SOURCES OF PEAK SIGNAL VARIATION FROM A THIN PLATE IMPACT SENSOR MEASURING PARTICLE RADIUS

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by
Joseph John Cibere
Saskatoon, Saskatchewan

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This thesis is dedicated to my family
Jolanda my wife
David my son and Julia my daughter
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Head of the Department of Electrical Engineering,
University of Saskatchewan,
Saskatoon, Canada S7N 0W0
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Abstract

This thesis identifies and examines the sources responsible for the variations in the peak signal from a thin plate impact sensor used to measure particle radius. Steel spherical particles, dropped onto a thin aluminium plate that is simply supported on all sides, cause plate vibrations that are detected by a piezo-electric transducer. The peak positive signal from the sensor is used as a measure of the radius of the impacting particle. The thin plate impact sensor is a simplified version of a sensor from a commercial device known as the grain loss monitor. Earlier experiments using the grain loss monitor sensor to measure the radius of particles, reported an unexplainable peak sensor signal variation. An Experimental and a theoretical analysis of this problem using the thin plate impact sensor, shows that small changes in impact position result in large changes in the peak sensor signal.

Experiments using six steel spheres, ranging in radius from 3/32 inches to 8/32 inches, impacting onto a square 1/32 inch thick aluminium plate, show that the peak sensor signal and the degree of sensitivity of the peak signal to changes in impact position depends on the location of the impact site. Specifically, for the impact locations tested, this sensitivity exhibits a minimum at approximately 2 cm from the transducer and a maximum sensitivity at the transducer site. Therefore, using these results, a hypothetical change in the impact position of approximately 0.1% of the lateral dimension of the plate, or about 0.5 mm, results in the peak sensor signal exhibiting changes, for all the particles tested, that average 4.7%, 3.9%, 1.9%, and 4.5% of the peak at impact locations at approximately 0 cm, 1 cm, 2 cm, and 3 cm from the transducer, respectively. The peak sensor signal also shows an almost linear
response to changes in particle radius at all impact positions tested, indicating that the peak signal of this particular sensor configuration provides a measure of particle radius.

A theoretical model and analysis of the thin plate impact sensor identifies the factors that influence the generation of the peak sensor signal and indicates that changes in the impact position are the most likely cause for changes in the peak sensor signal. Numeric simulations, using the theoretical model to approximate the experimental thin plate impact sensor, confirm that the sensor signal and its spectra change with impact location and particle radius to a degree that is similar to that seen in the experimental results. However, the simulated peak sensor signals are not accurately modelled and is due to the simple transducer model having difficulty reproducing the large signal response of the transducer.

An extension of the theoretical and experimental results to the grain loss monitor sensor in sizing steel spheres suggests that unperceived changes in impact position of approximately 0.4% of the sensor’s lateral dimension, or changes in distance of about 0.25 mm and 0.5 mm in the shortest and longest lateral plate dimensions, respectively, could result in a standard deviation of 16% of the mean peak sensor signal.
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\( a \) \hspace{1cm} Acceleration, \((m/s^2)\).

\( a_n \) \hspace{1cm} Taut string real coefficients.

\( A \) \hspace{1cm} Taut string constant, \( \frac{2\rho_n}{m}, (s)^{-1} \).
- Magnitude term for the taut string normal mode solution.

\( \mathbf{A} \) \hspace{1cm} Taut string complex amplitude coefficients.

\( \mathbf{A}_n \) \hspace{1cm} \( n^{th} \) complex amplitude coefficients.

\( \mathbf{A}_{mn} \) \hspace{1cm} Thin plate impact position dependent terms, \( \sin \frac{m\pi x_i}{l_x} \sin \frac{n\pi y_i}{l_y} \).

\( b_n \) \hspace{1cm} Taut string imaginary coefficients.

\( b_t \) \hspace{1cm} Damping coefficient for the transducer model.

\( B \) \hspace{1cm} Taut string constant, \( 2cv\rho_1, (N) \).

\( \mathbf{B} \) \hspace{1cm} Taut string complex amplitude coefficients.

\( \mathbf{B}_n \) \hspace{1cm} Taut string \( n^{th} \) complex amplitude coefficients.

\( \mathbf{B}_l \) \hspace{1cm} Damping coefficient for the taut string boundary condition model.

\( \mathbf{B}_p \) \hspace{1cm} Force pulse constant, \( \sqrt{\frac{3\rho_p}{E_p}} \left( \frac{1}{16\rho_p l_x^2} \right), (s/kg) \).

\( \mathbf{B}_{mn} \) \hspace{1cm} Thin plate Fourier coefficients.

\( \mathbf{B}_t \) \hspace{1cm} Bandwidth of the transducer response, (Hz).

\( \mathbf{B}_s \) \hspace{1cm} Bandwidth of the plate displacement motion, (Hz).

\( c \) \hspace{1cm} Phase speed for a taut string, \( \sqrt{\frac{\rho_1}{\rho_1}} \), (m/s).

\( c_r \) \hspace{1cm} Phase speed of Rayleigh waves in a thin plate, (m/s).

\( c_{tr} \) \hspace{1cm} Phase speed of the transverse components in a thin plate, (m/s).

\( C_t \) \hspace{1cm} Proportionality constant for the transducer material.

\( C_{mn} \) \hspace{1cm} Thin plate time dependent Fourier coefficients.

\( d_{31} \) \hspace{1cm} Piezo-electric constant of the transducer material.

\( D \) \hspace{1cm} Thin plate equation constant, \( \frac{8E_p^2}{12(1-\nu_p^2)}, (N \cdot m) \).

\( D_t \) \hspace{1cm} Electric displacement inside the transducer.

\( E_b \) \hspace{1cm} Modulus of elasticity of a beam used in the transducer model, \( (N/m^2) \).
$E_p$  
Modulus of elasticity of the thin plate, (N/m$^2$).

$E'_p$  
Modified modulus of elasticity of the thin plate, $\frac{E_p}{(1-v^2)}$, (N/m$^2$).

$E_s$  
Modulus of elasticity of the sphere. (N/m$^2$)

$E'_s$  
Modified modulus of elasticity of the sphere, $\frac{E_s}{(1-v^2)}$, (N/m$^2$).

$E_t$  
Electrical field within the transducer, (V/m).

$f_a$  
Force pulse function.

$f_c$  
Force required to compress the equivalent transducer spring, (N).

$f$  
Frequency, (Hz).

$f_k$  
Frequency sample points, (Hz).

$f'_k$  
Scaled dimensionless frequency sample points, $f_k T$.

$f_s$  
Sampling frequency, (samples/s).

$f'_s$  
Scaled dimensionless sampling frequency, $f_s T$.

$F$  
Normalized frequency variable, $\frac{f}{f_s}$.

$F_a$  
Force pulse spectrum.

$F_l$  
Initial string displacement function at $t = 0$.

$F_{pm}$  
Force pulse normalizing factor, $K_f \left( \frac{15v^2(E'_s - 1 + E'_p - 1)m_s}{16\sqrt{r_s}} \right)^{\frac{5}{2}}$, (N).

$g$  
- Earth's gravitational constant, (9.81 m/s$^2$).
- arbitrary function.

$G_l$  
Initial string velocity function at $t = 0$.

$h$  
Time increment.

$h_d$  
Discrete time unit impulse response of a differentiator shifted in time by $(N - 1)/2 + 1/2$ samples.

$h'_d$  
Discrete time unit impulse response of a differentiator.

$h_p$  
Thin plate impulse function.

$h_t$  
Transducer impulse function.

$H_d$  
Ideal differentiator transfer function.

$H_p$  
Thin plate transfer function.

$H_t$  
Transducer transfer function.
\( I_b \)  
Moment of inertia for the beam used in the transducer model, (kg·m²).

\( j \)  
Imaginary number, \( \sqrt{-1} \).

\( k \)  
- Integer.
  - Taut string wave number.
  - Spring constant, (N/m).

\( k_b \)  
Equivalent spring constant for the beam used in the transducer model, (N/m).

\( k_{d}(z) \)  
Linear proportionality constant for the transducer.

\( k_n \)  
Wave number for a taut string.

\( k_t \)  
Spring constant for the transducer model.

\( k_{mn} \)  
Wave number for a thin plate.

\( K \)  
Integer.

\( K_l \)  
Spring constant for the taut string boundary condition model, (N/m).

\( K_f \)  
Spring constant for two solids in compression, \( \frac{4}{3} \sqrt{\frac{\rho}{E_s}} \frac{E'_s E'_p}{E'_s + E'_p} \), (N/m\(^{3}\)).

\( K_p \)  
Thin plate constant, \( \frac{2}{\rho_p l_y l_z} \), (kg\(^{-1}\)), (N/m).

\( K_t \)  
Gain constant for transducer model, (V/m).

\( K_{mn} \)  
Thin plate transducer position dependent terms, \( \sin \frac{m \pi x}{l_y} \sin \frac{n \pi y}{l_y} \).

\( l \)  
- Length of taut string, (m).
  - Integer.

\( l_b \)  
Length of a beam used in the transducer model.

\( l_x \)  
Length of thin plate in the \( x \) direction, (m).

\( l_y \)  
Length of thin plate in the \( y \) direction, (m).

\( l_z \)  
Half thickness of thin plate, (m).

\( l_t \)  
Half thickness of transducer, (m).

\( L \)  
Integer.

\( L_k \)  
Taut string term, \( \frac{B}{A^2 + \omega_n^2} (A \sin \omega_n t - \omega_n \cos \omega_n t) \).

\( m \)  
- Integer.
  - Mass of a point mass, (kg).
$m_b$ Mass of a beam used in the transducer model, (kg).

$m_s$ Mass of spherical particles, (kg).

$m_i$ Mass of the mass element of transducer model, (kg).

$M_i$ Mass element for the taut string boundary condition model.

$M_k$ Taut string eigenfunctions.

$n$ Integer.

$N$ Integer.

$N_k$ Time dependent Fourier coefficients for the taut string.

$r_s$ Radius of spherical particles, (m).

$r_i$ Distance between the transducer and impact site, (m).

$t$ Time, (s).

$t'$ Time needed for a disturbance to propagate a distance $\Delta x$, (s).

$t_n$ Discrete time, $nt_s$, (s).

$t_{end}$ The time at the end of some interval, (s).

$T$ Normalizing time factor, $\left(\frac{m_p}{R_s/\sqrt{\nu}}\right)^{\frac{1}{2}}$, (s).

$T_D$ Measured duration of a light pulse, (s).

$T_{D_e}$ Expected duration of measured light pulse, (s).

$T_i$ Tension of taut string, (N).

$T_s$ Sampling interval, (s/sample).

$T_s'$ Scaled sampling interval, $T_s/T$ (1/sample).

$T_i(z)$ Stress at the point $z$ in a transducer cross section, (N).

$U_{mn}$ Normalized thin plate eigenfunctions.

$v$ Velocity of point mass, (m/s).

$v_o$ Output signal voltage of the sensor, (V).

$v_p$ $\sqrt{E_p/\rho_p}$, (m/s).

$v_s$ Spherical particle impact velocity, (m/s).

$v_{s_A}$ Calculated spherical particle velocity at point A, (m/s).

$v_{s_B}$ Calculated spherical particle velocity at point B, (m/s).
$v_{sc}$ Calculated spherical particle velocity at point C, (m/s).

$v_{sB}$ Expected spherical particle velocity at point B, (m/s).

$v_{scE}$ Expected spherical particle velocity at point C, (m/s).

$v_t$ Output voltage of the transducer, (V).

$v_{pk}$ Peak positive output voltage, (V).

$V_0$ Spectrum of sensor output signal.

$w$ Discrete time taut string function.

$w_t$ Discrete time plate displacement function at the transducer location.

$w_{mn}$ Discrete time version of a single sinusoid from the displacement function.

$x$ - Coordinate direction.

$y_a$ Continuous time function.

$y_{an}$ $n^{th}$ component of a continuous time function.

$x_1$ Discrete time force function.

$x_2$ Discrete time sinusoidal function.

$x_a$ Continuous time function.

$x_i$ $x$ coordinate of impact site. (m).

$x_t$ $x$ coordinate of transducer location. (m).

$X$ Discrete Fourier transform of a discrete time function.

$X_a$ Fourier transform of a continuous time function.

$X_{DFT}$ Discrete Fourier transform of a discrete time function.

$y$ Coordinate direction.

$y_i$ $y$ coordinate of impact site. (m).

$y_t$ $y$ coordinate of transducer location. (m).

$z$ Coordinate direction.

$z_c$ Compression of spring in transducer model, $z_0 - z_t$, (m).

$z_d$ Relative displacement between sphere and thin plate, $z_s - z_p$, (m).

$z_{dn}$ Discrete time version of $z_d$, (m).
\( z_h \)  
Drop height of spherical particles (m).

\( z_0 \)  
Motion of mass element in transducer model, (m).

\( z_p \)  
Displacement of plate, referenced from the point of impact, (m).

\( z_s \)  
Displacement of sphere, referenced from the point of impact, (m).

\( z_t \)  
Transverse plate displacement at the transducer site, (m).

\( z_{mn} \)  
Transverse plate displacement component due to mode \( \{m, n\} \).

\( Z_c \)  
Fourier or Laplace transform of \( z_c \).

\( Z_t \)  
Fourier or Laplace transform of \( z_t \).

\( \alpha \)  
Zener's epi-centre displacement integration constant.

\( \delta \)  
- Dummy variable.
- Dirac delta function.

\( \delta_k \)  
Kronecker delta function.

\( \eta \)  
Dummy variable.

\( \Theta \)  
General phase angle.

\( \Delta \)  
- Difference symbol.
- Factor change symbol.

\( \epsilon_s \)  
Dielectric constant of the transducer piezo-material.

\( \lambda \)  
Inelasticity parameter.

\( \Lambda \)  
Wavelength, (m).

\( \Lambda_{mn} \)  
Wavelength of the thin plate component at the frequency \( \omega_{mn} \), (m).

\( \nu_p \)  
Poisson ratio of thin plate.

\( \nu_s \)  
Poisson ratio of spheres.

\( \xi \)  
- Dummy variable.
- Dimensionless wave number.

\( \rho_l \)  
Linear density of taut string, (kg/m^3).

\( \rho_p \)  
Density of thin plate, (kg/m^3).

\( \rho_s \)  
Density of spheres, (kg/m^3).

\( \sigma \)  
Dimensionless relative displacement, \( z_d/(Tv_s) \).
\( \sigma_n \) Discrete version of \( \sigma \).

\( \sigma_p \) Dimensionless relative displacement of the plate, \( z_p/(Tv_p) \).

\( \tau \) Dimensionless time, \( t/T \).

\( \omega \) Frequency, \( (\text{rad/s}) \).

\( \omega_b \) Resonant frequency of a beam used in the transducer model, \( (\text{rad/s}) \).

\( \omega_c \) Break point frequency for the transducer electrical element model, \( (\text{rad/s}) \).

\( \omega_k \) Taut string vibrational frequencies, \( \omega_k = \omega_n, (\text{rad/s}) \).

\( \omega_m \) Resonant frequency for the transducer mechanical element model, \( (\text{rad/s}) \).

\( \omega_n \) Taut string vibrational frequencies, \( (\text{rad/s}) \).

\( \omega_s \) Sampling frequency, \( (2\pi \text{samples/s}) \).

\( \omega_t \) Piezo-electric transducer resonant frequency, \( (\text{rad/s}) \).

\( \omega_{mn} \) Frequency of plate vibration at mode \( \{m, n\} \).

\( \omega_{mn_{max}} \) The maximum plate frequency, \( (\text{rad/s}) \).

\( \Omega \) - Normalized frequency variable, \( \omega/\omega_s \).

- Dimensionless frequency variable.

- unit of Ohms.

Subscripts

\( a \) Continuous time function.

\( b \) Beam in transducer model.

\( c \) Spring in transducer model.

\( e \) Electrical component.

\( f \) Force.

\( i \) Impact.

\( j \) Integer.

\( k \) Integer.

\( l \) Taut string component.
Plate mode component in the $z$ direction.
Plate mode component in the $y$ direction.
Output.
Plate.
Sphere.
- Sampling.
Transducer.
Directional component.
Directional component.
Directional component.
End.
Minimum.
Maximum.
Peak.

Operators

$\nabla$ Laplacian.
$\partial$ Partial derivative.
$*$ Convolution.
$d$ Differentiation.
$F$ Fourier transform.
$F^{-1}$ Inverse Fourier transform.
$L$ Laplace transform.
$L^{-1}$ Inverse Laplace transform.
$\Im$ Imaginary component.
$\Re$ Real component.
### Dimensions

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>ampere.</td>
</tr>
<tr>
<td>dB</td>
<td>decibel.</td>
</tr>
<tr>
<td>F</td>
<td>farad.</td>
</tr>
<tr>
<td>Hz</td>
<td>hertz.</td>
</tr>
<tr>
<td>kg</td>
<td>kilogram.</td>
</tr>
<tr>
<td>m</td>
<td>metre.</td>
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<tr>
<td>N</td>
<td>newton.</td>
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<tr>
<td>Ω</td>
<td>ohms.</td>
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<tr>
<td>Pa</td>
<td>pascal.</td>
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<tr>
<td>rad</td>
<td>radian.</td>
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<tr>
<td>s, sec.</td>
<td>second.</td>
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<tr>
<td>V</td>
<td>volt.</td>
</tr>
<tr>
<td>W</td>
<td>watt.</td>
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</tbody>
</table>

### Dimensional Prefixes

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Prefix</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>pico, 10⁻¹².</td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>nano, 10⁻⁹.</td>
<td></td>
</tr>
<tr>
<td>μ</td>
<td>micro, 10⁻⁶.</td>
<td></td>
</tr>
<tr>
<td>m</td>
<td>milli, 10⁻³.</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>centi, 10⁻².</td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>kilo, 10³.</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>mega, 10⁶.</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>giga, 10⁹.</td>
<td></td>
</tr>
</tbody>
</table>
1. Introduction

1.1 The Problem

The measurement of various characteristics, such as size or density, of granular sized particles, those with features greater than approximately 50 micrometers, is important to many industrial processes. In particular, industry believes that improvements in measuring these characteristics results in lower cost, better control or more consistent products. A specific example includes the potash mining industry, where a mechanically based sieving system separates a sample of potash particles into groups of different size using screens to pass or retain the particles. The weight distribution of each group of particles, as a percentage of the total weight of the sample, provides an indication of the quantity of each group of particles. This information determines the quality of the processed potash. Operation of the sieving system is a time and labour intensive task and improvements to this process generate considerable interest.

