the SM.

Moving away from historical developments to the current SM, we need to explore the current overall structure of the theory. There are three forces mediated by the force carrying, spin-1 gauge bosons which make up the first category of particles within the SM: the gluon (g), the photon (γ), and the W and Z bosons. Of these force carriers, two are massless (g, γ) while the rest are massive. The second category of particles do not interact via the strong force — they are the spin-1/2 leptons further broken up into the electrically charged particles (e, μ, τ) and the corresponding neutrinos (ν_e, ν_μ, ν_τ). The quarks from the conventional quark model (u, d, s, c, b, t) make up the next category of spin-1/2 particles. This group does experience the strong force via gluons; this type of interaction makes up most of what will be covered within this thesis. Finally, the Higgs boson forms its own class of spin-0 particles.

Yet, despite all the successes and internal consistency of the SM, there are still many questions left to be answered and many phenomenon to be explained. Perhaps most conspicuously, the force of gravity (and thus general relativity) is not fully incorporated into the SM. This void fundamentally prevents the SM from being a functional "Theory of Everything". On a less broad scale, experimental results from the last forty years have provided new information about the physical world that unsurprisingly is simply not accounted for by the SM (although in some cases extensions to the model can be made to deal with these

results). Neutrino oscillations are a prime example of this: the MNS mixing matrix has no foundations in the base SM. Similarly dark matter is external to the standard model. The conventional quark model also isn't complete — exotic hadron states cannot be explained by the two basic structures (baryons and mesons) and represent new structure. For the purposes of this paper, the last of these cases is the most interesting and will be overarching focus of the rest of the text.

1.4 Exotic States

1.4.1 Background

As mentioned in the motivation, the 2003 discovery of the $X(3872)$ particle by Belle [17] and its subsequent confirmation by a host of other collaborations [1, 3, 5, 7] was the first piece of evidence that the conventional quark model wasn't telling the whole story about hadron composition. The particle $X(3872)$ is almost certainly a hadron — its decay chains speak strongly to this fact. But, at the same time, its mass and J^{PC} value of 1^{++} don't seem to fit into the conventional quark model. And $X(3872)$ is only the first of many states (often referred to as "XYZ" states) that have been discovered over the last decade (including $Y(4630), Z_2(4250), Y(3940)$ and $X(3940)$ [54]) where the measurements from Babar, Belle, CLEO, D0, and LHCb seem to exclude traditional triple-quark baryon or quark-antiquark meson configurations. Several of these states can be seen relative to the charmonium spectrum in figure 1.5.

Now, on a theoretical level this isn't particularly a problem for QCD; all QCD requires

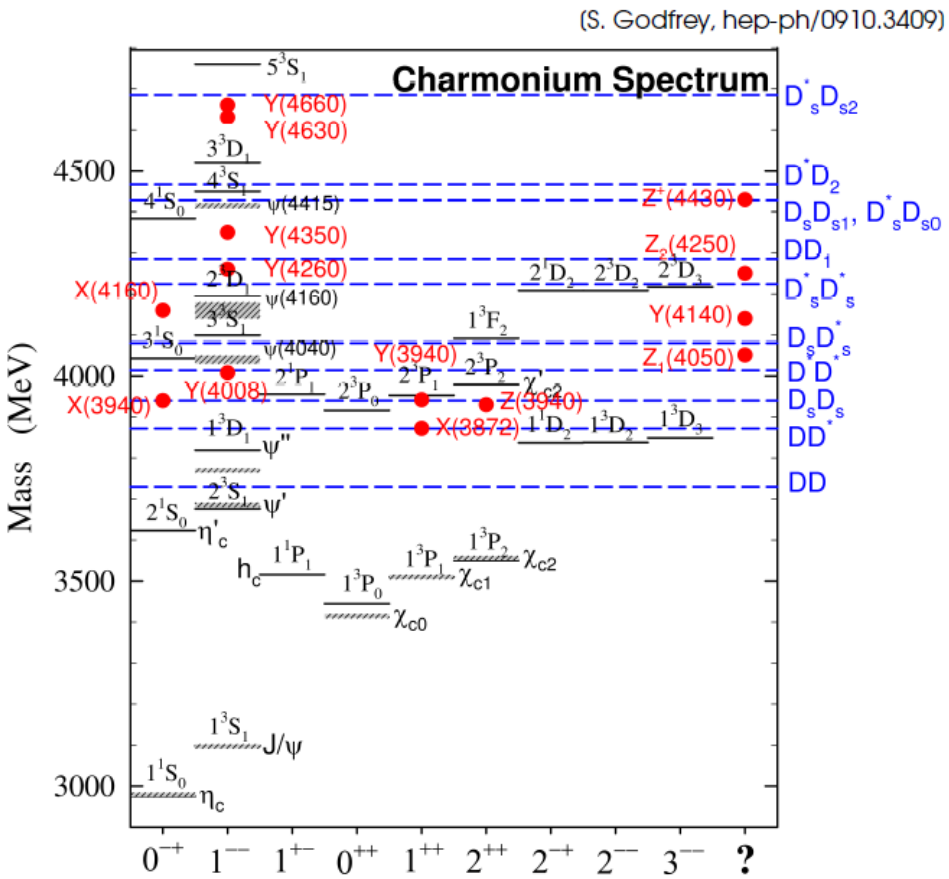

Figure 1.5: Charmonium Spectrum. Black lines indicate charmonium states, blue lines indicate thresholds for various decays, and red dots indicate charmonium-like (possibly exotic) states. The horizontal axis is J^{PC} values. [31]

is that a hadronic state is colour singlet — the actual internal structures is unrestricted. Of course, at this current point there is only hard evidence for the traditional baryon and meson states, however theoretical speculations include: di-meson molecular states, tetraquarks, pentaquarks, hybrid charmonium, conventional charmonium, and more [38, 41]. These proposed states make up a subsection ("exotic hadrons") of so-called "exotic matter": particles outside the Standard Model and the conventional quark model that occasionally have non-standard properties. In the most general (and colloquial) sense, exotic matter includes exotic hadrons, dark matter, tachyons, and more. Here, we only care about the first of these and ignore the remaining (*more* exotic) elements of the list.

Furthermore, the category of exotic hadrons can be broken up into three subsections: multiquarks (such as tetraquarks and pentaquarks), hybrids, and glueballs. The "more" exotic elements of this list, hybrids and glueballs, consist of a excited gluon stuck onto a conventional meson ($qG\bar{q}$) and interacting gluons (GG and GGG), respectively. These states are allowed due to the fact that gluons both carry colour charge (allowing them to help form a colour singlet) and are capable of self-interaction. Although both of these sub-categories are intriguing, the multiquark states is what this paper focuses on: specifically four-quark hadrons($qq\bar{q}\bar{q}$).

Although there have been many different proposed internal configurations for four-quark states, there are two which have emerged as front runners (especially in the context of $X(3872)$): a tetraquark configuration ($[qq][\bar{q}\bar{q}]$) which is made up of two diquark clusters (each of which have a net colour charge) and a di-meson (also referred to as a molecular) configuration ($[q\bar{q}][q\bar{q}]$) where two colour singlet mesons are weakly bound together [6, 19, 46]. Examples of both these states can be seen in figure 1.6.

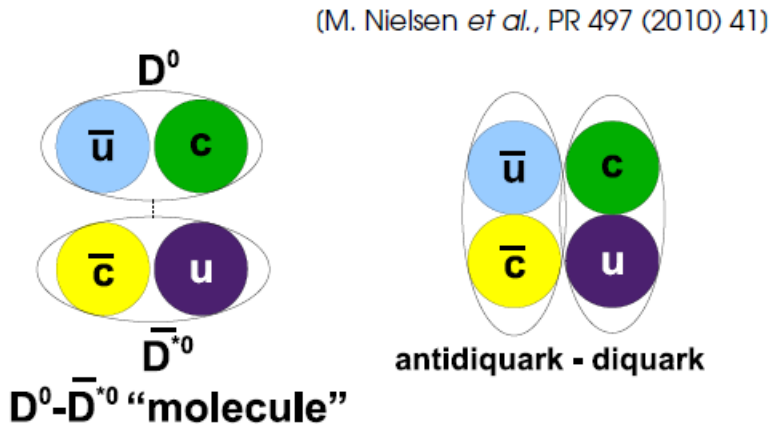

Figure 1.6: Potential structures for $X(3872)$: the double diquark molecule and the tetraquark (antidiquark-diquark) configuration [53].

Before moving on, a brief detour must be taken to discuss heavy quarkonia. As mentioned in the preliminary discussion of the quark model, the up, down, and strange quarks are referred to as the light quarks while the remaining three (charm, bottom, and top) are called the heavy quarks. This is due to the huge difference in mass scale between these groups, as seen in tables 1.1 and 1.2. The top quark is immensely heavy, but decays incredibly quickly through weak interactions and thus does not form bound states [57]. The other two heavy quarks do, however, form bound states — charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$). The various different ways these quarks combine together form the charmonium and bottomonium spectra, respectively. The four-quark states being examined in this thesis have masses right in the middle of the charmonium spectrum (masses beginning around 3.8 GeV) and thus are often referred to as charmonium-like states.

At this point, it's best to return to back to $X(3872)$. I mentioned earlier that the decay chains indicate that this particle is not only an exotic hadron, but some type of multi-quark. This suggestions must be expanded on, albeit briefly. First, however, it should be noted that the most direct way of demonstrating that the particle is exotic, its J^{PC} value, doesn't

Flavour	Mass (MeV)
u	2.3
d	4.8
s	95

Table 1.1: Light Quark Masses. Data from Particle Data Group Summary Tables [54].

work in this case: 1^{++} is a perfectly normal quantum number that can be achieved by conventional quark configurations (contrast this to $J^{PC} = 0^{--}, 0^{+-}$, or 1^{-+}; none of these values can exist within conventional quarkonia). But, crucially, $X(3872)$ (and many other XYZ states) has several decay modes involving the J/ψ particle [17]. As this is the lightest spin-1 charmonium state, at absolute minimum $X(3872)$ must contain $c\bar{c}$. At the same time, the mass of this state makes it impossible for it to have a bottom quark component. Even more compellingly (as shown in figure 1.5), 3872 MeV is extraordinarily close to the mass of a D^0 meson combined with a $\bar{D}^{0*}$ meson. These results have led some [6,46] to give credence to molecular interpretation mentioned earlier in this section. On the other hand, the decay width points to a more tightly bound configuration (such as the tetraquark) [24, 25]. As of yet, neither structure is the clear winner. Other XYZ states have been interpreted in similar fashions. Overall, the experimental results seem to heavily favour the existence of exotic hadrons.

1.4.2 Diquarks and Exotic States

On the most basic level, diquarks (qq) can be viewed as a strong correlation between two quarks within a hadron. Although they are a purely hypothetical structure, their existence

has been postulated by Gell-Mann from the very beginnings of the quark model [29]. Traditionally, diquarks were used to simplify the rather complex three-body problem present in baryons where all quarks are treated individually to a two-body where only a diquark and quark treated [47]. Additionally, the use of diquarks allows for the simplification of four quark states. But, in the case of the diquark-antidiquark interpretation of four quark states, diquarks are crucial to any type of analysis — if a particle is made up of diquarks, you must be able to work with diquarks!

Before I fully unpack this subject, a glance at the constituent quark model must be taken. Hadrons, such as the proton, are made up of valance quarks (uud for a proton) from QCD. However, it quickly becomes apparent that this is not the whole story: the masses of the three quarks that make up a proton only account for about 1% of the proton's mass! The rest of the mass comes form the constant interactions taking place within the particle. Gluons interact with the valance quarks and each other, while virtual quark-antiquark pairs wink into and out of existence continually. Thus, although QCD is our theory of quark interactions it is not effective when used to study hadrons as the sum of their quark make-up. On a more technical level, this is due to the reliance of QCD on perturbation theory; a theory that only holds when the coupling is small. The coupling within hadrons is massive and thus non-QCD approaches must be used to interpret hadronic states. The constituent quark model is one such approach [40]. The constituent quark model operates under many of the same assumptions as QCD: hadrons form colour singlets and $SU(3)$ is taken to be the symmetry group for flavour. The difference lies in the treatment of quarks; here quarks are given effective constituent masses that take into account both the valence quarks and the virtual quark-gluon sea that surrounds them. Going back to the proton example, the total

mass of the proton, 938 MeV, would lead to effective masses of around 300 MeV for the up and down quarks.

This model, though not rooted in QCD, provides a sound way of describing the hadrons in most cases. The light scalar mesons have shown to be an exception to this rule, however. It's a difficult struggle to fit the light $J^{PC} = 0^{++}$ particles into the quark-antiquark nonet that would be expected under the conventional quark model [54]. There are nineteen states under 1800 MeV — far too many to explain using an $SU(3)$ constituent quark structure. Just like with $X(3872)$, four-quark states have been proposed in order to explain the existence of these particles. And, just like with $X(3872)$, four-quark states based off of diquarks have been proposed [26].

At this point, diquarks appear useful; but how does one go about using them to obtain meaningful information (or predictions) about the hadrons they compose? The answer to this question lies in QCD sum-rules (QSR). QSR can be used to calculate the mass of the diquark [68] which can be used as an approximation of the effective diquark mass, with this a Hamiltonian of the form [68]:

$$H = \sum_i m_i + \sum_{i<j} 2\kappa_{ij} \left(S_i \cdot S_j \right) \tag{1.1}$$

can be used to determine a mass prediction for a four-quark state. Above, m_i is the constituent quark mass and the second term is the spin-dependent interactions between constituent quarks with κ_{ij} being the interaction coefficient. Diquarks are taken to be in the colour triplet configuration (i.e Jaffe's "good diquarks" [40]) and the colour force represented by κ is then taken from the quark-quark interaction because the force between colour triples is assumed to be universal. With the mass of the four-quark state in hand, we've accom-

plished our goal of making physical predictions for an exotic state. Plus, we have the added bonus of studying the diquark though QCD directly (via QSR) as opposed to using a phenomenological model. Although this thesis' work only revolves around the computation of diquark correlation functions, the road to four-quark mass predictions has been outlined here to demonstrate the phenomenological applicability of the dressed field results.

Flavour	Mass (GeV)
c	1.275
b	4.18
t	173.21

Table 1.2: Heavy Quark Masses. Data from Particle Data Group Summary Tables [54].

Chapter 2
The Background

2.1 Quantum Field Theory

Before we can progress to any form of actual computation, the standard field theory framework must be laid out. The concepts of this chapter are rooted in the derivations found in Ryder's "Quantum Field Theory" (QFT) [59], Peskin and Schroeder's "An Introduction to Quantum Field Theory" [55], and Matthew Schwartz's "Quantum Field Theory and the Standard Model" [60].

2.1.1 Relativistic Quantum Mechanics

The motivations behind quantum field theory are rooted in the need for a quantum theory consistent with relativity. The search for such a theory began in earnest in the mid 1920s with work by Heisenberg, Born, Jordan, Pauli, and more building towards the development of the first proper quantum field equations: the Klein-Gordon and Dirac equations.

In order to discuss these ideas, familiarity with relativistic notation is crucial. First and foremost, we adopt the convention that $\hbar = c = 1$ in order to simplify expressions. Secondly, the notion of distance in Minkowski space time is taken to have the following form:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,. \tag{2.1}$$

Here, x, y, z and t are the coordinates of a Cartesian system. The above metric tenser leads us to our definition of 4-vectors:

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) \text{ and } x_\mu = (x_0, x_1, x_2, x_3) = (t, -x, -y, -z)\,. \tag{2.2}$$

These vectors are related via the metric tensor of Minkowski space-time:

$$x_\mu = g_{\mu\nu} x^\nu\,. \tag{2.3}$$

Thus, the metric tensor of the space is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{2.4}$$

Finally, we can define energy-momentum 4-vector

$$p^\mu = (E, \mathbf{p})\,. \tag{2.5}$$

2.1.2 The Klein-Gordon Equation

With the foundational notation in place, the relativistic equation for scalar particles, the Klein-Gordon equation, can be determined. Combination of the contravariant and covariant energy momentum tensors leads to the Lorentz-invariant expression:

$$p^2 = p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2 c^2 = m^2\,. \tag{2.6}$$

Building from this base, we replace the energy and momentum components with their respective operators $\hat{E} = i\frac{\partial}{\partial t}$ and $\hat{p} = -i\frac{\partial}{\partial x}$ acting on a field, ψ:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi + m^2\psi = 0\,. \tag{2.7}$$

We can make use of the d'Alembertian operator

$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2, \tag{2.8}$$

that allows us to write the Klein-Gordon equation in its standard Lorentz-invariant form:

$$(\Box + m^2)\psi = 0. \tag{2.9}$$

If, instead, the non-relativistic approximation of (2.6) is used, the Schrödinger free-particle equation results. Further, the probability density obeys the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0\,. \tag{2.10}$$

The $\mathbf{j}$ about is the conserved current of the system. This leads to a probability density of:

$$\rho = \frac{i}{2m}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\,. \tag{2.11}$$

Originally, this result led a rather underwhelming response to the Klein-Gordon equation: it didn't deal with spin-1/2 particles, lacked a positive definite result for ρ that could be interpreted as a probability distribution, and allowed negative energy states — all large problems to pre-field theory quantum mechanics. It would not be until significantly later that the Klein-Gordon equation was understood to correctly describe the spin-0 particles.

2.1.3 The Dirac Equations

The Dirac equation was first derived in 1928 and was the first theory to fully account for both special relativity and the principles of quantum mechanics. Differing from the Klein-Gordon equation, the Dirac equation deals with massive (the zero mass limit results in the Weyl equations) spin-1/2 particles and includes electromagnetic interactions. Critically,

Dirac realized that in order to entertain the possibility of a positive definite probability distribution an equation of first order in space and time (as opposed to second order, like the Klein-Gordon equation) must be provided. Furthermore, Dirac's calculations led him to recognize that 4 by 4 matrices were required to set up a system with the desired properties of:

1. Correct energy-momentum for a free particle
2. Positive-definite probability density
3. Covariance under Lorentz transformations.

This moved away from the one component wave function theorized by Schrödinger and the two component function suggested by Pauli, replacing both in favour of a four component wave function. The Dirac equation is:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{2.12}$$

where the γ^μ represents one of the 4 x 4 matrices derived by Dirac. These matrices are based on the 2 x 2 Pauli spin matrices, which are, in turn, rooted in the mathematics of the SU(2) group. The first four gamma matrices are defined as follows:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \text{ for } i = 1, 2, 3\,. \tag{2.13}$$

Additionally, for the purpose of helicity projections, a γ^5 matrix is also defined:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{2.14}$$

Within the equations above, 0 represents a 2 x 2 zero matrix and 1 represents a 2 x 2 identity matrix. One crucial feature of these gamma matrices is that they satisfy the following anti-commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{2.15}$$

As the $g^{\mu\nu}$ present in the above equation metric tensor of Minkowski space-time (2.4), these relations indicate the correct relativistic energy-momentum relation is present.

One of the key results obtained through the study of the Dirac's equation is the existence of anti-particles. If we replace $i\partial_\mu$ with p_μ in (2.12)

$$(\gamma^\mu p_\mu - m)\psi = 0 \tag{2.16}$$

After writing out the 4 components of the above equation, it can be shown that the appropriate eigenvalues (each appearing twice) for E are:

$$E = (m^2 + p^2)^{\frac{1}{2}} \text{ and } E = -(m^2 + p^2)^{\frac{1}{2}} \tag{2.17}$$

Although Dirac interpreted this result, somewhat incorrectly, as indicating there was a vast sea of negative energy states already completely filled (thus preventing additional electrons from entering the negative energy states), the equation still lays the theoretical framework for the understanding of anti-particles — an accomplishment that would be recognized as a major success beginning with the discovery of the first anti-particles by Carl Anderson in 1932.

