Weighted Hypergroups
and some questions in
Abstract Harmonic Analysis

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Abstract

Weighted group algebras have been studied extensively in Abstract Harmonic Analysis.Complete characterizations have been found for some important properties of weighted group algebras, namely, amenability and Arens regularity. Also studies on some other features of these algebras, say weak amenability and isomorphism to operator algebras, have attracted attention.

Hypergroups are generalized versions of locally compact groups. When a discrete group has all its conjugacy classes finite, the set of all conjugacy classes forms a discrete commutative hypergroup. Also the set of equivalence classes of irreducible unitary representations of a compact group forms a discrete commutative hypergroup. Other examples of discrete commutative hypergroups come from families of orthogonal polynomials.

The center of the group algebra of a discrete finite conjugacy (FC) group can be identified with a hypergroup algebra. For a specific class of discrete FC groups, the restricted direct products of finite groups (RDPF), we study some properties of the center of the group algebra including amenability, maximal ideal space, and existence of a bounded approximate identity of maximal ideals.

One of the generalizations of weighted group algebras which may be considered is weighted hypergroup algebras. Defining weighted hypergroups, analogous to weighted groups, we study a variety of examples, features and applications of weighted hypergroup algebras. We investigate some properties of these algebras including: dual Banach algebra structure, Arens regularity, and isomorphism with operator algebras.

We define and study Følner type conditions for hypergroups. We study the relation of the Følner type conditions with other amenability properties of hypergroups. We also demonstrate some results obtained from the Leptin condition for Fourier algebras of certain hypergroups. Highlighting these tools, we specially study the Leptin condition on duals of compact groups for some specific compact groups. An application is given to Segal algebras on compact groups.

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The understanding of *Mathematics* is necessary for a sound grasp of *Ethics*.

Socrates (469 BC–399 BC)

In life it is never a mathematical proposition which we need, but we use mathematical propositions *only* in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics.

(In philosophy the question "Why do we really use that word, that proposition?" constantly leads to valuable results.)

Ludwig Wittgenstein (1889 AD–1951 AD)
Tractatus Logico-Philosophicus 6.211

Contents

Introduction

Roughly speaking, a *hypergroup* is a topological space equipped with an extra structure, which leads to the construction of a Banach algebra on the Banach space of all bounded complex Radon measures on the hypergroup. This binary operation takes the Dirac measures of each two elements of the hypergroup to a compactly supported probability measure and therefore has probabilistic taste, since one may roughly express that the outcome of the action of two elements of a hypergroup is chosen 'randomly'.

A consistent definition of hypergroups was presented in different manuscripts. Dunkl in [22, 21], Spector in [71], and Jewett in [38] defined hypergroups in different ways. Although the ideas are essentially the same, these definitions are not exactly equivalent. The definition which has been widely studied afterwards is Jewett's in [38] wherein he calls hypergroups "convos". Apparently, the term 'hypergroup' first was used for different mathematical objects back in the 1930s, see [56]. Here, we follow Jewett's definition of hypergroups.

Not only were hypergroups defined as a generalization of locally compact groups, but also one may show that some objects related to locally compact groups may be studied as hypergroups. For instance, if G is a FC group (i.e. every conjugacy class is finite), then the set of all conjugacy classes of G, denoted by $\mathrm{Conj}(G)$, forms a commutative discrete hypergroup. Also, for a compact group G, the set of equivalence classes of irreducible unitary representations of G, denoted by $\widehat{G}$ and called the *dual of the group* G, is a commutative discrete hypergroup. On one hand, these examples together with hypergroups which are defined on orthogonal polynomials, connect the studies done on hypergroups to different topics in abstract harmonic analysis. On the other hand, the similarities of hypergroups with groups suggest that one may be able to generalize the studies on locally compact groups to hypergroups. For example, different amenability properties of hypergroups and hypergroup algebras have been studied extensively, [70, 62, 51, 46, 37].

After a brief review of preliminaries in Chapter 1, in Chapter 2, we want to know more about center of group algebras for FC groups. It is known that the center of the group algebra, $Z\ell^1(G)$, for an FC group G, is amenable if G', the derived subgroup of G, is finite (see [5]). In Section 2.2, for a specific class of FC groups, called *restricted direct products of finite groups*, we show that the other side is held. Let $\{G_i\}_{i\in\mathbf{I}}$ be a family of finite groups. Then

$$G := \{(x_i)_{i\in\mathbf{I}} : \ x_i = e_{G_i} \text{ for all except finitely many } i \in \mathbf{I}\ \}$$

is called the restricted direct product of $\{G_i\}_{i\in\mathbf{I}}$. In this section, we study various properties of $Z\ell^1(G)$, such as amenability and its Gelfand spectrum. We show that $Z\ell^1(G)$ is amenable if and only if G_i is abelian for all but finitely many i which proves the aforementioned conjecture for RDPF groups. Moreover, we characterize maximal ideals of $Z\ell^1(G)$ with bounded approximate identities. This section is based on a joint work with Professor Yemon Choi and Professor Ebrahim Samei.

In Chapter 3, we study three examples of hypergroups. First in Section 3.1, we introduce $\mathrm{Conj}(G)$ the set of all conjugacy classes of an FC group G as a hypergroup and characterize its hypergroup algebra. Features of duals of compact groups, as hypergroups, have strong relations to the properties of their corresponding compact groups. In Sections 3.2, we check the hypergroup definition for dual of compact groups and perform some observations on them. Eventually, we close this chapter by a polynomial hypergroup structure on $\mathbb{N}_0$ in Section 3.3. This class of hypergroups has been of interest for many studies on discrete hypergroups namely [27, 37, 47, 46, 50, 51].

One of the topics related to hypergroups which has been initiated based on a similar study on locally compact groups is “weighted hypergroups" and “weighted hypergroup algebras". One may note that, for the specific weight $\omega \equiv 1$, the weighted case is reduced back to regular hypergroups and their algebras. The first studies over weighted hypergroup algebras may be tracked back to [7, 32, 33]. Chapter 4 is devoted to weights on discrete hypergroups, their corresponding algebras, and their examples. In Section 4.1, we study weighted hypergroup algebras, $\ell^1(H,\omega)$, for commutative discrete hypergroups H and hypergroups weights ω. Subsequently, in Section 4.2, we introduce some weights which are related to the growing rate of finitely generated hypergroups.

To emphasize the importance of weighted hypergroup algebras in abstract harmonic analysis, we continue by studying some well-known Banach algebras on groups which are isomorphic to some weighted hypergroup algebras. First we study weights and their properties on $\mathrm{Conj}(G)$, as a hypergroup, for FC groups G. As examples of these weights, if (G,σ) is a weighted discrete FC group for some group weight σ, then $Z\ell^1(G,\sigma)$, the center of σ-weighted group algebra, is shown in Section 4.3 to be isometrically algebraic isomorphic to $\ell^1(\mathrm{Conj}(G),\omega_\sigma)$ for some hypergroup weight ω_σ which is generated using σ. We will introduce and study more examples of hypergroup weights on $\mathrm{Conj}(G)$ in Sections 4.4 and 4.5. Finally, we close the chapter by Section 4.6 in which we introduce and study some hypergroup weights on dual of compact groups.

The Fourier algebra of a general locally compact group was first studied by Eymard, [26]. There are several papers which defined Fourier space on hypergroups, as a Banach space, [78, 4, 60, 45]. But unfortunately, none of them could show that the Fourier space defined in the aforementioned references actually forms an algebra, unlike the group case. So although the definitions are mainly similar, strategies to study the hypergroups for which the Fourier space is an algebra are different. In [60], Muruganandam defined the Fourier space of a hypergroup H, as a Banach space denoted by $A(H)$, analogous to the Fourier algebra of groups. Applying some tools from character theory of hypergroups, he studied the pointwise multiplication of elements of $A(H)$ and the behaviour of the norm with respect to this multiplication. He could develop a machinery to study hypergroups whose Fourier space is a Banach algebra. He called a hypergroup H a *regular Fourier hypergroup* if $A(H)$ equipped with pointwise multiplication is a Banach algebra. Muruganandam also recognized a variety of regular Fourier hypergroups in [60, 61]. He proved that some polynomial hypergroups, double coset hypergroups, and the space of all orbits in a locally compact group G for some relatively compact subgroup of automorphisms of G including inner ones are regular Fourier hypergroups. Chapter 5 studies Fourier algebra of hypergroups. After studying some general properties of Fourier algebra of hypergroups in Section 5.1, in Section 5.2, we add the dual of compact groups to the list of regular Fourier hypergroups. Furthermore, we show that if G is a compact group, the Fourier algebra of $\widehat{G}$, $A(\widehat{G})$, is isometrically isomorphic to the center of the group algebra of G, $ZL^1(G)$. Furthermore, we prove that $ZA(G) := A(G) \cap L^1(G)$, as a subalgebra of $A(G)$, is isometrically isomorphic to the Banach algebra $\ell^1(\widehat{G})$.

In [29], it was proved that the Fourier algebra of a compact group G is weakly amenable if and only if the connected component of the identity, denoted by G_e, is abelian. We study the question of weak amenability for $ZA(G)$ of a compact group G in Section 5.3. Here, by constructing one non-zero bounded derivation on $ZA(SU(2))$ and similarly $ZA(SO(3))$, we prove the existence of a non-zero bounded derivation on $ZA(G)$ for every compact group G when G_e is not abelian; $ZA(G)$ for this class of compact groups is not weakly amenable.

Chapter 6, is an attempt to re-create some amenability features of locally compact groups for hypergroups and their relations with the Fourier algebra on regular Fourier hypergroups. We develop some definitions and observe some of their examples and applications in harmonic analysis.

For a locally compact group G, the *Leptin condition* was defined to characterize the existence of a bounded approximate identity of the Fourier algebra, $A(G)$; meanwhile, it is equivalent to the amenability of the group G. In an attempt to develop a similar machinery for hypergroups, we introduce a modified version of the Leptin condition for hypergroups called *D-Leptin condition* for some $D \geq 1$, in Subsection 6.1.1. This definition for $D = 1$ corresponds previous definitions of Leptin condition for locally compact groups in [54] as well as polynomial hypergroups in [37]. In Subsection 6.1.2, we show that the D-Leptin condition implies the existence of a bounded approximate identity of the Fourier algebra for regular Fourier hypergroups. Furthermore, the D-Leptin condition results some other amenability properties of hypergroups introduced and studied in [70], see Subsection 6.1.3. Furthermore, we study the D-Leptin condition of some Lie groups. As a result, the dual of $SU(2)$, the special unitary group of 2×2 matrices, satisfies the 1-Leptin condition. Further, based on some studies on representation theory of $SU(3)$, the special unitary group of 3×3 matrices, we show that the 3^8-Leptin condition is satisfied by the hypergroup of the dual of $SU(3)$. Also for every connected simply connected compact real Lie group $\mathbb{G}$, the hypergroup $\widehat{\mathbb{G}}$ satisfies the D-Leptin condition for some $D \geq 1$, as it is shown in Section 6.2.

Approximate amenability of a Banach algebra was defined in [31]. A Banach algebra $\mathcal{A}$ is said to be approximately amenable if every bounded derivation from $\mathcal{A}$ into the dual of any $\mathcal{A}$-bimodule can be approximated by a net of inner derivations. Reiter established classical *Segal algebras* in his monograph [64]. A Segal algebra $S^1(G)$ on a locally compact group G is a dense left ideal of $L^1(G)$ that satisfies some extra conditions. For example, the elements of $L^1(G)$ act on the Segal algebra as bounded multipliers. Approximate amenability of Segal algebras has been studied in several papers. Dales and Loy, in [18], studied approximate amenability of Segal algebras on the torus $\mathbb{T}$ and the group of real numbers $\mathbb{R}$. They showed that certain Segal algebras on $\mathbb{T}$ and $\mathbb{R}$ are not approximately amenable. It was further conjectured that no proper Segal algebra on $\mathbb{T}$ is approximately amenable. Choi and Ghahramani, in [14], have shown the stronger fact that no proper Segal algebra on $\mathbb{T}^d$ or $\mathbb{R}^d$ is approximately amenable. In Subsection 6.2.3, applying the D-Leptin condition of hypergroups, we study the approximate amenability of Segal algebras of compact groups. We prove that for every compact group G whose dual satisfies the D-Leptin condition for some $D \geq 1$, every proper Segal algebra is not approximately amenable. A version of this subsection has been published as a part of [2].

In Chapter 7, we study some properties of weighted hypergroup algebras, including Arens regularity and isomorphism with operator algebras. We also study these features for some of examples of weighted hypergroups introduced in the previous chapters.

Arens regularity of weighted group algebras has been studied by Craw and Young in [16]. They showed that a locally compact group G has a weight ω such that $L^1(G, \omega)$ is Arens regular if and only if G is discrete and countable. They also characterized the Arens regularity of weighted group algebras with respect to one feature of the weight, called 0-*clusterness* as described in [17]. In Section 7.1, the Arens regularity of weighted hypergroup algebras for discrete hypergroups is studied and it is shown that strong 0-clusterness of the corresponding hypergroup weight results in the Arens regularity of the weighted hypergroup algebra (strong 0-clusterness implies 0-clusterness, [17]).

A Banach algebra $\mathcal{A}$ is called an *operator algebra* if there is a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is a closed subalgebra of $\mathcal{B}(\mathcal{H})$. For a Banach algebra, one may ask about the existence of an algebra isomorphism from the algebra onto an operator algebra. Isomorphism of weighted group algebras to operator algebras has been studied before, see [52, 76]. In Section 7.2, studying the hypergroup case, we demonstrate that for hypergroup weights which are weakly additive and whose inverse is 2-summable over the hypergroup, an isomorphism to an operator algebra exists.

Chapter 1

Preliminaries

1.1 Hypergroups

To define hypergroups, we need to present the definition of the *Michael topology* as follows.

Definition 1.1.1. [8, 1.1.1]
Let $\mathfrak{C}(X)$ denote the space of nonvoid compact subsets of some locally compact space X. For $A, B \subset X$, set $\mathfrak{C}_A(B) := \{C \in \mathfrak{C}(X) : C \cap A \neq \emptyset \text{ and } C \subset B\}$. Then $\mathfrak{C}(X)$ is given the topology which is generated by the subbasis of all $\mathfrak{C}_U(V)$ for all U and V open subsets of X. Then $\mathfrak{C}(X)$ is a locally compact Hausdorff space. Moreover, if Ω is a compact subset of $\mathfrak{C}(X)$ then $B = \bigcup\{A : A \in \Omega\}$ is a compact subset of X.

For a locally compact space X, we use $M(X)$ to denote the Banach space of all bounded complex Radon measures on X. Recall that $M(X)$ can be identified as the dual of $C_0(X)$, the C^*-algebra of all continuous functions vanishing at infinity. For each $x \in X$, δ_x denotes the *Dirac measure* at x i.e. $\delta_x(f) = f(x)$ for each $f \in C_0(X)$. We denote the C^*-algebra of bounded continuous complex valued functions on X by $C(X)$. Also, $C_c(X)$ denotes the space of all compactly supported elements of $C(X)$ which is dense in $C_0(X)$.

Definition 1.1.2. [8, 1.1.2]
We call a locally compact space H a *hypergroup* if the following conditions hold.

(H1) There exists an associative binary operation $*$ called *convolution* on $M(H)$ under which $M(H)$ is an algebra. Moreover, for every x, y in H, $\delta_x * \delta_y$ is a positive measure with compact support and $\|\delta_x * \delta_y\|_{M(H)} = 1$.

(H2) The mapping $(x, y) \mapsto \delta_x * \delta_y$ is a continuous map from $H \times H$ into $M(H)$ equipped with the weak* topology that is $\sigma(M(H), C_c(H))$ where each $\mu \in M(H)$ is considered as a functional on $C_c(H)$ that is $\mu(f) := \int_H f d\mu$ for any $f \in C_c(H)$.

(H3) The mapping $(x,y) \to \operatorname{supp}(\delta_x * \delta_y)$ is a continuous mapping from $H \times H$ into $\mathfrak{C}(H)$ equipped with the Michael topology.

(H4) There exists an element (necessarily unique) e in H such that

$$\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$$

for all x in H.

(H5) There exists a (necessarily unique) homeomorphism $x \to \check{x}$ of H called *involution* satisfying the following:

(i) $(\check{x})\check{} = x$ for all $x \in H$.

(ii) If $\check{f}$ is defined by $\check{f}(t) := f(\check{t})$ for all $f \in C_c(H)$ and $t \in H$, one may define $\check{\mu}(f) := \mu(\check{f})$ for all $\mu \in M(H)$. Then

$$(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}} \quad \text{for all } x, y \in H.$$

(H6) e belongs to $\operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Remark 1.1.3. Since here we mainly work with discrete hypergroups, we may notice that for discrete hypergroup, two continuity conditions (H2) and (H3) are automatically satisfied.

Applying the convolution of $M(H)$, one may define an action between subsets of H. We denote it by $*$ again where the notation $A * B$ stands for

$$\bigcup\{\operatorname{supp}(\delta_x * \delta_y) : \text{ for all } x \in A \text{ and } y \in B\} \tag{1.1.1}$$

for A, B subsets of the hypergroup H. With abuse of notation, we use $x * A$ and $x * y$ to denote $\{x\} * A$ and $\{x\} * \{y\}$, respectively.

We call a hypergroup H *commutative* if $M(H)$ forms a commutative algebra. To facilitate the notation, for each pair $x, y \in H$ and $f \in C_c(H)$, the value of the measure of $\delta_x * \delta_y$ on f is denoted by $f(\delta_x * \delta_y)$. As mentioned in [8, 1.1.2], for each pair $\mu, \nu \in M(H)$ and $f \in C_c(H)$,

$$\mu * \nu(f) = \int_H \int_H f(\delta_x * \delta_y) d\mu(x) d\nu(y).$$

We can define a *left translation* on $C(H)$ by

$$L_x f : H \to \mathbb{C}, \quad L_x f(y) = f(\delta_x * \delta_y)$$

for each f in $C(H)$ and $x, y \in H$. Note that $L_x f \in C_c(H)$ for $f \in C_c(H)$. Similar to the group case a non-zero, positive, left invariant linear functional h (possibly unbounded) on $C_c(H)$ is called a *Haar measure* i.e. $h(L_x f) = h(f)$ for all $f \in C_c(H)$ and $x \in H$. Note that h is a Radon measure which satisfies the subinvariant translation on measurable sets i.e. $h(K) \le h(x * K)$ for each compact set $K \subseteq H$ and $x \in H$. The Haar measure is unique up to multiplication by a positive constant, [8].

Unlike the theory of locally compact groups, the existence of a Haar measure on hypergroups is not proven for general hypergroups[1]. But for specific cases of hypergroups including commutative hypergroups, discrete hypergroups, and compact hypergroups, always a Haar measure exists, [8, Section 1.3]. If there exists a Haar measure on a hypergroup H, it is unique up to a constant multiplier.

Theorem 1.1.4. *[8, Theorem 1.3.26]*
Let H be a discrete hypergroup. Then there exists a Haar measure $h : H \to (0, \infty)$. If we assume that $h(e) = 1$, $h(x) = (\delta_{\check{x}} * \delta_x(e))^{-1}$ for all $x \in H$.

Note that unlike groups, the Haar measure on discrete hypergroups is not necessarily a fixed multiplier of the counting measure. As an instance, one may look at the Haar measure on $\widehat{SU(2)}$ (see Example 3.2.2).

Since most of the hypergroups that we work with are discrete or commutative or both, from now on, we assume that a hypergroup H possesses a Haar measure. In this case, for each $1 \le p < \infty$, we define $L^p(H, h)$ (sometimes denoted by $L^p(H)$ if there is no risk of confusion) to be the Banach space of p-integrable functions on H with respect to the Haar measure h; hence,

$$\|f\|_p := \left(\int_H |f(x)|^p dh(x) \right)^{\frac{1}{p}} < \infty.$$

Furthermore, for each $f, g \in C_c(H)$ and $y \in H$, let us define

$$f *_h g(y) := \int_H f(x) g(\delta_{\check{x}} * \delta_y) dh(x) \quad \tilde{f}(x) := \overline{f(\check{x})}.$$

One may extend $*_h$ and $\sim$ to $L^1(H, h)$. $L^1(H, h)$ equipped with the convolution $*_h$ forms a Banach algebra, [8, Section 1.4]. To facilitate writing, we may use dx for integration with respect to the Haar measure i.e. $dh(x)$.

[1]Revising very last drafts of the thesis, Professor Yemon Choi directed me to the recent manuscript [11]. In that, it has been claimed that the existence of a left invariant measure on an arbitrary hypergroup is proven.

Let H be a discrete hypergroup. As we mentioned before, $\ell^1(H) = M(H)$ is a Banach algebra. One may easily show that in discrete case, for each pair $f, g \in \ell^1(H)$, we have

$$\|f\|_1 := \sum_{x\in H} |f(x)| \quad , \quad f * g(x) := \sum_{t\in H}\sum_{s\in H} \delta_t * \delta_s(x) f(t) g(s) \quad (x \in H). \tag{1.1.2}$$

On the other hand, for a discrete hypergroup H equipped with the Haar measure h, one gets

$$\|f\|_{L^1(H,h)} := \sum_{t\in H} |f(t)| h(t) \text{ and } f *_h g(x) := \sum_{t\in H} f(t) L_t g(x) h(t).$$

Proposition 1.1.5. *[48, Theorem 1.8]*
The map $f \to fh$, $L^1(H,h) \to \ell^1(H)$ is an isometric algebra isomorphism from the Banach algebra $L^1(H,h)$ onto the Banach algebra $\ell^1(H)$.

Lemma 1.1.6. *Let H be a discrete hypergroup equipped with a Haar measure h. For each pair $x, y \in H$,*

$$\delta_x *_h \delta_y(z) = \delta_x * \delta_y(z) \frac{h(x)h(y)}{h(z)} \quad (z \in H).$$

Proof. Let Θ be the inverse of the isomorphism defined in Proposition 1.1.5 i.e. $\Theta(f)(x) = h(x)^{-1} f(x)$ and $x, y \in H$ be arbitrary. For each $z \in H$, we have

$$\frac{1}{h(z)} \delta_x * \delta_y(z) = \Theta(\delta_x * \delta_y)(z) = \Theta(\delta_x) *_h \Theta(\delta_y)(z) = \frac{1}{h(x)h(y)} \delta_x *_h \delta_y(z).$$

□

The following proposition is a discrete case of [8, Proposition 1.2.16].

Proposition 1.1.7. *Let H be a discrete hypergroup. Then for every $\phi \in c_0(H)$ and $f \in \ell^1(H)$, the function*

$$x \mapsto \sum_{t\in H} f(t) \phi(\delta_{\tilde{t}} * \delta_x)$$

belongs to $c_0(H)$.

Proof. Let $\epsilon > 0$ be fixed. Therefore there is some $K \subset H$ finite such that for every $x \in H \setminus K$, $|\phi(x)| < \epsilon \|f\|_1^{-1}$. Also there is some $F \subseteq H$ finite such that

$$\sum_{x\in H\setminus F} |f(x)| < \epsilon \|\phi\|_\infty^{-1}.$$

Based on the definition of convolution between sets and (H1) in Definition 1.1.2, it is obvious that $C := F * K$ is a finite subset of H.

Let $x \in H \setminus C$, $t \in F$, and $s \in K$. If $\delta_{\tilde{t}} * \delta_x(s) \neq 0$, $s \in \tilde{t} * x$. Therefore, by (H6), $e \in \tilde{s} * \tilde{t} * x$. Again (H6) implies that $\tilde{x} \in \tilde{s} * \tilde{t}$ or equivalently $x \in t * s \subseteq F * K$ which is a contradiction. Hence, for $x \in H \setminus C$, $t \in F$, and $s \in K$, $\delta_{\tilde{t}} * \delta_x(s) = 0$. Consequently,

$$\sum_{t \in F} |f(t)| \sum_{s \in K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) = 0.$$

Therefore for $x \in H \setminus C$, one gets

$$\begin{aligned} \left| \sum_{t \in H} f(t)\phi(\delta_{\tilde{t}} * \delta_x) \right| &\leq \left| \sum_{t \in F} f(t)\phi(\delta_{\tilde{t}} * \delta_x) \right| + \left| \sum_{t \in H \setminus F} f(t)\phi(\delta_{\tilde{t}} * \delta_x) \right| \\ &\leq \sum_{t \in F} |f(t)||\phi(\delta_{\tilde{t}} * \delta_x)| + \sum_{t \in H \setminus F} |f(t)|\|\phi\|_\infty \\ &\leq \epsilon + \sum_{t \in F} |f(t)| \sum_{s \in H} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) \\ &= \epsilon + \sum_{t \in F} |f(t)| \sum_{s \in H \setminus K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) + \sum_{t \in F} |f(t)| \sum_{s \in K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) \\ &\leq \epsilon + \sup_{s \in H \setminus K} |\phi(s)|\|f\|_1 + \sum_{t \in F} |f(t)| \sum_{s \in K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) = 2\epsilon. \end{aligned}$$

And this finishes the proof. □

For two discrete hypergroups H_1 and H_2, $H := H_1 \times H_2$ forms a discrete hypergroup where

$$\delta_{(x_1, x_2)} *_H \delta_{(y_1, y_2)}(s, t) := \delta_{x_1} *_{H_1} \delta_{y_1}(s)\, \delta_{x_2} *_{H_2} \delta_{y_2}(t)$$

for all $x_1, y_1, s \in H_1$ and $x_2, y_2, t \in H_2$. As an extension of the previous product of hypergroups, let $\{H_i\}_{i \in \mathbf{I}}$ be a family of discrete hypergroups, then $H := \bigoplus_{i \in \mathbf{I}} H_i$ where for each $x \in H$, $x = (x_i)_{i \in \mathbf{I}}$ where x_i is the identity of the hypergroup H_i, e_{H_i}, for all $i \in \mathbf{I}$ except finitely many. H is called *restricted direct product* of $(H_i)_{i \in \mathbf{I}}$.

A hypergroup H is said to be *amenable* if there exists a left invariant positive linear functional of norm 1 on $C(H)$. Amenability of hypergroups has been studied widely in [70]. Skantharajah, in [70], showed that similar to amenable groups, all compact or commutative hypergroups are amenable. But unlike group case, the amenability of hypergroups does not necessarily imply the amenability of the hypergroup algebra as an algebra (defined in the following). The converse is always true i.e. the amenability of a hypergroup algebra (as a Banach algebra) implies the amenability of the corresponding hypergroup, [70, Proposition 4.9].

Let H be a commutative hypergroup equipped with a Haar measure h. Define

$$\widehat{H} := \{\alpha \in C_b(H) : \ \alpha(\delta_x * \delta_y) = \alpha(x)\alpha(y), \ \alpha(\check{x}) = \overline{\alpha(x)}, \text{ and } \alpha \neq 0\}. \tag{1.1.3}$$

Let us give $\widehat{H}$ the topology of uniform convergence on compact subsets of H. Every $\alpha \in \widehat{H}$ is called a *character* of H and the topological space $\widehat{H}$ is called the *dual* of the hypergroup H.

For each $\mu \in M(H)$, one may define the *Fourier-Stieltjes transform* $\widehat{\mu}$ (or $\mathcal{F}(\mu)$) on $\widehat{H}$ by

$$\widehat{\mu}(\alpha) := \int_H \overline{\alpha}(x) d\mu(x) \quad (\alpha \in \widehat{H}).$$

For $f \in L^1(H)$ this gives the *Fourier transform*

$$\widehat{f}(\alpha) = \int_H \overline{\alpha(x)} f(x) dx,$$

which is also denoted by $\mathcal{F}(f)$, see [8, Definition 2.2.3]. In the following we just summarize some of the main properties of $\widehat{H}$ from [8, Chapter 2].

Theorem 1.1.8. *[8, Theorem 2.2.4]*
Let H be a commutative hypergroup.

(1) Since the constant function 1 belongs to $\widehat{H}$, it is non empty.

(2) $\widehat{H}$ is a locally compact topological space.

(3) The Fourier-Stieltjes transform is a norm-decreasing linear mapping from $M(H)$ into $C_b(\widehat{H})$.

(4) The Fourier transform is a norm-decreasing linear mapping from $L^1(H)$ into $C_0(\widehat{H})$. Furthermore, $\mathcal{F}(L^1(H))$ is a dense subalgebra of $(C_0(\widehat{H}), \|\cdot\|_\infty)$.

Theorem 1.1.9. *[8, Theorem 2.2.13]*
Let H be a commutative hypergroup. Then there exists a non-negative measure π on $\widehat{H}$, called Plancherel measure *of $\widehat{H}$ such that*

$$\int_H |f(x)|^2 dx = \int_{\widehat{H}} |\widehat{f}(\alpha)|^2 d\pi(\alpha)$$

for all $f \in L^1(H) \cap L^2(H)$.

Note that for an arbitrary hypergroup H (unlike group case) the support of the Plancherel measure, $\text{supp}(\pi)$, may not be equal to $\widehat{H}$.

1.2 Group algebra and its center

Let G be a locally compact group which is a hypergroup that possesses a Haar measure λ such that $\lambda(xE) = \lambda(E)$ for each measurable set $E \subseteq G$. From now on, we use dx to denote $d\lambda(x)$ in our integrations over a group G.

For a group G, the *derived subgroup of* G (also called the *commutator subgroup of* G) is the closed normal subgroup of G generated by the set of all commutators of elements in G. We denote the derived subgroup of G by G'.

We denote the hypergroup algebra of G by $L^1(G)$ and call it the *group algebra* of G. Note that $L^1(G)$ is commutative if and only if G is a commutative group.

Definition 1.2.1. Let G be a locally compact group. The center of the group algebra is the subalgebra of $L^1(G)$ consisting of all elements which commute with all elements of the group algebra and is denoted by $ZL^1(G)$.

Theorem 1.2.2. *Let G be a locally compact group. $ZL^1(G)$ is the set of all elements of $L^1(G)$ which are almost everywhere constant on conjugacy classes of the group G i.e. $f(yxy^{-1}) = f(x)$ for almost all $x, y \in G$.*

For a proof, one may look at [57]. Liukkonen and Mosak in their paper, [57], studied some of the properties of $ZL^1(G)$. Namely they showed that $ZL^1(G)$ is a regular, Tauberian, symmetric Banach $*$-algebra and contains a bounded approximate identity. In [5], Azimifard, Samei, and Spronk studied some amenability properties of the center of group algebras for compact and finite groups.

1.3 Banach algebras

1.3.1 Characters of commutative Banach algebras

Let $\mathcal{A}$ be a commutative Banach algebra. We denote by $\sigma(\mathcal{A})$ the *(Gelfand) spectrum* of $\mathcal{A}$ which is the set of all non-zero multiplicative linear functionals on $\mathcal{A}$; that is also called the *maximal ideal space* or *character space* of $\mathcal{A}$. For each $\psi \in \sigma(\mathcal{A})$, ψ is called a *character* on the algebra $\mathcal{A}$.

1.3.2 Injective and projective tensor products

We use [69] as our reference of this subsection. Let X and Y be two linear spaces. There exists a linear space $X \otimes Y$, called the *tensor product* of X and Y and a canonical bilinear map $\varphi : X \times Y \to X \otimes Y$ with the following universal property. For each linear space E and an arbitrary bilinear map $B : X \times Y \to E$ there exists one and only one linear map $\tilde{B}$ such that

$B = \tilde{B} \circ \varphi$. A typical element u in $X \otimes Y$ can be represented in the form of

$$u = \sum_{i=1}^{n} x_i \otimes y_i \tag{1.3.1}$$

for $x_i \in X$ and $y_i \in Y$ and $x_i \otimes y_i = \varphi(x_i, y_i)$. Note that this representation for each element may not be unique. If X and Y are Banach spaces, we may apply their norms to norm $X \otimes Y$ and even complete it to a Banach space. In this case, a norm $\|\cdot\|$ on $X \otimes Y$ is called *cross norm* if $\|x \otimes y\| = \|x\|_X \|y\|_Y$ for all $x \in X$ and $y \in Y$. For each $u \in X \otimes Y$, in the form of (1.3.1), let

$$\|u\|_\gamma := \inf \left\{ \sum_{i=1}^{n} \|x_i\|_X \|y_i\|_Y \right\} \tag{1.3.2}$$

where the infimum is taken over all representations of u. This norm is called *projective tensor norm* and is the largest possible cross norm defined on $X \otimes Y$. The completion of $X \otimes Y$ with respect to $\|\cdot\|_\gamma$ is called *projective tensor product* of X and Y and denoted by $X \otimes_\gamma Y$. One may show that the dual of $X \otimes_\gamma Y$, as a Banach space, is isometrically isomorphic to $\mathcal{L}(X, Y^*)$, the space of all bounded operators from X into Y^*.

If $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras, there is a product on $\mathcal{A} \otimes \mathcal{B}$ which makes it an algebra such that $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. The projective norm on $\mathcal{A} \otimes \mathcal{B}$ is an algebra norm; hence, $\mathcal{A} \otimes_\gamma \mathcal{B}$ is a Banach algebra.

Let (S_1, μ_1) and (S_2, μ_2) be two measured spaces, then $L^1(S_1, \mu_1) \otimes_\gamma L^1(S_2, \mu_2)$, as a Banach space, is isometrically isomorphic to $L^1(S_1 \times S_2, \mu_1 \times \mu_2)$. Let H_1 and H_2 be two hypergroups. Then there exists an isometric isomorphism θ from the Banach algebra $L^1(H_1) \otimes_\gamma L^1(H_2)$ onto $L^1(H_1 \times H_2)$ such that

$$\theta(f \otimes g)(x, y) = f(x) g(y)$$

for all $f \in L^1(H_1)$, $g \in L^1(H_2)$, and almost all $x \in H_1$ and $y \in H_2$. A proof would be exactly similar to the group case, (see [43, Proposition 1.5.5]).

Moreover, for Banach spaces X and Y and $u \in X \otimes Y$,

$$\|u\|_\epsilon := \sup \left\{ \sum_{i=1}^{n} \psi(x_i)\phi(y_i) : \psi \in X^* \text{ and } \phi \in Y^* \text{ such that } \|\psi\| \le 1 \text{ and } \|\phi\| \le 1 \right\},$$

forms another norm called *injective tensor norm* which is the least cross norm one may define on $X \otimes Y$. The completion of $X \otimes Y$ with respect to the injective norm is called *injective tensor product* of X and Y and denoted by $X \otimes_\epsilon Y$.

Remark 1.3.1. For each two Banach spaces X and Y, the projective tensor norm or injective tensor norm are *cross norms.*

1.3.3 Amenability of Banach algebras

For amenability of Banach algebras, we use [68] as the main reference. The proof of the following results can be found there.

If $\mathcal{A}$ is a Banach algebra, a Banach space X is called a Banach $\mathcal{A}$-bimodule if there are bounded maps, a homomorphism $\mathcal{A} \to B(X) : a \mapsto (x \mapsto a \cdot x)$ and an anti-homomorphism $\mathcal{A} \to B(X) : a \mapsto (x \mapsto x \cdot a)$, with commuting ranges. A Banach $\mathcal{A}$-bimodule X is called symmetric if the left and right module actions coincide i.e. $a \cdot x = x \cdot a$ for all $x \in X$ and $a \in \mathcal{A}$. The adjoints of these actions make the dual space X^* into a dual Banach $\mathcal{A}$-bimodule.

A linear map $D : \mathcal{A} \to X$ is called a *derivation* if $D(ab) = a \cdot D(b) + D(a) \cdot b$ for a, b in $\mathcal{A}$. Inner derivations are those of the form $D(a) = a \cdot x - x \cdot a$ for some x in X. A Banach algebra $\mathcal{A}$ is *amenable* if, for every dual Banach $\mathcal{A}$-bimodule X^*, every bounded derivation $D : \mathcal{A} \to X^*$ is inner.

Note that, canonical actions of $\mathcal{A}$ into its dual make $\mathcal{A}^*$ into a dual Banach $\mathcal{A}$-bimodule. A Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation $D : \mathcal{A} \to \mathcal{A}^*$ is inner. If $\mathcal{A}$ is commutative, the weak amenability is equivalent to this fact that every bounded derivation $D : \mathcal{A} \to X$ for a symmetric Banach $\mathcal{A}$-bimodule X is constantly 0. Let $\phi \in \mathcal{A}^*$ be a character i.e. ϕ is non-zero and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$. A non-zero linear functional $d_\phi \in \mathcal{A}^*$ such that $d_\phi(ab) = \phi(a)d_\phi(b) + d_\phi(a)\phi(b)$ is called a *point derivation*. If a non-zero point derivation ϕ exists, the derivation $D : \mathcal{A} \to \mathcal{A}^*$ which is defined as $D(a) := d_\phi(a)\phi$ implies that $\mathcal{A}$ cannot be weakly amenable. If a Banach algebra $\mathcal{A}$ is amenable, it is clearly weakly amenable as well.

Let $\mathcal{A}$ be a Banach algebra. Then the Banach algebra $\mathcal{A} \otimes_\gamma \mathcal{A}$ forms a Banach $\mathcal{A}$-bimodule where

$$a \cdot (b \otimes c) := (ab) \otimes c \text{ and } (b \otimes c) \cdot a = b \otimes (ca)$$

for all $a, b, c \in \mathcal{A}$. Moreover, the mapping $\mathbf{m} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, where $\mathbf{m}(a, b) = ab$, has a continuous extension from $\mathcal{A} \otimes_\gamma \mathcal{A}$ into $\mathcal{A}$ which we denote by $\mathbf{m}$ again.

For a Banach algebra $\mathcal{A}$ there are a variety of conditions which equal the amenability of the Banach algebra. For example, $\mathcal{A}$ is amenable if and only if there is a norm bounded net $(m_\alpha)_\alpha$ in $\mathcal{A} \otimes_\gamma \mathcal{A}$ such that

(1) $\lim_\alpha a \cdot m_\alpha - m_\alpha \cdot a = 0$.

(2) $\lim_\alpha \|\mathbf{m}(m_\alpha)a - a\|_{\mathcal{A}} = 0$ and $\lim_\alpha \|a\mathbf{m}(m_\alpha) - a\|_{\mathcal{A}} = 0$.

Such a bounded net $(m_\alpha)_\alpha$ is called a *bounded approximate diagonal* of $\mathcal{A}$.

Note that the second adjoint $\mathbf{m}$ is a mapping $\mathbf{m}^{**} : (\mathcal{A} \otimes_\gamma \mathcal{A})^{**} \to \mathcal{A}^{**}$. Moreover, $\mathcal{A}^{**}$ and $(\mathcal{A} \otimes_\gamma \mathcal{A})^{**}$ are also $\mathcal{A}$-bimodules in canonical ways. An element $M \in (\mathcal{A} \otimes_\gamma \mathcal{A})^{**}$ is called a *virtual diagonal* for $\mathcal{A}$ if

$$a \cdot M = M \cdot a \text{ and } a \cdot \mathbf{m}^{**}(M) = \mathbf{m}^{**}(M) \cdot a = a.$$

The existence of a virtual diagonal also equals the amenability of $\mathcal{A}$.

The concept of *amenability constant* was developed by Johnson, in [39], as a tool to study the amenability of Fourier algebras. Roughly speaking, we can measure amenability of a Banach algebra via amenability constant.

Definition 1.3.2. For a Banach algebra $\mathcal{A}$, we denote the *amenability constant* of $\mathcal{A}$ by $\mathrm{AM}(\mathcal{A})$ and define it to be

$$\inf \left\{ \sup_\alpha \|m_\alpha\|_{\mathcal{A} \otimes_\gamma \mathcal{A}} \right\}$$

where the infimum is taken over all bounded approximate diagonals $(m_\alpha)_\alpha$ of $\mathcal{A}$. If the set of bounded approximate diagonals of $\mathcal{A}$ is empty, $\mathrm{AM}(\mathcal{A})$ is set to be $+\infty$.

For a Banach algebra $\mathcal{A}$, $\mathrm{AM}(\mathcal{A}) < \infty$ if and only if $\mathcal{A}$ is amenable. If $\mathcal{A}$ is a unital Banach algebra, $\mathrm{AM}(\mathcal{A}) \geq 1$.

Remark 1.3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism with dense range. It is well known that if $\mathcal{A}$ is amenable then so is $\mathcal{B}$; moreover, $\mathrm{AM}(\mathcal{B}) \leq \|\phi\|^2 \mathrm{AM}(\mathcal{A})$. Toward a proof, one may note that for every bounded approximate diagonal $(m_\alpha)_\alpha$ of $\mathcal{A}$, $(\phi \otimes \phi(m_\alpha))_\alpha$ is a bounded approximate diagonal of $\mathcal{B}$.

Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are two amenable Banach algebras, the Banach algebra $\mathcal{A} \otimes_\gamma \mathcal{B}$ is also amenable, by [40, Proposition 5.4].

Note that for every amenable Banach algebra $\mathcal{A}$, it has a bounded approximate identity. Let $\mathcal{A}$ be an amenable Banach algebra, and let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ with finite dimension or codimension. Then $\mathcal{I}$ is amenable. Specially for an amenable commutative Banach algebra $\mathcal{A}$, $\mathrm{Ker}(\psi)$ for all $\psi \in \sigma(\mathcal{A})$ is amenable and consequently has a bounded approximate identity.

1.4 More on locally compact groups

1.4.1 Representation theory of compact groups

For a locally compact group G, a *unitary representation* π from G into $\mathcal{U}(\mathcal{H}_\pi)$, the group of all unitary operators on a Hilbert space $\mathcal{H}_\pi$, is a group homomorphism which is continuous with respect to the topology of G and the strong operator topology on $\mathcal{U}(\mathcal{H}_\pi)$. For a sub-Hilbert space $\mathcal{K}_\pi$ of $\mathcal{H}_\pi$ that is invariant under the group action of $\pi(x)$ for all $x \in G$, the representation $\pi|_{\mathcal{K}_\pi} : G \to \mathcal{U}(\mathcal{K}_\pi)$ is called a *sub-representation* of π. If π has exactly two sub-representations corresponding to the Hilbert spaces $\{0\}$ and $\mathcal{H}_\pi$, then the representation π is said to be *irreducible*. In this manuscript we mainly care about irreducible unitary representations of locally compact groups which we call representations if there is no risk of confusion. Two unitary representations ρ and π on a locally compact group G are called to be equivalent if there is some unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$ such that $U^*\rho(x)U = \pi(x)$ for every $x \in G$.

Let G be a compact group, $\widehat{G}$ denotes a selection of continuous unitary irreducible representations of G, when from each class of equivalent irreducible unitary representations, we have one element in $\widehat{G}$. Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of a compact group G, it is well-known that $\mathcal{H}_\pi$ is a finite dimensional Hilbert space (see [36]). In this case, we denote the dimension of $\mathcal{H}_\pi$ by d_π and call it the *degree* (or dimension) of the representation π.

The trace of a matrix $A = (a_{i,j})_{i,j\in 1,\dots,n}$ is defined to be the sum of the coefficients on the diagonal of A, i.e. $\sum_{i\in 1,\dots,n} a_{i,i}$. We denote the trace of A by $\operatorname{Tr}(A)$. Let G be a compact group. Since for each irreducible representation of G, say π, d_π is finite, $\pi(x)$ is a matrix for each $x \in G$. So we may define a function $\chi_\pi : G \to \mathbb{C}$, called *a character* of G, by $\chi_\pi(x) := \operatorname{Tr}\pi(x)$ for $x \in G$. Note that $\chi_\pi(x^{-1}) = \overline{\chi_\pi(x)}$ for all $\pi \in \widehat{G}$ and $x \in G$. Moreover, $\chi_\pi(xyx^{-1}) = \operatorname{Tr}\pi(xyx^{-1}) = \operatorname{Tr}(\pi(x)\pi(y)\pi(x)^{-1}) = \operatorname{Tr}\pi(y) = \chi_\pi(y)$, since $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for all matrices A, B. Hence, χ_π for each representation π is a continuous class function i.e. it is constant over conjugacy classes of G.

For each two compact groups G_1 and G_2, $G := G_1 \times G_2$ forms a compact group. For two representations $\pi_i \in \widehat{G_i}$ with corresponding Hilbert space $\mathcal{H}_i$, $i = 1, 2$ one may define a unitary irreducible representation $\pi_1 \times \pi_2 \in \widehat{G}$ where

$$\pi_1 \times \pi_2(x, y)(\xi \otimes \eta) = \pi_1(x)\xi \otimes \pi_2(y)\eta$$

for $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$. Moreover, $d_{\pi_1\times\pi_2} = d_{\pi_1} d_{\pi_2}$ and one may show that $Tr(\pi_1 \otimes \pi_2(x, y)) = Tr(\pi_1(x))Tr(\pi_2(y))$. The inverse is true that is $\widehat{G} \equiv \widehat{G}_1 \times \widehat{G}_2$ i.e. for each $\pi \in \widehat{G}$, $\pi \equiv \pi_1 \times \pi_2$ for some $\pi_i \in \widehat{G}_i$ where $i = 1, 2$.

For each two representations $\pi_1, \pi_2 \in \widehat{G}$, for a compact group G, $\pi_1 \oplus \pi_2$ is a new representation of G with Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ where $\pi_1 \oplus \pi_2(x)(\xi \oplus \eta) = \pi_1(x)\xi \oplus \pi_2(x)\eta$ for all $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$. And $Tr(\pi_1 \oplus \pi_2(x)) = Tr(\pi_1(x)) + Tr(\pi_2(x))$.

The proof of the following results can be found in [28, Section 5.3].

Proposition 1.4.1. *For each character χ_π $(\pi \in \widehat{G})$, we have $\chi_\pi(e_G) = d_\pi$, $\|\chi_\pi\|_\infty = d_\pi$, and $\chi_\pi \in ZL^1(G)$.*

Proposition 1.4.2. *[28, (5.20)]*
Let G be a compact group and $\lambda(G)$ denotes the Haar measure of G. Then for all $\pi_1, \pi_2 \in \widehat{G}$,

$$\chi_{\pi_1} * \chi_{\pi_2}(x) = \begin{cases} \frac{\lambda(G)}{d_\pi}\chi_\pi(x) & \text{if } \pi = \pi_1 = \pi_2 \\ 0 & \text{if } \pi_1 \neq \pi_2 \end{cases}$$

For a compact group G, for each $1 \leq p < \infty$, $L^p(G) \subseteq L^1(G)$; furthermore, $(L^p(G), \|\cdot\|_p)$ equipped with the convolution forms a Banach algebra. Similarly, $C(G) \subseteq L^1(G)$ and therefore, $(C(G), \|\cdot\|_\infty)$ equipped with the convolution forms a Banach algebra.