A novel method to obtain the distribution of various sized potash particles was investigated at the University of Saskatchewan in the early 1980's. The method attempted to size particles by analysing the vibrations caused by their impact upon the surface of a sensor plate. It was hypothesized that the impact of different sized potash particles would produce measured plate vibrations unique enough to allow their size to be determined. If this were the case, potash particles could be counted and grouped into sizes equivalent to those obtained with a mechanical sieving system. This would allow the weight distribution, as a percentage of total sample weight, for each group of particles to be obtained. An instrument based on this technique was seen as the electronic equivalent of a mechanical sieving system and had the potential to size particles with a minimum amount of labour and more rapidly than
a mechanical system. The instrument was envisioned to be a small, self-contained, computer controlled system that could easily be transported to other sites or installed as a continuous measuring system.

These benefits were seen by the potash mining industry and their considerable resources were used to develop a particle sizing system using this method. The system consisted of a sensor to detect the impact of individual potash particles, a mechanical apparatus that dropped individual particles from a sample of potash, and electronic circuitry to measure the output of the sensor. The sensor consisted of a plastic plate having a piezo-electric transducer mounted on its underside. Appendix L provides a detailed description of the sensor. In operation, the mechanical apparatus would drop individual potash particles, at a controlled rate, onto the sensor. The resulting impacts would produce a measurable signal from the transducer that the circuitry would process to produce an indication of particle size.

To develop the appropriate circuitry the characteristics of the transducer signal that distinguished one particle size from another had to be determined. This was done by examining the time and spectral characteristics of the signals produced by dropping numerous potash particles of known size onto the sensor. After extensive empirical testing with a laboratory system, it was decided to use an amplified and filtered peak sensor signal as the indicator of particle size. Initially, this peak signal appeared to provide a repeatable measure of particle size but careful examination of the results showed that a random variation in the peak signal was present. In an attempt to reduce this variation, all the factors that were observed to influence the transducer signal were controlled or fixed. This included using spherical particles instead of potash particle, holding the particle drop height, impact location, and impact angle as constant as possible. In spite of these precautions, the variation was still present.

\footnote{In private conversations with the Potash Corporation of Saskatchewan, it was learned that the Division of Control Engineering of the College of Engineering at the University of Saskatchewan began an investigation into using this sensor for sizing potash particles in the early 1980's.}
Regardless of the peak signal variation, the laboratory system provided acceptable performance that lead to the decision to develop a field model. The results obtained from the field model were not repeatable and its performance did not compare well with the laboratory model. Efforts to conclusively identify the cause of these performance problems were not successful. Further work on this particle sizing system was then stopped.

The performance of the field model could possibly be the result of the peak signal variation seen in the laboratory being aggravated by design changes in the field model. Without knowing the cause of the peak signal variations seen in the laboratory model, it is not possible to assess the influence that field model design changes would have on the peak signal. In spite of these difficulties interest is still very strong in a device with obvious advantages over the mechanical sieving system. If the cause of this peak signal variation can be identified then it may be possible to develop a useful particle sizing system.

This thesis investigates the reasons for the observed variation of peak signal and explains its cause using experimental results from a simplified version of the sensor and a proposed theoretical model of an ideal sensor. Together, the results and model identify the possible factors that cause the variation in the peak signal.

1.2 Historical Background

The idea that the plate vibrations due to particle impacts could be used to characterize particles originated at the University of Saskatchewan in the mid 1960’s in the area of agriculture. During the harvesting of grains, it was noticed that grain seeds being lost by a combine would produce a distinct sound, different from the unwanted material or chaff, when it impacted on the outside of a closely following truck [1]. Using this observation, Reed, Grovum, and Krause [1] developed a system known as the grain loss monitor to measure the rate of grain being lost by the combine. The

\footnote{In private meeting with officials from the Potash Corporation of Saskatchewan, they provided the details on the operation and the performance of both the laboratory and field models of the particle sizing system.}
system consists of a sensor to detect the impact of particles and electronic circuitry to process the sensor signals. By mounting the sensor in an appropriate location on the combine, the system discriminated the signals, resulting from the impact upon the sensor, of grain seeds from that of chaff. The sensor portion of the grain loss monitor was the same sensor used to measure the size of potash particles and was described in section 1.1. In operation, the impact of a grain seed on the sensor, produces a short duration, large amplitude signal whereas the impact of chaff produces a low frequency, small amplitude signal. By suppressing the low frequency components of the transducer signal and detecting the amplitude of any resulting signal over a given threshold, the impact of grain seeds can be distinguished from that of chaff [1].

Since the introduction of the first commercial grain loss monitor system in 1968, Kirk [2] in 1977 provided the only recorded refinement to the grain loss monitor system. Kirk's improvement provided a finer degree of discrimination between the impact of grain and chaff upon the grain loss monitor sensor plate by noting that the signals had a characteristic distribution of power in a high and low frequency band. By detecting the change in the relative power in these particular frequency bands, smaller lighter seeds having the same density as the chaff, such as canola, could also be detected [3].

It became apparent that if the grain loss monitor system could distinguish seeds from chaff it could potentially be used to discriminate between the sizes of various particles. An instrument based on the sensor used in the grain loss monitor could provide the electronic equivalent of a mechanical sieving system. The potash industry expressed a great interest in developing such an instrument to complement the mechanical sieving system now in use and they expended considerable resources investigating this possibility in the early 1980's. The results of this research showed that the performance of this new instrument did not compare well with a mechanical sizing system and further work in this direction was stopped. Since this time no additional refinements or usage of the sensor in the grain loss monitor has been reported.
1.3 Existing Particle Sizing Techniques

The methods available to measure particle size are numerous [4] and are grouped by the size and nature of the particles to be measured. A simple and common method for measuring granular particles, those with the largest proportion of their features greater than 50 \( \mu m \), is the sieving method [5, 6]. This method consists of putting a mixed sample of different sized particles into a device containing sieves or screens of different sized apertures and mechanically agitating the device to allow the particles to be either passed or retained by the various sieves. The particle size distribution of the sample is expressed as a weight distribution of particles retained on each sieve. Other methods of sieving include wet sieving, and air-jet sieving used to extend the range of mechanically shaken sieving down below 50 \( \mu m \). In these systems air or water is drawn up through the sieves fluidizing the material. A small negative pressure at the bottom of the sieve removes the fine particles [5, 7, 8]. In all cases the sieving methods use screens or sieves to trap or pass particles. Though sieving is a simple technique, it is a labour intensive process and not well suited for real time analysis. Other methods more amenable to automation and real time analysis are those that use optical or ultrasonic principles, especially for particles with their largest features less than approximately 50 \( \mu m \). These methods typically use interferometry or absorption of externally supplied optic or acoustic signals to either count or size particles that are typically suspended in a liquid or gaseous medium.

Size analysis for granular particles using acoustic or optical principles is limited and few publications have been found. One optical method, that sizes particles in the 1 to 1000 \( \mu m \) range, uses a high powered laser beam to illuminate and measure the reflected light from suspended particles in a slurry mixture [9]. The instrument, known commercially, as the PAR-TEC 200 is effectively a scanning laser microscope. A high power, 2 MW per square inch, laser beam is focussed and scanned over a 0.8 by 2 \( \mu m \) area with a 8 \( \mu m \) depth of field. Particles within the scanning area and depth of field are detected and sized by the reflected light of the laser beam. Since
the laser beam scanning speed is fixed and large relative to the particle motion, the duration of the reflected light is proportional to the size of the particle.

An acoustic method reported in 1977 by Leach et al [10] describes sizing particles from 50 μm to 3 cm in diameter by analysing the acoustic emission from the particles impinging upon each other. In Leach's technique, metal or glass spheres are placed in a foam-lined cylindrical vessel that rotates causing the particles to impact upon each other. The acoustic emissions caused by the impacting particles are recorded using a wide band microphone and amplifier system. The resulting acoustic emissions combine to produce measured signals that exhibited frequency beating. The signals are then analysed in the time domain by noting the durations of characteristic beat patterns and oscillations within the beat patterns. Plotting the diameter of the particle versus the measured durations showed a strong linear relationship. Leach et al performed further investigations [11, 12, 13, 14] to determine the influence of the higher order modes of vibration in estimating size, the ability to estimate the proportions of various sized particles from a mixture, and the use of this method for irregularly shaped particles. In all these investigations Leach showed that the measured waveform parameters were good indicators of particle size.

In 1985, Kwan and Leach [15] applied Fourier theory to the recorded waveforms and examined their frequency spectra for particle shape and size estimators. Kwan concluded that for the glass cylindrical particles, the measured spectral peaks were related to diameter and length of the particles. Evidence also indicated that the spectral width of the peak frequency was related to the small size differences of mono sized particles and the intensity of the peak frequencies were indicating the relative proportion of different sized particles.

Another acoustic method used to measure the smaller granular particles, those between 25 and 1500 μm, was described by Bragg et al in 1981 [16]. In Bragg's method air borne particles become entrapped in an airflow that moves through a orifice. As the particle moves through the orifice an audible emission is detected that
is unique to the shape and size of the particle. Bragg derived a model that predicted the diameter of the particle using the peak emission measured.

A technique described by Buttle et al in 1990 [17], uses the acoustic emissions produced by spheres impacting on a thick plate to yield impact force and particle size. Buttle was interested in measuring on-line erosion rates of pipes or vessels due to the transport of dust entrained in high pressure and temperature gas. Buttle hypothesized that knowing the impact force or source function of particles, a correlation could be made to the erosion rate of the pipes or vessels. He described a method to obtain the impact force function by measuring the acoustic emissions produced by spheres impacting on a thick plate.

Buttle's experimental set-up places a wide bandwidth point contact piezo electric transducer, having a -6 dB point at 3 MHz, on the underside of the plate located at the impact epi-centre. Bronze and glass spheres, from 50 to 90 μm in diameter, are dropped from a fixed height, in a vacuum, onto either a mild steel or aluminium plate approximately six millimetres thick. The resulting disturbance from the impact propagates through the plate to the point contact piezo electric transducer, located on the underside, where the surface displacements, on the order of tens of picometres, are translated into electrical signals. Buttle relates the observed electrical signal or acoustic signal, \( v_o(t) \), to the time convolution of the particle impact force function, \( f_o(t) \), the plate propagation function, \( h_p(t) \), and the piezo transducer/amplifier function, \( h_t(t) \). This is expressed as

\[
v_o(t) = f_o(t) \ast h_p(t) \ast h_t(t) \quad ,
\]

where \( \ast \) represents convolution defined as

\[
a(t) \ast b(t) = \int_{-\infty}^{\infty} a(\tau)b(t - \tau)d\tau \quad .
\]

Buttle was able to obtain the impact force function by determining the transfer function of the plate for a propagating impulse and by measuring the transfer function for the detector/amplifier. By deconvolving these with the observed acoustic signals
he obtains the impact force function. He concluded that it was possible to determine the particle size from the impact force function using the acoustic emission of the elastic impacts.

Of all the reported particle sizing techniques, Buttle’s work represents the only resemblance to the techniques used in the grain loss monitor, in that plate vibrations due to particle impacts are used to estimate particle size. The most important observation in this resemblance is in Buttle use of Equation (1.2) to describe the transducer’s output voltage.

Buttle’s work also exhibits many specific and obvious differences. Firstly, Buttle uses glass and bronze spheres impacting onto an aluminium or mild steel plate. Potash particles have irregular shapes and impact onto a plastic plate. Secondly, Buttle was concerned with the impact of particles on a thick plate. A thick plate is one whose thickness has dimensions on the same order as its other lateral dimensions. The plate of the grain loss monitor sensor is thin relative to it’s lateral dimensions. In addition, the diameter of the particles impacting on Buttle’s plate were approximately 60 to 120 times smaller than the thickness of the plate. In the case of the potash particles their size varied between 5 to 10 times smaller and larger than the thickness of the plate. Finally, since it is difficult to imagine the grain loss monitor plate moving only tens of picometres and producing signal levels on the order of volts from the transducer, it appears that the measured plate motion in Buttle’s work is orders of magnitude smaller than the motion of the grain loss monitor plate. The significance of the resemblance and the differences seen in Buttle’s work will be made clear as the operation of the sensor is investigated.

1.4 Problem Analysis

In this section the nature of the problem and possible causes of the signal variation seen in the operation of the grain loss monitor sensor are introduced. First, the approach used in this thesis to solve this problem is discussed. Then the complexities of the problem and a possible solution are presented using a simple analogy
based on a particle falling into a pool of water. Finally, the results of an earlier work regarding spheres impacting onto a thin plate and another more substantial analogy using a taut string complete this preliminary analysis.

1.4.1 The General Approach

Consider the problem that was observed in the operation of the grain loss monitor sensor in sizing potash particles. Magnitude variations of the peak detected signal from trials that were assumed to be the same were seen. The factors that were observed to influence the magnitude of the peak signal included the particle shape, the size, the drop height, the impact location, the impact angle with the sensor, the plate characteristics, and the piezo-electric transducer’s response. In spite of controlling and fixing these factors and using the same spherical particle dropped from a fixed height onto the same point of the sensor, a variation of the peak signal was still measured.

There are two possible reasons for this variation. The first is that there is some unknown but significant factor or factors influencing the signal. The second is that there are no significant unknown factors but instead, there are small unperceived changes in the known factors that cause large signal variation. This implies that the sensor signal is sensitive to small changes in one or more of the factors.

Considerable time and effort could be required to determine, by experimental trial and error, the possible unknown factors or the sensitivity of the sensor signal to changes in the known factors. An alternate approach is to gain an understanding of the sensor’s operation using analysis to identify likely causes of the signal variation. Experiments could then test the predictions of the analysis and the results could suggest additional analysis and experiments. This process can be repeated until the answers are found.

In this thesis, a simple analysis is given that suggests the nature of the sensor’s response and factors that affect it. This leads to proposing specific sensor charac-
teristics and significant influencing factors that are tested in experiments. A more detailed examination is performed to substantiate the initial analysis and allows the results of experiments to be obtained analytically using numeric simulations. The results of the experiments and simulation are then compared and additional suggestions are put forward for further analysis and experiments. The conclusion of this work identifies the most significant factors that were possibly responsible for the observed signal variation seen in using the grain loss monitor sensor to size potash particles.

1.4.2 Using Analogous Examples

It is difficult to begin an analysis without understanding the principles of the sensor's operation. One approach to gain this insight is to find examples that appear to be analogous. Consider the characteristics of the sensor observed during the impact of a particle. The sound of the particle hitting the surface of the sensor plate and the corresponding measured sensor signal suggest that the plate of the sensor has been set into motion by the impact and that the transducer responded to this motion by producing an electrical signal. One analogy, that shares some of these observed characteristics and allows a gradual unrolling of events, is that of a particle falling into a pool of still water.

After the particle hits the surface of the water, waves appear on the surface. These waves begin at the point of impact and propagate radially outward. The wave crests decrease in amplitude and widen as the radius of the wavefront increases. The initial formation of the waves can be viewed as a deformation of the surface of the water by the impacting particle. If the water is contained in a small enclosure, these waves will reflect off the walls and in a short time the individual radial waves become indistinguishable in the apparent chaotic motion of the surface of the water.

By increasing the speed of the events for a particle falling into a pool of water, it is possible to envision the events that occur when a particle impacts with the surface of a plate. The previous analogy suggests the plate will be initially deformed by an impacting particle, followed by the outward propagation of a disturbance and the
eventual vibration of the entire plate. If this analogy is applicable, then the decreasing amplitude and the widening of the wave crests suggest that the measurement of the plate vibrations at any particular point is dependent on the relative distance between the impact point and the transducer location.

More substantial analogous examples, provided by the early work of Zener [18], are now considered. These examples provide the analytical basis and the insight to experimentally determine the factors responsible for the peak signal variation.

Zener was interested in explaining the experimental results showing the apparent inelastic behaviour of steel spheres dropped onto large glass plates. These experiments showed that as the radius of the spheres approached and exceeded the thickness of the plate, the measured coefficient of restitution decreased to a point where the spheres did not bounce back upward off the plate but simply stopped at the point of impact [19]. Zener developed expressions that not only agreed with the experimental results but provided an explanation for the apparent inelasticity of the impacts.