2.2 Quantization

Throughout section 2.1, several of the foundational equations for explaining for relativistic, non-interacting particles were outlined. In order to properly treat reality, however, more is required. Within this section the cornerstone of quantum field theory, the Lagrangian formulation, will be introduced and put to immediate use in deepening our understanding of the Dirac and Klein-Gordon equations.

2.2.1 Classical Fields

The concept of the Lagrangian originated in the classical sphere of physics in the late 18th century when Joseph-Louis Lagrange developed a reformulation of Newtonian mechanics. This powerful reformulation allowed for the easy induction of generalized coordinates into classical problems - making it far simpler to take constraints into consideration. The crux of this reconstruction of mechanics lies in the Lagrangian, defined as:

$$L = T - V \tag{2.18}$$

Additionally, the so-called Lagrangian density is written as:

$$L = \int \mathcal{L}\, \mathrm{d}^3x \tag{2.19}$$

The action S of a system is determined by integration of the Lagrangian:

$$S = \int L\, \mathrm{d}t \tag{2.20}$$

Now, in order to express this concept in a form more conducive to the fields used in quantum mechanics the Lagrangian density is substituted into (2.20):

$$S = \int \mathcal{L}(\psi, \partial_\mu \psi) \mathrm{d}^4 x. \tag{2.21}$$

Two important distinctions must be made at this point: firstly, the term "the Lagrangian" will be used in place of "the Lagrangian density" for brevity, and, secondly, it will be assumed that $\mathcal{L}$ is dependent only on ϕ and its first derivatives for two reasons: 1) this simplifies calculations and 2) this reflects the actual Lagrangians we use.

Perhaps the most important result arising out of the Lagrangian formulation grows out of applying variational parameter methods to the action of a system — the result is the equation of motion for the field ϕ, known as the Euler-Lagrange equation.

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right] = 0. \tag{2.22}$$

2.2.2 Noether's Theorem

Beyond the mere construction of a host of different Lagrangians, the variational techniques used above can be applied to further explore the nature of the Lagrangian formulation of relativistic mechanics. The most crucial result is Noether's theorem; a fundamental result that demonstrates physical symmetries correspond to conserved currents. Explicitly: any symmetry of the action of a system has a corresponding conservation law.

The derivation of the theorem again begins with introduction of variational techniques. If we deform the field ϕ:

$$\phi \Rightarrow \phi^{'} = \phi + \alpha \Delta \phi, \tag{2.23}$$

this transform is a symmetry if it leaves the equations of motion invariant — this is guaranteed if the Lagrangian is invariant up to a 4-divergence:

$$\mathcal{L} \Rightarrow \mathcal{L}^{'} = \mathcal{L} + \alpha \partial_\mu J^\mu. \tag{2.24}$$

Variation of the fields leads to the final result:

$$\partial_\mu j^\mu = 0 \text{ and } j^\mu = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] \delta\phi - J^\mu. \tag{2.25}$$

This result indicates that current densities are conserved through transformations. More specifically, this result accounts for the conservation of energy, momentum, angular momentum, and conserved quantum numbers, making its existence all the more vital.

2.2.3 The Lagrangian Formulation of the Klein-Gordon Equation

With the tools of Lagrangian mechanics in hand, field interpretations of the relativistic particle equations can begin in earnest. The Lagrangian of a scalar particle field is taken to be:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \,. \tag{2.26}$$

Applying partial derivative operators results in:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \text{ and } \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \,. \tag{2.27}$$

Plugging these results into (2.22) leads directly to the Klein-Gordon equation:

$$\partial_\mu \partial^\mu \phi + m^2 \phi = \Box \phi + m^2 \phi = 0. \tag{2.28}$$

Critically, this reveals that the Euler-Lagrange equation is equivalent to the Klein-Gordon equation — indicating that either the Lagrangian formulation does apply to relativistic fields or that it has not been formulated properly.

This principle can be extended beyond the domain of just a real scalar field. A generalization leads to the Lagrangian for a complex scalar field with electromagnetic interactions:

$$\mathcal{L} = (\partial_\mu \phi + ieA_\mu \phi)(\partial^\mu \phi^* - ieA^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{2.29}$$

Again, applying equation (2.22) to this leads directly to the complex Klein-Gordon equation for particles interacting with an electromagnetic field. Similar to both Klein-Gordon Lagrangians, there exists a Lagrangian for fields of spin-1/2 particles:

$$\mathcal{L} = \bar{\phi}(i\gamma^\mu \partial_\mu - m)\phi \tag{2.30}$$

This, once again, leads directly to the Dirac equation upon being plugged into (2.22).

2.2.4 The Klein-Gordon Fields

The starting point of the canonical quantization of the Klein-Gordon equation rests in the Hamiltonian of the classical Klein-Gordon field. If the Klein-Gordon equation is viewed as a many particle equation its Hamiltonian is given by:

$$H = \int \theta^{00} \mathrm{d}^3 x \,, \tag{2.31}$$

where $\theta^{\mu\nu}$ is the energy-momentum tensor:

$$\theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \,. \tag{2.32}$$

Plugging in appropriately, this leads to a Hamiltonian of:

$$H = \frac{1}{2} \int [(\partial_0 \phi)^2 + \nabla\phi \cdot \nabla\phi + m^2 \phi^2] \mathrm{d}^3 x \tag{2.33}$$

for the real Klein-Gordon field.

Moving to the complex formulation, the Hamiltonian has the following form:

$$H = \frac{1}{2}\int[(\partial_0\phi^*)(\partial_0\phi) + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi]\mathrm{d}^3x. \tag{2.34}$$

Introducing operators $a(k)$ and $a^\dagger(k)$, the variable $\omega_k = (\mathbf{k}^2 + m^2)$, regarding $\phi(x)$ as a Hermitian operator, and performing a Fourier transform results in:

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}]. \tag{2.35}$$

From this position, the next step is to determine the commutator relations for the field. In order for this to take place, an additional field is introduced, the canonical momentum:

$$\pi(\mathbf{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\mathbf{x}, t)}\,. \tag{2.36}$$

This, coupled with the ϕ field lead to the equal-time commutation relation (ETCR):

$$[\phi(x,t), \pi(x',t)] = i\delta^{(3)}(x - x') \tag{2.37}$$

$$[\phi(x,t), \phi(x',t)] = [\pi(x,t), \pi(x',t)] = 0. \tag{2.38}$$

These commutation relations form the underlying structure of the quantized field and, combined with the Hamiltonian and key field equations demonstrate the many-particle interpretation of the equation, remove the negative energy issues and demand that the particles described are Bosons.

The quantization of the complex field is similar, and the results indicate (just as in the pure Lagrangian formulation) the existence of antiparticles.

2.3 Foundations of QCD

Quantum Chromodynamics is a fundamental theory that describes strong interactions. Just like its predecessor, Quantum Electrodynamics (which describes electromagnetic interac-

tions), it's a quantum field theory. And, just like its predecessor, QCD is ultimately concerned with describing quantum mechanical processes that involve the creation or annihilation of particles. The fundamental particles of the theory are quarks, which act as the building blocks for hadronic matter, and gluons, which act as the force carriers of the theory. On a more technical level, QCD is non-Abelian gauge theory with an SU(3) symmetry group. Experimental evidence exhibits both asymptotic freedom and confinement for strongly interacting particles: these rather unique properties turn out to be key defining elements of QCD. Two different (but equivalent) approaches can be used to develop QCD: canonical quantization and the path integral formulation. Within this section, the canonical approach is primarily used to develop the QCD content required for understanding the research done with dressed fields (although there is a small bit included about the path integral formulation, for the sake of completeness).

2.3.1 Canonical Quantization of Quark Fields

Thus far the formulations used to describe particle creation and annihilation have been lacking; in order to achieve a more accurate picture we transition to canonical quantization. This transition requires three choices to made regarding the nature of the particle equations:

1. They act as field equations.

2. Fields are to be viewed as operators.

3. Fields are quantized.

These rules are all well and fine, however since these rules apply to any quantum field theory they leave too broad a scope if left alone. In order to begin building up QCD we need to be

a little more specific in what we are looking for: quarks are massive spin-1/2 fermions — so to describe them we must restrict fields that fit this case. Fortunately, we've already dealt with such a field; it's simply the Dirac field that we touched on earlier in the chapter:

$$(i\gamma^\mu \partial_\mu - m)q(x) = 0. \tag{2.39}$$

Here we have shifted to using q to represent the Dirac field, as this is the convention that mimics that used by TARCER (the loop integral calculation software used for this project, further details can be viewed in section 2.4.1), making later calculations simpler to input. The Dirac field now needs to be given the same treatment as the Klein-Gordon field received in the previous section. The simplest possible Lagrangian of the Dirac field is (after applying the definition $\bar{q} = q^\dagger \gamma^0$):

$$\mathcal{L} = \bar{q}\left(i\partial\!\!\!/ - m\right) q. \tag{2.40}$$

We can take this and use the relationship between the Hamiltonian and the Lagrangian densities to find:

$$\mathcal{H} = \pi \dot{q} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial[\partial_0 q(x)]}\partial_0 q(x) - \mathcal{L} = i\bar{q}(x)\gamma^0 \partial_0 q(x). \tag{2.41}$$

This, of course, leads to the full Hamiltonian:

$$H = \int \mathrm{d}^3x\, \mathcal{H} = i\int \mathrm{d}^3x\, \bar{q}(x)\gamma^0 \partial_0 q(x). \tag{2.42}$$

With this in hand, we can start to apply some quantum mechanics to learn more about the Dirac field. If we approach our field from the Heisenberg picture, operators carry time dependence according to the Heisenberg equation of motion:

$$i\partial_0 q(x) = [q(x), H]\,. \tag{2.43}$$

The only way to progress from this stage is to take the plunge and substitute (2.42) into Heisenberg's Equation:

$$\begin{aligned} i\partial_0 q(x) &= i\frac{\partial}{\partial x^0}q(x) = [q(x), H] \\ &= \left[q(x), i\int \mathrm{d}^3y\, \bar{q}(y)\gamma^0\partial_0 q(y)\right] \\ &= i\int \mathrm{d}^3y\, \left[q(x), \bar{q}(y)\gamma^0\partial_0 q(y)\right] \\ &= i\int \mathrm{d}^3y\, \left(\{q(x), \bar{q}(y)\}\gamma^0\partial_0 q(y) - \bar{q}(y)\{q(x), \gamma^0\partial_0 q(y)\}\right) \\ &= i\int \mathrm{d}^3y\, \left(\{q(x), \bar{q}(y)\}\gamma^0\frac{\partial}{\partial y^0}q(y) - \bar{q}(y)\gamma^0\frac{\partial}{\partial y^0}\{q(x), q(y)\}\right). \end{aligned} \tag{2.44}$$

In the third line we have made use of the commutator identity [X,YZ] = {X,Y}Z - Y{X,Z}. A similar process can be used to manipulate $\bar{q}$. In order to get the left hand side of the equation to match up with the right hand side, we're forced to impose the following anti-commutator relations:

$$\{q_i(x), \bar{q}_j(y)\} = \delta^{(3)}(x-y)\gamma^0_{ij} \tag{2.45}$$

$$\{q_i(x), q_j(y)\} = \{\bar{q}_i(x), \bar{q}_j(y)\} = 0\,. \tag{2.46}$$

From a mathematical perspective, commutator relations could also have been used to satisfy equation (2.44) — but this violates the spin-statistics theorem: fields of integer spin (bosons) commute (and thus obey a commutator algebra) while fields of half-integer spin (fermions) anti-commute (and thus obey an anti-commutator algebra). Spinor indices i and j have been included for completeness.

As a final bit of exploration of the Dirac equation, we take a look at the general plane wave solution (with the standard $E_k = k^0$):

$$q_i(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_\alpha [a^\alpha(\mathbf{k})u_i^\alpha(k)e^{-ik\cdot x} + b^\alpha(\mathbf{k})v_i^\alpha(k)e^{ik\cdot x}] \tag{2.47}$$

The solution for $\bar{q}$ can also be found:

$$\bar{q}_j(y) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \sum_\alpha [b^{\alpha\dagger}(\mathbf{k})\bar{v}_j^\alpha(k)e^{-ik\cdot y} + a^{\alpha\dagger}(\mathbf{k})\bar{u}_j^\alpha(k)e^{ik\cdot y}] \tag{2.48}$$

If we take these two equations and put them through the anti-commutator relations (2.45) and (2.46), we come to a new set of anti-commutators:

$$\{a_i^\alpha(k), a_j^{\beta\dagger}(p)\} = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})\delta_{ij}\delta^{\alpha\beta} \tag{2.49}$$

$$\{b_i^\alpha(k), b_j^{\beta\dagger}(p)\} = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})\delta_{ij}\delta^{\alpha\beta} \tag{2.50}$$

$$\{a_i^\alpha(k), b_j^{\beta\dagger}(p)\} = \{a_i^\alpha(k), b_j^\beta(p)\} = 0 \tag{2.51}$$

$$\{a_i^{\alpha\dagger}(k), b_j^{\beta\dagger}(p)\} = \{a_i^{\alpha\dagger}(k), b_j^\beta(p)\} = 0. \tag{2.52}$$

2.3.2 QFT Operators

At this point, we need to take a little bit of a detour away from working with quark fields and towards laying out a little bit more of the formalism that will be used to explore these fields. The first bit of this is truly a throw back to quantum mechanics — Dirac notation (bra-ket notation). In the loosest sense, bra-ket notation is nothing more than a way of representing states, operators, and the interplay between the two. Within QFT this notation appears most often in the form of some sort of transition amplitude

$$\langle \mathrm{Final\,State}|H_{Int}|\mathrm{Initial\,State}\rangle \ , \tag{2.53}$$

where H_{Int} is the interaction piece of the Hamiltonian. Taking a more rigorous glance, the bra (the leftmost piece of (2.53)) is an element of the dual space of Hilbert space. This makes the inner product of a bra and a ket (the rightmost piece of (2.53)) an action of a linear functional on a vector in a complex vector space and, due to the bra's relationship with

Hilbert space, gifts the inner-product the properties of linearity, associativity, and Hermitian conjugation; and clearly establishes the isomorphism present between L2 vector operators and the vector space itself.

As fantastic as the underlying mathematics are, the core physical concepts that can be drawn out of the mathematics are what truly intrigues us. In this context, everything truly begins with the the vacuum, $|0\rangle$. This vacuum is non-interacting and represents a ground state with no field quanta. Raising operators $a_i^\alpha(k), b_j^{\beta\dagger}(p)$ represent an excitation of the field and the creation of a particle while lowering operators $a_i^\alpha(k), b_j^\beta(p)$ represent the annihilation of a particle. This, of course, means that any lowering operator acting on the ground state leads to a value of zero: $a_i^\alpha(k)|0\rangle = b_j^\beta(p)|0\rangle = 0$. As in quantum mechanics, wave functions can be represented in the form:

$$A\Psi(r) = \langle r|A|\Psi\rangle. \tag{2.54}$$

Perhaps the most important thing that we can build out of these basic concepts is the propagation amplitudes for the Dirac field — or, in other words, the probability amplitude for a quark to propagate from one location to another distinct location in space time:

$$\langle 0|T\left[\, q_a^\alpha(x)\, \bar{q}_b^\beta(y)\,\right]|0\rangle = S_f(x-y) = i\delta_{\alpha\beta}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, S_{ab}(p)\mathrm{e}^{-ip\cdot(x-y)}\,, \tag{2.55}$$

with:

$$S_{ab}(p) = \frac{\not{p}+m}{p^2-m^2+i\epsilon},\ \epsilon\to 0^+. \tag{2.56}$$

In order to approach these expressions, there is a tiny bit of architecture that must be added. First off, we introduce Feynman slash notation where $\not{p} = \gamma^\mu p_\mu$. Second, we have introduced a term, ϵ into the denominator of the propagator in order to avoid poles in the integration contours and ensure time ordering is obeyed. Explicitly, the S term can be referred to as a

Green's function or, alternatively, as a two-point function. These expressions ultimately grow out of the anti-commutation relations (2.45) and (2.45) and the general planewave solutions (2.47) and (2.48). Finally, the time-ordering operator has been used:

$$T\left[q_a^\alpha(x)\,\bar{q}_b^\beta(y)\right] = \begin{cases} q_a^\alpha(x)\,\bar{q}_b^\beta(y) \text{ if } x^0 > y^0 \\ \\ -\bar{q}_b^\beta(y)\,q_a^\alpha(x) \text{ if } x^0 < y^0 \end{cases} . \tag{2.57}$$

This time ordering operator makes the operators act on the ground state in chronological order (they appear is reverse chronological order when viewed from left to right), ensuring that particles only propagate forward in time. The time ordering operator represents a discrete symmetry transformation that stands to compliment the continuous symmetry transformations of the Lorentz group that leave correlation functions invariant. There are other discrete transformation operators present within QFT — namely the parity, P, and charge conjugation, C operators. Strictly speaking, the P operator reverses the momentum of a particle *without* flipping its spin, while the C operator takes a fermion (anti-fermion) with a given spin and converts it to an anti-fermion (fermion) with the same spin. Although these operators are crucial to understanding quantum field theory, they don't play a significant role in the computations completed within this thesis and are thus, for the most part, ignored. One additional operator must be introduced before progressing forward, and that is the normal ordering operator:

$$N\left[a_i^\alpha(k)b_j^{\beta\dagger}(p)\right] =: a_i^\alpha(k)b_j^{\beta\dagger}(p) := -b_j^{\beta\dagger}(p)a_i^\alpha(k) \tag{2.58}$$

or

$$N\left[a_i^\alpha(k)a_j^{\beta\dagger}(p)\right] =: a_i^\alpha(k)a_j^{\beta\dagger}(p) := -a_j^{\beta\dagger}(p)a_i^\alpha(k). \tag{2.59}$$

Essentially, what the normal ordering operator does is take all annihilation operators and push them to the right (automatically pushing all creation operators to the left). This gives an interesting and very general result:

$$\langle 0| : ... : |0\rangle. \tag{2.60}$$

Thus, any quantum field (or combination of quantum fields) that is sandwiched by free ground states and acted on by a normal ordering operator gives a value of zero. Although this operator doesn't represent a crucial discrete symmetry, it does allow for the simplification of fairly complicated functions through its central role within the application of Wick's theorem.

2.3.3 Wick's Theorem

One of the most crucial techniques used within our manipulation of the diquark correlation functions is the contraction of quark states being acted on by bra and ket operators in accordance with the results of Wick's theorem. On a strictly mathematical level, Wick's theorem is quite general and essentially demonstrates that higher order derivatives can be converted into a combinatorics problem. The general concept of the theorem, when applied to creation and annihilation operators can be stated as follows [55]:

A collection of n creation and annihilation operators can be rewritten as the normal-ordered product of the collection, plus the normal-ordered product after all single contractions among operator pairs, plus all double contractions, ..., plus all $\frac{n}{2}$ contractions.