Definition 1.4.3. For a compact group G, define $ZL^p(G) := ZL^1(G) \cap L^p(G)$ $(1 \leq p < \infty)$ and $ZC(G) = ZL^1(G) \cap C(G)$ i.e. $ZL^p(G) = \{f \in L^p(G) : \ f * h = h * f \ \forall h \in L^1(G)\}$ and $ZC(G) = \{f \in C(G) : \ f * h = h * f \ \forall h \in L^1(G)\}$.

The Banach spaces $ZL^p(G)$ $(1 \leq p < \infty)$ and $ZC(G)$ equipped with the convolution form commutative Banach algebras. Furthermore, $ZL^p(G)$ and $ZC(G)$ are the closure of the linear span of $\{\chi_\pi : \pi \in \widehat{G}\}$ for all $1 \leq p \leq \infty$ with respect to $\|\cdot\|_p$ and $\|\cdot\|_\infty$, respectively. Let the Haar measure on compact group G be normalized i.e. $\lambda(G) = 1$. Then $\{\chi_\pi\}_{\pi \in \widehat{G}}$ even forms an orthonormal basis of $ZL^2(G)$.

Theorem 1.4.4. *[28, Theorem 5.26]*
Let G be a compact group. Then $\pi \mapsto \psi_\pi$ forms a bijection from $\widehat{G}$ onto $\sigma(\mathcal{A})$, where $\mathcal{A}$ is one of $ZL^p(G)$ $(1 \leq p < \infty)$ or $ZC(G)$ as a commutative algebra with convolution and

$$\psi_\pi(f) = \frac{1}{d_\pi}\int_G f(y)\overline{\chi_\pi(y)}dy$$

for each $f \in \mathcal{A}$.

For each $\pi \in \widehat{G}$, let us define

$$\langle \widehat{f}(\pi)\xi, \eta \rangle := \int_G f(x)\langle \pi(x^{-1})\xi, \eta \rangle dx \quad (\xi, \eta \in \mathcal{B}(\mathcal{H}_\pi)).$$

for each $f \in L^1(G)$. The Fourier transform, $\mathcal{F}$, of some $f \in L^1(G)$ is defined to be $(\widehat{f}(\pi))_{\pi\in\widehat{G}}$. For each pair $\pi, \sigma \in \widehat{G}$, as a result of Schur orthogonality relations ([28, Section 5.2]), one gets that

$$\widehat{\chi_\pi}(\sigma) = \begin{cases} \frac{1}{d_\pi} I_{d_\pi} & \text{if } \pi = \sigma \\ 0 & \text{if } \pi \neq \sigma. \end{cases} \tag{1.4.1}$$

For each compact group G, $f \mapsto \widehat{f}(\pi)$ is an algebra homomorphism of $L^1(G)$ onto the algebra of $d_\pi \times d_\pi$ matrices. In this case, $A(G)$, the Fourier algebra of the compact group G is the set of all functions $f \in L^1(G)$ such that

$$\|f\|_{A(G)} := \sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{\mathcal{S}_1} < \infty \tag{1.4.2}$$

where $\|A\|_{\mathcal{S}_1}$ for a matrix A denotes the trace class norm i.e. $\|A\|_{\mathcal{S}_1} = \mathrm{Tr}(|A|)$. One can show that $A(G)$ equipped with pointwise multiplication and the norm $\|\cdot\|_{A(G)}$ defined above forms a Banach algebra (see [36, Theorem 34.18]). By (1.4.2), one can show that $\mathrm{lin}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $A(G)$ and $\|\chi_\pi\|_{A(G)} = d_\pi$ for every $\pi \in \widehat{G}$.

Similarly, for each $f \in L^2(G)$,

$$\|f\|_2^2 = \sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{\mathcal{S}_2}^2 \tag{1.4.3}$$

where $\|A\|_{\mathcal{S}_2}^2$ is the the *Hilbert-Schmidt norm* of a matrix $A = [a_{i,j}]_{1\leq i,j\leq n}$ that is $\left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2}$.

Unfortunately, the word 'character' is used in both Banach algebra theory and in the representation theory of finite groups and means two slightly different things. To prevent ambiguity, we may occasionally use the phrase *algebra character* to mean a character in the sense of Gelfand theory for the center of group algebra.

1.4.2 Segal algebras

Abstract Segal algebras first was defined in [10] as generalization of Segal algebras. We say that the Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is an *abstract Segal algebra* of a Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ if

1. $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$.

2. There exists $M > 0$ such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$ for each $b \in \mathcal{B}$.

3. There exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$ for all $a, b \in \mathcal{B}$.

If $\mathcal{B}$ is a proper subalgebra of $\mathcal{A}$, we call it a *proper* abstract Segal algebra of $\mathcal{A}$.

The definition of Segal algebra and most of examples below are from [64]. Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$, the group algebra of G, is said to be a *Segal algebra* on G, if it satisfies the following conditions:

1. $S^1(G)$ is dense in $L^1(G)$.
2. $S^1(G)$ is a Banach space under some norm $\|\cdot\|_{S^1}$ and $\|f\|_{S^1} \geq \|f\|_1$ for all $f \in S^1(G)$.
3. $S^1(G)$ is left translation invariant and the map $x \mapsto L_x f$ of G into $S^1(G)$ is continuous where $L_x f(y) = f(x^{-1}y)$.
4. $\|L_x f\|_{S^1} = \|f\|_{S^1}$ for all $f \in S^1(G)$ and $x \in G$.

Note that every Segal algebra on G is an abstract Segal algebra of $L^1(G)$ with convolution product. Similarly, we call a Segal algebra on G *proper* if it is a proper subalgebra of $L^1(G)$.

Example 1.4.5.

- Let $\mathcal{L}A(G) := L^1(G) \cap A(G)$ and $|||h||| := \|h\|_1 + \|h\|_{A(G)}$ for $h \in \mathcal{L}A(G)$. Then $\mathcal{L}A(G)$ with norm $|||\cdot|||$ is a Banach space; this space was studied extensively by Ghahramani and Lau in [30]. They have shown that $\mathcal{L}A(G)$ with the convolution product is a Banach algebra called the *Lebesgue-Fourier algebra* of G; moreover, it is a Segal algebra on the locally compact group G. $\mathcal{L}A(G)$ is a proper Segal algebra on G if and only if G is not discrete.

  Also, $\mathcal{L}A(G)$ with pointwise multiplication is a Banach algebra and even an abstract Segal algebra of $A(G)$. Similarly, $\mathcal{L}A(G)$ is a proper subset of $A(G)$ if and only if G is not compact.

- The convolution algebra $L^1(G) \cap L^p(G)$ for $1 \leq p < \infty$ equipped with the norm $\|f\|_1 + \|f\|_p$ is a Segal algebra.

- Similarly, $L^1(G) \cap C_0(G)$ with respect to the norm $\|f\|_1 + \|f\|_\infty$ is a Segal algebra.

- Let G be a compact group, $\mathcal{F}$ denote the Fourier transform, and $\mathcal{L}^p(\widehat{G})$ be the space which will be defined in (3.2.1). We can see that $\mathcal{F}^{-1}\big(\mathcal{L}^p(\widehat{G})\big)$, which we denote by $\mathfrak{C}^p(G)$, equipped with convolution is a subalgebra of $L^1(G)$. For $\|f\|_{\mathfrak{C}^p(G)} := \|\mathcal{F}f\|_{\mathcal{L}^p(\widehat{G})}$, one can show that for each $1 \leq p \leq 2$, $\big(\mathfrak{C}^p(G), \|\cdot\|_{\mathfrak{C}^p(G)}\big)$ is a Segal algebra of G.

Chapter 2

Restricted direct products of finite groups

In this chapter, we are interested in knowing more about the center of discrete group algebras. Where G is a discrete group, if some conjugacy class $C_x = \{yxy^{-1} : y \in G\}$ is infinite for each function $f \in Z\ell^1(G)$, it is easy to verify that $f(C_x) = 0$; therefore, the characteristic function of the set C_x, denoted by 1_{C_x}, does not belong to $Z\ell^1(G)$. This is the main reason that we restrict our study to discrete groups with finite conjugacy classes (including finite groups) which are called *finite conjugacy groups* or in short *FC groups*. For a FC group G, we denote the set of all conjugacy classes of G by $\mathrm{Conj}(G)$. In this chapter, we study a specific class of FC groups, called RDPF groups and some properties of the center of their group algebras.

This chapter is based on a joint project with Professor Yemon Choi and Professor Ebrahim Samei; a version of that has been written in the manuscript [3].

2.1 General properties of $\mathrm{Z}\ell^1(G)$ for product groups

It is well known that when G is finite, the space of maximal ideals of $\mathrm{Z}\ell^1(G)$ corresponds to the set of irreducible group characters of G. As a particular class of compact groups, one may re-write Theorem 1.4.4 for finite groups, as follows.

Lemma 2.1.1. *Let G be a finite group. If ψ is an algebra character on $\mathrm{Z}\ell^1(G)$, then there is a unique $\pi \in \widehat{G}$ such that for the corresponding group character χ_π,*

$$\psi(f) = \sum_{x \in G} f(x) {d_{\chi_\pi}}^{-1} \chi_\pi(x^{-1}) \qquad \text{for all } f \in \mathrm{Z}\ell^1(G). \tag{2.1.1}$$

Conversely, for each group character χ_π of G, $\pi \in \widehat{G}$, the formula (2.1.1) defines an algebra character on $\mathrm{Z}\ell^1(G)$.

The following lemma will be used later, in several places.

Lemma 2.1.2. *Let H and K be (discrete) FC-groups. Then the canonical, isometric isomorphism of Banach algebras $\ell^1(H) \otimes_\gamma \ell^1(K) \cong \ell^1(H \times K)$ restricts to an isometric isomorphism of Banach algebras $\mathrm{Z}\ell^1(H) \otimes_\gamma \mathrm{Z}\ell^1(K) \cong \mathrm{Z}\ell^1(H \times K)$.*

Proof. For any FC group H, we can define an averaging operator $P_H : \ell^1(H) \to \mathrm{Z}\ell^1(H)$ by

$$P_H(f)(x) = \frac{1}{|C_x|} \sum_{t \in C_x} f(t)$$

where C_x denotes the conjugacy class of x in H. Since H is an FC-group, P_H is well-defined and P_H leaves elements of $\mathrm{Z}\ell^1(H)$ fixed. For two FC groups H and K, we define $P_H \otimes P_K : \ell^1(H) \otimes_\gamma \ell^1(K) \to Z\ell^1(H) \otimes_\gamma Z\ell^1(K)$ where $P_H \otimes P_K(f \otimes g) = P_H(f) \otimes P_K(g)$ for each $f \otimes g \in \ell^1(H) \otimes_\gamma \ell^1(K)$.

Now let H and K be FC groups, and let $\theta : \ell^1(H)\otimes_\gamma \ell^1(K) \to \ell^1(H \times K)$ be the canonical isometrical isomorphism of Banach algebras, which satisfies $\theta(f \otimes g)(x, y) = f(x)g(y)$ for all $f \in \ell^1(H)$, $g \in \ell^1(K)$, $x \in H$ and $y \in K$. We claim that

$$P_{H\times K} \circ \theta = \theta \circ (P_H \otimes P_K). \tag{2.1.2}$$

Note that for every pair H, K of FC groups, $H \times K$ is a FC group. Let $f \in \ell^1(H)$ and $g \in \ell^1(K)$. If $(x, y) \in H \times K$, then since $C_{(x,y)}$ may be identified with $C_x \times C_y$, we have

$$\begin{aligned} P_{H\times K} \circ \theta(f \otimes g)(x, y) &= P_{H\times K}(f(x)g(y)) \\ &= |C_{(x,y)}|^{-1} \sum_{(t,s)\in C_x\times C_y} f(t)g(s) \\ &= |C_x|^{-1} \sum_{t\in C_x} f(t)\, |C_y|^{-1} \sum_{s\in C_y} g(s) \\ &= P_H(f)(x)P_K(g)(y) \qquad = \theta \circ (P_H \otimes P_K)(f \otimes g)(x, y), \end{aligned}$$

so $P_{H\times K} \circ \theta(f \otimes g) = \theta \circ (P_H \otimes P_K)(f \otimes g)$. Let $f \in \mathrm{Z}\ell^1(H)$ and $g \in \mathrm{Z}\ell^1(K)$, then for each $h_1 \otimes h_2 \in \ell^1(H)\otimes_\gamma \ell^1(K)$, note that

$$\begin{aligned} \theta(f \otimes g) *_{\ell^1(H\times K)} \theta(h_1 \otimes h_2) &= \theta\big((f *_{\ell^1(H)} h_1) \otimes (g *_{\ell^1(K)} h_2)\big) \\ &= \theta\big((h_1 *_{\ell^1(H)} f) \otimes (h_2 *_{\ell^1(K)} g)\big) \\ &= \theta(h_1 \otimes h_2) *_{\ell^1(H\times K)} \theta(f \otimes g). \end{aligned}$$

Since the space generated by the set of all $h_1 \otimes h_2$ is dense in $\ell^1(H)\otimes_\gamma \ell^1(K)$, $\theta(f \otimes g) \in \mathrm{Z}\ell^1(H\times K)$. Hence, $\theta(\mathrm{Z}\ell^1(H)\otimes_\gamma \mathrm{Z}\ell^1(K)) \subseteq \mathrm{Z}\ell^1(H\times K)$. To prove the converse inclusion: let $u \in Zl^1(H\times K)$; then $\theta^{-1}(u) \in \ell^1(H)\otimes_\gamma \ell^1(K)$, and so

$$u = P_{H\times K}\theta(\theta^{-1}(u)) = \theta(P_H \otimes P_K)(\theta^{-1}(u)) \in Z\ell^1(H \times K),$$

since $P_H \otimes P_K(\theta^{-1}(u)) \in Z\ell^1(H) \otimes_\gamma Z\ell^1(K)$. Moreover, since θ^{-1} and $P_H \otimes P_K$ both have norm 1, this shows that

$$\theta|_{Zl^1(H)\otimes_\gamma Zl^1(K)} : Zl^1(H)\otimes_\gamma Zl^1(K) \to Zl^1(H \times K)$$

is not just surjective, but is an isometry, as claimed. ☐

2.2 The restricted direct product of finite groups

Let $\mathbf{I}$ be an indexing set and $(G_i)_{i\in\mathbf{I}}$ a family of finite groups; the *restricted direct product* of the family (G_i), abbrevated RDPF here, is defined to be the group

$$\bigoplus_{i\in\mathbf{I}} G_i := \left\{(x_i)_{i\in\mathbf{I}} \in \prod_{i\in\mathbf{I}} G_i \colon x_i = e_{G_i} \text{ for all but finitely many } i.\right\}$$

which is a group version of the definition mentioned in Section 1.1. Note that if $\mathbf{I}$ is finite, the restricted direct product agrees with the usual direct product of groups.

Proposition 2.2.1. *Let $(G_i)_{i\in\mathbf{I}}$ be a family of finite groups and G their restricted direct product. Then G is a FC group.*

Proof. Let $x = (x_i)_{i\in\mathbf{I}} \in G$ and let C_x denote the conjugacy class of x. Since $x_i = e_{G_i}$ for all but finitely many $i \in \mathbf{I}$, for any $y = (y_i)_{i\in\mathbf{I}}$, we have $y_i x_i y_i^{-1} = e_{G_i} \in G$ for all but finitely many $i \in \mathbf{I}$.

Define $\mathbf{I}_x := \{i \in \mathbf{I} : x_i \neq e_{G_i}\}$, which is a finite subset of $\mathbf{I}$. Then

$$|C_x| = \prod_{i\in\mathbf{I}_x} |C_{x_i}| \leq \prod_{i\in\mathbf{I}_x} |G_i| < \infty,$$

and since x was chosen arbitrarily, G is a FC group. ☐

Recall that for a group G, the center of the group is defined to be the set of all $x \in G$ such that $xy = yx$ for all $y \in G$ and denoted by $Z(G)$.

Proposition 2.2.2. *Let $(G_i)_{i\in\mathbf{I}}$ be a family of finite groups and let $G = \bigoplus_{i\in\mathbf{I}} G_i$.*

(i) *$Z(G) = \bigoplus_{i\in\mathbf{I}} Z(G_i)$.*

(ii) *$G' = \bigoplus_{i\in\mathbf{I}} G_i'$.*

(iii) *If each G_i is nilpotent of class n, then so is G.*

(iv) *If each G_i is solvable of length n, then so is G.*

Proof. (*i*). Let $x = (x_i)_{i\in\mathbf{I}} \in Z(G)$. For some $i_0 \in \mathbf{I}_x$, suppose that $x_{i_0} \notin Z(G_{i_0})$. Therefore, there exists some $y_{i_0} \in G_{i_0}$ such that $y_{i_0} x_{i_0} \neq x_{i_0} y_{i_0}$. So for $y := (y_i)$ where $y_i = e_{G_i}$ for all $i \in \mathbf{I} \setminus \{i_0\}$ and $y_i = y_{i_0}$ as defined for $i = i_0$. Hence, the i_0th coordinate of xy which is $x_{i_0} y_{i_0}$ is not equal to the i_0th coordinate of yx which is $y_{i_0} x_{i_0}$. So, $xy \neq yx$ which is a contradiction. So,

$Z(G) \subseteq \bigoplus_{i\in\mathbf{I}} Z(G_i)$. Conversely, for each $x = (x_i)_{i\in\mathbf{I}} \in \bigoplus_{i\in\mathbf{I}} Z(G_i)$ and each $y = (y_i)_{i\in\mathbf{I}} \in G$, for each $i \in \mathbf{I}$, $x_i y_i = y_i x_i$; hence, $xy = yx$.

(ii). Note that for each commutator $[x,y] \in G'$, $[x,y] = ([x_i,y_i])_{i\in\mathbf{I}} \in \bigoplus_{i\in\mathbf{I}} G_i'$. Therefore, $G' \subseteq \bigoplus_{i\in\mathbf{I}} G_i'$. On the other hand, for each $x \in \bigoplus_{i\in\mathbf{I}} G_i'$, we know $x = x^{(1)}\cdots x^{(n)}$ such that for $j \in 1,\cdots,n$, $I_{x(j)}$ is a singleton. So without loss of generality, we assume that $x \in \bigoplus_{i\in\mathbf{I}} G_i'$ and $\mathbf{I}_x = \{i_0\}$ is a singleton; hence, $x_{i_0} = y_{i_0}^1 \cdots y_{i_0}^m$ where for each $j \in 1,\cdots,m$, $y_{i_0}^j$ is a commutator in G_{i_0}. For each $j \in 1,\cdots,m$, define $y^j = (y_i^j)_{i\in\mathbf{I}} \in G$ such that y_i^j is the mentioned commutator $y_{i_0}^j$ for $i = i_0$ and e_{G_i} for $i \neq i_0$. Therefore, $y^j \in G'$ for each $j \in 1,\cdots,m$, and consequently, $x = y^1 \cdots y^m \in G'$.

(iii) and (iv). First, note that if for each $i \in \mathbf{I}$, N_i is a normal subgroup of G_i such that G_i/N_i is commutative, then $N = \bigoplus_{i\in\mathbf{I}} N_i$ is a normal subgroup of $G = \bigoplus_{i\in\mathbf{I}} G_i$ and G/N is commutative. If for each i, G_i is nilpotent of class n, one may find a central series $\{e_{G_i}\} = N_i^1 \lhd N_i^2 \lhd \cdots \lhd N_i^n = G_i$. So, $\{e_G\} = \bigoplus_{i\in\mathbf{I}} N_i^1 \lhd \cdots \lhd \bigoplus_{i\in\mathbf{I}} N_i^n = G$ is a central series. Similarly, a set of subnormal series $\{e_{G_i}\} = N_i^1 \lhd N_i^2 \lhd \cdots \lhd N_i^n = G_i$ such that N_{i+1}/N_i is commutative implies the subnormal series $\{e_G\} = N^1 = \bigoplus_{i\in\mathbf{I}} N_i^1 \lhd \cdots \lhd N^n = \bigoplus_{i\in\mathbf{I}} N_i^n = G$ such that N^{j+1}/N^j is commutative. □

Let (G_i) be a family of finite groups, and let $F \subset \mathbf{I}$; write F^c for $\mathbf{I} \setminus F$. Since

$$\bigoplus_{i\in\mathbf{I}} G_i \cong \Big(\bigoplus_{i\in F} G_i\Big) \times \Big(\bigoplus_{i\in F^c} G_i\Big)$$

by Lemma 2.1.2, we obtain an isometric isomorphism of Banach algebras

$$\mathrm{Z}\ell^1\Big(\bigoplus_{i\in\mathbf{I}} G_i\Big) \cong \mathrm{Z}\ell^1\Big(\bigoplus_{i\in F} G_i\Big) \otimes_\gamma \mathrm{Z}\ell^1\Big(\bigoplus_{i\in F^c} G_i\Big). \tag{2.2.1}$$

Hence, if we write E_F^c for the identity of $\mathrm{Z}\ell^1(\bigoplus_{i\in F^c} G_i)$, there is a unital, isometric, homomorphism of Banach algebras

$$\imath_F : \mathrm{Z}\ell^1\Big(\bigoplus_{i\in F} G_i\Big) \to \mathrm{Z}\ell^1\Big(\bigoplus_{i\in\mathbf{I}} G_i\Big) \left(\cong Z\ell^1\Big(\bigoplus_{i\in F} G_i\Big) \otimes_\gamma Z\ell^1\Big(\bigoplus_{i\in F^c} G_i\Big)\right) \tag{2.2.2}$$

defined by $\imath_F(f) = f \otimes E_F^c$. When F is a singleton, say $\{j\}$, we denote $\imath_F$ by $\imath_j$. Let ε_{F^c} denote the *augmentation character* on $\mathrm{Z}\ell^1(\bigoplus_{i\in F^c} G_i)$ i.e. for each $f \in \mathrm{Z}\ell^1(\bigoplus_{i\in F^c} G_i)$,

$$\varepsilon_{F^c}(f) = \sum_{x\in\bigoplus_{i\in F^c} G_i} f(x).$$

If we denote by id_F the identity homomorphism on $\mathrm{Z}\ell^1(\bigoplus_{i\in F} G_i)$, then there is a unital, surjective homomorphism of Banach algebras

$$\mathbf{P}_F = \mathrm{id}_F \otimes \varepsilon_{F^c} : \mathrm{Z}\ell^1\Big(\bigoplus_{i\in\mathbf{I}} G_i\Big) \to \mathrm{Z}\ell^1\Big(\bigoplus_{i\in F} G_i\Big) \tag{2.2.3}$$

which satisfies $\mathbf{P}_F(f \otimes g) = \varepsilon_{F^c}(g) f$ for all $f \in \mathrm{Z}\ell^1(\bigoplus_{i\in F} G_i)$ and all $g \in \mathrm{Z}\ell^1(\bigoplus_{i\in F^c} G_i)$.

2.3 The Gelfand spectrum of $Z\ell^1(G)$

Let G be a RDPF group. Since $\mathrm{Z}\ell^1(G)$ is a commutative Banach algebra, it is natural to ask for a description of its Gelfand spectrum. This is given by the following result (recall that Lemma 2.1.1 gives a description of the spectrum when G is finite).

Theorem 2.3.1. *Let $(G_i)_{i\in\mathbf{I}}$ be a family of finite groups and G their restricted direct product. Then there is a homeomorphism from $\sigma(\mathrm{Z}\ell^1(G))$ onto*

$$\prod_{i\in\mathbf{I}}\sigma(\mathrm{Z}\ell^1(G_i)) := \{(\psi_i)_{i\in\mathbf{I}} : \psi_i \in \sigma(\mathrm{Z}\ell^1(G_i))\ \forall i\in\mathbf{I}\}$$

equipped with the product topology. In particular, $\sigma(\mathrm{Z}\ell^1(G))$ is totally disconnected.

Proof. For each $\omega\in\sigma(\mathrm{Z}\ell^1(G))$, we can define $\psi_i := \omega\circ\imath_i$ for each $i\in\mathbf{I}$. Note that ψ_i is a functional on $\mathrm{Z}\ell^1(G_i)$. Moreover, since $\|\imath_i\|\leq 1$, ψ_i is also continuous. Moreover, note that for all $f,g\in\mathrm{Z}\ell^1(G_i)$,

$$\psi_i(f*g) = \omega\circ\imath_i(f*g) = \omega(f\otimes E_i^c * g\otimes E_i^c) = \omega(f\otimes E_i^c)\omega(g\otimes E_i^c) = \omega\circ\imath_i(f)\omega\circ\imath_i(g) = \psi_i(f)\psi_i(g).$$

So, ψ_i is an algebra character for $\mathrm{Z}\ell^1(G_i)$ for each $i\in\mathbf{I}$. Conversely, let

$$Zc_c(G) = \{f\in c_c(G) : f \text{ is constant on the conjugacy classes of G}\}.$$

For each set $(\psi_i)_{i\in\mathbf{I}}$, we define ω on $Zc_c(G)$ as follows: given $f\in Zc_c(G)$, since $\mathrm{supp}(f)$ is a finite subset of G. There is some $F\subseteq\mathbf{I}$ such that $\mathrm{supp}(f)\subseteq\bigoplus_{i\in F}G_i\times E_F^c$. It is clear that

$$\omega_F(x) := \prod_{i\in F}\psi_i(x_i), \qquad x=(x_i)_{i\in F}\in\bigoplus_{i\in F}G_i$$

for ψ_i's defined above. It will define a character group for finite group $\bigoplus_{i\in F}G_i$. Let $\omega(f) := \omega_F(\mathbf{P}_F(f))$ ($\mathbf{P}_F$ was defined in (2.2.3)). We show that ω is well-defined. If for some $F\subseteq F'$ for some $F,F'\subseteq\mathbf{I}$, and $x=(x_i)_{i\in F}\in\bigoplus_{i\in F}G_i$, then for $y=(y_i)_{i\in F'}\in\bigoplus_{i\in F'}$ such that $y_i = x_i$ for all $i\in F$ and $y_i = e_{G_i}$ for $i\notin F$, $\omega_F(x) = \omega_{F'}(y)$. Also, $\mathbf{P}(\delta_x) = \delta_x\otimes E_F^c$. Clearly, for each pair F,F' of subsets of $\mathbf{I}$, if $\mathrm{supp}(f)\subseteq\bigoplus_{i\in F}G_i\times E_F^c$ and $\mathrm{supp}(f)\subseteq\bigoplus_{i\in F'}G_i\times E_{F'}^c$, $\mathrm{supp}(f)\subseteq\bigoplus_{i\in F\cap F'}G_i\times E_{F\cap F'}^c$. Moreover, since f is finitely supported, $\omega_F(\mathbf{P}_F(f)) = \omega_{F'}(\mathbf{P}_{F'}(f)) = \omega_{F\cap F'}(\mathbf{P}_{F\cap F'}(f))$.

Since ω is a bounded linear map, we can extend it to $\mathrm{Z}\ell^1(G)$. On the other hand, for all $f,g\in Zc_c(G)$ we have $\omega(f*g)=\omega(f)\omega(g)$ and so ω belongs to $\sigma(\mathrm{Z}\ell^1(G))$.

Let $\mathbf{j}:\sigma(\mathrm{Z}\ell^1(G))\to\prod_{i\in\mathbf{I}}\sigma(\mathrm{Z}\ell^1(G_i))$ be the map defined by

$$\mathbf{j}(\omega) = (\omega\circ\imath_i)_{i\in\mathbf{I}}.$$

We have just seen that $\mathbf{j}$ is a bijection. It remains only to show that $\mathbf{j}$ is continuous. Let $U = \prod_{i\in\mathbf{I}} U_i \subseteq \prod_{i\in\mathbf{I}} \sigma(Z\ell^1(G_i))$ be a sub-basic open set i.e $U_i = \sigma(Z\ell^1(G_i))$ for all $i \in \mathbf{I}$ but one i_0 when

$$U_{i_0} = \{\psi \in \sigma(Z\ell^1(G_{i_0})) \text{ so that } |\langle \psi - \phi, f\rangle| < \varepsilon\}$$

for some $f \in Z\ell^1(G_{i_0})$, $\epsilon > 0$, and $\phi \in \sigma(Z\ell^1(G_{i_0}))$. We show that $\mathbf{j}^{-1}(U)$ is open, by showing that each $\omega \in \mathbf{j}^{-1}(U)$ has a Gelfand-open neighbourhood contained in $\mathbf{j}^{-1}(U)$. Since $\omega \in \mathbf{j}^{-1}(U)$, $\delta := \varepsilon - |\langle \omega \circ \imath_{i_0} - \phi, f\rangle| > 0$. Define

$$V := \{\omega' \in \sigma(Z\ell^1(G)) \text{ so that } |\langle \omega' - \omega, \imath_{i_0}(f)\rangle| < \delta\}$$

an open neighborhood of ω in Gelfand topology on $\sigma(Z\ell^1(G))$. So, for each $\omega' \in V$,

$$\langle \omega' - \omega, \imath_{i_0}(f)\rangle = \langle (\omega' - \omega) \circ \imath_{i_0}, f\rangle = \langle \omega'\imath_{i_0} - \omega\imath_{i_0}, f\rangle.$$

Hence,

$$|\langle \omega'\imath_{i_0} - \phi, f\rangle| \le |\langle \omega'\imath_{i_0} - \omega\imath_{i_0}, f\rangle| + |\langle \omega\imath_{i_0} - \phi, f\rangle| < \epsilon;$$

thus, $\omega' \circ \imath_{i_0} \in U_{i_0}$. Therefore, $\mathbf{j}(\omega') \in U$. Since i_0 is arbitrary and f is an arbitrary element in $Z\ell^1(G_{i_0})$, Thus $\mathbf{j}$ is continuous; since it is bijective from a compact space onto a Hausdorff one, we conclude that $\mathbf{j}^{-1}$ is also continuous. □

2.4 Characterizing ZL-amenability of RDPF groups

The amenability of the group algebra of a locally compact group G is equivalent to the amenability of the group G, [68]. But, the amenability of $ZL^1(G)$, the centre of $L^1(G)$, is not characterized completely. If $ZL^1(G)$ is amenable for a locally compact group G, we call G *ZL-amenable*. As we mentioned before, amenability constant of a Banach algebra is a tool to quantify the amenability of the algebra. The amenability constant of $ZL^1(G)$ for a locally compact group G is called the *ZL-amenability constant* of G.

As a conjecture for a discrete FC group G, ZL-amenability of G is equivalent to the finiteness of G'. In this section, we prove this conjecture for RDPF groups.

The following proposition is a result by Rider, [67], about the norm one of a class of idempotents in group algebras of compact groups, presented in the following. Note that Rider's result is stated for the case where the Haar measure on G is normalized i.e. $\lambda(G) = 1$. However, a rescaling argument, based on Proposition 1.4.2, shows that this is equivalent to the formulation we have given.

Proposition 2.4.1. *[67, Lemma 5.2]*

Let G be a compact group with a Haar measure λ and $\psi = \sum_{\pi\in F} \frac{d_\pi}{\lambda(G)}\chi_\pi$ for a finite subset $F \subseteq \widehat{G}$ such that $\|\psi\|_{L^1(G,\lambda)} > 1$. Then $\|\psi\|_{L^1(G,\lambda)} \geq 301/300$.

Remark 2.4.2. For a finite group G, let μ be the normalized Haar measure on G i.e. $\mu(x) = |G|^{-1}$ for every $x \in G$. Let us denote the group algebra generated by the normalized Haar measure μ by $\ell^1(G,\mu)$ and $\|\cdot\|_{1,\mu}$ and $*_\mu$ denote the corresponding norm and the convolution respectively. While $\ell^1(G,\lambda)(= \ell^1(G))$ denotes the center of the group algebra with the regular counting measure i.e. $\lambda(x)$ for every element $x \in G$ is 1. Hence, $\lambda = |G|\mu$. Then one may define an isometric linear map $\theta : \ell^1(G,\mu) \to \ell^1(G,\lambda)$ where $\theta(f) = |G|f$ for every $f \in \ell^1(G,\mu)$. Note that

$$\begin{aligned}
\theta(f) *_\mu \theta(g)(x) &= \sum_{y\in G} \theta(f)(y)\theta(y^{-1}x)\mu(y) \\
&= |G|^2 \sum_{y\in G} f(y)g(y^{-1}x)\frac{\lambda(y)}{|G|} \\
&= |G|(f *_\lambda g)(x) \\
&= \theta(f *_\lambda g)(x).
\end{aligned}$$

Therefore, two algebras $\ell^1(G,\mu)$ and $\ell^1(G)$ are isomorphic. In many studies of finite groups as a special case of compact groups, they are equipped with a normalized Haar measure. Here we mainly work with the counting measure. So all results which are mentioned in the following have been modified for the counting measure, in particular the following summary from [5].

Azimifard, Spronk, and Samei in [5] studied the ZL-amenability constant of the center of the group algebra of finite groups. In [5, Theorem 1.8] they have shown that for a finite group G,

$$M = \sum_{\pi\in\widehat{G}} \frac{d_\pi^2}{|G|^2}\chi_\pi \otimes \chi_\pi \tag{2.4.1}$$

is a virtual diagonal of $Z\ell^1(G)$ which is actually an idempotent of the form mentioned in Proposition 2.4.1 that belongs to $Z\ell^1(G) \otimes_\gamma Z\ell^1(G)$. Computing the norm of M where they consider G equipped with normalized Haar measure, they achieved a formula for ZL-amenability constant of G. One may note that by some simplifications in the ZL-amenability developed in [5], for an abelian finite group G, since $Z\ell^1(G) = \ell^1(G)$, $AM(Z\ell^1(G)) = 1$. The following proposition is based an observation which is done in the proof of [5, Theorem 1.10] applying Proposition 2.4.1 and computing norm of M.

Proposition 2.4.3. *Let G be a non-abelian finite group equipped with the normalized Haar measure λ generated by the counting measure. Then the ZL-amenability constant of G is always greater than or equal to $1 + 1/300$ i.e. $AM(Z\ell^1(G)) \geq 301/300$.*

Proof. Let us consider M as an element in $\ell^1(G \times G)(= \ell^1(G) \otimes_\gamma \ell^1(G))$. Note that, for each representation $\pi \in \widehat{G}$, $\pi \otimes \pi \in \widehat{G} \times \widehat{G} = \widehat{G \times G}$ where $\pi \otimes \pi(x, y) = \pi(x) \otimes \pi(y)$ and $d_{\pi\otimes\pi} = d_\pi^2$. On the other hand,

$$\chi_\pi \otimes \chi_\pi(x, y) = Tr(\pi(x))Tr(\pi(y)) = Tr(\pi \otimes \pi(x, y)) = \chi_{\pi\otimes\pi}(x, y).$$

Therefore, $M = |G|^{-2} \sum_{\pi\in\widehat{G}} d_{\pi\otimes\pi}\chi_{\pi\otimes\pi}$ forms an idempotent which is a finite combination of characters of the group $G \times G$, based on Proposition 1.4.2. But by [5, Corollary 1.9], if G is non-abelian $\|M\|_{\ell^1(G\times G)} = AM(Z\ell^1(G)) > 1$. Therefore, by Proposition 2.4.1, $\|M\|_1 \geq 301/300$. $\square$

The following theorem is the main result of this section.

Theorem 2.4.4. *Let $(G_i)_{i\in\mathbf{I}}$ be a family of finite groups and let $G = \bigoplus_{i\in\mathbf{I}} G_i$. Then the followings are equivalent:*

(i) $Z\ell^1(G)$ *is amenable;*

(ii) G_i *is abelian for all but finitely many i;*

(iii) G *is isomorphic to the product of a finite group with an abelian group;*

(iv) *the derived subgroup of G is finite.*

Proof. We start by defining $N = \{i \in \mathbf{I} : G_i \text{ is non-abelian}\}$.

(i) $\Longrightarrow$ (ii). By Proposition 2.4.3, $\mathrm{AM}(Z\ell^1(H)) \geq 1 + 1/300$ whenever H is a finite non-abelian group. Now suppose $Z\ell^1(G)$ is amenable, and let F be a finite subset of N. Recall that we have a quotient homomorphism of Banach algebras $\mathbf{P}_F : Z\ell^1(G) \to Z\ell^1\big(\bigoplus_{i\in F} G_i\big)$, as defined earlier in (2.2.3). Moreover, note that if $Z\ell^1(G)$ is equipped with the regular counting measure, one may compute an upper bound for $\|\mathbf{P}_F\|$. For each $f \otimes g \in Z\ell^1(G) \cong \big(Z\ell^1(\bigoplus_{i\in F} G_i)\big) \otimes_\gamma \big(Z\ell^1(\bigoplus_{i\in F^c} G_i)\big)$, one gets that $\|f \otimes g\|_1 = \|f\|_1\|g\|_1$. Therefore,

$$\|\mathbf{P}_F(f \otimes g)\|_1 \leq |\varepsilon_{F^c}(g)|\|f\|_1 \leq \|f\|_1\|g\|_1. \tag{2.4.2}$$

Note that according to the definition of projective tensor product and its norm, (2.4.2) implies that $\|\mathbf{P}_F\| \leq 1$.

It follows from Remark 1.3.3 that

$$\infty > \mathrm{AM}(\mathrm{Z}\ell^1(G)) \geq \|\mathbf{P}_F\|^2\,\mathrm{AM}(\mathrm{Z}\ell^1(G)) \geq \mathrm{AM}(\mathrm{Z}\ell^1(\bigoplus_{i\in F} G_i)).$$

Moreover, it was proven in [5] that

$$\mathrm{AM}(\mathrm{Z}\ell^1(\bigoplus_{i\in F} G_i)) = \prod_{i\in F} \mathrm{AM}(\mathrm{Z}\ell^1(G_i))$$

Hence, by Proposition 2.4.3,

$$\infty > \mathrm{AM}(\mathrm{Z}\ell^1(G)) \geq \prod_{i\in F} \mathrm{AM}(\mathrm{Z}\ell^1(G_i)) \geq (1+1/300)^{|F|}.$$

Since F was an arbitrary finite subset of N, this shows N is finite.

(ii) $\Longrightarrow$ (iii). Let $\mathbf{I}_A = \{i\in\mathbf{I} : G_i \text{ is abelian}\}$; thus, $A := \bigoplus_{i\in\mathbf{I}_A} G_i$ is abelian. Since $F := \mathbf{I}\setminus\mathbf{I}_A$ is finite, so $K = \bigoplus_{i\in F} G_i$ is a finite group and $G = A\times K$.

(iii) $\Longrightarrow$ (i). If K is finite and A is abelian, then by Lemma 2.1.2,

$$\mathrm{Z}\ell^1(K\times A) \cong \mathrm{Z}\ell^1(K)\otimes_\gamma \mathrm{Z}\ell^1(A) = \mathrm{Z}\ell^1(K)\otimes_\gamma \ell^1(A)$$

which is the projective tensor product of two amenable Banach algebras, hence is amenable by [40, Proposition 5.4].

(ii) $\Longleftrightarrow$ (iv). It is pointed out in Proposition 2.2.2(ii) that $G' = \bigoplus_{i\in I} G_i'$. Note that $G_i' = \{e_{G_i}\}$ if and only if G_i is abelian. Therefore G' is finite if and only if G_i is abelian for all but finitely many i.

□

2.5 Bounded approximate identities in maximal ideals of $\mathrm{Z}\ell^1(G)$

Stegmeir, in [72], studied the center of the group algebra $\mathrm{Z}\ell^1(G)$ for a RDPF group G where $\mathrm{Z}\ell^1(G)$ has a maximal ideal without a bounded approximate identity, and therefore, G is not ZL-amenable. In this section, we study the existence of a bounded approximate identity of maximal ideals of $\mathrm{Z}\ell^1(G)$ for RDPF groups with respect to a characterization of the corresponding character $\psi\in\sigma(\mathrm{Z}\ell^1(G))$.

We need some preliminary observations, which all follow from basic properties of the non-abelian Fourier transform for finite groups. Note that for a finite group G, the linear span of $\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $\mathrm{Z}\ell^1(G)$.

Let ψ_σ be the algebra character on $\mathrm{Z}\ell^1(G)$ that corresponds to the irreducible group representation $\sigma \in \widehat{G}$. Let us recall a finite version of Proposition 1.4.2 as follows where $*$ denotes the convolution with respect to the counting measure:

$$\frac{d_\sigma}{|G|}\chi_\sigma * \frac{d_\pi}{|G|}\chi_\pi = \begin{cases} \frac{d_\sigma}{|G|}\chi_\sigma & \sigma = \pi \\ 0 & \sigma \neq \pi \end{cases} \tag{2.5.1}$$

Thus if $\pi \neq \sigma$, then

$$\psi_\sigma(\chi_\pi) = \sum_{x\in G}\chi_\pi(x)d_\sigma^{-1}\chi_\sigma(x^{-1}) = d_\sigma^{-1}\chi_\pi * \chi_\sigma(e_G) = 0.$$

Therefore, $\mathrm{Ker}(\psi_\sigma)$ is the closure of the linear span $\{\chi_\pi : \pi \in \widehat{G}\setminus\{\sigma\}\}$. Note that $\mathrm{Z}\ell^1(G)$ has the identity δ_e. We show that $\mathrm{Ker}(\psi_\sigma)$ has the identity element

$$u_\sigma := \delta_e - \frac{d_\sigma}{|G|}\chi_\sigma. \tag{2.5.2}$$

To observe that, let $f = \sum_{\pi\in\widehat{G},\pi\neq\sigma}\alpha_\pi\chi_\pi$ (which belongs to $\mathrm{Ker}(\psi_\sigma)$), for some finite set $\{\alpha_\pi\}_{\pi\in\widehat{G},\pi\neq\sigma} \subseteq \mathbb{C}$. Therefore

$$\begin{aligned} f * u_\sigma &= f - \sum_{\pi\in\widehat{G},\pi\neq\sigma}\alpha_\pi\chi_\pi * \frac{1}{|G|}d_\sigma\chi_\sigma \\ &= f - \sum_{\pi\in\widehat{G},\pi\neq\sigma}\alpha_\pi\frac{d_\sigma}{|G|}\chi_\sigma * \chi_\pi = f \end{aligned}$$

by (2.5.1).

Now suppose $G = G_1\times\cdots\times G_n$. Then the Banach algebra $\ell^1(G)$ is isometrically isomorphic with $\ell^1(G_1)\otimes_\gamma\ldots\otimes_\gamma\ell^1(G_n)$. Let $\sigma,\pi \in \widehat{G}$. Then for $i = 1,\ldots,n$, there exists $\sigma_i,\pi_i \in \widehat{G_i}$ such that $\sigma = \sigma_1\times\cdots\times\sigma_n$ and $\pi = \pi_1\times\cdots\times\pi_n$, by Theorem 2.3.1; and since $d_\sigma = \prod_{i=1}^n d_{\sigma_i}$ and $d_\pi = \prod_{i=1}^n d_{\pi_i}$, one may conclude from (2.5.1) that

$$\left(\bigotimes_{i=1}^n\frac{d_{\sigma_i}}{|G_i|}\chi_{\sigma_i}\right) * \left(\bigotimes_{i=1}^n\frac{d_{\pi_i}}{|G_i|}\chi_{\pi_i}\right) = \bigotimes_{i=1}^n\left(\frac{d_{\sigma_i}}{|G_i|}\chi_{\pi_i} * \frac{d_{\pi_i}}{|G_i|}\chi_{\pi_i})\right) = \begin{cases} \otimes_{i=1}^n\frac{d_{\sigma_i}}{|G_i|}\chi_{\sigma_i} & \sigma = \pi \\ 0 & \sigma\neq\pi \end{cases}$$

Hence, one may rewrite the whole story of the identity of kernel of some character ψ_σ corresponding to a representation $\sigma \in \widehat{G}$. Consequently, (2.5.2) implies that

$$u_\sigma = \delta_e - \left(\frac{1}{|G_1|}d_{\sigma_1}\chi_{\sigma_1}\right)\otimes\cdots\otimes\left(\frac{1}{|G_n|}d_{\sigma_n}\chi_{\sigma_n}\right) \in \ell^1(G_1)\otimes_\gamma\ldots\otimes_\gamma\ell^1(G_n), \tag{2.5.3}$$

is an identity for $\mathrm{Ker}(\psi_\sigma)$.

Theorem 2.5.1. *Let $(G_i)_{i\in\mathbf{I}}$ be a family of finite groups, and let $G = \bigoplus_{i\in\mathbf{I}}G_i$. Let $\omega \in \sigma(\mathrm{Z}\ell^1(G))$, and let $(\chi_i)_{i\in\mathbf{I}}$ be the corresponding family of group characters. Then* $\mathrm{Ker}\,\omega$ *has a bounded approximate identity, if and only if $d_{\chi_i}\|\chi_i\|_1 = |G_i|$ for all but finitely many $i \in \mathbf{I}$.*

Proof. Define for each $F \subseteq \mathbf{I}$ finite, $H_F = \bigoplus_{i\in F} G_i$ and $H_{F^c} = \bigoplus_{i\in \mathbf{I}\setminus F} G_i$, so that $\mathrm{Z}\ell^1(G) = \mathrm{Z}\ell^1(H_F \times H_{F^c})$. Write E_{F^c} for the identity element of $\mathrm{Z}\ell^1(H_{F^c})$, i.e. the Dirac measure at the identity element of H_{F^c}, and likewise write E_F for the identity element of $\mathrm{Z}\ell^1(H_F)$. We also write d_i for the degree of χ_i.