In deriving his expressions, Zener assumed a plate large enough in the lateral dimensions that reflections, caused by the initial impact, would return only after the impact was completed. This assumption simplified his analysis so that the motion of the plate at the impact point could be determined. In explaining the apparent inelasticity of the thin plate, Zener indicated that thin plates react to an impulsive force in a manner that is similar to that of an infinitely long taut string. Specifically, at the point of application of an impulsive force, the string or plate moves downward to a maximum point and then remain stationary until the return of a reflection. The description of this motion, at the point of application, is given by equations having the same form for both the string and the plate. Deferring the discussion of the assumptions used in deriving this expression and its physical significance, it is of the form

\[ z(t) = \alpha \int_0^t f_a(\xi) d\xi , \tag{1.3} \]

where \( z(t) \) is the transverse motion of the plate at the point of application of the
force $f(t)$ and $\alpha$ is a constant that depends on the factors particular to a string or thin plate [18]. Using this expression for the thin plate, Zener then determined the equation describing the relative motion of the sphere and plate and the forces generated during impact. These were used to show how the coefficient of restitution changed as the radius of the spheres varied.

An interesting result of Zener's work is his derivation of an expression that relates the plate and sphere parameters to a single variable that determines the form of the resulting impact force. This expression is given as

$$\lambda = \frac{\pi^\frac{1}{2}}{\sqrt{3}} \left( \frac{r_s}{2I_s} \right)^2 \left( \frac{v_s}{v_p} \right)^\frac{1}{2} \left( \frac{\rho_s}{\rho_p} \right)^\frac{1}{2} \left( \frac{E_s'}{E_s' + E_p'} \right)^\frac{1}{2},$$  \hspace{2cm} (1.4)

where $\lambda$ is referred to by Zener as the inelasticity parameter, $r_s$ is the radius of the sphere, $I_s$ is the half thickness of the plate, $v_s$ is the sphere's impact velocity, $v_p$ is defined as

$$v_p = \sqrt{\frac{E_p'}{\rho_p}},$$  \hspace{2cm} (1.5)

$\rho_s$ and $\rho_p$ are the sphere and plate densities respectively, and where

$$E_s' = \frac{E_s}{1 - v_s^2}, \quad \text{and} \quad E_p' = \frac{E_p}{1 - v_p^2},$$  \hspace{2cm} (1.6)

uses Young's modulus, $E$, and Poisson's ratio, $\nu$, for the sphere and plate denoted with the subscripts $s$ and $p$, respectively. These expressions are discussed in a detailed theoretical analysis given in a latter part of this thesis. The important point to emphasize at this time is that a change of an appropriate magnitude, in any one parameter of Equation (1.4), can result in a change in $\lambda$ that is the same. Equivalently, a change in any parameter can result in identical changes in the force pulse. This behaviour will be discussed later during a detailed theoretical analysis of the impact force.

One major difference between the string and plate behaviour, briefly mentioned by Zener, is in the propagation of a disturbance. In the string, a disturbance that is formed over a portion of an infinitely long string will not change its shape as it propagates. In a thin plate, a disturbance that is formed over a portion of the surface
of an infinitely large plate will change its shape as it propagates [18]. The change in
the shape of the disturbance is due to dispersion that is examined later.

When the sensor is considered simply as a thin plate having spherical particle
impacting onto it, Zener’s work is seen to provide a number of important developments
and ideas. In particular, his use of the taut string as an analogy for thin plate
behaviour, the analytical methods he used to develop models of the events during a
particle impact, and the derivation of the expressions that explicitly show the factors
involved in the impact, all help to provide a better understanding of the sensor’s
operation. One important aspect that Zener’s work does not deal with, is the motion
of the plate at points other than the impact location.

Before using Zener’s work to understand the sensor’s operation, the taut string
analogy is examined first since it is mathematically less complex than a thin plate.
The characteristics of a vibrating string and its response to an impulsive point force
provides a basis to understand the equivalent characteristics in a thin plate.

1.4.3 The Taut String as a Thin Plate Analogy

A detailed examination of the behaviour of a taut string is given in Ap­
pendix B. In this section the important results of this analysis are reviewed. The
characteristics of a taut string of finite length, exhibiting no damping, are listed below.

1. The method used to hold the ends of the string affect its motion. These are
known as the boundary conditions and can vary from being fixed or allowing
no motion, to being free, allowing unrestricted motion.

2. When the string is in motion it tends to vibrate at discrete frequencies called
resonant frequencies. These frequencies are determined by the physical param­
eters of the string and the boundary conditions.

3. When the string vibrates or oscillates at one frequency, its motion is confined to
a particular stationary shaped envelope referred to as a standing wave. Points
along the envelope having no string motion are called nodes and points having
maximum string motion are called anti-nodes. The boundary conditions deter­
mine the form of the envelope and the location of the nodes and anti-nodes.
For a string fixed at both ends, a sinusoidal shaped envelope with nodes at each
end is formed. The number of sinusoidal shaped half cycle envelopes along the length of the string increases, as do the corresponding number of nodes and anti-nodes, with each increase in resonant frequency.

4. The string does not disperse a propagating disturbance or equivalently, the disturbance does not change shape as it moves along string.

5. The string can support any number of oscillation at different frequencies simultaneously. This implies that the principle of superposition of sinusoids can be used to understand the formation of a disturbance. A disturbance on the string can be viewed as a unique summation of these individual oscillations such that, at one instant of time, they sum to zero outside the region of the disturbance. A propagating disturbance is the result of summing, at a many different instants of time, these oscillations such that the disturbance appears to move over a different regions of the string.

6. The form of the reflection of a disturbance at a boundary depends on the boundary conditions. The possible complexities of the boundary conditions can result in equally complex reflections that change the shape and magnitude of the incident disturbance. Simple boundary conditions result in simple reflections. For example, fixed ends result in reflections that are an inverted form of the incident disturbance.

Using these string characteristics as an analogy for a thin plates, then it is reasonable to assume that a thin plate exhibits discrete frequencies of oscillations that are confined to a two dimensional envelope with nodal and anti-nodal locations. A propagating disturbance can also be viewed as a time varying summation of these oscillations. Finally, reflections will be a complex version of the incident disturbance for complex boundary conditions. The propagation characteristic of a disturbance, as noted by Zener, is dispersive for a thin plate.

Consider now the behaviour of the same taut string to a force that is equivalent to a point mass having a fixed velocity impacting onto the string. This is examined in detail in Appendix B. The interesting characteristics in this case are listed below.

1. The force function during the impact is obtained using the techniques of Zener. In this particular case, a point force, equivalent to that experienced in the impact of a point mass on the string, decays exponentially from an initial value. The decay rate and initial value depend on the mass, impact velocity and on the string parameters.
2. The string motion at the point of application of the force is of form given by Equation (1.3). In this particular case, the string moves downward in the direction of the applied force and approaches, with exponentially decreasing velocity, an asymptotic point.

3. The propagating disturbance due to this force appears as a moving wavefront having the same exponential-like shape produced at the impact location. The motion of the string, as seen at a single point other than the impact location, will then resemble the motion seen at the impact location. This is true only if reflections do not interfere or pass by the same point while the disturbance or a significant portion of it is passing by. This illustrates, in a different manner, the non-dispersive propagation characteristics of a taut string.

4. If, at some measurement point and over some interval of time, reflections do not interfere with the disturbance, then the impact and measurement locations on the string will not affect the form of the motion seen at the measurement point over that interval of time. In addition, the boundary conditions are of no significance to the motion over this same interval of time.

5. If, at some measurement point and over some interval of time, reflections do interfere with the disturbance, or if the duration is long enough to see reflections, then the impact and measurement point on the string will affect the form of the motion seen over that interval of time. In addition the boundary conditions will influence the form of the measured motion.

6. If the motion of the string at an arbitrary point is examined over a period of time greater than the period of the lowest resonant frequency, allowing reflections to be seen, the spectral content of that motion will reveal that it is a frequency sampled version of the force spectrum. The frequency samples are at the string’s resonant frequencies and the magnitude at the frequency sample is dependent on both the measurement point and impact location. The characteristics of this dependency on location is determined by the mode shape at that frequency.

These characteristics suggest that the plate motion, measured at a point, could be complicated by reflections in certain time intervals and impact and transducer locations. These intervals and locations should be avoided to simplify an analysis. Most importantly, the analogy is suggesting that the force pulse and plate motion spectra are related and that this relationship is dependent on the location of the measurement and impact locations.
1.5 Proposed Sensor Behaviour and Influencing Factors

1.5.1 The sensor system description

The output signal of the transducer can be viewed as the time convolution of the force function, the plate propagation function, and the transducer function [17]. This is expressed as

\[ v_o(t) = f_a(t) \ast h_p(t) \ast h_t(t) \]  \hspace{1cm} (1.7)

where \( v_o(t) \) is the transducer's output, \( f_a(t) \) is the force pulse function, \( h_p(t) \) is the plate impulse response, and \( h_t(t) \) is the transducer impulse function. The terms identifying these functions reflect usage common in communication analysis [20, 21].

A Fourier transform of Equation (1.7) results in

\[ V_o(\omega) = F_a(\omega)H_p(\omega)H_t(\omega) \]  \hspace{1cm} (1.8)

where \( V_o(\omega) \) is the transducer output spectrum, \( F_a(\omega) \) is the force pulse spectrum, \( H_p(\omega) \) is the plate transfer function, and \( H_t(\omega) \) is the transducer transfer function [20, 22, 23].

A simpler form of either Equation (1.7) or Equation (1.8) is postulated by assuming that the transducer, located at the point denoted \((x_i, y_i)\), measures the plate's transverse motion, or displacement, \( z_t(t) \). This results in expressing the output as

\[ v_o(t) = z_t(t) \ast h_t(t) \]  \hspace{1cm} (1.9)

or in the frequency domain as

\[ V_o(\omega) = Z_t(\omega)H_t(\omega) \]  \hspace{1cm} (1.10)

where \( Z_t(\omega) \) is the plate displacement spectrum as measured at the transducer location. This implies that

\[ z_t(t) = f_a(t) \ast h_p(t) \]  \hspace{1cm} (1.11)

or

\[ Z_t(\omega) = F_a(\omega)H_p(\omega) \]  \hspace{1cm} (1.12)
Figure 1.1: The proposed block diagram of the thin plate impact sensor system.

The frequency domain representation suggests the operational arrangement shown in Figure 1.1. This arrangement represents the system block diagram for a sensor known as the thin plate impact sensor.\(^3\)

Instead of immediately attempting an analysis of the processes shown in Figure 1.1, it is easier to build upon this work, use the previous taut string analysis and analogies to hypothesize the operation of the sensor. This is approached by first examining the force of impact and the motion of the plate at the impact site, then the propagation of the disturbance and plate motion at other locations are considered. Finally, the translation of this motion by the transducer is examined.

1.5.2 The initial force and plate motion

To use Zener's work, his assumptions are applied to the operation of the sensor. This restricts the particle forms to spheres and requires the sensor to be large enough in its lateral dimensions to prevent reflections from returning until after the impact is completed.

The force generated during impact is, according to Zener, dependent on the sphere and plate material properties as given by Equation (1.4). Depending on the values of the parameters, the force, as a function of time, can resemble a sinusoidal half

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\(^3\)For convenience, the thin plate impact sensor will be simply referred to as the sensor. The grain loss monitor sensor is explicitly identified whenever it is referred to.
pulse or a sinusoidal half pulse that decays almost exponentially during its latter half. In particular, the results indicate that values of $\lambda$ between zero and approximately 0.5 produce force pulses resembling sinusoidal half pulse with the resemblance increasing as $\lambda$ approaches zero. As $\lambda$ increases above 0.5, the force pulse begins to resemble a sinusoidal half pulse that decays exponentially during its latter half.\(^4\) This decay interval increases with increasing $\lambda$ but the interval of the initial rise of the sinusoidal pulse has small increases relative to the decay interval.

Since $\lambda$ is the product of a number of parameters, as indicated by Equation (1.4), a change in its value could be due to a change in any of its parameters. This leads to the observation that a change in one parameter can be expressed as an equivalent change in another parameter. For example, doubling the radius of a sphere is equivalent to increasing the impact velocity of the sphere by $2^{(2)(5/3)} \approx 10$ times or the density of the sphere by $2^{(2)(5/3)} \approx 10$ times. All these changes result in the same change in $\lambda$. Therefore, no change in the form or shape of the force pulse would be observed.\(^5\)

The motion of the plate at the impact site is not explicitly considered by Zener but it can be estimated by using Equation (1.3) and noting that it also describes the motion of a taut string at the impact site. In the case of the string, the impact force $f_a(t)$ decays exponentially from an initial value resulting in a motion that is of the form $1 - e^{-t}$. For a thin plate, the force $f_a(t)$, generated with an impacting sphere, is similar. This is especially true when $\lambda$ is large resulting in the duration of the latter half of the force pulse being much longer than its initial rise. Therefore, this force will cause the plate motion to also be of the form $1 - e^{-t}$. With the force pulse changing and becoming more of a sinusoidal half pulse, the plate motion becomes increasingly step-like. This can be seen by integrating a sinusoidal half pulse over its duration.

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\(^4\)The $\lambda = 0.5$ point was subjectively chosen to differentiate between the two forms of the force pulse.

\(^5\)Though no change in the form or shape of the force pulse would be seen there will be a change in the magnitude and duration of the force pulse. The degree of this change is discussed during a theoretical analysis of the force.
The plate motion at the impact can be viewed with additional insight by considering the spectral content of the force pulse. A Fourier transform of ideal versions of the exponential and half sinusoidal-like force pulse will provide their spectrum. In particular, a sinusoidal half pulse of duration \( t_1 \) has a wider bandwidth than the exponential decaying pulse of duration \( t_2 > t_1 \). A greater high frequency content in the force pulse could result in motion characterized by more abrupt changes.

Changes in the force pulse, due to changes in any sphere or plate parameter, will change the plate motion as measured at the impact site. In the case of the same sphere being dropped onto the same plate, it is the impact velocity of the sphere that is the most likely parameter to change. The other parameters, such as the density, can for practical purposes, be considered constant.

1.5.3 The propagation of the initial disturbance

The pool of water analogy showed that the wave crests decrease in amplitude and widen as they propagate outward. Zener indicated that dispersion causes a disturbance to change its shape as it propagates. Both these observations suggest that factors that change the shape of the propagating disturbance result in the measurement of the plate motion, at particular position, to be dependent on the relative distance between the impact and measurement point. Therefore, it is important to hypothesize how the characteristics, which describe the shape of the disturbance, change during propagation. This is achieved by considering the influence that factors, such as dispersion, have on a propagating disturbance. To simplify the discussion, it is assumed that reflections are not interfering with the propagating disturbance.

Before the shape of the disturbance is discussed, it is important to distinguish between the two representations used to describe a disturbance. A disturbance is shown either as a function of distance \( x \) or as a function of time \( t \). When the disturbance is shown as a function of distance, it is at a specific instant in time. Alternately, when the disturbance is shown as a function of time, it is at a specified point in space. It is convenient to select the latter representation when discussing the motion of the
plate measured by a transducer and the former representation when discussing the effects of dispersion.

When the disturbance is shown as a function of time, the two characteristics used to describe its shape are the magnitude and the time rate of change, or slope with respect to time, of the plate motion at a particular instant. When the disturbance is shown as a function of distance, the same two characteristics are used to describe the shape except that the rate of change, or slope is with respect to distance at a specific location. The slope of the motion in either case is also referred to as abruptness of the motion. Without an analytical bases, the change in these characteristic will be described qualitatively as either increasing or decreasing relative to a previous comparison. The effect of dispersion on the disturbance is now considered.

It was shown in the taut string analogy that a disturbance, shown as a function of distance, can be considered a unique summation of sinusoids having a specific set of amplitudes and phase relationships. Changing either or both the magnitudes and phase relationships of the sinusoids causes the shape of the disturbance to change. Dispersion results when the various frequency components or sinusoids travel at different phase speeds. This results in a continually changing phase relationship that consequently changes the shape of the disturbance [24, 25, 26].

The possible effect that dispersion has on the shape of a disturbance is illustrated with a simple example. Consider Figure 1.2 where over the region $0 < z < 1$ and at time $t = t_0 = 0$, two sinusoids having equal amplitudes are shown. The results of dispersion at a time $t = t_1 > 0$ is shown in Figure 1.3 where sinusoid 2, having the faster phase speed, has a constant phase point moving a greater distance than that of sinusoid 1. A hypothetical disturbance, consisting of the sum of both sinusoids, identified with a solid line in the figures, has changed its shape from that shown at $t_0$ to that given at $t_1$.

Consider the shape of the disturbance over the region $0 < x < 0.5$, and at $t_0$ and at $t_1$. The slopes and magnitudes of the disturbance at time $t_1$ have decreased relative
Figure 1.2: The state at $t = 0$ of a hypothetical disturbance consisting of the sum of sinusoids.

Figure 1.3: The effect of dispersion at a time $t > 0$ of a hypothetical disturbance consisting of the sum of two sinusoids.
to those at \( t_0 \). A decrease in the magnitude and slopes also exists if the disturbance is examined as a function of time. This is realized by noting that the only difference between the two representations of the disturbance is that the summation of sinusoids is given over time or distance.

The reduction in magnitudes and slopes is true only during particular intervals of time and over appropriate regions in \( x \). This is seen by noting that the relative phase between the two sinusoids eventually returns to that given at \( t_0 \). For a disturbance that is created by the impact of a particle, this time interval is likely to begin after the impact and be of a short duration. A short time interval also implies that the location of the appropriate region is near the impact site. If these assumptions are true, then immediately after the impact and within regions near the impact site, the disturbance will exhibit decreases in magnitude and slope as time advances, or equivalently, as the disturbance propagates. Specifically, the magnitude and slope of the disturbance, examined as a function of time, is expected to decrease as an impact site increases its distance from a fixed measurement point.