Or, in more traditional notation:

$$T\left[q(x_1)\bar{q}(x_2)q(x_3)...\bar{q}(x_i)\right] =: q(x_1)\bar{q}(x_2)q(x_3)...\bar{q}(x_i) + \text{All Possible Contractions} : \tag{2.61}$$

Now, before demonstrating what this actually looks like in practice, let's first look at what a contraction actually is — the quark propagator introduced in the previous section:

$$q(x)\bar{q}(y) = S_f(x-y). \tag{2.62}$$

Furthermore, if there are additional fields present we can see that the normal ordering of a contraction gives the following result

$$: q_1\bar{q}_2 q_3\bar{q}_4 := S_f(x_1 - x_4) \cdot : \bar{q}_2 q_3 :, \tag{2.63}$$

with $q_1 = q(x_1)$. Running with this idea, we can do a quick example of applying Wick's theorem to a four field time ordered product.

$$T\left[q_1\bar{q}_2 q_3\bar{q}_4\right] =: q_1\bar{q}_2 q_3\bar{q}_4 + \text{All Possible Contractions} : \tag{2.64}$$

$$=: q_1\bar{q}_2 q_3\bar{q}_4 : + : q_1\bar{q}_2 q_3\bar{q}_4 : + : q_1\bar{q}_2 q_3\bar{q}_4 : + : q_1\bar{q}_2 q_3\bar{q}_4 : + : q_1\bar{q}_2 q_3\bar{q}_4 :$$

$$+ : q_1\bar{q}_2 q_3\bar{q}_4 : + : q_1\bar{q}_2 q_3\bar{q}_4 :$$

After applying equations (2.62) and (2.63), we arrive at:

$$T\left[q_1\bar{q}_2 q_3\bar{q}_4\right] =: q_1\bar{q}_2 q_3\bar{q}_4 : + S_f(x_1 - x_2) \cdot : q_3\bar{q}_4 : + S_f(x_3 - x_4) \cdot : q_1\bar{q}_2 : + S_f(x_2 - x_3) \cdot : q_1\bar{q}_4 : \tag{2.65}$$

$$+ S_f(x_1 - x_4) \cdot : \bar{q}_2 q_3 : + S_f(x_1 - x_2)\ S_f(x_3 - x_4) + S_f(x_1 - x_4)\ S_f(x_2 - x_3).$$

If we were to sandwich this expression in-between vacuum state bra-kets, all terms with normal ordered components drop out in accordance with equation (2.60),

$$\langle 0|T\left[q_1\bar{q}_2 q_3\bar{q}_4\right]|0\rangle = S_f(x_1 - x_2)\ S_f(x_3 - x_4) + S_f(x_1 - x_4)\ S_f(x_2 - x_3). \tag{2.66}$$

The application of Wick's theorem to the four quark field provides several key insights into the nature of non-interacting vacuum correlation functions that hold for any number of quark fields:

1. contractions correspond to propagators,

2. only terms where every field contracts contribute,

3. and there must be an even number of q and $\bar{q}$ field operators for a non-zero expectation value.

The second (and thus third) results are a direct consequence of the normal ordered ground state (2.60) — any time a field operator is not contracted it must, in accordance with Wick's theorem, be normal ordered. This elimination of non-contracted terms is a huge boon for calculation, making it far simpler to determine correlation function values; the quark fields can be simplified to mere propagators. Flashing back to the two-point function, we can see that these results hold:

$$\langle 0|T\left[\, q_a^\alpha(x)\, \bar{q}_b^\beta(y)\,\right]|0\rangle = S_f(x-y) = i\delta_{\alpha\beta}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, S_{ab}(p)\mathrm{e}^{-ip\cdot(x-y)}\,. \tag{2.67}$$

This relation will be made use of time and time again over the course of the diquark function computation. However, beyond its base usefulness, Wick's theorem is important on a much deeper level than is obviously apparent: the fact that it transmutes field operators to propagators allows for the use of Feynman diagrams and is thus crucial for our understanding of any type of QCD.

2.3.4 Perturbation Theory

In the previous two sections we introduced n-point functions within the framework of a non-interacting vacuum — in essence building up a technical foundation for a free theory. However, interactions between quantum fields are required if we are to construct a physical

theory. In order to deal with this, we must modify the n-point correlation function to a new form dependant on the interacting vacuum, $|\Omega\rangle$, rather than the free vacuum, $|0\rangle$. If we look to the simplest useful correlator, the two-point Green's function, it can be demonstrated (with a fair amount of effort) that [55]:

$$\langle\Omega|T\left[q(x)\bar{q}(y)\right]|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|T\left[q(x)\bar{q}(y)e^{iS_{int}}\right]|0\rangle}{\langle 0|T\left[e^{iS_{int}}\right]|0\rangle}. \tag{2.68}$$

This equation informs us that the interacting correlator is a time-dependent perturbation of the free correlator. This is fairly simple to see after recalling the series expansion of e^{ix},

$$e^{ix} = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \frac{ix^4}{120} + O(x^5) \tag{2.69}$$

Applying this to the actual exponential term gives us:

$$e^{iS_{int}} = \exp\left[i\int \mathcal{L}_{int}\,\mathrm{d}^4z\right] = 1 + i\int \mathcal{L}_{int}(z)\,\mathrm{d}^4z + \frac{i^2}{2}\int \mathcal{L}_{int}(z)\,\mathrm{d}^4z \int \mathcal{L}_{int}(w)\,\mathrm{d}^4w + \ldots. \tag{2.70}$$

Crucially, the first term in the series expansion of the numerator of equation (2.68) gives us $\langle 0|T\left[q(x)\bar{q}(y)\right]|0\rangle$; this is, of course, just the propagator for the free theory. One piece of the two-point function remains to be defined: the action of the interacting theory,

$$S_{int} = \int d^4x\,\mathcal{L}_{int}. \tag{2.71}$$

Here, we use $\mathcal{L}_{int}$ to represent the Lagrangian of the interacting theory — or, in other words, the Lagrangian of whatever theory defines the local interaction between the given quantum fields. Since this project is entirely within the scope of QCD, the Lagrangian we work with is the QCD Lagrangian. This will be looked at in more detail in section 2.3.6, but for now let us take a quick glance at a small piece of the QCD Lagrangian:

$$\mathcal{L} = \frac{g}{2}\bar{q}(x)\lambda^a\gamma^\mu A^a_\mu(x)q(x). \tag{2.72}$$

This is the definition of the interaction between quark and gluon fields within QCD ($A^a_\mu(x)$ denotes a gluon field and λ^a represents one of the Gell-Mann matrices). There are several things that can be noted from this relationship. First, the local nature of quantum field interaction is established by each field operator taking on the same space-time location, x. Second, the term has a coupling constant, g, which represents the strength of the interaction. Due to this, the perturbative expansion of $e^{iS_{int}}$ can be viewed as a power series expansion in g. Finally, Wick's theorem can be used to simplify the correlation function calculation to any given order. This leads to a general process for calculating an interacting correlator: expand out to the perturbative order desired, apply Wick's theorem to each level of perturbation, then calculate the simplified "free" results. As a side note, sadly not all normal ordered terms drop to zero in the interacting vacuum. These extra non-perturbative contributions are referred to as QCD condensates and come into play when dealing with light quarks. Although some work was done with condensates for this project, the technical difficulties proved too much to work through (this will be discussed in a bit more detail in appendix D). As such, condensates will be ignored for the rest of the body of this thesis.

2.3.5 Feynman Diagrams

Understanding exactly what is going on within a given interaction is often incredibly difficult when faced purely with a wall of mathematical notation. (Note: although much of what is discussed in this section uses the more natural language of the path integral formulation, the concepts introduced here function just fine in the canonical quantization setting.) Further, when attempting to determine all possible intermediate actions to a certain perturbative order, it can be difficult to consider all possible options making skipping calculations a very

real possibility. In order to deal with these issues, Richard Feynman introduced the concept of Feynman diagrams in the 1950s. Essentially, a Feynman diagram represents all the mathematics of an often complex interaction in a simple picture that allows for easy determination of perturbative order and intermediate components. This is done by the creation of so called "Feynman rules" — mathematical expressions are equated with diagram components and can be mixed and matched together to create a digram (and corresponding equation) that matches any possible interaction being studied. Diagrams, however, correspond to unique mathematical expressions for each individual quantum field theories: the Feynman rules when working with QCD are completely different than those present in scalar field theory or ϕ^4 theory.

Since we are interested in the interactions between gluons and quarks, we will only explore the Feynman rules for quantum chromodynamics. Although we avoid actually deriving the rules here, full computation of the rules can be found within Peskin and Schroeder [55]. Some examples of basic Feynman diagram components can be seen in figure 2.1. In these particular cases, only propagators are present (although the photon propagator is not used within the diquark correlation function, it is kept in for the sake of completeness) — interaction vertices make up the other fundamental Feynman diagram components. The use of the terms propagator and interaction here is no accident: these refer directly to the corresponding mathematical machinery defined in the previous sections. The quark propagator, figure 2.1 a), corresponds to the quark propagator defined in section 2.3.2:

$$\langle 0|T\left[\, q_a^\alpha(x)\, \bar{q}_b^\beta(y)\,\right]|0\rangle = S_f(x-y) = i\delta_{\alpha\beta}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, S_{ab}(p)\mathrm{e}^{-ip\cdot(x-y)}\,. \tag{2.73}$$

Figure 2.1: Feynman Diagrams for a) quark propagation b) photon propagation and c) gluon propagation

Similarly, the gluon propagator in figure 2.1 c) has the mathematical form:

$$\langle 0|T\left[\,A^a_\mu(x)\,A^b_\nu(y)\,\right]|0\rangle = -i\delta_{ab}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,D_{\mu\nu}(k)\frac{\mathrm{e}^{-ik\cdot(x-y)}}{k^2+i\eta}\,, \tag{2.74}$$

with

$$D_{\mu\nu}(k) = \left[g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2+i\eta}\right]. \tag{2.75}$$

In most prior work the gauge, ξ, takes on a convenient value: the Feynman gauge ($\xi = 1$) and the Landau gauge ($\xi = 0$). Of course, this doesn't remove the gauge dependence from any final result — it just pushes off the problem temporarily.

On a different note, a vertex between two quarks and a gluon will have mathematics corresponding to the interaction term between gluon and quark fields,

$$\mathcal{L} = \frac{g}{2}\bar{q}(x)\lambda^a\gamma^\mu A^a_\mu(x)q(x). \tag{2.76}$$

Each Feynman diagram component is related to a mathematical term; all these terms (for a given diagram) multiply together to give a full expression describing the system. The actual Feynman rules for a given diagram chunk depend on which field one is working with. For example, in ϕ^4 theory a basic propagator (like Figure 2.1 a) correspond to: $\frac{i}{p^2-m^2+i\epsilon}$. On the other hand, a spin 1/2 Dirac particle has a propagator diagram with backing math that consists of: $\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$. The key Feynman rules for this thesis will be discussed when they come into use.

For the case of the diquark correlation function there are two set states: the incoming diquark and the outgoing diquark. All other interactions from gluon emission and re-absorption

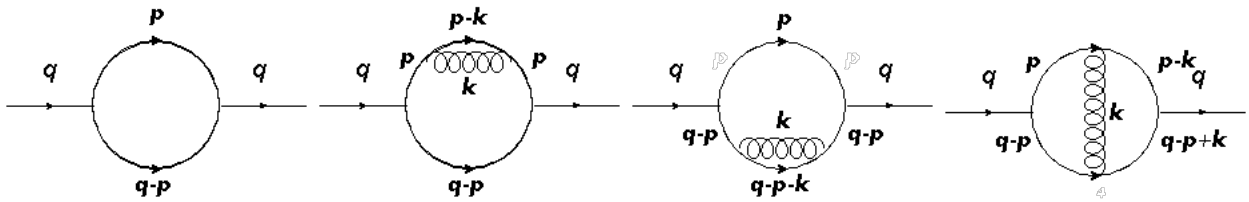

Figure 2.2: Feynman Diagrams for least order perturbation and next to least order perturbation for the diquark correlator

to simple quark propagation are not set in stone and merely represent a possible intermediate path that could be taken by the interacting particles. However, although an infinite number of paths are allowed, the further a path deviates from the simplest interaction possible the less probable that path is taken. Paths (or field configurations) can be classified according to perturbative order - 1st order is the simplest interaction path, 2nd order has two interaction vertices between the initial and final states, 3rd has four interaction vertices, etc. For the purposes of this paper, the highest order explored will be second order perturbation.

Some of the Feynamn diagrams being examined are demonstrated in figure 2.2. Within these diagrams four key parts can be seen: diquark propagators coming into and going out of the interaction, quark propagators, gluon propagators, and quark-gluon vertices. Triple or quadruple gluon vertices are not present, nor are ghost propagators (and, as a result, their interaction vertices), as these interaction all come into play at a higher order than is being studied here. In addition, one should note that there are no disconnected diagrams present — these are cancelled by the denominator term of equation (2.68).

2.3.6 Non-Abelian Gauge Theory

As the formulating of the scalar and Dirac fields have demonstrated, the Lagrangian formulation is an incredibly effective way of describing field dynamics. However, the most important

Lagrangian for our purposes has yet to be introduced: the QCD Lagrangian.

Although the full-form, rigorous derivation of the QCD Lagrangian is beyond the scope of this thesis (after all, ghosts, for example, play no part in our dressed field calculations), it still must be sketched out for the sake of completeness. Like other interaction Lagrangians, we require that the QCD Lagrangian by locally $SU(N)$ (specifically, $SU3$) gauge invariant. In this case, the $SU(N)$ gauge theory used for this thesis is a non-Abelian gauge theory (or a Yang-Mills theory). In order to properly describe QCD scenarios we need terms that describe all the possible interactions of the theory. Below, we give the game away early; here is the QCD Lagrangian that we must build up:

$$\begin{aligned}\mathcal{L}_{QCD} = \bar{q}[i\partial\!\!\!/ - m]q - \frac{1}{4}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu][\partial^\mu A^\nu_a - \partial^\nu A^\mu_a] + \frac{1}{2}g\bar{q}\gamma^\mu A^a_\mu \lambda_a q - \frac{1}{2}g f^{abc} A^a_\mu A^b_\nu [\partial^\mu A^\nu_a - \partial^\nu A^\mu_a] \\ - \frac{1}{4}g^2 f^{abc} f^{cef} A^a_\mu A^b_\nu A^\mu_a A^\nu_b - \frac{1}{2\xi}(\partial^\mu A^a_\mu \partial^\nu A^a_\nu) - \partial^\mu \bar{c}^a \partial_\mu c^a + g f^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b. \quad (2.77)\end{aligned}$$

Term wise, we can break this down as follows: the first term introduces the quark propagator; the second introduce the gluon propagator; the third allows for us to derive rules for quark-gluon vertices; the fourth gives us the 3 gluon interaction, while the fifth offers up the 4 gluon interactions; the sixth is a gauge fixing term that is technically part of the quark propagator; the seventh term is essentially a ghost propagators; and the final term gives information about ghost-gluon field interactions. Once we have this, equations of motion can be obtained by applying the Euler-Lagrange equation (2.22) to the above results. Further, any QCD correlation function can be calculated (to any order in g) that contains a legal combination of quark, gluon and ghost propagators. This can be done by applying perturbation theory and Wick's theorem, as covered in the last several sections.

Now the question becomes how to push forward to find this expression. The answer is

to take a step back to basic quark fields and begin building the Lagrangian from the ground up. The quark field is a 4-component Dirac-spinor that we can label with explicit colour and flavour indices:

$$q = q_a^\alpha. \tag{2.78}$$

Here α is a colour index that can take on a value of 1, 2, or 3 (representing the colours red, green, and blue), while a is the flavour index that can take on any of the six quark flavours (u, d, s, c, b, or t). We can introduce a local $SU(3)_c$ (the c stands for colour) gauge transformation,

$$q(x) \to \tilde{q}(x) = Uq(x). \tag{2.79}$$

If we let U take on the values,

$$U = e^{-i\theta_a(x)\frac{\lambda_a}{2}}, \tag{2.80}$$

for several $\theta_a(x)$ functions (8 to be exact). We need the physics of our system to be invariant under these types of gauge transformations. If we flash back to the Lagrangian for the free quark field, we'll see that we have:

$$\mathcal{L} = \bar{q}\left(i\partial\!\!\!/ - m\right)q. \tag{2.81}$$

This, sadly, is decidedly not invariant under the above gauge transformations. In order to deal with this, the regular derivative (∂_μ) is replaced with a covariant derivative (D_μ),

$$D_\mu = \partial_\mu - igA_\mu(x). \tag{2.82}$$

As mentioned earlier, g is the coupling constant representing the strength of the interaction. $A_\mu(x)$ is a vector field. More formally,

$$A_\mu(x) = \frac{\lambda_a}{2}A_\mu^a(x) \tag{2.83}$$

Again, this is a throwback to the perturbation theory section's equation (2.72) — $A^a_\mu(x)$ is a gluon field (crucially, this is an $SU(3)_c$ gauge field). This leads us to the first piece of our Lagrangian,

$$\mathcal{L}_{quarks} = \bar{q}[i\not{D} - m]q. \tag{2.84}$$

This result is, of course, gauge invariant:

$$\widetilde{D_\mu q} = \partial^\mu \tilde{q} - ig\tilde{A}^\mu \tilde{q} = U(D^\mu U) \tag{2.85}$$

When we combine equations 2.84 and 2.85, we get an expanded quark Lagrangian:

$$\mathcal{L}_{quarks} = \bar{q}[i\not{\partial} - m]q + \frac{1}{2}g\bar{q}\gamma^\mu A^a_\mu \lambda_a q. \tag{2.86}$$

Thinking back to the QCD Lagrangian (2.77), these are the first and third terms — the ones governing the quark propagator and the quark-gluon intersection.

The next step in our construction is to set up the gluon vertex pieces. To do this, we go back to the gauge transformation we introduced earlier (3.8), and do a series expansion of its infinitesimal form:

$$U = e^{-i\theta_a(x)\frac{\lambda_a}{2}} = 1 - i\theta_a(x)\frac{\lambda_a}{2} + \dots. \tag{2.87}$$

If we apply this to our gluonic gauge field (2.83) and expand to 1st order in θ_a we get a transformation,

$$A^\mu_a(x) \to \tilde{A}^\mu_a(x) = A^\mu_a(x) - \frac{\partial^\mu \theta_a(x)}{g} + f_{abc}\theta_b(x)A^\mu_c(x) \tag{2.88}$$

Here, f_{abc} is the structure constant for $SU(3)$. For this gauge transformation, the simplest Lagrangian we can construct for the gluon fields is:

$$\mathcal{L}_{glue} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a. \tag{2.89}$$

Where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f_{abc} A^b_\mu(x) A^c_\nu(x), \tag{2.90}$$

is the gauge invariant gluon field tensor. If we use this definition along with our glue Lagrangian, it can be demonstrated that we get the pieces of the QCD Lagrangian concerned with 3 and 4 gluon vertices as well as the gluon propagator:

$$\mathcal{L}_{glue} = -\frac{1}{4}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu][\partial^\mu A^\nu_a - \partial^\nu A^\mu_a] - \frac{1}{2} g f^{abc} A^a_\mu A^b_\nu [\partial^\mu A^\nu_a - \partial^\nu A^\mu_a] - \frac{1}{4} g^2 f^{abc} f^{cef} A^a_\mu A^b_\nu A^\mu_a A^\nu_b. \tag{2.91}$$

The final pieces of the QCD Lagrangian (ghosts and gauge fixing) are much simpler to obtain via the path integral formulation and ultimately grow out of the generating functional for the gluon field:

$$Z[J] = \int DA \exp\left[i \int d^4x [\mathcal{L}_{glue} + J^\mu_a J^a_\mu\right] \tag{2.92}$$

It's quite tricky to quantize the gluon fields from this functional. However, by removing the redundancy in the integration over A^a_μ via the introduction of a gauge fixing function, a more workable expression is reached:

$$Z[J] = \int DA \det\left(\frac{\delta G}{\delta \theta}\right) \delta G \, \exp\left[i \int d^4x [\mathcal{L}_{glue} + J^\mu_a J^a_\mu\right] \tag{2.93}$$

The functional determinant is given by:

$$\det\left(\frac{\delta G}{\delta \theta}\right) = \int Dc \, D\bar{c} \exp\left[i \int d^4x \, \mathcal{L}_{ghosts}\right], \tag{2.94}$$

where

$$\mathcal{L}_{ghosts} = -\partial^\mu \bar{c}^a \partial_\mu c^a + g f^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b \tag{2.95}$$

is the ghost Lagrangian where the first term describes the ghost propagator and the second describes the interplay of the ghost and gluon fields. This technique was introduced by

Faddeev and Popov [28] and results in the inclusion of the unphysical c fields in the QCD Lagrangian. There's no way around this, however, as these fields are required to create the generating functional. That said, in certain cases the determinant is trivial and ghosts can avoid inclusion.