First suppose that $d_i\|\chi_i\|_1 = |G_i|$ for all but finitely many $i \in \mathbf{I}$. As in Equation (2.5.3), define $u_F \in \mathrm{Z}\ell^1(H_F)$ by

$$u_F = E_F - \bigotimes_{i\in F} \frac{d_i}{|G_i|}\chi_i\,.$$

Let $\omega_F := \omega\circ\imath_F$, where $\imath_F : \mathrm{Z}\ell^1(H_F) \to \mathrm{Z}\ell^1(G)$ was defined in (2.2.2). Then u_F is the identity element of $\operatorname{Ker}\omega_F$. As $\otimes_\gamma$ is a cross-norm,

$$\sup_F \|u_F\|_1 = \sup_F \|E_F - \bigotimes_{i\in F} \frac{d_i}{|G_i|}\chi_i\| \le 1 + \sup_F \|\bigotimes_{i\in F} \frac{d_i}{|G_i|}\chi_i\| = 1 + \sup_F \prod_{i\in F}\left(\frac{d_i}{|G_i|}\|\chi_i\|_1\right) < \infty\,.$$

Moreover, since $\|\imath_F(u_F)\|_1 = \|u_F \otimes E_{F^c}\|_1 = \|u_F\|_1$, the family $(\imath_F(u_F))_{F\subset\mathbf{I},|F|<\infty}$ is bounded. We claim that it is, when ordered by inclusion of finite subsets, a bounded approximate identity for $\operatorname{Ker}\omega$. To prove this, it is enough to prove that

$$\lim_{F\subseteq\mathbf{I},|F|<\infty} (\imath_F(u_F)) * f = f \quad \text{for all } f \in \operatorname{Ker}(\omega); \tag{2.5.4}$$

and by a standard approximation argument, we may assume without loss of generality that f has a finite support. Thus, for $f \in Zc_c(G)\cap\operatorname{Ker}(\omega)$ and S the support of f; if F is any finite subset of $\mathbf{I}$ such that $S \subseteq \otimes_{i\in F} G_i \otimes E_{F^c}$, then

$$f = \imath_F \mathbf{P}_F(f)$$

(where $\mathbf{P}_F$ is the homomorphisms defined in (2.2.3)). So, $0 = \omega(f) = \omega\circ\iota_F\circ\mathbf{P}_F(f) = \omega_F(\mathbf{P}_F(f))$. Consequently $\mathbf{P}_F(f) \in \operatorname{Ker}(\omega_F)$, and thus

$$f * \imath_F(u_F) = \imath_F(\mathbf{P}_F(f) * u_F) = \imath_F(\mathbf{P}_F(f)) = f.$$

This proves Equation (2.5.4).

Conversely, suppose that $\operatorname{Ker}\omega$ has a bounded approximate identity say $(h_\alpha)_\alpha$. For each $F \subseteq \mathbf{I}$ when $|F| < \infty$ define

$$\Lambda_F^\omega(f\otimes g) = \omega_{F^c}(g) f \quad \text{for all } f \in \mathrm{Z}\ell^1(H_F) \text{ and } g \in \mathrm{Z}\ell^1(H_{F^c}).$$

Since $\omega_{F^c} = \omega\circ\imath_{F^c}$ has norm 1, being an algebra character, we have

$$\|\Lambda_F^\omega(f\otimes g)\|_1 \le \|f\|_1\|g\|_1,$$

and so Λ_F^ω defines a linear contraction from $Z\ell^1(H_F \times H_{F^c})$ onto $Z\ell^1(H_F)$, using Lemma 2.1.2.

Moreover, given $f_1, f_2 \in Z\ell^1(H_F)$ and $g_1, g_2 \in Z\ell^1(H_F^c)$,

$$\begin{aligned}\Lambda_F^\omega\big((f_1 \otimes g_1) * (f_2 \otimes g_2)\big) &= \Lambda_F^\omega\big((f_1 * f_2) \otimes (g_1 * g_2)\big) = \omega_{F^c}(g_1 * g_2) f_1 * f_2 \\ &= \omega_{F^c}(g_1) f_1 * \omega_{F^c}(g_1) f_2 = \Lambda_F^\omega(f_1 \otimes g_1) * \Lambda_F^\omega(f_2 \otimes g_2).\end{aligned}$$

Hence, by linearity and continuity, Λ_F^ω is an algebra homomorphism. Moreover,

$$\omega_F \Lambda_F^\omega(f \otimes g) = \omega_F(\omega_{F^c}(g) f) = \omega_F(f)\omega_{F^c}(g) = \omega(f \otimes g)$$

and

$$\Lambda_F^\omega \imath_F(f) = \Lambda_F^\omega(f \otimes E_{F^c}) = \omega_{F^c}(E_{F^c}) f = f$$

for all $f \in Z\ell^1(H_F)$ and $g \in Z\ell^1(H_{F^c})$.

Observe, since $\omega_F \Lambda_F^\omega = \omega$, that $\Lambda_F^\omega(\mathrm{Ker}(\omega)) \subseteq \mathrm{Ker}(\omega_F)$. Moreover, for each $f \in \mathrm{Ker}(\omega_F)$, $\omega(\iota_F(f)) = \omega(f \otimes E_{F^c}) = \omega_F(f)$; therefore, $\imath_F(f) \in \mathrm{Ker}(\omega)$. Since $u_F \in \mathrm{Ker}(\omega_F)$ and (h_α) is a bounded approximate identity for $\mathrm{Ker}(\omega_F)$,

$$\begin{aligned}\|\Lambda_F^\omega(h_\alpha) - u_F\|_1 &= \|\Lambda_F^\omega(h_\alpha) * u_F - u_F\|_1 \\ &= \|\Lambda_F^\omega(h_\alpha) * \Lambda_F^\omega \imath_F(u_F) - \Lambda_F^\omega \imath_F(u_F)\|_1 \\ &= \|\Lambda_F^\omega\left(h_\alpha * \imath_F(u_F) - \imath_F(u_F)\right)\|_1 \qquad \to 0\,.\end{aligned}$$

Because $\|\Lambda_F^\omega\| \leq 1$,

$$\sup_{F \subseteq \mathbf{I}, |F| < \infty} \sup_\alpha \|\Lambda_F^\omega(h_\alpha)\|_1 \leq \sup_\alpha \|h_\alpha\|_1 \leq M$$

for some $M > 0$, thus $\|u_F\|_1 \leq M$ for all finite subsets $F \subseteq \mathbf{I}$. Hence

$$\prod_{i \in F} \frac{d_i}{|G_i|} \|\chi_i\|_1 = \|E_F - u_F\|_1 \leq M + 1 \tag{2.5.5}$$

Let $i \in \mathbf{I}$. For each i, $|G_i|^{-1} d_i \chi_i$ is a central idempotent in the group algebra $\ell^1(G_i)$; in particular it has ℓ^1-norm ≥ 1. Moreover, by Proposition 2.4.1,

$$\text{either} \quad |G_i|^{-1} d_i \|\chi_i\|_1 = 1 \qquad \text{or} \quad |G_i|^{-1} d_i \|\chi_i\|_1 \geq \frac{301}{300}\,.$$

But if $\{i \in \mathbf{I} : d_i \|\chi_i\| > |G_i|\}$ is infinite, we may find a subset $F \subseteq \mathbf{I}$ such that $(301/300)^{|F|} > M + 1$. But by (2.5.5),

$$\left(\frac{301}{300}\right)^{|F|} \leq \prod_{i \in F} \frac{d_i}{|G_i|} \|\chi_i\|_1 = \|E_F - u_F\|_1 \leq M + 1$$

which is a contradiction. Therefore, $d_i \|\chi_i\| = |G_i|$ for all but finitely many i. □

Although the following theorem is not resulted from Theorem 2.5.1 directly, its proof is analogous to some parts of the proof of Theorem 2.5.1.

Theorem 2.5.2. *Let $(G_i)_{i\in\mathbf{I}}$ be a family of finite groups and let $G = \bigoplus_{i\in\mathbf{I}} G_i$. Then the following are equivalent:*

(i) *every maximal ideal in $\mathrm{Z}\ell^1(G)$ has a bounded approximate identity;*

(ii) *there is a finite subset $F \subset \mathbf{I}$ such that, for each $i \in \mathbf{I} \setminus F$ and each irreducible group character χ of G_i, we have $d_\chi \|\chi\|_1 = |G_i|$.*

(iii) *there exists a constant $M > 0$ such that each maximal ideal in $\mathrm{Z}\ell^1(G)$ has a bounded approximate identity of norm $\le M$.*

Proof.

(iii) $\Longrightarrow$ (i). This is trivial.

(ii) $\Longrightarrow$ (iii). Let F be as assumed in (ii), and define

$$M := \prod_{i\in F} \sup_{\pi\in\widehat{G}} \frac{d_\pi}{|G_n|} \|\chi_\pi\|_1 < \infty.$$

Given $\omega \in \sigma(\mathrm{Z}\ell^1(G))$, let (χ_i) be the corresponding family of (irreducible) group characters, and let d_i denote the degree of χ_i. For each finite subset $T \subset \mathbf{I}$, define $u_T \in \mathrm{Z}\ell^1(\bigoplus_{i\in T} G_i) = \widehat{\bigotimes}_{i\in T} \mathrm{Z}\ell^1(G_i)$ by

$$u_T = \delta_e - \bigotimes_{i\in T} \frac{d_i}{|G_i|}\chi_i$$

Order the net $(\imath_T(u_T))$, where T ranges over all finite subsets of $\mathbf{I}$, by inclusion. Then by an argument like that in the proof of Theorem 2.5.1, $(\imath_T(u_T))$ is a bounded approximate identity for $\operatorname{Ker}\omega$, with $\sup_T \|\imath_T(u_T)\| \le M + 1$.

(i) $\Longrightarrow$ (ii). Suppose that (ii) does not hold. Then there exists an infinite set $S \subset \mathbf{I}$, and for each $j \in S$, an irreducible group character ϕ_j on G_j such that $d_j\|\phi_j\|_1 \ne |G_j|$. Since $|G_j|^{-1} d_j \phi_j$ is an idempotent in $\mathrm{Z}\ell^1(G_j)$, Proposition 2.4.1 implies that $|G_j| d_j \|\phi_j\|_1 \ge 301/300$. Now let ω in $\sigma(\mathrm{Z}\ell^1(G))$ be such that the corresponding family (χ_i) of group characters satisfies $\chi_j = \phi_j$ for all $j \in S$ and $\chi_j \equiv 1$ for $j \in \mathbf{I} \setminus S$. Then as in the last part of the proof of Theorem 2.5.1, we can show that $\operatorname{Ker}(\omega)$ does not have a bounded approximate identity. □

For any irreducible group character χ on a finite group G, $d_\chi \|\chi\|_1 \geq |G|$ implies a lower bound. In the rest of this section, we investigate this property a bit deeper.

Lemma 2.5.3. *Let G be a finite group and χ an irreducible character on G. Then $d_\chi \|\chi\|_1 \geq |G|$. Moreover, equality holds if and only if*

$$|\chi(x)| \in \{0, d_\chi\} \quad \textit{for all } x \in G. \tag{2.5.6}$$

Proof. Since χ is irreducible, $\sum_{x \in G} |\chi(x)|^2 = |G|$, [28, Proposition 5.23]. Moreover, since $\|\chi\|_\infty \leq d_\chi$,

$$|G| = \sum_{x \in G} |\chi(x)|^2 \leq d_\chi \sum_{x \in G} |\chi(x)| = d_\chi \|\chi\|_1. \tag{$*$}$$

For the second statement, we need to show that equality holds in ($*$) if and only if (2.5.6) is satisfied. If $|\chi(x)| \in \{0, d_\chi\}$ for all $x \in G$, clearly the inequality in ($*$) is replaced by an equality. Conversely, if (2.5.6) is not satisfied, pick $y \in G$ such that $0 < |\chi(y)| < d_\chi$. Then $|\chi(y)|^2 < d_\chi |\chi(y)|$, so that

$$\sum_{x \in G} |\chi(x)|^2 = |\chi(y)|^2 + \sum_{x \in G \setminus \{y\}} |\chi(x)|^2 < d_\chi |\chi(y)| + d_\chi \sum_{x \in G \setminus \{y\}} |\chi(x)| = d_\chi \sum_{x \in G} |\chi(x)|$$

as required. □

Following [42], in [3], a character satisfying (2.5.6) is called *absolutely idempotent character*, abbreviated as AIC. Clearly each linear character is AIC. We also call a finite group G to be *AIC* if each irreducible character of G is AIC. It follows from the definition that quotients of AIC groups and from Theorem 2.3.1 that products of AIC groups are also AIC. F. Ladisch proves the following result about AIC groups, [3].

Theorem 2.5.4. *Every finite AIC group is nilpotent.*

Chapter 3

Three families of examples

In this chapter we introduce three main classes of discrete commutative hypergroups. In the rest of this manuscript, we will study more properties of these classes of hypergroups.

3.1 $\mathrm{Conj}(G)$ as a hypergroup

In this section, G is a discrete finite conjugacy group with the group algebra $(\ell^1(G), \circledast, \|\cdot\|_1)$ where $\circledast$ denotes the regular convolution of the group algebra. Furthermore, $\mathrm{Conj}(G)$ is the set of conjugacy classes of G. By Theorem 1.2.2, we know that for each two conjugacy classes C and D, 1_C and 1_D, the characteristic functions of C and D respectively, belong to $\mathrm{Z}\ell^1(G)$; thus, $1_C \circledast 1_D \in \mathrm{Z}\ell^1(G)$, for $\circledast$ the convolution of $\ell^1(G)$. Let Ψ denotes a linear mapping from $c_c(G) \cap Z\ell^1(G)$ to $c_c(\mathrm{Conj}(G))$ such that for each $f \in Z\ell^1(G) \cap c_c(G)$, $\Psi(f)(C) = |C| f(C)$ for $C \in \mathrm{Conj}(G)$ where

$$f(C) := f(x) \quad \text{for (every) } x \in C. \tag{3.1.1}$$

First note that

$$\|\Psi(f)\|_1 = \sum_{C \in \mathrm{Conj}(G)} |\Psi(f)(C)| = \sum_{C \in \mathrm{Conj}(G)} |f(C)||C| = \sum_{t \in G} |f(t)| = \|f\|_1,$$

which shows that Ψ is an isometry. On the other hand, for each $C \in \mathrm{Conj}(G)$, $\Psi(1_C) = |C|\delta_C$. Therefore Ψ is surjective. Since $c_c(G) \cap Z\ell^1(G)$ and $c_c(\mathrm{Conj}(G))$ are dense in $Z\ell^1(G)$ and $\ell^1(\mathrm{Conj}(G))$ respectively, Ψ can be extended as an isometric linear mapping from $Z\ell^1(G)$ onto $\ell^1(\mathrm{Conj}(G))$.

Applying Ψ, let us define an associative binary operation $*$ on $c_c(\mathrm{Conj}(G))$, where

$$\delta_C * \delta_D := \frac{1}{|C||D|} \Psi(1_C \circledast 1_D) \quad C, D \in \mathrm{Conj}(G). \tag{3.1.2}$$

A simple approximation argument lets us to extend $*$, called convolution, as a continuous bilinear action on $\ell^1(\mathrm{Conj}(G))$; hence, $(\ell^1(\mathrm{Conj}(G)), *, \|\cdot\|_1)$ forms a Banach algebra. Therefore, from now on we identify each function $f \in \ell^1(\mathrm{Conj}(G))$ with its pre-image with respect to Ψ. The following theorem is an immediate result of the previous observations.

Theorem 3.1.1. *Let G be a FC group. Then the Banach algebra $(\ell^1(\mathrm{Conj}(G)), *)$ is isometrically isomorphic to $(Z\ell^1(G), \circledast)$.*

Remark 3.1.2. Note that for each $z \in G$ and $C, D \in \mathrm{Conj}(G)$ such that $1_C \circledast 1_D(z) \neq 0$,

$$1_C \circledast 1_D(z) = \sum_{t\in G} 1_C(t)1_D(t^{-1}z) = \sum_{t\in C} 1_{tD}(z);$$

therefore, $z \in CD$. Moreover, $\mathrm{supp}(1_C \circledast 1_D)$ is a subset of G which is invariant with respect to inner automorphisms, so for some $\alpha_E^{C,D} \geq 0$, we have

$$1_C \circledast 1_D = \sum_{E\in\mathrm{Conj}(G), E\subseteq CD} \alpha_E^{C,D} 1_{E_r}.$$

So for all $C, D \in \mathrm{Conj}(G)$,

$$\Psi(1_C \circledast 1_D) = \Psi(\sum_{E\in\mathrm{Conj}(G), E\subseteq CD} \alpha_E^{C,D} 1_E) = \sum_{E\in\mathrm{Conj}(G), E\subseteq CD} \alpha_E^{C,D} |E|\delta_E.$$

Therefore

$$\delta_C * \delta_D = \frac{1}{|C||D|} \sum_{E\in\mathrm{Conj}(G), E\subseteq CD} \alpha_E^{C,D} |E|\delta_E. \tag{3.1.3}$$

We show that $\mathrm{Conj}(G)$ is a discrete hypergroup with the convolution defined in (3.1.2) with respect to Definition 1.1.2. We observed that $(\ell^1(\mathrm{Conj}(G)), *, \|\cdot\|_1)$ forms a Banach algebra. Moreover, for each $C, D \in \mathrm{Conj}(G)$, one can write

$$\begin{aligned}\sum_{E\in\mathrm{Conj}(G), E\subseteq CD} \alpha_E^{C,D}|E| &= \sum_{t\in G} 1_C \circledast 1_D(t) \\ &= \sum_{t\in G}\sum_{s\in G} 1_C(s)1_D(s^{-1}t) \\ &= \sum_{s\in G} 1_C(s) \sum_{t\in G} 1_D(t) = |C||D|.\end{aligned}$$

Hence, $\|\delta_C * \delta_D\|_1 = 1$ for the positive measure $\delta_C * \delta_D$. So $\ell^1(\mathrm{Conj}(G))$ satisfies $(H1)$. For e, the identity of the group G, let us denote the conjugacy class $\{e\}$ by e as well. Therefore, $1_e \circledast 1_C = 1_C$ for each $C \in \mathrm{Conj}(G)$; hence, $\delta_e * \delta_C = \delta_C * \delta_e = \delta_C$ and therefore $\mathrm{Conj}(G)$ satisfies $(H4)$.

If for $C \in \mathrm{Conj}(G)$, $\check{C} := C^{-1}$ where $C^{-1} = \{x^{-1} : x \in C\}$, one has that $e \in CC^{-1}$; hence, $e \in \mathrm{supp}(1_C \star 1_{C^{-1}})$ and consequently $e \in \mathrm{supp}(\delta_C * \delta_{C^{-1}})$. On the other hand, suppose that $e \in \mathrm{supp}(\delta_C * \delta_D)$ for some $C, D \in \mathrm{Conj}(G)$. Therefore,

$$0 \neq 1_C \circledast 1_D(e) = \sum_{t\in G} 1_C(t)1_D(t^{-1}) = \sum_{t\in C} 1_D(t^{-1}).$$

It implies that at least for one $x \in C$, $x^{-1} \in D$. Then by a simple argument about conjugacy classes of groups, one may verify that $D = C_{x^{-1}} = \check{C}_x = \check{C}$. This implies $(H6)$. For each

$f \in \ell^1(G)$, we define $\tilde{f}(x) = \overline{f(x^{-1})}$ for all $x \in G$. A brief study of the properties of $\circledast$ verifies that $(f \circledast g)\check{} = \tilde{g} \circledast \tilde{f}$. Also, $\tilde{1}_C = 1_{\check{C}}$ for each $C \in \mathrm{Conj}(G)$. Since $|C| = |\check{C}|$ for all $C \in \mathrm{Conj}(G)$,

$$\begin{aligned}
(\delta_C * \delta_D)\check{} &= \Big(\frac{1}{|C||D|} \sum_{E\in\mathrm{Conj}(G), E\subseteq CD} \alpha_E^{C,D} |E| \delta_E\Big)\check{} \\
&= \Big(\frac{1}{|\check{C}||\check{D}|} \sum_{\check{E}\subseteq\check{C}\check{D}} \alpha_E^{C,D} |\check{E}| \delta_{\check{E}}\Big) \\
&= \Big(\frac{1}{|\check{C}||\check{D}|} \sum_{\check{E}\subseteq\check{C}\check{D}} 1_C \circledast 1_D(E) |\check{E}| \delta_{\check{E}}\Big) \\
&= \Big(\frac{1}{|\check{C}||\check{D}|} \sum_{\check{E}\subseteq\check{C}\check{D}} 1_{\check{D}} \circledast 1_{\check{C}}(\check{E}) |\check{E}| \delta_{\check{E}}\Big) = \delta_{\check{D}} * \delta_{\check{C}}.
\end{aligned}$$

So $\mathrm{Conj}(G)$ satisfies $(H5)$. This implies that $\mathrm{Conj}(G)$ is a commutative discrete hypergroup.

Note that to avoid conflict between our notations, in the section $\circledast$ denotes the convolution of the group algebra of G, $\ell^1(G)$, while $*$ denotes the convolution of the hypergroup algebra of $\mathrm{Conj}(G)$, $\ell^1(\mathrm{Conj}(G))$.

Example 3.1.3. For a family of finite groups $(G_i)_{i\in\mathbf{I}}$, let $G := \bigoplus_{i\in\mathbf{I}} G_i$ be the restricted direct product of $(G_i)_{i\in\mathbf{I}}$. Then G is a discrete FC group, by Proposition 2.2.1. For each $C \in \mathrm{Conj}(G)$, C can be seen as the conjugacy class of some $x = (x_i)_{i\in\mathbf{I}} \in C$. On the other hand, for each $i \in \mathbf{I}_x = \{i \in \mathbf{I} : x_i \neq e_{G_i}\}$, $C_{x_i} \neq e_{G_i}$. Therefore, $C = \prod_{i\in\mathbf{I}_x} C_{x_i} \times E^c_{\mathbf{I}_x}$ where $E^c_{\mathbf{I}_x}$ is the identity of the group $\bigoplus_{i\in\mathbf{I}\setminus\mathbf{I}_x} G_i$; hence, $C \in \bigoplus_{i\in\mathbf{I}} \mathrm{Conj}(G_i)$. Conversely, for each $C \in \bigoplus_{i\in\mathbf{I}} \mathrm{Conj}(G_i)$, $C = (C_i)_{i\in\mathbf{I}}$ where $C_i = e_{G_i}$ for all $i \in \mathbf{I}$ except finitely many. We denote the set of all $i \in \mathbf{I}$ for them $C_i \neq e_{G_i}$ by $\mathbf{I}_C$. For each $i \in \mathbf{I}_C$, $C_i = C_{x_i}$ for some $x_i \in G_i$. Define $y = (y_i)_{i\in\mathbf{I}} \in G$ where $y_i = x_i$ for each $i \in \mathbf{I}_C$ and $x_i = e_{G_i}$ otherwise. It is not hard to show that $C_x = C \in \mathrm{Conj}(G)$. This argument implies that the hypergroup $\mathrm{Conj}(G)$ equals the hypergroup generated by the restricted direct product of $(\mathrm{Conj}(G_i))_{i\in\mathbf{I}}$,

$$\mathrm{Conj}(G) = \bigoplus_{i\in\mathbf{I}} \mathrm{Conj}(G_i),$$

as defined in Section 1.1.

Remark 3.1.4. By Theorem 1.1.4, for h, the Haar measure on $\mathrm{Conj}(G)$,

$$h(C) = \big(\delta_{\check{C}} * \delta_C(e)\big)^{-1} = |C| \ (C \in \mathrm{Conj}(G))$$

To prove that, note

$$\delta_{\check{C}} * \delta_C(e) = \frac{1}{|\check{C}|\,|C|} \sum_{E\subseteq\check{C}C} \alpha_E^{C,\check{C}} |E| \delta_E(e) = \frac{1}{|C|^2} 1_{C^{-1}} \circledast 1_C(e) = \frac{1}{|C|^2} \sum_{t\in G} 1_{C^{-1}}(t) 1_C(t^{-1}) = |C|^{-1}.$$

3.2 Dual of compact groups as hypergroups

In this section, G is a compact group. Let $\widehat{G}$ denote the set of all irreducible unitary representations of a compact group G, up to equivalence relation, as defined in Section 1.4. Here we follow the notation of [23] for the dual of compact groups and apply many results of [36, Section 27] about representation theory of compact groups.

Let $\phi = \{\phi_\pi : \pi \in \widehat{G}\}$ if $\phi_\pi \in \mathcal{B}(\mathcal{H}_\pi)$ for each $\pi \in \widehat{G}$ and define $\|\phi\|_{\mathcal{L}^\infty(\widehat{G})} := \sup_\pi \|\phi_\pi\|_\infty$ for $\|\cdot\|_\infty$ to be the operator norm of a matrix. The set of all those ϕ's with $\|\phi\|_{\mathcal{L}^\infty(\widehat{G})} < \infty$ forms a C^*-algebra denoted by $\mathcal{L}^\infty(\widehat{G})$. It is well known that $\mathcal{L}^\infty(\widehat{G})$ is isomorphic to the group von Neumann algebra of G i.e. the dual of $A(G)$, see [23, 8.4.17]. We define

$$\mathcal{L}^p(\widehat{G}) = \{\phi \in \mathcal{L}^\infty(\widehat{G}) : \|\phi\|^p_{\mathcal{L}^p(\widehat{G})} := \sum_{\pi\in\widehat{G}} d_\pi \|\phi_\pi\|^p_{\mathcal{S}_p} < \infty\}, \tag{3.2.1}$$

for $\|\cdot\|_{\mathcal{S}_p}$, the p-Schatten norm[1] that is $\|A\|^p_{\mathcal{S}_p} := \sum_{n\geq 1} s_n^p(A)$ for $s_1(A) \geq s_2(A) \geq \cdots s_n(A) \geq 0$ the singular values of a matrix A, i.e. the eigenvalues of the Hermitian matrix $|A| := \sqrt{(A^*A)}$. For each p, $\mathcal{L}^p(\widehat{G})$ is an ideal of $\mathcal{L}^\infty(\widehat{G})$, see [23, 8.3]. Moreover, we define

$$\mathcal{C}_0(\widehat{G}) = \{\phi \in \mathcal{L}^\infty(\widehat{G}) : \lim_{\pi\to\infty} \|\phi_\pi\|_\infty = 0\}.$$

For each $f \in L^1(G)$, $\mathcal{F}(f) = (\hat{f}(\pi))_{\pi\in\widehat{G}}$ belongs to $\mathcal{C}_0(\widehat{G})$, where $\mathcal{F}$ denotes the Fourier transform and

$$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx.$$

Indeed, $\mathcal{F}(L^1(G))$ is a dense subset of $\mathcal{C}_0(\widehat{G})$ and $\mathcal{F}$ is an algebra isomorphism from $L^1(G)$ onto its image where $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$ for all $f, g \in L^1(G)$. Moreover, $\mathcal{F}(A(G))$ is isometrically isomorphic to the Banach space $\mathcal{L}^1(\widehat{G})$, [23].

For each two unitary irreducible representations $\pi, \sigma \in \widehat{G}$, we know that $\pi \otimes \sigma$ forms a new unitary representation of G whose dimension is $d_\pi d_\sigma$. This new representation can be decomposed as a direct product of a finite set of irreducible unitary representations $\pi_1, \ldots, \pi_n$ with respective positive constants $m_1^{\pi,\sigma}, \ldots, m_n^{\pi,\sigma} \in \mathbb{N}$, i.e.

$$\pi \otimes \sigma \cong \bigoplus_{i=1}^n m_i^{\pi,\sigma} \pi_i.$$

[1] Note that , as described in Chapter 1, for $p = 1$ and $p = 2$, p-Schatten norm is called Trace norm and Hilbert-Schmidt norm respectively.

consequently,

$$\chi_\pi\chi_\sigma = \mathrm{Tr}(\pi)\,\mathrm{Tr}(\sigma) = \mathrm{Tr}(\pi\otimes\sigma) = \mathrm{Tr}\left(\bigoplus_{i=1}^{n} m_i^{\pi,\sigma}\pi_i\right) = \sum_{i=1}^{n} m_i^{\pi,\sigma}\,\mathrm{Tr}(\pi_i) = \sum_{i=1}^{n} m_i^{\pi,\sigma}\chi_{\pi_i}. \tag{3.2.2}$$

Therefore the dimension of $\pi\otimes\sigma$ which is $d_\pi d_\sigma$ is equivalent to $\sum_{i=1}^{n} m_i^{\pi,\sigma} d_{\pi_i}$. We define a convolution on $c_c(\widehat{G})$ by

$$\delta_\pi * \delta_\sigma := \sum_{i=1}^{n} \frac{m_i^{\pi,\sigma} d_{\pi_i}}{d_\pi d_\sigma}\delta_{\pi_i}. \tag{3.2.3}$$

Note that

$$\Big\|\sum_{i=1}^{n} \frac{m_i^{\pi,\sigma} d_{\pi_i}}{d_\pi d_\sigma}\delta_{\pi_i}\Big\|_{\ell^1(\widehat{G})} = \sum_{i=1}^{n} \frac{m_i^{\pi,\sigma} d_{\pi_i}}{d_\pi d_\sigma} = 1 \le \|\delta_\pi\|_{\ell^1(\widehat{G})}\|\delta_\sigma\|_{\ell^1(\widehat{G})}. \tag{3.2.4}$$

By (3.2.3) the convolution is submultiplicative on $c_c(\widehat{G})$, and since $c_c(\widehat{G})$ is dense in $\ell^1(\widehat{G})$; we can extend the convolution defined in (3.2.3) to $\ell^1(\widehat{G})$. For each representation $\pi\in\widehat{G}$, $\overline{\pi}$ denotes the complex conjugate of π, [36, Definition 27.27].

Theorem 3.2.1. *Let G be a compact group. Then $\widehat{G}$ equipped with discrete topology, the convolution (3.2.3), and the involution resulting from complex conjugate forms a discrete commutative hypergroup.*

Proof. (3.2.4) implies $(H1)$ while the associativity is resulted from associativity of tensor products of the representations i.e. $\pi_1\otimes(\pi_2\otimes\pi_3)\cong(\pi_1\otimes\pi_2)\otimes\pi_3$. Commutativity is a direct result of this fact that $\pi\otimes\sigma$ is equivalent to $\sigma\otimes\pi$. The trivial representation $\pi_0: G\to\mathcal{U}(\mathbb{C})$ where $\pi_0(x)=1$ for all $x\in G$ always belongs to $\widehat{G}$. Also, $\pi\otimes\pi_0\cong\pi$ for all $\pi\in\widehat{G}$. So δ_{π_0} plays the role of the identity of $\ell^1(\widehat{G})$ and so $\widehat{G}$ satisfies $(H4)$. Since $\overline{\overline{\pi}}\cong\pi$ and $\overline{\pi\otimes\sigma}\cong\overline{\pi}\otimes\overline{\sigma}$, $(H5)$ is held. One side of $(H6)$ is directly resulted from [36, (27.34)]. On the other hand, suppose that for two representations $\pi,\sigma\in\widehat{G}$, $\chi_\pi\chi_\sigma=\sum_{i=1}^{n} m_i^{\pi,\sigma}\chi_{\pi_i}$ such that for one i, say $i=1$, $\pi_1=\pi_0$, the trivial representation of G, and $m_1^{\pi,\sigma}>0$. Then

$$\int_G \sum_{i=1}^{n} m_i^{\pi,\sigma}\overline{\chi_{\pi_0}(x)}\chi_{\pi_i}(x)dx = \sum_{i=1}^{n} m_i^{\pi,\sigma}\int_G \overline{\chi_{\pi_0}(x)}\chi_{\pi_i}(x)dx = \sum_{i=1}^{n} m_i^{\pi,\sigma}\langle\chi_{\pi_i},\chi_{\pi_0}\rangle_{L^2(G)} = m_1^{\pi,\sigma}$$

since $\{\chi_\pi\}_{\pi\in\widehat{G}}$ forms an orthogonal basis for $ZL^2(G)$ (see [36, Theorem 27.24]). Note that $\chi_{\pi_0}(x)\equiv 1$ and $\overline{\chi_\pi(x)}=\chi_{\overline{\pi}}(x)$; hence similarly,

$$m_1^{\pi,\sigma} = \int_G \chi_\pi(x)\chi_\sigma(x)\overline{\chi_{\pi_0}(x)}dx = \int_G \chi_\pi(x)\overline{\chi_{\overline{\sigma}}(x)}dx = \langle\chi_\pi,\chi_{\overline{\sigma}}\rangle_{L^2(G)} = \delta_{\pi,\overline{\sigma}},$$

Therefore $\pi=\overline{\sigma}$. □

To calculate the Haar measure of $\widehat{G}$, we apply Theorem 1.1.4. By [36, (27.34)], the multiplicity of π_0 in the irreducible decomposition of $\pi\otimes\overline{\pi}$ is 1. So for each $\pi\in\widehat{G}$, the Haar measure is defined by

$$h(\pi) = \left(\delta_\pi * \delta_{\overline{\pi}}(\pi_0)\right)^{-1} = \frac{d_\pi d_{\overline{\pi}}}{1} = d_\pi^2,$$

since $d_{\overline{\pi}} = d_\pi$.

Example 3.2.2. Let $SU(2)$ denote the compact Lie group of 2×2 special unitary matrices on $\mathbb{C}$, and let $\widehat{SU(2)}$ be the hypergroup of all irreducible representations on $SU(2)$. It is known that

$$\widehat{SU(2)} = (\pi_\ell)_{\ell \in 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots}$$

where the dimension of π_ℓ is $2\ell + 1$, see [36, 29.13]. Moreover, for all ℓ, ℓ', $\overline{\pi}_\ell = \pi_\ell$ and

$$\pi_\ell \otimes \pi_{\ell'} \cong \bigoplus_{r=|\ell-\ell'|}^{\ell+\ell'} \pi_r = \pi_{|\ell-\ell'|} \oplus \pi_{|\ell-\ell'|+1} \oplus \cdots \oplus \pi_{\ell+\ell'} \quad \text{[36, Theorem 29.26]}$$

called "*Clebsch-Gordan*" decomposition formula. So using Definition 3.2.3, we have that

$$\delta_{\pi_\ell} * \delta_{\pi_{\ell'}} = \sum_{r=|\ell-\ell'|}^{\ell+\ell'} \frac{(2r+1)}{(2\ell+1)(2\ell'+1)} \delta_{\pi_r}.$$

Also $\overline{\pi}_\ell = \pi_\ell$ and $h(\pi_\ell) = (2\ell+1)^2$ for all ℓ.

Remark 3.2.3. Indeed, the hypergroup structure of $\widehat{SU(2)}$ can be rendered by a family of Chebyshev polynomials as a polynomial hypergroup structure on $\mathbb{N}_0$. We will define polynomial hypergroups in the following section.

Example 3.2.4. Suppose that $\{G_i\}_{i \in \mathbf{I}}$ is a non-empty family of compact groups for arbitrary indexing set $\mathbf{I}$. Let $G := \prod_{i \in \mathbf{I}} G_i$ be the product of $\{G_i\}_{i \in \mathbf{I}}$ i.e.

$$G := \{(x_i)_{i \in \mathbf{I}} : \ x_i \in G_i\}$$

equipped with the product topology. Then G is a compact group and [36, Theorem 27.43] implies that $\widehat{G}$ is nothing but

$$\{\pi = \bigotimes_{i \in \mathbf{I}} \pi_i : \text{ such that } \pi_i \in \widehat{G}_i \text{ and } \pi_i = \pi_0 \text{ except for finitely many } i \in \mathbf{I}\}$$

equipped with the discrete topology. Moreover, for each $\pi = \otimes_{i \in \mathbf{I}} \pi_i \in \widehat{G}$, $d_\pi = \prod_{i \in \mathbf{I}} d_{\pi_i}$. When $\pi_k = \otimes_{i \in \mathbf{I}} \pi_i^{(k)} \in \widehat{G}$ for $k = 1, 2$, one can show that

$$\delta_{\pi_1} * \delta_{\pi_2}(\pi) = \prod_{i \in \mathbf{I}} \delta_{\pi_i^{(1)}} *_{\widehat{G}_i} \delta_{\pi_i^{(2)}}(\pi_i) \quad \text{for all } \pi = \bigotimes_{i \in \mathbf{I}} \pi_i \in \widehat{G},$$

where $*_{\widehat{G}_i}$ is the hypergroup convolution of $\widehat{G}_i$ for each $i \in \mathbf{I}$. Also, each character χ of G is related to a family of characters $(\chi_i)_{i\in\mathbf{I}}$ such that χ_i is a character of G_i and

$$\chi(x) = \prod_{i\in\mathbf{I}} \chi_i(x_i)$$

for each $x = (x_i)_{i\in\mathbf{I}} \in G$. Note that $\chi_i \equiv 1$ for all of $i \in \mathbf{I}$ except finitely many; therefore, χ is well-defined.

3.3 Polynomial hypergroups on $\mathbb{N}_0$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(a_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$ be sequences of non-zero real numbers and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence of real numbers with the property

$$\begin{aligned} a_0 + b_0 &= 1 \\ a_n + b_n + c_n &= 1, \quad n \geq 1. \end{aligned}$$

If $(R_n)_{n\in\mathbb{N}_0}$ is a sequence of polynomials defined by

$$\begin{aligned} R_0(x) &= 1, \\ R_1(x) &= \tfrac{1}{a_0}(x - b_0), \\ R_1(x)R_n(x) &= a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \qquad n \geq 1, \end{aligned} \tag{3.3.1}$$

then it is proven in [12] that there exists a probability measure π on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} R_n(x)R_m(x)d\pi(x) = \delta_{n,m}\mu_m \tag{3.3.2}$$

where $(\mu_n)_{n\in\mathbb{N}_0}$ is a sequence of positive numbers. The sequence $(R_n)_{n\in\mathbb{N}_0}$ satisfying (3.3.2) is called an *orthogonal polynomial sequence*. By induction, one can see that $R_n(1) = 1$ for each $n \in \mathbb{N}_0$. Moreover,

$$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m;k)R_k(x) \tag{3.3.3}$$

where $g(n,m;k) \in \mathbb{R}$ for all $|n-m| \leq k \leq n+m$. Moreover, $g(n,m;|n-m|)$ and $g(n,m;n+m)$ are non-zero. The following theorem summarizes some of the main results of [49, Section 5].

Theorem 3.3.1. *Let $(R_n)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence defined by (3.3.1). Assume that*

$$g(n,m;k) \geq 0 \quad \forall n,m \in \mathbb{N}_0, \ |n-m| \leq k \leq n+m.$$

Let $*$ to be defined on $\mathbb{N}_0$ to $\ell^1(\mathbb{N}_0)$ such that

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g(n,m;k)\delta_k$$

and $\check{n} = n$. *Then* $(\mathbb{N}_0, *, \check{\ })$ *is a discrete commutative hypergroup with the unit element* 0 *which is called the polynomial hypergroup on* $\mathbb{N}_0$ *induced by* $(R_n)_{n\in\mathbb{N}_0}$.

[49, Section 5] is a good reference to observe almost all of the facts mentioned above. Here, we should mention that there are plenty of concrete examples of polynomials that satisfy the conditions of Theorem 3.3.1, namely *Chebyshev polynomial* of the first and second kinds, *cosh polynomials*, *ultraspherical polynomials*, *Jacobi polynomials*, *Karlin–McGregor polynomials*, and *little q-Legendre polynomials*.

Remark 3.3.2. For a polynomial hypergroup the left translation is defined by

$$L_n f(m) = \sum_{k=|n-m|}^{n+m} g(n, m; k) f(k).$$

Moreover, the Haar function is defined by

$$h(n) = (\delta_n * \delta_n)(0))^{-1} = g(n, n; 0)^{-1} = \mu_n^{-1}.$$

Chapter 4

Weighted discrete hypergroups

In this chapter we study weights on discrete hypergroups, their corresponding algebras, and their examples. Specially we are interested to see concrete examples of weights defined on the classes of commutative discrete hypergroups which were introduced in Chapter 3.

4.1 Weighted hypergroups and their algebras

Definition 4.1.1. Let H be a discrete hypergroup. We call a function $\omega : H \to (0,\infty)$ a *weight* if, for every $x, y \in H$,

$$\omega(\delta_x * \delta_y) \leq \omega(x)\omega(y)$$

where $\omega(\delta_x * \delta_y) = \sum_{t\in H} \omega(t)\delta_x * \delta_y(t)$ is as defined in Section 1.1. We call (H,ω) a *weighted hypergroup*. Let $\ell^1(H,\omega)$ be the set of all complex functions on H such that

$$\|f\|_{\ell^1(H,\omega)} := \sum_{t\in H} |f(t)|\omega(t) < \infty.$$

Then one can easily observe that $(\ell^1(H,\omega), \|\cdot\|_{\ell^1(H,\omega)})$ forms a Banach space.

This definition is a specific case of the weighted hypergroups defined in [32]; here, we focus mainly on discrete hypergroups.

Definition 4.1.2. A function $\omega_z : H \to (0,\infty)$ is called a *central weight* if

$$\omega_z(t) \leq \omega_z(x)\omega_z(y)$$

for all $t, x, y \in H$ where $t \in x * y (= \operatorname{supp}(\delta_x * \delta_y))$, in the sense of (1.1.1).

Since, $\delta_x * \delta_y$ is a positive probability measure, for each central weight ω_z, one gets that

$$\sum_{t\in H} \omega_z(t)\delta_x * \delta_y(t) \leq \sum_{t\in H} \omega_z(x)\omega_z(y)\delta_x * \delta_y(t) \leq \omega_z(x)\omega_z(y)\|\delta_x * \delta_y\| = \omega_z(x)\omega_z(y).$$

Hence, ω_z is a weight over a hypergroup H. Although, most of the hypergroup weight studied in here are central, in Subsection 4.4.1 and Section 4.6, we will see some examples of weights which are not central weights.

For all $f, g \in c_c(H)$, we have

$$\begin{aligned}
\|f * g\|_{\ell^1(H,\omega)} &= \sum_{t\in H} |f * g(t)|\omega(t) \\
&= \sum_{t\in H} |\sum_{x\in H}\sum_{y\in H} \delta_x * \delta_y(t) f(x)g(y)|\omega(t) \\
&\le \sum_{x\in H}\sum_{y\in H}\sum_{t\in H} \delta_x * \delta_y(t)\omega(t)|f(x)g(y)| \\
&= \sum_{x\in H}\sum_{y\in H} \omega(\delta_x * \delta_y)|f(x)|\,|g(y)| \\
&\le \sum_{x\in H} \omega(x)|f(x)| \sum_{y\in H} \omega(y)|g(y)| = \|f\|_{\ell^1(H,\omega)}\|g\|_{\ell^1(H,\omega)}.
\end{aligned}$$

Since $c_c(H)$ is dense in $\ell^1(H,\omega)$ and the convolution is continuous with respect to $\|\cdot\|_{\ell^1(H,\omega)}$, one may extend the convolution to $\ell^1(H,\omega)$. Therefore, $(\ell^1(H,\omega), *, \|\cdot\|_{\ell^1(H,\omega)})$ is actually a Banach algebra; we call it *weighted hypergroup algebra* of H with respect to the weight ω.

Moreover, we can see that the dual of $\ell^1(H,\omega)$ is nothing but $\ell^\infty(H,\omega^{-1})$ which is the set of all functions $\phi : H \to \mathbb{C}$ such that

$$\|\phi\|_{\ell^\infty(H,\omega^{-1})} := \sup_{t\in H} |f(t)|\omega(t)^{-1} < \infty.$$

We may easily see that $\ell^\infty(H,\omega^{-1})$ equipped with the norm $\|\cdot\|_{\ell^\infty(H,\omega^{-1})}$ forms a Banach space.

Definition 4.1.3. [17, Definition 2.6]
Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is a *dual Banach algebra* with respect to E, if E is a closed sub-bimodule of the dual $\mathcal{A}$-bimodule A^* that if for every $\phi \in E$ and $f \in \mathcal{A}$, $f\cdot\phi$ and $\phi\cdot f$ belong to E such that $\mathcal{A} = E^*$.

Let $c_0(H,\omega^{-1})$ be the set all elements ϕ in $\ell^\infty(H,\omega^{-1})$ such that $\phi\omega^{-1}$ is vanishing at infinity. Clearly $c_0(H,\omega^{-1})$ is Banach subspace of $\ell^\infty(H,\omega^{-1})$.

Proposition 4.1.4. *Let (H,ω) be a weighted discrete hypergroup for a central weight ω. Then $\ell^1(H,\omega)$ is a dual Banach algebra with respect to $c_0(H,\omega^{-1})$.*

Proof. We know that $\ell^1(H,\omega)$ can be considered as the dual of $c_0(H,1/\omega) = \{f : H \to \mathbb{C} : f/\omega \in c_0(H)\}$, by Riesz representation theorem; the second dual of $c_0(H,1/\omega)$ is $\ell^\infty(H,1/\omega)$. Let us define the Banach space isomorphism $\kappa : \ell^1(H,\omega) \to \ell^1(H)$ where $\kappa(f) = f\omega$ for each $f \in \ell^1(H,\omega)$. Then $\kappa^* : \ell^\infty(H) \to \ell^\infty(H,1/\omega)$ where

$$\langle\kappa^*(\phi), \delta_x\rangle = \langle\phi, \kappa(\delta_x)\rangle = \sum_{t\in H} \phi(t)\omega(t)\delta_x(t) = \phi(x)\omega(x)$$

for all $x \in H$. So, $\kappa^*(\phi) = \phi\omega$. One may easily show that $\kappa^*(c_0(H)) = c_0(H,1/\omega)$.