It is clear that the phase, the amplitudes, and the phase speeds of the sinusoids forming the disturbance are factors that determine how the shape of the disturbance changes as it propagates. For example, a disturbance will change its shape more slowly if it consists of sinusoids having small differences in phase speeds than if it consisted of sinusoids having large differences in phase speed. This result is seen by noting that a longer period of time is needed to form a significant change in the phase relationships between sinusoids having smaller differences in phase speed.

A disturbance consisting of sinusoids with small differences in phase speed can also be described as having a narrow bandwidth. Bandwidth refers to the range of frequencies were sinusoids of significant magnitude exist.\(^6\) This assumes that the phase speed increases or decreases with frequency. In either case, if the difference

\(^6\)References to bandwidth in communication usage typically define the frequencies were the magnitudes are no less than 3 dB or 6 dB of a maximum [20, 21] The 3 dB or 6 dB limitation on the magnitudes is not appropriate for this thesis and a precise measure of bandwidth is provided later.
in phase between sinusoids at the lowest and highest frequency of one disturbance is smaller than that of another disturbance, then the former disturbance has a narrower bandwidth than the latter disturbance. Therefore, a decrease in the magnitude and the slope of a disturbance, due to an increase in the distance between the impact and measurement point, is less noticeable as the bandwidth of the disturbance decreases. This is a direct result of the observation that a disturbance travels a greater distance with smaller changes in shape as its bandwidth decreases.

The bandwidth is obtained from the spectrum of the disturbance. To calculate the spectrum with complete certainty requires measuring the disturbance for an infinite time, or at least until the motion repeats itself. If this measurement is not possible, then the spectrum becomes an estimate that improves as the time interval of the measurement increases [22, 27]. Since short measurement intervals and no reflections are assumed, the spectrum is always an estimate.

An estimate of the disturbance's spectrum is obtained using the motion hypothesized in section 1.5.2. This motion resembles an exponential of the form $1 - e^{-t}$ and becomes more abrupt as the particle size decreases. A Fourier transform in the limit of $1 - e^{-t}$ and an ideal step function shows that the bandwidth of the step-like function is wider than that of the exponential function [22, 23]. Therefore, impacts of smaller particles increase the bandwidth and decrease the magnitude of the disturbance.

This estimate of the spectrum will change as the distance between the measurement and impact point increases. This assumes no interference from reflections, that the magnitudes and abruptness of the disturbance decreases as the distance between the measurement and impact point increases, and that the measurement time is short. The estimate of the spectrum then exhibits a reduction in magnitude and a narrower bandwidth. The reduction in the magnitude of the spectrum is due directly to the reduced magnitudes of the measured disturbance and the narrowing of the bandwidth is the result of less abrupt motion in the measured disturbance which reduces the high frequency spectral estimates.
Another factor that can change the shape of the disturbance is attenuation which results when the energy of vibration is dissipated, for example as heat, into the material itself or into the surrounding medium [24, 25]. Since a disturbance can be viewed as the summation of sinusoids, changes in the amplitudes will change the shape of the disturbance. If the sinusoids are attenuated by the same proportion, then the disturbance is simply scaled in magnitude. If the attenuation varies with each sinusoids, or equivalently, is dependent on frequency, then estimating the change in shape requires knowing this dependency.

Typically, attenuation increases with frequency and is dependent on the material, the medium, and the boundary conditions [22, 23]. The change in the shape of the disturbance, due to attenuation, may be small relative to the change caused by dispersion, if short measurement intervals are used or if the bandwidth of the disturbance is narrow. Shorter measurement intervals, implying smaller distances of propagation, directly result in less attenuation. Decreasing the bandwidth of a disturbance produces smaller differences in attenuation between the lowest and highest frequencies of the disturbance, thereby reducing the change in shape.

Summarizing, the shape of a propagating disturbance changes when the phase relationships or amplitudes of the sinusoids forming the disturbance are changed. Dispersion changes the phase relationship and attenuation reduces the magnitude of these sinusoids. Assuming that small intervals of time or small propagation distances are used to measure the disturbance, then it is expected that as the distance between the impact and measurement points increases, dispersion and attenuation will reduce the magnitude and slope of the disturbance. It is assumed that the effects of attenuation are small relative to that of dispersion. Decreasing the bandwidth of the disturbance is expected to reduce the influence of both dispersion and attenuation.

In terms of affecting the resulting sensor output, it is clear that changes in the distance between transducer and impact site affect the shape of the disturbance measured by the transducer. This consequently changes the estimate of its spectrum.
1.5.4 The transducer influence

It is convenient to use the frequency domain representation, given by Equation (1.10), to examine the influence of the transducer. The following discussions assume that the transducer has negligible effect on the motion of the plate.

Assume that a significant response of the transducer's transfer function occurs over a bandwidth $B_t$ defined over the frequencies given by $\omega_t \leq \omega \leq \omega_{th}$. Assume also that the bandwidth of the disturbance $B_s$ is defined over the frequencies given by $\omega_s \leq \omega \leq \omega_{sh}$. The frequencies, $\omega_t$, $\omega_{th}$, $\omega_s$, and $\omega_{sh}$, determine if the spectrum of $V_o(\omega)$ resembles either $H_t(\omega)$ or $Z_t(\omega)$. Three simple cases are considered. Firstly, if $B_s$ is wide relative to $B_t$ and also encompasses $B_t$, or equivalently, if $\omega_t \leq \omega_{th} \leq \omega_{sh}$, then $V_o(\omega)$ will resemble $H_t(\omega)$. Secondly, if $B_s$ is narrow relative to $B_t$ and $B_t$ encompasses $B_s$, or equivalently, if $\omega_{th} \leq \omega_{sh} \leq \omega_t$, then $V_o(\omega)$ will resemble $Z_t(\omega)$. Finally, regardless of whether $B_s$ is wider or narrower than $B_t$, if they do not overlap, or equivalently, if $\omega_{sh} < \omega_t$ or if $\omega_{th} < \omega_s$, then assigning a resemblance to $V_o(\omega)$ is meaningless. In this case, $V_o(\omega)$, being the product of $H_t(\omega)$ and $Z_t(\omega)$, could be the result of either one or both of $H_t(\omega)$ and $Z_t(\omega)$ having insignificant values or values corrupted with noise. The more difficult cases involving an overlap, to some degree, of $B_t$, or $B_s$ are assumed to result in $V_o(\omega)$ resembling, more or less, one of the three previous cases. The strength of the resemblance to any one of the three cases depends on the degree of overlap.

Using the three previous cases, a change in $V_o(\omega)$ due to a change in $Z_t(\omega)$ can be estimated. A change in $Z_t(\omega)$ or $V_o(\omega)$ is defined to be a change in bandwidth only. Two simple cases are considered. Firstly, assume that either one or both of the conditions $\omega_{th} \leq \omega_t$ and $\omega_{sh} \leq \omega_{sh}$ are true, then a change in one or both of $\omega_{th}$ and $\omega_{sh}$, such that previous conditions remain as they were, will result in a no change in the bandwidth of $V_o(\omega)$. Secondly, assume that one or both of the conditions $\omega_t < \omega_{th}$ and $\omega_{sh} < \omega_{sh}$ are true, then a change in one or both of $\omega_{th}$ and $\omega_{sh}$, such that previous conditions remain as they were, will always change the bandwidth of $V_o(\omega)$. 
1.5.5 Analysis Summary

The operation of the sensor is described using Equation (1.7) or Equation (1.8). The motion of the plate at the impact epi-centre, resembles the exponential function $1 - e^{-t}$. It was also shown that as the particle radius decreases the disturbance becomes more abrupt or step-like and is associated with an increase in the bandwidth of the disturbance. From all the known sphere and plate parameters influencing the force of impact, it is the particle impact velocity that is the most likely to vary.

The influence of dispersion and attenuation change the shape of this initial disturbance as it propagation outwards. It is expected that over small intervals of time and distances near the impact site, that the magnitude and slope of the disturbance will decrease as the distance between the measurement and impact points increases. Therefore, changes in the impact location are the most probable cause of changes in the measured disturbance and in the estimate of its spectrum. Finally, the response of the transducer will be a factor in determining whether or not particular changes in $Z_i(\omega)$ are measured in $V_\omega(\omega)$.

1.6 Proposed Experimental and Theoretical Analysis

1.6.1 Experimental Considerations

To demonstrate the behaviour predicted by the previous analysis, the experiments should attempt to meet the underlining assumptions of the analysis and facilitate the development of a detailed examination. This places constraints on the experiments. To insure that the force pulse is of the form given by Zener, the material parameters of the plate and sphere are required and the lateral dimensions of the plate must be large enough to prevent reflections from returning until after the completion of the impact. To simplify the solution to the thin plate equations, simple geometric shapes and easily described boundary conditions are needed. Finally, the same transducer used in the original grain loss monitor sensor is required.
To meet these constraints the experiments uses steel spheres impacting onto a large rectangular simply supported aluminium plate. The meaning of a simply supported plate is defined later. The material parameters required in Zener’s analysis are commonly given for most metals such as steel and aluminium and are obtained from texts [28]. Additional analysis is needed to estimate the size of the plate and radii of the spheres to insure that reflections do not return until after the completion of the impact. This requires approximate values for the force pulse duration and propagation speed of the disturbance. Finally, the transducer is mounted in the centre of the plate so that it is as far from all boundaries as possible to reduce the possibility of returning reflections. It is also mounted in the same manner as it was on the sensor in the grain loss monitor.

To study the effects of changes in the impact locations on the transducer output, the steel spheres are dropped at various locations on the plate that include the transducer site. To determine the effects of changes in the force pulse, spheres of various radii are used. Varying the radius of a sphere is easier to control than changing the impact velocity. Changes in radius provide a greater change in the form or shape of the force pulse than the same proportional change in the velocity, as indicated by Equation (1.4). In addition, varying the radii of the spheres provides insight into the particle sizing capability of the sensor. Therefore, the particle impact velocities are kept constant.

From this data, the magnitudes and slopes of the sensor output are examined for changes due to variations in the impact location and particle radius. An estimate of spectrum of the sensor signal is also calculated. Finally, the sensitivity of the peak sensor signal to changes in the impact location is determined by plotting the peak signal versus impact location and examining the slopes of the resulting curves. Similarly, the sensitivity of the peak sensor signal to changes in the particle radius, is examined by plotting the peak signal versus particle radius and examining the slopes of the resulting curves.
1.6.2 Detailed Analysis Considerations

The reasons for performing a detailed analysis are to provide additional insight into the operation of the sensor, to determine if the analysis can predict the results of the experiments, and to attempt to extend the experimental results to the sensor of the grain loss monitor. Equation (1.7) or Equation (1.8), are used to direct detailed analysis. Therefore, the terms $F_a(\omega)$, $H_p(\omega)$, and $H_t(\omega)$ in Equation (1.8) are examined separately and then as complete system, using numeric simulations.

The analysis required for each term differs because the extent of previous work varies greatly. The force of impact has been developed by Zener [18]. Additional work is needed to describe the changes in $f_a(t)$ and $F_a(\omega)$ given changes in the radius of the particle. The analysis of $H_p(\omega)$ requires solving the thin plate equation using the configuration of the experimental setup. Similar solutions have been demonstrated in a number of texts but an unconventional frequency domain representation is needed to fit into the requirements of Equation (1.8). Finally, the analysis of $H_t(\omega)$ requires the most work as no previous results are available. Two approaches are used to understand the characteristics of the transducer; the first is to measure its magnitude and phase response over a range of frequencies and the second is to attempt to develop a model of the transducer that includes only the simplest of approximations.

The analysis of the sensor model is done using numerical simulation and the results of both the simulation and experiments are compared. In particular, comparisons are made between the predicted and experimental sensor outputs in both the time and frequency domain. In addition, the simulated peak sensor signals as a function of impact location and particle radius are also examined. Two aspect of the simulations that can not be compared to experiments, are the form of the force pulse and the plate motion at the transducer location. If the model of the transducer is too simplistic and inaccurate, the simulation of $V_a(\omega)$ may not be revealing.
1.6.3 Additional Considerations

The use of experiments and a detailed analysis can suggest the cause of the signal variation in the grain loss monitor sensor but it is not conclusive proof since this sensor was not the centre of the experiments or analysis. A number of reasons are given for this. Firstly, the intent of the experiments and detailed analysis is to validate the results of the initial analysis and to provide a basis to develop more complex configurations, of which the grain loss monitor sensor is one. Secondly, going from the initial analysis to examine the grain loss monitor sensor is of questionable value at this point. The initial analysis uses a number of simplifying assumptions for the plate and the spheres. These assumptions may not be satisfied by the grain loss monitor sensor. The parameters needed for Zener’s analysis are available for metal plates but could be of questionable use in a plastic plate. Also, the dimension of the plastic plate may allow reflections to return before the impact is completed. Further, the shape of this plastic plate is not a simple rectangle and the boundary conditions are possibly too difficult to specify.

Another approach to determine the cause of the signal variations is to measure the force pulse and the plate motion and to compare these to results to a detailed analysis. Similarly, experiments and analysis could also be performed on the transducer alone and the transducer mounted on the plate. The difficulty in using this approach at this time is that it assumes a well developed understanding of the basic processes that have not yet been demonstrated. The proposed experiments and analysis attempt to provide this basis.

1.7 Thesis Organization

The remainder of this thesis is presented in five chapters. Chapter 2 presents the results of experiments and analysis of the data as outlined in Chapter 1. The results are presented immediately for three reasons. Firstly, it represents the chronological order of the investigation. Secondly, it provides immediate confirmation
of the preliminary analysis given in Chapter 1. Finally, it provides results that can be used to understand the detailed theoretical analysis. Chapter 3 presents a detailed theoretical analysis of an ideal sensor. It develops the thin plate model, the force model, and the transducer model used in the sensor system. The behaviour of each element, as impact location and particle radius are varied, is examined. Chapter 3 may be read prior to Chapter 2, if desired, without loss of continuity. Chapter 4 presents the results of numeric simulations of the sensor system and examines its output as the impact location and particle radius is varied. These results are then compared to the experimental results. Chapter 5 discusses the significance of errors and the consequence of the assumptions used in the experiments and theoretical analysis. It then extends the experimental results, using the theoretical analysis, to the sensor in the grain loss monitor. Finally, it suggests additional analysis and experimental work. Chapter 6 concludes the thesis by summarizing the results reached in the experimental and theoretical work. A number of appendices are provided for the derivation of model equations and discussions on related topics.

1.8 Thesis Objectives

The primary objective of this thesis is to provide experimental and theoretical evidence, using the thin plate impact sensor, that suggests the cause and extent of the observed signal variation in using the sensor of the grain loss monitor to size potash particles. A secondary objective is to provide a theoretical basis to describe and model the processes involved in the operation of an ideal sensor which can be used to develop more complex models.
2. Experimental Results and Analysis

2.1 Overview

This chapter presents the results of a set of experiments using a thin plate impact sensor that is a simplified version of the sensor used in the grain loss monitor. The objectives of these experiments and the analysis are to measure the variation in the peak sensor output, given changes in particle radius and impact location, and to determine if the corresponding sensor signal spectra changes as predicted in Chapter 1.

The experimental setup used for these experiments is discussed first. This is followed by the results of measuring the sensor output for various particle radii and impact locations. Next, an analysis of these results examines the peak sensor signal as a function of particle radius and impact location. The spectra of the sensor signal, given the previous changes in radius and location, is then presented. Finally, a summary of the results and analysis is given.

2.2 Experimental Setup and Measurements

Figure 2.1 shows a diagram of the experimental setup used to measure and collect the data generated by particles impacting onto the thin plate. A large aluminium plate has the piezo-electric transducer mounted in the centre and underside of the plate. The magnetic-holding mechanism allows the height and impact position of the steel spheres to be controlled. The piezo-electric transducer’s output is fed directly into a digital storage oscilloscope that allows the stored transducer waveforms to be sent to a computer workstation as a data file for later processing.
Figure 2.1: A simplified diagram of the experimental setup used to collect the data produced by particles impacting upon a thin plate

The lateral dimensions of the plate needed to be large enough to prevent reflections from interfering with the force of impact. This required knowing the fastest propagation speed of the disturbance and the longest duration of any impact. There are two difficulties in determining these quantities. Firstly, the duration of the force pulse is dependent on the plate and sphere parameters which is obtained by solving Zener's equations describing the relative motion between the sphere and plate. Secondly, the propagation velocity is a function of frequency that requires the spectrum of the force pulse to be examined for the most significant highest frequency components.

Instead of performing a detailed calculation to determine accurate values for these quantities, an approximation is used. This approximation is based on spheres impacting onto a massively thick plate to obtain the duration of impact [26] and in assuming that the propagation velocity is independent of frequency. To determine the duration of the impact, the radius of the spheres and impact velocity were chosen on the bases of convenience and availability. Therefore, six steel spherical particles with radii varying from 1.191 to 3.175 mm, (3/64 to 8/64 inches, in increments of 1/64 inches), were chosen and a drop height of 10 cm was used resulting in an impact
velocity of about 1.4 m/s. Using the previous approximations, and as shown in Appendix C, the duration of impact for the largest particle is then 10 \( \mu s \). With the assumed propagation velocity of 6300 m/s, which is typical for bulk aluminium [25], a propagation distance of approximately 6.3 cm results. Allowing for errors, the lateral dimensions of the plate were then set to eight times this distance or about 0.5 m. The thickness of the plate was chosen on the basis of availability of material resulting in the plate's final dimension being (53 X 47 X 0.079) cm. Appendix K presents the measured dimensions and material characteristics of the plate.