Finally, to complete the QCD Lagrangian, a gauge fixing term is needed — the exact form of this term depends on the ghost results for the generating functional of the gluon field. Here, our choices makes the gauge fixing term have the form:

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial^\mu A^a_\mu \partial^\nu A^a_\nu). \tag{2.96}$$

This breaks the gauge symmetry and removes the divergence in the functional integral. Without this piece, the theory cannot be quantized and thus fails. With that, we've constructed the final piece Lagrangian puzzle.

A final note about the QCD Lagrangian (2.77) must be made before moving on. Intriguingly, the addition of the ghost fields leads to an extension (which we shall not delve into) of the gauge transformations — BRST gauge symmetry [14]. This symmetry is enough to prove the Slavnov-Taylor identities (which relate QCD correlation functions).

2.4 Computational Techniques

Up until this point, the focus has been almost exclusively on the underlying physics of this project. This section deviates a bit from this course, instead focusing on the actual process used during the computation of the various diquark correlation functions. Thus, from a physics standpoint the following pages can be omitted — they can, however, be used as an aid in understanding exactly how the results of chapters 4 and 5 are obtained.

2.4.1 Integration

Every Feynman diagram present within the diquark correlator perturbation requires at least one loop integration to be completed in order to arrive at an actual answer. These integrations can be done in a host of different ways: for this project TARCER (an integral reduction package for Mathematica) [52] was used for all loop integrations. TARCER functions by taking two-loop propagator integrals with various numerators and arbitrary masses and reducing them to simpler basis integrals. The possibility of reducing integrals to an expression involving only some finite set of basic integrals was demonstrated (using De Rham homology's implication that there are only a finite number of inequivalent forms on a given manifold) in "Battelle Rencontres: 1967 Lectures in mathematics and physics" [43]. The actual reduction algorithms used by TARCER are, however, designed by Tarasov [62, 63].

On the Mathematica level, TARCER takes in loop integrals in TFI form (more detail can be found the algorithm appendix) — this form can either be entered manually or a FeynCalc (the core Mathematica extension for working with QFTs [51, 61]) expression "ToTFI" can be used to automatically obtain the TFI format from an unformatted expression. From this TFI form, the application of the "TarcerRecurse" command results in the integral being broken down into an expression made up of various combinations of six types of master integrals (denoted by the letters **F**, **V**, **J**, **K**, **B**, and **A**). Many of these simpler integrals (and all massless single loop integrals) can be then evaluated explicitly within TARCER. If all the component integrals of a given expression (post TarcerRecurse) can be simplified by TARCER, then "TarcerExpand" provides the explicit results for the expression according to the dimensional-regularization protocol being used. Typically, TARCER's dimensional-

regularization protocol is to compute integrals in $\mathrm{d} \to 4+\epsilon$ dimensions. This can be changed to fit the user's preferences; however, for this project, the value was left unadjusted.

For the first set of computations (the undressed doubly light diquark at leading and next-to-leading order), every variable was written into Mathematica code line by line. This is, of course, an error prone way to go about things and required numerous hours of double checking. To increase the efficiency of work, several semi-automated scripts were written (and tested) that automatically takes integrals, puts them into TARCER usable form, simplifies the integrals, explicitly calculates said integrals (if possible), and sums the results. This was used first to double check the preliminary results of the doubly light, undressed diquark, then applied to the dressed doubly light diquark calculations and all heavy-light and doubly heavy diquark calculations. Pieces of the script will be included in the algorithm appendix for reference.

Further technical information can be found either online at www.feryncalc.org/tarcer or [52].

2.4.2 Renormalization Scheme

Renormalization is a large and dense topic that won't be dealt with in detail here — no proper renormalization needs to be undertaken to study the dressed field's impact on the computation of the gauge dependence in the diquark correlator. Nevertheless, a bit of insight must be shined on a couple of renormalization schemes as they provide a nice, concise way of presenting the results of the doubly light diquark.

Once the various $\mathrm{d} \to 4+\epsilon$ dimensional integrations have been completed via TARCER, expansion must be done around small ϵ values in order to isolate divergent terms. During

this process, different renormalization schemes can be chosen in order to cancel the divergences. The two most common schemse are MS and $\overline{\text{MS}}$. The minimal-subtraction (MS) renormalization scheme chooses renormalization coefficients such that the $\frac{1}{\epsilon}$ term is cancelled in divergent loop calculations. This scheme is effective at removing divergences at the poles $\epsilon = 0$, however additional terms containing $\log 4\pi$ and the Euler-Mecheroni constant, γ, tend to appear and complicate matters. In order to deal with these extra terms, the $\overline{\text{MS}}$ (modified minimal-subtraction scheme) can be introduced to cancel the combination naturally arising in expansions of Γ functions,

$$\frac{1}{\epsilon} - \ln 4\pi + \gamma, \tag{2.97}$$

and thus arrive at a cleaner result with none of the prior mentioned additional terms. For this paper, $\overline{\text{MS}}$ was the scheme of choice. When TARCER is used to expand (via the "TarcerExpand" command) the results of the two-loop integrations the term

$$\left(\frac{1}{4\pi}\right)^{\epsilon} \left(\frac{1}{\nu^2}\right)^{\epsilon} p^2 \left(-p^2\right)^{\epsilon} \mathbf{S}_{\epsilon}{}^{2} \tag{2.98}$$

appears in front of every component. Here, ν is the arbitrary renormalization scale and+ $\mathbf{S}_{\epsilon}$ is defined as:

$$\mathbf{S}_{\epsilon} = e^{\gamma_E (d-4)/2}. \tag{2.99}$$

When a series expansion in epsilon is undertake for expression 2.98, we obtain:

$$p^2 + \epsilon p^2 \left(\gamma \ln(e) + \ln\left(-p^2\right) + \ln\left(\frac{1}{\nu^2}\right) - \ln(4\pi)\right) + O(\epsilon^2). \tag{2.100}$$

This can be simplified to obtain a much nicer expression,

$$p^2 + \epsilon p^2 \left(\ln\left(\frac{-p^2 e^{\gamma}}{4\pi\nu^2}\right)\right) + O(\epsilon^2). \tag{2.101}$$

Thus the $\overline{\text{MS}}$ scheme enters the computation when the substitution:

$$\mathrm{L} = \ln\left[\frac{-p^2 e^{\gamma}}{4\pi\nu^2}\right] \tag{2.102}$$

is made.

Chapter 3

Dressed Fields

With the textbook work out of the way, it's time to take the QCD skeleton constructed in the last chapter and dress it up a bit. This is done by first providing a quick overview of the concepts and QCD realization of gauge, gauge transformations, and gauge theories. Second, the theoretical framework of the dressed field is discussed: specifically, we focus on how this allows for the construction of locally gauge invariant fields from a gauge dependent field and an inverse operator (Green's function). Finally, the dressing of QCD fields will be explored and put in a way that allows for the calculation of the diquark correlation functions present in chapters 4 and 5.

3.1 General Gauge Concepts

The concept of gauge is quite an abstract one that can prove quite tricky if approached incorrectly — so, for a bit of clarity, we take a quick glance back to the first field theory: Maxwell's electrodynamics.

- Gauss's law for magnetism:

$$\nabla \cdot \mathbf{B} = 0\,, \tag{3.1}$$

- Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,, \tag{3.2}$$

- Maxwell-Faraday equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{3.3}$$

- Ampère's circuital law:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)\,. \tag{3.4}$$

For our current discussion, we care about the first of these laws. The fact that the divergence of $\mathbf{B}$ is zero everywhere indicates that $\mathbf{B}(\mathbf{x})$ must be the curl of the vector field $\mathbf{A}(\mathbf{x})$:

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}). \tag{3.5}$$

This is referred to as the magnetic vector potential. It is perhaps simpler to first show how this leads to the concept of gauge within the context of magnetostatics — after which the proper, fully time-dependent situation can be assessed. As demonstrated in Jackson [39], the magnetostatic magnetic field can be written as an integration of a current density $\mathbf{J}(\mathbf{x})$,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3y. \tag{3.6}$$

Using equation (3.5) and some vector calculus we can arrive at an expression for the vector potential:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3y \; + \nabla \Psi(\mathbf{x}). \tag{3.7}$$

Within this expression, Ψ is an arbitrary scalar function that can take on any value at any point with no repercussions on $\mathbf{A}$; it is the gauge of the vector potential. Of course, this means that the vector potential can be transformed in the following fashion,

$$\mathbf{A} \to \mathbf{A} + \nabla \Psi, \tag{3.8}$$

referred to as a gauge transformation. Any transformation of this form leaves the physical magnetic field invariant and thus doesn't effect the actual physics. What gauge transformations do allow for is mathematical reconfiguration of problems, sometimes allowing for huge reductions (especially in the cases of the Lorentz and Coulomb gauges) in a problem's computational intensity.

Now, let us return to Maxwell's equations and determine the nature of their potential formulation (this time with time-dependence included!). Equation 3.5 still holds as this arises directly from the fact that $\mathbf{B}$ is divergenceless. Taking this result and substituting for $\mathbf{B}$ in equation 3.3 gives us:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{A}\right). \tag{3.9}$$

This can be rearranged into the form

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{3.10}$$

Since the curl in the above expression vanishes, we can write the bracketed quantity as the gradient of some scalar potential,

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi, \tag{3.11}$$

or,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi. \tag{3.12}$$

These results for $\mathbf{E}$ and $\mathbf{B}$ are left invariant under the following gauge transformations:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \Lambda \tag{3.13}$$

and

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}, \tag{3.14}$$

for some scalar function Λ.

Although our above work is wholly done within the context of the classical electromagnetic field, there are some general gauge concepts that can be built up through the generalization and extension of the electrodynamic ideas. First off, the gauge itself is actually redundant degrees of freedom within the Lagrangian of any given theory. Further, transformations between a theory's possible gauges (a generalization of the transformation shown in equation (3.8)) actually form the Lie symmetry group for that theory (in the case of electrodynamics this is the U(1) group). These generalities, as the word suggests, hold for all gauge theories including QCD and can thus be put to use. Additionally, there are two more pieces of vocabulary that should be included before pressing forward: 1) the quanta of quantized gauge theories can be referred to as gauge bosons and, 2) theories with a non-commutative symmetry group are referred to as non-abelian.

The general rules and principles of gauge and gauge theories are all well and fine, but when it really comes down to it, QCD is what we care about. Within the context of this project, gauge is introduced to the diquark correlator (this will be discussed in more detail in chapter 4) via the gluon propagator (2.74) [41, 55].

$$\langle 0|T\left[\,A^a_\mu(x)\,A^b_\nu(y)\,\right]|0\rangle = -i\delta_{ab}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,D_{\mu\nu}(k)\frac{\mathrm{e}^{-ik\cdot(x-y)}}{k^2+i\eta}\,. \tag{3.15}$$

Where:

$$D_{\mu\nu}(k) = g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2+i\eta}\,. \tag{3.16}$$

Here, the ξ term is the gauge being used within the diquark calculations. Just like any other gauge, ξ can take on an arbitrary value. As such, one would hope that any calculations that

give physical results (the diquark correlation function, for example) would experience a gauge cancellation of some type and thus give gauge independent results. After all, results that can be shifted arbitrarily don't make for good physics! Tragically, if the diquark correlation function is just pushed through using standard QCD techniques a glaring ξ dependence rears its ugly head. As a result, past work [41] has danced around the issue in primarily two ways: the gauge has always been either carried into the final result (a specific choice of gauge would then be made if further computations were needed) or eliminated through the use of the Schwinger string [23],

$$S_{\alpha\omega}[x,0] = P\exp\left[ig\frac{\lambda^a_{\alpha\omega}}{2}\int \mathrm{d}z^\mu A^a_\mu(z)\right]. \tag{3.17}$$

Both of these choices leave much to be desired, however. Leaving the gauge in the final result (or arbitrarily choosing a gauge) is simply not good physics while the Schwinger string introduces massive complications into the calculation of the diquark correlation function, making calculation of higher order diagrams infeasible. Due to these issues, this project moves away from both these approaches, instead introducing the dressed field technique.

3.2 Theoretical Framework

The concept of the dressed field formalism, introduced by Lavelle *et al* [10–13, 37] (from here on just Lavelle) is very simple to state: a locally gauge invariant field ψ_h can be found by operating on a gauge dependent field ψ by the inverse of a field-dependent operator $h(x)$,

$$\psi_h(x) = h^{-1}(x)\psi(x). \tag{3.18}$$

Understanding what this means and how to apply it within the diquark context is, however, a fair bit more complicated. In order to unpack this fully, I begin where Lavelle and his colleagues did in their paper, *Charges from Dressed Matter: Physics and Renormalisation* [13]; specifically the potential gauge problems that can be encountered within QED.

For unbroken gauge theories (which possess massless gauge bosons) there are non-negatable interactions that effect particles at large distances [18, 22]. If the interaction coupling does not vanish, this leads to a non-trivial gauge transformation and matter fields that are ultimately gauge dependent. Of course, this suggests that objects of the theory are non-physical. And, as QED and QCD are theories mediated by massless gauge bosons, this is a problem that impacts both theories. Thus, the task at hand is thus to build up fields that are locally gauge invariant — this is done by "dressing" the gauge dependent fields with vector potentials in order to "manufacture" locally gauge independent fields. Through this dressing process, many gauge invariant fields can be constructed; unfortunately, not all of them correspond to physically significant fields. In the words of Lavelle, *The [...] task is thus two-fold: one must identify which of the gauge invariant fields correspond to physically significant variables and carry out the construction of these fields.* [13]

The actual formal process of driving such results [12] is well beyond the scope of this thesis, but a (comparatively) brief sketch of the core ideas will be completed here so the calculations in chapters 4 and 5 avoid any aura of mystery. Within Lavelle's prior paper [10], it was argued that due to the massless nature of the photon (QED's gauge boson) there is a slow fall-off for QED interactions and thus setting the QED coupling, e, to zero at extreme times during scattering processes is non-justifiable. This results in an infra-red divergence problem — specifically Lavelle showed that although the QED creation and annhilation operators do

correspond to gauge invariant (and thus physical) particles, they do not obey the asymptotic limits of the Lagrangian matter field. Further, Lavelle was able to demonstrate that by applying a field dressing particles that both were gauge invariant and obeyed free asymptotic dynamics. The dressing applied is operator mentioned earlier in the equation 3.18:

$$\psi_h(x) = h^{-1}(x)\psi(x).$$

This Green's function h takes the following form:

$$h^{-1}(x)\psi(x) = e^{-ig\chi(x)}\psi(x)\,. \tag{3.19}$$

The general form of $\chi\,(x)$ is given by Lavelle to be:

$$\chi(x) = \frac{\mathcal{G}\cdot A}{\mathcal{G}\cdot\partial}\,. \tag{3.20}$$

Here $\mathcal{G}^\mu = (\eta + v)^\mu(\eta - v)\cdot\partial - \partial^\mu$ represents an on-shell charged particle moving with 4-velocity $u = \gamma(\eta + v)$ where η is a time-like 4-vector with $\eta^2 = 1$, γ is the Lorentz factor, and v is a space-like 4-vector with $v^2 = -|\mathbf{v}|^2$. This interpretation is limited to describing physical particles at the on-shell point $p = m\gamma(\eta + v)$.

This formalism, when applied to unbroken gauge theories, has several desirable qualities including: good pole structure [9] and dressed interactions between asymptotic fields being free of on-shell infra-red divergences for QED [11] and anti-screening behaviour in QCD [45]. These extra benefits make the dressed field formalism an even more appealing tool to retrieve gauge invariant information from gauge dependent results.

3.3 QCD Dressing

3.3.1 Motivations for QCD Dressing

While the general ideas behind field dressings were introduced in the previous section, some effort must be made to specify the appeal and promise of dressing explicitly within the QCD realm. Lavelle and McMullan's paper, *Constituent Quarks From QCD* [44], explores this concept is extreme detail — here I will only break down the key issues. QCD, much like QED, has shown itself to be an excellent theoretical description of strong interactions over the last 40 years or so: on a phenomenological level QCD predictions have agreed with experiments again and again, scaling violations predicted by QCD have been demonstrated, asymptotic freedom exists, and theoretically predicted jets have been observed [55].

However, in spite of all these triumphs, there are still many puzzles left to solve. Perhaps the problem that defines QCD the most is that of confinement. The QCD predictions are excellent, yes; but the building blocks of the theory, the quarks and the gluon, have never been observed in isolation. This absence of detection was taken to indicate that colour charge (as mentioned in the introduction) is confining. In other words, only colour singlet particles, such as hadrons, can exist in nature and particles (including composite particles) with a net colour charge simply cannot exist. Another difficulty of QCD is the gap between constituent quarks (quarks from the quark model that explain hadronic spectroscopy) and the quarks from the QCD Lagrangian. Typically, there's a large difference in magnitude between the (current) quark mass scale and hadronic mass scale — take a proton for instance: up quarks have a mass of $2.3^{+0.7}_{-0.5}\,MeV$ [54], down have a mass of $4.8^{+0.5}_{-0.3}\,MeV$ [54], while a proton (two

up quarks and a down) has a mass of $\approx 938.27\,MeV$ [54]. Additionally, a significant portion of a proton's momentum is carried by particles that have no electric charge; as every quark flavour is electrically charged, it is thought that gluons are responsible for the rest of this momentum.

So, where is this extra mass hiding? How do we start with our current quarks from QCD and arrive at the constituent quarks required to explain hadrons? Lavelle [13] argues that the three valence quarks of, say, a proton are "dressed" with gluons and quark-antiquark pairs such that they become much heavier. In order for this to be the case, the dressing needs to be able to explain the correct elements of the constituent quark model: the quantum numbers (colour charge, spin, etc) of hadrons absolutely must be reproduced accurately. As an additional constraint, confinement must also be respected by the dressed formalism.

Above, we have stated what we hope a dressed field can accomplish, but have not outlined a good physical reason behind the dressed formalism. Although there are several of these outlined in detail by Lavelle [13] including removal of infra-red problems and an identification of confinement, here we focus on the core issue: the gauge symmetry present in the QCD Lagrangian indicates that several fields within the theory lack physical significance. More explicitly, QCD Lagrangian quarks are not gauge invariant. This is where the dressing technique shines — through the use of perturbative methods, a coloured glounic dressing can be constructed such that these Lagrangian quarks are gifted both a well-defined colour charge and (at minimum, local) gauge invariance.

This last bit is the crucial result for this project. As soon as the quark fields are gauge invariant the correlation function can also be made gauge invariant — regardless of the size, order, or elements present. The focus then moves from whether or not the extraction of

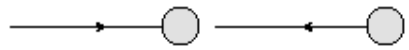

Figure 3.1: Quark propagator vertex pre-dressing

gauge invariant information from a given correlator is possible to whether or not such an extraction is feasible. This truly is the question that this thesis seeks to answer: can a correlation function using the dressed field formalism be calculated in the diquark scenario (doubly-light, light-heavy, and doubly-heavy)?

3.3.2 Feynman Rules for QCD Dressing

After much preparation, sketching of justifications, and establishment of theoretical framework, it's time to look at the actual application of the dressed field in the context of QCD; and more specifically in the context of this project.

The most natural place to begin this explanation is the Feynmann rules for dressing an interaction vertex. Sample interaction vertices can be seen in figure 3.1. These are external propagators and, as we want to be able to view the impact of the dressing, are both granted a value of 1.