On the other hand, we show that $c_0(H,1/\omega)$ is an $\ell^1(H,\omega)$-bimodule. To do so, let $f,g \in \ell^1(H,\omega)$ and $\phi \in c_0(H,1/\omega)$. Hence

$$\begin{aligned}
\langle g, \phi\cdot f\rangle = \langle f*g,\phi\rangle &= \sum_{y\in H} f*g(y)\phi(y)\\
&= \sum_{y\in H}\sum_{t\in H}\sum_{s\in H} \delta_t*\delta_s(y)f(t)g(s)\phi(y)\\
&= \sum_{s\in H} g(s)\sum_{t\in H} f(t)\sum_{y\in H}\delta_t*\delta_s(y)\phi(y)\\
&= \sum_{s\in H} g(s)\sum_{t\in H} f(t)\phi(\delta_t*\delta_s).
\end{aligned}$$

Therefore,

$$\phi\cdot f(x) = \sum_{t\in H} f(t)\phi(\delta_t*\delta_x).$$

And similarly,

$$f\cdot\phi(x) = \sum_{t\in H} f(t)\phi(\delta_x*\delta_t).$$

Here we show that if $\phi\in c_0(H,1/\omega)$ and ω is central, $f\cdot\phi$ and $\phi\cdot f$ also belong to $c_0(H,1/\omega)$ for all $f\in\ell^1(H,\omega)$. To do so, let us recall from Proposition 1.1.7 that for every $\varphi\in c_0(H)$ and $\psi\in\ell^1(H)$,

$$x\mapsto \sum_{t\in H}\psi(t)\varphi(\delta_{\tilde{t}}*\delta_x) \tag{4.1.1}$$

belongs to $c_0(H)$.

Let ω be a central weight on H as defined in Definition 4.1.2. Note that for $\phi\in c_0(H,1/\omega)$ and $f\in\ell^1(H,\omega)$, $\kappa^{*-1}(|\phi|) = |\phi|\omega^{-1}\in c_0(H)$. Moreover, since $\sim\colon H\to H$ is a bijection, $\tilde{\kappa}(|f|) := |\tilde{f}|\tilde{\omega}\in\ell^1(H)$. Therefore,

$$\begin{aligned}
|\phi\cdot f(x)| &= \left|\sum_{t\in H} f(t)\phi(\delta_t*\delta_x)\right|\\
&\le \sum_{t\in H}|f|(t)|\phi|(\delta_t*\delta_x)\\
&= \sum_{t\in H}|f|(t)\sum_{s\in H}|\phi|(s)\delta_t*\delta_x(s)\\
&= \sum_{t\in H}\frac{\kappa(|f|)(t)}{\omega(t)}\sum_{s\in H}\kappa^{*-1}(|\phi|)(s)\,\omega(s)\,\delta_t*\delta_x(s)\\
&\le \sum_{t\in H}\frac{\kappa(|f|)(t)}{\omega(t)}\sum_{s\in t*x}\kappa^{*-1}(|\phi|)(s)\,\omega(x)\omega(t)\,\delta_t*\delta_x(s) \quad (\star)\\
&= \omega(x)\sum_{t\in H}\kappa(|f|)(t)\kappa^{*-1}(|\phi|)(\delta_t*\delta_x)\\
&= \omega(x)\sum_{t\in H}\tilde{\kappa}(|f|)(t)\kappa^{*-1}(|\phi|)(\delta_{\tilde{t}}*\delta_x). \quad (\star\star)
\end{aligned}$$

Note that $(\star)$ is because ω is central. By (4.1.1), $(\star\star)$ belongs to $c_0(H,1/\omega)$. Consequently, $\phi\cdot f$ whose absolute value is dominated by a function in $c_0(H,1/\omega)$ belongs to $c_0(H,1/\omega)$ as well.

Similarly, one may show that $f \cdot \phi$ lies in $c_0(H, 1/\omega)$. Therefore, $c_0(H, 1/\omega)$ is a sub-bimodule of $\ell^1(H,\omega)$-bimodule $\ell^\infty(H,\omega)$ while $c_0(H,1/\omega)^* = \ell^1(H,\omega)$. Consequently, $\ell^1(H,\omega)$ is a dual Banach algebra. □

It would be interesting to know if the theorem remains true for arbitrary weights, not just the central weights.

Definition 4.1.5. A hypergroup weight ω on H is called *weakly additive*, if for some $C > 0$, $\omega(\delta_x * \delta_y) \leq C(\omega(x) + \omega(y))$ for all $x, y \in H$. Also ω is said to be *centrally additive* if $\omega(t) \leq C(\omega(x) + \omega(y))$ for some $C > 0$ and all $x, y, t \in H$ such that $t \in x * y$.

Note that for all $x, y \in H$, $\sum_{t\in H} \delta_x * \delta_y(t) = 1$; therfore, for each weight ω which is centrally additive, it is weakly additive as well.

Definition 4.1.6. Let H be a hypergroup and ω_1 and ω_2 are two weights on H. Then ω_1 and ω_2 are two *equivalent* weights if there are two constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\omega_1 \leq \omega_2 \leq C_2\omega_1$.

It is straightforward to check that if two weights ω_1 and ω_2 on a hypergroup H are equivalent with respect to two constants C_1 and C_2 as defined in Definition 4.1.6,

$$C_1\|\cdot\|_{\ell^1(H,\omega_1)} \leq \|\cdot\|_{\ell^1(H,\omega_1)} \leq C_2\|\cdot\|_{\ell^1(H,\omega_1)};$$

hence, $\ell^1(H,\omega_1)$ is isomorphic to $\ell^1(H,\omega_2)$.

Remark 4.1.7. Let H be a discrete hypergroup equipped with the Haar measure h. For a weight $\omega : H \to (0,\infty)$, let $L^1(H,\omega)$ be the set of all complex functions on H such that

$$\|f\|_{L^1(H,\omega)} := \sum_{t\in H} |f(t)|\omega(t)h(t) < \infty.$$

Then $(L^1(H,\omega), \|\cdot\|_{\ell^1(H,\omega)})$ equipped with the convolution forms a Banach algebra .

4.1.1 Product of weighted hypergroups

Let (H_1,ω_1) and (H_2,ω_2) be two weighted hypergroups. For the hypergroup $H := H_1 \times H_2$ and the function $\omega := \omega_1 \times \omega_2 : H_1 \times H_2 \to (0,\infty)$, we have that

$$\begin{aligned}
\sum_{(s,t)\in H_1\times H_2} \omega(s,t)\delta_{(x_1,x_2)} *_H \delta_{(y_1,y_2)}(s,t) &= \sum_{s\in H_1} \delta_{x_1} *_{H_1} \delta_{y_1}(s)\omega_1(s) \sum_{t\in H_2} \delta_{x_2} *_{H_2} \delta_{y_2}(t)\omega_2(t) \\
&\leq \omega_1(x_1)\omega_1(y_1)\,\omega_2(x_2)\omega_2(y_2) \\
&= \omega(x_1,x_2)\omega(y_1,y_2).
\end{aligned}$$

Therefore, ω forms a weight on the hypergroup H. An argument similar to the group case implies that $\ell^1(H_1 \times H_2, \omega_1 \times \omega_2) = \ell^1(H_1, \omega_1) \otimes_\gamma \ell^1(H_2, \omega_2)$.

As an extension of the previous example, let $\{H_i\}_{i\in\mathbf{I}}$ be a family of discrete hypergroups with corresponding weights $\{\omega_i\}_{i\in\mathbf{I}}$ such that $\omega_i(e_{H_i}) = 1$ for all $i \in \mathbf{I}$ except finitely many. Let us recall from Section 1.1 that the restricted direct product of $\{H_i\}_{i\in\mathbf{I}}$, denoted by $H := \bigoplus_{i\in\mathbf{I}} H_i$, is

$$\{(x_i)_{i\in\mathbf{I}} : \ x_i = e_{H_i} \text{ for all } i \in \mathbf{I} \text{ but finitely many}\}.$$

We can define

$$\delta_{(x_i)_{i\in\mathbf{I}}} * \delta_{(y_i)_{i\in\mathbf{I}}}(s_i)_{i\in\mathbf{I}} := \prod_{i\in\mathbf{I}} \delta_{x_i} *_{H_i} \delta_{y_i}(s_i)$$

and

$$\omega(x_i)_{i\in\mathbf{I}} := \prod_{i\in\mathbf{I}} \omega_i(x_i)$$

which are well-defined. Using the properties of finite product of hypergroups, one may easily verify that H forms a discrete hypergroup. Moreover,

$$\begin{aligned} \sum_{(s_i)_{i\in\mathbf{I}}\in H} \omega(s_i)_{i\in\mathbf{I}}\, \delta_{(x_i)_{i\in\mathbf{I}}} * \delta_{(y_i)_{i\in\mathbf{I}}}(s_i)_{i\in\mathbf{I}} &= \prod_{i\in\mathbf{I}} \sum_{s_i\in H_i} \omega_i(s_i)\, \delta_{x_i} *_{H_i} \delta_{y_i}(s_i) \\ &\le \prod_{i\in\mathbf{I}} \omega_i(x_i)\, \omega_i(y_i) \\ &= \omega(x_i)_{i\in\mathbf{I}}\, \omega(y_i)_{i\in\mathbf{I}} \end{aligned}$$

for all $(x_i)_{i\in\mathbf{I}}$ and $(y_i)_{i\in\mathbf{I}}$ in H. Therefore, (H, ω) is a weighted hypergroup. Note that since in the aforementioned equations, only for finitely many indices, the corresponding values may not be 1, the calculations are well-defined.

4.2 Some weights related to the growth of hypergroups

If $a, b \geq 0$ and $\beta \geq 0$, then

$$(a+b)^\beta \leq C(a^\beta + b^\beta) \tag{4.2.1}$$

where $C = \min\{1, 2^{\beta-1}\}$. We will use this inequality in the following.

Let H be a discrete hypergroup. For each finite subset F of H, we define

$$F^n := \bigcup \{x_1 * \cdots * x_n : \ \text{ for all } x_1, \ldots, x_n \in F\}.$$

Definition 4.2.1. A hypergroup H is called a *finitely generated hypergroup* if there exists a finite subset $F \subseteq H$, called a *generator*, such that

$$H = \bigcup_{n\in\mathbb{N}} F^n.$$

Let F be a *finite symmetric generator* of H i.e. $x \in F$ implies that $\check{x} \in F$. Then we define

$$\tau_F : H \to \mathbb{N} \cup \{0\} \tag{4.2.2}$$

by $\tau_F(x) = \inf\{n \in \mathbb{N} : \ x \in F^n\}$ for all $x \neq e$ and $\tau_F(e) = 0$. Moreover, since F is symmetric, $\tau_F(\check{x}) = \tau_F(x)$. It is straightforward to verify that if F' is another finite symmetric generator of H, then for some constants C_1, C_2, $C_1 \tau_{F'} \leq \tau_F \leq C_2 \tau_{F'}$.

If there is no risk of confusion, we may just use τ instead of τ_F. For each pair $x, y \in H$, for each $t \in x * y (= \mathrm{supp}(\delta_x * \delta_y))$, t belongs to $F^{\tau(x)+\tau(y)}$, so

$$\tau(t) \leq \tau(x) + \tau(y) \quad \text{where } t \in x * y. \tag{4.2.3}$$

- **Polynomial weight**. For a given $\beta \geq 0$, $\omega_\beta(x) := (1 + \tau(x))^\beta$ is a central weight on H.

  Proof. We know that for each $t \in x * y$, $\omega_\beta(t) = (1 + \tau(t))^\beta \leq (1 + \tau(x) + \tau(y))^\beta$. On the other hand,

  $$\begin{aligned} \ln(1 + \tau(x) + \tau(y)) &\leq \ln(1 + \tau(x) + \tau(y) + \tau(x)\tau(y)) \\ &= \ln(1 + \tau(x)) + \ln(1 + \tau(y)). \end{aligned}$$

  Therefore $\omega_\beta(t) \leq \omega_\beta(x)\omega_\beta(y)$. □

Remark 4.2.2. ω_β is centrally additive (and consequently weakly additive). For a generator F of H, note that

$$\begin{aligned} \omega_\beta(t) = (1 + \tau_F(t))^\beta \leq (1 + \tau_F(x) + \tau_F(y))^\beta &\leq C(1 + \tau_F(x))^\beta + C(1 + \tau_F(y))^\beta \\ &= C(\omega_\beta(x) + \omega_\beta(y)) \end{aligned}$$

where $C = \min\{1, 2^{\beta-1}\}$ based on the inequality (4.2.1).

- **Exponential weight**. For given $C > 0$ and $0 \leq \alpha \leq 1$, $\sigma_{\alpha,C}(x) := e^{C\tau(x)^\alpha}$ is a central weight on H.

  Proof. For $x, y \in H$ and $t \in x * y$, $\sigma_{\alpha,C}(t) = e^{C\tau(t)^\alpha} \leq e^{C(\tau(x)+\tau(y))^\alpha}$. Also $e^{C(\tau(x)+\tau(y))^\alpha} \leq e^{C\tau(x)^\alpha} e^{C\tau(y)^\alpha}$, since $\tau(x) \geq 1$ for all $x \neq e$. □

Example 4.2.3. [49] implies that $F = \{0, 1\}$ is a symmetric generator of $\mathbb{N}_0$. Using induction and since $1 \in F^1$, suppose that $\tau_F(n) = n$. Then by [49, Proposition 5.2], $g(n, 1; n+1) \neq 0$;

therefore, $n+1 \in \operatorname{supp}(\delta_n * \delta_1) \subseteq F^{n+1}$. But $n+1 \notin F^n$. Thus $\tau_F(n) = n$ for all $n \in \mathbb{N}_0$ for the map τ_F defined in (4.2.2). In particular, $\mathbb{N}_0$ is a finitely generated hypergroup.

Consequently, for each $\beta \geq 0$, we can define a polynomial weight ω_β on $\mathbb{N}_0$ where

$$\omega_\beta(n) = (1+n)^\beta \qquad (n \in \mathbb{N}_0).$$

Also, for each $0 \leq \alpha \leq 1$ and $C > 0$, we can define an exponential weight $\sigma_{\alpha,C}$ on $\mathbb{N}_0$ where

$$\sigma_{\alpha,C}(n) = e^{Cn^\alpha} \qquad (n \in \mathbb{N}_0).$$

Using these two classes of weights we can generate a variety of weighted hypergroup algebras.

4.3 Weights on Conj(G) derived from group weights

For a (discrete) group G, as a hypergroup, a weight is a mapping $\sigma : G \to (0,\infty)$ such that $\sigma(xy) \leq \sigma(x)\sigma(y)$ for all $x, y \in G$, since $\delta_x \circledast \delta_y = \delta_{xy}$. Then (G, σ) is called a *weighted group*. Therefore, $\ell^1(G,\sigma)$ equipped by the convolution and the weighted norm i.e.

$$\|f\|_{\ell^1(G,\sigma)} = \sum_{t\in G} |f(t)|\sigma(t). \tag{4.3.1}$$

is a Banach algebra called *weighted group algebra*.

To prove the main result of this section, we need the following lemma.

Lemma 4.3.1. *Let* $\omega : \operatorname{Conj}(G) \to (0,\infty)$ *be defined on* $\operatorname{Conj}(G)$. *Then*

$$\omega(\delta_C * \delta_D) = \frac{1}{|C||D|}\sum_{t\in C}\sum_{s\in D}\omega(C_{ts}) \quad (C, D \in \operatorname{Conj}(G)).$$

Proof. The proof is a straightforward calculation based on (3.1.3) as follows:

$$\begin{aligned}
\sum_{E\in\operatorname{Conj}(G)} \omega(E)\delta_C * \delta_D(E) &= \sum_{E\in\operatorname{Conj}(G)} \omega(E)\frac{|E|}{|C||D|}\alpha_E^{C,D} \\
&= \sum_{C_t\in\operatorname{Conj}(G), C_t\subseteq CD} \frac{1}{|C||D|}\alpha_{C_t}^{C,D}\omega(C_t) \\
&= \sum_{t\in G}\frac{\omega(C_t)}{|C||D|} 1_C \circledast 1_D(t) \\
&= \sum_{t\in G}\frac{\omega(C_t)}{|C||D|}\sum_{s\in G} 1_C(s)1_D(s^{-1}t) \\
&= \sum_{t\in G}\sum_{s\in G}\frac{\omega(C_{st})}{|C||D|}1_C(s)1_D(t) \\
&= \frac{1}{|C||D|}\sum_{t\in D}\sum_{s\in C}\omega(C_{st}).
\end{aligned}$$

∎

The following proposition, as the main result of this section, lets us apply group cases which are constant on conjugacy classes to generate hypergroup weights on Conj(G).

Proposition 4.3.2. *Let G be a FC group possessing a weight σ. Then the mean function ω_σ defined as*

$$\omega_\sigma(C) = \frac{1}{|C|}\sum_{t\in C}\sigma(t) \quad \text{for every } C \in \mathrm{Conj}(G) \tag{4.3.2}$$

is a weight on the hypergroup Conj(G).

Proof. By Lemma 4.3.1, it suffices to show that

$$\frac{1}{|C||D|}\sum_{t\in C}\sum_{s\in D}\omega_\sigma(C_{ts}) \le \omega_\sigma(C)\omega_\sigma(D)$$

for all $C, D \in \mathrm{Conj}(G)$. To do so, using weighted group algebra $\ell^1(G,\sigma)$, one gets

$$\begin{aligned}
\sum_{t\in D}\sum_{s\in C}\omega_\sigma(C_{st}) &= \sum_{t\in G}\sum_{s\in G}1_C(s)1_D(t)\omega_\sigma(C_{st})\\
&= \sum_{t\in G}\sum_{s\in G}1_C(s)1_D(s^{-1}t)\omega_\sigma(C_t)\\
&= \sum_{t\in G}1_C \circledast 1_D(C_t)\omega_\sigma(C_t)\\
&= \sum_{E\in\mathrm{Conj}(G)}1_C \circledast 1_D(E)|E|\omega_\sigma(E)\\
&= \sum_{E\in\mathrm{Conj}(G)}1_C \circledast 1_D(E)\sum_{s\in E}\sigma(s)\\
&= \sum_{t\in G}1_C \circledast 1_D(t)\sigma(t) = \|1_C \circledast 1_D\|_{\ell^1(G,\sigma)}\\
&\le \|1_C\|_{\ell^1(G,\sigma)}\|1_D\|_{\ell^1(G,\sigma)} = |C|\omega_\sigma(C)|D|\omega_\sigma(D),
\end{aligned}$$

because $\|1_E\|_{\ell^1(G,\sigma)} = \sum_{t\in E}\sigma(t) = \omega_\sigma(E)|E|$ for every $E \in \mathrm{Conj}(G)$.

□

We call ω_σ the *weight derived* from σ. When (G,σ) is a weighted FC group, we define

$$Z\ell^1(G,\sigma) = \{f \in \ell^1(G,\sigma),\ f(yxy^{-1}) = f(x)\ \forall x,y \in G\}$$

which is the center of the Banach algebra $\ell^1(G,\sigma)$; hence, it is a commutative Banach algebra.

Corollary 4.3.3. *Let (G,σ) be a weighted FC group, and ω_σ on* Conj(G) *be the weight derived from σ. Then the weighted hypergroup algebra $\ell^1(\mathrm{Conj}(G),\omega_\sigma)$ is isometrically isomorphic to $Z\ell^1(G,\sigma)$.*

Proof. By Proposition 4.3.2, $\ell^1(\mathrm{Conj}(G),\omega_\sigma)$ is a weighted hypergroup algebra. Similar to the proof of Theorem 3.1.1, we define $\Psi : Z\ell^1(G,\sigma) \to \ell^1(\mathrm{Conj}(G),\omega_\sigma)$ such that for each $f \in$

$Z\ell^1(G,\sigma)$, $\Psi(f)(C) = |C|f(C)$ for all $C \in \mathrm{Conj}(G)$. Note that Ψ is an algebra homomorphism, since the convolution is the same of the hypergroup algebra restricted to $Zc_c(G)(= c_c(G) \cap \ell^1(G))$. To see that Ψ is an isometry, note that for every $f \in \mathrm{Z}\ell^1(G,\sigma)$,

$$\begin{aligned}\|\Psi(f)\|_{\ell^1(\mathrm{Conj}(G),\omega_\sigma)} &= \sum_{C\in\mathrm{Conj}(G)} |\Psi(f)(C)|\omega_\sigma(C) = \sum_{C\in\mathrm{Conj}(G)} |C||f(C)|\omega_\sigma(C)\\ &= \sum_{C\in\mathrm{Conj}(G)} |f(C)| \sum_{s\in C}\sigma(s) = \sum_{t\in G}|f(t)|\sigma(t) = \|f\|_{\ell^1(G,\sigma)}.\end{aligned}$$

Here, $F(C)$ is as defined in (3.1.1). □

4.4 Central weights on $\mathrm{Conj}(G)$

Let ω be a mapping from $\mathrm{Conj}(G)$ to $\mathbb{R}^+$ such that $\omega(E) \le \omega(C)\omega(D)$ for all conjugacy classes $E, C, D \in \mathrm{Conj}(G)$ where $E \subseteq CD$. Then it is immediate that ω forms a central weight on the hypergroup $\mathrm{Conj}(G)$ as defined in Definition 4.1.2.

Remark 4.4.1. Let G be a FC group and ω be a central weight on $\mathrm{Conj}(G)$. Then the mapping σ_ω is defined on G by $\sigma_\omega(x) := \omega(C_x)$ that forms a group weight on G. And $\ell^1(\mathrm{Conj}(G),\omega)$ as a Banach algebra is isometrically isomorphic to $Z\ell^1(G,\sigma_\omega)$.

Example 4.4.2. Let G be a discrete FC group. The mapping $\omega(C) = |C|$, for $C \in \mathrm{Conj}(G)$, is a central weight on $\mathrm{Conj}(G)$. Clearly, if $E \subseteq CD$, we have $|E| \le |C||D|$.

Example 4.4.3. Let $G = \bigoplus_{i\in\mathbf{I}} G_i$ for a family of finite groups $(G_i)_{i\in\mathbf{I}}$ as studied in Example 3.1.3. Given $C \in \mathrm{Conj}(G)$, for each $\alpha > 0$, we define a mapping

$$\omega_\alpha(C) := (1 + |C_{i_1}| + \cdots + |C_{i_n}|)^\alpha$$

where $i_j \in \mathbf{I}_C$. We show that ω_α is a central weight on $\mathrm{Conj}(G)$. Let $E \subseteq CD$ for some $E, C, D \in \mathrm{Conj}(G)$. One can easily show that for each $i \in \mathbf{I}$, $E_i \subseteq C_iD_i$; $\mathbf{I}_E \subseteq \mathbf{I}_C \cup \mathbf{I}_D$. Therefore,

$$\begin{aligned}\omega_\alpha(C) &= (1 + \sum_{i\in\mathbf{I}_E}|E_i|)^\alpha \le (1 + \sum_{i\in\mathbf{I}_E}|C_i||D_i|)^\alpha \qquad \text{by Example 4.4.2}\\ &\le \left(1 + \sum_{i\in\mathbf{I}_C}|C_i|\right)^\alpha \left(1 + \sum_{i\in\mathbf{I}_D}|D_i|\right)^\alpha = \omega_\alpha(C)\omega_\alpha(D).\end{aligned}$$

Theorem 4.4.4. *Let (G,σ) be a weighted FC group such that $M = \sup_{C\in\mathrm{Conj}(G)}|C| < \infty$. Then the hypergroup weight $\omega_z(C) := M^2\omega_\sigma(C)$, for $C \in \mathrm{Conj}(G)$, forms a central weight.*

Proof. Let $E \subseteq CD$ for some $C, D, E \in \text{Conj}(G)$. Note that for each $t \in E$, there are some $x \in C$ and $y \in D$ such that $t = xy$, so one gets that

$$\omega_\sigma(E) = \frac{1}{|E|}\sum_{t \in E}\sigma(t) \le \frac{1}{|E|}\sum_{t\in E}\sum_{x \in C}\sigma(x)\sum_{y \in D}\sigma(y) \le \sum_{x\in C}\sigma(x)\sum_{y\in D}\sigma(y).$$

Hence,

$$\omega_\sigma(E) \le |C|\omega_\sigma(C)\,|D|\omega_\sigma(D) \le M^2\omega_\sigma(C)\omega_\sigma(D),$$

and so, $\omega_z(E) \le \omega_z(C)\omega_z(D)$. □

Theorem 4.4.4 implies that for discrete groups whose conjugacy classes are uniformly finite, every weight on G leads to a central weight on $\text{Conj}(G)$. A group G is called a group with *finite commutator group* or *FD* if its derived subgroup is finite. Let G be an FD group. For every $C \in \text{Conj}(G)$, $C = \{zxz^{-1} : z \in G\}$ for some $x \in C$. Hence, $Cx^{-1} = \{zxz^{-1}x^{-1} : z \in G\} \subseteq G'$ and therefore $|C| = |Cx^{-1}| \le |G'|$. Therefore, the order of conjugacy classes of an FD group are uniformly bounded by $|G'|$.

4.4.1 An example: $\text{Conj}(S_3)$

The natural question that one may ask would be the existence of a weight over $\text{Conj}(G)$, for some discrete FC group G, which is not equivalent to any central weight with respect to the equivalency defined in Definition 4.1.6. In this subsection, we generate a class function which is satisfying Lemma 4.3.1 but is not equivalent to any central weight.

Let S_n be the symmetric group of degree n. First, we study $\text{Conj}(S_n)$ for $n = 3$ and some possible weights on the finite hypergroup $\text{Conj}(S_3)$. Recall that for any element $x = (i_1^{(1)}\cdots i_{k_1}^{(1)})\cdots(i_1^{(m)}\cdots i_{k_m}^{(m)}) \in S_n$, where $(i_1^{(j)}\cdots i_{k_j}^{(j)})$'s are pairwise commute cycles, the conjugacy class of x is the set of all elements of S_n which can be written in the same cycle structure [19, Section 1.3].

Figure 4.1 summarizes the support of $\delta_C * \delta_D$ for all $C, D \in \text{Conj}(S_3)$. As an example, $\text{supp}(\delta_{C_{(12)}} * \delta_{C_{(123)}}) = C_{(12)}$. To check this note that for each $z \in G$ where $1_{C_{(12)}} \circledast 1_{C_{(123)}}(z) \neq 0$, one gets that

$$1_{C_{(12)}} \circledast 1_{C_{(123)}}(z) = \sum_{t\in S_3} 1_{C_{(12)}}(t)1_{C_{(123)}}(t^{-1}z) = \sum_{t\in S_3} 1_{C_{(12)}}(t)1_{tC_{(123)}}(z) = \sum_{t \in C_{(12)}C_{(123)}}(t).$$

So, by the definition of the convolution on $\text{Conj}(S_3)$, $\text{supp}(\delta_{C_{(12)}} * \delta_{C_{(123)}}) = C_{(12)}C_{(123)}$ which is $C_{(12)}$. Clearly, since $\text{Conj}(S_3)$ forms a commutative hypergroup the table is symmetric, and hence, one may complete the other half symmetrically.

$*$	C_e	$C_{(12)}$	$C_{(123)}$
C_e	$\{C_e\}$	$\{C_{(12)}\}$	$\{C_{(123)}\}$
$C_{(12)}$		$\{C_e, C_{(123)}\}$	$\{C_{(12)}\}$
$C_{(123)}$			$\{C_e, C_{(123)}\}$

Figure 4.1: Convolution action on conjugacy classes of S_3

Example 4.4.5. Using the table, one may easily verify that the weight ω, defined in the following, forms a central weight on $\mathrm{Conj}(S_3)$.

	C_e	$C_{(12)}$	$C_{(123)}$
ω	1	3	5

Applying Lemma 4.3.1, it is sufficient to check that the following inequalities hold for ω to be a weight on $\mathrm{Conj}(S_3)$.

(i) $1/3\,\omega(C_e) + 2/3\,\omega(C_{(123)}) \leq \omega(C_{(12)})^2$.

(ii) $1 \leq \omega(C_e), \omega(C_{(123)})$.

(iii) $1/2\omega(C_{(123)}) + 1/2\omega(C_e) \leq \omega(C_{(123)})^2$.

Remark 4.4.6. Some long computations on the previous equations imply that for each weight ω on $\mathrm{Conj}(S_3)$, one may show that $\omega_z = \alpha\omega$ will be a central weight for all $\alpha > 5/4$.

Example 4.4.7. Considering equations (i), (ii), and (iii), one may verify that the weight ω as defined below is a weight on $\mathrm{Conj}(S_3)$.

	C_e	$C_{(12)}$	$C_{(123)}$
ω	1	2	5

On the other hand, since $5 = \omega(C_{(123)}) \nleq \omega(C_{(12)})^2 = 4$, ω is not a central weight.

Question. Can one generate a group weight σ on S_3 for which ω_σ is not a central weight on $\mathrm{Conj}(S_3)$?

Example 4.4.8. We generate the RDPF group $G = \bigoplus_{n \in \mathbb{N}} S_3$ as defined in Section 2.2. Let us define the weight $\omega' := \prod_{n \in \mathbb{N}} \omega$ on $\mathrm{Conj}(G)$ where ω is the hypergroup weight on $\mathrm{Conj}(S_3)$ defined in Example 4.4.7. For each $N \in \mathbb{N}$, define $D_N := \prod_{n \in \mathbb{N}} D_n^{(N)} \in \mathrm{Conj}(G)$ where $D_n^{(N)} = C_{(123)}$ for all $n \in 1, \ldots, N$ and $D_n^{(N)} = C_e$ otherwise. One can verify that $D_N \in \mathrm{supp}(\delta_{E_N} * \delta_{E_N})$ for

$E_N = \prod_{n\in\mathbb{N}} E_n^{(N)} \in \mathrm{Conj}(G)$ with $E_n^{(N)} = C_{(12)}$ for all $n \in 1,\dots,N$ and $E_n^{(N)} = C_e$ otherwise. Therefore

$$\frac{\omega'(D_N)}{\omega'(E_N)^2} = \prod_{n=1}^{N} \frac{\omega(C_{(123)})}{\omega(C_{(12)})^2} = (5/4)^N \to \infty$$

where $N \to \infty$. We claim that ω' is a weight which cannot equal any central weight i.e. there is not a constant M such that $M\omega'$ is a central weight. To prove this claim, let ω_z be a central weight and α_1 and α_2 two positive constants such that $\alpha_1\omega' \le \omega_z \le \alpha_2\omega'$. Hence,

$$\frac{\omega'(D_N)}{\omega'(E_N)^2} \le \frac{\alpha_2^2\omega_z(D_N)}{\alpha_1\omega_z(E_N)^2} < \frac{\alpha_2^2}{\alpha_1}$$

which is a contradiction.

4.5 Weights related to quotient groups

Let G be a group, N a normal subgroup of G, and $T : \ell^1(G) \to \ell^1(G/N)$ the Reiter's map as defined in [65, (3.4.10)],

$$Tf(xN) = \sum_{t\in N} f(xt) \quad \text{for } f \in \ell^1(G)$$

which is an onto algebra homomorphism. For each $f \in Z\ell^1(G)$ and $g \in \ell^1(G)$, note that $Tf \circledast Tg = T(f \circledast g) = T(g \circledast f) = Tg \circledast Tf$. Since T is onto, this implies that $T(Z\ell^1(G)) \subseteq Z\ell^1(G/H)$. Let us denote the restriction of T to $Z\ell^1(G)$ by T again; hence, $T : Z\ell^1(G) \to Z\ell^1(G/N)$. By Theorem 3.1.1, T can be seen as a mapping $T : \ell^1(\mathrm{Conj}(G)) \to \ell^1(\mathrm{Conj}(G/N))$.

Claim. We claim that for each $x \in G$, $T(\delta_{C_x}) = \alpha_{C_{xN}}\delta_{C_{xN}}$ for some $0 < \alpha_{C_{xN}} \le 1$.

Proof of Claim. For each $C_x \in \mathrm{Conj}(G)$ and $C_{zN} \in \mathrm{Conj}(G/N)$, applying Ψ defined to prove Theorem 3.1.1, one gets

$$T(\delta_{C_x})(C_{zN}) = T \circ \Psi(\frac{1}{|C_x|}1_{C_x})(zN) = \frac{1}{|C_x|}\sum_{t\in N} 1_{C_x}(zt). \tag{4.5.1}$$

First, assume that $C_{xN} \neq C_{zN}$. Toward a contradiction, one may assume that for some $t \in N$, $zt \in C_x$. Without loss of generality, let $zt = x$. But, this implies that $zN = xN$ which is a contradiction. Therefore, if $C_{xN} \neq C_{zN}$, $T(\delta_{C_x})(C_{zN}) = 0$.

On the other hand, one may complete the equation (4.5.1) as follows.

$$T(\delta_{C_x})(C_{zN}) = \frac{1}{|C_x|}\sum_{t\in N} 1_{C_x}(zt) = \frac{1}{|C_x|}\sum_{y\in C_x}\sum_{t\in N}\delta_y(zt).$$

So, if for some $y \in C_x$, $zt = y$, $\sum_{t\in N}\delta_y(zt) = 1$;

$$0 < \frac{1}{|C_x|} \le \frac{1}{|C_x|}\sum_{y\in C_x}\sum_{t\in N}\delta_y(zt) \le 1 = \delta_{C_{xN}}(C_{xN}).$$

For some $0 < \alpha_{C_{xN}} \leq 1$, we have proved the claim. □

Suppose that ω be a weight on $\mathrm{Conj}(G)$ which is bounded away from zero i.e. for some $\delta > 0$, $\omega(C) > \delta$ for all $C \in \mathrm{Conj}(G)$. Since ω is away from zero by some $\delta > 0$, for each $f \in \ell^1(\mathrm{Conj}(G), \omega)$,

$$\delta \|f\|_{\ell^1(\mathrm{Conj}(G))} = \sum_{C \in \mathrm{Conj}(G)} \delta |f(C)| \leq \sum_{C \in \mathrm{Conj}(G)} |f(C)| \omega(C) = \|f\|_{\ell^1(\mathrm{Conj}(G), \omega)}.$$

Therefore, $\ell^1(\mathrm{Conj}(G), \omega)$ is a subalgebra of $\ell^1(\mathrm{Conj}(G))$. So, let us define the restricted map T_ω by

$$T_\omega := T|_{\ell^1(\mathrm{Conj}(G), \omega)} : \ell^1(G, \omega) \to \mathcal{A}$$

where $\mathcal{A} = \mathrm{Im}(T_\omega)$ is equipped with the quotient norm i.e. for each $f \in \ell^1(\mathrm{Conj}(G), \omega)$,

$$\|T_\omega(f)\|_q = \inf\{\|f - k\|_{\ell^1(\mathrm{Conj}(G), \omega)}, k \in \mathrm{Ker}\, T_\omega\}.$$

Some arguments in the proof of the following proposition are similar to the ones in [65, Proposition 3.6.11].

Proposition 4.5.1. *The mapping* $\tilde{\omega} : \mathrm{Conj}(G/N) \to \mathbb{R}^+$ *defined as*

$$\tilde{\omega}(C_{xN}) = \inf\{\omega(C_{xy}) : \ y \in N\} \quad (C_{xN} \in \mathrm{Conj}(G/N))$$

forms a weight on $\mathrm{Conj}(G/N)$.

Proof. Note that, $T_\omega(\delta_{C_x} - \delta_{C_{xy}}) = \alpha_{C_{xN}}(\delta_{C_{xN}} - \delta_{C_{xyN}}) = 0$ for all $x \in G$ and $y \in N$. For each $x \in G$ and

$$\begin{aligned} \|T_\omega(\delta_{C_x})\|_q &= \inf\{\|\alpha_{C_{xN}} \delta_{C_x} - k\|_{\ell^1(\mathrm{Conj}(G), \omega)} : \quad k \in \mathrm{Ker}\, T_\omega\} \\ &\leq \inf\{\|\alpha_{C_{xN}} \delta_{C_x} + \alpha_{C_{xN}} \delta_{C_{xy}} - \alpha_{C_{xN}} \delta_{C_x}\|_{\ell^1(\mathrm{Conj}(G), \omega)} : \quad y \in N\} \\ &= \alpha_{C_{xN}} \inf\{\omega(C_{xy}) : \quad y \in N\} = \alpha_{C_{xN}} \tilde{\omega}(C_{xN}). \end{aligned}$$

Let us study the dual of the map T_ω which is denoted by $T_\omega^* : \mathcal{A}^* \to \ell^\infty(\mathrm{Conj}(G), \omega^{-1})$. So for each $\varphi \in \mathcal{A}^*$, $T_\omega^*(\varphi) \in \ell^\infty(\mathrm{Conj}(G), \omega^{-1})$. Hence,

$$\|\varphi\|_{\mathcal{A}^*} \geq \sup_{C \in \mathrm{Conj}(G)} \frac{|\langle \varphi, T_\omega(\delta_{C_x}) \rangle|}{\|T_\omega(\delta_{C_x})\|_q} \geq \sup_{C_{xN} \in \mathrm{Conj}(G/N)} \frac{\alpha_{C_{xN}} |\varphi(C_{xN})|}{\alpha_{C_{xN}} \tilde{\omega}(C_{xN})} = \|\varphi\|_{\ell^\infty(\mathrm{Conj}(G/N), \tilde{\omega}^{-1})}.$$

Also, since $\mathcal{A}$ is equipped with the quotient topology, T_ω^* is an isometry. Hence,

$$\begin{aligned}
\|\varphi\|_{\mathcal{A}^*} &= \|T_\omega^*(\varphi)\|_{\ell^\infty(\mathrm{Conj}(G),\omega^{-1})} \\
&= \sup_{C_x\in\mathrm{Conj}(G)} \frac{|T_\omega^*(\varphi)(C_x)|}{\omega(C_x)} \\
&\le \sup_{C_x\in\mathrm{Conj}(G)} \frac{|T_\omega^*(\varphi)(C_x)|}{\tilde{\omega}(C_{xN})} \\
&= \sup_{C_x\in\mathrm{Conj}(G)} \frac{|\langle\varphi, T_\omega(\delta_{C_x})\rangle|}{\tilde{\omega}(C_{xN})} \\
&= \sup_{x\in G} \alpha_{C_{xN}} \frac{|\varphi(C_{xN})|}{\tilde{\omega}(C_{xN})} \le \|\varphi\|_{\ell^\infty(\mathrm{Conj}(G/N),\tilde{\omega}^{-1})}.
\end{aligned}$$

So $\|\cdot\|_{\mathcal{A}^*} = \|\cdot\|_{\ell^\infty(\mathrm{Conj}(G/N),\tilde{\omega}^{-1})}$. Consequently, as two Banach algebras,

$$\ell^1(\mathrm{Conj}(G/N),\tilde{\omega}) \cong \ell^1(\mathrm{Conj}(G),\omega)/\operatorname{Ker} T_\omega.$$

Thus,

$$\|\delta_{C_{xN}} * \delta_{C_{yN}}\|_{\ell^1(\mathrm{Conj}(G/N),\tilde{\omega})} \le \|\delta_{C_{xN}}\|_{\ell^1(\mathrm{Conj}(G/N),\tilde{\omega})}\|\delta_{C_{yN}}\|_{\ell^1(\mathrm{Conj}(G/N),\tilde{\omega})}$$

which equals to this fact that $\tilde{\omega}(\delta_{C_{xN}} * \delta_{C_{yN}}) \le \tilde{\omega}(C_{xN})\tilde{\omega}(C_{yN})$. This shows that $\tilde{\omega}$ is a weight on $\mathrm{Conj}(G/N)$. □

4.6 Weights on duals of compact groups

Corollary 4.6.1. *Let G be a compact group and $\widehat{G}$ be the set of all irreducible representations of G as a discrete commutative hypergroup. Then $\omega_\beta(\pi) = d_\pi^\beta = h(d_\pi)^{\beta/2}$ is a central weight for each $\beta \ge 0$.*

Proof. Since for every pair $\pi,\sigma \in \widehat{G}$, the dimension of $\pi\otimes\sigma$ which is $d_\pi d_\sigma$ is equivalent to $\sum_{i=1}^n m_i^{\pi,\sigma} d_{\pi_i}$ for some $m_i^{\pi,\sigma} > 0$ and $(\pi_i)_{i=1}^n \subseteq \widehat{G}$, for each $\pi_{i_0} \in \pi * \sigma$ one gets that

$$d_\pi d_\sigma = \sum_{i=1}^n m_i^{\pi,\sigma} d_{\pi_i} \ge m_{i_0}^{\pi,\sigma} d_{\pi_i} \ge d_{\pi_{i_0}}.$$

□

Example 4.6.2. Let $\widehat{SU(2)}$ be the hypergroup of all irreducible representations on the compact group $SU(2)$. Let $F = \{\pi_0, \pi_{1/2}\}$. We claim that F is a generator for $\widehat{SU(2)}$. For π_1 we know that

$$\delta_{\pi_{1/2}} * \delta_{\pi_{1/2}} = \frac{3}{4}\delta_{\pi_1} + \delta_{\pi_0}.$$

So $F^2 = \{\pi_0, \pi_{1/2}, \pi_1\}$. We claim that $F^{2k} = \{\pi_\ell\}_{\ell=0}^k$. Suppose that this claim is correct for $k - \frac{1}{2}$ that is $F^{2k-1} = \{\pi_i\}_{i=0}^{k-1/2}$. Therefore, for each $\pi_i \in F^{2k-1}$, $\pi_i * \pi_0 = \pi_i$ and for each $\pi_i \ne \pi_0$,

$\pi_i * \pi_{1/2} = \{\pi_{i-1/2}, \pi_{i+1/2}\}$. So $F^{2k} = F^{2k-1} \cup \{\pi_k\}$. Thus, $|F^n| = n+1$ and for each $\ell \in \widehat{SU(2)}$, $\tau_F(\pi_\ell) = 2\ell$ for all $\ell > 0$ and $\tau_F(\pi_0) = 0$ where τ_F is the map defined in (4.2.2).

Consequently, for each $\beta \geq 0$, we can define a polynomial weight ω_β on $\widehat{SU(2)}$ when

$$\omega_\beta(\pi_\ell) = (1+2\ell)^\beta \qquad \pi_\ell \in \widehat{SU(2)}. \tag{4.6.1}$$

Note that (4.6.1) implies that $\omega_\beta(\pi) = h(\pi)^{\beta/2} = d_\pi^\beta$ corresponds to the weight defined in Corollary 4.6.1.

Also, for each $0 \leq \alpha \leq 1$ and $C > 0$, we can define an exponential weight $\sigma_{\alpha,C}$ on $\widehat{SU(2)}$ where

$$\sigma_{\alpha,C}(\ell) = e^{C(2\ell)^\alpha} \qquad \ell \in \widehat{SU(2)}.$$

Example 4.6.3. Let us define $\omega_a : \widehat{SU(2)} \to \mathbb{R}^+$ such that

$$\omega_a(\pi_\ell) = \frac{a^{2\ell+1}}{2\ell+1}$$

for a fixed constant $a \geq (\sqrt{5}+1)/2$. We show that ω_a is a weight on $\widehat{SU(2)}$. For a pair of ℓ, ℓ' in $\frac{1}{2}\mathbb{Z}^+ := \{0, 1/2, 1, 3/2, \ldots\}$, without loss of generality suppose that $\ell \geq \ell'$. So

$$\begin{aligned}
\sum_{r=\ell-\ell'}^{\ell+\ell'} (2r+1)\omega_a(\pi_r) &= \sum_{r=\ell-\ell'}^{\ell+\ell'} a^{2r+1} \\
&= a^{2\ell-2\ell'+1} \sum_{r=0}^{2\ell'} a^{2r} \\
&= a^{2\ell-2\ell'+1} \frac{a^{4\ell'+2}-1}{a^2-1} \\
&= \frac{a^{2\ell+2\ell'+3}}{a^2-1} - \frac{a^{2\ell+2\ell'+1}}{a^2-1} \\
&\leq a^{2\ell+2\ell'+2}\left(\frac{a}{a^2-1}\right) \\
&\leq \frac{a}{a^2-1}\omega_a(\ell)(2\ell+1)\omega_a(\ell')(2\ell'+1).
\end{aligned}$$

But since $a \geq (\sqrt{5}+1)/2$, $a/(a^2-1) \leq 1$; therefore,

$$\omega_a(\delta_{\pi_\ell} * \delta_{\pi_{\ell'}}) \leq \omega_a(\pi_\ell)\omega_a(\pi_{\ell'}).$$

Note that

$$\frac{\omega_a(\pi_{2\ell})}{\omega_a(\pi_\ell)^2} = \frac{a^{4\ell+1}}{4\ell+1} \Big/ \left(\frac{a^{2\ell+1}}{2\ell+1}\right)^2 \to \infty$$

where $\ell \to \infty$; while $\pi_{2\ell} \in \pi_\ell * \pi_\ell$. Hence, not only is ω_a not a central weight but also it is not equivalent to any central weight. To show the second claim, let ω_z be a central weight and α_1 and α_2 two constants such that $\alpha_1\omega_a \leq \omega_z \leq \alpha_2\omega_a$. Hence,

$$\frac{\omega_a(\pi_{2\ell})}{\omega_a(\pi_\ell)^2} \leq \frac{\alpha_2^2\omega_z(\pi_{2\ell})}{\alpha_1\omega_z(\pi_\ell)^2} < \frac{\alpha_2^2}{\alpha_1}$$

which is a contradiction.

Example 4.6.4. Let σ be a weight on the group $\mathbb{Z}$ i.e. $\sigma(m+n) \leq \sigma(m)\sigma(n)$. We define

$$\omega_\sigma(\pi_\ell) = \frac{1}{2\ell+1}\sum_{r=-2\ell}^{2\ell}\sigma(r) \qquad (\ell \in \frac{1}{2}\mathbb{Z}^+). \tag{4.6.2}$$

Recall that elements of $\widehat{SU(2)}$ can be regarded as π_k when $k \in \frac{1}{2}\mathbb{Z}^+$. Suppose that $m, n \in \frac{1}{2}\mathbb{Z}^+$ and without loss of generality $n \geq m$. Then,

$$\begin{aligned}
\omega_\sigma(\pi_m)\omega_\sigma(\pi_n) &= \frac{1}{2m+1}\sum_{t=-2m}^{2m}\sigma(t)\,\frac{1}{2n+1}\sum_{s=-2n}^{2n}\sigma(s)\\
&\geq \frac{1}{(2m+1)(2n+1)}\sum_{t=-2m}^{2m}\sum_{s=-2n}^{2n}\sigma(t+s)\\
&= \frac{1}{(2m+1)(2n+1)}\sum_{t=2n-2m}^{2n+2m}\sum_{s=-2n}^{2n}\sigma(t-2n+s)\\
&= \frac{1}{(2m+1)(2n+1)}\sum_{t=2n-2m}^{2n+2m}\sum_{s=t-4n}^{t}\sigma(s) \qquad\qquad (\star)\\
&\geq \frac{1}{(2m+1)(2n+1)}\sum_{t=2n-2m}^{2n+2m}\sum_{s=-t}^{t}\sigma(s)\\
&= \sum_{r=n-m}^{n+m}\frac{(2r+1)}{(2m+1)(2n+1)}\left(\frac{1}{2r+1}\sum_{s=-2r}^{2r}\sigma(r)\right)\\
&= \sum_{r=n-m}^{n+m}\frac{(2r+1)}{(2m+1)(2n+1)}\omega_\sigma(\pi_r)\\
&= \omega_\sigma(\delta_{\pi_m} * \delta_{\pi_n}).
\end{aligned}$$

We note that, in $(\star)$ above, since if $2n-2m \leq t \leq 2n+2m$ and $-2(n+m) \leq t-4n$; therefore $-2n \leq t$, one gets that $t-4n \leq -t$. Therefore, ω_σ forms a weight on $\widehat{SU(2)}$ as a hypergroup.