The plate was mounted along its edges in a manner that approximated a simply supported boundary condition. As shown in Figure 2.2(a), an ideal simply supported boundary does not allow transverse motion but does allow rotation of the member. This condition was approximated by firmly pressing spherically rounded clamps along the edge of the plate as shown in Figure 2.2(b).

In operation, the location of the impact site and the drop height of the particle are set using the particle holding mechanism. The digital storage oscilloscope is set to record a maximum of 500 samples with 8 bits of resolution over a duration of 500 \( \mu s \) resulting in a sampling frequency of 1 MHz. When the transducer's output exceeds a preset threshold or trigger level, the storage oscilloscope begins to capture
50 pre-trigger and 450 post-trigger sample points. After recording the transducer output waveform, it is transferred to the computer workstation as a data file.

The drop height of 10 cm produced transducer signals of appreciable magnitude for all the particles. An axis with 1 cm reference marks was drawn on the surface of the plate parallel to an edge such that the origin coincided with the location of the transducer mounted on the underside. This axis is used to localize the particle impact sites. Figure 2.3 shows the relative location of the plate’s coordinate system and the drawn reference axis.

The steel spherical particles were dropped a number of times with the holding mechanism at two locations referred to as site A and site B. These sites are located near the 0 cm axis mark drawn on the plate. Additional drops were made at approximately -1 cm, -2 cm, and -3 cm from impact site B. Figure 2.4 shows the regions that are considered to be the location of sites A and B. Appendix K provides the measured coordinates of all the impact locations, the reference axis, and the transducer location.

2.3 Experimental Results

2.3.1 Sensor Output due to Impacts at the Transducer Location

This section presents a set of the results obtained from dropping each steel sphere ten times at site A. The raw transducer signal is examined for behaviour that was hypothesized in Chapter 1 and for characteristics that are unusual or appear significant.

The proposed behaviour of the sensor given in Chapter 1 indicated that the sensor’s output can be expressed as

$$v_o(t) = z_i(t) \ast h_i(t)$$ \hspace{1cm} (2.1)

or in the frequency domain as

$$V_o(\omega) = Z_i(\omega)H_i(\omega)$$ \hspace{1cm} (2.2)
Figure 2.3: The approximate dimensions of the plate, the coordinate system and the location of the reference axis used to localize particle impact sites. Drawing is not to scale.

Figure 2.4: The regions indicating the approximate locations of the transducer, impact site A, and impact site B.
Chapter 1 hypothesized that $z_i(t)$ would resemble the function $1 - e^{-t}$. The response of the transducer\(^1\) to this motion is expected to result in $v_o(t)$ resembling the response of a damped resonant system \(^{[29]}\). With the impact of larger particles reducing the abruptness of this motion, the bandwidth of the estimate of $Z_i(\omega)$ is expected to narrow. This should result in $v_o(t)$ exhibiting decreased abruptness. The magnitudes of $v_o(t)$ or $V_o(\omega)$ are also expected to increase as the particles radius is increased.

Figure 2.5 shows a set of sensor signals due to ten impacts of the largest sphere. The expected damped sinusoidal response,\(^2\) decaying to negligible levels after about 400 $\mu$s, is present.

The differences between the signals from consecutive impacts are attributed to small changes in the impact location caused by the holding mechanism. Though the location of the impact sites appeared to be identical, careful observation revealed that

\(^1\)Appendix A presents the results of measuring the magnitude and phase response of the transducer

\(^2\)The response of the sensor signal resembles the impulse response of an under-damped resonant system \(^{[29]}\). For convenience this type of response is referred to as a damped sinusoidal response.
changes of approximately ±1 mm were occurring. This change in impact position is also responsible for the two different but distinct waveforms that appear between 100 μs and 200 μs. This is attributed to the holding device releasing the particles from two slightly different positions. Whenever the particle was released by the holding device from the same position, the second waveform was not seen. In this case, the variation in the impact site was estimated to be on the order of ±0.5 mm.

The ensemble average or the mean and the standard deviations of the signals are shown in Figure 2.6. The standard deviation near the peak at about 10 μs is of particular importance as it represents the variation of the peak signal reported for the grain loss monitor sensor.

To examine this variation, the standard deviation as a percentage of the mean signal near the peak of the ensemble average is shown in Figure 2.7. The expanded

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Section 5.3.1 discusses the method used to locate the particle impact sites.

Reference to the peak value or peak of the ensemble average of the sensor output implies the first positive peak of the sensor signal unless explicitly stated otherwise.
Figure 2.7: The standard deviation of the samples near the maximum mean peak sensor signal shown as a percentage of the mean sensor signal from the impacts of a 3.175 mm radius particle at impact site A.

The time scale in this figure shows the standard deviation of the signal, as a percentage of the mean, at each 1 μs time sample. The vertical set of symbolic stars at 11 μs indicates the time sample locating the peak of the ensemble average and shows that a standard deviation of 2.5% of the mean exists at this point.

The sensor signals resulting from the impacts of the smaller particles all resembled the sensor signals shown in Figure 2.5. Specific differences included a reduction in the magnitudes and an increase in the abruptness of the signal. Both these characteristics were predicted in Chapter 1.

To illustrate these characteristics, the signals due to the impacts of the smallest particle are shown in Figure 2.8. The reduced magnitudes, the increased abruptness, and the appearance of two waveforms between 25 μs and 100 μs are clearly apparent. Again, the second waveform is attributed to the holding device having two slightly different release positions for the particle. The variation in the impact location for
the smallest particle was also estimated to be about ±2 mm.\textsuperscript{5} Whenever the particle was released from the same position, the second waveform was not observed and the variation in the impact site was estimated to be on the order of ±0.5 mm.

The ensemble average or the mean and the standard deviations of the signal are shown in Figure 2.9. Again, the standard deviation of the signal near the peak at about 10 $\mu$s is of particular importance as it represents the variation of the peak signal reported for the grain loss monitor sensor. Therefore, the standard deviation as a percentage of the mean near the peak value is shown in Figure 2.10. In this case, the standard deviation has increased to 8% of the mean at peak of the ensemble average which is located at the 7 $\mu$s time sample. The majority of the increase in the standard deviation is possibly due to an increased variation in the impact position observed for the smallest particle.

The increased variation in the impact location for the smallest particle and the corresponding increase in the standard deviation of the peak signal, suggests that the\textsuperscript{5}The variation in the impact location for all the other particles was approximately ±1 mm.
Figure 2.9: The mean and standard deviation about the mean sensor output from the impacts of a 1.191 mm radius particle at impact site A.

Figure 2.10: The standard deviation of the samples near the maximum mean peak sensor signal shown as a percentage of the mean sensor signal from the impacts of a 1.191 mm radius particle at impact site A.
peak sensor signal is sensitive to changes in the impact location. The next section presents the results of dropping the particles at locations other than impact site A and with smaller variations in positions.

2.3.2 Sensor Output due to Impacts at Other Locations

With the sensor output exhibiting sensitivity to the impact location, a set of impact positions were chosen to have changes greater than the ±2 mm maximum variation observed at impact site A. In addition, the particles were carefully dropped to reduce the variation about any impact site to an estimated ±0.5 mm. The impact locations were selected to be near the 0 cm, -1 cm, -2 cm, and -3 cm marks of the reference axis, as shown in Figure 2.3 and Figure 2.4. These experiments were carried out after the experiments presented in section 2.3.1 and required the experimental setup to be reassembled. Repositioning of the particle holding device resulted in the impacts near the 0 cm mark to be at a location slightly different than that of impact site A. This location is referred to as impact site B and all impact sites for these experiments are referenced from this point.

The sensor signals, due to impacts near impact site B, are expected to resemble those seen at impact site A. With impact site B being approximately 1.5 mm further away from the transducer than impact site A, the magnitude and abruptness of the signals are expected to be reduced. Figure 2.11 shows the result of three impacts of a 3.175 mm particle at impact site B. The magnitude of the peak sensor signal is more than 30 volts less than the peak measured at impact site A. The abruptness of the signal is difficult to assess but it is shown later that the spectra of this signal does exhibit a reduced high frequency content.

Figure 2.12 shows the results of three impacts of a 1.191 mm particle at impact site B. It is evident that the peak of the signals, at about 17 volts is within the range of the peaks obtained at impact site A. With the assumed increase in distance between impact site B and transducer, a decrease in the magnitude of the signals was expected. It is possible that the large variation of the impact locations for the
Figure 2.11: The sensor output due to the impact of a 3.175 mm particle at impact site B.

The smallest particle at impact site A, resulted in the actual distance between its impact site and the transducer to be at least equal to the distances involved in the impacts at site B. This would produce signals of comparable magnitudes.

Also apparent in Figure 2.11 and Figure 2.12 is the reduction in the differences between the signals and the absence of the second waveform. This is the result of procedural changes made in response to the earlier experiments. By carefully operating the holding mechanism, the release of the particles from a second holding position was eliminated. In combination with reducing the motion of the holding mechanism during release of the particles, the variation in the impact location was reduced and estimated to be on the order of ±0.5 mm.

The standard deviation of the signals, as a percentage of the mean, near the peak of the ensemble average are shown in Figure 2.13 and Figure 2.14 for the largest and smallest particles, respectively. It is evident that the standard deviation is less than that measured at site A and has been reduced by a factor of four and two, for the smallest and largest particle, respectively. This reduction and the corresponding
observed reduction in the impact location, further substantiates that the impact location is a significant factor influencing the peak magnitude of the signal.

The discussions given in Chapter 1, indicated that a reduction in the magnitude and abruptness of the signal should be apparent as distance between the impact and measurement point increases. Figure 2.15 shows one signal caused by the largest particle impacting at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B. The change in the signal shape is significant and the expected reduction in the magnitude and the abruptness of the signal is present. An interesting feature of these signals, as the impacts move further away from the transducer, is the broadening of the main peak which reduces its definition. This is probably due to dispersion of the disturbance resulting in the transducer measuring a less abrupt change in the plate motion. Another interesting feature, as the impacts move further away from the transducer, is the tendency to have an increasing amplitude sinusoidal waveform preceding the peak of the signal, which also appears to occur later. Again, this behaviour could be interpreted as the result of dispersion and the propagation delay of the disturbance.

Figure 2.12: The sensor output due to the impact of a 1.191 mm particle at impact site B.
Figure 2.13: The standard deviation of the samples near the maximum mean peak sensor signal shown as a percentage of the mean sensor signal from the impacts of a 3.175 mm radius particle at impact site B.

Figure 2.14: The standard deviation of the samples near the maximum mean peak sensor signal shown as a percentage of the mean sensor signal from the impacts of a 1.191 mm radius particle at impact site B.
Figure 2.15: The sensor output due to the impact of a 3.175 mm radius particle at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B.

The signal due to the impacts of the smallest particle at 0 cm, -1 cm, -2 cm, and -3 cm marks from impact site B are shown in Figure 2.16. The expected reduction in the magnitudes and abruptness of the signal as the impacts move further away from the transducer is present. In addition, the signals have, as expected for a decrease in radius, an increase in the abruptness of the signals. These characteristics were predicted in Chapter 1. The signals, due to the other particles impacting at these locations, resembled the signals presented in this section.

2.4 Analysis of Results

The peak sensor signal is examined as a function of particle radius and distance from the transducer in this section. The spectra of the sensor output due to changes in particle radius and impact location are also presented.

6The continuous curves shown in all the figures in this section were obtained by applying a cubic spline, consisting of approximately 200 points, to the small set of data points. These curves suggest one possible form of the intermediate values or provide a means to distinguish one set of data from another.
2.4.1 Sensor Output as a Function of Particle Radius

The measured peak sensor signals, as a function of radius at impact sites A and B, are shown in Figure 2.17 and Figure 2.18, respectively. The curves representing the mean and the standard deviations about the mean for these data are also given. The figures reveal that the relationship between the peak signal and particle radius is almost linear at both impact sites. This suggests that the peak sensor signal from this particular sensor configuration provides an indication of particle radius.

The standard deviation of the peak signal as a percentage of the mean is shown in Figure 2.19. The reduction, from site A to site B, in the standard deviation of the peak signals is apparent. The reasons for this reduction were discussed in section 2.3.1.

Figure 2.19 also provides evidence that the standard deviation of the peak signal is proportional to changes in the impact location. From section 2.3.1 and section 2.3.2, the change in the impact positions at site A were estimated at ±1 mm for all but the smallest particle which was twice this estimate or about ±2 mm. The data
Figure 2.17: The peak sensor output as a function of particle radius due to impacts at impact site A.

Figure 2.18: The peak sensor output as a function of particle radius due to impacts at impact site B.
The standard deviation of the peak sensor output as a percentage of the mean shown as a function of particle radius due to impacts at sites A and B. Shown in Figure 2.19 for these impacts also show the standard deviation of the peak signal for the smallest particle is approximately twice that of the other particles. In addition, the changes in the locations for the impacts at site B were all estimated to be near ±0.5 mm. The variation in the impact location for all the particles, except the smallest, is about one half of that measured at impact site A. The variation in the impact location of the smallest particle has been reduced by a factor of four. Figure 2.19 shows that the standard deviation of the peak signal for the impacts at site B has been reduced by about a factor of four for the smallest particle and by a factor or two for the other particles. This observation suggests that a reduction in peak signal variation can be achieved by reducing the changes in impact position.

Another characteristic of the peak sensor signal, shown in Figure 2.17 and Figure 2.18, is the change in the peak magnitudes. Considering that a difference of about 1.5 mm exists between the centre of site A and B, the change in the magnitude of the peak signal is significant for the largest particle and becomes less significant for
the smaller particles. Specifically, the change in the mean peak signal values from site A to B are 30, 26, 23, 11, 6, and -0.5 volts, for the largest to smallest particles respectively. As discussed in section 2.3.2, this could be the result of the smaller particles, directed at site A and site B, both having approximately the same distance between the impact site and the transducer. This is a possibility for the smallest particle since it was observed to have a ±2 mm variation in impact position but it is more difficult to apply this explanation to particles having smaller changes in impact positions. A more likely explanations follows.

Another explanation for this behaviour is based on the discussions in Chapter 1 regarding the change in the spectral content of the disturbance and the expected change in the output spectrum of the sensor. Chapter 1 hypothesized that impacts of smaller particles produce more step-like disturbance having a spectra $Z_t(\omega)$ with a wider bandwidth than the spectra produced by the impacts of larger particles. Considering the resonant response of the transducer, it is possible that much of the change in the bandwidth of the spectra $Z_t(\omega)$ for the smaller particles, occurs outside the resonant response of the transducer. Therefore, small changes in the particle radius for small particles result in insignificant changes in the sensor output spectra $V_0(\omega)$. As the particle radius is increased, more of the changes or reduction in the bandwidth of the spectra $Z_t(\omega)$ occurs nearer or within the transducer's resonant response. This results in larger changes in the sensor output spectra $V_0(\omega)$. Evidence of this behaviour is shown later when the spectral content of the sensor signal is examined.

The almost linear relationship of the peak sensor signal to radius is also present at the other impact locations tested. Figure 2.20 shows the peak sensor signal as a function of radius at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B. The mean and two standard deviations limits about the mean for this data are also given. The other feature that is apparent as the impact locations move further from the transducer is the reduction in the sensitivity of the peak signal to changes in the radius. This is indicated by the reduction in the slope of the curves which are approximately 20, 12,
6, and 4 volts per millimetre of radius for the impact locations nearest to and furthest from the transducer, respectively. These slope values are important in determining a particle radius given a peak signal value which is discussed in the next section.

The standard deviation as a percentage of the mean for the data shown in Figure 2.20 are shown in Figure 2.21. Again, the data show that the standard deviations for each particle at any one impact location are about equal. One feature that appears significant is that the standard deviations at -2 cm from site B are less than the standard deviations at the other locations. This feature is examined more carefully in the next section.

As a final note, it is difficult to explain the linear-like relationship between the peak signal and particle radius without knowing the dynamics of the plate motion and the measurement the transducer is making. Arguments based only on the energy of both the impacting particle and the plate vibration do not provide an adequate

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Footnote: These slopes were obtained by placing a straight edge against the curve until it appears to represent, qualitatively, a good approximation. Error in these slope values can easily approach ±2 volts.
Figure 2.21: The standard deviation of the peak sensor output as a percentage of the mean shown as a function of particle radius due to impacts at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B.

This inadequacy is easily shown by assuming that the energy of a vibrating elemental volume of the plate is proportional to both the square of the velocity of that volume and to the energy of the impacting particle. With the energy of the particles being proportional to the cube of its radius, the velocity of an elemental volume of the vibrating plate becomes proportional to the radius raised to the $3/2$ power. Therefore, to arrive at the linear relationship between the peak signal and particle radius requires additional assumptions that are not obvious.

2.4.2 Sensor Output as a Function of Distance

The data from section 2.4.1 are given as a function of distance from impact site B and are shown in Figure 2.22 and Figure 2.23. It is difficult to gauge the magnitude of the standard deviation in the peak signal from these figures. Therefore, Figure 2.24 and Figure 2.25 show the standard deviation as a percentage of the mean given as a function of distance from impact site B.\textsuperscript{8}

\textsuperscript{8}The actual measured distances from impact site B are used to plot these curves.
Figure 2.22: The peak sensor output as a function of distance from impact site B for particle radii of 3.175 mm, 2.778 mm, and 2.381 mm.