Now, we take these interaction vertices and dress them using the technique referenced in equation 3.19,

$$\psi_h(x) = e^{-ig\chi(x)}\psi(x)\,, \tag{3.21}$$

and use it to transform vertices into so-called "dressing vertices". There is an explicit dependence on the coupling constant (in this case, g), and an expansion of the Green's function can be done in terms of the coupling:

$$e^{-ig\chi} = \sum_n \frac{(-ig\chi)^n}{n!}. \tag{3.22}$$

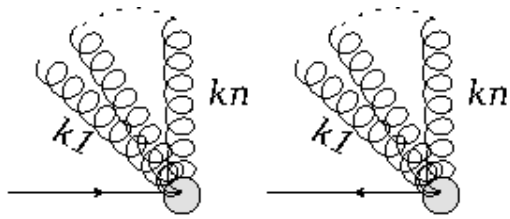

Figure 3.2: Quark propagator vertex post-dressing

In this case, the $n!$ will cancel with the contraction of the photon fields. The dressed field can be drawn diagrammatically as seen in figure 3.2. In these diagrams, there are n dressing gluon fields that have been added. Mathematically, this Feynman rule gives the diagrams the value:

$$\frac{gV_1^{\mu_1}}{V_1 \cdot k_1}\frac{gV_2^{\mu_2}}{V_2 \cdot k_2}\cdots\frac{gV_n^{\mu_n}}{V_n \cdot k_n}. \tag{3.23}$$

Typically,

$$V_m^{\mu_r} = (\eta + v)^{\mu_r}(\eta - v)\cdot k_r - k_r^{\mu_r}, \tag{3.24}$$

where $\eta^\mu = (1, \mathbf{0})$ and $v^\mu = (0, \mathbf{v})$. Although this is the general term [12], the choice of V for on-shell particles must be modified for the analysis of the off-shell diquark correlator. Thus, for the purposes of this project, we limit ourselves to the case where the first term in the equation for $\mathcal{G}$ (as used in 3.20) drops out completely. Applying this restriction to $\mathcal{G}$, gives us a value of:

$$\mathcal{G} = \partial^\mu. \tag{3.25}$$

We then naturally arrive at our simplified χ value,

$$\chi = \frac{\partial_\mu A_\mu}{\partial_x^2}\,. \tag{3.26}$$

To progress from here it easier to break the operator into two parts (numerator and denominator), solve the individual parts, and then couple them back together again. Beginning with

the denominator, we apply these operators to a test function $f(x)$ and complete a Fourier transform — the result gives us Feynman rules we can use within this project.

$$\frac{f(x)}{\partial_x^2} = g(x) \rightarrow f(x) = \partial_x^2 g(x), \tag{3.27}$$

leads us to:

$$\tilde{f}(k) = \partial_x^2 \int \frac{d^4k}{(2\pi)^4} \tilde{g}(k) e^{-ik\cdot x} = -k^2 \int \frac{d^4k}{(2\pi)^4} \tilde{g}(k) e^{-ik\cdot x}. \tag{3.28}$$

This contributes a $\frac{1}{-k^2}$ to our Feynman rule and settles the bottom half of our operator. If we look to the numerator of our expression we have a $\partial_\mu A_\mu$. A similar evaluation of this expression leaves a final result of $-ik_\mu A_\mu$. Combining this with our denominator piece, we arrive at our χ transformation:

$$\chi = \frac{\partial_\mu A_\mu}{\partial_x^2} \rightarrow \chi = \frac{-ik_\mu A_\mu}{k^2}. \tag{3.29}$$

The full Green's function thus conspires to give us our full Feynman rule for our dressed vertices seen in figure 3.2:

$$\frac{gk_{\mu_1}}{k_1^2} \frac{gk_{\mu_2}}{k_2^2} \cdots \frac{gk_{\mu_n}}{k_n^2}. \tag{3.30}$$

This is, in essence, all there is to the application of field dressings in the context of this thesis.

Chapter 4

The Massless Diquark

Back in section 1.4.2, the concept of the diquark was introduced and given some justification for being a potential role-player in the analysis of exotic states. However, if the diquark is to be used (and we want results that are to some degree physical), there must be some way of extracting gauge invariant information from this construction. The goal of these next two chapters is to demonstrate that the dressed field formalism allows for the discovery of gauge-invariant results — this chapter focuses on the simpler, doubly-light (massless) case, whereas the next focuses on massive diquarks.

4.1 Diquark Correlation Function Basics

First and foremost, we must define the fundamental pieces of the diquark correlation function. Notationally, Greek letters are used for colour indices and Latin denote spinor indices. Flavour of the incoming and outgoing diquarks are left in as general a form as possible: q and Q may refer to any light quark (up, u; down, d; or strange, s). Finally, C is the charge conjugation operator mentioned in chapter 2.

The incoming and outgoing diquark currents are respectively defined as follows:

$$J_\alpha(x) = \epsilon_{\alpha\beta\gamma} q_a^\beta(x) \left[C\gamma^5\right]_{ab} Q_b^\gamma(x)\,, \;\; J_\zeta^\dagger(x) = \epsilon_{\zeta\eta\theta} \bar{Q}_c^\eta(0) \left[C\gamma^5\right]_{cd} \bar{q}_d^\theta(0). \tag{4.1}$$

These currents can be plugged into the actual correlation function:

$$\Pi\left(q^2\right) = i\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\langle 0|T\left[J_\alpha(x)\, J_\zeta^\dagger(0)e^{i\int \mathrm{d}^4z\mathcal{L}_{int}(z)}\right]|0\rangle\,. \tag{4.2}$$

As per usual, in order to deal with this function in a reasonable way, approximation via perturbation over various orders is necessary. What the perturbation is in the case of (4.2) is the expansion of the Lagrangian term:

$$\exp\left[i\int \mathcal{L}\,\mathrm{d}^4z\right] = 1 + i\int \mathcal{L}(z)\,\mathrm{d}^4z + \frac{i^2}{2}\int \mathcal{L}(z)\,\mathrm{d}^4z\int \mathcal{L}(w)\,\mathrm{d}^4w + \ldots \tag{4.3}$$

In order to keep complexity of computation down to a manageable level, only perturbations up to second order are examined here,

$$\Pi\left(q^2\right) = \Pi_1\left(q^2\right) + \Pi_2\left(q^2\right) + \mathcal{O}\left(\alpha^2\right). \tag{4.4}$$

We have the lowest order perturbation:

$$\Pi_1\left(q^2\right) = i\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\langle 0|T\left[J_\alpha(x)\, J_\zeta^\dagger(0)\right]|0\rangle\,, \tag{4.5}$$

and second order perturbation:

$$\Pi_2\left(q^2\right) = i\frac{i^2}{2!}\int \mathrm{d}^4y\int \mathrm{d}^4z\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\langle 0|T\left[J_\alpha(x)\, J_\zeta^\dagger(0)\mathcal{L}_{int}(y)\mathcal{L}_{int}(z)\right]|0\rangle\,, \tag{4.6}$$

Visually, these approximations of the diquark correlation function can be represented by Feynman diagrams, as seen in figure 4.1.

4.2 Simplification

As an example of how these correlation functions are simplified, here is a step by step simplification of the 1st order approximation. Due to space constraints higher order computations

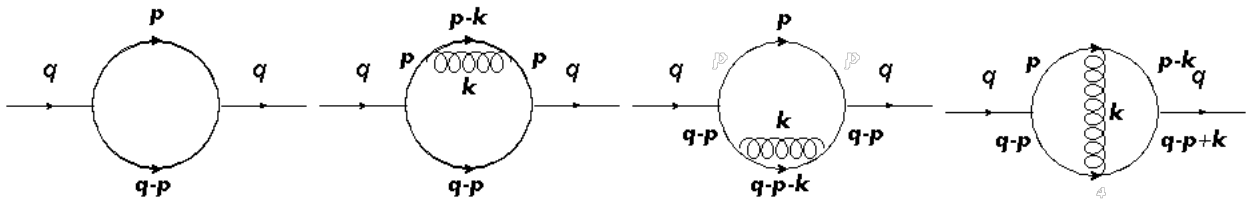

Figure 4.1: Feynman Diagrams for leading order perturbation and next-to-leading order perturbation for the diquark correlator

will be omitted — their results will, however, be posted in later in the chapter. Colour constants will be dealt with afterwards and are contained in the C_0 term.

Firstly, the values of $J_\alpha(x)$ and $J_\zeta^\dagger(0)$ are put in place. The constant values that are invariant under the ground state bra, ket, and time ordering operator are then pulled outside the angled brackets.

$$\Pi_1\left(q^2\right) = i\mathrm{C}_0 \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x} \langle 0|T\left[q_a(x)\left[C\gamma^5\right]_{ab} Q_b(x)\bar{Q}_c(0)\left[C\gamma^5\right]_{cd}\bar{q}_d(0)\right]|0\rangle\,, \tag{4.7}$$

$$\Pi_1\left(q^2\right) = (-1)^2 i\mathrm{C}_0 \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x} \left[C\gamma^5\right]_{ab}\left[C\gamma^5\right]_{cd} \langle 0|T\left[q_a(x)\bar{q}_d(0)Q_b(x)\bar{Q}_c(0)\right]|0\rangle\,, \tag{4.8}$$

Contractions can now be completed in accord with Wick's theorem:

$$q_a(x)\bar{q}_d(0) \to i\int \frac{\mathrm{d}^4p}{(2\pi)^4} S_{ad}(p)e^{-ip\cdot x}\,,\ Q_b(x)\bar{Q}_c(0) \to i\int \frac{\mathrm{d}^4(q-p)}{(2\pi)^4}\tilde{S}_{bc}(q-p)e^{i(q-p)\cdot x}. \tag{4.9}$$

A relabelling of variables and move towards trace form creates a correctly ordered and easier manipulated equation.

$$\Pi_1\left(q^2\right) = i^3\mathrm{C}_0 \int \frac{\mathrm{d}^4p_1}{(2\pi)^4}\int \frac{\mathrm{d}^4p_2}{(2\pi)^4}\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\mathrm{Tr}\left[\left[C\gamma^5\right]_{ab}\tilde{S}_{bc}(p_2)e^{-ip_2\cdot x}\left[C\gamma^5\right]_{cd} S^T_{da}(p_1)e^{-ip_1\cdot x}\right] \tag{4.10}$$

Introduction of the Dirac delta results in the conversion to momentum space and allows for further manipulation.

$$\Pi_1\left(q^2\right) = i^3\mathrm{C}_0 \int \frac{\mathrm{d}^4p_1}{(2\pi)^4}\int \frac{\mathrm{d}^4p_2}{(2\pi)^4}(2\pi)^4\delta^4(q-p_1-p_2)\,\mathrm{Tr}\left[\left[C\gamma^5\right]_{ab}\tilde{S}_{bc}(p_2)\left[C\gamma^5\right]_{cd} S^T_{da}(p_1)\right] \tag{4.11}$$

The application of the Dirac delta simplifies the equation to a simple state, where a conversion to slash notation is both simple and useful.

$$\Pi_1\left(q^2\right) = i^3 \mathrm{C}_0 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \operatorname{Tr}\left[\left[C\gamma^5\right] \tilde{S}(q-p) \left[C\gamma^5\right] S^T(p)\right] \tag{4.12}$$

$$\Pi_1\left(q^2\right) = i^3 \mathrm{C}_0 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\operatorname{Tr}\left[C\gamma^5(\not{q}-\not{p})C\gamma^5(\not{p})\right]}{(p)^2(q-p)^2} \tag{4.13}$$

At this point, the integral has nearly been prepared to the point that actual computation is feasible — only the trace simplification remains. Here the identities $[C, \gamma^5] = 0$, $C(\not{p}^T)C = \not{p}$, $\gamma^5\gamma^5 = 1$, and $\left\{\not{p}, \gamma^5\right\} = 0$ are used.

$$\operatorname{Tr}\left[C\gamma^5(\not{q}-\not{p})C\gamma^5(\not{p}^T)\right] = \operatorname{Tr}\left[C\gamma^5(\not{p}^T)C\gamma^5(\not{q}-\not{p})\right] = \operatorname{Tr}\left[\gamma^5(\not{p})\gamma^5(\not{q}-\not{p})\right] = -\operatorname{Tr}\left[(\not{p})(\not{q}-\not{p})\right] \tag{4.14}$$

This trace can be expanded to give:

$$-\operatorname{Tr}\left[(\not{p})(\not{q}-\not{p})\right] = 4(p^2 - p\cdot q) = 2p^2 - 2q^2 + 2(p-q)^2 \tag{4.15}$$

Plugging this back into the original equation, we arrive at:

$$\Pi_1\left(q^2\right) = -i\mathrm{C}_0 \int \frac{\mathrm{d}^D p}{(2\pi)^D} \left[\frac{2}{(q-p)^2} + \frac{2}{p^2} - \frac{2q^2}{p^2(p-q)^2}\right] \frac{1}{\nu^\epsilon} \tag{4.16}$$

Here the equation has been shifted into $D \to 4+\epsilon$ dimensions. The first two integrations are massless tadpoles and are thus zero. The third can be evaluated via TARCER, leaving us with:

$$\Pi_1\left(q^2\right) = 2iq^2 \frac{C_0}{\nu^\epsilon (4\pi)^{2+\frac{\epsilon}{2}}} \left[(-q^2)^{\frac{\epsilon}{2}} \mathbf{S}_\epsilon\right] \left[2i - \frac{2i}{\epsilon}\right] \tag{4.17}$$

From here, a Taylor expansion was undertaken for epsilon. Additionally, a move to the $\overline{MS}$ scheme is applied through the change of variables $\mathrm{L} = \ln\left[\frac{-q^2 e^\gamma}{4\pi\nu^2}\right]$. Finally, all terms dropping

quickly to zero with epsilon (higher order terms) and all L independent terms are ignored. This gives us a final result for the leading order diagram of:

$$\Pi_1\left(q^2\right) = -2q^2 \frac{C_0}{(4\pi)^2}\left[-\mathrm{L} + \epsilon\left(-\frac{\mathrm{L}^2}{4} + \mathrm{L}\right)\right] \tag{4.18}$$

Determination of the colour charge constant is fairly straight forward in this case as only two types of colour factors need to be considered. The first of these are the epsilon terms which arise from the incoming and outgoing diquark currents; the second are the Kronecker delta terms which come out of the Wick contractions. Through the quick application of some simple mathematical identities we find the colour charge constant to be:

$$\mathrm{C}_0 = \epsilon_{\alpha\beta\gamma}\epsilon_{\zeta\eta\theta}\delta_{\beta\theta}\delta_{\gamma\eta} = \epsilon_{\alpha\beta\gamma}\epsilon_{\zeta\gamma\beta} = -\epsilon_{\beta\gamma\alpha}\epsilon_{\beta\zeta\gamma} = -\left[\delta_{\gamma\zeta}\delta_{\alpha\gamma} - \delta_{\alpha\zeta}\delta_{\gamma\gamma}\right] = -\left[\delta_{\alpha\zeta} - 3\delta_{\alpha\zeta}\right] = 2\delta_{\alpha\zeta} \tag{4.19}$$

With the lowest order perturbation complete, the groundwork is in place to compute the second order perturbation. The techniques used in the lowest order case can easily be extended and applied in a very similar manner to the next-to-leading order computation. The only true additional content is the technique used to deal with the gluon propagation term:

$$A^B_\mu(s)A^C_\nu(w) \to \int \frac{\mathrm{d}^4k}{(2\pi)^4} D_{\mu\nu}(k)e^{-ik\cdot(s-w)}. \tag{4.20}$$

There are three second order cases — here one is worked out in detail (the case where a gluon is emitted from one quark and is absorbed by another); the other two cases are similar.

Step by step solution to the 2nd order approximation (3). Colour constant C_3 will, once

again, be dealt with afterwards.

$$\Pi_2^{(3)}\left(q^2\right) = i\frac{i^2}{2!}\mathrm{C}_3 \int \mathrm{d}^4 s \int \mathrm{d}^4 w \int \mathrm{d}^4 x\, \mathrm{e}^{-iq\cdot x}$$
$$\langle 0|T\left[q_a(x)\left[C\gamma^5\right]_{ab} Q_b(x)\bar{Q}_c(0)\left[C\gamma^5\right]_{cd}\bar{q}_d(0)\frac{g}{2}\bar{q}_e(s)\gamma^\mu_{ef}A^B_\mu(s)q_f(s)\frac{g}{2}\bar{Q}_g(w)\gamma^\nu_{gh}A^C_\nu(w)Q_h(w)\right]|0\rangle \tag{4.21}$$

This correlation function can be rearranged into more workable form:

$$\Pi_2^{(3)}\left(q^2\right) = (-1)^9\frac{i^3g^2}{8}\mathrm{C}_3 \int \mathrm{d}^4 s \int \mathrm{d}^4 w \int \mathrm{d}^4 x\, \mathrm{e}^{iq\cdot x}\left[C\gamma^5\right]_{ab}\left[C\gamma^5\right]_{cd}$$
$$\langle 0|T\left[Q_b(x)\bar{Q}_g(w)Q_h(w)\bar{Q}_c(0)q_a(x)\bar{q}_e(s)q_f(s)\bar{q}_d(0)\gamma^\nu_{gh}\gamma^\mu_{ef}A^B_\mu(s)A^C_\nu(w)\right]|0\rangle\,. \tag{4.22}$$

Wick contractions take place in a fashion similar to the lowest order perturbation, however there is an additional contraction type present for the gluon propagation term:

$$A^B_\mu(s)A^C_\nu(w) \to \int \frac{\mathrm{d}^4 k}{(2\pi)^4} D_{\mu\nu}(k)e^{-ik\cdot(s-w)}. \tag{4.23}$$

Again, we relabel the variables and convert to trace form:

$$\Pi_2^{(3)}\left(q^2\right) = -\frac{i^3g^2}{8}\mathrm{C}_3 \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\cdots\int \frac{\mathrm{d}^4 p_4}{(2\pi)^4}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\int \mathrm{d}^4 s \int \mathrm{d}^4 w \int \mathrm{d}^4 x\, \mathrm{e}^{iq\cdot x}$$
$$\mathrm{Tr}\left[S_{ae}(p_1)e^{-ip_1\cdot(x-s)}\gamma^\mu S_{fd}(p_2)e^{-i(p_2)\cdot(s)}\left[C\gamma^5\right]^T_{dc}\tilde{S}^T_{ch}(p_4)e^{-ip_4\cdot w}\gamma^{\nu T}\tilde{S}^T_{gb}(p_3)e^{-i(p_3)\cdot(x-w)}\left[C\gamma^5\right]^T_{ba}\right]$$
$$D_{\mu\nu}(k)e^{-ik\cdot(s-w)}. \tag{4.24}$$

Computing the Fourier transform of the exponential terms allows us to both obtain a pure momentum space expression and introduce the Dirac delta,

$$\Pi_2^{(3)}\left(q^2\right) = -\frac{i^3g^2}{8}\mathrm{C}_3 \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\cdots\int \frac{\mathrm{d}^4 p_4}{(2\pi)^4}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}(2\pi)^{12}\delta^4(q-p_1-p_3)\delta^4(p_2-p_1+k)$$
$$\delta^4(p_4-p_3-k)\mathrm{Tr}\left[S(p_1)\gamma^\mu S(p_2)\left[C\gamma^5\right]^T\tilde{S}^T(p_4)\gamma^{\nu T}\tilde{S}^T(p_3)\left[C\gamma^5\right]^T\right]D_{\mu\nu}(k). \tag{4.25}$$

Post simplification:

$$\Pi_2^{(3)}\left(q^2\right) = -\frac{i^3 g^2}{8}\mathrm{C}_3 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4}$$
$$\mathrm{Tr}\left[S(p)\gamma^\mu S(p-k)\left[C\gamma^5\right]^T \tilde{S}^T(q-p+k)\gamma^{\nu T}\tilde{S}^T(q-p)\left[C\gamma^5\right]^T\right] D_{\mu\nu}(k)\,, \quad (4.26)$$

$$\Pi_2^{(3)}\left(q^2\right) = -\frac{i^3 g^2}{8}\mathrm{C}_3 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4}$$
$$\frac{\mathrm{Tr}\left[\not{p}\gamma^\mu(\not{p}-\not{k})\left[C\gamma^5\right]^T(\not{q}-\not{p}+\not{k})^T\gamma^{\nu T}(\not{q}-\not{p})^T\left[C\gamma^5\right]^T\right]}{(p)^2(p-k)^2(q-p+k)^2(q-p)^2} D_{\mu\nu}(k)\,. \quad (4.27)$$