Remark 4.6.5. Let $\sigma(n) = a^n$ for some $a \geq 1$ on $\mathbb{Z}$. Clearly, σ is a weight on $\mathbb{Z}$ and therefore, one may consider the weight ω_σ as defined in Example 4.6.4. For each $\ell \in \frac{1}{2}\mathbb{Z}^+$ and $a \geq (1+\sqrt{5})/2$,

$$\begin{aligned}
\omega_\sigma(\pi_\ell) = \frac{1}{2\ell+1}\sum_{r=-2\ell}^{2\ell}a^r &= \frac{a^{-2\ell}}{2\ell+1}(1+a^2+\cdots+a^{4\ell})\\
&= \frac{a^{-2\ell}}{2\ell+1}\,\frac{a^{4\ell+2}-1}{a^2-1}\\
&= \omega_a(\pi_\ell)\,\frac{a^{4\ell+2}-1}{a^{4\ell+1}(a^2-1)}
\end{aligned}$$

where ω_a is the weight defined in Examples 4.6.3. But some simple calculations verify that for every $\ell \in \frac{1}{2}\mathbb{Z}^+$ and $a > 1$,

$$\frac{1}{a} \leq \frac{a^{4\ell+2}-1}{a^{4\ell+1}(a^2-1)} \leq \frac{a}{a^2-1}.$$

This implies that

$$\frac{1}{a}\omega_a \leq \omega_\sigma \leq \frac{a}{a^2-1}\omega_a$$

for each $a \geq (\sqrt{5}+1)/2$; hence, the weights ω_a and ω_σ are equivalent according to Definition 4.1.6.

Example 4.6.6. [58, Proposition 4.11]

Let G be a compact group and ω be a central weight on the hypergroup $\widehat{G}$. Then for each closed subgroup N of G, we may define ω_N on $\widehat{N}$ such that

$$\omega_N(\sigma) = \inf_{\pi \in \widehat{G}, \sigma \leq \pi|_N} \omega(\pi) \quad (\sigma \in \widehat{N})$$

where $\sigma \leq \pi|N$ means that σ is equivalent to one of the representations of irreducible decomposition of $\pi|N$. Given $\epsilon > 0$, note that if $\sigma_1, \sigma_2 \in \widehat{N}$ there are $\pi_1, \pi_2 \in \widehat{G}$ such that $\sigma_i \leq \pi_i$ and $\omega(\pi_i) < \omega_N(\sigma_i) + \epsilon$ for $i = 1, 2$. Note that for each $\sigma \in \sigma_1 * \sigma_2$, $\sigma \leq \sigma_1 \otimes \sigma_2 \leq \pi_1|_N \otimes \pi_2|_N$; hence,

$$\omega_N(\sigma) = \inf_{\pi \in \widehat{G}, \sigma \leq \pi|_N} \omega(\pi) \leq \inf_{\pi \leq \pi_1 \otimes \pi_2, \sigma \leq \pi|_N} \omega(\pi) \leq \omega(\pi_1)\omega(\pi_2) \leq (\omega_N(\sigma_1) + \epsilon)(\omega_N(\sigma_2) + \epsilon).$$

Since $\epsilon > 0$ is arbitrary, it implies that ω_N is a central weight on $\widehat{N}$.

Chapter 5

The Fourier algebra of a regular Fourier hypergroup

For a hypergroup H, Muruganandam, [60], gave a definition of the *Fourier space*, $A(H)$, and showed that $A(H)$ is a Banach algebra with pointwise product for certain commutative hypergroups. In this chapter first we study Fourier space of hypergroups in general. Then we focus on the Fourier algebra of dual of compact groups. We finishes the chapter by studying the amenability of $ZA(G)$ for a compact group G in Section 5.3.

A version of some results of Sections 5.1 and 5.2 has appeared in [2].

5.1 Background

In this section, we review main properties of $A(H)$ from [60]; the proof of all unproven results mentioned in the following may be found there.

For a compact hypergroup H, Vrem in [78] defined the *Fourier space* similar to the Fourier algebra of a compact group. Subsequently, Muruganandam, [60], defined the *Fourier-Stieltjes space* on an arbitrary (not necessary compact) hypergroup H using irreducible representations of H analogous to the Fourier-Stieltjes algebra on locally compact groups. Subsequently, he defined the Fourier space of a hypergroup H, as a closed subspace of the Fourier-Stieltjes algebra, generated by $\{f *_h \tilde{f} : f \in L^2(H,h)\}$ or equivalently generated by $\{f *_h \tilde{f} : f \in C_c(H)\}$; hence, $A(H) \cap C_c(H)$ is dense in $A(H)$.

Proposition 5.1.1. *Let H be a hypergroup. Then*

(1) $A(H) \cap C_c(H)$ is dense in $A(H)$,

(2) $A(H) \subseteq C_0(H)$, by [60, Corollary 2.13],

(3) $\|\cdot\|_\infty \leq \|\cdot\|_{A(H)}$, by [60, Remark 2.9],

(4) for every $u \in A(H)$, $L_x u$, $\check{u}$, and $\overline{u}$ belong to $A(H)$, [60, Proposition 2.16].

In [60], Muruganandam showed that when H is commutative, $A(H)$ can be characterized as follows. This argument first was appeared in [13].

Theorem 5.1.2. *[60, Section 4]*
Let H be a commutative hypergroup. Then $A(H) = \{f *_h \tilde{g} : f, g \in L^2(H, h)\}$ and $\|u\|_{A(H)} = \inf \|f\|_2 \|g\|_2$ for all $f, g \in L^2(H, h)$ such that $u = f * \tilde{g}$.

Remark 5.1.3. The key point for this advantage of commutative hypergroups, as it was proven in [60, Proposition 4.2] and [13, Section 2], is this fact that $\mathcal{F}(A(H))$, where $\mathcal{F}$ is the (extension of the) Fourier transform, is $L^1(S, \pi)$ where S, as a subset of $\widehat{H}$, is the support of the Plancherel measure π (see [8, Chapter 2]). Note that the Fourier transform $\mathcal{F} : L^1(H) \to C_0(\widehat{H})$ is an algebra isomorphism i.e. $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$. Moreover, similar to the group case, $\mathcal{F}|_{L^2(H)\cap L^1(H)}$ is an isometry which can be extended as an isometric isomorphism from $L^2(H)$ onto the Banach space $L^2(S, \pi)$ (see [8, Theorem 2.2.22]). Therefore, by taking care of some details, one may obtain that for each $u \in A(H)$, $\mathcal{F}(u) \in L^1(S, \pi)$ and since $\mathcal{F}(u) = fg$ for some $f, g \in L^2(S, \pi)$,

$$u = \mathcal{I} \circ \mathcal{F}(u) = \mathcal{I}(f) * \mathcal{I}(g).$$

But note that $\mathcal{I}(f), \mathcal{I}(g) \in L^2(H)$. The implication of the norm is now obvious applying the mapping $\mathcal{I}$ for every $f \in A(H)$.

For a hypergroup H, it is known that for every $x \in H$ and $f \in L^2(H)$, $L_x f \in L^2(H)$ while $\|L_x f\|_2 = \|f\|_2$ (see [8, (1.3.18)]). Therefore, L_x is an operator in $\mathcal{B}(L^2(H))$ which we denote it by $\lambda(x)$. The von Neumann sub-algebra of $\mathcal{B}(L^2(H))$ generated by $(\lambda(x))_{x\in H}$ is called the *hypergroup von Neumann algebra of H* and denoted by $VN(H)$.

On the other hand, for each $f \in L^1(H)$, $f * g \in L^2(H)$ for $g \in L^2(H)$ while

$$\|f * g\|_2 \le \|f\|_1 \|g\|_2 \tag{5.1.1}$$

(see [8, (1.4.12)]). So the operator $\lambda(f)$ which carries g to $f * g$ belongs to $\mathcal{B}(L^2(H))$. The C^*-algebra generated by $(\lambda(f))_{f\in L^1(H)}$ in $\mathcal{B}(L^2(H))$ is called *reduced C^*-algebra of H* and denoted by $C^*_\lambda(H)$. It is proven in [60] that $C^*_\lambda(H)$ is actually a C^*-subalgebra of $VN(H)$.

Let $B_\lambda(H)$ denote the set of all continuous, bounded functions ϕ on H such that

$$\|u\|_{B_\lambda(H)} := \sup\left\{\int_H u(x)f(x)dx : \quad f \in L^1(H), \|f(x)\|_{C^*_\lambda(H)} \le 1\right\} < \infty.$$

Similar to the group case, it is proved that $(B_\lambda(H), \|\cdot\|_{B_\lambda(H)})$ forms a Banach space which is isomorphic to the dual of $C^*_\lambda(H)$. Moreover, $A(H) \subseteq B_\lambda(H)$ and for every $u \in A(H)$, $\|u\|_{A(H)} = \|u\|_{B_\lambda(H)}$.

Remark 5.1.4. Note that based on the inequality (5.1.1), for each $f \in L^1(H)$, $\|f\|_{VN(H)} = \|f\|_{C^*_\lambda(H)} \le \|f\|_1$. Hence, for every $F \in VN(H)^*$ while $L^1(H)$ is observed as a subalgebra of $VN(H)$, $F|_{L^1(H)}$ can be considered as a functional on $L^1(H)$ or equivalently, $F|_{L^1(H)}$ can be represented by an element in $L^\infty(H)$, because

$$|\langle F, f\rangle| \le \|F\|\|f\|_{VN(H)} \le \|F\|\|f\|_{L^1(H)}.$$

For a net $(T_\alpha)_\alpha \subseteq \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space, T_α converges to 0 in σ-weak topology if

$$\lim_\alpha \sum_n \langle T_\alpha \xi_n, \eta_n\rangle = 0$$

for all sequences of $(\xi_n, \eta_n)_n \subseteq \mathcal{H}$ where $\sum_n (\|\xi_n\|^2_{\mathcal{H}} + \|\eta_n\|^2_{\mathcal{H}}) < \infty$.

Theorem 5.1.5. *[60, Theorem 2.19]*
Let H be a hypergroup. For every $T \in VN(H)$ there exists a unique continuous linear functional ϕ_T on $A(H)$ satisfying $\phi_T(u) = \langle T(f), g\rangle_{L^2(H)}$ where $\check{u} = f * \tilde{g}$. The mapping $T \mapsto \phi_T$ is a Banach space isomorphism between $VN(H)$ and $A(H)^*$. Moreover, the above mapping is also a homeomorphism when $VN(H)$ is given the σ-weak topology and $A(H)^*$ is given weak* topology.

Abusing the notation, for every $T \in VN(H)$ let us denote $\phi_T \in A(H)^*$ by T from now on. One may show that for each $\mu \in M(H)$, μ can be considered as an element in $VN(H)$. In this case for each $\mu \in M(H)$ and the corresponding operator $T_\mu \in VN(H)$,

$$\langle T_\mu, u\rangle = \int_H u(x) d\mu(x) \quad (u \in A(H)), \tag{5.1.2}$$

by [60, Proposition 2.21]. In particular, for each $u \in A(H)$,

$$\lambda(x)(u) = u(x). \tag{5.1.3}$$

Remark 5.1.6. For each $f \in L^1(H)$ and $u, v \in A(H)$, note that f can be considered as a function in $VN(H)$; therefore, by (5.1.2),

$$\langle f \cdot u, v\rangle = \langle f, uv\rangle = \int_H f(x)u(x)v(x) dh(x).$$

Hence, $f \cdot u$ and similarly $u \cdot f$ equals pointwise multiplication of u and f on almost every $x \in H$. Moreover, the reduced C^*-algebra of H, $C^*_\lambda(H)$, is a closed $A(H)$-submodule of $VN(H)$.

The last part of Theorem 5.1.5 and the characterization of $A(H)$ for a commutative hypergroup H in Theorem 5.1.2 result the following corollary.

Corollary 5.1.7. *Let H be a commutative hypergroup. Then for a net $(T_\alpha)_\alpha \subseteq VN(H)$, T_α converges to 0 in σ-weak topology if for every pair $f, g \in L^2(H)$, $\langle T_\alpha f, g\rangle \to 0$.*

Recall that a *state* on a C^*-algebra is a positive linear functional of norm 1. Moreover, if $\mathcal{A}$ is a von Neumann algebra with predual $\mathcal{A}_*$, every state of $\mathcal{A}$ can be approximated by a net of states of the elements of pre-dual in the weak* topology. Therefore, for a commutative hypergroup H each state u on $VN(H)$ which belongs to $A(H)$ is in the form of $f *_h \tilde{f}$ for some $f \in L^2(H)$ such that

$$1 = \|u\|_{A(H)} = u(e) = \|f\|_2^2. \tag{5.1.4}$$

In [60], Muruganandam calls the hypergroup H a *regular Fourier hypergroup*, if the Banach space $(A(H), \|\cdot\|_{A(H)})$ equipped with pointwise product is a Banach algebra. He studied this property for a variety of commutative hypergroups in [60]. He showed that some polynomial hypergroups including Jacobi polynomial hypergroups and Chebyshev polynomial hypergroups are regular Fourier hypergroups. Furthermore, in [61], he pursued this study for *double coset hypergroups* (which are not necessarily commutative). He showed that the hypergroup of the double coset of a locally compact group G with respect to some compact subgroup H, usually denoted by $G//H$, is also a regular Fourier hypergroup. One may see [8, Section 1.5] for more information about coset hypergroups.

We prove a hypergroup version of [26, Lemma 3.2] which shows some important properties of the Banach space $A(H)$ for an arbitrary hypergroup H (not necessarily a regular Fourier hypergroup). Some parts of the following Lemma have already been shown in [78] for compact hypergroups where proof is applicable to general hypergroups. Here we present a complete proof for the lemma. Note that for each $A \subseteq H$ for a hypergroup H, we define $\check{A} = \{\check{x} : x \in A\}$.

Lemma 5.1.8. *Let H be a hypergroup, K a compact subset of H and U an open subset of H such that $K \subset U$. Then for each relatively compact open set V such that $\overline{K * V * \check{V}} \subseteq U$, there exists some $u_V \in A(H) \cap C_c(H)$ such that:*

1. *$u_V(H) \geq 0$.*
2. *$u_V|_K = 1$.*
3. *$\mathrm{supp}(u_V) \subseteq U$.*

4. $\|u_V\|_{A(H)} \le \big(h_H(K*V)/h_H(V)\big)^{\frac{1}{2}}$.

Proof. Let us define

$$u_V := \frac{1}{h_H(V)} 1_{K*V} *_h \check{1}_V.$$

Since, for every $x,t \in H$, $1_{K*V}(t)\check{1}_V(\delta_{\check{t}} * \delta_x) \ge 0$, $u_V \ge 0$. Moreover, for each $x \in K$,

$$\begin{aligned}
h_H(V)u_V(x) &= 1_{K*V} *_h \check{1}_V(x) \\
&= \int_H 1_{K*V}(t)\check{1}_V(\delta_{\check{t}} * \delta_x)dh_H(t) \\
&= \int_H 1_{K*V}(t)1_V(\delta_{\check{x}} * \delta_t)dh_H(t) \\
&= \int_{t\in H} 1_{K*V}(\delta_x * \delta_t)1_V(t)dh_H(t) \quad \text{(by [8, Theorem 1.3.21])} \\
&= \int_V \langle 1_{K*V}, \delta_x * \delta_t\rangle dh_H(t) \\
&= h_H(V).
\end{aligned}$$

Also [8, Proposition 1.2.12] implies that

$$\operatorname{supp}(1_{K*V} *_h \check{1}_V) \subseteq \overline{(K * V * \check{V})} \subseteq U$$

which implies that u_V has compact support, [8, Proposition 1.2.12]. Finally, by [60, Proposition 2.8], we know that

$$\|u_V\|_{A(H)} \le \frac{\|1_{K*V}\|_2 \|1_V\|_2}{h_H(V)} = \frac{h_H(K*V)^{\frac{1}{2}} h_H(1_V)^{\frac{1}{2}}}{h_H(V)} = \frac{h_H(K*V)^{\frac{1}{2}}}{h_H(V)^{\frac{1}{2}}}.$$

□

Remark 5.1.9. For each pair K, U such that $K \subset U$, we can always find a relatively compact neighborhood V of e_H that satisfies the conditions in Lemma 5.1.8. The existence is a result of continuity of the mapping $(x,y) \mapsto x * y$ with respect to the locally compact topology of $H \times H$ into the Michael topology on $\mathfrak{C}(H)$, $(H3)$. Since H is locally compact, there exists some relatively compact open set W such that $K \subseteq W \subseteq \overline{W} \subseteq U$; $K \in \mathfrak{C}_{H\setminus\overline{W}}(W)$ as an open set in the Michael topology and consequently for each $x \in K$, $x * e \in \mathfrak{C}_{H\setminus\overline{W}}(W)$. Since, the mapping $e \to x * e$ is continuous, there is some neighborhood V_1^x of e such that for each $y \in V_1^x$, $x * y \in \mathfrak{C}_{H\setminus\overline{W}}(W)$ i.e. $x * y \subseteq W$ and $x * y \cap H \setminus \overline{W} = \varnothing$. Let us define

$$V^{(1)} = \cup_{x\in K}(V_1^x \cap \check{V}_1^x).$$

Clearly, $\check{V}^{(1)} = V^{(1)}$. Moreover,

$$K * V^{(1)} = \cup_{y\in V^{(1)}} \cup_{x\in K} x * y \subseteq \cup_{x\in K} x * V_1^x \subseteq W$$

and $K*V^{(1)}\cap H\setminus\overline{W}=\varnothing$ since $(x*y)\cap(H\setminus\overline{W})=\varnothing$ for all $x\in K$ and $y\in V^{(1)}$. Now let us replace K by the compact set $\overline{K*V^{(1)}}$. Therefore, similar to the previous argument, for some relatively compact open set W' such that $\overline{K*V^{(1)}}\subseteq W'\subseteq\overline{W'}\subseteq U$, one may find some $V^{(2)}$ a neighborhood of e such that $V^{(2)}=\check{V}^{(2)}$, $\overline{K*V^{(1)}}*V^{(2)}\subseteq W'$, and $(\overline{K*V^{(1)}}*V^{(2)})\cap(H\setminus\overline{W'})=\varnothing$. Hence, for the relatively compact open set $V:=V^{(1)}\cap V^{(2)}$, one gets that $V=\check{V}$ and

$$K*V*\check{V}\subseteq\overline{K*V^{(1)}}*V^{(2)}\subseteq\overline{W'}.$$

So $\overline{K*V*\check{V}}\subseteq U$.

Remark 5.1.10. Let H be a regular Fourier hypergroup. Then if $(e_\alpha)_\alpha$ is an approximate identity of $A(H)$, for each compact set $K\subseteq H$, by Lemma 5.1.8, there is some $u_K\in A(H)$ such that $u_K|K\equiv 1$. Therefore, for each $x\in K$, and based on Proposition 5.1.1,

$$\begin{aligned}\lim_\alpha|1-e_\alpha(x)|=\lim_\alpha|u_K(x)-u_K(x)e_\alpha(x)| &\le \lim_\alpha\|u_K-u_Ke_\alpha\|_\infty\\ &\le \lim_\alpha\|u_K-u_Ke_\alpha\|_{A(H)}=0.\end{aligned}$$

Therefore, $e_\alpha\to 1$ uniformly on compact subsets of H.

Remark 5.1.11. Let H be a discrete hypergroup. Then $\delta_x\in L^2(H)$ (and $\delta_x\neq 0$ almost everywhere); $\delta_x*\delta_e\in L^2(H)*\tilde{L^2(H)}\subseteq A(H)$. Hence, $c_c(H)\subseteq A(H)$ and equivalently, $A(H)\cap c_c(H)=c_c(H)$. Therefore, applying Proposition 5.1.1, $c_c(H)$ is dense in $A(H)$. Moreover, if H is a regular Fourier hypergroup, as it is proven in [60, Theorem 5.13], the space of maximal ideals of the Banach algebra $A(H)$ is homemorphic to H as a discrete topological set.

5.2 The dual of a compact group

Given a commutative hypergroup, it is not immediate that it is a regular Fourier hypergroup or not. We will show that when G is a compact group, the hypergroup $\widehat{G}$ is a regular Fourier hypergroup.

Theorem 5.2.1. *Let G be a compact group. Then $\widehat{G}$ is a regular Fourier hypergroup and $A(\widehat{G})$, equipped with pointwise multiplication, is isometrically isomorphic with the center of the group algebra G, i.e. $A(\widehat{G})\cong ZL^1(G)$.*

Proof. Let $\mathcal{F}$ be the Fourier transform on $L^1(G)$. By [23, Proposition 8.4.3], $\mathcal{F}|_{L^2(G)}$ is an isometric isomorphism from Banach space $L^2(G)$ onto $\mathcal{L}^2(\widehat{G})$. Recall that $ZL^2(G)=ZL^1(G)\cap$

$L^2(G)$). By the properties of the Fourier transform, [23, Proposition 4.2], for each $f \in ZL^2(G)$, $g \in L^1(G)$, and $\pi \in \widehat{G}$ we have

$$\mathcal{F}(f)(\pi) \circ \mathcal{F}(g)(\pi) = \mathcal{F}(f * g)(\pi) = \mathcal{F}(g * f)(\pi) = \mathcal{F}(g)(\pi) \circ \mathcal{F}(f)(\pi). \tag{5.2.1}$$

So $\mathcal{F}(f)(\pi)$ commutes with all $\mathcal{F}(g)(\pi) := \widehat{f}(\pi)$ for all $f \in L^1(G)$. Since π is an irreducible unitary representation into semisimple Banach algebra $\mathbb{M}_{d_\pi}(\mathbb{C})$; we conclude that $\mathcal{F}(f)(\pi) = \alpha_\pi I_{d_\pi}$ for some scalar $\alpha_\pi \in \mathbb{C}$ where I_{d_π} is the identity matrix. Because π was arbitrary, it implies that $\mathcal{F}(f) = (\alpha_\pi I_{d_\pi \times d_\pi})_{\pi\in\widehat{G}}$ for a family of scalars $(\alpha_\pi)_{\pi\in\widehat{G}}$ in $\mathbb{C}$. Hence, by (1.4.3),

$$\|f\|_{L^2(G)} = \|\mathcal{F}(f)\|^2_{\mathcal{L}^2(\widehat{G})} = \sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|^2_{\mathcal{S}_2} = \sum_{\pi\in\widehat{G}} d_\pi \alpha_\pi^2 \|I_{d_\pi}\|^2_{\mathcal{S}_2} = \sum_{\pi\in\widehat{G}} \alpha_\pi^2 {d_\pi}^2 = \sum_{\pi\in\widehat{G}} \alpha_\pi^2 h(\pi). \tag{5.2.2}$$

Note that $\{\chi_\pi\}_{\pi\in\widehat{G}}$ forms an orthonormal basis for $ZL^2(G)$ and $d_\pi\chi_\pi$ is a non-zero idempotent with respect to the convolution for every $\pi \in \widehat{G}$, [28]. Therefore $\mathcal{F}(d_\pi\chi_\pi)(\pi)$ is the only non-zero idempotent of the center $\mathbb{M}_{d_\pi}(\mathbb{C})$ i.e. I_{d_π}. Hence $\widehat{\chi_\pi}(\pi) = d_\pi^{-1} I_{d_\pi}$ and $\widehat{\chi_\pi}(\sigma) = 0$ for all $\sigma \neq \pi$. Similarly, $\mathcal{F}(\chi_{\overline{\pi}}) = d_{\overline{\pi}}^{-1} I_{d_{\overline{\pi}}}$. Let us define $\overline{f}(x) := \overline{f(x)}$; hence, $\overline{\chi_\pi} = \chi_{\overline{\pi}}$. Using (5.2.2), we define $\mathcal{T} : \operatorname{span}\{\chi_\pi\}_{\pi\in\widehat{G}} \to c_c(\widehat{G})$ by $\mathcal{T}(\chi_\pi) = d_\pi^{-1}\delta_\pi$. So,

$$\mathcal{T}(\overline{\chi_\pi}) = \mathcal{T}(\chi_{\overline{\pi}}) = d_{\overline{\pi}}^{-1}\delta_{\overline{\pi}} = d_{\overline{\pi}}^{-1}\check{\delta}_\pi = \mathcal{T}(\chi_\pi)\check{}.$$

Note that $\operatorname{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $ZL^2(G)$ and

$$\|d_\pi\chi_\pi\|_{L^2(G)} = d_\pi = h(\pi)^{1/2} = \|\delta_\pi\|_{L^2(\widehat{G},h)}, \text{ [28].}$$

Furthermore, for each $f \in \operatorname{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ say $f = \sum_{i=1}^n \alpha_i \chi_{\pi_i}$ for some $\alpha_i \in \mathbb{C}$ and $\pi_i \in \widehat{G}$, one may apply (5.2.2) to observe that $\|f\|_{L^2(G)} = \|\mathcal{T}(f)\|_{\mathcal{L}^2(G)}$. So, since $\mathcal{T}$ acts as an isometry on $\operatorname{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$; moreover, since $c_c(\widehat{G})$ is dense in $L^2(\widehat{G},h)$ and $\operatorname{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $ZL^2(G)$ (see [28, Chapter 5]), $\mathcal{T}$ can be extended as a mapping from $ZL^2(G)$ onto $L^2(\widehat{G},h)$ which is an isometrically isomorphism and takes complex conjugate to the involution.

We claim that $\mathcal{T}(f\overline{g}) = \mathcal{T}(f) *_h \mathcal{T}(g)\check{}$ for all $f, g \in ZL^2(G)$. To prove our claim it is enough to show that $\mathcal{T}(\chi_{\pi_1}\chi_{\pi_2}) = \mathcal{T}(\chi_{\pi_1}) *_h \mathcal{T}(\chi_{\pi_2})$ for $\pi_1, \pi_2 \in \widehat{G}$. Using Lemma 1.1.6 and (3.2.2), for each two representations $\pi, \sigma \in \widehat{G}$, we have

$$\begin{aligned}
\mathcal{T}(\chi_\pi\chi_\sigma) &= \mathcal{T}\left(\sum_{i=1}^n m_i^{\pi,\sigma}\chi_{\pi_i}\right) \\
&= \sum_{i=1}^n m_i^{\pi,\sigma}\mathcal{T}(\chi_{\pi_i}) \\
&= \sum_{i=1}^n m_i^{\pi,\sigma} d_{\pi_i}^{-1}\delta_{\pi_i} \\
&= d_\pi^{-1}\delta_\pi *_h d_\sigma^{-1}\delta_\sigma \\
&= \mathcal{T}(\chi_\pi) *_h \mathcal{T}(\chi_\sigma).
\end{aligned}$$

Now we can define a surjective extension $\mathcal{T}: ZL^1(G) \to A(\widehat{G})$, using the fact that $\text{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $ZL^1(G)$ and $\|f\|_1 = \inf \|g_1\|_2\|g_2\|_2$ for all $g_1, g_2 \in ZL^2(G)$ such that $f = g_1\overline{g_2}$ which are straightforward results. Using the definition of the norm of $A(\widehat{G})$,

$$\begin{aligned}\|\mathcal{T}(f)\|_{A(\widehat{G})} &= \inf\{\|\mathcal{T}(g_1)\|_{L^2(\widehat{G},h)}\|\mathcal{T}(g_2)\|_{L^2(\widehat{G},h)} : \mathcal{T}(f) = \mathcal{T}(g_1) * \mathcal{T}(g_2)\check{}\,\} \\ &= \inf\{\|g_1\|_{L^2(G)}\|g_2\|_{L^2(G)} : f = g_1\overline{g_2}\} = \|f\|_1\end{aligned}$$

for each $f \in ZL^1(G)$. To show that the extension of $\mathcal{T}$ is onto, for each pair $g_1, g_2 \in ZL^2(G)$, we note that $g_1\tilde{g}_2 \in ZL^1(G)$.

So, $\mathcal{T}$ is an isometric isomorphism between Banach spaces. Hence, $A(\widehat{G})$ is a Banach algebra with the product which is carried through the mapping $\mathcal{T}$. In the remaining we show that this product is actually the pointwise multiplication of functions in $A(\widehat{G})$ by using this fact that $\text{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $ZL^1(G)$ and $\mathcal{T}(d_\pi\chi_\pi) = \delta_\pi$, [28, Proposition 5.25]. Recall that for each pair $\pi, \sigma \in \widehat{G}$,

$$d_\pi\chi_\pi * d_\sigma\chi_\sigma = \begin{cases} d_\pi\chi_\pi & \pi = \sigma \\ 0 & \pi \neq \sigma \end{cases}$$

by Proposition 1.4.2. On the other hand,

$$\delta_\pi\delta_\sigma = \begin{cases} \delta_\pi & \pi = \sigma \\ 0 & \pi \neq \sigma \end{cases}$$

Since $\text{span}\{\delta_\pi\}_{\pi\in\widehat{G}}$ is dense in $A(\widehat{G})$, the algebraic action of $A(\widehat{G})$, inherited from $ZL^1(G)$ through $\mathcal{T}$, is corresponding to the pointwise multiplication of $A(\widehat{G})$. So one may conclude that $\big(A(\widehat{G}), \cdot, \|\cdot\|_{A(\widehat{G})}\big) \cong \big(ZL^1(G), *, \|\cdot\|_1\big)$. □

Remark 5.2.2. For a compact group G, we define $ZA(G) := A(G) \cap ZL^1(G)$. It is straightforward to check that

$$ZA(G) = \{f \in A(G) : f(yxy^{-1}) = f(x) \text{ for all } x, y \in G\}.$$

Furthermore, $ZA(G)$ forms a subalgebra of $A(G)$ with respect to the pointwise multiplication.

Theorem 5.2.3. *Let G be a compact group. Then the hypergroup algebra of $\widehat{G}$, $L^1(\widehat{G}, h)$, is isometrically isomorphic with the Banach algebra $ZA(G)$.*

Proof. Note that the Fourier transform $\mathcal{F}$ is an isometric isomorphism from Banach space $A(G)$ onto Banach space $\mathcal{L}^1(\widehat{G})$, [23, Theorem 8.4.16]. Since G is a compact group $A(G) \subseteq L^1(G)$;

therefore, for each $f \in ZA(G)$ and $g \in L^1(G)$, $f * g = g * f$; $f \in ZL^1(G)$. By an argument similar to the one after (5.2.1), for each $\pi \in \widehat{G}$, $\widehat{f}(\pi) = \alpha_\pi I_\pi$ for some $\alpha_\pi \in \mathbb{C}$ such that, by (1.4.2),

$$\begin{aligned} \|f\|_{A(G)} &= \sum_{\pi\in\widehat{G}} d_\pi |\alpha_\pi| \|I_\pi\|_{\mathcal{S}_1} = \sum_{\pi\in\widehat{G}} \alpha_\pi d_\pi^2 \\ &= \sum_{\pi\in\widehat{G}} \alpha_\pi h(\pi) = \|\mathcal{T}(f)\|_{L^1(\widehat{G})}. \end{aligned}$$

So $\mathcal{T}|_{ZA(G)}$ is an isometry into $L^1(\widehat{G}, h)$.

Conversely, for each $g = (\alpha_\pi)_{\pi\in\widehat{G}} \in c_c(\widehat{G})$, define $\phi_g = (\alpha_\pi I_\pi)_{\pi\in\widehat{G}} \in \mathcal{L}^1(\widehat{G})$. Since $\mathcal{F} : A(G) \to \mathcal{L}^1(\widehat{G})$ is surjective, by [23, Definition 8.4.12], for such a ϕ_g, there exists some $f \in \text{span}\{\chi_\pi\}_{\pi\in\widehat{G}} \subseteq A(G) \cap ZL^1(G)$ such that $\mathcal{F}(f) = \phi_g$; hence, $\mathcal{T}(f) = g \in c_c(\widehat{G}) \subseteq L^1(\widehat{G})$. Applying this fact that $\mathcal{T}$ is an isometric mapping and $c_c(\widehat{G})$ is dense in $L^1(\widehat{G})$, $\mathcal{T}$ is a surjective mapping. Note that this argument consequently implies that $\text{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $ZA(G)$.

We claim that $\mathcal{T}$ is an algebra isomorphism. To prove our claim, we just note that

$$\mathcal{T}(\chi_{\pi_1}\chi_{\pi_2}) = \mathcal{T}(\sum_{i=1}^n m_i \chi_{\sigma_i}) = \sum_{i=1}^n m_i d_{\sigma_i}^{-1} \delta_{\sigma_i} = d_{\pi_1}^{-1}\delta_{\pi_1} *_h d_{\pi_2}^{-1}\delta_{\pi_2} = \mathcal{T}(\chi_{\pi_1}) *_h \mathcal{T}(\chi_{\pi_2}).$$

One may extend it to the whole $ZA(G)$, using this fact that $\text{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is dense in $ZA(G)$. □

Although the following corollary is a well-known result, we mention it here since we apply it more often in the next section. Besides the classic proof, it may be implied by the argument in the proof of Theorem 5.2.3.

Corollary 5.2.4. *Let G be a compact group. Then $ZA(G)$ is the closure of* $\text{span}\{\chi_\pi\}_{\pi\in\widehat{G}}$.

5.3 Amenability of $ZA(G)$

First let us briefly mention some results from [36, (29.25)] which characterize group characters of $SU(2)$. In this section, we present the torus $\mathbb{T}$ by the interval $[0, 2\pi]$ where each $\theta \in [0, 2\pi]$ represents $e^{i\theta}$.

Let $u \in SU(2)$. There is a matrix $a \in SU(2)$ such that

$$a^{-1}ua = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \tag{5.3.1}$$

for some $\theta \in [0, 2\pi]$. As we saw in Example 3.2.2, the irreducible unitary representations of $SU(2)$ can be represented by π_ℓ where $\ell \in \frac{1}{2}\mathbb{Z}^+ := \{0, 1/2, 1, 3/2, 2, \ldots\}$. Also for each χ_{π_ℓ} the group character generated by the representation π_ℓ, $\chi_{\pi_\ell}(a^{-1}ua) = \chi_{\pi_\ell}(u)$. So, we will know χ_ℓ if we compute it on the matrices of the form (5.3.1) for each $\theta \in [0, 2\pi]$. To facilitate the writing, we denote $\chi_{\pi_\ell}(u)$ for some u corresponding to θ by $\chi_\ell(\theta)$.

By [36, (29.25)], the irreducible group characters of $SU(2)$ represented by $\{\chi_k\}_{k\in\frac{1}{2}\mathbb{Z}^+}$ can be characterized by the functions[1]

$$\chi_k(\theta) = \begin{cases} \frac{\sin((2k+1)\theta)}{\sin(\theta)} & \theta \in (0,\pi) \cup (\pi, 2\pi) \\ (2k+1)e^{2i\theta k} & \theta = 0, \pi \end{cases} \tag{5.3.2}$$

where $k \in \frac{1}{2}\mathbb{Z}^+$ and $\theta \in \mathbb{T}$.

For each $f = \sum_{j=1}^n \alpha_j \chi_{k_j} \in \text{lin}\{\chi_k\}_{k\in\mathbb{N}_0}$, by (1.4.2) and (1.4.1),

$$\|f\|_{A(SU(2))} = \sum_{j=1}^n d_{k_j}|\alpha_j| \|\widehat{\chi}_{k_j}(\pi_{k_j})\|_{\mathcal{S}_1} = \sum_{j=1}^n d_{k_j}|\alpha_j| \|d_{k_j}^{-1} I_{\pi_{k_j}}\|_{\mathcal{S}_1} = \sum_{j=1}^n |\alpha_j|(2k_j+1). \tag{5.3.3}$$

Let us define a mapping $\mathcal{I} : \text{lin}\{\chi_k\}_{\mathbb{N}_0} \to C^1(\mathbb{T})$ to be the restriction map to the the torus $\mathbb{T}$ as defined in (5.3.2) where $C^1(\mathbb{T})$ denotes the set of all differentiable functions on $\mathbb{T}$. We claim that $\mathcal{I}$ can be extended to a continuous mapping on $ZA(SU(2))$. Doing so, it is enough to show that $\mathcal{I}(ZA(SU(2))) \subseteq A(\mathbb{T})$ where $A(\mathbb{T})$ is the Fourier algebra of the torus.

Proving our claim about $\mathcal{I}$, note that for each $k \in \frac{1}{2}\mathbb{Z}^+$ and $\theta \in \mathbb{T} \setminus \{0, \pi\}$,

$$\mathcal{I}(\chi_k)(\theta) = \frac{\sin((2k+1)\theta)}{\sin(\theta)} = \frac{e^{i(2k+1)\theta} - e^{-i(2k+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \sum_{\ell=0}^{2k} e^{i(2k-2\ell)\theta} = \sum_{\ell=-2k}^{2k} e^{i\ell\theta}.$$

A similar argument works for $\theta = 0, \pi$ as well. Note that by (1.4.2),

$$\|\mathcal{I}(\chi_k)\|_{A(\mathbb{T})} = \|\theta \mapsto \sum_{\ell=-2k}^{2k} e^{i\ell\theta}\|_{A(\mathbb{T})} = \sum_{\ell=-2k}^{2k} 1 = (2k+1).$$

Therefore, $\|\mathcal{I}(\chi_k)\|_{A(\mathbb{T})} = d_k$ which is equal to $\|\chi_k\|_{A(SU(2))}$. For each $f \in \text{lin}\{\chi_k\}_{k\in\mathbb{N}_0}$ say $f = \sum_{j=0}^n \alpha_j \chi_j$,

$$\|f\|_{A(SU(2))} = \sum_{\ell=0}^n |\alpha_\ell|(2\ell+1) \geq \|\mathcal{I}(f)\|_{A(\mathbb{T})}.$$

Hence, one may extend $\mathcal{I}$ as a continuous linear mapping from $ZA(SU(2))$ into $A(\mathbb{T})$. Note that $\mathcal{I}$ is the restriction mapping on $\mathbb{T}$; therefore, $\mathcal{I}(fg) = \mathcal{I}(f)\mathcal{I}(g)$ for all $f, g \in ZA(SU(2))$. Furthermore, if for some $f \in ZA(SU(2))$, $\mathcal{I}(f) = 0$, it means that for each conjugacy class C of $SU(2)$, $f(C) = 0$; $f \equiv 0$. So $\mathcal{I}$ is injective. We denote the image of $\mathcal{I}$ here by $\mathcal{A}_{\mathcal{I}}$.

[1] Applying this representation of the elements of $SU(2)$ with respect to their eigenvalues as elements in $[0, 2\pi]$, each conjugacy class will be represented twice by the angle θ. But since, we want to study $ZA(SU(2))$ as functions restricted on $\mathbb{T}$ as the maximal torus of $SU(2)$, we rely on this representation.

Remark 5.3.1. One may show that $\mathcal{I}$ is not surjective to the closure of its image, $\mathcal{A}_{\mathcal{I}}$. If $\mathcal{I}$ is surjective, since it is one to one as well, $\mathcal{I}^{-1}$ should form a bounded mapping from $\mathcal{A}_{\mathcal{I}}$ into $ZA(SU(2))$. But note that on one hand for each $k \in \mathbb{N}_0$; $\|\chi_k - \chi_{k-1}\|_{ZA(SU(2))} = (2k+1)+(2k-1) = 4k$. On the other hand, $\|\mathcal{I}(\chi_k - \chi_{k-1})\|_{A(\mathbb{T})} = \|e^{i2k\theta} + e^{-i2k\theta}\|_{A(\mathbb{T})} = 2$. Therefore, $\mathcal{I}$ cannot be invertible and $\mathcal{A}_{\mathcal{I}}$ is not a closed subalgebra of $A(\mathbb{T})$.

Proposition 5.3.2. *$ZA(SU(2))$ is not (weakly) amenable.*

Proof. Let us consider $\mathcal{A}_{\mathcal{I}} := \mathcal{I}(ZA(SU(2)) \subseteq C^1(\mathbb{T})$ the restriction of the functions in $ZA(SU(2))$ on the maximal torus. We will prove the existence of a non-zero continuous point derivation on $\mathcal{A}_{\mathcal{I}}$. For $\theta \in (0,\pi)$, one may define $D_\theta : \mathcal{A}_{\mathcal{I}} \to \mathbb{C}$ where D_θ is the point derivation on functions of $C^1(\mathbb{T})$ evaluated at θ. Therefore

$$\begin{aligned} D_\theta(\chi_k) &= \sum_{\ell=-2k}^{2k} i\ell e^{i\ell\theta} = \sum_{\ell=1}^{2k} i\ell e^{i\ell\theta} - i\ell e^{-i\ell\theta} = -2\sum_{\ell=1}^{2k} \ell \sin(\ell\theta) \\ &= \frac{2}{4\sin^2(\theta/2)}(2k\sin((2k+1)\theta) - (2k+1)\sin(2k\theta)). \end{aligned}$$

Moreover, D_θ is non zero, for example $D_{\pi/2}(\chi_{1/2}) = -2$; further,

$$|D_\theta(\chi_k)| \le \frac{1}{\sin^2(\theta/2)}(2k+1) = \frac{1}{\sin^2(\theta/2)}\|\chi_k\|_{A(SU(2))}.$$

One may apply (5.3.3) to verify that for each $f = \sum_{j=1}^n \alpha_j \chi_{k_j} \in \mathrm{lin}\{\chi_k\}_{k\in\mathbb{N}_0}$,

$$|D_\theta(f)| \le \sum_{j=1}^n \frac{|\alpha_j|}{\sin^2(\theta/2)}\|\chi_k\|_{A(SU(2))} = \frac{1}{\sin^2(\theta/2)}\|f\|_{A(SU(2))},$$

which implies that D_θ can be extended as a norm bounded linear map on $ZA(SU(2))$, because $\mathrm{lin}\{\chi_k\}_{k\in\mathbb{N}_0}$ is dense in $ZA(G)$. Hence, D_θ is a non-zero bounded derivation on $\mathcal{A}_{\mathcal{I}}$, so $\mathcal{A}_{\mathcal{I}}$ is not weakly amenable. □

Remark 5.3.3. Let $SO(3)$ be the compact Lie group of 3×3 special orthogonal group. $SO(3)$ actually forms the set all of rotations in $\mathbb{R}^3$ which preserves the length and orientation. In fact, there is a two-to-one continuous homomorphism τ from $SU(2)$ onto $SO(3)$ such that $\mathrm{Ker}(\tau) = \{\pm I\}$, [36, Theorem 29.36]. For

$$u_\theta := \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \quad \theta \in [0,2\pi]$$

one gets that

$$\tau(u_\theta) = \begin{bmatrix} \cos(2\theta) & -\sin 2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying τ, one may show that for every integer $\ell \in \mathbb{Z}^+ := \{0, 1, 2, \cdots\}$, there is a representation $\pi_\ell \in \widehat{SO(3)}$ and vice versa. Moreover, for all $\ell, \ell' \in \mathbb{Z}^+$ $(\ell \geq \ell')$, $\pi_\ell \otimes \pi_{\ell'}$ is equivalent to $\pi_{\ell-\ell'} \oplus \pi_{\ell-\ell'+1} \oplus \cdots \oplus \pi_{\ell+\ell'}$. Moreover, $\chi_\ell(u_\theta)$ is defined exactly as one defined χ_ℓ on θ in (5.3.2). So one may rewrite this section for $SO(3)$ and all the results would be still valid. Specially, $ZA(SO(3))$ is not (weakly) amenable.

In the remaining, we prove that not only are $ZA(SO(3))$ and $ZA(SU(2))$ not (weakly) amenable, but also for fro a wider class of compact groups G, $ZA(G)$ cannot be (weakly) amenable.

To prove the main result of this section, we need to prove a few results first. Although the following proposition is known for the experts, because of completeness we prove it here.

Proposition 5.3.4. *Let G be a compact group. Then the space of maximal ideals of $ZA(G)$ is homeomorphic to* $\mathrm{Conj}(G)$ *equipped with the quotient topology of G through the mapping $\iota : G \to \mathrm{Conj}(G)$ where $x \mapsto C_x$. Moreover, $ZA(G)$ separates conjugacy classes of G i.e. for $C, D \in \mathrm{Conj}(G)$, $C \neq D$, there is some $f \in ZA(G)$ such that $f(C) = 0$ and $f(D) = 1$.*

Proof. Clearly for each $C \in \mathrm{Conj}(G)$, $\psi_C : ZA(G) \to \mathbb{C}$ forms a multiplicative bounded functional on $ZA(G)$ where $\psi_C(f) = f(x)$ for some $x \in C$. Hence, $\mathrm{Ker}\,\psi_C$ is a maximal ideal space of $ZA(G)$.

Conversely, for each $\phi \in \sigma(ZA(G))$ and $f \in \mathrm{Ker}\,\phi$, f is not invertible. Theorem 3.6.15 and Theorem 3.7.1 of [66] imply that for a commutative regular Banach algebra $\mathcal{A}$ and a closed subset E of $\sigma(\mathcal{A})$ equipped with the Gelfand spectrum topology, if $a \in \mathcal{A}$ such that $|\varphi(a)| \geq \delta > 0$ for every $\varphi \in E$, then there exists some $a' \in \mathcal{A}$ such that $\varphi(aa') = 1$ for every $\varphi \in E$. In particular, this applies to $E = \sigma(\mathcal{A})$. It is known that $A(G)$ is a commutative regular Banach algebra and its Gelfand spectrum is homeomorphic to G.