Figure 2.23: The peak sensor output as a function of distance from impact site B for particle radii of 1.984 mm, 1.588 mm, and 1.191 mm.
Figure 2.24: The standard deviation as a percentage of the mean peak sensor output as a function of distance from impact site B for particle radii of 3.175 mm, 2.778 mm, and 2.381 mm.

Figure 2.25: The standard deviation as a percentage of the mean peak sensor output as a function of distance from impact site B for particle radii of 1.984 mm, 1.588 mm, and 1.191 mm.
The standard deviations, shown in Figure 2.24 and Figure 2.25, reveal an almost random change in value from one impact location to another. This characteristic can be understood by noting that these standard deviations are based on the results of three impacts and are only estimates. This is apparent when considering the unlikely standard deviation value of 0% associated with the third smallest particle at the 0 cm from impact site B. This value is simply due to the statistical nature of the impacts which in this case resulted in the peak signal having the same quantized value for all three impacts.

Instead of examining the standard deviations, the relationship of the peak sensor signal to changes in the impact location, as shown in Figure 2.22 and Figure 2.23 is considered. A significant feature of any one of the curves in Figure 2.22 and Figure 2.23, is the increase in the slope of the curves as the impact locations move closer to the transducer site. The value of the slope, at any particular impact location, indicates the rate of change of the peak sensor signal with respect to distance. Large slope values are therefore an undesirable attribute since they indicate that a small change in impact position results in large changes in the peak sensor signal. Both Figure 2.22 and Figure 2.23 then suggest that impacts at or near the transducer result in the peak sensor signal being the most sensitive to changes in impact position. Alternately, there also appears to be impact locations were small values of slopes exist. These points represent more desirable impact locations since a small change in impact position would produce a small change in the peak sensor signal.

To more accurately consider the value of these slopes at any impact location, the curves representing the means in Figure 2.22 and Figure 2.23 have been differentiated and the results shown as a function of distance in Figure 2.26.\(^9\) Figure 2.26 reveals

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\(^9\)The slopes were calculated by applying the waveform representing the mean sensor signal to a digital differentiator. The waveform was applied from left to right and offset so that the initial value of the waveform was at zero volts thus preventing large initial transients in the output of the differentiator. The delay through the differentiator prevented obtaining outputs near the -3 cm point. These slopes could be obtained by applying a suitably offset waveform to the differentiator in reverse order or from right to left but this was not done because the value of the slopes near the 3 cm point are not critical and can be easily linearly extrapolated.
that a minimum for all the curves occurs between the -2 cm and the -2.7 cm impact locations. This supports the observation in section 2.4.1 where a minimum in the standard deviations was seen at the -2 cm location.

The relevance of the change in slope of these curves, especially at the minimum and maximum values, is shown explicitly by assuming a small fixed change in the impact position at each of the separate impact locations. These curves are then used to estimate the change in the peak sensor signal due to this fixed change in the impact position. The change in the peak signal can then be related to a change in the radius measure by using the curves relating the peak signal to particle radius.

To begin this estimation, considering the values of the slopes for all the curves at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B. Table 2.1 lists these values for all the particles. In addition, consider the approximate sensitivity of the sensor output to changes in the particle radius $dv_{pk}/dr_e$ at these same locations. From section 2.4.1, these values are listed in Table 2.2. Now assume a hypothetical change in the impact position of $\Delta x = \pm 0.5$ mm about the 0 cm location. From Table 2.1, the slope

Figure 2.26: The rate of change of the mean peak sensor output as a function of distance from impact site B for all particles.
Table 2.1: The sensitivity of the peak sensor signal to changes in impact location at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B.

\[ dv_{pk}/dx \] for this particle at the 0 cm location is 65 V/cm. This results in a peak sensor signal

\[
\Delta v_{pk} = \Delta x \frac{dv_{pk}}{dx}, \quad x = 0 \text{ cm}, \ r_0 = 3.175 \text{ mm}, \quad (2.3)
\]

\[
= (\pm 0.050 \text{ cm})(65 \text{ V/cm}) ,
\]

\[
= \pm 3.3 \text{ V} ,
\]

and assumes that \( dv_{pk}/dx \) is constant over \( \Delta x \). Given this change in the peak sensor signal, the variation in the measure of the particle radius is found by using the slope \( dv_{pk}/dr_0 \) associated at this impact location. From Table 2.2, this results in

\[
\Delta r_0 = \frac{\Delta v_{pk}}{dv_{pk}/dr_0}, \quad x = 0 \text{ cm}, \ r_0 = 3.175 \text{ mm}, \quad (2.4)
\]

\[
= (\pm 3.3 \text{ V})/(20 \text{ V/mm}) ,
\]

\[
= \pm 0.16 \text{ mm} ,
\]

<table>
<thead>
<tr>
<th>Particle Radius (mm)</th>
<th>( x = 0 \text{ cm} ) (V/cm)</th>
<th>( x = -1 \text{ cm} ) (V/cm)</th>
<th>( x = -2 \text{ cm} ) (V/cm)</th>
<th>( x = -3 \text{ cm} ) (V/cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.175</td>
<td>65</td>
<td>24</td>
<td>4.6</td>
<td>3.7</td>
</tr>
<tr>
<td>2.778</td>
<td>54</td>
<td>21</td>
<td>4.8</td>
<td>4.4</td>
</tr>
<tr>
<td>2.381</td>
<td>46</td>
<td>17</td>
<td>5.0</td>
<td>6.3</td>
</tr>
<tr>
<td>1.984</td>
<td>38</td>
<td>14</td>
<td>4.6</td>
<td>8.3</td>
</tr>
<tr>
<td>1.588</td>
<td>28</td>
<td>11</td>
<td>3.6</td>
<td>5.7</td>
</tr>
<tr>
<td>1.191</td>
<td>18</td>
<td>5.8</td>
<td>1.4</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Table 2.2: The approximate sensitivity of the peak sensor signal to changes in particle radius at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B.
and also assumes that \( \frac{dv_{pk}}{dr} \) is constant over \( \Delta r \). This change in radius represents a ±5.1 percent variation in the radius measure for the largest particle. In a similar manner, the variation in the radius measure, due to this same fixed change in the impact position, is calculated for all the other particles at each impact location and are listed in Table 2.3 through to Table 2.6. It is clear that a minimum in the average variation of the peak sensor signal and the radius measure for all the particles occurs near the -2 cm point.

These results indicate that dropping the particles at the transducer location is not the best choice for measuring radius even though it provides the largest output signals. These results also suggest that dropping potash particles at the transducer location on the grain loss monitor sensor could possibly result in a maximum variation of the signal due to small changes in impact position. It is possible, as indicated by these results, that other locations may exhibit less sensitivity to changes in impact position.

The results of this section show that changes in the impact location are responsible for the variation in the peak sensor signal. It has also indicated that the sensitivity of the peak sensor signal to changes in impact location depends on the location of the impact site and to some extent on the particle radius.

### 2.4.3 The Sensor Output Spectrum

The sensor output spectra was calculated by zero padding the 500 point data sequence representing the sensor signal, to a length of \( 2^{14} \) or 16384 points and performing a fast Fourier transform on the extended sequence.\(^{10}\) The resulting complex terms were then scaled by the inverse of the sampling frequency to obtain magnitudes equivalent to those obtained from a continuous Fourier transform of the signal.\(^{11}\) The power spectral density was then obtained by multiplying these scaled complex terms

---

\(^{10}\)The spectra showed no significant change in the regions of interest when the zero padding was appended or prefixed to the original data sequence.

\(^{11}\)Appendix J discusses the relationship between the fast Fourier transform of a discrete time signal and the Fourier transform of the continuous time signal.
<table>
<thead>
<tr>
<th>Particle Radius $r_s$ (mm)</th>
<th>Mean $v_{pk}$ (V)</th>
<th>$\Delta v_{pk}$ (V)</th>
<th>$\Delta v_{pk}$ % of mean</th>
<th>$\Delta r_s$ (mm)</th>
<th>$\Delta r_s$ % of radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.175</td>
<td>68.6</td>
<td>3.3</td>
<td>4.7</td>
<td>0.16</td>
<td>5.1</td>
</tr>
<tr>
<td>2.778</td>
<td>59.5</td>
<td>2.7</td>
<td>4.5</td>
<td>0.14</td>
<td>4.9</td>
</tr>
<tr>
<td>2.381</td>
<td>51.1</td>
<td>2.3</td>
<td>4.5</td>
<td>0.11</td>
<td>4.8</td>
</tr>
<tr>
<td>1.984</td>
<td>41.8</td>
<td>1.9</td>
<td>4.6</td>
<td>0.096</td>
<td>4.8</td>
</tr>
<tr>
<td>1.588</td>
<td>30.8</td>
<td>1.4</td>
<td>4.6</td>
<td>0.071</td>
<td>4.5</td>
</tr>
<tr>
<td>1.191</td>
<td>17.45</td>
<td>0.89</td>
<td>5.1</td>
<td>0.045</td>
<td>3.7</td>
</tr>
<tr>
<td>Averages</td>
<td>—</td>
<td>—</td>
<td>4.7</td>
<td>—</td>
<td>4.6</td>
</tr>
</tbody>
</table>

**Table 2.3:** The expected variation of the peak sensor signal and the corresponding radius measure for all the particles given a hypothetical 0.5 mm change in the impact location at 0 cm from impact site B.

<table>
<thead>
<tr>
<th>Particle Radius $r_s$ (mm)</th>
<th>Mean $v_{pk}$ (V)</th>
<th>$\Delta v_{pk}$ (V)</th>
<th>$\Delta v_{pk}$ % of mean</th>
<th>$\Delta r_s$ (mm)</th>
<th>$\Delta r_s$ % of radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.175</td>
<td>28.4</td>
<td>1.2</td>
<td>4.2</td>
<td>0.10</td>
<td>3.1</td>
</tr>
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<td>2.778</td>
<td>26.1</td>
<td>1.0</td>
<td>3.9</td>
<td>0.085</td>
<td>3.1</td>
</tr>
<tr>
<td>2.381</td>
<td>23.1</td>
<td>0.86</td>
<td>3.7</td>
<td>0.072</td>
<td>3.0</td>
</tr>
<tr>
<td>1.984</td>
<td>18.81</td>
<td>0.69</td>
<td>3.7</td>
<td>0.058</td>
<td>2.9</td>
</tr>
<tr>
<td>1.588</td>
<td>13.68</td>
<td>0.53</td>
<td>3.9</td>
<td>0.044</td>
<td>2.8</td>
</tr>
<tr>
<td>1.191</td>
<td>6.91</td>
<td>0.29</td>
<td>4.2</td>
<td>0.024</td>
<td>2.0</td>
</tr>
<tr>
<td>Averages</td>
<td>—</td>
<td>—</td>
<td>3.9</td>
<td>—</td>
<td>2.8</td>
</tr>
</tbody>
</table>

**Table 2.4:** The expected variation of the peak sensor signal and the corresponding radius measure for all the particles given a hypothetical 0.5 mm change in the impact location at -1 cm from impact site B.
<table>
<thead>
<tr>
<th>Particle Radius $r_s$ (mm)</th>
<th>Mean $v_{pk}$ (V)</th>
<th>Impact location, $x = -2$ cm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta v_{pk}$ (V)</td>
<td>$\Delta v_{pk}$ % of mean</td>
</tr>
<tr>
<td>3.175</td>
<td>15.40</td>
<td>0.23</td>
</tr>
<tr>
<td>2.778</td>
<td>14.74</td>
<td>0.24</td>
</tr>
<tr>
<td>2.381</td>
<td>13.60</td>
<td>0.25</td>
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<tr>
<td>1.984</td>
<td>11.29</td>
<td>0.23</td>
</tr>
<tr>
<td>1.588</td>
<td>7.85</td>
<td>0.18</td>
</tr>
<tr>
<td>1.191</td>
<td>4.00</td>
<td>0.071</td>
</tr>
<tr>
<td>Averages</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>

Table 2.5: The expected variation of the peak sensor signal and the corresponding radius measure for all the particles given a hypothetical 0.5 mm change in the impact location at -2 cm from impact site B.

<table>
<thead>
<tr>
<th>Particle Radius $r_s$ (mm)</th>
<th>Mean $v_{pk}$ (V)</th>
<th>Impact location, $x = -3$ cm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta v_{pk}$ (V)</td>
<td>$\Delta v_{pk}$ % of mean</td>
</tr>
<tr>
<td>3.175</td>
<td>12.69</td>
<td>0.18</td>
</tr>
<tr>
<td>2.778</td>
<td>11.45</td>
<td>0.22</td>
</tr>
<tr>
<td>2.381</td>
<td>9.44</td>
<td>0.32</td>
</tr>
<tr>
<td>1.984</td>
<td>6.57</td>
<td>0.41</td>
</tr>
<tr>
<td>1.588</td>
<td>4.41</td>
<td>0.28</td>
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<tr>
<td>1.191</td>
<td>2.35</td>
<td>0.18</td>
</tr>
<tr>
<td>Averages</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>

Table 2.6: The expected variation of the peak sensor signal and the corresponding radius measure for all the particles given a hypothetical 0.5 mm change in the impact location at -3 cm from impact site B.
by their conjugate. All the power spectral densities are referenced to a one ohm resistor and have the units of micro-watts/hertz.

Consider the spectra due to the impacts near site A. Figure 2.27 and Figure 2.28 show the mean and the one standard deviation limits based on ten power spectra from each of the particles. The characteristics that are prominent are the two peaks that occur near 17 kHz and 25 kHz. These peaks correspond, within a few kilohertz, to the resonant peaks measured in the transducer's magnitude response, given in Figure A.2.

Chapter 1 postulated that the behaviour of $V_o(\omega)$ can be predicted using the expression

$$V_o(\omega) = Z_t(\omega)H_t(\omega)$$  \hspace{1cm} (2.5)

To determine if the relationship of Equation (2.5) holds for the experimental results, the change in the ratio of the power densities at the peaks near 17 kHz and 25 kHz are compared. This ratio will indicate whether the power densities of $Z_t(\omega)$ are increasing or decreasing as the particle radius and impact location are changed. The change in the power densities can then be compared to the change predicted in the discussions of Chapter 1 which is based on Equation (2.5).

According to the discussions in Chapter 1, the smallest particle should produce a $Z_t(\omega)$ with the widest bandwidth but with the smallest magnitudes. As the particles become larger the bandwidth narrows and grows in magnitude. If the bandwidth of $Z_t(\omega)$ narrows but still encompasses $H_t(\omega)$, then the ratios of the power densities of $V_o(\omega)$ are expected to remain constant. If the bandwidth of $Z_t(\omega)$ narrows so that it no longer encompasses $H_t(\omega)$, the ratios of the power densities of $V_o(\omega)$ are expected to change. A change in the power densities is also expected as the impact location moves away from the transducer. In both these cases, the change in the power densities is expected to reflect an increase in power at the lower frequency relative to the power at the higher frequency.

The power spectral densities measured at impact site A are expressed as a ratio of the mean power densities at the peak near 25 kHz to peak near 17 kHz. These
Figure 2.27: The mean and standard deviation about the mean of the power spectra of the sensor output due to impacts of particles with radii of 1.984 mm, 1.588 mm, and 1.191 mm at impact site A.

Figure 2.28: The mean and standard deviation about the mean of the power spectra of the sensor output due to impacts of particles with radii of 3.175 mm, 2.778 mm, and 2.381 mm at impact site A.
Table 2.7: The mean power spectral densities near the peaks at 17 kHz and 25 kHz for the signals produced by the particles at impact site A.

<table>
<thead>
<tr>
<th>Particle Radius ( r_s ) (mm)</th>
<th>Mean power spectral densities at ( \approx 17 \text{ kHz} ) ( P_1 ) (( \mu \text{W/Hz} ))</th>
<th>Mean power spectral densities at ( \approx 25 \text{ kHz} ) ( P_2 ) (( \mu \text{W/Hz} ))</th>
<th>( P_2/P_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.175</td>
<td>5.5</td>
<td>3.3</td>
<td>0.60</td>
</tr>
<tr>
<td>2.778</td>
<td>3.2</td>
<td>3.6</td>
<td>1.13</td>
</tr>
<tr>
<td>2.381</td>
<td>1.87</td>
<td>3.5</td>
<td>1.87</td>
</tr>
<tr>
<td>1.984</td>
<td>0.94</td>
<td>1.82</td>
<td>1.94</td>
</tr>
<tr>
<td>1.588</td>
<td>0.39</td>
<td>0.74</td>
<td>1.90</td>
</tr>
<tr>
<td>1.191</td>
<td>0.101</td>
<td>0.128</td>
<td>1.27</td>
</tr>
</tbody>
</table>

Power densities and ratios are listed in Table 2.7. As a comparison to these ratios, the power densities, at the corresponding peaks in the actual measured transducer response, varies from approximately 3 dB in trial three to 8 dB in trial one. This corresponds to ratios in the power density of about 2 to 6, respectively. Therefore, the measured ratio of about 2 in Table 2.7 suggests that the bandwidth of \( Z_i(\omega) \) is at least as wide as the 25 kHz bandwidth of the signal used to excite the transducer during the measurement of its response. Therefore, power ratios of less than 2 may indicate that the bandwidth of \( Z_i(\omega) \) is less than 25 kHz. Similarly, power ratios less than one indicate that there is more power at the low frequency point relative to that at the high frequency point.