At this point, TARCER was again used to determine the loop integral result:

$$\Pi_2^{(3)}(q^2) = \frac{2C_3\pi\alpha q^2}{(4\pi)^4}\left[\frac{12\mathrm{L}}{\epsilon} + 6\mathrm{L}^2 - 29\mathrm{L} + \xi\left[2\mathrm{L}^2 - 9\mathrm{L} + \frac{4\mathrm{L}}{\epsilon}\right]\right] \quad (4.28)$$

Now, we determine the colour factor:

$$\mathrm{C}_3 = \epsilon_{\alpha\beta\gamma}\epsilon_{\zeta\eta\theta}\delta_{\beta\iota}\delta_{\kappa\lambda}\delta_{\xi\theta}\delta_{\gamma\eta}\delta^{BC}\lambda^B_{\iota\kappa}\lambda^C_{\lambda\xi} = \lambda^B_{\beta\theta}\lambda^B_{\gamma\eta}\epsilon_{\alpha\beta\gamma}\epsilon_{\zeta\eta\theta} = \frac{16}{3}\delta_{\alpha\zeta}. \quad (4.29)$$

The other two second order approximations are incredibly similar in terms of calculation, so I will not explicitly work them out here. That said, the core differences are as follows: the order of elements in the trace and the type of contractions present changes from the prior calculation. For instance, only 1 S contraction between x and 0 occurs, but three $\tilde{S}$ take place: x to s, s to w, and w to 0. The gluon component remains unchanged.

$$\Pi_2^{(1)}\left(q^2\right) = -\frac{i^3 g^2}{8}\mathrm{C}_1 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\mathrm{Tr}\left[S(p)\gamma^\mu S(p-k)\gamma^\nu S(p)\left[C\gamma^5\right]^T \tilde{S}^T(q-p)\left[C\gamma^5\right]^T\right] D_{\mu\nu}(k) \quad (4.30)$$

The other two second order cases are symmetric to each other, as we are treating both quarks as massless. Due to this, the colour factor for both is identical.

$$\mathrm{C}_2 = \mathrm{C}_1 = \epsilon_{\alpha\beta\gamma}\epsilon_{\zeta\eta\theta}\delta_{\beta\iota}\delta_{\kappa\lambda}\delta_{\xi\theta}\delta_{\gamma\eta}\delta^{BC}\lambda^B_{\iota\kappa}\lambda^C_{\lambda\xi} = \frac{32}{3}\delta_{\alpha\zeta} \quad (4.31)$$

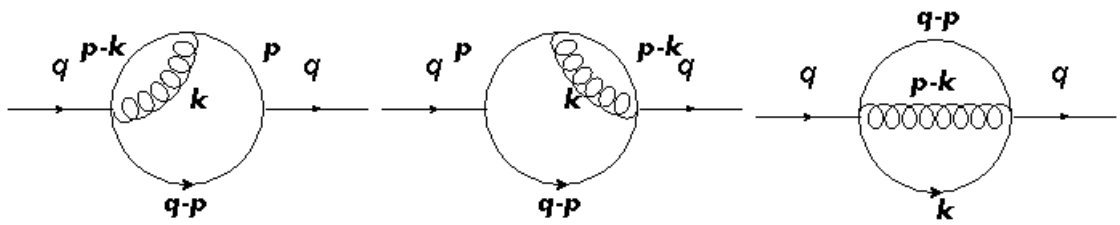

Figure 4.2: Feynman diagrams of leading order dressed field geometries

Once all diagrams upto next to leading order have been calculated, the results can be summed to give:

$$\Pi_1\left(q^2\right) = -2q^2\frac{C_0}{(4\pi)^2}\left[-\mathrm{L}+\epsilon\left(-\frac{\mathrm{L}^2}{4}+\mathrm{L}\right)\right]+\frac{(C_1+C_2)\pi\alpha q^2}{(4\pi)^4}\left[\frac{4\mathrm{L}}{\epsilon}+2\mathrm{L}^2-9\mathrm{L}\right]\xi+ \\ \frac{2C_3\pi\alpha q^2}{(4\pi)^4}\left[\frac{12\mathrm{L}}{\epsilon}+6\mathrm{L}^2-29\mathrm{L}+\xi\left[2\mathrm{L}^2-9\mathrm{L}+\frac{4\mathrm{L}}{\epsilon}\right]\right]. \quad (4.32)$$

Here our gauge dependency problem fully reveals itself: there's ξ dependence all over the place! This must be dealt with, so we turn to the dressed field formalism to obtain the gauge invariant information.

4.3 Dressed Components

Last but not least, we turn towards the basics of a dressed field interaction. Dressing the field results in 5 different diagrams: 2 symmetric diagrams where the gluon is emitted from the starting vertex and is annihilated along one of the quark lines, 2 symmetric diagrams where the gluon is emitted along a quark line and annihilated at the final vertex, and 1 diagram where the gluon propagates from the starting vertex to the ending vertex. The first of these options corresponds to the Feynman diagram in figure 4.2.

The crucial piece to be added on here is the $h^{-1}(x)$ function outlined in section 3.6.

$$h^{-1}(x) = e^{-ie\frac{\partial_\mu \cdot A}{\partial_x^2}} \quad (4.33)$$

When we expand this, the lowest order contribution doesn't change anything out of our original expression. However, when we go to the next order perturbation one of the correlation function we obtains has the form:

$$\Pi_D^{(1)}\left(q^2\right) = i^2 \mathrm{C}_{D1} \int \mathrm{d}^4 s \int \mathrm{d}^4 x \, \mathrm{e}^{iq\cdot x}$$
$$\langle 0|T\left[q_a(x)\left[C\gamma^5\right]_{ab} Q_b(x)\frac{g}{2}\frac{\partial_\mu \cdot A_\mu^B}{\partial_x^2}\bar{Q}_c(0)\left[C\gamma^5\right]_{cd}\bar{q}_d(0)\frac{g}{2}\bar{q}_e(s)\gamma^\nu A_\nu^C(s)q_f(s)\right]|0\rangle \,. \quad (4.34)$$

The other dressed diagrams add similar equations. Further reduction of this equation can be carried out in a similar way to the example above. When completing these dressed field calculations, the mathematical machinery is nearly identical to that of the undressed diagrams. The only addition is that the derivative terms, $(\frac{\partial_\mu \cdot A_\mu}{\partial_x^2})$, must be dealt with. It can be shown explicitly that the derivative terms do lead to additional terms being multiplied into the expression once the Fourier transform has taken place:

$$\frac{\partial_\mu \cdot A_\mu}{\partial_x^2} \rightarrow \frac{k_\mu}{k^2} A_\mu \quad (4.35)$$

Since the dressed computations are so similar to those already outlined earlier in this chapter, the explicit calculation will be omitted here — the results are in the next section. However, for future reference the Mathematica code will be included in the algorithm appendix.

4.4 Results

The results of the study of the diquark correlation can be broken into two sections: standard perturbation results and dressed field results. All standard perturbation results computed (leading order and next-to-leading order) were verified against the work of R. Kleiv *et al* [41]. As the dressed computations have not been completed previously, no verification was possible

- save, perhaps, for a strict double checking. All together, the nine Feynman diagrams give a summed result of:

$$\Pi_1\left(q^2\right) = -2q^2\frac{C_0}{(4\pi)^2}\left[-\mathrm{L}+\epsilon\left(-\frac{\mathrm{L}^2}{4}+\mathrm{L}\right)\right] + \frac{(C_1+C_2)\pi\alpha q^2}{(4\pi)^4}\left[\frac{4\mathrm{L}}{\epsilon}+2\mathrm{L}^2-9\mathrm{L}\right]\xi + \frac{2C_3\pi\alpha q^2}{(4\pi)^4}\left[\frac{12\mathrm{L}}{\epsilon}+6\mathrm{L}^2-29\mathrm{L}+\xi\left[2\mathrm{L}^2-9\mathrm{L}+\frac{4\mathrm{L}}{\epsilon}\right]\right] - 2\frac{(C_{D1}+C_{D2})\pi\alpha q^2}{(4\pi)^4}\left[\frac{4\mathrm{L}}{\epsilon}+2\mathrm{L}^2-9\mathrm{L}\right]\xi - \frac{C_{D3}\pi\alpha q^2}{(4\pi)^4}\left[\frac{4\mathrm{L}}{\epsilon}+2\mathrm{L}^2-9\mathrm{L}\right]\xi \quad (4.36)$$

Here, the first two lines are the result from the undressed field, as seen in equation 4.32; in contrast, the last line contains all the content from the dressed contributions. At this point, things don't look much better: there still seems to be dependence on ξ. Fortunately, some quick algebra reveals the truth. Rearranging and adding together like terms results in:

$$\Pi_1\left(q^2\right) = 4q^2\frac{C_0\delta_{\alpha\delta}}{(4\pi)^2}\left[-\mathrm{L}+\epsilon\left(-\frac{\mathrm{L}^2}{4}+\mathrm{L}\right)\right] + \frac{32C_3\pi\alpha q^2\delta_{\alpha\delta}}{3(4\pi)^4}\left[\frac{12\mathrm{L}}{\epsilon}+6\mathrm{L}^2-29\mathrm{L}\right] + \left(2C_1+2C_3-2C_{D1}-2C_{D2}-C_{D3}\right)\frac{\pi\alpha q^2}{(4\pi)^4}\left[\frac{4\mathrm{L}}{\epsilon}+2\mathrm{L}^2-9\mathrm{L}\right]\xi \quad (4.37)$$

Subbing in calculated colour constants for the third term gives a coefficient result of:

$$\left(2C_1+2C_3-2C_{D1}-2C_{D2}-C_{D3}\right) = \left[2\left(\frac{-32}{3}\right)+2\left(\frac{16}{3}\right)-2\left(\frac{-4}{3}\right)-2\left(\frac{-4}{3}\right)-\left(\frac{-16}{3}\right)\right]\delta_{\alpha\delta} = [0]\delta_{\alpha\delta} \quad (4.38)$$

Thus the ξ dependent term drops out and the final result for the diquark correlation function is gauge invariant — exactly what we were hoping to show.

Chapter 5

Heavy-Light and Doubly-Heavy Diquarks

Although the case of the doubly-light diquark is intriguing in its own right, the XYZ states that inspired us to look into diquarks in the first place have masses right in the middle of the charmonium spectrum. And, if that wasn't enough, our prime example of charmonium-like states, $X(3872)$, decays into J/ψ — indicating $c\bar{c}$ content. As mentioned in the last chapter, the quarks up to the strange have masses low enough to be ignored within the diquark correlation function; this restriction tragically doesn't hold up for charm or bottom quarks. Thus, if we want to explore possible internal structures for the charmonium-like particles we must include mass within our calculations: either for one or both of the quarks within the diquark. At its core, the calculations to be done here are very similar to those done in the last chapter with a small bit of complexity added in.

As a side note, although the heavy-light calculations were completed as an intermediate step to working on the doubly-heavy case only the doubly-heavy case will be discussed for the remainder of this chapter. This is because the demonstration of gauge invariance for a diquark with two arbitrary masses (as is present in the doubly-heavy case) also gifts us gauge invariance for a diquark where one or both of the masses are set to zero (the light-heavy and doubly-light scenarios, respectively).

On a mathematical level, the only real change to the structure of the problem enters the

calculation via the quark propagator:

$$S_{ab}(p) = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}, \; \epsilon \to 0^+. \tag{5.1}$$

In the previous chapter, the mass, m, was taken to be zero — here the variable is not assigned a value and simply kept within the equation. Going back to the calculation of the leading-order diagram in chapter 4 the process here is the same upto equation 4.13:

$$\Pi_1\left(q^2\right) = i^3 \mathrm{C}_0 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[C\gamma^5(\not{q} - \not{p} + m_1)C\gamma^5(\not{p} + m_2)\right]}{[(p)^2 - m_2^2][(q-p)^2 - m_1^2]}. \tag{5.2}$$

From here, the trace can be evaluated fairly easily, giving the following result:

$$\Pi_1\left(q^2\right) = i^3 \mathrm{C}_0 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{[4(p \cdot q - p^2 + m_1 m_2)]}{[(p)^2 - m_2^2][(q-p)^2 - m_1^2]}. \tag{5.3}$$

However, here is where the problem becomes more difficult: TARCER is the integral reduction package for Mathematica used for all the loop integrations within this project [52]. And, while TARCER can take in massless integrals, reduce them to simpler basis integrals, and finally evaluate them under a chosen dimensional regulation scheme, it cannot do the final step for massive integrals. Fortunately, the first two steps are enough to demonstrate the effectiveness of the dressed field formalism.

As I mentioned in section 2.4, the way TARCER breaks down loop integrals is as follows: TARCER takes in loop integrals in TFI form; from this TFI form, the application of the "TarcerRecurse" command results in the integral being broken down into an expression made up of various combinations of six types of master integrals (denoted by the letters **F**, **V**, **J**, **K**, **B**, and **A**). If all the component integrals of a given expression (post TarcerRecurse) can be simplified by TARCER, then "TarcerExpand" provides the explicit results for the expression according to the dimensional-regularization protocol being used. Here the

component integrals cannot be specified, so "TarcerRecurse" leaves us with a combination of master integrals. In full, the master integrals have the following forms:

1.

$$F^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \frac{1}{\pi^d}\int\int \frac{d^dk_1 d^dk_2}{[k_1^2-m_1^2]^{\nu_1}[k_2^2-m_2^2]^{\nu_2}[k_3^2-m_3^2]^{\nu_3}[k_4^2-m_4^2]^{\nu_4}[k_5^2-m_5^2]^{\nu_5}}; \quad (5.4)$$

2.

$$V^{(d)}_{\nu_1\nu_2\nu_3\nu_4} = \frac{1}{\pi^d}\int\int \frac{d^dk_1 d^dk_2}{[k_5^2-m_1^2]^{\nu_1}[k_2^2-m_2^2]^{\nu_2}[k_3^2-m_3^2]^{\nu_3}[k_4^2-m_4^2]^{\nu_4}}; \quad (5.5)$$

3.

$$J^{(d)}_{\nu_1\nu_2\nu_3} = \frac{1}{\pi^d}\int\int \frac{d^dk_1 d^dk_2}{[k_1^2-m_1^2]^{\nu_1}[k_5^2-m_2^2]^{\nu_2}[k_4^2-m_3^2]^{\nu_3}}; \quad (5.6)$$

4.

$$K^{(d)}_{\nu_1\nu_2\nu_3} = \frac{1}{\pi^d}\int\int \frac{d^dk_1 d^dk_2}{[k_1^2-m_1^2]^{\nu_1}[k_5^2-m_2^2]^{\nu_2}[k_2^2-m_3^2]^{\nu_3}}; \quad (5.7)$$

5.

$$B^{(d)}_{\nu_1\nu_2} = \frac{1}{\pi^{d/2}}\int \frac{d^dk_1}{[k_1^2-m_1^2]^{\nu_1}[k_3^2-m_2^2]^{\nu_2}}; \quad (5.8)$$

6.

$$A^{(d)}_{\nu_1} = \frac{1}{\pi^{d/2}}\int \frac{d^dk_1}{[k_1^2-m_1^2]^{\nu_1}}. \quad (5.9)$$

Within these master integrals p is the incoming momentum, k_1 and k_2 are the loop momentum and $k_3 = k_1 - p$, $k_4 = k_2 - p$, and $k_5 = k_1 - k_2$.

The full result, in master integral form, for the leading order Feynman diagram is as follows:

$$\Pi_1(q^2) = C_0\left(4B^{(d)}_{(1,m_2),(1,m_1)} - 4B^{(d)}_{(1,m_2),(1,m_1)} + 4m_1m_2B^{(d)}_{(1,m_2),(1,m_1)}\right) = C_0 4m_1m_2B^{(d)}_{(1,m_2),(1,m_1)} \quad (5.10)$$

```
afterTFI2 = ToTFI[preTFI2 * preCalcDenom2, k1, k2, p]
```

$$
\begin{aligned}
&-4\,m^4\,\mathbf{F}^{(D)\,0\;\;1\;\;0\;\;0\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - 8\,m^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;2}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - 8\,m^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;1\;\;1}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + \\
&\quad 4\,m^2\,\mathbf{F}^{(D)\,0\;\;1\;\;0\;\;0\;\;1}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + 4\,m^2\,\mathbf{F}^{(D)\,0\;\;1\;\;0\;\;1\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - 4\,m^2\,\mathbf{F}^{(D)\,0\;\;1\;\;1\;\;0\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + \\
&\quad 8\,m^2\,\mathbf{F}^{(D)\,1\;\;1\;\;0\;\;0\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - 16\,\mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;2}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + 8\,\mathbf{F}^{(D)\,0\;\;1\;\;1\;\;0\;\;1}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + \\
&\quad 8\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;2}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + 8\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;1\;\;1}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - 4\,\mathbf{F}^{(D)\,1\;\;1\;\;0\;\;0\;\;1}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - \\
&\quad 4\,\mathbf{F}^{(D)\,1\;\;1\;\;0\;\;1\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} + 4\,\mathbf{F}^{(D)\,1\;\;1\;\;1\;\;0\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}} - 4\,\mathbf{F}^{(D)\,2\;\;1\;\;0\;\;0\;\;0}_{\{2,m\}\{2,0\}\{1,m\}\{0,0\}\{1,m\}}
\end{aligned}
$$

```
masterIntegrals2 = TarcerRecurse[afterTFI2]
```

$$
\begin{aligned}
&-\frac{2\,(D-2)^2\left(\mathbf{A}^{(D)}_{\{1,m\}}\right)^2}{(D-4)\left(4\,m^2-p^2\right)} - \\
&\quad \left(2\left(2\,D^2\,m^2+D^2\,p^2-18\,D\,m^2-4\,D\,p^2+32\,m^2+4\,p^2\right)\mathbf{J}^{(D)}_{\{1,m\}\{1,m\}\{1,0\}}\right)\Big/\left((D-4)\left(4\,m^2-p^2\right)\right) + \\
&\quad \frac{8\,m^2\left(2\,D\,m^2+D\,p^2-10\,m^2-2\,p^2\right)\mathbf{J}^{(D)}_{\{2,m\}\{1,m\}\{1,0\}}}{(D-4)\left(4\,m^2-p^2\right)}
\end{aligned}
$$

Figure 5.1: Master integral results for a doubly-heavy, next-to-leading order, undressed diagram. Note that this is only the gauge dependent piece — the entire expression here would be multiplied by ξ in the actual correlation function.

Whereas the gauge dependent part of the first next-to-leading order diagram gives the result seen in figure 5.1.

Unfortunately, the expansion of the next-to-leading order diagrams is quite expansive — writing out the final, unsimplified result takes pages. As such, the final expression will not be expressed here: the code used for evaluations of the doubly heavy integrals will be included in the algorithms appendix for future reference. Once each diagram (both dressed and undressed) was evaluated via "TarcerRecurse", the final results were summed via Mathematica. Just like the doubly-light case, the gauge dependence disappears, demonstrating that

the dress field formalism is an effective technique for extracting gauge-invariant information from general diquarks.

Chapter 6

Conclusion

Ultimately, the dressed field formalism has demonstrated to be an effective means of dealing with the gauge dependency problems that arise within the computation of the diquark correlator. Additionally, the work done here has shown that the dress field technique can be feasibly performed using conventional methodology regardless of quark mass — this is a stark contrast to the Schwinger string which becomes less and less feasible in calculations involving heavy quarks. Of course with such an unambiguous result, the question becomes "where does the research go from here?" To this end, there are two equal, but fairly divergent paths: on a theoretical level, work can be done towards including non-perturbative gluon condensate contributions; or, on a phenomenological level, QCD sum-rule calculations of the diquark mass can be made.