Now assume that $f(x) \neq 0$ for all $x \in G$; therefore, f as an element in $A(G)$ is invertible i.e. there exists some $f' \in A(G)$ such that $ff'(x) = 1$ for all $x \in G$. Clearly since f is a class function so is f'; $f' \in ZA(G)$ which violates our assumption. Therefore, $f(x) = 0$ for some $x \in G$. Therefore, $\mathrm{Ker}\,\phi \subseteq \mathrm{Ker}\,\psi_{C_x}$ as two maximal ideals. So, $\phi = \psi_{C_x}$.

Let us note that for each $f \in A(G)$, $L_y R_y f(x) = f(y^{-1}xy)$ also belongs to $A(G)$. Therefore, we may get a Bochner integral of $L_y R_y f$ for all $y \in G$; consequently, $P : A(G) \to ZA(G)$ where

$$P(f)(x) = \int_G f(y^{-1}xy)dy.$$

For $x \in G$, note that $C_x \in \mathrm{Conj}(G)$ is the image of the compact set G through the continuous mapping $y \mapsto yxy^{-1}$; C_x is a compact and hence closed subset of G. Applying regularity of the

Banach algebra $A(G)$, for each $C, D \in \mathrm{Conj}(G)$ where $C \neq D$, there is some $f \in A(G)$ such that $f(C) \equiv 1$ and $f(D) \equiv 0$. The existence of such a f is proven in [26, Lemma 3.2], since the conjugacy classes are closed as mentioned. Hence, $P(f)(C) \equiv 1$ and $P(f)(D) = 0$ when $\lambda(G) = 1$.

Let $\Psi : \mathrm{Conj}(G) \to \sigma(ZA(G))$ be the mapping such that $\Psi(C) = \psi_C$. Ψ is an onto mapping. In the following we demonstrate that Ψ is a homeomorphism where $\mathrm{Conj}(G)$ is equipped with the quotient topology of G through the mapping $\iota : G \to \mathrm{Conj}(G)$ where $x \mapsto C_x$. By [59, Theorem 22.2], for each $f \in ZA(G)$ as a continuous class function, f can be regarded as a function in $C(\mathrm{Conj}(G))$; consequently the quotient topology of $\mathrm{Conj}(G)$ is finer that the Gelfand topology $\sigma(ZA(G))$. Note that $\sigma(ZA(G))$ is compact, because $ZA(G)$ is a unital algebra. Therefore, $\Psi : \sigma(ZA(G)) \to \mathrm{Conj}(G)$ forms a continuous bijection from a compact space to another compact space; hence, Ψ^{-1} is continuous. □

The following theorem is the main result of this section.

Theorem 5.3.5. *Let G be a compact group such that G_e, the connected component of the identity, is not abelian. Then $ZA(G)$ is not weakly amenable.*

Proof. Since G_e is not abelian, by a result in the proof of Theorem 2.1 in [29], G_e has a closed subgroup H such that is isomorphic to the topological group $SU(2)$ or $SO(3)$.

For each $f \in A(G)$, let $\iota_H(f)$ denotes the restriction of the function f to the subgroup H. As it was proven in [20], for the closed subgroup H of G, $\iota_H(A(G)) = A(H)$; further, $\|\iota_H(f)\|_{A(H)} \leq \|f\|_{A(G)}$. Therefore, $ZA(G)|_H \subseteq A(H)$ where $ZA(G)|_H = \iota_H(ZA(G))$.

Moreover, for each $f \in ZA(G)$, $f(xyx^{-1}) = f(y)$ for all $x, y \in G$. So for each $x, y \in H$, $f(xyx^{-1}) = f(y)$. In other words, $f|_H$ is a class function on H as well. Hence $ZA(G)|_H \subseteq ZA(H)$.

For each $\pi \in \widehat{G}$, note that $\pi|_H$ is a unitary representation and it may be decomposed applying finitely many representations $\sigma_i \in \widehat{H}$ such that $\pi|_H = \oplus_{i=1}^n m_i \sigma_i$; hence,

$$\chi_\pi(y) = \mathrm{Tr}(\pi|_H(y)) = \sum_{i=1}^m m_i \chi_{\sigma_i}(y) \quad \text{for all } y \in H,$$

where m_i denotes the number of redundant of each representation σ_i in the irreducible decomposition of $\pi|_H$. Hence, $\chi_\pi|_H \in \mathrm{lin}\{\chi_\sigma\}_{\sigma \in \widehat{H}}$.

If for each $\pi \in \widehat{G}$, χ_π is a constant function on H; therefore, $\mathrm{lin}\{\chi_\pi\}_{\pi \in \widehat{G}}$ and consequently $ZA(G)$ are also constant on H. In this case, Proposition 5.3.4 implies that H is contained in just one conjugacy class of G i.e. $H = \{e\}$. Hence, there is some $\pi \in \widehat{H}$ such that $\chi_\pi|_H (= \iota_H(\chi_\pi))$ is not a constant function on H; $\chi_\pi|_H = \sum_{k=0}^n \alpha_k \chi_k$ for $n \in \mathbb{K}$ where $\mathbb{K} = \frac{1}{2}\mathbb{Z}^+ \,(= \{0, 1/2, 1, \cdots\})$ if

$H = SU(2)$ or $\mathbb{K} = \mathbb{Z}^+ (= \{0, 1, \cdots\})$ if $H = SO(3)$, $\alpha_k \in \mathbb{C}$, and $\{\chi_k\}_{k\in\mathbb{K}}$ are the group characters of H. Note that $\alpha_k \neq 0$ for at least one $k > 0$. Therefore, for the restriction mapping $\mathcal{I}$ defined earlier, we get that $\mathcal{I}(\chi_k) \in A(\mathbb{T})$ is not a constant function on $\mathbb{T}$ (because all constant functions in $A(\mathbb{T})$ are the elements of the subalgebra generated by the constant function 1, by [28, Proposition 5.23], and clearly $\mathcal{I}(\chi_k)$ does not belong to that subalgebra while $\alpha_k \neq 0$ for some $k > 0$). Therefore,

$$\mathcal{I}(\chi_\pi|H)(\theta) = \sum_{k=0}^{n} \alpha_k \mathcal{I}(\chi_k)(\theta) = \sum_{k=0}^{n} \alpha_k \sum_{\ell=-2k}^{2k} e^{i\ell\theta} \in \operatorname{lin}\{\chi_k\}_{k\in\mathbb{Z}^+/2} \quad (\theta \in \mathbb{T})$$

is not a constant function on $\mathbb{T}$. Hence, there is some $\theta \in (0, \pi)$ such that $D_\theta(\chi_\pi|H) \neq 0$ for the continuous point derivation D_θ defined in the proof of Proposition 5.3.2. Note that $\|\iota_H(f)\|_{A(H)} \leq \|f\|_{A(G)}$ for all $f \in ZA(G)$; hence, $D_\theta \circ \iota_H$ forms a non-zero bounded derivation on $ZA(G)$ and therefore, $ZA(G)$, as a commutative algebra, is not weakly amenable. □

Question. In [29], it was proved that the Fourier algebra of a compact group G is weakly amenable if and only if G_e is abelian. Theorem 5.3.5 shows that for such a G one side of such a result holds for $ZA(G)$ as well. One may conjecture that the other side can be proven for $ZA(G)$ as well where G_e is abelian.

Chapter 6

Følner type conditions on hypergroups

6.1 Amenability properties of regular Fourier hypergroups

Amenability of hypergroups has different levels. The concept of amenability can be defined on hypergroups as the existence of a left invariant mean, analogous to groups. In this sense lots of hypergroups that we know are amenable, say all commutative hypergroups and compact hypergroups. This notion of amenability was mainly studied in [70]. In that paper, the author also showed that the amenability of a hypergroup is equivalent to the property (P_1) which is defined in the following. In this chapter, we introduce more amenability properties of hypergroups and study them.

6.1.1 Følner type conditions on Hypergroups

Amenable locally compact groups are characterized by a variety of properties including Følner type conditions. These conditions have been studied extensively, [25, 54]. Not only have Følner conditions attracted attention for locally compact groups, but Følner conditions have also been interesting and useful in the study of semigroups, [73]. They relate the concept of "amenability" to the structure of the group or semigroup. In this section, we look at a generalization of Følner type conditions over hypergroups.

In [2], I introduced the Leptin condition for hypergroups. Here, we define more Følner type conditions for hypergroups and we study their relations. To recall, for each two subsets A and B of some set X, we denote their symmetric difference, $(A \setminus B) \cup (B \setminus A)$, by $A \triangle B$.

Definition 6.1.1. Let H be a hypergroup and $D \geq 1$ an integer. We define the following properties:

(L_D) We say that H satisfies the *D-Leptin condition* if for every compact subset K of H and $\epsilon > 0$, there exists a measurable set V in H such that $0 < h(V) < \infty$ and $h(K * V)/h(V) < D + \epsilon$.

(F) We say that H satisfies the *Følner condition* if for every compact subset K of H and $\epsilon > 0$, there exists a measurable set V in H such that $0 < h(V) < \infty$ and $h(x * V \triangle V)/h(V) < \epsilon$ for every $x \in K$.

(SF) We say that H satisfies the *Strong Følner condition* if for every compact subset K of H and $\epsilon > 0$, there exists a measurable set V in H such that $0 < h(V) < \infty$ and $h(K*V \triangle V)/h(V) < \epsilon$.

Remark 6.1.2. If a hypergroup H satisfies the 1-Leptin condition, H is said to satisfy the *Leptin condition* as defined in [2, Definition 4.1]. From now on, we may use the Leptin condition instead of the 1-Leptin condition and we denote it by (L).

Proposition 6.1.3. *For every compact hypergroup H, H satisfies all conditions (SF), (F), and (L).*

Proof. The proof is a direct result of finiteness of the Haar measure on compact hypergroups, [8], by replacing $V = H$ for all conditions in Definition 6.1.1. □

Remark 6.1.4. In Definition 6.1.1 of the Leptin condition, (L_D), we can suppose that V is compact. To show this fact suppose that H satisfies the D-Leptin condition. For compact subset K of H and $\epsilon > 0$, there exists a measurable set V such that $h(K * V)/h(V) < D + \epsilon$. Using regularity of h, as a measure, for each positive integer n, we can find compact set $V_1 \subseteq V$ such that $h(V \setminus V_1) < h(V)/n$. This implies that $0 < h(V_1)$ and $h(V)/h(V_1) < n/(n-1)$. Therefore

$$\frac{h(K * V_1)}{h(V_1)} \leq \frac{h(V)}{h(V_1)} \left(\frac{h(K * V_1)}{h(V)} \right) < \frac{n}{n-1}(D + \epsilon).$$

So we can add compactness of V to the definition of the Leptin condition.

Proposition 6.1.5. *For every hypergroup H, (SF) implies (L).*

Proof. For a compact set K and $\epsilon > 0$, let V be a measurable set such that $h(K*V \triangle V) < \epsilon h(V)$. Hence

$$\begin{aligned}
\frac{h(K * V)}{h(V)} - 1 &\leq \frac{h(K * V) - h(V)}{h(V)} \\
&\leq \frac{h(K * V) + h(V) - 2h((K * V) \cap V)}{h(V)} \\
&= \frac{h((K * V) \triangle V)}{h(V)} < \epsilon.
\end{aligned}$$

□

Proposition 6.1.6. *For every discrete hypergroup H, (F) implies (SF). And consequently, (F) implies (L).*

Proof. We should just show that $(F) \Rightarrow (SF)$ the rest is obtained by Proposition 6.1.5. Let K be a compact subset of H. Since for discrete hypergroups, each compact set is finite, we may suppose that $K = \{x_i\}_{i=1}^n$. Therefore, for $\epsilon > 0$ there is a finite set V such that $0 < h(V)$ and

$$\frac{h((x * V) \triangle V)}{h(V)} < \frac{\epsilon}{|K|} \quad (x \in K).$$

So

$$\begin{aligned} \frac{h((\bigcup_{i=1}^n x_i) * V \triangle V)}{h(V)} &= \frac{h(\bigcup_{i=1}^n (x_i * V) \triangle V)}{h(V)} \\ &\leq \sum_{i=1}^n \frac{h(x_i * V \triangle V)}{h(V)} = \epsilon. \end{aligned}$$

The last inequality is a result of the following fact about arbitrary sets B_1, B_2, C:

$$((B_1 \cup B_2) \triangle C) \subseteq \big(B_1 \triangle C)\big) \cup \big(B_2 \triangle C)\big).$$

□

Remark 6.1.7. If H is a locally compact group, all the conditions (F), (SF), and (L) are equivalent and they equal the amenability of the group H. If one tries to adapt the rest of relations of (F), (SF), and (L) from the group case, [63], one may notice that in almost all of the arguments, the inclusion $x(A \setminus B) \subseteq xA \setminus xB$ is crucially applied where A, B are subsets of the group H and x is one arbitrary element.[1] But this inclusion does not necessarily hold for a general hypergroup. As an example, one may consider $A := \{\pi_0, \pi_{\frac{1}{2}}, \ldots, \pi_{k-\frac{1}{2}}, \pi_k\}$ and $B := \{\pi_0, \pi_{\frac{1}{2}}, \ldots, \pi_{k-1}, \pi_{k-\frac{1}{2}}\}$ as two subsets of $\widehat{SU(2)}$ for some $k \in \frac{1}{2}\mathbb{Z}^+$ (see Example 3.2.2). Therefore, one gets $\pi_k * A = \{0, \pi_{1/2}, \ldots, \pi_{2k}\}$ and $\pi_k * B = \{\pi_{1/2}, \ldots, \pi_{2k-\frac{1}{2}}\}$; hence, $(\pi_k * A) \setminus (\pi_k * B) = \{\pi_0, \pi_{2k}\}$. But $\pi_k * (A \setminus B) = \pi_k * \pi_k = \{\pi_0, \pi_1, \ldots, \pi_{2k-1}, \pi_{2k}\}$.

6.1.2 The existence of a bounded approximate identity of Fourier algebra

For a regular Fourier hypergroup H, we denote the existence of a $\|\cdot\|_{A(H)}$-bounded approximate identity by some $D \geq 1$ by (B_D) and we call it D-bounded approximate identity.

Theorem 6.1.8. *Let H be a regular Fourier hypergroup which satisfies the D-Leptin condition. Then $A(H)$ has a D-bounded approximate identity.*

[1]Note that in general the equality holds, but this side of the inclusion suffices.

Proof. Fix $\epsilon > 0$. Using the D-Leptin condition on H, for every arbitrary non-void compact set K in H, we can find a measurable subset V_K of H with $0 < h(V_K) < \infty$ such that $h(K * V_K)/h(V_K) < D^2(1+\epsilon)^2$. Define

$$v_K := \frac{1}{h(V_K)} 1_{K * V_K} *_h \tilde{1}_{V_K}.$$

As in the proof of Lemma 5.1.8, we have $\|v_K\|_{A(H)} < D(1+\epsilon)$ and $v_K|_K \equiv 1$. We consider the net

$$\{a_{\epsilon,K} : K \subseteq H \text{ compact, and } 0 < \epsilon < 1\}$$

in $A(H)$ where $a_{\epsilon,K} := (1+\epsilon)^{-1} v_K$ and $a_{\epsilon_1,K_1} \preccurlyeq a_{\epsilon_2,K_2}$ whenever $\operatorname{supp}(v_{K_1}) \subseteq K_2$ and $\epsilon_2 < \epsilon_1$. So $(a_{\epsilon,K})_{K \subseteq H, 0<\epsilon<1}$ forms a $\|\cdot\|_{A(H)}$-norm D-bounded net in $A(H) \cap C_c(H)$. Let $f \in A(H) \cap C_c(H)$ with $K_0 = \operatorname{supp} f$. Then $v_K f = f$ where $K_0 \subseteq K$. Therefore

$$\lim_{\epsilon \to 0} \lim_{K_0 \subseteq K \to H} \|a_{\epsilon,K} f - f\|_{A(H)} = \lim_{\epsilon \to 0} \|\frac{f}{1+\epsilon} - f\|_{A(H)} = \|f\|_{A(H)} \lim_{\epsilon \to 0} \frac{\epsilon}{1+\epsilon} = 0.$$

Since, by Proposition 5.1.1, $A(H) \cap C_c(H)$ is dense in $A(H)$, $(a_{\epsilon,K})_{0<\epsilon<1, K \subseteq H}$ is a D-bounded approximate identity of $A(H)$. □

Remark 6.1.9. Let G be a compact group. Then the Fourier algebra of $\widehat{G}$, $A(\widehat{G})$, is algebraically isometrically isomorphic to $ZL^1(G)$, by Theorem 5.2.1. Also since every compact group G is a SIN-group, [35], $ZL^1(G)$ always has a 1-bounded approximate identity. Therefore, $A(\widehat{G})$ has a 1-bounded approximate identity.

6.1.3 Reiter condition

In [70, Theorem 4.1], it was shown that the amenability of a hypergroup is equivalent to the property (P_1) which is defined as follows.

Definition 6.1.10. [70, p32]
We say that H satisfies (P_r), $r = 1$ or 2, if whenever $\epsilon > 0$ and a compact set $E \subseteq H$ are given, then there exists $f \in L^r(H)$, $f \geq 0$, $\|f\|_r = 1$ such that

$$\|\delta_x * f - f\|_r < \epsilon \quad x \in E.$$

We say that K satisfies the *Reiter condition* if it has property (P_1).

[70, Theorem 4.3] showed that (P_2) implies (P_1). Furthermore, for every hypergroup H, (P_1) is equivalent to the amenability of H, [70, Theorem 4.1].

Example 6.1.11. Every commutative or compact hypergroup H, as an amenable hypergroup, satisfies condition (P_1), see [70].

We rely on the following lemma which is from [70] to characterize (P_2).

Lemma 6.1.12. *[70, Lemma 4.4]*
Let H be a hypergroup. Then H satisfies (P_2) if and only if there is a net $(f_\alpha)_\alpha \subseteq L^2(H)$ such that $\|f_\alpha\|_2 = 1$ and $f_\alpha * \tilde{f}_\alpha$ converges to 1 uniformly on compact subsets of H.

Remark 6.1.13. Note that by Lemma 6.1.12, (P_2) implies the existence of a net (g_α) (in the form of $g_\alpha := f_\alpha * \tilde{f}_\alpha$) which belongs to $A(H)$ while, by Theorem 5.1.2, $\|g_\alpha\|_{A(H)} \le \|f_\alpha\|_2^2 = 1$.

Remark 6.1.14. Note that in hypergroup case, (P_2) is not necessarily equivalent to the amenability of the hypergroup, though it implies the amenability of the hypergroup. As a counterexample, one may consider the *Naimark hypergroup*, see [8, (3.5.66)] and [70, Example 4.6], that is a commutative hypergroup structure on $\mathbb{R}^*$ $(= [0, +\infty))$ where

$$\delta_x * \delta_y := \frac{1}{\sinh(x)\sinh(y)} \int_{|x-y|}^{x+y} \sinh(t)\delta_t \, dt \quad (x, y \in \mathbb{R}^*),$$

$\check{x} := x$, and 0 is the identity. For this hypergroup, constant character 1 does not belong to the support of the Plancherel measure. But [70, Lemma 4.5] shows that for a commutative hypergroup H, the constant character 1 belongs to the support of the Plancherel measure if and only if H satisfies (P_2). Therefore, the Naimark hypergroup does not satisfy condition (P_2) while as a commutative hypergroup it does satisfy (P_1).

The following theorem resembles the Leptin theorem for commutative regular Fourier hypergroups. In the proof, some techniques of the proof of group case (see [68, Theorem 7.1.3]) have been applied. Some properties of the Fourier algebra which are applied here have been mentioned briefly in Section 5.1.

Let us recall that a *state* on a C^*-algebra is a positive linear functional of norm 1. Moreover, if $\mathcal{A}$ is a von Neumann algebra with predual $\mathcal{A}_*$, every state of $\mathcal{A}$ can be approximated by a net of states of the elements of pre-dual in the weak* topology. Therefore, for a commutative hypergroup H, a state u on $VN(H)$ which belongs to $A(H)$, is in the form of $g * \tilde{g}$ for some $g \in L^2(H)$ (see Section 5.1).

Theorem 6.1.15. *Let H be a commutative regular Fourier hypergroup. Then the following conditions are equivalent.*

(B_1) *$A(H)$ has a 1-bounded approximate identity.*

(B_D) *$A(H)$ has a D-bounded approximate identity for some $D \geq 1$.*

(P_2) *H satisfies (P_2).*

Proof. $(B_1) \Rightarrow (B_D)$ is trivial.

$(B_D) \Rightarrow (P_2)$.

For $(e_\alpha)_\alpha$ a D-bounded approximate identity of $A(H)$, there exists a w^*-cluster point $F \in VN(H)^*$. Note that for each $x \in H$, $\langle \lambda(x), F\rangle = \lim_\alpha \langle \lambda(x), e_\alpha\rangle = \lim_\alpha e_\alpha(x) = 1$. So $F|_{L^1(H,h)}$ may be interpreted as the constant function 1 on H (where $L^1(H,h)$ is observed as a subalgebra of $VN(H)$). Therefore, for each $f, g \in L^1(H,h)$, one gets that $\langle F, f * g\rangle = \langle F, f\rangle \langle F, g\rangle$. Hence $F|_{L^1(H,h)}$ is a multiplicative functional on $L^1(H,h)$. Therefore, for each $f \in L^1(H,h)$, $\langle F, \tilde{f} *_h f\rangle = \langle F, \tilde{f}\rangle\langle F, f\rangle = |\langle F, f\rangle|^2 \geq 0$. But $L^1(H,h)$ is dense in the C^*-algebra $C^*_\lambda(H)$; hence, $F|_{C^*_\lambda(H)}$ is a positive functional on $C^*_\lambda(H)$ that is $\langle F, f * \tilde{f}\rangle \geq 0$ for every $f \in C^*_\lambda(H)$. Also as a multiplicative functional, $\|F|_{C^*_\lambda(H)}\| = 1$. But as a positive norm 1 functional, $F|_{C^*_\lambda(H)}$ is a state. Thus, by [55, Corollary 2.3.12], $F|_{C^*_\lambda(H)}$ is extendible to a state E on $VN(H)$. Because states of $VN(H)$ which belong to $A(H)$ are weak* dense in the set of all states of $VN(H)$, we may find a net $(f_\beta)_\beta$ in $\{f *_h \tilde{f} : f \in L^2(H,h)\}$ such that $f_\beta = g_\beta *_h \tilde{g_\beta} \to E$ in weak* topology for a net $(g_\beta)_\beta \subseteq L^2(H,h)$. Moreover, $1 = \|f_\beta\|_{A(H)} = f_\beta(e) = g_\beta *_h \tilde{g}_\beta(e) = \|g_\beta\|_2^2$. Since $F|_{C^*_\lambda(H)} = E|_{C^*_\lambda(H)}$, for each $u \in A(H)$ and $f \in L^1(H)$, since $uf \in L^1(H)$, we have

$$\lim_\beta \langle u f_\beta, f\rangle = \langle u \cdot E, f\rangle = \langle F, uf\rangle = \lim_\alpha \langle e_\alpha, uf\rangle = \langle u, f\rangle. \tag{6.1.1}$$

Therefore, $uf_\beta \to u$ with respect to the topology $\sigma(A(H), L^1(H))$. Recall that $L^1(H)$ is dense in $C^*_\lambda(H)$ while $A(H) \subseteq B_\lambda(H)$ and $B_\lambda(H) = C^*_\lambda(H)^*$. Let us fix $u \in A(H)$. Therefore, for some given $\epsilon > 0$ and $f \in C^*_\lambda(H)$, there is a $g \in L^1(G)$ such that $\|g - f\|_{C^*_\lambda(H)} < \epsilon$. Also there is some β_0 such that for each $\beta \succcurlyeq \beta_0$, $|\langle uf_\beta - u, g\rangle| < \epsilon$. So,

$$\begin{aligned} |\langle f_\beta u - u, f\rangle| &\leq |\langle f_\beta u - u, f - g\rangle| + |\langle f_\beta u - u, g\rangle| \\ &\leq \|u\|_{A(H)}(\|f_\beta\|_{A(H)} + 1)\|f - g\|_{C^*_\lambda(H)} + \epsilon < (2\|u\|_{A(H)} + 1)\epsilon. \end{aligned}$$

Therefore, $uf_\beta \to f$ with respect to the topology $\sigma(A(H), C^*_\lambda(H))$ which corresponds to the weak topology on $B_\lambda(H)$. It is a well-known result of functional analysis that the weak closure of a convex set coincides with its norm closure, so that for every $\epsilon > 0$, there exists $\varphi_{\{u_1,\ldots,u_n\},\epsilon} = \varphi \in \mathrm{conv}\{f_\beta\}$ such that $u_i \in A(H)$ for $i = 1, \ldots, n$ and $\|u_i\varphi - u_i\|_{A(H)} < \epsilon$. Moreover,

$$1 = \varphi(e) \leq \|\varphi\|_\infty \leq \|\varphi\|_{A(H)} \leq 1.$$

Note that φ is also is a positive functional in the cone of positive functionals on $VN(H)$; therefore, φ is actually a state and since H is commutative, $\varphi = \psi * \tilde{\psi}$ for some $\psi \in L^2(H)$.

To make the set of all such φ's a net, let $I := \{(S, \epsilon) : S \subseteq A(H) \text{ is finite}, \ \epsilon > 0\}$ become a directed set by $(S, \epsilon) \leq (S', \epsilon')$ if $S \subseteq S'$ and $\epsilon \geq \epsilon'$. This lets us to render the net $(\varphi_\alpha)_\alpha \subseteq \operatorname{conv}\{f_\beta\}$ that is a bounded approximate identity of $A(H)$. On the other hand, for each compact set $K \subseteq H$, by Lemma 5.1.8, there is some $u_K \in A(H)$ such that $u_K|K \equiv 1$. Therefore, for each $x \in K$,

$$\begin{aligned} \lim_\alpha |1 - \varphi_\alpha(x)| = \lim_\alpha |u_K(x) - u_K(x)\varphi_\alpha(x)| &\leq \lim_\alpha \|u_K - u_K\varphi_\alpha\|_\infty \\ &\leq \lim_\alpha \|u_K - u_K\varphi_\alpha\|_{A(H)} = 0. \end{aligned}$$

So $\varphi_\alpha \to 1$ uniformly on compact subsets of H. Consequently, by Lemma 6.1.12, the existence of the net $(\varphi_\alpha)_\alpha$ implies (P_2).

$(P_2) \Rightarrow (B_1)$.

Let $(g_\beta)_\beta$ be the net generated by (P_2) in Lemma 6.1.12, that is $g_\beta = f_\beta * \tilde{f}_\beta$ for some $f_\beta \in L^2(H)$ while $\|f_\beta\|_2 = 1$ for every β and $g_\beta \to 1$ uniformly on compact sets. Therefore,

$$1 = \|f_\beta\|_2^2 = g_\beta(e) \leq \|g_\beta\|_\infty \leq \|g_\beta\|_{A(H)} \leq \|f_\beta\|_2^2 \leq 1.$$

Also for each $u \in A(H) \cap C_c(H)$ and $f \in L^1(H)$,

$$\begin{aligned} \lim_\beta |\langle ug_\beta - u, f\rangle| &\leq \lim_\beta \int_H |u(x)||g_\beta(x) - 1||f(x)|dx \\ &= \int_{\operatorname{supp}(u)} |u(x)||g_\beta(x) - 1||f(x)|dx = 0. \end{aligned}$$

Let us fix $u \in A(H)$. For given $\epsilon > 0$ and $f \in L^1(H)$, there is some $v \in A(H) \cap C_c(H)$ such that $\|u - v\|_{A(H)} < \epsilon$ and β_0 such that for any $\beta \succcurlyeq \beta_0$, $|\langle vg_\beta - v, f\rangle| < \epsilon$. So for any $\beta \succcurlyeq \beta_0$,

$$\begin{aligned} |\langle ug_\beta - u, f\rangle| &\leq |\langle ug_\beta - vg_\beta, f\rangle| + |\langle vg_\beta - v, f\rangle| + |\langle v - u, f\rangle| \\ &\leq \|u - v\|_{A(H)}\|g_\beta\|_{A(H)}\|f\|_1 + \epsilon + \|v - u\|_{A(H)}\|f\|_1 \\ &< \epsilon(2\|f\|_1 + 1). \end{aligned}$$

Therefore, by one generalization to arbitrary functions on $A(H)$, $\lim_\beta ug_\beta = u$ in the topology $\sigma(A(H), L^1(H))$. But indeed $A(H) \subseteq B_\lambda(H)$ and this topology on bounded subsets of $A(H)$ coincides to the weak topology on $B_\lambda(H)$ i.e. $\sigma(B_\lambda(H), C^*_\lambda(H))$. So similar to the previous part, there is a $(e_\alpha)_\alpha \subset \operatorname{conv}\{g_\beta\}_\beta$ such that

$$\lim_\alpha \|ue_\alpha - e_\alpha\|_{A(H)} = 0$$

for every $u \in A(H)$. Also note that for each α,

$$1 = e_\alpha(e) \leq \|e_\alpha\|_\infty \leq \|e_\alpha\|_{A(H)} \leq 1.$$

□

Remark 6.1.16. Let G be a locally compact group. Then G satisfies the D-Leptin condition for each $D > 1$ if and only if it satisfies the Leptin condition. To observe this fact, note that the existence of a bounded approximate identity for $A(G)$ is equivalent to satisfaction of the Leptin condition by the group G, [68, Theorem 7.1.3].

6.1.4 Summary

Theorem 6.1.17. *Let H be a commutative regular Fourier hypergroup. Where*

(SF) *H satisfies the strong Følner condition.*

(L_D) *H satisfies the D-Leptin condition for some $D \geq 1$.*

(B_D) *$A(H)$ has a D-bounded approximate identity for some $D \geq 1$.*

(P_2) *H satisfies (P_2).*

Then

$$(SF) \Longrightarrow (L_1) \Longrightarrow (L_D) \Longrightarrow (B_D) \Longleftrightarrow (B_1) \Longleftrightarrow (P_2)$$

Proof. $(SF) \Rightarrow (L_1)$ by Proposition 6.1.5. While $(L_1) \Rightarrow (L_D)$ is trivial, $(L_D) \Rightarrow (B_D)$ by Theorem 6.1.8. $(B_1) \Longleftrightarrow (B_D) \Longleftrightarrow (P_2)$, by Theorem 6.1.15.

□

Note. In Theorem 6.1.17, note that we suppose H to be a commutative hypergroup; hence, H is amenable, and equivalently, H satisfies the Reiter condition.

Note. For a locally compact group H in Theorem 6.1.17, all aforementioned conditions are equivalent and equal the amenability of the group (see [63] and [68]).

6.2 D-Leptin condition on dual of compact groups

In Theorem 5.2.1, it was proven that the duals of compact groups, as discrete commutative hypergroups, are regular Fourier hypergroups. We will apply this fact to study some properties of compact groups, using the Fourier algebra of the dual of compact groups. In the following we study the D-Leptin condition for some hypergroups which are the dual of compact groups. We calculate D, for $\widehat{SU(3)}$ and $\widehat{SU(2)}$, relying on representation theory of the corresponding compact groups.

Note that since the duals of compact groups are commutative, they are all amenable hypergroups, [70], but this amenability does not say anything about the Følner condition on these hypergroups (unlike groups). So the next question is: for which compact groups G do the hypergroups $\widehat{G}$ satisfy the D-Leptin condition?

A version of some results of this section has been published in [2].

Proposition 6.2.1. *The hypergroup $\widehat{SU(2)}$ satisfies the Leptin condition.*

Proof. Take a finite subset K of $\widehat{SU(2)}$ and $\epsilon > 0$. Let $k := \sup\{\ell :\ \pi_\ell \in K\}$. Recall that for the Haar measure h, $h(\pi_\ell) = d_{\pi_\ell}^2 = (2\ell+1)^2$ for every $\ell \in \frac{1}{2}\mathbb{Z}^+$ (see Example 3.2.2). We select $m \geq k$ such that for $V = \{\pi_\ell\}_{\ell=0}^m$,

$$\begin{aligned} \frac{h(\pi_k * V)}{h(V)} &= \frac{\sum_{j=1}^{2m+2k+1} j^2}{\sum_{j=1}^{2m+1} j^2} \qquad (6.2.1)\\ &= \frac{\frac{1}{3}(2m+2k+1)^3 + \frac{1}{2}(2m+2k+1)^2 + \frac{1}{6}(2m+2k+1)}{\frac{1}{3}(2m+1)^3 + \frac{1}{2}(2m+1)^2 + \frac{1}{6}(2m+1)} < 1+\epsilon. \end{aligned}$$

Note that for every $\pi_{\ell_1} \in K$, $\ell_1 \leq k \leq m$ and for every $\pi_{\ell_2} \in V$, $\ell_2 \leq m$. Therefore, $\pi_{\ell_1} * \pi_{\ell_2} = \bigcup_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\{\pi_\ell\}$. Let us fix $\pi_{\ell_0} \in \pi_{\ell_1} * \pi_{\ell_2}$; $0 \leq \ell_0 \leq \ell_1+\ell_2 \leq k+m$. By splitting the possibilities of ℓ_0 with respect to m and k in the following, we show that $\pi_{\ell_0} \in \pi_k * V$ and since ℓ_0, ℓ_1, and ℓ_2 are arbitrary, this implies that $\pi_{\ell_1} * V \subseteq \pi_k * V$ for every $\pi_{\ell_1} \in K$.

(i) If $k \leq \ell_0 \leq m+k$. Then for $t := \ell_0 - k \leq m$, one gets that $\pi_k * \pi_t = \bigcup_{\ell=|\ell_0-2k|}^{\ell_0}\{\pi_\ell\}$ which clearly contains π_{ℓ_0}.

(ii) If $0 \leq \ell_0 < k$ and $\ell_0 \in \mathbb{N}^+$. Then $\pi_k * \pi_k = \bigcup_{\ell=0}^{2k}\{\pi_\ell\}$ contains π_{ℓ_0}.

(iii) If $0 < \ell_0 < k$ and $\ell_0 \notin \mathbb{N}^+$. Then $\pi_k * \pi_{k-\frac{1}{2}} = \bigcup_{\ell=\frac{1}{2}}^{2k-\frac{1}{2}}\{\pi_\ell\}$ contains π_{ℓ_0}.

Therefore, for each $x \in K$, $K * V = \bigcup_{x\in K} x * V \subseteq \pi_k * V$. So by using (6.2.1),

$$\frac{h(K*V)}{h(V)} \leq \frac{h(\pi_k * V)}{h(V)} < 1+\epsilon.$$

□

In the following theorem, we study the D-Leptin condition for the hypergroups defined in Example 3.2.4.

Theorem 6.2.2. *Let $G = \prod_{i\in\mathbf{I}} G_i$ for a family of compact groups $(G_i)_{i\in\mathbf{I}}$ such that for each $i \in \mathbf{I}$, $\widehat{G}_i$ satisfies the D_i-Leptin condition. Then if $D := \prod_{i\in\mathbf{I}} D_i$ exists, $\widehat{G}$ satisfies the D-Leptin condition.*

Proof. Given finite subset K of $\widehat{G}$ and $\epsilon > 0$ there exists some finite set $F \subseteq \mathbf{I}$ such that $K \subseteq \bigotimes_{i\in F} K_i \otimes E_F^c$ where K_i is a finite subset of $\widehat{G}_i$ and $E_F^c = \bigotimes_{i\in\mathbf{I}\setminus F} \pi_0$ where π_0's are the identities of the corresponding hypergroup $\widehat{G}_i$. If $D := \prod_{i\in\mathbf{I}} D_i < \infty$, given $\epsilon > 0$, one may find an $\epsilon' > 0$ such that $\prod_{i\in F}(D_i + \epsilon') < D + \epsilon$. Using the D_i-Leptin condition for each $\widehat{G}_i$, there exists some finite set V_i such that $h_{\widehat{G}_i}(K_i * V_i)/h_{\widehat{G}_i}(V_i) < D_i + \epsilon'$. Therefore, for the finite set $V = (\bigotimes_{i\in F} V_i) \otimes E_F^c$,

$$\frac{h(K * V)}{h(V)} \le \prod_{i\in F} \frac{h_{G_i}(K_i * V_i)}{h_{G_i}(V_i)} < \prod_{i\in F}(D_i + \epsilon') < D + \epsilon.$$

□

6.2.1 D-Leptin condition for dual of Lie groups

Let $\mathbb{G}$ be a connected simply connected compact real Lie group, (e.g. $SU(n)$). Then, $\widehat{\mathbb{G}}$, as the dual object of a compact Lie groups, forms a finitely generated hypergroup (see [9, 77]). Suppose that F is a finite generator of $\widehat{\mathbb{G}}$; therefore, by [6, Theorem 2.1], there exists positive integers $0 < \alpha, \beta < \infty$ such that

$$\alpha < \frac{h_{\widehat{\mathbb{G}}}(F^k)}{k^{d_{\mathbb{G}}}} < \beta \tag{6.2.2}$$

for all $k \in \mathbb{N}$ where $d_{\mathbb{G}}$ is the dimension of the group $\mathbb{G}$ as a Lie group over $\mathbb{R}$. According to the following theorem, this estimation for the growth rate of $\widehat{\mathbb{G}}$ results in the satisfaction of D-Leptin condition for $\widehat{\mathbb{G}}$.

Theorem 6.2.3. *Let $\mathbb{G}$ be a connected simply connected compact real Lie group. Then $\widehat{\mathbb{G}}$, as a hypergroup, satisfies the D-Leptin condition for some $D \ge 1$.*

Proof. Take a finite set $K \subseteq \widehat{\mathbb{G}}$. Suppose that F is a finite generator of $\widehat{\mathbb{G}}$. For some $k \in \mathbb{N}$, $K \subseteq F^k$. Moreover, for each $\ell \in \mathbb{N}$, $F^\ell * F^k \subseteq F^{\ell+k}$. By applying (6.2.2),

$$\limsup_{\ell\to\infty} \frac{h_{\widehat{\mathbb{G}}}(K * F^\ell)}{h_{\widehat{\mathbb{G}}}(F^\ell)} \le \limsup_{\ell\to\infty} \frac{h_{\widehat{\mathbb{G}}}(F^{\ell+k})}{h_{\widehat{\mathbb{G}}}(F^\ell)} = \limsup_{\ell\to\infty} \frac{h_{\widehat{\mathbb{G}}}(F^{\ell+k})}{(\ell+k)^{d_{\mathbb{G}}}} \frac{\ell^{d_{\mathbb{G}}}}{h_{\widehat{\mathbb{G}}}(F^\ell)} \frac{(\ell+k)^{d_{\mathbb{G}}}}{\ell^{d_{\mathbb{G}}}} \le \beta/\alpha.$$

Therefore, $\widehat{\mathbb{G}}$ satisfies the D-Leptin condition for some $1 \le D < \infty$. □

6.2.2 D-Leptin condition of $\widehat{SU(3)}$

Let $SU(3)$ denote the special group of 3×3 unitary matrices which is a connected simply connected compact real Lie group. Although by Theorem 6.2.3, we may verify the satisfaction of the D-Leptin condition for $SU(3)$, we found it difficult to calculate the constants α and β in the proof of Theorem 6.2.3 for $\mathbb{G} = SU(3)$. Here we may apply some studies on the representation theory of $SU(3)$ to find a concrete answer for D.

Proposition 6.2.1 proves that Leptin condition is satisfied for the dual of 2×2 special unitary matrices group. We have not been able to prove the same result for the dual of the 3×3 special unitary matrix group, $\widehat{SU(3)}$. Instead, we may show that $\widehat{SU(3)}$ satisfies the 6561-Leptin condition. Our main reference to study $\widehat{SU(3)}$ is [79]. In this paper, the irreducible decomposition of the tensor product of irreducible representations of $SU(3)$ has been studied. The author would like to thank Professor Wesslén for the constructive communication about this subsection. Here we recall the following brief background from [79] as well.

One may present the irreducible representations of $SU(3)$ by $\{(p,q)\}_{p,q\in\mathbb{N}\cup\{0\}}$ where for each representation (p,q) the dimension of the representation is $(p+1)(q+1)(p+q+2)/2$. Although the precise decomposition of tensor product of irreducible representation studied in [79] is fairly complicated and we do not have a simple formula similar to the "Clebsch-Gordan" decomposition formula for $SU(2)$, introduced in Example 3.2.2, the work of [79, Section E] shows that

$$(p,q)*(p',q') \subseteq \{(i,j): 0\leq i,j\leq 3\max\{p,q\}+3\max\{p',q'\}+1\}. \tag{6.2.3}$$

Proposition 6.2.4. *The hypergroup $\widehat{SU(3)}$ satisfies the 6561-Leptin condition.*

Proof. Fix a finite set K of $\widehat{SU(3)}$ and $\epsilon>0$. Given $K_k := \{(i,j)\}_{i,j=0}^{k}$ for some $k\in\mathbb{N}$ such that $K\subseteq K_k$, for each $n\in\mathbb{N}$, define $V_n := \{(i,j)\}_{i,j=0}^{n}$. Hence, by (6.2.3),

$$K_k * V_n \subseteq U_{k,n} := \{(i,j): i,j\in 0,\dots,3k+3n+1\}.$$

Therefore,

$$\begin{aligned}\frac{h(K*V_n)}{h(V_n)} &\leq \frac{h(K_k*V_n)}{h(V_n)} \leq \frac{h(U_{n,k})}{h(V_n)}\\ &= \frac{\sum_{i,j=0}^{3k+3n+1} h((i,j))}{\sum_{i,j=0}^{n} h((i,j))} = \frac{\sum_{i,j=0}^{3k+3n+1}\frac{1}{4}(i+1)^2(j+1)^2(i+j+2)^2}{\sum_{i,j=0}^{n}\frac{1}{4}(i+1)^2(j+1)^2(i+j+2)^2}\end{aligned}$$

which approaches $6561=3^8$ when $n\to\infty$, by some simple calculations. □

6.2.3 An application: approximate amenability of Segal algebras of compact groups

The notion of approximate amenability of a Banach algebra was introduced by Ghahramani and Loy in [31]. A Banach algebra $\mathcal{A}$ is said to be *approximately amenable* if for every $\mathcal{A}$-bimodule X and every bounded derivation $D : \mathcal{A} \to X$, there exists a net (D_α) of inner derivations such that

$$\lim_\alpha D_\alpha(a) = D(a) \quad \text{for all } a \in \mathcal{A}.$$

This is not the original definition but it is equivalent. In [31], it is observed that approximately amenable algebras have approximate identities; moreover, closed ideals with a bounded approximate identity and quotient algebras of approximately amenable Banach algebras are approximately amenable.

In this subsection, we study the approximate amenability of proper Segal algebras of compact groups. Approximate amenability of Segal algebras has been studied in several papers. Dales and Loy, in [18], studied approximate amenability of Segal algebras on $\mathbb{T}$ and $\mathbb{R}$. They showed that certain Segal algebras on $\mathbb{T}$ and $\mathbb{R}$ are not approximately amenable. It was further conjectured that no proper Segal algebra on $\mathbb{T}$ is approximately amenable. Choi and Ghahramani, in [14], have shown the stronger fact that no proper Segal algebra on $\mathbb{T}^d$ or $\mathbb{R}^d$ is approximately amenable.

Remark 6.2.5. I extend the result of Choi and Ghahramani to apply to all locally compact abelian groups, not just $\mathbb{T}^d$ and $\mathbb{R}^d$ in [2]. My approach, like that of Choi-Ghahramani and Dales-Loy, was to apply the Fourier transform and work with abstract Segal subalgebras of the Fourier algebra of a locally compact abelian group.

In [14], a nice criterion is developed to prove the non-approximate amenability of Banach algebras. We will rely crucially on this criterion. For this reason, we present a version of the criterion below. Recall that for a Banach algebra $\mathcal{A}$, a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ is called *multiplier-bounded* if, for some $M > 0$, $\sup_{n\in\mathbb{N}} \|a_n b\| \leq M\|b\|$ and $\sup_{n\in\mathbb{N}} \|ba_n\| \leq M\|b\|$ for all $b \in \mathcal{A}$. If $\mathcal{S}$ is an abstract Segal algebra of a Banach algebra $\mathcal{A}$, each element $a \in \mathcal{A}$ acts on $\mathcal{A}$ as a bounded multiplier on $\mathcal{S}$.

Theorem 6.2.6. *(Choi-Ghahramani)*
Let $\mathcal{A}$ be a Banach algebra. Suppose that there exists an unbounded but multiplier-bounded sequence $(a_n)_{n\geq 1} \subseteq \mathcal{A}$ such that

$$a_n a_{n+1} = a_n = a_{n+1} a_n$$

for all n. Then $\mathcal{A}$ is not approximately amenable.

To prove the main theorem we need the following lemma. The proof of the following lemma is adapted from [44, Lemma 1].

Lemma 6.2.7. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{J}$ be a dense left ideal of $\mathcal{A}$. Then for each idempotent element p in the center of algebra $\mathcal{A}$ i.e. $p^2 = p \in Z(\mathcal{A})$, p belongs to $\mathcal{J}$.*

Proof. Since $\mathcal{J}$ is dense in $\mathcal{A}$, there exists an element $a \in \mathcal{J}$ such that $\|p-a\|_{\mathcal{A}} < 1$. Let us define

$$b := p + \sum_{n=1}^{\infty} (p-a)^n.$$

One can check that $pb - pb(p-a) = pba$, which is an element in $\mathcal{J}$. On the other hand,

$$\begin{aligned} pb - pb(p-a) &= p\left(p + \sum_{n=1}^{\infty}(p-a)^n\right) - p\left(p + \sum_{n=1}^{\infty}(p-a)^n\right)(p-a) \\ &= p + p\sum_{n=1}^{\infty}(p-a)^n - p\sum_{n=2}^{\infty}(p-a)^n - p(p-a) \\ &= p + p(p-a) - p(p-a) = p. \end{aligned}$$

□

Remark 6.2.8. As an alternative proof for Lemma 6.2.7, let us assume that (if $\mathcal{A}$ is not unital) $\mathcal{A}_e$ is the unitalized algebra of $\mathcal{A}$ with the identity e. Therefore, for the idempotent $p \in \mathcal{A}$, there is some $a \in \mathcal{J}$ such that $\|p-a\|_{\mathcal{A}} < 1$. Therefore, by a well-known argument in spectral theory of Banach algebras, for $x := p - a$, $e - x$ is invertible and

$$(e-x)^{-1} = e + x + x^2 + \cdots.$$

Note that $a = p - x$; therefore $pa = p - px = p(e-x) = (e-x)p$ since $p \in Z(\mathcal{A})$. So $p = (e-x)^{-1}pa \in \mathcal{J}$.