The four smallest particles, with the exception of the smallest particle, all exhibit power densities ratios that are reasonably constant and close to the ratio of 2 seen in the actual measured transducer response. Again, this suggests that the bandwidth of \( Z_i(\omega) \) is at least 25 kHz. The exception in the smallest particle is difficult to explain since it indicates that there is a more relative power in \( Z_i(\omega) \) at the peak near 17 kHz than there is for the other larger particles. This is contrary to what is expected and, as discussed later when examining the spectra due to the impacts near site B, is attributed to the large variation in its impact position.
For the largest two particles, it appears that $Z_t(\omega)$ has narrowed to a point where the response in $H_t(\omega)$ at the peak near 25 kHz is being excited with less energy than the peak near 17 kHz. This is seen by a reduction in the power densities ratios. The largest particle, producing a $Z_t(\omega)$ with the most narrow bandwidth, has resulted in $V_0(\omega)$ displaying more power at the peak near 17 kHz than the peak near 25 kHz. This is indicated by the power ratio becoming less than one.

Now consider the spectra due to the impacts near site B that are approximately 1.5 mm from impact site A. The discussions in Chapter 1 indicated that an increase in the distance between the impact and measurement point should reduce the high frequency estimate in the spectra of $Z_t(\omega)$. Therefore, a reduction in the ratio of the power densities is expected for all the previous particles having bandwidths for $Z_t(\omega)$ being near or less than 25 kHz.

Figure 2.29 and Figure 2.30 show the sensor signal spectra for particles impacting at 0 cm from impact site B. The spectra show a resemblance to those seen at site A and exhibit a reduction in standard deviation of the power densities. The reduction in the standard deviation is attributed to the reduced variation in the impact positions for the site B experiments.

The ratio of the power densities are listed in Table 2.8. These values indicate that the estimate for the higher frequency power densities have decreased for all but the smallest and second largest particles. This is indicated by the smaller ratio values.

The cause of these exceptions could be the possibility that the impact locations for these two particles were slightly different than those of the other particles. For example, the smallest particle was observed to have the largest variation in position for impacts directed at site A. If these impact were further from the transducer than those at site B, then the site A results would produce a lower high frequency spectral estimate. The slight increase in the ratio for the second largest particle at site B could also be the result of impacts at site B being slightly closer to the transducer than those at site A.
Figure 2.29: The mean and standard deviation about the mean of the power spectra of the sensor output due to impacts of particles with radii of 1.984 mm, 1.588 mm, and 1.191 mm at the location 0 cm from impact site B.

Figure 2.30: The mean and standard deviation about the mean of the power spectra of the sensor output due to impacts of particles with radii of 3.175 mm, 2.778 mm, and 2.381 mm at the location 0 cm from impact site B.
Figure 2.31: The mean and standard deviation about the mean of the power spectra of the sensor output due to impacts of particles with radii of 1.984 mm, 1.588 mm, and 1.191 mm at the location -3 cm from impact site B.

Figure 2.32: The mean and standard deviation about the mean of the power spectra of the sensor output due to impacts of particles with radii of 3.175 mm, 2.778 mm, and 2.381 mm at the location -3 cm from impact site B.
Table 2.8: The mean power spectral densities near the peaks at 17 kHz and 25 kHz for the signals produced by the particles at impact site B.

Examining the spectra due to impacts at distances further from the transducer reveals that the high frequency spectral estimates decrease with increases in distance. Figure 2.31 and Figure 2.32 shows the spectra of the sensor signal due to impacts of particles at -3 cm from impact site B. The reduction in the high frequency power densities estimates are significant. Alternately, the low frequency power densities estimates have increased, especially for the larger particles.

The spectra of the sensor signals resulting from the particles impacting at -1 cm and -2 cm from impact site B also revealed reduction in both the magnitudes and the high frequency spectral estimates.

2.5 Summary of the Experimental Results and Analysis

This chapter has determined that the peak sensor signal is dependent on the impact location and particle radius. In particular, using the peak sensor signal, the sensor responds almost linearly to changes in particle radius at every impact location tested. In addition, the sensitivity of the peak sensor signal to changes in particle radius decreases with distance from the transducer site. Similarly, the sensitivity of the peak sensor signal to changes in impact position generally decreases as the dis-
tance is increased. For these experiments a maximum and minimum sensitivity were measured at the transducer site and at about 2 cm from the transducer, respectively. The change in the spectrum of $V_{o}(\omega)$, given changes in particle radius and impact location, is also as predicted in Chapter 1.

These results strongly suggest that the variation in the peak sensor signal seen in the sensor of the grain loss monitor is primarily due to small unobserved variations in the impact location. With the impacts occurring very near the transducer these variation are also likely to be at a maximum. The extent of the applicability of these results to the sensor in the grain loss monitor is difficult to determine at this point of the analysis because of the different configuration of the sensor. A theoretical analysis of the ideal sensor will determine the validity of extending the experimental results to the grain loss monitor. It will also explicitly reveal the factors of the force, plate, and transducer that influence the output signal. This theoretical analysis is presented in the next chapter.
3. Theoretical Analysis and Results

3.1 Overview

This chapter presents the results of a detailed theoretical analysis of an ideal version of the experimental thin plate impact sensor. The theoretical analysis begins by introducing the configuration and nomenclature of the ideal sensor. The analysis then examines the behaviour of a rectangular simply supported thin plate to a point force by solving the classical thin plate equation and representing the solution in the frequency domain. The relevant characteristics of the force pulse and its spectra are then considered. Finally, a simple model of the transducer used in the grain loss monitor sensor is developed.

3.2 The Ideal Sensor Model

The theoretical analysis is based on using an ideal form of the experimental thin plate impact sensor discussed in Chapter 2 and is directed by

\[ V_0(\omega) = F_0(\omega)H_p(\omega)H_t(\omega) \], \hspace{1cm} (3.1)

to describe its operation. To model these processes, the physical elements and the coordinate system shown in Figure 3.1 are assumed. The mid-plane of the plate is at \( z = 0 \).

A spherical particle of radius \( r \) and velocity \( v \) impacts at the point denoted \((x_i, y_i)\) on a rectangular plate that is simply supported on all sides. Not shown in the figure is a transducer that is placed on the plate's underside at an arbitrary point denoted \((x_t, y_t)\) and the simply supported mounting of the plate.
The thin homogeneous plate has lateral dimensions, $l_x$, $l_y$ and thickness $2l_z$ with a density $\rho_p$, modulus of elasticity $E_p$, and a Poisson ratio of $\nu_p$. The spherical particle is also homogeneous and has a density $\rho_s$, modulus of elasticity $E_s$, and a Poisson ratio of $\nu_s$. The transducer is also assumed to have negligible influence on the motion of the plate.

In operation, the impact of the particle with the plate, using the parameters given above, generates a force pulse $f_a(t)$ with a spectrum $F_a(\omega)$ at the point $(x_i, y_i)$. This force then causes the plate to move as described by the thin plate transfer function $H_p(\omega)$. The spectrum $Z_i(\omega)$ of the measured motion $z_i(t)$, at the point $(x_i, y_i)$, is transformed by $H_i(\omega)$ into the output signal spectrum $V_o(\omega)$.

The analysis begins by first developing the thin plate transfer function. Next the equations describing the form of the force pulse and the force pulse spectrum are presented and finally the transducer transfer function is developed.

### 3.3 The Thin Plate Transfer Function

In this section an expression for the thin plate transfer function is obtained by solving the non-homogeneous equation for a rectangular thin plate, simply supported on all sides, having a point force applied to it. The solution is then expressed
in the frequency domain allowing it to be used in Equation (3.1). This solution explicitly shows that the plate dimensions, boundary conditions, material parameters, and the location of the transducer and impact sites affect the plate transfer function $H_p(\omega)$ that consequently influences the measured plate displacements $z_t(t)$. Of all these parameters, the impact location is the one factor most likely to vary and affect $H_p(\omega)$ and consequently $z_t(t)$.

3.3.1 The General Solution of the Thin Plate Equation

The non-homogeneous equation for a thin plate is given by [30, 24]

$$D \nabla^4 z(x, y, t) + (2\rho_p l_z) \frac{\partial^2 z(x, y, t)}{\partial t^2} = f_s(x, y, t) ,$$

where $f_s(x, y, t)$ is the surface density of the applied force,

$$D = \frac{8E_p l_z^3}{12(1 - \nu^2)} ,$$

and the squared Laplacian operator is

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} .$$

Equation (3.2) is also referred to as the classical thin plate equation and is the result of approximating the expressions describing the propagation of waves through a solid [30, 24]. When the thickness is small relative to the solid's other lateral dimensions, the solid is referred to as a thin plate [24, 28]. As an approximation, Equation (3.2) provides an accurate description of plate behaviour when the propagating waves, with wavelengths $\Lambda$, are long relative to $2l_z$, the thickness of the plate [24, 28, 26]. Equation (3.2) also assumes that the plate deforms by flexing in the transverse direction, that there are no in-plane loads, and that rotary inertia and shear deformations do not exist.\(^1\) The flexural deformations are assumed to be normal to the mid-plane of the plate and small in magnitude relative to the thickness of

\(^1\)A force that is applied parallel to the surface of the plate is referred to as an in-plane load. Shear deformation can be viewed as an infinite number of internal planes parallel to the surface of the plate sliding over each other. Rotary inertia refers to the inertia associated with the rotation of an elemental volume of material about an axis.
the plate [28]. Generally a factor of ten or more is sufficient to express the difference between small and large dimensions or long and short waves.

Using a point force $f_a(x_i, y_i, t)$ and a rectangular simply supported plate, the general solution to Equation (3.2), as reviewed in Appendix D, becomes

$$z_i(t) = \frac{2}{\rho_p l_x l_y} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right)$$

$$\times \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right) \int_0^t f_a(\tau) \sin \omega_{mn}(t - \tau) d\tau,$$

where

$$\omega_{mn} = \sqrt{\frac{E_p}{3\rho_p(1 - \nu^2)}} \left[ \left( \frac{m\pi}{l_x} \right)^2 + \left( \frac{n\pi}{l_y} \right)^2 \right],$$

are the plate resonant frequencies and the indices $m$ and $n$ the plate mode numbers [18, 26, 31]. It should be noted that the plate resonant frequencies are not multiples of lowest resonant frequency as they were in the taut string.

The solution given by Equation (3.5) is actually a function of $(x, y, t)$ but since it is evaluated at the fixed transducer location $(x = x_i, y = y_i)$, Equation (3.5) is written conveniently as a function of $t$ with the understanding that using the notation $z_i(t)$ implies $z(x_i, y_i, t)$. Equation (3.5) is now written more compactly as

$$z_i(t) = K_p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn} A_{mn} \int_0^t f_a(\tau) \sin \omega_{mn}(t - \tau) d\tau,$$

where

$$K_p = \frac{2}{\rho_p l_x l_y},$$

$$K_{mn} = \frac{1}{\omega_{mn}} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right),$$

and

$$A_{mn} = \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right).$$

3.3.2 The Thin Plate Transfer Function

A frequency domain representation of Equation (3.7) is more useful in determining an expression for the thin plate model. By taking the Fourier transform
in the limit of each term in Equation (3.7), the result, as given in Appendix H, becomes

\[ Z_t(\omega) = K_p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn} A_{mn} F_a(\omega) j\pi \left[ \delta_k(\omega + \omega_{mn}) - \delta_k(\omega - \omega_{mn}) \right], \]  

(3.11)

where \( \delta_k(x) \) is the Kronecker delta function defined as [21]

\[ \delta_k(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases} \]  

(3.12)

which is used instead of the Dirac delta function. This results in

\[ Z_t(\omega) = \begin{cases} -j\pi K_p K_{mn} A_{mn} F_a(\omega) & \text{for } \omega = \omega_{mn}, \\ j\pi K_p K_{mn} A_{mn} F_a(\omega) & \text{for } \omega = -\omega_{mn}, \\ 0 & \text{for } \omega \neq \omega_{mn}. \end{cases} \]  

(3.13)

Therefore, \( Z_t(\omega) \) has non-zero values only at the discrete resonant frequencies \( \omega_{mn} \). This result assumes that the plate is not square so that there is a unique set of constants \( K_{mn} A_{mn} \) for each \( \omega_{mn} \) producing a single term for \( Z_t(\omega) \) at each discrete frequency. A square plate has \( \omega_{mn} = \omega_{nn} \) resulting in the sum of two terms for \( Z_t(\omega) \) at each discrete frequency. Since \( Z_t(\omega) \) is zero everywhere except at \( \omega = \omega_{mn} \), it is written as a function of \( \omega_{mn} \), or as

\[ Z_t(\omega_{mn}) = \begin{cases} -j\pi K_p K_{mn} A_{mn} F_a(\omega_{mn}), & \omega_{mn} > 0, \\ j\pi K_p K_{mn} A_{mn} F_a(\omega_{mn}), & \omega_{mn} < 0. \end{cases} \]  

(3.14)

Taking the magnitude of Equation (3.14) and knowing that it is even about \( \omega_{mn} = 0 \), it becomes

\[ |Z_t(\omega_{mn})| = \pi K_p |K_{mn} A_{mn} F_a(\omega_{mn})|, \]  

(3.15)

\[ = |H_p(\omega_{mn})| F_a(\omega_{mn})|, \]  

(3.16)

where

\[ H_p(\omega_{mn}) = \pi K_p K_{mn} A_{mn}, \]  

(3.17)

is defined as the thin plate transfer function and represents the expression used in the thin plate model. Equation (3.16) can also be interpreted as consisting of the input \( F_a(\omega_{mn}) \) being modified by the transfer function \( H_p(\omega_{mn}) \) to produce the output
$Z_t(\omega_{mn})$. This expression also has a subtle assumption that requires $z_t(t)$ to be measured for a duration that is greater than the period of the lowest plate resonant frequency. If this is not the case, $Z_t(\omega_{mn})$ will not resemble a discrete version of $F_t(\omega_{mn})$ but instead will be an estimate. This estimate is based on the Fourier transform of a truncated version of $z_t(t)$, where the form and duration of $z_t(t)$ will depend on the particular interval of time that was used to recorded $z_t(t)$.

The thin plate transfer function, in Equation (3.17), is simply a series of real numbers or constants that depend not only on the plate resonant frequency but also on the transducer and impact location, as an examination of $K_{mn}$ or $A_{mn}$ shows. Equation (3.16) is more revealing and indicates that $Z_t(\omega_{mn})$ consists of the weighted samples of $F_t(\omega)$ taken at the resonant frequencies $\omega_{mn}$ of the plate. The distribution of these weighting factors is given by $H_p(\omega_{mn})$ and depends on the position of both the transducer, and more importantly, the impact site. With the transducer location and the other plate parameters fixed, changes in the impact location are the most likely to vary and change the magnitude distribution of the spectral components of $Z_t(\omega_{mn})$, that in turn directly affects the form of the displacement $z_t(t)$.

### 3.3.3 Behaviour of the Thin Plate Transfer Function

$H_p(\omega_{mn})$ has both a frequency and positional dependency that affects $Z_t(\omega_{mn})$. To understand this behaviour, consider all the factors forming $H_p(\omega_{mn})$, except $A_{mn}$, being fixed in value. The terms $x_i$ and $y_i$ within $A_{mn}$, giving the coordinates of the impact site, are allowed to vary through the range $0 \leq x_i \leq 1$ and $0 \leq y_i \leq 1$, respectively, for a given set of mode numbers $m$ and $n$. Each set of mode numbers represents a different plate resonant frequency. The resulting behaviour of $H_p(\omega_{mn})$ can be visualized as a surface over the $xy$ plane that changes for each set of mode numbers or resonant frequency.

Figure (3.2) shows the shape of this surface normalized by $\pi K_p K_{mn}$ for $\{m, n\}$ equal to $\{1, 1\}, \{1, 2\} \{2, 2\}$, and $\{3, 3\}$. As is seen, increasing $m$ or $n$, or equivalently the plate resonant frequency, causes more sine cycles to form over the surface.
Figure 3.2: Variation of $H_p(\omega_{mn})$ normalized by $\pi K_p K_{mn}$ with $A_{mn}$ having $0 \leq x_i \leq 1$ and $0 \leq y_i \leq 1$ for $\{m,n\} = \{1,1\}, \{1,2\}, \{2,2\},$ and $\{3,3\}$ for the surfaces shown from left to right and top to bottom, respectively.

The surfaces in Figure (3.2) represent the normalized values of $H_p(\omega_{mn})$ as a function of impact position for the frequencies $\omega_{11}, \omega_{12}, \omega_{22},$ and $\omega_{33}$. As the impact position varies, the normalized values of $H_p(\omega_{mn})$ vary from -1 to +1, inclusively. As the frequency increases, the sensitivity of the normalized values of $H_p(\omega_{mn})$ to changes in impact location also increases. This is realized by noting that any one surface consists of a simple two dimensional sine function. The sensitivity to changes in the impact location are then proportional to the slope of a plane tangent to that surface. As the plate resonant frequency increases the slope of the surface also increases. The points that correspond to the locations where the normalized values of $H_p(\omega_{mn}) = \pm 1$ represent points that have zero slope or are the least sensitivity to changes in impact location. Alternately, points where the normalized values of $H_p(\omega_{mn}) = 0$ have the maximum value of slope and are the most sensitive to changes in impact location.