The former of these extensions was attempted during the course of this project; unfortunately unforeseen difficulties plagued the gluon condensate calculations. Most notably, the inclusion of the dressed condensate diagrams led to the presence of divergent pieces that likely require higher-order terms from the dressed field operator. On a purely speculative level, it is quite possible that these divergent pieces could be cancelled out by higher order condensate contributions — tragically the time line for finishing this thesis prevented finishing these additional calculations. Exploring this direction further is necessary for the full

understanding of the dressed field formalism as applied to diquarks. Hopefully, the appropriate cancellations occur and the condensate contributions have no deeper issues. That said, only further research can determine this.

Turning to the the phenomenological side, the explicit computation of a diquark mass via QCD sum rule provides the crucial link between QCD and the effective potential models currently in use. If the diquark masses for a particular effective potential model (there are various potential models, each with different predictions that can be compared to) match those obtained via QSR, we have theoretical evidence for the internal structure suggested by said model. Through the work of this paper, the framework has been laid to truly start this approach. As with the condensate contributions, only time (and further calculations) will tell whether diquarks truly do make up the internal structure of these unknown states.

The core question that motivated this thesis, "what exactly are the XYZ states?" still remains, and hopefully the next 5-10 years of experiments, phenomenology, and theory can provide a decisive answer. Whether this answer is exotic states in the form of multi-quarks, or hybrids, or something else entirely, certainty will provide charmonium spectroscopy with far, far better rest than it has ever known; or, at the very least, the best since the day Belle killed the conventional quark model.

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Appendix A

Background Mathematics

A.1 Group Theory Basics

At the core of all QCD lies the mathematical framework of group theory. A group here is defined in the typical manner: a set of elements paired with an operator. Groups are held to 4 axioms:

For a group G with elements $a, b, c \epsilon G$ and operation $\cdot$:

1. **Closure under operation:** $a \cdot b = d \epsilon G$
2. **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. **Existence of the identity:** There exists $0 \epsilon G$ such that $x \cdot 0 = 0 \cdot x = x$ for every $x \epsilon G$
4. **Existence of an inverse element:** For every $x \epsilon G$ there exists $y \epsilon G$ such that $x \cdot y = y \cdot x = 0$

The group theory required to understand QCD has many other crucial definitions. Groups can be classified according to whether operations commute — commuting groups are referred to as Abelian. Additionally, a subset H of G that forms its own group under operation $\cdot$ is known as a subgroup. Finally, for any given subset of G, S, is called the generator of the smallest subgroup of G that contains S - conversely, this subgroup is "generated by S". Continuous groups that have an infinite number of elements are known as Lie groups.

For the purposes of QCD, Lie groups that are non-Abelian are the most important as the special unitary (3) group represents the symmetries present within the standard model (as displayed by the Yang-Mills field). Technically, the SU(3) group is defined as the set of unitary three by three matrices with determinant one coupled with the matrix multiplication operator.

As with any other group, SU(3) has an infinite number of representations; the most useful of these, however, are irreducible representations — in particular the singlet (1), fundamental (3), and adjoint (8) representations. Here, the representations take the form of square matrices where the number in brackets declares the dimension (number or rows/columns) present.

A.2 The Dirac Delta

Along with Wick's theorem, the other key technique used within the evaluation of correlation functions within this paper is that of the Dirac Delta. Similar to classical quantum mechanics, the Dirac delta function is defined as follows:

$$\delta(x) = \infty \text{ for } x = 0 \quad \text{and} \quad \delta(x) = 0 \text{ elsewhere} \tag{A.1}$$

If the variable x is integrated over:

$$\int_{-\infty}^{\infty} \delta(x)\, \mathrm{d}x = 1 \tag{A.2}$$

The Dirac delta can also be simply generalized to four dimensions:

$$\int_{-\infty}^{\infty} \delta^{(4)}(x)\, f(a - x)\, \mathrm{d}^4 x = f(a) \tag{A.3}$$

The Dirac delta can be used in many contexts and, often times, allows for the reduction in the number of variables present within a given integration. Within the computations undertaken throughout the document, the most important result is the Fourier transform between momentum and position space:

$$\int \frac{\mathrm{d}^4 x}{(2\pi)^4}\, \mathrm{e}^{-ik\cdot x} = \delta^{(4)}(k) \tag{A.4}$$

Appendix B
Algorithms

Calculation of the massive diquark - leading order diagram.

```
$LoadTARCER = True
```

True

```
<< HighEnergyPhysics`FeynCalc`
```

Loading FeynCalc from /home/paul/.Mathematica/Applications/HighEnergyPhysics
Loading TARCER
/home/paul/.Mathematica/Applications/HighEnergyPhysics/Tarcer/tarcerLinuxx8664bit25.mx
FeynCalc 8.2.0 For help, type ?FeynCalc, open `FeynCalcRef8.nb` or visit www.feyncalc.org
Loading FeynArts, see www.feynarts.de for documentation

FeynArts not found. Please install FeynArts, e.g., in
/home/paul/.Mathematica
and reload FeynCalc
FeynArts can be downloaded from www.feynarts.de

```
preCalcDenom = FAD[{k1, m2}, {(p - k1), m1}]
```

$$\frac{1}{\left([\mathrm{k1}^2-\mathrm{m2}^2]\right)\left([(p-\mathrm{k1})^2-\mathrm{m1}^2]\right)}$$

```
preTFI = preCalcDenom * TR[(GSD[k1] - m2).(GSD[(p - k1)] - m1)]
```

$$4\left(\mathrm{k1}\cdot p-\mathrm{k1}^2+\mathrm{m1}\,\mathrm{m2}\right)\frac{1}{\left([\mathrm{k1}^2-\mathrm{m2}^2]\right)\left([(p-\mathrm{k1})^2-\mathrm{m1}^2]\right)}$$

It appears as if the ToTFI notation will not work here since it is designed to convert 2-loop integrals into TFI form. Due to this, evaluation shall have to be done semi-manually - fortunately there are only 3 terms!

```
master1 = 4 TBI[d, p^2, {0, 0, 1, 0, 0}, {{1, m2}, {1, m1}}]
```

$$4\,\mathbf{B}^{(d)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}}$$

```
master2 = -4 TBI[d, p^2, {1, 0, 0, 0, 0}, {{1, m2}, {1, m1}}]
```

$$-4\,\mathbf{B}^{(d)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}}$$

```
master3 = 4 * m1 * m2 * TBI[d, p^2, {0, 0, 0, 0, 0}, {{1, m2}, {1, m1}}]
```

$$4\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{B}^{(d)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}}$$

Calculation of the massive diquark - next to leading order, diagram 1.

```
$LoadTARCER = True
```

True

```
<< HighEnergyPhysics`FeynCalc`
```

Loading FeynCalc from /home/paul/.Mathematica/Applications/HighEnergyPhysics
Loading TARCER
/home/paul/.Mathematica/Applications/HighEnergyPhysics/Tarcer/tarcerLinuxx8664bit25.mx
FeynCalc 8.2.0 For help, type ?FeynCalc, open `FeynCalcRef8.nb` or visit www.feyncalc.org
Loading FeynArts, see www.feynarts.de for documentation

FeynArts not found. Please install FeynArts, e.g., in
/home/paul/.Mathematica
and reload FeynCalc
FeynArts can be downloaded from www.feynarts.de

Metric Tensor Component

```
preCalcDenom1 = FAD[{(k1 - p), m2}, {k1, m1}, {(k1 - k2), m1}, {k1, m1}, {k2, 0}]
```

$$\frac{1}{([\mathrm{k2}^2-0])\left([\mathrm{k1}^2-\mathrm{m1}^2]\right)^2\left([(\mathrm{k1}-\mathrm{k2})^2-\mathrm{m1}^2]\right)\left([(\mathrm{k1}-p)^2-\mathrm{m2}^2]\right)}$$

```
preCalcFrac1 = TR[
  (GSD[k1] - m1).(GSD[(k1 - p)] + m2).(GSD[k1] - m1).GAD[μ].(GSD[(k2 - k1)] - m1).GAD[ν]]
```

$$\begin{aligned}4\Big(&-\mathrm{k1}^2 g^{\mu\nu}\,\mathrm{k1}\cdot\mathrm{k2} - \mathrm{m1}^2 g^{\mu\nu}\,\mathrm{k1}\cdot\mathrm{k2} + 2\,\mathrm{m1}\,\mathrm{m2}\,g^{\mu\nu}\,\mathrm{k1}\cdot\mathrm{k2} + 2\,g^{\mu\nu}\,\mathrm{k1}\cdot\mathrm{k2}\,\mathrm{k1}\cdot p - \mathrm{k1}^2 g^{\mu\nu}\,\mathrm{k2}\cdot p + \mathrm{k1}^{2^2} g^{\mu\nu} + \\ &3\,\mathrm{k1}^2\,\mathrm{m1}^2 g^{\mu\nu} - 3\,\mathrm{m1}^2 g^{\mu\nu}\,\mathrm{k1}\cdot p - 3\,\mathrm{k1}^2\,\mathrm{m1}\,\mathrm{m2}\,g^{\mu\nu} - \mathrm{k1}^2 g^{\mu\nu}\,\mathrm{k1}\cdot p + \mathrm{m1}^2 g^{\mu\nu}\,\mathrm{k2}\cdot p + \mathrm{m1}^3(-\mathrm{m2})\,g^{\mu\nu} + \\ &\mathrm{k1}^2\,\mathrm{k1}^\nu\,\mathrm{k2}^\mu + \mathrm{k1}^2\,\mathrm{k1}^\mu\,\mathrm{k2}^\nu + \mathrm{m1}^2\,\mathrm{k1}^\nu\,\mathrm{k2}^\mu + \mathrm{m1}^2\,\mathrm{k1}^\mu\,\mathrm{k2}^\nu - 2\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}^\nu\,\mathrm{k2}^\mu - 2\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}^\mu\,\mathrm{k2}^\nu + \\ &\mathrm{k1}^2\,\mathrm{k2}^\nu p^\mu + \mathrm{k1}^2\,\mathrm{k2}^\mu p^\nu - 2\,\mathrm{k1}^\nu\,\mathrm{k2}^\mu\,\mathrm{k1}\cdot p - 2\,\mathrm{k1}^\mu\,\mathrm{k2}^\nu\,\mathrm{k1}\cdot p - 2\,\mathrm{k1}^2\,\mathrm{k1}^\mu\,\mathrm{k1}^\nu - 2\,\mathrm{m1}^2\,\mathrm{k1}^\mu\,\mathrm{k1}^\nu + \mathrm{m1}^2\,\mathrm{k1}^\nu p^\mu + \\ &\mathrm{m1}^2\,\mathrm{k1}^\mu p^\nu + 4\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}^\mu\,\mathrm{k1}^\nu - \mathrm{k1}^2\,\mathrm{k1}^\nu p^\mu - \mathrm{k1}^2\,\mathrm{k1}^\mu p^\nu + 4\,\mathrm{k1}^\mu\,\mathrm{k1}^\nu\,\mathrm{k1}\cdot p - \mathrm{m1}^2\,\mathrm{k2}^\nu p^\mu - \mathrm{m1}^2\,\mathrm{k2}^\mu p^\nu\Big)\end{aligned}$$

```
preTFI1 = Expand[Contract[preCalcFrac1 * (MTD[μ, ν] )]]
```

$$\begin{aligned}&-4\,D\,\mathrm{m1}^2\,\mathrm{k1}\cdot\mathrm{k2} + 8\,D\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}\cdot\mathrm{k2} + 8\,D\,\mathrm{k1}\cdot\mathrm{k2}\,\mathrm{k1}\cdot p - 4\,D\,\mathrm{k1}^2\,\mathrm{k2}\cdot p - \\ &4\,D\,\mathrm{k1}^2\,\mathrm{k1}\cdot\mathrm{k2} - 12\,D\,\mathrm{m1}^2\,\mathrm{k1}\cdot p + 12\,D\,\mathrm{k1}^2\,\mathrm{m1}^2 - 12\,D\,\mathrm{k1}^2\,\mathrm{m1}\,\mathrm{m2} - 4\,D\,\mathrm{k1}^2\,\mathrm{k1}\cdot p + 4\,D\,\mathrm{k1}^{2^2} + \\ &4\,D\,\mathrm{m1}^2\,\mathrm{k2}\cdot p - 4\,D\,\mathrm{m1}^3\,\mathrm{m2} + 8\,\mathrm{m1}^2\,\mathrm{k1}\cdot\mathrm{k2} - 16\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}\cdot\mathrm{k2} - 16\,\mathrm{k1}\cdot\mathrm{k2}\,\mathrm{k1}\cdot p + 8\,\mathrm{k1}^2\,\mathrm{k2}\cdot p + \\ &8\,\mathrm{k1}^2\,\mathrm{k1}\cdot\mathrm{k2} + 8\,\mathrm{m1}^2\,\mathrm{k1}\cdot p - 8\,\mathrm{k1}^2\,\mathrm{m1}^2 + 16\,\mathrm{k1}^2\,\mathrm{m1}\,\mathrm{m2} + 8\,\mathrm{k1}^2\,\mathrm{k1}\cdot p - 8\,\mathrm{k1}^{2^2} - 8\,\mathrm{m1}^2\,\mathrm{k2}\cdot p\end{aligned}$$

```
afterTFI1 = ToTFI[preTFI1 * preCalcDenom1, k1, k2, p]
```

$$-4\,D\,\mathrm{m1}^3\,\mathrm{m2}\,\mathbf{F}^{(D)}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 4\,D\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 8\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} +$$
$$4\,D\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;1\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 8\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;1\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 12\,D\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} +$$
$$8\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 12\,D\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 8\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} +$$
$$8\,D\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 16\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 12\,D\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} +$$
$$16\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 8\,D\,\mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 16\,\mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} -$$
$$4\,D\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 8\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;1}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 4\,D\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;1\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 8\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;1\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} -$$
$$4\,D\,\mathbf{F}^{(D)\,1\;\;0\;\;1\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 8\,\mathbf{F}^{(D)\,1\;\;0\;\;1\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} + 4\,D\,\mathbf{F}^{(D)\,2\;\;0\;\;0\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}} - 8\,\mathbf{F}^{(D)\,2\;\;0\;\;0\;\;0\;\;0}_{\{2,\mathrm{m1}\}\{1,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}$$

```
masterIntegrals1 = TarcerRecurse[afterTFI1]
```

$$\Big((D+1)(D-2)\big(2\,D\,\mathrm{m1}^4 - 2\,D\,\mathrm{m1}^2\,\mathrm{m2}^2 - 2\,D\,\mathrm{m1}^2\,p^2 - 5\,\mathrm{m1}^4 + 2\,\mathrm{m1}^3\,\mathrm{m2} +$$
$$6\,\mathrm{m1}^2\,\mathrm{m2}^2 + 6\,\mathrm{m1}^2\,p^2 - 2\,\mathrm{m1}\,\mathrm{m2}^3 + 2\,\mathrm{m1}\,\mathrm{m2}\,p^2 - \mathrm{m2}^4 + 2\,\mathrm{m2}^2\,p^2 - p^4\big)\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{B}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}}\Big)\Big/$$
$$\big((D-3)\,\mathrm{m1}^2\big(\mathrm{m1}^2 + 2\,\mathrm{m1}\,\mathrm{m2} + \mathrm{m2}^2 - p^2\big)\big) + \frac{2\,(D+2)\,(D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}}{(D-4)\big(\mathrm{m1}^2 + 2\,\mathrm{m1}\,\mathrm{m2} + \mathrm{m2}^2 - p^2\big)} +$$
$$\frac{(D+1)\,(D-2)\big(2\,D\,\mathrm{m1}\,\mathrm{m2} - \mathrm{m1}^2 - 6\,\mathrm{m1}\,\mathrm{m2} - \mathrm{m2}^2 + p^2\big)\big(\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\big)^2}{(D-3)\,\mathrm{m1}^2\big(\mathrm{m1}^2 + 2\,\mathrm{m1}\,\mathrm{m2} + \mathrm{m2}^2 - p^2\big)} -$$
$$\Big(2\big(4\,D^2\,\mathrm{m1}^2 - 4\,D^2\,\mathrm{m1}\,\mathrm{m2} + 4\,D^2\,\mathrm{m2}^2 - 4\,D^2\,p^2 - 9\,D\,\mathrm{m1}^2 + 26\,D\,\mathrm{m1}\,\mathrm{m2} - 9\,D\,\mathrm{m2}^2 + 9\,D\,p^2 -$$
$$4\,\mathrm{m1}^2 - 40\,\mathrm{m1}\,\mathrm{m2} - 4\,\mathrm{m2}^2 + 4\,p^2\big)\,\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\Big)\Big/\big((D-4)\big(\mathrm{m1}^2 + 2\,\mathrm{m1}\,\mathrm{m2} + \mathrm{m2}^2 - p^2\big)\big) +$$
$$\Big(2\big(5\,D^2\,\mathrm{m1}^4 - 6\,D^2\,\mathrm{m1}^3\,\mathrm{m2} - 2\,D^2\,\mathrm{m1}^2\,\mathrm{m2}^2 - 6\,D^2\,\mathrm{m1}^2\,p^2 + 2\,D^2\,\mathrm{m1}\,\mathrm{m2}^3 - 2\,D^2\,\mathrm{m1}\,\mathrm{m2}\,p^2 + D^2\,\mathrm{m2}^4 -$$
$$2\,D^2\,\mathrm{m2}^2\,p^2 + D^2\,p^4 - 15\,D\,\mathrm{m1}^4 + 34\,D\,\mathrm{m1}^3\,\mathrm{m2} + 14\,D\,\mathrm{m1}^2\,\mathrm{m2}^2 + 18\,D\,\mathrm{m1}^2\,p^2 - 6\,D\,\mathrm{m1}\,\mathrm{m2}^3 +$$
$$6\,D\,\mathrm{m1}\,\mathrm{m2}\,p^2 - 3\,D\,\mathrm{m2}^4 + 6\,D\,\mathrm{m2}^2\,p^2 - 3\,D\,p^4 + 4\,\mathrm{m1}^4 - 40\,\mathrm{m1}^3\,\mathrm{m2} - 24\,\mathrm{m1}^2\,\mathrm{m2}^2 - 8\,\mathrm{m1}\,\mathrm{m2}^3 +$$
$$8\,\mathrm{m1}\,\mathrm{m2}\,p^2 - 4\,\mathrm{m2}^4 + 8\,\mathrm{m2}^2\,p^2 - 4\,p^4\big)\,\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}\Big)\Big/\big((D-4)\,(D-3)\big(\mathrm{m1}^2 + 2\,\mathrm{m1}\,\mathrm{m2} + \mathrm{m2}^2 - p^2\big)\big) -$$
$$\frac{4\,\mathrm{m2}^2\big(D\,\mathrm{m1}^2 + 2\,D\,\mathrm{m1}\,\mathrm{m2} - 3\,D\,\mathrm{m2}^2 + 3\,D\,p^2 - 4\,\mathrm{m1}^2 - 8\,\mathrm{m1}\,\mathrm{m2}\big)\,\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}}{(D-4)\big(\mathrm{m1}^2 + 2\,\mathrm{m1}\,\mathrm{m2} + \mathrm{m2}^2 - p^2\big)}$$