Corollary 6.2.9. *Let G be a compact group. Then $\operatorname{lin}\{\chi_\pi\}_{\pi\in\widehat{G}}$ is $\|\cdot\|_1$-dense in $ZL^1(G)$ and for every Segal algebra $S^1(G)$, its center, $ZS^1(G)$, contains $\operatorname{lin}\{\chi_\pi\}_{\pi\in\widehat{G}}$.*

Proof. Let $\mathcal{T}$ be the map defined in the proof of Theorem 5.2.1. Then, $\mathcal{T}(ZL^1(G)) = A(\widehat{G})$. Also for the discrete hypergroup $\widehat{G}$, $A(\widehat{G})$ equals the $\|\cdot\|_{A(\widehat{G})}$-closure of $\operatorname{lin}\{\delta_\pi\}_{\pi\in\widehat{G}}$, [60]. Therefore, $ZL^1(G)$ is the $\|\cdot\|_1$-closure of $\operatorname{lin}\{\chi_\pi\}_{\pi\in\widehat{G}}$, since $\mathcal{T}(\chi_\pi) = d_\pi^{-1}\delta_\pi$ for each $\pi \in \widehat{G}$. Also by Lemma 6.2.7, $S^1(G)$ contains all central idempotents $d_\pi\chi_\pi$ for each $\pi \in \widehat{G}$.

□

The main theorem of this section is as follows.

Theorem 6.2.10. *Let G be a compact group such that $\widehat{G}$ satisfies the D-Leptin condition for some $D \geq 1$. Then every proper Segal algebra on G is not approximately amenable.*

Proof. Let $S^1(G)$ be a proper Segal algebra on G. Fix $\varepsilon > 0$. Using the D-Leptin condition on $\widehat{G}$, for every arbitrary non-void finite set K in $\widehat{G}$, we can find a finite subset V_K of $\widehat{G}$ such that $h(K * V_K)/h(V_K) < (D+\varepsilon)^2$. Using the proof of Lemma 5.1.8, for

$$v_K := \frac{1}{h(V_K)} 1_{K*V_K} *_h \tilde{1}_{V_K} \tag{6.2.4}$$

we have $\|v_K\|_{A(\widehat{G})} < (D+\varepsilon)$, $v_K|_K \equiv 1$, and support of v_K is compact. We consider the net $\{v_K : K \subseteq \widehat{G} \text{ compact}\}$ in $A(\widehat{G})$ where $v_{K_2} \geq v_{K_1}$ whenever $\mathrm{supp}(v_{K_1}) \subseteq K_2$. So $(v_K)_{K\subseteq\widehat{G}}$ forms a $\|\cdot\|_{A(\widehat{G})}$-bounded net in $A(\widehat{G}) \cap c_c(\widehat{G})$. Let $f \in A(\widehat{G}) \cap c_c(\widehat{G})$ with $K = \mathrm{supp}\, f$. Then $v_K f = f$. Therefore, $(v_K)_{K\subseteq\widehat{G}}$ is a $(D+\varepsilon)$-bounded approximate identity of $A(\widehat{G})$, since $A(\widehat{G}) \cap c_c(\widehat{G})(= c_c(\widehat{G}))$ is dense in $A(\widehat{G})$, by Remark 5.1.11.

Using $\mathcal{T}$ defined in the proof of Theorem 5.2.1, we can define the net $(u_K)_{K\subseteq\widehat{G}}$ in $S^1(G)$ by $u_K := \mathcal{T}^{-1}(v_K)$. Fix a finite set $K_0 \subseteq \widehat{G}$. We show that $(u_K)_{K\subseteq\widehat{G}}$ satisfies some conditions. First of all, since $\mathcal{T}$ is an isometry from $ZL^1(G)$ onto $A(\widehat{G})$, $(u_K)_{K\subseteq\widehat{G}}$ is a $\|\cdot\|_1$-bounded net in $S^1(G)$, by Lemma 6.2.7. Moreover, since $\mathcal{T}$ is an isomorphism,

$$u_{K_1} * u_{K_2} = \mathcal{T}^{-1}(v_{K_1}) * \mathcal{T}^{-1}(v_{K_2}) = \mathcal{T}^{-1}(v_{K_1} v_{K_2}) = \mathcal{T}^{-1}(v_{K_1}) = u_{K_1}$$

for $v_{K_2} \geq v_{K_1}$ which we equivalently denote by $u_{K_2} \geq u_{K_1}$. Let $(u_K)_{K\subseteq\widehat{G}}$ be the net constructed.

Claim. For every $K_0 \subseteq \widehat{G}$, K_0 finite, the net $\{u_K : u_K \geq u_{K_0}\}$ is unbounded in the norm of $S^1(G)$.

To prove the claim, assume towards a contradiction that there exists K_0 finite and $C > 0$ such that $\|u_K\|_{S^1(G)} \leq C$ for all $u_K \geq u_{K_0}$. Since G is compact and $S^1(G)$ is a Segal algebra, we know, [44], that $S^1(G)$ has a central approximate identity which is bounded in L^1-norm. Denote this by $(e_\alpha)_\alpha$. By Corollary 6.2.9, let us generate a net $(e'_{\alpha,\epsilon})_{\alpha,1>\epsilon>0}$ in $\mathrm{lin}\{\chi_\pi\}_{\pi\in\widehat{G}}$ where $\|e'_{\alpha,\epsilon} - e_\alpha\|_1 < \epsilon$ for each pair of (α,ϵ) while $\alpha \nearrow$ on the given order and $\epsilon \to 0$; therefore, $(e'_{\alpha,\epsilon})_{\alpha,1>\epsilon>0} \subseteq ZS^1(G)$. We show that $(e'_{\alpha,\epsilon})_{\alpha,1>\epsilon>0}$ is still a central $\|\cdot\|_1$-bounded approximate identity of $S^1(G)$. To do so, for each $f \in S^1(G)$, note that

$$\begin{aligned}\|e'_{\alpha,\epsilon} * f - f\|_{S^1(G)} &\leq \|e'_{\alpha,\epsilon} * f - e_\alpha * f\|_{S^1(G)} + \|e_\alpha * f - f\|_{S^1(G)} \\ &\leq \|e'_{\alpha,\epsilon} - e_\alpha\|_1 \|f\|_{S^1(G)} + \|e_\alpha * f - f\|_{S^1(G)}\end{aligned}$$

which goes toward 0 where α grows and $\epsilon \to 0$. Moreover, for each (α,ϵ), there exists some finite set K' such that $K_0 \subseteq K' \subseteq \widehat{G}$ and $\mathcal{T}(e'_{\alpha,\epsilon})v_K = \mathcal{T}(e'_{\alpha,\epsilon})$ for each $v_{K'} \preccurlyeq v_K$; hence,

$\|e'_{\alpha,\epsilon}\|_{S^1(G)} = \lim_K \|e'_{\alpha,\epsilon} * u_K\|_{S^1(G)}$. Consequently,

$$\|e'_{\alpha,\epsilon}\|_{S^1(G)} = \lim_K \|e'_{\alpha,\epsilon} * u_K\|_{S^1(G)} \le \sup_{K_0 \subset K \subseteq \widehat{G}} \|e'_{\alpha,\epsilon}\|_1 \|u_K\|_{S^1(G)} \le C\|e'_{\alpha,\epsilon}\|_1.$$

This implies that $(e'_{\alpha,\epsilon})_{\alpha,1>\epsilon>0}$ is $\|\cdot\|_{S^1(G)}$-bounded. But, a Segal algebra cannot have a bounded approximate identity unless it coincides with the group algebra, [10], which contradicts the properness of $S^1(G)$. Hence, the claim is proved.

To generate a sequence which satisfies the conditions of Theorem 6.2.6, fix a non-empty finite set $K_0 \subseteq \widehat{G}$. By our claim, we inductively construct a sequence of finite sets $K_0 \subset K_1 \subset \cdots$ in $\widehat{G}$ such that $u_{K_n} \ge u_{K_{n-1}}$ and $\|u_{K_n}\|_{S^1(G)} \ge n$ for all $n \in \mathbb{N}$. Since $u_{K_n} * u_{K_{n-1}} = u_{K_{n-1}}$, by Theorem 6.2.6, $S^1(G)$ is not approximately amenable.

□

6.3 Leptin condition for Polynomial hypergroups

In [37] the authors try to render the notion of Følner conditions on polynomial hypergroups. With this motivation, summing sequences in the context of polynomial hypergroups are defined as follows.

Definition 6.3.1. [37, Definition 2.1]
Let $\mathbb{N}_0$ denote the polynomial hypergroup defined in Section 3.3 and h its Haar measure. A sequence $(A_n)_{n\in\mathbb{N}_0}$ where $A_n \subseteq \mathbb{N}_0$ for all $n \in \mathbb{N}$ is called *summing sequence* on the polynomial hypergroup $\mathbb{N}_0$ if it satisfies

(1) $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}_0$,

(2) $\mathbb{N}_0 = \bigcup_{n\in\mathbb{N}_0} A_n$,

(3) $h(A_n) < \infty$ for every $n \in \mathbb{N}_0$,

(4) $\lim_{n\to\infty} \dfrac{h((k * A_n)\Delta A_n)}{h(A_n)} = 0$ for all $k \in \mathbb{N}$.

In [37], a polynomial hypergroup $\mathbb{N}_0$ is said to satisfy property (H) if

$$\lim_{n\to\infty} \frac{h(n)}{\sum_{i=0}^{n} h(i)} = 0. \tag{6.3.1}$$

[37, Theorem 2.5] shows that a polynomial hypergroup on $\mathbb{N}_0$ satisfies condition (H) if and only if the sequence $(S_n)_{n\in\mathbb{N}}$ is a summing sequence where for each $n \in \mathbb{N}$, $S_n := \{0, 1, \ldots, n\}$.

Proposition 6.3.2. *Let* $\mathbb{N}_0$ *be a polynomial hypergroup which has a summing sequence* $(A_n)_{n\in\mathbb{N}_0}$. *Then it satisfies all the Leptin, Strong Følner, and Følner conditions.*

Proof. If we just show that the existence of a summing sequence implies the Følner condition, the rest would be proven based on Proposition 6.1.6, since $\mathbb{N}_0$ is a discrete commutative hypergroup. Let $K \subseteq \mathbb{N}_0$ be finite. Since $(A_n)_{n\in\mathbb{N}}$ is a summing sequence, for given $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$\frac{h(k * A_N \Delta A_N)}{h(A_N)} < \epsilon$$

for every $k \in K$. Therefore, $\mathbb{N}_0$ satisfies (F) and consequently (SF) and (L). □

Remark 6.3.3. Note that if $\mathbb{N}_0$ satisfies (H), the canonical sequence $(S_n)_{n\in\mathbb{N}_0}$ is a summing sequence. Therefore by Proposition 6.3.2, every polynomial hypergroup which satisfies condition (H) satisfies (L), (F), and (SF). As an example in [37], it was shown that Jacobi polynomials satisfy condition (H) and consequently have the canonical sets $(S_n)_{n\in\mathbb{N}}$ as a summing sequence. So this class of polynomial hypergroups includes Jacobi polynomials.

Chapter 7

Arens regularity and operator alegbras

In Chapters 3 and 4, we saw a family of hypergroups which have applications to some Banach algebras on locally compact groups. In this chapter, we pursue our studying on properties of weighted hypergroup algebras. Therefore the abstract results obtained in this chapter can be applied to the Banach algebras mentioned in the previous chapters. Doing so, we enrich this chapter with a variety of examples.

7.1 Arens regularity

7.1.1 General theory

This subsection is a brief report of the general theory of Arens regularity of Banach algebras which is a summary of a part of [17, Chapter 2]. So all unproven results can be found there.

Let $\mathcal{A}$ be a Banach algebra. For $\phi \in \mathcal{A}^*$ and $f, g \in \mathcal{A}$, one may define $\phi \cdot f$ and $f \cdot \phi$ in $\mathcal{A}^*$ by

$$\langle f \cdot \phi, g \rangle := \langle \phi, gf \rangle, \quad \langle \phi \cdot f, g \rangle := \langle \phi, fg \rangle.$$

Note that this implies that $\mathcal{A}^*$ is actually an $\mathcal{A}$-bimodule with respect to the maps

$$(f, \phi) \mapsto \phi \cdot f, \qquad (f, \phi) \mapsto f \cdot \phi, \quad \mathcal{A} \times \mathcal{A}^* \to \mathcal{A}^*.$$

For each $\phi \in \mathcal{A}^*$ and $F \in \mathcal{A}^{**}$, let us define $\phi \cdot F$ and $F \cdot \phi$ in $\mathcal{A}^*$ by their action on $\mathcal{A}$ where

$$\langle f, \phi \cdot F \rangle = \langle F, f \cdot \phi \rangle, \quad \langle f, F \cdot \phi \rangle = \langle F, \phi \cdot f \rangle$$

for all $f \in \mathcal{A}$. Eventually, for each pair $F, G \in \mathcal{A}^{**}$, one may define

$$\langle F \square G, \phi \rangle = \langle F, G \cdot \phi \rangle, \quad \langle F \diamond G, \phi \rangle = \langle G, \phi \cdot F \rangle$$

for all $\phi \in \mathcal{A}^*$.

Theorem 7.1.1. *[17, Theorem 2.8]*
The Banach space $\mathcal{A}^{**}$ equipped with multiplication $\square$ (with multiplication $\diamond$) forms a Banach algebra.

Definition 7.1.2. The Banach algebra $\mathcal{A}$ is called *Arens regular* if two actions $\Box$ and $\Diamond$ coincide.

Let $\mathcal{A}$ be a Banach algebra and $F, G \in \mathcal{A}^{**}$, we know that there are nets $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ in $\mathcal{A}$ such that $f_\alpha \to F$ and $g_\beta \to G$ in weak* topology. One may show that for products $\Box$ and $\Diamond$ of $\mathcal{A}^{**}$,

$$F \Box G = w^* - \lim_{\alpha} w^* - \lim_{\beta} f_\alpha g_\beta \quad \text{and} \quad F \Diamond G = w^* - \lim_{\beta} w^* - \lim_{\alpha} f_\alpha g_\beta.$$

Note that since $\mathcal{A}$ is a closed subalgebra of $(\mathcal{A}^{**}, \Box)$ and $(\mathcal{A}^{**}, \Diamond)$, by identifying each element of $\mathcal{A}$ by its image in the second dual, one gets that

$$f \cdot F := f \Box F = f \Diamond F \text{ and } F \cdot f := F \Box f = F \Diamond f \ \ (F \in \mathcal{A}^{**}, f \in \mathcal{A}).$$

Let us recall from Definition 4.1.3, that a Banach algebra $\mathcal{A}$ is called a *dual Banach algebra* with respect to E, if E is a closed sub-bimodule of the dual $\mathcal{A}$-bimodule A^* that if for every $\phi \in E$ and $f \in \mathcal{A}$, $f \cdot \phi$ and $\phi \cdot f$ belong to E such that $\mathcal{A} = E^*$. Also by Proposition 4.1.4, for a central weight ω, $\ell^1(H, \omega)$ is a dual Banach algebra.

For Banach algebra $\mathcal{A}$, let $\mathcal{A}_*$ be a Banach space such that $\mathcal{A}$ is (isometrically isomorphic to) $(\mathcal{A}_*)^*$ as its dual Banach algebra that is the dual space of $\mathcal{A}_*$ as a Banach space, and such that the multiplication becomes separately weak*-continuous. Therefore, for every $f, g \in \mathcal{A}$ and $\phi \in \mathcal{A}_*$,

$$\langle fg, \phi \rangle = \langle g, \phi \cdot f \rangle.$$

Moreover, let $\mathcal{A}$ be a dual Banach algebra with respect to $\mathcal{A}_*$. Let $\iota : \mathcal{A}_* \to (\mathcal{A}_*)^{**}$ be the canonical embedding which identifies every elements $\phi \in \mathcal{A}_*$ as a linear functional in $\mathcal{A}^*$. For every $\phi \in \mathcal{A}_*$, $F \in \mathcal{A}^{**}$, $\langle P(F), \phi \rangle = \langle F, \phi \rangle$. Further, for each $f \in \mathcal{A}$, $F \in \mathcal{A}^{**}$, and $\phi \in \mathcal{A}_*$, recall that F is the weak * limit of a net $(f_\alpha)_\alpha \subseteq \mathcal{A}$; hence,

$$\begin{aligned}
\langle P(f\cdot F),\phi\rangle &= \langle f\cdot F,\iota(\phi)\rangle \\
&= \langle f\cdot F,\iota(\phi)\rangle \\
&= \langle F,\iota(\phi)\cdot f\rangle \\
&= \lim_\alpha \langle f_\alpha,\iota(\phi)\cdot f\rangle \\
&= \lim_\alpha \langle f f_\alpha,\iota(\phi)\rangle \\
&= \lim_\alpha \langle f_\alpha,\phi\cdot f\rangle \quad (\star) \\
&= \lim_\alpha \langle f_\alpha,\iota(\phi\cdot f)\rangle \\
&= \langle F,\iota(\phi\cdot f)\rangle \\
&= \langle P(F),\phi\cdot f\rangle \\
&= \langle fP(F),\phi\rangle.
\end{aligned}$$

Note that $(\star)$ is correct based on this fact that $\mathcal{A}$ is a dual Banach algebra of $\mathcal{A}_*$. But since $\mathcal{A}_*$ is weak* dense in $\mathcal{A}^* = (\mathcal{A}_*)^{**}$, one may conclude that $P(f\cdot F) = fP(F)$ and similarly, $P(F\cdot f) = P(F)f$ for all $f\in\mathcal{A}$ and $F\in\mathcal{A}^{**}$.

Let us define $\mathcal{A}_*^\perp := \{F\in\mathcal{A}^{**} : \langle F,\phi\rangle = 0 \text{ for all } \phi\in\mathcal{A}_*\}$.

Proposition 7.1.3. *[17, Proposition 2.16]*
Let $\mathcal{A}$ be a dual Banach algebra with respect to some Banach space $\mathcal{A}_*$ where $\mathcal{A} = (\mathcal{A}_*)^*$. Then $\mathcal{A}$ is Arens regular if $\Phi\,\Box\,\Psi = \Phi\diamond\Psi = 0$ for all $\Phi,\Psi\in\mathcal{A}_*^\perp$.

Proof. Note that for each $F\in\mathcal{A}^{**}$, $\langle F-P(F),\phi\rangle = 0$; $F-P(F)\in\mathcal{A}_*^\perp$. Hence, $\mathcal{A}^{**} = \mathcal{A}\oplus\mathcal{A}_*^\perp$ as a direct sum of Banach spaces where every $F\in\mathcal{A}^{**}$ can be decomposed canonically as $(P(F),F-P(F))$. Furthermore, note that for every $F,G\in\mathcal{A}^{**}$,

$$\begin{aligned}
F\,\Box\,G &= (P(F)+(F-P(F)))\,\Box\,(P(G)+(G-P(G))) \\
&= P(F)P(G)+P(F)\cdot(G-P(G))+(F-P(F))\cdot P(G) \\
&+ (F-P(F))\,\Box\,(G-P(G)).
\end{aligned}$$

Therefore, $(\mathcal{A}^{**},\Box)$ can be identified as a semidirect product $\mathcal{A}\rtimes\mathcal{A}_*^\perp$ if for $F=(f,\Phi)$ and $G=(g,\Psi)$ in $\mathcal{A}^{**}$,

$$F\,\Box\,G = (fg, f\cdot\Psi+\Phi\cdot g+\Psi\,\Box\,\Phi). \tag{7.1.1}$$

Similar argument works for $\diamond$-action as well; hence, $(\mathcal{A}^{**},\diamond) = A\rtimes\mathcal{A}_*^\perp$ where for each $F\in\mathcal{A}^{**}$, if $\Phi := F-P(F)\in\mathcal{A}_*^\perp$ and $f = P(F)$, $F = (P(F),\Psi)$. Therefore, for each $F=(f,\Phi)$ and $G=(g,\Psi)$

in $\mathcal{A}^{**}$,

$$F \diamond G = (fg, f \cdot \Psi + \Phi \cdot g + \Psi \diamond \Phi). \tag{7.1.2}$$

Therefore if $\Phi \square \Psi = \Phi \diamond \Psi = 0$, (7.1.1) and (7.1.2) finish the proof. □

7.1.2 Arens regularity of weighted hypergroup algebras

In [41, Chaptetr 4], Kamyabi-Gol applied the topological center of hypergroup algebras to prove some results about the hypergroup algebras and their second duals. For example, in [41, Corollary 4.27], he showed that for a (not necessarily discrete and commutative) hypergroup H (which possesses a Haar measure), $L^1(H)$ is Arens regular if and only if H is finite.

Arens regularity of weighted group algebras has been studied by Craw and Young in [16]. They showed that a locally compact group G has a weight ω such that $L^1(G,\omega)$ is Arens regular if and only if G is discrete and countable. [17] presents a thorough report of the Arens regularity of weighted group algebras. In the following we adapt the machinery developed in [17, Section 8] for weighted hypergroups. [17, Section 3] studies repeated limit conditions and gives a rich variety of results for them. Here, we will use some of these results. We define 0-cluster functions as presented in [17, Definition 3.2] and [17, Definition 3.6].

Definition 7.1.4. Let X and Y be non-empty sets, and let $f : X \times Y \to \mathbb{C}$ be a function. Then f 0-*clusters* on $X \times Y$ if

$$\lim_n \lim_m f(x_m, y_n) = \lim_m \lim_n f(x_m, y_n) = 0$$

whenever (x_m) and (y_n) are sequences in X and Y, respectively, each consisting of distinct points and both repeated limits exist.

If f is a bounded continuous function on $X \times Y$ into $\mathbb{C}$. Then f 0-*clusters strongly* on $X \times Y$ if

$$\lim_{x\to\infty} \limsup_{y\to\infty} f(x,y) = \lim_{y\to\infty} \limsup_{x\to\infty} f(x,y) = 0.$$

Let X and Y be non-empty sets, and let $f : X \times Y \to \mathbb{C}$ be a continuous bounded function. Then [17, Proposition 3.8] shows that if f 0-clusters strongly on $X \times Y$, it 0-clusters on $X \times Y$.

Note that for $\kappa^{**} : \ell^1(H,\omega)^{**} \to \ell^1(H)^{**}$ and $\Phi \in c_0(H,\omega)^\perp$, one gets

$$\langle \kappa^{**}(\Phi), \phi\rangle = \langle \Phi, \kappa^*(\phi)\rangle$$

which is equal to 0 for all $\phi \in c_0(H)$. Therefore $\kappa^{**}(\Phi) \in c_0(H)^\perp$. The converse is also true and straightforward to show (which we do not use here so we do not mention).

Let us define the bounded function $\Omega : H \times H \to (0,1]$ by

$$\Omega(x,y) := \frac{\omega(\delta_x * \delta_y)}{\omega(x)\omega(y)} \quad (x,y \in H). \tag{7.1.3}$$

The following theorem is a generalization of [17, Theorem 8.8]. In the proof we use some techniques of the proof for [53, Theorem 3.16].

Theorem 7.1.5. *Let (H,ω) be a weighted hypergroup and let Ω 0-cluster strongly on $H \times H$. Then $\Phi \,\Box\, \Psi = 0$ and $\Phi \diamond \Psi = 0$ whenever $\Phi, \Psi \in c_0(H, 1/\omega)^\perp$.*

Proof. Let us show the theorem for $\Phi \,\Box\, \Psi$, the proof for the other action is similar. Let $\Phi, \Psi \in c_0(H,1/\omega)^\perp$. There are nets $(f_\alpha)_\alpha, (g_\beta)_\beta \subseteq \ell^1(H)$ such that $f_\alpha \to \kappa^{**}(\Phi)$ and $g_\beta \to \kappa^{**}(\Psi)$ in weak* topology of $\ell^1(H)^{**}$. Without loss of generality, let $\|\kappa^{**}(\Phi)\| = \|\kappa^{**}(\Psi)\| = 1$; hence, by a standard corollary of Goldstine's theorem, [24, Theorem 9.7.14], $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ may be chosen such that $\sup_\alpha \|f_\alpha\|_1 \le 1$ and $\sup_\beta \|g_\beta\|_1 \le 1$.

So for each $\psi \in \ell^\infty(H)$, $\kappa^*(\psi) = \psi\omega \in \ell^\infty(H, 1/\omega)$ and $\Phi \,\Box\, \Psi \in \ell^1(H,\omega)^{**}$; hence,

$$\begin{aligned}
\langle \psi\omega, \kappa^{**}(\Phi \,\Box\, \Psi)\rangle &= \langle \kappa^*(\psi), \Phi \,\Box\, \Psi\rangle \\
&= \lim_\alpha \lim_\beta \langle \psi\omega, \kappa^{-1}(f_\alpha) * \kappa^{-1}(g_\beta)\rangle \\
&= \lim_\alpha \lim_\beta \langle \psi\omega, f_\alpha/\omega * g_\beta/\omega\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
|\langle \psi\omega, \kappa^{**}(\Phi \,\Box\, \Psi)\rangle| &= \lim_\alpha \lim_\beta |\langle \psi\omega, f_\alpha/\omega * g_\beta/\omega\rangle| \\
&= \lim_\alpha \lim_\beta \left| \sum_{y \in H} \psi(y)\omega(y) \sum_{x,z \in H} \frac{f_\alpha(x)}{\omega(x)} \frac{g_\beta(z)}{\omega(z)} \delta_x * \delta_z(y) \right| \\
&\le \limsup_\alpha \limsup_\beta \sum_{x,z \in H} \frac{|f_\alpha(x)|}{\omega(x)} \frac{|g_\beta(z)|}{\omega(z)} \sum_{y \in H} |\psi(y)|\omega(y)\delta_x * \delta_z(y) \\
&\le \limsup_\alpha \limsup_\beta \|\psi\|_{\ell^\infty(H)} \sum_{x,z \in H} |f_\alpha(x)||g_\beta(z)| \sum_{y \in H} \frac{\omega(y)}{\omega(x)\omega(z)} \delta_x * \delta_z(y) \\
&= \limsup_\alpha \limsup_\beta \|\psi\|_{\ell^\infty(H)} \sum_{x,z \in H} |f_\alpha(x)||g_\beta(z)|\Omega(x,z).
\end{aligned}$$

For a given $\epsilon > 0$, since by the hypothesis $\lim_x \limsup_z \Omega(x,z) = 0$, there is a finite set $A \subseteq H$ such that for each $x \in A^c := H \setminus A$ there exists a finite set $B_x \subseteq H$ such that for each $z \in B_x^c := H \setminus B$, $|\Omega(x,z)| \le \epsilon$.

First note that

$$\limsup_\alpha \limsup_\beta \sum_{x \in A^c} \sum_{z \in B_x^c} |f_\alpha(x)||g_\beta(z)|\Omega(x,z) \le \limsup_\alpha \limsup_\beta \epsilon \|f_\alpha\|_1 \|g_\beta\|_1 \le \epsilon.$$

Also according to our assumption about Φ and Ψ and since for each $x \in H$, $\delta_x \in c_0(H, 1/\omega)$,

$$\lim_\alpha f_\alpha(x) = \langle \Phi, \delta_x \rangle = 0, \quad \lim_\beta g_\beta(x) = \langle \Psi, \delta_x \rangle = 0.$$

So for the given $\epsilon > 0$ there is α_0 such that for all $\alpha_0 \preccurlyeq \alpha$, $|f_\alpha(x)| < \epsilon/|A|$ for all $x \in A$. Moreover, for each $x \in A^c$ there is some β_0^x such that for all β where $\beta_0^x \preccurlyeq \beta$, $|g_\beta(z)| < \epsilon/|B_x|$ for all $z \in B_x$ (this is possible since A and B_x are finite). Therefore, since $|\Omega(x,z)| \le 1$,

$$\limsup_\alpha \limsup_\beta \sum_{x\in A} \sum_{z\in H} |f_\alpha(x)||g_\beta(z)|\Omega(x,z) \le \limsup_\beta \epsilon \|g_\beta\|_1 = \epsilon$$

and

$$\begin{aligned}\limsup_\alpha \limsup_\beta \sum_{x\in A^c} \sum_{z\in B_x} |f_\alpha(x)||g_\beta(z)|\Omega(x,z) &\le \limsup_\alpha \sum_{x\in A^c} |f_\alpha(x)| \limsup_\beta \sum_{z\in B_x} |g_\beta(z)| \\ &\le \limsup_\alpha \epsilon \|f_\alpha\|_1 = \epsilon.\end{aligned}$$

But

$$\begin{aligned}\sum_{x,z\in H} |f_\alpha(x)||g_\beta(z)|\Omega(x,z) &= \sum_{x\in A^c, z\in B_x^c} |f_\alpha(x)||g_\beta(z)|\Omega(x,z) \\ &+ \sum_{x\in A, z\in H} |f_\alpha(x)||g_\beta(z)|\Omega(x,z) \\ &+ \sum_{x\in A^c, z\in B_x} |f_\alpha(x)||g_\beta(z)|\Omega(x,z),\end{aligned}$$

and so, one gets that $|\langle \psi\omega, \kappa^{**}(\Phi \square \Psi)\rangle| \le 3\epsilon \|\psi\|_\infty$. This implies that $\Phi \square \Psi = 0$.

□

Theorem 7.1.6. *Let (H, ω) be a discrete weighted hypergroup with a central weight ω and consider the following conditions:*

(1) *Ω 0-clusters strongly on $H \times H$.*

(2) *$\Phi \square \Psi = \Phi \diamond \Psi = 0$ for all $\Phi, \Psi \in c_0(H, 1/\omega)^\perp$.*

(3) *$\ell^1(H, \omega)$ is Arens regular.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$ by Theorem 7.1.5. $(2) \Rightarrow (3)$ is implied from Proposition 7.1.3 and Proposition 4.1.4. □

Remark 7.1.7. Since in hypergroups, the cancellation does not necessarily exist, the argument of [16, Theorem 1] cannot be applied to show ((3)) implies ((1)).

Corollary 7.1.8. *Let (H, ω) be a weighted discrete hypergroup such that ω is a weakly additive central weight. If $1/\omega \in c_0(H)$, then $\ell^1(H, \omega)$ is Arens regular.*

Proof. We have

$$\begin{aligned}\lim_{x\to\infty} \limsup_{y\to\infty} \frac{\omega(\delta_x * \delta_y)}{\omega(x)\omega(y)} &\leq \limsup_{x\to\infty} \limsup_{y\to\infty} C\frac{\omega(x)+\omega(y)}{\omega(x)\omega(y)} \\ &= C \limsup_{x\to\infty} \limsup_{y\to\infty} \frac{1}{\omega(x)} + \frac{1}{\omega(y)} = 0.\end{aligned}$$

Therefore Ω 0-clusters on $H \times H$ and hence $\ell^1(H, \omega)$ is Arens regular by Theorem 7.1.6.

□

Corollary 7.1.9. *Let H be an infinite finitely generated hypergroup. Then for each polynomial weight ω_β ($\beta > 0$) on H, $\ell^1(H, \omega_\beta)$ is Arens regular.*

Proof. Let F be a finite generator of the hypergroup H containing the identity of H rendering the central weight ω_β. First, note that by Remark 4.2.2, ω_β is centrally additive (and consequently weakly additive) with constant $C = \min\{1, 2^{\beta-1}\}$. Moreover, for each $N \in \mathbb{N}$, F^N is a finite subset of H such that for each $x \in H \setminus F^N$, $\tau_F(x) \geq N$; hence,

$$\omega_\beta(x) = (1 + \tau_F(x))^\beta \geq (1 + N)^\beta.$$

Hence $1/\omega_\beta \in c_0(H)$ and therefore $\ell^1(H, \omega_\beta)$ is Arens regular, by Corollary 7.1.8. □

Example 7.1.10. In Section 3.3, we introduced polynomial hypergroup structure of $\mathbb{N}_0$. Further, as a finitely generated hypergroup, we defined the polynomial weight ω_β on that where $\omega_\beta(n) = (1+n)^\beta$ for every $n \in \mathbb{N}_0$. Therefore by Corollary 7.1.9, $\ell^1(\mathbb{N}_0, \omega_\beta)$ is Arens regular.

Remark 7.1.11. Every infinite finitely generated hypergroup H admits a weight for which the corresponding weighted algebra is Arens regular. An argument similar to [16, Corollary 1] may apply to show that for every uncountable discrete hypergroup H, H does not have any central weight ω which 0-clusters.

7.1.3 Arens regularity of weighted hypergroup algebra of Conj(G) for some special G

Remark 7.1.12. Let ω be a central weight on Conj(G) for some FC group G. Then there is a group weight σ_ω, as defined in Remark 4.4.1, such that $\ell^1(\text{Conj}(G), \omega)$ is isometrically Banach algebra isomorphic to $Z\ell^1(G, \sigma_\omega)$. So one may use the embedding $\ell^1(\text{Conj}(G), \omega) \hookrightarrow \ell^1(G, \sigma_\omega)$

to prove the results of this subsection applying the theorems which are characterizing weighted group algebras..

Example 7.1.13. Let $\mathrm{Aff}_p := \mathbb{Z}_p \rtimes \mathbb{Z}_p^*$ be the affine group generated with $\mathbb{Z}_p(:= \mathbb{Z}/p\mathbb{Z})$ for a prime number p, when for each $(a,b),(a',b') \in \mathrm{Aff}_p$ we define $(a,b)(a',b') = (a+ba', bb')$. Based on the calculations of [74, p 274], in the following table, we re-present the structure of conjugacy classes of Aff_p.

Conjugacy classes	$C_{(0,1)}$	$C_{(1,1)}$	$C_{(0,y)}\ _{y\in 2,\cdots,p-1}$
Number	1	1	$p-2$
Size	1	$p-1$	p

As a direct result of the above table, for each three conjugacy classes say $C_1, C_2, D \in \mathrm{Conj}(\mathrm{Aff}_p)$, $|D| \le (|C_1| + |C_2|)$ if $D \subseteq C_1C_2$ for each prime number $p \ge 3$. In other words, the weight $\omega_p(C) := |C|$, defined in Example 4.4.2, forms a central additive weight on Aff_p. Let $\mathcal{P}$ be the set of all prime numbers greater than or equal to 3. Define $G = \oplus_{p\in\mathcal{P}} \mathrm{Aff}_p$ and ω_α is the weight defined in Example 3.1.3 for some $\alpha > 0$. For $C \in D * E$ for $C, D, E \in \mathrm{Conj}(G)$, we have

$$\begin{aligned}\omega_\alpha(C) &= (1 + |C_{i_1}| + \cdots + |C_{i_n}|)^\alpha \\ &\le (1 + |D_{i_1}| + \cdots + |D_{i_n}| + 1 + |E_{i_1}| + \cdots + |E_{i_n}|)^\alpha \\ &\le M\left((1 + |D_{i_1}| + \cdots + |D_{i_n}|)^\alpha + (1 + |E_{i_1}| + \cdots + |E_{i_n}|)^\alpha\right) \le C(\omega_\alpha(D) + \omega_\alpha(E))\end{aligned}$$

for $M = \min\{1, 2^{\alpha-1}\}$, by (4.2.1), where $i_j \in \mathbf{I}_C$ and $\mathbf{I}_C$ is the set of all indexes $i \in \mathbf{I}$ such that $x_i \ne e_{G_i}$ for some $x = (x_i)_{i\in\mathbf{I}} \in C$ as defined before in Example 3.1.3.

Hence, ω_α is centrally additive (see Definition4.1.5). Moreover, since for each $p \in \mathcal{P}$, $|C| \ge p-1$ for any non-trivial element $C \in \mathrm{Conj}(\mathrm{Aff}_p)$, $\lim_n \omega_\alpha(C_n) = \infty$ for each sequence of distinct elements of $\mathrm{Conj}(G)$. Therefore by Corollary 7.1.8, $\ell^1(\mathrm{Conj}(G), \omega_\alpha)$ is Arens regular.

Example 7.1.14. Let $SL(2, 2^n)$ denote the finite group of special linear matrices over the field $\mathbb{F}_{2^n}$ with cardinal 2^n, for given $n \in \mathbb{N}$. The character table of $SL(2, 2^n)$ can be found at [1]. In the following we just present the part of the character table related to the conjugacy classes of $SL(2, 2^n)$.

Conjugacy classes	e	N	$c_3(x)$	$c_4(z)$
Number	1	1	$(2^n-2)/2$	2^{n-1}
Size	1	$2^{2n}-1$	$2^{2n}+2^n$	$2^{2n}-2^n$

As a direct result of the above table, for each three conjugacy classes say $C_1, C_2, D \in \mathrm{Conj}(SL(2,2^n))$, $|D| \le 2(|C_1| + |C_2|)$ if $D \subseteq C_1C_2$ for all n. Let us define the FC group G to be the RDPF of $(SL(2,2^n))_{n\in\mathbb{N}}$ i.e.

$$G := \bigoplus_{n=1}^{\infty} SL(2,2^n).$$

Therefore, similar to the previous example, for the hypergroup $\mathrm{Conj}(G)$, the weight ω_α, as defined as in Example 3.1.3, is centrally additive for the constant $M = 2^\alpha \min\{1, 2^{\alpha-1}\}$, and consequently, weakly additive. Moreover, since $\lim_{C\to\infty} \omega_\alpha(C) = \infty$, $\ell^1(\mathrm{Conj}(G), \omega_\alpha)$ is Arens regular, by Corollary 7.1.8.

Remark 7.1.15. Let G be an FC group and σ a group weight on G. We defined ω_σ, the derived weight on $\mathrm{Conj}(G)$ from σ in Definition 4.3.2. Recall that in this case $Z\ell^1(G,\sigma)$ is isomorphic to the Banach algebra $\ell^1(\mathrm{Conj}(G), \omega_\sigma)$, by Corollary 4.3.3. If N is a normal subgroup of G, in Section 4.5, we defined a quotient mapping $T_{\omega_\sigma} : \ell^1(\mathrm{Conj}(G), \omega_\sigma) \to \ell^1(\mathrm{Conj}(G/N), \tilde{\omega}_\sigma)$ in Proposition 4.5.1 where

$$\tilde{\omega}_\sigma(C_{xN}) = \inf\{\omega_\sigma(C_{xy}) : \ y \in N\} \quad (C_{xN} \in \mathrm{Conj}(G/N)).$$

Let us note that for an Arens regular Banach algebra $\mathcal{A}$, every quotient algebra $\mathcal{A}/\mathcal{I}$ where $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ is Arens regular as well (see [17, Corollary 3.15]). Therefore, if $\ell^1(\mathrm{Conj}(G), \omega_\sigma)$ is Arens regular, for every normal subgroup N, $\ell^1(\mathrm{Conj}(G/N), \tilde{\omega}_\sigma)$, which is isomorphic to $\ell^1(\mathrm{Conj}(G), \omega_\sigma)/\mathrm{Ker}(T_{\omega_\sigma})$, is Arens regular.

7.1.4 Arens regularity of $\ell^1(\widehat{SU(n)}, \omega)$

Example 7.1.16. In Example 4.6.2, we observed that for each $\beta > 0$, $\omega_\beta(\ell) = (2\ell+1)^\beta$ is a polynomial weight on the finitely generated hypergroup $\widehat{SU(2)}$. Therefore $\ell^1(\widehat{SU(2)}, \omega_\beta)$ is Arens regular. Note that ω_β also corresponds the weight on the dual of $SU(2)$ which is generated by the degree of representations. See Corollary 4.6.1.

In this subsection, we may generalize the result of Example 7.1.16 for all $SU(n)$ and ω_β for $\beta > 0$ base on some recent studies on the representation theory of $SU(n)$. As an example for Corollary 4.6.1, $(\widehat{SU(n)}, \omega_\beta)$ is a discrete commutative hypergroup where $\omega_\beta(\pi) = d_\pi^\beta$ for some $\beta \ge 0$. It is known that there is a one-to-one correspondence between $\widehat{SU(n)}$ and n-tuples $(\pi_1, \ldots, \pi_n) \in \mathbb{Z}_+^n$ such that

$$\pi_1 \ge \pi_2 \ge \cdots \ge \pi_{n-1} \ge \pi_n = 0.$$

This presentation of the representation theory of $SU(n)$ is called *dominant weight*. Using this presentation, we have the following formula which gives the dimension of each representation.

$$d_\pi = \prod_{1\le i<j\le n} \frac{\pi_i - \pi_j + j - i}{j - i} \tag{7.1.4}$$

where π is the representation corresponding to $(\pi_1, \ldots, \pi_n)$. Suppose that π, ν, μ are representations corresponding to $(\pi_1, \ldots, \pi_n)$, $(\nu_1, \ldots, \nu_n)$, and $(\mu_1, \ldots, \mu_n)$, respectively, such that $\pi \in \nu * \mu$. By a new result of Collins, Lee, and Ŝniady [15, Corollary 1.2], for each $n \in \mathbb{N}$, there exists some $D_n > 0$ such that

$$\frac{d_\pi}{d_\mu d_\nu} \le D_n(1/\mu_1 + 1/\nu_1). \tag{7.1.5}$$

when neither ν nor μ is the trivial representation of $SU(n)$. Hence in general if $C_n = \max\{D_n, 1/2\}$, then

$$\frac{d_\pi}{d_\mu d_\nu} \le 2C_n\left(\frac{1}{1+\mu_1} + \frac{1}{1+\nu_1}\right). \tag{7.1.6}$$

Applying (7.1.6), we prove that ω_β 0-clusters on $\widehat{SU(2)}$.

Theorem 7.1.17. *For every $\beta > 0$, $\ell^1(\widehat{SU(n)}, \omega_\beta)$ is Arens regular.*

Proof. Let $(\mu_m)_{m\in\mathbb{N}}$ and $(\nu_k)_{k\in\mathbb{N}}$ are two arbitrary sequence of distinct elements of $\widehat{SU(n)}$. Since, the elements of $(\mu_m)_{m\in\mathbb{N}}$ $((\nu_k)_{k\in\mathbb{N}})$ are distinct, $\lim_{m\to\infty} \mu_1^{(m)} = \infty$ $(\lim_{k\to\infty} \nu_1^{(k)} = \infty$ $)$ where $\mu_m = (\mu_1^{(m)}, \ldots, \mu_n^{(m)})$ $(\nu_k = (\nu_1^{(k)}, \ldots, \nu_n^{(k)}))$. For each arbitrary pair $(m, k) \in \mathbb{N} \times \mathbb{N}$, if $\pi \in \mu_m * \nu_k$, we have

$$d_\pi \le 2C_n\left(\frac{1}{1+\mu_1^{(m)}} + \frac{1}{1+\nu_1^{(k)}}\right) d_{\mu_m} d_{\nu_k}.$$

Hence

$$\omega_\beta(\pi) \le (2C_n)^\beta \left(\frac{1}{1+\mu_1^{(m)}} + \frac{1}{1+\nu_1^{(k)}}\right)^\beta \omega_\beta(\mu_m)\omega_\beta(\nu_k).$$

Therefore

$$\omega_\beta(\delta_{\mu_m} * \delta_{\nu_k}) = \sum_{\pi\in\widehat{SU(n)}} \delta_{\mu_m} * \delta_{\nu_k}(\pi)\omega_\beta(\pi) \le (2C_n)^\beta \left(\frac{1}{1+\mu_1^{(m)}} + \frac{1}{1+\nu_1^{(k)}}\right)^\beta \omega_\beta(\mu_m)\omega_\beta(\nu_k).$$

Or equivalently

$$\Omega_\beta(\mu_m, \nu_k) := \frac{\omega_\beta(\delta_{\mu_m} * \delta_{\nu_k})}{\omega_\beta(\mu_m)\omega_\beta(\nu_k)} \le (2C_n)^\beta \left(\frac{1}{1+\mu_1^{(m)}} + \frac{1}{1+\nu_1^{(k)}}\right)^\beta.$$

And,

$$\lim_{m\to\infty} \limsup_{k\to\infty} \Omega_\beta(\mu_m, \nu_k) = \lim_{k\to\infty} \limsup_{m\to\infty} \Omega_\beta(\mu_m, \nu_k) = 0.$$

Since $\widehat{SU(n)}$ is countable, this argument implies that Ω_β 0-clusters strongly on $\widehat{SU(n)} \times \widehat{SU(n)}$ and, by Theorem 7.1.6, $\ell^1(\widehat{SU(n)}, \omega_\beta)$ is Arens regular. □

7.1.5 Some weighted hypergroup algebras are not Arens regular

In the following theorem, we apply some techniques of [16] to show that for restricted direct product of hypergroups many weights fail to give Arens regular algebras.

Theorem 7.1.18. *Let $(H_i)_{i\in\mathbf{I}}$ be an infinite family of non-trivial hypergroups and for each $i\in\mathbf{I}$, ω_i is a weight on H_i such that $\omega_i(e_{H_i})=1$ for all except finitely many $i\in\mathbf{I}$. Let $H=\bigoplus_{i\in\mathbf{I}}H_i$ and $\omega=\prod_{i\in\mathbf{I}}\omega_i$, as defined in Subsection 4.1.1. Then $\ell^1(H,\omega)$ is not Arens regular.*

Proof. Since $\mathbf{I}$ is infinite, suppose that $\mathbb{N}_0\times\mathbb{N}_0\subseteq\mathbf{I}$. Define $v_n=(x_i)_{i\in\mathbf{I}}$ ($u_m=(x_i)_{i\in\mathbf{I}}$) where $x_i=e_{H_i}$ for all $i\in\mathbf{I}\setminus(n,0)$ ($i\in\mathbf{I}\setminus(0,m)$) and $x_{(n,0)}$ ($x_{(0,m)}$) be a non-identity element of $H_{(n,0)}$ ($H_{(0,m)}$) for all $n\in\mathbb{N}$ ($m\in\mathbb{N}$). Note that for each pair of elements $(n,m)\in\mathbb{N}\times\mathbb{N}$, v_n*u_m forms a singleton in H; moreover, $\omega(v_n*u_m)=\omega(v_n)\omega(u_m)$. Hence, $(v_n*u_m)_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ forms a sequence of distinct elements in H.

Let us define $f_n=\delta_{v_n}$ and $g_m=\delta_{u_m}$ for all $n,m\in\mathbb{N}$; hence, $f_n*g_m(t)=\delta_{v_n*u_m}$. Suppose that $A:=\{(v_n,u_m):\ n>m\}$ and $\phi\in\ell^\infty(H)$ is the characteristic function of the subset A. Clearly, $\kappa^{-1}(f_n)=\omega^{-1}f_n$ and $\kappa^{-1}(g_m)=\omega^{-1}g_m$ belong to $\ell^1(H,\omega)$ for all n,m and $\kappa^*(\phi)=\omega\phi\in\ell^\infty(H,\omega^{-1})$, for κ define in Subsection 7.1.2. Note that

$$\begin{aligned}\langle\omega^{-1}f_n*\omega^{-1}g_m,\kappa^*(\phi)\rangle &= \langle\omega^{-1}f_n*\omega^{-1}g_m,\omega\phi\rangle\\ &= \sum_{t\in H}(\omega^{-1}f_n*\omega^{-1}g_m)(t)\omega(t)\phi(t)\\ &= \frac{\omega(v_n*u_m)}{\omega(v_n)\omega(u_m)}\phi(v_n*u_m)\\ &= \phi(v_n*u_m)=\begin{cases}1 & \text{if } n>m\\ 0 & \text{if } n\le m\end{cases}\end{aligned}$$

Let us recall that for each n and m, $\|f_n\|_{\ell^1(H,\omega)}=1$ and $\|g_m\|_{\ell^1(H,\omega)}=1$. So $(f_n)_{n\in\mathbb{N}}$ and $(g_m)_{m\in\mathbb{N}}$, as two nets in the unit ball of $\ell^1(H,\omega)^{**}$ which is weak* compact, have two subnets $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ such that f_α and g_β converge weakly* to some F and G in $\ell^1(H,\omega)^{**}$, respectively.

Note that for the specific element ϕ, defined above,

$$\langle F\,\square\,G,\phi\rangle=\lim_\alpha\lim_\beta\langle\omega^{-1}f_\alpha*\omega^{-1}g_\beta,\kappa^*(\phi)\rangle=0$$

while

$$\langle F\diamond G,\phi\rangle=\lim_\beta\lim_\alpha\langle\langle\omega^{-1}f_\alpha*\omega^{-1}g_\beta,\kappa^*(\phi)\rangle=1.$$

Hence $F\,\square\,G\neq F\diamond G$ and $\ell^1(H,\omega)$ is not Arens regular. □

Example 7.1.19. Let G be the restricted direct product of an infinite family of finite groups $(G_i)_i$. By Example 3.1.3, $\mathrm{Conj}(G) = \bigoplus_{i\in\mathbf{I}} \mathrm{Conj}(G_i)$. Also for $\omega(C_x) = \prod_i |C_{x_i}|$, ω is a weight such that $\ell^1(\mathrm{Conj}(G), \omega)$ is not Arens regular, by Theorem 7.1.18. One may compare this weight with the examples in Subsection 7.1.3.

7.2 Operator algebra property of weighted hypergroup algebras

Let (H, ω) be a weighted discrete hypergroup. In this section, we study the existence of an algebra isomorphism from $\ell^1(H, \omega)$ onto an operator algebra. A Banach algebra $\mathcal{A}$ is called an operator algebra if there is a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is a closed subalgebra of $\mathcal{B}(\mathcal{H})$.

Definition 7.2.1. Let $\mathcal{A}$ be a Banach algebra and $m : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is bilinear mapping $m(f, g) = fg$. Then $\mathcal{A}$ is called *injective*, if m has a bounded extension from the injective tensor product $\mathcal{A} \otimes_\epsilon \mathcal{A}$ into $\mathcal{A}$, where $\otimes_\epsilon$ is the injective tensor product. In this case, we denote the norm of m by $\|m\|_\epsilon$.

We present the following theorem from [52, Corollary 2.2.] without a proof.

Theorem 7.2.2. *Let $\mathcal{A}$ be an injective Banach algebra. Then A is isomorphic to an operator algebra.*

Injectiveness of weighted group algebras has been studied before. Initially Varopoulos, in [76], studied the group $\mathbb{Z}$ equipped with the weight $\sigma_\alpha(n) = (1 + |n|)^\alpha$ for all $\alpha \geq 0$. This study looked at injectiveness of $\ell^1(\mathbb{Z}, \sigma_\alpha)$. He showed that $\ell^1(\mathbb{Z}, \sigma_\alpha)$ is injective if and only if $\alpha > 1/2$. The manuscript [52], which studied the injectiveness question for a wider family of weighted group algebras, developed a machinery applying Littlewood multipliers. In particular, they partially extended Varopoulos's result in [76] to finitely generated groups with polynomial growth. Following the structure of [52], in this section, we study the injective property of weighted hypergroup algebras.

We know that $\ell^1(H, \omega) \otimes_\gamma \ell^1(H, \omega)$ is isometrically isomorphic with $\ell^1(H\times H, \omega\times\omega)$. Moreover, $\ell^1(H\times H, \omega\times\omega)^*$ is nothing but $\ell^\infty(H\times H, \omega^{-1}\times\omega^{-1})$. Since the injective tensor product is minimal among all Banach space tensor products, the identity map $\iota : \ell^1(H)\times\ell^1(H) \to \ell^1(H)\times\ell^1(H)$ may extend to a contractive mapping

$$\iota : \ell^1(H) \otimes_\gamma \ell^1(H) \to \ell^1(H) \otimes_\epsilon \ell^1(H).$$

Since, ι has a dense range,

$$\iota^* : (\ell^1(H) \otimes_\epsilon \ell^1(H))^* \to (\ell^1(H) \otimes_\gamma \ell^1(H))^* = \ell^\infty(H \times H) \tag{7.2.1}$$

is an injective mapping. Therefore, applying ι^*, one may embed $(\ell^1(H) \otimes_\epsilon \ell^1(H))^*$ into $\ell^\infty(H \times H)$, as a sub vector space of $\ell^\infty(H \times H)$.

7.2.1 Littlewood multipliers for hypergroups

Let H be a discrete hypergroup. We define *Littlewood multipliers* of H to be set of all functions $f : H \times H \to \mathbb{C}$ such that there exist functions $f_1, f_2 : H \times H \to \mathbb{C}$ where

$$f(x,y) = f_1(x,y) + f_2(x,y) \quad (x, y \in G)$$

and

$$\sup_{y \in H} \sum_{x \in H} |f_1(x,y)|^2 < \infty, \quad \sup_{x \in H} \sum_{y \in H} |f_2(x,y)|^2 < \infty.$$

We denote the set of all Littlewood multipliers by $T^2(H)$ and define the following norm on $T^2(H)$

$$\|f\|_{T^2(H)} = \inf \left\{ \sup_{y \in H} \left(\sum_{x \in H} |f_1(x,y)|^2 \right)^{1/2} + \sup_{x \in H} \left(\sum_{y \in H} |f_2(x,y)|^2 \right)^{1/2} \right\}.$$

where the infimum is taken over all possible decompositions f_1, f_2. Note that for each $f \in T^2(H)$ and a decomposition f_1, f_2 of that,

$$\begin{aligned} \|f\|_{\ell^\infty(H \times H)} = \sup_{x,y \in H} |f(x,y)| &\leq \sup_{x,y \in H} |f_1(x,y)| + \sup_{x,y \in H} |f_2(x,y)| \\ &\leq \sup_{y \in H} \left(\sum_{x \in H} |f_1(x,y)|^2 \right)^{1/2} + \sup_{x \in H} \left(\sum_{y \in H} |f_2(x,y)|^2 \right)^{1/2} < \infty, \end{aligned}$$

since for discrete space H, $\ell^2(H) \subseteq \ell^\infty(H)$ and $\|\cdot\|_\infty \leq \|\cdot\|_2$. Since f_1, f_2, in the previous equation are arbitrary, $\|f\|_{\ell^\infty(H \times H)} \leq \|f\|_{T^2(H)}$. Hence $T^2(H) \subseteq \ell^\infty(H \times H)$. Furthermore, for each $\phi \in \ell^\infty(H \times H)$ and $f \in T^2(H)$, $f\phi \in T^2(H)$ and

$$\|f\phi\|_{T^2(H)} \leq \|f\|_{T^2(H)} \|\phi\|_\infty. \tag{7.2.2}$$

Theorem 7.2.3. *Let $I : T^2(H) \to (\ell^1(H) \otimes_\gamma \ell^1(H))^* = \ell^\infty(H \times H)$ be the mapping which takes every element of $T^2(H)$ to itself as a bounded function on $H \times H$. Then $I(T^2(H)) \subseteq \iota^*((\ell^1(H) \otimes_\epsilon \ell^1(H))^*)$ for the mapping ι^* defined in (7.2.1).*

Moreover, for $J := \iota^{*-1} \circ I$ i.e. $J : T^2(H) \to (\ell^1(H) \otimes_\epsilon \ell^1(H))^*$ is well-defined and $\|J\| \leq K_{\mathfrak{G}}$ where $K_{\mathfrak{G}}$ is Grothendieck's constant.

The proof and its preliminaries are given in Appendix A. From now on, we identify $(\ell^1(H)\otimes_\epsilon \ell^1(H))^*)$ with its image with respect to the mapping ι^*; hence, J is the identity mapping which takes $T^2(H)$ into $(\ell^1(H)\otimes_\epsilon \ell^1(H))^*$.

7.2.2 The operator algebra property of weighted hypergroup algebras

Theorem 7.2.4. *Let H be a discrete hypergroup and ω is a weight on H such that Ω, defined in (7.1.3), belongs to $T^2(H)$. Then $\ell^1(H,\omega)$ is injective. Moreover, for m as defined in Definition 7.2.1,*

$$\|m\|_\epsilon \le K_{\mathfrak{G}}\|\Omega\|_{T^2(H)}.$$

Proof. Let

$$\Gamma_\omega : \ell^1(H\times H,\omega\times\omega) \to \ell^1(H,\omega)$$

such that

$$\Gamma_\omega(f\otimes g) = f * g \tag{7.2.3}$$

for $f,g\in\ell^1(H,\omega)$. The adjoint of Γ_ω, Γ_ω^*, can be characterized as follows.

$$\Gamma_\omega^*(\phi)(x,y) = \langle\Gamma_\omega^*(\phi),\delta_x\otimes\delta_y\rangle = \langle\phi,\Gamma_\omega(\delta_x\otimes\delta_y)\rangle = \langle\phi,\delta_x*\delta_y\rangle$$

for all $\phi\in\ell^\infty(H,\omega^{-1})$ and $x,y\in H$. Now we define L from $\ell^\infty(H)$ to $\ell^\infty(H\times H)$ such that the following diagram commutes.

$$\begin{array}{ccc}
\ell^\infty(H,\omega^{-1}) & \xrightarrow{\ \Gamma_\omega^*\ } & \ell^\infty(H\times H,\omega^{-1}\times\omega^{-1}) \\
P\uparrow & & \downarrow R \\
\ell^\infty(H) & \xrightarrow{\ \ L\ \ } & \ell^\infty(H\times H)
\end{array}$$

where $P(\varphi)(x) = \varphi(x)\omega(x)$ for $\varphi\in\ell^\infty(H)$ and $R(\phi)(x,y) = \phi(x,y)\omega^{-1}(x)\omega^{-1}(y)$ for $\phi\in\ell^\infty(H\times H,\omega^{-1}\times\omega^{-1})$ and $x,y\in H$. Hence, one gets

$$\begin{aligned}
L(\varphi)(x,y) = R\left(\Gamma_\omega^*\circ P(\varphi)\right)(x,y) &= \frac{\left(\Gamma_\omega^*\circ P(\varphi)\right)(x,y)}{\omega(x)\omega(y)} \\
&= \frac{\Gamma_\omega^*\left(\omega\varphi\right)(x,y)}{\omega(x)\omega(y)} \\
&= \frac{\langle\varphi\omega,\delta_x*\delta_y\rangle}{\omega(x)\omega(y)} \\
&= \sum_{t\in H}\delta_x*\delta_y(t)\frac{\omega(t)}{\omega(x)\omega(y)}\varphi(t).
\end{aligned}$$

for all $\varphi\in\ell^\infty(H)$. Hence,

$$\left|\sum_{t\in H}\delta_x*\delta_y(t)\frac{\omega(t)}{\omega(x)\omega(y)}\varphi(t)\right| \le \sum_{t\in H}\delta_x*\delta_y(t)\frac{\omega(t)}{\omega(x)\omega(y)}|\varphi(t)| \le \|\varphi\|_\infty\Omega(x,y)$$

So there is a function $v_\varphi : H \times H \to \mathbb{C}$ such that

$$\frac{\langle \delta_x * \delta_y, \omega\varphi \rangle}{\omega(x)\omega(y)} = v_\varphi(x,y)\|\varphi\|_\infty \Omega(x,y)$$

and $\|v_\varphi\|_\infty \leq 1$. Therefore

$$L(\varphi) = \Lambda(\varphi)\Omega$$

where $\Lambda(\varphi)(x,y) := v_\varphi(x,y)\|\phi\|_\infty$ for all $\varphi \in \ell^\infty(H)$. Since Ω belongs to $T^2(H)$ and $T^2(H)$ is an $\ell^\infty(H \times H)$-module, $L(\varphi) \in T^2(H)$ and $\|L(\varphi)\|_{T^2(H)} \leq \|\varphi\|_\infty \|\Omega\|_{T^2(H)}$. Therefore $L(\ell^\infty(H)) \subseteq T^2(H) \subseteq (\ell^1(H) \otimes_\epsilon \ell^1(H))^*$.

In this case, using the following diagram with $\mathcal{A} = R^{-1}((\ell^1(H) \otimes_\epsilon \ell^1(H))^*)$,

$$\begin{array}{ccccc}
\ell^\infty(H,\omega^{-1}) & \xrightarrow{\Gamma_\omega^*} & \mathcal{A} & \xrightarrow{\iota} & \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1}) \\
P \uparrow & & \downarrow R|_r & & R \downarrow \\
\ell^\infty(H) & \xrightarrow{L} & (\ell^1(H) \otimes_\epsilon \ell^1(H))^* & \xrightarrow{\iota} & \ell^\infty(H \times H)
\end{array}$$

Cliam. $\mathcal{A} = (\ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega))^*$.

Proof of Claim. Note that R is the adjoint of R_* an isomorphism from $\ell^1(H) \otimes_\gamma \ell^1(H)$ into $\ell^1(H,\omega) \otimes_\gamma \ell^1(H,\gamma)$ such that $R_*(f \otimes g) = f\omega^{-1} \otimes g\omega^{-1}$. Similarly, one may define the isometry R_*^ϵ such that $R_*^\epsilon(f \otimes g) = f\omega^{-1} \otimes g\omega^{-1}$. Therefore, $R_*^\epsilon : \ell^1(H) \otimes_\epsilon \ell^1(H) \to \ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega)$ is a Banach space isomorphism. Let us define, similar to ι, $\iota_\omega : \ell^1(H,\omega) \otimes_\gamma \ell^1(H,\omega) \to \ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega)$. Therefore, clearly the following diagram commutes (one may study the maps on elementary elements).

$$\begin{array}{ccc}
\ell^1(H,\omega) \otimes_\gamma \ell^1(H,\omega) & \xrightarrow{\iota_\omega} & \ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega) \\
R_* \uparrow & & R_*^\epsilon \uparrow \\
\ell^1(H) \otimes_\gamma \ell^1(H) & \xrightarrow{\iota} & \ell^1(H) \otimes_\epsilon \ell^1(H)
\end{array}$$

For $R_\epsilon = (R_*^\epsilon)^*$, we get

$$\begin{array}{ccc}
(\ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega))^* & \xrightarrow{\iota_\omega^*} & \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1}) \\
\downarrow R_\epsilon & & \downarrow R \\
(\ell^1(H) \otimes_\epsilon \ell^1(H))^* & \xrightarrow{\iota^*} & \ell^\infty(H \times H)
\end{array}$$

Therefore, for each $\psi \in \iota_\omega^*((\ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega))^*)$, $R(\psi) \in \iota^*((\ell^1(H) \otimes_\epsilon \ell^1(H))^*)$. Similarly, if we may identify $\iota_\omega^*((\ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega))^*)$ with $(\ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega))^*$ as a subspace of $\ell^\infty(H \times H, \omega^{-1} \times \omega^{-1})$,

$$R((\ell^1(H,\omega) \otimes_\epsilon \ell^1(H,\omega))^*) = (\ell^1(H) \otimes_\epsilon \ell^1(H))^*.$$

So, we have shown that Γ^* is a map projecting $\ell^\infty(H)$ into $(\ell^1(H)\otimes_\epsilon \ell^1(H))^*$ as a subset of $\ell^\infty(H\times H)$. we see that Γ_ω^* is a map projecting $\ell^\infty(H,\omega^{-1})$ into $(\ell^1(H,\omega)\otimes_\epsilon \ell^1(H,\omega))^*$. Hence, $\Gamma_\omega^* = m^*$, where

$$m : \ell^1(H,\omega)\otimes_\epsilon \ell^1(H,\omega) \to \ell^1(H,\omega).$$

and therefore m is bounded while $\|m\| = \|\Gamma_\omega\| = \|R\Gamma_\omega P\| = \|L\|$. Moreover,

$$\begin{aligned}\|L(\varphi)\|_{(\ell^1(H)\otimes^\epsilon\ell^1(H))^*} &\leq \|J\|\,\|\Gamma^*(\varphi)\|_{T^2(H)} \leq K_{\mathfrak{G}}\,\|\Omega\|_{T^2(H)}\,\|\Lambda(\varphi)\|_{\ell^\infty(H\times H)}\\ &\leq K_{\mathfrak{G}}\,\|\Omega\|_{T^2(H)}\,\|\varphi\|_{\ell^\infty(H)}\end{aligned}$$

for all $\varphi\in\ell^\infty(H)$. Consequently, $\|m\|_\epsilon \leq K_{\mathfrak{G}}\|\Omega\|_{T^2(H)}$. □

Example 7.2.5. Let ω_β be the weight defined on $\widehat{SU(n)}$ in Corollary 4.6.1. As we have shown in the proof of Theorem 7.1.17, for polynomial weight ω_β, $\beta\geq 0$, and $\mu,\nu\in\widehat{SU(n)}$,

$$\Omega_\beta(\mu,\nu) \leq (2C_n)^\beta\left(\frac{1}{1+\mu_1}+\frac{1}{1+\nu_1}\right)^\beta \leq A_\beta(2C_n)^\beta\left(\frac{1}{(1+\mu_1)^\beta}+\frac{1}{(1+\nu_1)^\beta}\right),$$

where $A_\beta = \min\{1,2^{\beta-1}\}$. To study $\|\cdot\|_{T^2(\widehat{SU(2)})}$ for Ω_β, let us note that for each $k\in\mathbb{N}\cup\{0\}$, there are less than $(1+k)^{n-2}$ many $\lambda\in\widehat{SU(n)}$ such that $\lambda_1 = k$. Therefore

$$\sum_{\lambda\in\widehat{SU(n)}}\frac{1}{(1+\lambda_1)^{2\beta}} \leq \sum_{k=0}^{\infty}\frac{(1+k)^{n-2}}{(1+k)^{2\beta}}$$

which is convergent if and only if $2\beta-n+2>1$. Therefore, for $\beta>(n-1)/2$, $\Omega_\beta\in T^2(\widehat{SU(n)})$ and by Theorem 7.2.4, $\ell^1(\widehat{SU(2)},\omega_\beta)$ is injective. Moreover, note that

$$\begin{aligned}\|\Omega_\beta\|_{T^2(\widehat{SU(n)})} &\leq \left\|(\mu,\nu)\mapsto\frac{A_\beta(2C_n)^\beta}{1+\mu_1}+\frac{A_\beta(2C_n)^\beta}{1+\nu_1}\right\|_{T^2(H)}\\ &\leq \left(\sup_{\nu\in\widehat{SU(n)}}\left(\sum_{\mu\in\widehat{SU(n)}}\left|\frac{A_\beta(2C_n)^\beta}{1+\mu_1}\right|^2\right)^{1/2}+\sup_{\mu\in\widehat{SU(n)}}\left(\sum_{\nu\in\widehat{SU(n)}}\left|\frac{A_\beta(2C_n)^\beta}{1+\nu_1}\right|^2\right)^{1/2}\right)\\ &\leq A_\beta(2C_n)^\beta 2\left(\sum_{k=0}^{\infty}\frac{1}{(1+k)^{2\beta-n+2}}\right)^{1/2}.\end{aligned}$$

Hence

$$\|m\|_\epsilon \leq K_{\mathfrak{G}}A_\beta 2^{\beta+1}C_n^\beta\left(\sum_{k=0}^{\infty}\frac{1}{(1+k)^{2\beta-n+2}}\right)^{1/2}$$

for $A_\beta = \min\{1,2^{\beta-1}\}$.

Let us recall the definition of weakly additive weights on hypergroups from Definition 4.1.5. ω is a weakly additive weight on a hypergroup H if for all $x,y\in H$, $\omega(\delta_x*\delta_y)\leq C(\omega(x)+\omega(y))$ for some fixed $C>0$.

Corollary 7.2.6. *Let H be a discrete hypergroup and ω is a weakly additive weight on H with corresponding constant $C > 0$. Then $\ell^1(H,\omega)$ is injective if*

$$\sum_{x\in H} \frac{1}{\omega(x)^2} < \infty.$$

Moreover, for m as defined in Definition 7.2.1,

$$\|m\|_\epsilon \le 2CK_{\mathfrak{G}} \left(\sum_{x\in H} \frac{1}{\omega(x)^2} \right)^{1/2}.$$

Proof. Suppose that $\sum_{x\in H} \omega(x)^{-2} < \infty$. Note that for each $t \in x * y$,

$$\frac{\omega(t)}{\omega(x)\omega(y)} \le C\frac{\omega(x)+\omega(y)}{\omega(x)\omega(y)} = \frac{C}{\omega(x)} + \frac{C}{\omega(y)}.$$

Thus based on (7.2.2) and for functions $f_1(x,y) = \omega(x)^{-1}$ and $f_2(x,y) = \omega(y)^{-1}$,

$$\begin{aligned}
\|\Omega\|_{T^2(H)} &\le \left\| (x,y) \mapsto \frac{C}{\omega(x)} + \frac{C}{\omega(y)} \right\|_{T^2(H)} \\
&\le \left(\sup_{y\in H} \left(\sum_{x\in H} \left| \frac{C}{\omega(x)} \right|^2 \right)^{1/2} + \sup_{x\in H} \left(\sum_{y\in H} \left| \frac{C}{\omega(y)} \right|^2 \right)^{1/2} \right) \\
&\le 2C \left(\sum_{x\in H} \frac{1}{\omega(x)^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently, by Theorem 7.2.4, $\Omega \in T^2(H)$ and

$$\|m\|_\epsilon \le 2CK_{\mathfrak{G}} \left(\sum_{x\in H} \frac{1}{\omega(x)^2} \right)^{1/2}.$$

□

Remark 7.2.7. In Example 7.1.14, we introduced a hypergroup which results from conjugacy classes of a specific group, $G = \oplus_{n=2}^{\infty} SL(2,2^n)$. For the weight ω_α defined on $\mathrm{Conj}(G)$ by Example 3.1.3, we observed that $\ell^1(\mathrm{Conj}(G),\omega_\alpha)$ is Arens regular. Moreover, as mentioned in Example 7.1.14, ω_α forms a weakly additive weight on $\mathrm{Conj}(G)$. But we may show that $\sum_{C\in\mathrm{Conj}(G)} \omega(C)^{-2} = \infty$. Doing so, let us define E_m to be the set of all $C = \oplus_{n\in\mathbb{N}} C_n \in \mathrm{Conj}(G)$ such that $\mathbf{I}_C = \{1,2,\ldots,m\}$ where $\mathbf{I}_C := \{n \in \mathbb{N} : C_n \ne e_{SL(2,2^n)}\}$ for each n in $\mathbf{I}_C$. Moreover, for each $n \in \mathbf{I}_C$, let $C_n = c_4(z)$ for $c_4(z)$ denoted in the conjugacy table of $SL(2,2^n)$ in Example 7.1.14. Therefore,

$$\begin{aligned}
\sum_{x\in\mathrm{Conj}(G)} \frac{1}{\omega(C)^2} &\ge \sum_{m=2}^{\infty} \sum_{C\in E_m} \frac{1}{\omega(C)^2} \\
&\ge \sum_{m=2}^{\infty} \frac{\prod_{i=1}^{m-1} 2^i}{(1+4^1+\cdots+4^m)^2} \\
&= \sum_{m=2}^{\infty} \frac{2^{m(m-1)/2}}{(4^{m+1}-1)^2/9} = \infty.
\end{aligned}$$

Hence, not all weakly additive weights are satisfying the other condition mentioned in Corollary 7.2.6.

For finitely generated hypergroups, we have introduced two classes of weights in Section 4.2, namely, polynomial growth weights and exponential weights. Applying this fact that polynomial weights are weakly additive, in the following, we study operator algebra isomorphism for weighted hypergroup algebras with polynomial weights. Developing a machinery which relates exponential weights to polynomial ones, we also study exponential weights in Subsection 7.2.3. For the case that H is a group, this has been achieved in [52]

Corollary 7.2.8. *Let H be a finitely generated hypergroup. If F is a generator of H such that $|F^n| \le Dn^d$ for some $d, D > 0$ and ω_β is the polynomial weight on H associated to F (see Section 4.2). Then $\ell^1(H,\omega_\beta)$ is injective if $2\beta > d+1$. Moreover,*

$$\|m\|_\epsilon \le 2CK_{\mathfrak{G}}\left(1+\sum_{n=1}^{\infty}\frac{Dn^d}{(1+n)^{2\beta}}\right)^{1/2}$$

for $C = \min\{1, 2^{\beta-1}\}$.

Proof. To show this corollary, we mainly rely on Corollary 7.2.6. By Remark 4.2.2, ω_β is weakly additive whose constant is $C = \min\{1, 2^{\beta-1}\}$.

To show the desired bound for $\|m\|_\epsilon$, note that

$$\begin{aligned}\sum_{x\in H}\frac{1}{\omega_\beta(x)^2} &= \sum_{x\in H}\frac{1}{(1+\tau(x))^{2\beta}} = \sum_{n=0}^{\infty}\sum_{\{x\in F^n\setminus F^{n-1}\}}\frac{1}{(1+n)^{2\beta}}\\ &\le 1+\sum_{n=1}^{\infty}\frac{|F^n|}{(1+n)^{2\beta}} \le 1+\sum_{n=1}^{\infty}\frac{Dn^d}{(1+n)^{2\beta}}\end{aligned}$$

which is convergent if $2\beta > d+1$. Furthermore, by Corollary 7.2.6,

$$\|m\|_\epsilon \le 2CK_{\mathfrak{G}}\left(\sum_{x\in H}\frac{1}{\omega_\beta(x)^2}\right)^{1/2} \le 2CK_{\mathfrak{G}}\left(1+\sum_{n=1}^{\infty}\frac{Dn^d}{(1+n)^{2\beta}}\right)^{1/2}.$$

□

Example 7.2.9. As we have seen in Example 4.6.2, for each $\beta \ge 0$, ω_β defined in (4.6.1) is the polynomial weight on $\widehat{SU(2)}$ associated to $F = \{\pi_0, \pi_{1/2}\}$. Therefore, by Remark 4.2.2, ω_β is weakly additive on $\widehat{SU(2)}$. On the other hand,

$$\sum_{\pi\in\widehat{SU(2)}}\frac{1}{\omega_\beta^2(\pi)} = \sum_{\ell\in\frac{\mathbb{Z}^+}{2}}\frac{1}{(2\ell+1)^{2\beta}} = \sum_{n\in\mathbb{N}}\frac{1}{n^{2\beta}}$$

which is convergent if $\beta > 1/2$. Note that this bound for β verifies the bound which was found in Example 7.2.5 by using a different property of $\widehat{SU(2)}$. Furthermore, Corollary 7.2.6 implies that

$$\|m\|_\epsilon \leq 2CK_{\mathfrak{G}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2\beta}} \right)^{1/2}$$

for $C = \min\{1, 2^{\beta-1}\}$.

Example 7.2.10. For a polynomial hypergroup $\mathbb{N}_0$, as a finitely generated hypergroup with the generator $F = \{0,1\}$, we have $|F^n| = n+1 \leq 2n$, as we have seen in Section 3.3. Hence for $d = 1$ and $D = 2$, $|F^n| \leq Dn^d$. Recall that by Remark 3.2.3, $\widehat{SU(2)}$ can be regarded as a specific case in this example.

By Corollary 7.2.8, for the polynomial weight ω_β with $\beta > 1$ associated to F, $\ell^1(\mathbb{N}_0, \omega_\beta)$ is injective. In this case also, an argument similar to Example 7.2.9 implies that

$$\|m\|_\epsilon \leq 2CK_{\mathfrak{G}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2\beta}} \right)^{1/2}$$

for $C = \min\{1, 2^{\beta-1}\}$.

7.2.3 Hypergroups with exponential weights

The other class of weights introduced for finitely generated hypergroups in Section 4.2 is the class of exponential weights. As we mentioned before, unlike polynomial weights, exponential weights are not necessarily weakly additive. In this subsection , following [52], we develop a machinery to study operator algebra isomorphism of these weights. The following lemma is narrated from [52, Lemma 3.2] without its proof.

Lemma 7.2.11. *Let* $0 < \alpha < 1$*,* $C > 0$*, and* $\beta \geq \max\left\{1, \frac{6}{C\alpha(1-\alpha)}\right\}$*. Define the functions* $p : [0,\infty) \to \mathbb{R}$ *and* $q : (0,\infty) \to \mathbb{R}$ *by*

$$p(x) := Cx^\alpha - \beta \ln(1+x), q(x) := \frac{p(x)}{x}.$$

Then on $\left[\left(\frac{\beta^2}{C\alpha(1-\alpha)}\right)^{1/\alpha}, \infty\right)$*,* p *is increasing and* q *is decreasing.*

The following lemma is a hypergroup adaptation for [52, Theorem 3.3] and the proof is similar to [34, Lemma B.2].

Lemma 7.2.12. *Suppose that* $0 < \alpha < 1$*,* $C > 0$*, and* $\beta \geq \max\left\{1, \frac{6}{C\alpha(1-\alpha)}\right\}$*. Let* $p : [0,\infty) \to \mathbb{R}$ *and* $q : (0,\infty) \to \mathbb{R}$ *be the functions defined in Lemma 7.2.11. Let* H *be a finitely generated hypergroup with a symmetric generator* F *and* $\omega : H \to (0,\infty)$ *such that*

$$\omega(x) = e^{p(\tau_F(x))} = e^{\tau_F(x) q(\tau_F(x))} \quad \text{for all } x \in H.$$

Then $\omega(t) \leq M\omega(x)\omega(y)$ *for all* $t, x, y \in H$ *such that* $t \in x * y$ *where*

$$M = \max\{e^{p(z_1)-p(z_2)-p(z_3)} : z_1 \in [0, 4K] \cap \mathbb{N}_0,\ \ z_2, z_3 \in [0, 2K] \cap \mathbb{N}_0\}$$

and

$$K = \left(\frac{\beta^2}{C\alpha(1-\alpha)}\right)^{1/\alpha}.$$

Proof. We split the proof into four cases with respect to possibilities of $\tau_F(x)$, $\tau_F(y)$, and $\tau_F(t)$ for $t \in x * y$. In each case, we apply Lemma 7.2.11 and this fact that $e^{-p(0)} = 1$, and $\tau_F(t) \leq \tau_F(x) + \tau_F(y)$. In particular note that $M \geq 1$.

Case I: $\max\{\tau_F(x), \tau_F(y)\} \leq 2K$. In this case, note that $\tau_F(t) \leq \tau_F(x) + \tau_F(y) \leq 4K$ for every $t \in x * y$. Therefore,

$$\frac{\omega(t)}{\omega(x)\omega(y)} = e^{p(\tau_F(t))-p(\tau_F(x))-p(\tau_F(y))} \leq M.$$

Case II: $\max\{\tau_F(x), \tau_F(y)\} > 2K$ and $\min\{\tau_F(x), \tau_F(y)\} \leq K$. Without loss of generality, we may suppose that $\tau_F(x) > 2K$ and $\tau_F(y) \leq K$. Since H is a discrete commutative hypergroup, for each $t \in x * y$, $x \in t * \check{y}$ (see [49, Lemma 1.2]). Therefore, for the symmetric generator F (where $\tau_F(y) = \tau_F(\check{y})$), one gets that $\tau_F(x) + \tau_F(y) \geq \tau_F(t) \geq \tau_F(x) - \tau_F(y) \geq 2K - K = K$. Hence,

$$\begin{aligned}
e^{p(\tau_F(t))} &\leq e^{p(\tau_F(x)+\tau_F(y))} \\
&= e^{(\tau_F(x)+\tau_F(y))q(\tau_F(x)+\tau_F(y))} \\
&= e^{\tau_F(x)q(\tau_F(x)+\tau_F(y))} e^{\tau_F(y)q(\tau_F(x)+\tau_F(y))} \\
&\leq e^{\tau_F(x)q(\tau_F(x)+\tau_F(y))} e^{Kq(K)} \\
&\leq e^{\tau_F(x)q(\tau_F(x))} e^{Kq(K)} \quad (\star) \\
&= e^{p(\tau_F(x))} e^{p(\tau_F(y))} e^{p(K)-p(\tau_F(y))} \leq M\omega(x)\omega(y).
\end{aligned}$$

Note that $(\star)$ is implied by this fact that q is a decreasing function on that specific interval.

Case III: $\min\{\tau_F(x), \tau_F(y), \tau_F(t)\} > K$ for some $t \in x * y$. In this case, note that $K < \tau_F(x), \tau_F(y) < \tau_F(x) + \tau_F(y)$. Hence,

$$\begin{aligned}
e^{p(\tau_F(t))} &\leq e^{p(\tau_F(x)+\tau_F(y))} \\
&= e^{\tau_F(x)q(\tau_F(x)+\tau_F(y))} e^{\tau_F(y)q(\tau_F(x)+\tau_F(y))} \\
&\leq e^{\tau_F(x)q(\tau_F(x))} e^{\tau_F(y)q(\tau_F(y))} = \omega(x)\omega(y) \leq M\omega(x)\omega(y).
\end{aligned}$$

Case IV: Finally let $\min\{\tau_F(x), \tau_F(y)\} > K$ while $\tau_F(t) \le K$ for some $t \in x * y$. So

$$\begin{aligned} \omega(x)\omega(y) &= e^{p(\tau_F(x))+p(\tau_F(y))} \\ &\ge e^{2p(K)} \\ &= e^{2p(K)-p(\tau_F(t))}\omega(t) \ge \frac{1}{M}\omega(t). \end{aligned}$$

In other words, $\omega(t) \le M\omega(x)\omega(y)$. ☐

Theorem 7.2.13. *Let H be a finitely generated hypergroup. If F is a symmetric generator of H such that $|F^n| \le Dn^d$ for some $d, D > 0$ and $\sigma_{\alpha,C}$ is an exponential weight on H for some $0 < \alpha < 1$ and $C > 0$. Then $\ell^1(H, \sigma_{\alpha,C})$ is injective.*

Proof. Let ω_β be the weight defined in Lemma 7.2.12. We define a function $\omega : H \to (0, \infty)$ by

$$\omega(x) := \frac{\sigma_{\alpha,C}(x)}{\omega_\beta(x)} = e^{C\tau_F(x)^\alpha - \beta\ln(1+\tau_F(x))} \quad (x \in H)$$

where ω_β is the polynomial weight defined on H associated to F and $\beta > \max\{1, \frac{6}{C\alpha(1-\alpha)}, \frac{d+1}{2}\}$. Therefore, by Lemma 7.2.12, $\omega(t) \le M\omega(x)\omega(y)$ for some $M > 0$ and all $t, x, y \in H$ such that $t \in x * y$. Therefore

$$\frac{\sigma_{\alpha,C}(t)}{\sigma_{\alpha,C}(x)\sigma_{\alpha,C}(y)} \le M\frac{\omega_\beta(t)}{\omega_\beta(x)\omega_\beta(y)}.$$

Hence it follows from Remark 4.2.2 that

$$\frac{\sigma_{\alpha,C}(t)}{\sigma_{\alpha,C}(x)\sigma_{\alpha,C}(y)} \le M'\left(\frac{1}{(1+\tau(x))^\beta} + \frac{1}{(1+\tau(y))^\beta}\right)$$

for a modified constant $M' > 0$. Therefore by the proof of Corollary 7.2.8, $\Omega_{\sigma_{\alpha,C}} \in T^2(H)$. Hence $\ell^1(H, \sigma_{\alpha,C})$ is injective by Theorem 7.2.4. ☐

Example 7.2.14. As a result of Theorem 7.2.13, and to follow Example 7.2.9 and Example 7.2.10, if H is a polynomial hypergroup on $\mathbb{N}_0$, for each exponential weight $\sigma_{\alpha,C}$ for $0 < \alpha < 1$ and $C > 0$, $\ell^1(H, \sigma_{\alpha,C})$ is injective. Note that by Remark 3.2.3, this class of hypergroups includes $\widehat{SU(2)}$.

Appendix A

p-summing operators and dual of injective tensor products

Recall that in Subsection 1.3.2, we briefly have mentioned the definitions and some basic facts about tensor products of Banach spaces.

Definition A.0.15. [75, Section 9]
Let X and Y be Banach spaces. An operator $T : X \to Y$ is called *p-summing* if there exists a constant $C \geq 0$ such that for all finite sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ one gets

$$\left(\sum_n \|T(x_n)\|^p\right)^{1/p} \leq C \sup_{\phi\in X^*:\ \|\phi\|_{X^*}\leq 1} \left(\sum_n |\langle \phi, x_n\rangle|^p\right)^{1/p}.$$

The infimum of all such C is denoted by $\pi_p(T)$ and is called *p-summing norm* of T.

If an operator T is not p-summing, we may define $\pi_p(T) = \infty$. The set of all p-summing operators from X into Y is denoted by $\Pi_p(X,Y)$ after [75] and (Π_p, π_p) forms a normed operator ideal in $\mathcal{L}(X,Y)$ (the space of all bounded operators from X into Y).

Definition A.0.16. [69, pp63-64] and [75, p42]
Let X and Y be Banach spaces. An operator $T : X \to Y$ is called *integral* if there exists a compact Hausdorff space K and a probability measure μ on K and two operators $W_1 : X \to C(K)$ and $W_2 : L^1(K,\mu) \to Y^{**}$ such that the following diagram commutes i.e. $id \circ T = W_2 \circ I \circ W_1$.

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{id} & Y^{**} \\ {\scriptstyle W_1}\downarrow & & & & \uparrow{\scriptstyle W_2} \\ C(K) & & \xrightarrow{I} & & L^1(K,\mu) \end{array}$$

where id is the canonical identity from Y into its second dual and I is the identity map from $C(K)$ into $L^1(K,\mu)$. Then $\|T\|_i := \inf \|W_1\|\|W_2\|$, where W_1 and W_2 are changing between all possible factorizations, defines a norm called *integral norm*. The space of all integral norms of X into Y denoted by $\mathcal{I}(X,Y)$ equipped with $\|\cdot\|_i$ is a normed operator ideal in $\mathcal{L}(X,Y)$.

Proposition A.0.17. *Let X and Y be two Banach spaces. Then $(X \otimes_\epsilon Y)^*$ is isometrically isomorphic to $\Pi_1(X,Y^*)$.*

Proof. By [69, Proposition 3.14], we know that $(X \otimes_\epsilon Y)^*$ is isometrically isomorphic to the space of all *integral operators* from X into Y^*, denoted by $\mathcal{I}(X,Y^*)$. By [75, p50], $T : X \to Y$ is an integral operator if and only if T is 1-summing. Moreover, $\pi_1(T) = \|T\|_i$ where $\|\cdot\|_i$ implies the integral norm of T. □

Theorem A.0.18. *[75, Theorem 10.11]*
Let $L^1(S)$ be the Banach space of all μ-integrable functions on a measure space (S, Σ, μ) and let $\mathcal{H}$ be a Hilbert space. Then every operator $T \in \mathcal{L}(L^1(S), \mathcal{H})$ is 1-summing and $\pi_1(T) \le K_{\mathfrak{G}}\|T\|$ where $K_{\mathfrak{G}}$ is Grothendieck's constant.

Lemma A.0.19. *Let $T_1 : X \to Y$ and $T_2 : Y \to Z$ be bounded operators. Then $\pi_1(T_1T_2) \le \|T_1\|\pi_p(T_2)$ for each p.*

Proof. Just note that

$$\left(\sum_n \|T_1T_2(x_n)\|^p\right)^{1/p} \le \|T_1\| \left(\sum_n \|T_2(x_n)\|^p\right)^{1/p} \|T_1\|C \sup_{\phi\in X^*:\ \|\phi\|_{X^*}\le 1} \left(\sum_n |\langle \phi, x_n\rangle|^p\right)^{1/p}.$$

□

Proof of Theorem 7.2.3. For each $f \in T^2(H)$, let $f = f_1 + f_2$ be an arbitrary Littlewood decomposition of f. Clearly, f_1, f_2, and f belong to $\ell^\infty(H \times H)$. Note that

$$\ell^\infty(H \times H) = (\ell^1(H) \otimes_\gamma \ell^1(H))^* = \mathcal{L}(\ell^1(H), \ell^\infty(H)) \text{ [69, Section 2.2]}.$$

So f_i may be represented by some $T_{f_i} \in \mathcal{L}(\ell^1(H), \ell^\infty(H))$ $i = 1, 2$, if we prove that $T_{f_i} \in \Pi_1(\ell^1(H), \ell^\infty(H))$, then we are done, by Proposition A.0.17.

For f_1, note that $t \mapsto f_1(\cdot, t) : H \to \ell^2(H)$ is a function in $\ell^\infty(H, \ell^2(H))$. So, for each $g \in \ell^1(H)$ define $T_{f_1}(g) = \sum_{t\in H} g(t) f_1(\cdot, t) \in \ell^2(H)$. So T_{f_1} is an operator from $\ell^1(H)$ into $\ell^2(H)$. Furthermore,

$$\begin{aligned} \|T_{f_1}(g)\|_2 &= \left\| \sum_{t\in H} g(t) f_1(\cdot, t) \right\|_2 \\ &\le \sum_{t\in H} |g(t)| \|f_1(\cdot, t)\|_2 \le \|g\|_1 \sup_{t\in H} \left(\sum_{s\in H} |f_1(s,t)|^2\right)^{1/2}. \end{aligned}$$

Therefore, by Theorem A.0.18, for T_{f_1} as an operator from $\ell^1(H)$ into $\ell^2(H)$,

$$\pi_1(F_{f_1}) \le K_{\mathfrak{G}} \|T_{f_1}\| \le K_{\mathfrak{G}} \sup_{t\in H} \left(\sum_{s\in H} |f_1(s,t)|^2\right)^{1/2}.$$

For f_2, similarly, note that $s \mapsto f_2(s, \cdot) : H \to \ell^2(H)$ is a function in $\ell^\infty(H, \ell^2(H))$. So, for each $g \in \ell^1(H)$ define $T_{f_2}(g) = \sum_{s\in H} g(s) f_2(s, \cdot) \in \ell^2(H)$. So T_{f_2} is an operator from $\ell^1(H)$ into

$\ell^2(H)$. Furthermore,

$$\begin{aligned}\|T_{f_2}(g)\|_2 &= \left\|\sum_{s\in H} g(s) f_1(s,\cdot)\right\|_2 \\ &\leq \sum_{s\in H} |g(s)| \|f_1(s,\cdot)\|_2 \leq \|g\|_1 \sup_{s\in H}\left(\sum_{t\in H} |f_2(s,t)|^2\right)^{1/2}.\end{aligned}$$

For T_{f_2} as an operator from $\ell^1(H)$ into $\ell^2(H)$,

$$\pi_1(T_{f_2}) \leq K_{\mathfrak{G}} \|T_{f_2}\| \leq K_{\mathfrak{G}} \sup_{s\in H}\left(\sum_{t\in H} |f_2(s,t)|^2\right)^{1/2}.$$

Also, note that $id_{2,\infty} \circ T_{f_i}$ is an operator from $\ell^1(H)$ into $\ell^\infty(H)$. Lemma A.0.19 implies that

$$\pi_1(id_{2,\infty} \circ T_{f_i}) \leq \|id_{2,\infty}\| \pi_1(T_{f_i}) \quad (i = 1, 2).$$

Note that since $\|\cdot\|_\infty \leq \|\cdot\|_2$, $\|id_{,\infty}\| \leq 1$. To conclude, one may apply Lemma A.0.19 to conclude

$$\begin{aligned}\|T_f\|_i = \pi_1(f) &\leq \pi_1(f_1) + \pi_1(f_2) \\ &\leq \sum_{i=1,2} \pi_1(id_{2,\infty} \circ T_{f_i}) \\ &\leq \sum_{i=1,2} \|id_{2,\infty}\| \pi_1(T_{f_i}) \\ &\leq K_{\mathfrak{G}}\left(\sup_{t\in H}\left(\sum_{s\in H} |f_1(s,t)|^2\right)^{1/2} + \sup_{s\in H}\left(\sum_{t\in H} |f_2(s,t)|^2\right)^{1/2}\right).\end{aligned}$$

Since, the choice of f_1 and f_2 was arbitrary $\|T_f\|_i \leq K_{\mathfrak{G}} \|f\|_{T^2(H)}$. □

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