Gauge Component

```
preCalcDenom2 =
 FAD[{(k1 - p), m2}, {k1, m1}, {(k1 - k2), m1}, {k1, m1}, {k2, 0}, {k2, 0}]
```

$$\frac{1}{\big([\mathrm{k2}^2-0]\big)^2\,\big([\mathrm{k1}^2-m^2]\big)^2\,\big([(\mathrm{k1}-\mathrm{k2})^2-m^2]\big)\,\big([(\mathrm{k1}-p)^2-m^2]\big)}$$

```
preCalcFrac2 = TR[
  (GSD[k1] - m1).(GSD[(k1 - p)] + m2).(GSD[k1] - m1).GAD[μ].(GSD[(k1 - k2)] - m1).GAD[ν]]
```

$$4\Big(\mathrm{k1}^2\, g^{\mu\nu}\, \mathrm{k1}\cdot\mathrm{k2} - m^2\, g^{\mu\nu}\, \mathrm{k1}\cdot\mathrm{k2} - 2\, g^{\mu\nu}\, \mathrm{k1}\cdot\mathrm{k2}\, \mathrm{k1}\cdot p + \mathrm{k1}^2\, g^{\mu\nu}\, \mathrm{k2}\cdot p - \mathrm{k1}^{2^2}\, g^{\mu\nu} + 2\, \mathrm{k1}^2\, m^2\, g^{\mu\nu} - m^2\, g^{\mu\nu}\, \mathrm{k1}\cdot p + \mathrm{k1}^2\, g^{\mu\nu}\, \mathrm{k1}\cdot p - m^2\, g^{\mu\nu}\, \mathrm{k2}\cdot p + m^4\, (-g^{\mu\nu}) - \mathrm{k1}^2\, \mathrm{k1}^\nu\, \mathrm{k2}^\mu - \mathrm{k1}^2\, \mathrm{k1}^\mu\, \mathrm{k2}^\nu + m^2\, \mathrm{k1}^\nu\, \mathrm{k2}^\mu + m^2\, \mathrm{k1}^\mu\, \mathrm{k2}^\nu - \mathrm{k1}^2\, \mathrm{k2}^\nu\, p^\mu - \mathrm{k1}^2\, \mathrm{k2}^\mu\, p^\nu + 2\, \mathrm{k1}^\nu\, \mathrm{k2}^\mu\, \mathrm{k1}\cdot p + 2\, \mathrm{k1}^\mu\, \mathrm{k2}^\nu\, \mathrm{k1}\cdot p + 2\, \mathrm{k1}^2\, \mathrm{k1}^\mu\, \mathrm{k1}^\nu - 2\, m^2\, \mathrm{k1}^\mu\, \mathrm{k1}^\nu - m^2\, \mathrm{k1}^\nu\, p^\mu - m^2\, \mathrm{k1}^\mu\, p^\nu + \mathrm{k1}^2\, \mathrm{k1}^\nu\, p^\mu + \mathrm{k1}^2\, \mathrm{k1}^\mu\, p^\nu - 4\, \mathrm{k1}^\mu\, \mathrm{k1}^\nu\, \mathrm{k1}\cdot p + m^2\, \mathrm{k2}^\nu\, p^\mu + m^2\, \mathrm{k2}^\mu\, p^\nu\Big)$$

```
preTFI2 = Expand[Contract[preCalcFrac2 * ((FVD[k2, μ].FVD[k2, ν]))]]
```

$$-4\, \mathrm{k2}^2\, m^2\, \mathrm{k1}\cdot p - 8\, m^2\, \mathrm{k1}\cdot\mathrm{k2}\, \mathrm{k2}\cdot p + 8\, \mathrm{k1}^2\, \mathrm{k2}^2\, m^2 - 8\, m^2\, \mathrm{k1}\cdot\mathrm{k2}^2 + 4\, \mathrm{k2}^2\, m^2\, \mathrm{k1}\cdot\mathrm{k2} - 16\, \mathrm{k1}\cdot\mathrm{k2}^2\, \mathrm{k1}\cdot p + 4\, \mathrm{k1}^2\, \mathrm{k2}^2\, \mathrm{k1}\cdot p + 8\, \mathrm{k2}^2\, \mathrm{k1}\cdot\mathrm{k2}\, \mathrm{k1}\cdot p + 8\, \mathrm{k1}^2\, \mathrm{k1}\cdot\mathrm{k2}\, \mathrm{k2}\cdot p - 4\, \mathrm{k1}^2\, \mathrm{k2}^2\, \mathrm{k2}\cdot p - 4\, \mathrm{k1}^{2^2}\, \mathrm{k2}^2 + 8\, \mathrm{k1}^2\, \mathrm{k1}\cdot\mathrm{k2}^2 - 4\, \mathrm{k1}^2\, \mathrm{k2}^2\, \mathrm{k1}\cdot\mathrm{k2} - 4\, \mathrm{k2}^2\, m^4 + 4\, \mathrm{k2}^2\, m^2\, \mathrm{k2}\cdot p$$

```
afterTFI2 = ToTFI[preTFI2 * preCalcDenom2, k1, k2, p]
```

$$-4\, m^4\, \mathbf{F}^{(D)\,0\;\;1\;\;0\;\;0\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 8\, m^2\, \mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;2}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 8\, m^2\, \mathbf{F}^{(D)\,0\;\;0\;\;0\;\;1\;\;1}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 4\, m^2\, \mathbf{F}^{(D)\,0\;\;1\;\;0\;\;0\;\;1}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 4\, m^2\, \mathbf{F}^{(D)\,0\;\;1\;\;0\;\;1\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 4\, m^2\, \mathbf{F}^{(D)\,0\;\;1\;\;1\;\;0\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 8\, m^2\, \mathbf{F}^{(D)\,1\;\;1\;\;0\;\;0\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 16\, \mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;2}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 8\, \mathbf{F}^{(D)\,0\;\;1\;\;1\;\;0\;\;1}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 8\, \mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;2}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 8\, \mathbf{F}^{(D)\,1\;\;0\;\;0\;\;1\;\;1}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 4\, \mathbf{F}^{(D)\,1\;\;1\;\;0\;\;0\;\;1}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 4\, \mathbf{F}^{(D)\,1\;\;1\;\;0\;\;1\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} + 4\, \mathbf{F}^{(D)\,1\;\;1\;\;1\;\;0\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}} - 4\, \mathbf{F}^{(D)\,2\;\;1\;\;0\;\;0\;\;0}_{\{2,m\}\,\{2,0\}\,\{1,m\}\,\{0,0\}\,\{1,m\}}$$

```
masterIntegrals2 = TarcerRecurse[afterTFI2]
```

$$-\frac{2\,(D-2)^2\,\left(\mathbf{A}^{(D)}_{\{1,m\}}\right)^2}{(D-4)\left(4\,m^2-p^2\right)} - \frac{2\left(2\,D^2\,m^2 + D^2\,p^2 - 18\,D\,m^2 - 4\,D\,p^2 + 32\,m^2 + 4\,p^2\right)\mathbf{J}^{(D)}_{\{1,m\}\,\{1,m\}\,\{1,0\}}}{(D-4)\left(4\,m^2-p^2\right)} + \frac{8\,m^2\left(2\,D\,m^2 + D\,p^2 - 10\,m^2 - 2\,p^2\right)\mathbf{J}^{(D)}_{\{2,m\}\,\{1,m\}\,\{1,0\}}}{(D-4)\left(4\,m^2-p^2\right)}$$

```
masterU1 = masterIntegrals2;
```

Calculation of the massive diquark - dressed, diagrams 1 and 2.

```
$LoadTARCER = True
```

True

```
<< HighEnergyPhysics`FeynCalc`
```

Loading FeynCalc from /home/paul/.Mathematica/Applications/HighEnergyPhysics
Loading TARCER
/home/paul/.Mathematica/Applications/HighEnergyPhysics/Tarcer/tarcerLinuxx8664bit25.mx
FeynCalc 8.2.0 For help, type ?FeynCalc, open `FeynCalcRef8.nb` or visit www.feyncalc.org
Loading FeynArts, see www.feynarts.de for documentation

FeynArts not found. Please install FeynArts, e.g., in
/home/paul/.Mathematica
and reload FeynCalc
FeynArts can be downloaded from www.feynarts.de

Diagram 1

```
preCalcDenom1 = FAD[{k1, m1}, {k2, 0}, {k2, 0}, {(k1 - p), m2}, {(k1 - k2), m1}]
```

$$\frac{1}{\left([k2^2-0]\right)^2 \left([k1^2-m1^2]\right) \left([(k1-k2)^2-m1^2]\right) \left([(k1-p)^2-m2^2]\right)}$$

```
preCalcFrac1 = TR[(GSD[(k1 - p)] + m2).(GSD[k1] - m1).GAD[ν].(GSD[(k1 - k2)] - m1)]
```

$$4\left(-k1^2\, k2^\nu - p^\nu\, k1\cdot k2 + k2^\nu\, k1\cdot p + k1^\nu\, k2\cdot p + k1^2\, k1^\nu + m1^2\, k1^\nu - 2\, m1\, m2\, k1^\nu + k1^2\, p^\nu - 2\, k1^\nu\, k1\cdot p + m1\, m2\, k2^\nu - m1^2\, p^\nu\right)$$

```
preTFI1 = Expand[Contract[preCalcFrac1 * (FVD[k2, ν] )]]
```

$$4\, m1^2\, k1\cdot k2 - 8\, m1\, m2\, k1\cdot k2 - 8\, k1\cdot k2\, k1\cdot p + 4\, k2^2\, k1\cdot p + 4\, k1^2\, k2\cdot p - 4\, k1^2\, k2^2 + 4\, k1^2\, k1\cdot k2 - 4\, m1^2\, k2\cdot p + 4\, k2^2\, m1\, m2$$

```
afterTFI1 = ToTFI[preTFI1 * preCalcDenom1, k1, k2, p]
```

$$4\, m1^2\, \mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} - 4\, m1^2\, \mathbf{F}^{(D)\,0\;\;0\;\;0\;\;1\;\;0}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} - 8\, m1\, m2\, \mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} + 4\, m1\, m2\, \mathbf{F}^{(D)\,0\;\;1\;\;0\;\;0\;\;0}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} - 8\, \mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;1}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} + 4\, \mathbf{F}^{(D)\,0\;\;1\;\;1\;\;0\;\;0}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} + 4\, \mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;1}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} + 4\, \mathbf{F}^{(D)\,1\;\;0\;\;0\;\;1\;\;0}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}} - 4\, \mathbf{F}^{(D)\,1\;\;1\;\;0\;\;0\;\;0}_{\{1,m1\}\,\{2,0\}\,\{1,m2\}\,\{0,0\}\,\{1,m1\}}$$

```
masterIntegrals1 = TarcerRecurse[afterTFI1]
```

$$-\frac{2\,(D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}}{(D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)}+$$
$$\left(2\left(D^2\,\mathrm{m1}^2-4\,D^2\,\mathrm{m1}\,\mathrm{m2}+D^2\,\mathrm{m2}^2-D^2\,p^2-4\,D\,\mathrm{m1}^2+26\,D\,\mathrm{m1}\,\mathrm{m2}-4\,D\,\mathrm{m2}^2+4\,D\,p^2+\right.\right.$$
$$\left.\left.4\,\mathrm{m1}^2-40\,\mathrm{m1}\,\mathrm{m2}+4\,\mathrm{m2}^2-4\,p^2\right)\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\Big/\left((D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)\right)-$$
$$\frac{4\,\mathrm{m1}^2\left(D\,\mathrm{m1}^2-2\,D\,\mathrm{m1}\,\mathrm{m2}-D\,\mathrm{m2}^2-D\,p^2-2\,\mathrm{m1}^2+8\,\mathrm{m1}\,\mathrm{m2}+4\,\mathrm{m2}^2+2\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}}{(D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)}+$$
$$\frac{4\,\mathrm{m2}^2\left(D\,\mathrm{m1}^2+2\,D\,\mathrm{m1}\,\mathrm{m2}-D\,\mathrm{m2}^2+D\,p^2-4\,\mathrm{m1}^2-8\,\mathrm{m1}\,\mathrm{m2}+2\,\mathrm{m2}^2-2\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}}{(D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)}$$

```
masterD1 = masterIntegrals1;
```

Diagram 2

```
preCalcDenom2 = FAD[{k1, m1}, {k2, 0}, {k2, 0}, {(k1 - p), m2}, {(k1 - k2), m1}]
```

$$\frac{1}{\left([\mathrm{k2}^2-0]\right)^2\left([\mathrm{k1}^2-\mathrm{m1}^2]\right)\left([(\mathrm{k1}-\mathrm{k2})^2-\mathrm{m1}^2]\right)\left([(\mathrm{k1}-p)^2-\mathrm{m2}^2]\right)}$$

```
preCalcFrac2 = TR[(GSD[(k1 - p)] + m2).(GSD[(k1 - k2)] - m1).GAD[ν].(GSD[k1] - m1)]
```

$$4\left(-\mathrm{k1}^2\,\mathrm{k2}^\nu-p^\nu\,\mathrm{k1}\cdot\mathrm{k2}+\mathrm{k2}^\nu\,\mathrm{k1}\cdot p+\mathrm{k1}^\nu\,\mathrm{k2}\cdot p+\right.$$
$$\left.\mathrm{k1}^2\,\mathrm{k1}^\nu+\mathrm{m1}^2\,\mathrm{k1}^\nu-2\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}^\nu+\mathrm{k1}^2\,p^\nu-2\,\mathrm{k1}^\nu\,\mathrm{k1}\cdot p+\mathrm{m1}\,\mathrm{m2}\,\mathrm{k2}^\nu-\mathrm{m1}^2\,p^\nu\right)$$

```
preTFI2 = Expand[Contract[preCalcFrac2 * (FVD[(k2), ν])]]
```

$$4\,\mathrm{m1}^2\,\mathrm{k1}\cdot\mathrm{k2}-8\,\mathrm{m1}\,\mathrm{m2}\,\mathrm{k1}\cdot\mathrm{k2}-8\,\mathrm{k1}\cdot\mathrm{k2}\,\mathrm{k1}\cdot p+4\,\mathrm{k2}^2\,\mathrm{k1}\cdot p+$$
$$4\,\mathrm{k1}^2\,\mathrm{k2}\cdot p-4\,\mathrm{k1}^2\,\mathrm{k2}^2+4\,\mathrm{k1}^2\,\mathrm{k1}\cdot\mathrm{k2}-4\,\mathrm{m1}^2\,\mathrm{k2}\cdot p+4\,\mathrm{k2}^2\,\mathrm{m1}\,\mathrm{m2}$$

```
afterTFI2 = ToTFI[preTFI2 * preCalcDenom2, k1, k2, p]
```

$$4\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}-4\,\mathrm{m1}^2\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;1\;\;0}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}-8\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{F}^{(D)\,0\;\;0\;\;0\;\;0\;\;1}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}+$$
$$4\,\mathrm{m1}\,\mathrm{m2}\,\mathbf{F}^{(D)\,0\;\;1\;\;0\;\;0\;\;0}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}-8\,\mathbf{F}^{(D)\,0\;\;0\;\;1\;\;0\;\;1}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}+4\,\mathbf{F}^{(D)\,0\;\;1\;\;1\;\;0\;\;0}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}+$$
$$4\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;0\;\;1}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}+4\,\mathbf{F}^{(D)\,1\;\;0\;\;0\;\;1\;\;0}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}-4\,\mathbf{F}^{(D)\,1\;\;1\;\;0\;\;0\;\;0}_{\{1,\mathrm{m1}\}\{2,0\}\{1,\mathrm{m2}\}\{0,0\}\{1,\mathrm{m1}\}}$$

```
masterIntegrals2 = TarcerRecurse[afterTFI2]
```

$$-\frac{2\,(D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}}{(D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)}+$$
$$\Big(2\left(D^2\,\mathrm{m1}^2-4\,D^2\,\mathrm{m1}\,\mathrm{m2}+D^2\,\mathrm{m2}^2-D^2\,p^2-4\,D\,\mathrm{m1}^2+26\,D\,\mathrm{m1}\,\mathrm{m2}-4\,D\,\mathrm{m2}^2+4\,D\,p^2+4\,\mathrm{m1}^2-40\,\mathrm{m1}\,\mathrm{m2}+4\,\mathrm{m2}^2-4\,p^2\right)\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\Big)\Big/\left((D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)\right)-$$
$$\Big(4\,\mathrm{m1}^2\left(D\,\mathrm{m1}^2-2\,D\,\mathrm{m1}\,\mathrm{m2}-D\,\mathrm{m2}^2-D\,p^2-2\,\mathrm{m1}^2+8\,\mathrm{m1}\,\mathrm{m2}+4\,\mathrm{m2}^2+2\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}\Big)\Big/\left((D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)\right)+$$
$$\Big(4\,\mathrm{m2}^2\left(D\,\mathrm{m1}^2+2\,D\,\mathrm{m1}\,\mathrm{m2}-D\,\mathrm{m2}^2+D\,p^2-4\,\mathrm{m1}^2-8\,\mathrm{m1}\,\mathrm{m2}+2\,\mathrm{m2}^2-2\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\Big)\Big/\left((D-4)\left(\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-p^2\right)\right)$$

```
masterD2 = masterIntegrals2;
```

Summation of gauge dependent pieces.

```
simpU1 = Simplify[masterU1]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}$$
$$2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\right.$$
$$\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\right.$$
$$\left.\left.\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpU2 = Simplify[masterU2]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}$$
$$2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\right.$$
$$\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\right.$$
$$\left.\left.\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpU3 = Simplify[masterU3]
```

$$\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}$$
$$4\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\right.$$
$$\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\right.$$
$$\left.\left.\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpD1 = Simplify[masterD1]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}$$
$$2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\right.$$
$$\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\right.$$
$$\left.\left.\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpD2 = Simplify[masterD2]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}\Big(2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpD3 = Simplify[masterD3]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}\Big(2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpD4 = Simplify[masterD4]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}\Big(2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

```
simpD5 = Simplify[masterD5]
```

$$-\frac{1}{(D-4)\left((\mathrm{m1}+\mathrm{m2})^2-p^2\right)}\Big(2\left((D-2)^2\,\mathbf{A}^{(D)}_{\{1,\mathrm{m1}\}}\,\mathbf{A}^{(D)}_{\{1,\mathrm{m2}\}}+\left(2\left(2\,D^2-13\,D+20\right)\mathrm{m1}\,\mathrm{m2}+(D-2)^2\left(-\mathrm{m1}^2\right)-(D-2)^2\,\mathrm{m2}^2+(D-2)^2\,p^2\right)\mathbf{J}^{(D)}_{\{1,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}+2\left(\mathrm{m1}^2\left((D-2)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}-(D-4)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m1}\}\{1,\mathrm{m2}\}\{1,0\}}+\mathrm{m2}^2\left(-(D-4)\,\mathrm{m1}^2-2\,(D-4)\,\mathrm{m1}\,\mathrm{m2}+(D-2)\,\mathrm{m2}^2-(D-2)\,p^2\right)\mathbf{J}^{(D)}_{\{2,\mathrm{m2}\}\{1,\mathrm{m1}\}\{1,0\}}\right)\right)$$

Test and see if the functions are integer multiples of each other.

```
simpU3 == simpU1 * -2
```

True

```
simpD1 == simpD2 == simpD3 == simpD4 == simpD5
```

True

```
simpD3 == simpU1
```

True

```
colourU1 = colourU2 = -32 / 3
```

$$-\frac{32}{3}$$

```
colourU3 = 16 / 3
```

$$\frac{16}{3}$$

```
colourD1 = colourD2 = colourD4 = colourD5 = 4 / 3
```

$$\frac{4}{3}$$

```
colourD3 = -16 / 3
```

$$-\frac{16}{3}$$

```
coeU1 = coeU2 = -1
```

$$-1$$

```
coeU3 = 1
```

$$1$$

```
coeD1 = coeD2 = coeD4 = coeD5 = -1
```

$$-1$$

```
coeD3 = 1
```

$$1$$

```
total = (coeU1 * colourU1 * α) + (coeU2 * colourU2 * α) +
  (coeU3 * colourU3 * (-2 α)) + (coeD1 * colourD1 * α) + (coeD2 * colourD2 * α) +
  (coeD3 * colourD3 * α) + (coeD4 * colourD4 * α) + (coeD5 * colourD5 * α)
```

$$0$$