These surfaces also indicate the shape of the vibrational modes of the thin plate. For example, if the plate was vibrating at one frequency, $\omega_{11}$, the envelope
of the positive half of the vibration is confined to the shape given by the surface associated with the mode \( \{ m, n \} = \{ 1, 1 \} \). The negative half of the vibration is symmetrical to the positive half. This implies that points where the normalized values of \( H_p(\omega_{mn}) = \pm 1 \) and \( H_p(\omega_{mn}) = 0 \) correspond to the anti-nodal and nodal points of the vibrating plate, respectively. The sensitivity of \( H_p(\omega_{mn}) \) to changes in impact location is then determined by the mode shapes of the thin plate that itself is determined by the boundary conditions and plate dimensions. In this case, the mode shapes are simple sine functions over the plate dimensions.

As a final note, changing the transducer position affects \( H_p(\omega_{mn}) \) in the same manner as changes in the impact position. This is due to the sine terms in \( K_{mn} \) being of the same form as those in \( A_{mn} \). The only difference between \( K_{mn} \) and \( A_{mn} \) is the additional \( 1/\omega_{mn} \) term causing \( K_{mn} \) to be inversely proportional to the resonant frequency of the thin plate.

### 3.3.4 Plate Displacement Behaviour

It is difficult to visualize the motion of the plate \( z_i(t) \) by interpreting the time domain expression of Equation (3.7). The summation of an infinite number of convolutions of sinusoids with the force pulse does not lend itself to simple or obvious forms for the solution. The frequency domain expression, given by Equation (3.14), also does not easily allow \( z_i(t) \) to be visualized but it can provide a simpler interpretation of the factors that combine to form \( z_i(t) \). This interpretation uses the linear superposition of sinusoids.

Consider each one of the terms of Equation (3.14) at the frequencies \( \pm \omega_{11} \). These two terms represent a sinusoid at the frequency \( \omega_{11} \) expressed as

\[
z_{\omega_{11}}(t) = |Z_i(\omega_{11})| \frac{e^{j\omega_{11}t} - e^{-j\omega_{11}t}}{2j},
\]

where \( |Z_i(\omega_{11})| \) is the magnitude of the sinusoid. Each positive and corresponding negative term in Equation (3.14) represent a sinusoid at a different frequency and phase. If the phase of these sinusoids are taken relative to the phase of the sinusoid
at the frequency $\omega_{11}$, then the summation of these sinusoids results in the plate displacement $z_i(t)$. Therefore, instead of attempting to form the sum of convolutions to visualize $z_i(t)$, the sum of sinusoids can be used to provide a simpler view of the factors that combine to form $z_i(t)$.

Section 3.3.3 showed that a change in the impact position results in $H_p(\omega_{mn})$ expanding out into a different series of real numbers. From Equation (3.16), this changes $Z_i(\omega_{mn})$ and consequently the magnitudes of the sinusoids in Equation (3.18). Therefore, a change in the impact position can be viewed as affecting the magnitudes of each sinusoid forming $z_i(t)$.

Section 3.3.3 also showed that the sensitivity of $H_p(\omega_{mn})$ to changes in the impact position increases with frequency. This is also true for the magnitude of the sinusoids. A given change in the impact position changes the magnitudes of the high frequency sinusoids more than the magnitudes of the low frequency sinusoids. This results in the summation of low frequency sinusoids not changing significantly. Alternately, the summation of the high frequencies does exhibit greater changes than the lower frequency summations. This suggests that the characteristics of $z_i(t)$, attributed to the lower frequencies, changes less than the characteristic attributed to the higher frequencies. This could result in $z_i(t)$ exhibiting less abrupt changes and reduced magnitudes as the distance between the impact site and the transducer increases. A simple example shows how $z_i(t)$ changes with a change in impact position.

Consider a hypothetical ideal rectangular force pulse defined as

$$f_a(t) = \begin{cases} A, & 0 \leq t \leq t_{end}, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transform is [20]

$$F_a(\omega) = A t_{end} \sin(\omega t_{end}/2) e^{-j\omega t_{end}/2}.$$  

To evaluate $z_i(t)$, assume$^3$ that $A = 1$ N, $t_{end} = 1$ ms, $l_x = l_y = 1$ m, $l_z = 1 \times 10^{-4}$ m,

---

$^2$Appendix I discusses this development in detail.

$^3$The values assumed for the plate parameters are reasonably similar to those of the actual thin plate used in the experiments. In particular, from Appendix K, these are $l_x \approx l_y \approx 0.5$ m, $l_z \approx 0.8$ cm, $\rho \approx 1.54$ kg$^{-1}$, and $\omega_{mn} \approx 1.2 \pi[(2n)^2 + (2m)^2]$ rad/s.
Figure 3.3: The motion at the point \((x_t, y_t) = (0.5, 0.5)\) m of a thin plate over the duration \(0 \leq t \leq 8\) ms due to a 1 ms rectangular force pulse of 1 N applied at \((x_i, y_i) = \{(0.5, 0.5), (0.4, 0.5), (0.3, 0.5), (0.2, 0.5)\}\) m.

\[ K_p = 1 \text{ kg}^{-1}, \text{ and } \omega_{mn} = \pi^2 (m^2 + n^2) \text{ rad/s}. \] If the transducer or measurement point, \((x_t, y_t)\), is assumed to be at the centre of the plate or at \((0.5, 0.5)\) m and the application of the force pulse \(f_a(t)\) occurs at \((x_i, y_i) = \{(0.5, 0.5), (0.4, 0.5), (0.3, 0.5), (0.2, 0.5)\}\) m, then \(z_i(t)\) can be determined numerically.

The results of the numeric evaluation are shown in Figure 3.3. The approximation of \(z_i(t)\) used the summation of more than 3200 sinusoids that included frequencies up to about 10 kHz. Sinusoids with magnitudes greater than \(0.02 |F_a(0)|\) were used in the summation.\(^4\)

Figure 3.3 shows that the plate displacements \(z_i(t)\), measured at the impact location \((x_i = x_t, y_i = y_t)\), have the largest magnitudes and the most abrupt motion of the set. As the impact site moves away from the measurement point \((x_t, y_t)\), the magnitudes decrease and the motion becomes less abrupt. This is especially apparent

\(^4\)The details of evaluating \(z_i(t)\) using the summation of sinusoids is discussed in Appendix I.
when the motion, due to the impact at (0.5, 0.5) m, is compared to the motion due to the impact at (0.4, 0.5) m. This change in the motion is directly attributed to a smaller change in the summation of the low frequency sinusoids. Also apparent in Figure 3.3, as the impacts move further from the measurement point, is the delay in the start of the motion and its gradual increase in magnitude. Both these observations are physically attributed to the finite time required for a disturbance to propagate to the measurement point and to dispersion.

3.3.5 Summary of the Thin Plate Theoretical Analysis

It has been shown that the plate dimensions, boundary conditions, material parameters, and the location of the transducer and impact sites influence the behaviour of the plate transfer function \( H_p(\omega_{mn}) \) that consequently affects the resulting measured plate displacements \( z_t(t) \). When considering the factors that could vary in the experimental results, it is the impact location that is the most likely to vary and produce changes in \( H_p(\omega_{mn}) \) and consequently in \( z_t(t) \). The sensitivity of \( H_p(\omega_{mn}) \) to changes in impact location increases with the plate resonant frequencies and the magnitude of the sensitivity is proportional to the slope of the vibrational mode shapes.

If \( z_t(t) \) is measured over a duration greater than the period of the lowest plate resonant frequency, the resulting plate displacement spectrum \( Z_t(\omega_{mn}) \) resembles a frequency sampled version of the force pulse spectrum \( F_s(\omega_{mn}) \). The frequency sample points correspond to the resonant frequencies of the plate that have magnitudes that varies with the location of both the measurement and impact point.

If \( z_t(t) \) is measured over a durations that is less than the period of the lowest plate resonant frequency, as was the case in the experiments, then using the summation of sinusoids provides insight into the factors that influence \( z_t(t) \) when the impact position varies. Evaluating \( z_t(t) \) in this manner shows that the plate motion is reduced in magnitude and abruptness when the distance between the measurement and impact point is increased.
3.4 The Force Model

This section reviews the equations, developed by Zener, that describe the relative motion between the sphere and the thin plate and the force generated during their impact. In particular, the dimensionless form of the equations are developed to show how the characteristics of the force pulse and force pulse spectrum varies with the inelasticity parameter $\lambda$.

3.4.1 The Normal Force Model

The development of the force model centres on determining the equation describing the motion of the plate during a particle impact. This equation was developed in 1941 by Zener, who assumed a plate so large that reflections from boundaries would return to the point of application only after the end of the force impulse [18]. This assumption results in a simple expression that describes the motion of the plate at the point of the application of a force. Using a relationship, known as the force deformation law, that explicitly expresses the force in terms of the relative motion of two solids in compression, and the equation describing the motion of the particle, a non-linear second order differential equation is developed to describe the relative motion between the plate and particle.

In the impact of a solid sphere with a solid plate, both objects deform. If the deformation does not exceed the elastic limits of either solid, both will return to their original shape [26]. A simple spring is a useful analogy of this process. Applying a force $f_a$ to the free end of fixed spring causes it to deform according to the linear relationship $f_a = k \times$, where $k$ is the spring constant and $x$ the deformation or displacement of the spring from its original undeformed position. In removing the force $f_a$, the spring returns to its original position. If two springs are pressed together, their deformation is still linearly proportional to the force but the equivalent spring constant $k$ using both springs in this case is given by $\frac{k_1k_2}{k_1+k_2}$, where $k_1$ and $k_2$ are the spring constants for each of the two springs, respectively [29].
For two solids pressed together, the deformation is not a simple linear relationship to force. It is dependent on the shape of the solids in the region of contact and is expressed using the force-deformation law [26]

\[ f_a(t) = K_f z_a(t)^{\frac{3}{2}} \tag{3.21} \]

where for a solid sphere and a plane solid surface

\[ K_f = \frac{4}{3} \sqrt{\frac{E_s E_p'}{E_s' + E_p'}} \tag{3.22} \]

With both the sphere and plate moving during compression, \( z_a(t) \) becomes the relative distance between the two objects or

\[ z_a(t) = z_s(t) - z_p(t) \tag{3.23} \]

where \( z_s(t) \) and \( z_p(t) \) represent the displacement of the sphere and the plate referenced from the point of impact, respectively. If Equation (3.23) is differentiated twice with respect to time the expression

\[ \frac{d^2 z_a(t)}{dt^2} = \frac{d^2 z_s(t)}{dt^2} - \frac{d^2 z_p(t)}{dt^2} \tag{3.24} \]

is obtained [18].

To complete this expression, the motion of the plate, \( z_p(t) \), and that of the sphere, \( z_s(t) \), are required. The sphere's motion is expressed as

\[ \frac{d^2 z_s(t)}{dt^2} = -\frac{f_a(t)}{m_s} \tag{3.25} \]

The motion of the plate at the point of contact is more complex and is obtained by solving the classical thin plate equation with the assumption that reflections do not return to the point of impact until after the impact is completed [18]. As reviewed in appendix E, this expression is given by

\[ z_p(t) = B_p \int_0^t f_a(\tau) d\tau \tag{3.26} \]

where

\[ B_p = \sqrt{\frac{3\rho_p}{E_p} \left( \frac{1}{16\rho_p l_z^2} \right)} \tag{3.27} \]
Differentiating Equation (3.26) twice with respect to time and substituting both it and Equation (3.25), back into Equation (3.24), the equation

$$\frac{d^2 z_d(t)}{dt^2} + B_p \frac{df_a(t)}{dt} + \frac{f_a(t)}{m_s} = 0 \quad (3.28)$$

results. With the force-deformation law given by Equation (3.21), Equation (3.28), as shown in appendix 2, as

$$\frac{d^2 z_d(t)}{dt^2} + 3 \frac{3}{2} K_f B_p \sqrt{z_d(t)} \frac{dz_d(t)}{dt} + \frac{K_f}{m_s} z_d(t)^{\frac{3}{2}} = 0 \quad (3.29)$$

Equation (3.29) together with Equation (3.21) form the normal force model. Solving Equation (3.29) for $z_d(t)$ and substituting it into Equation (3.21) yields the force of impact. This force can then be substituted into Equation (3.26) to find the plate motion $z_p(t)$ at the impact site.

### 3.4.2 The Dimensionless Force Model

The dimensionless form of the force model equations allows the use of a single parameter $\lambda$ to specify the form of the solutions for the relative motion between the sphere and plate.

Equation (3.29) can be made dimensionless by an appropriate change of variables [18] using

$$\sigma(\tau) = \frac{z_d(\tau)}{T v_s} \quad (3.30)$$

$$\tau = \frac{t}{T} \quad (3.31)$$

where $\sigma(\tau)$ and $\tau$ represent the dimensionless relative displacement and time, respectively and where $z_d(t)$ is written as $z_d(\tau)$ for notational convenience. Equation (3.29) is then written, as developed in appendix F, in the form

$$\frac{d^2 \sigma(\tau)}{d\tau^2} + \lambda \left( \frac{3}{2} \sqrt{\sigma(\tau)} \right) \frac{d\sigma(\tau)}{d\tau} + \sigma(\tau)^{\frac{3}{2}} = 0 \quad (3.32)$$

The constant $T$ is chosen so that the coefficients of Equation (3.29) for the second and zero order terms are unity. This is achieved by setting

$$T = \left( \frac{m_s}{\nu_{n,n,f}} \right)^{\frac{2}{3}} \quad (3.33)$$

$$T = \left( \frac{m_s}{\nu_{n,n,f}} \right)^{\frac{2}{3}} \quad (3.33)$$

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$$T = \left( \frac{m_s}{\nu_{n,n,f}} \right)^{\frac{2}{3}} \quad (3.33)$$
and defining

\[ \lambda = \frac{B_p m_s}{T} \quad (3.34) \]

As noted previously, the coefficient \( \lambda \) is referred to by Zener as the inelasticity parameter and determines the form of the solution \( \sigma(\tau) \) [18]. Having found \( \sigma(\tau) \), it is a simple matter to obtain \( f_a(\tau) \) from Equation (3.21). It should be noted that \( f_a(\tau) \) is not dimensionless.

By rearranging the terms forming \( \lambda \), its relationship to the plate and sphere parameters becomes apparent [18] and as developed in Appendix F, is given by

\[ \lambda = \frac{\pi^{\frac{3}{2}}}{\sqrt{3}} \left( \frac{r_s}{2l_s} \right)^{\frac{1}{2}} \left( \frac{v_s}{v_p} \right) \left( \frac{\rho_s}{\rho_p} \right)^{\frac{1}{6}} \left( \frac{E'}{E_s + E'} \right)^{\frac{3}{2}} \quad (3.35) \]

where \( v_p \) is defined as

\[ v_p = \sqrt{\frac{E_p}{\rho_p}} \quad (3.36) \]

Equation (3.35) indicates that a change in any one of the sphere and plate parameters changes \( \lambda \) and therefore the resulting form of the solution. For example, doubling the radius of a sphere is equivalent to increasing the impact velocity of the sphere by \( 2^{(2)(5)} = 1024 \) times or the density of the sphere by \( 2^{(2)(5/3)} \approx 10 \) times. In all these instances the change in the form of the solution is identical since the changed values of \( \lambda \) are the same. This indicates that the solution, and consequently the form of the force pulse, is most sensitive to changes in the sphere radius or plate thickness. The sensitivity of \( \lambda \), to changes in the factors forming it, increases as the power of the factors increases.

The convenience of using \( \lambda \) to study how the force pulse changes is apparent. Instead of varying individual sphere and plate parameters in the normal force model, it is simpler to vary \( \lambda \) to determine the form of the force pulse. The values of \( \lambda \) can always be related back to the actual sphere and plate parameters.
3.4.3 Behaviour of the Force Pulse

To determine the form of the force pulse requires that the relative displacement \( z_d(t) \) or dimensionless relative displacement \( \sigma(\tau) \) be solved using the differential equations given by Equation (3.29) or Equation (3.32), respectively. Since both these equations are non-linear second order differential equations, they have no closed form solution and require numerical methods to solve [18].

To compare the shape of the force pulse for various values of \( \lambda \), without regard to the actual magnitudes and time, it is convenient to plot the dimensionless force versus dimensionless time \( \tau \). The normalizing factor producing a dimensionless force [18] is given by

\[
F_m = K_f \left( \frac{15v_o^2(E_s^{-1} + E_p^{-1})m_o}{16\sqrt{\rho_s}} \right)^{\frac{2}{5}}
\]

and represents the maximum force generated during the impact of sphere of radius \( r_s \) upon a plate of infinite thickness [26]. With an infinitely thick plate, \( \lambda = 0 \). It is shown in Appendix F that the quantity given by \( f_\sigma(\tau)/Fm \) is equivalent to \( 0.8^{0.5}\sigma(\tau)^{1.5} \) and represents the dimensionless force model.

The shape of the force pulse using \( f_\sigma(\tau)/Fm \) versus \( \tau \) is shown in Figure 3.4 for \( \lambda = \{0, 0.1, 0.2, 0.4\} \) and in Figure 3.5 for \( \lambda = \{0.5, 1, 1.5, 2, 3, 4, 6, 8\} \). As discussed in Chapter 1, values of \( \lambda \leq 0.5 \) result in curves resembling a half sinusoidal pulse and values of \( \lambda > 0.5 \) result in the latter half of the force pulse to decay more exponentially as \( \lambda \) increases. Multiplying the dimensionless force and time axes by \( Fm \) and \( T \) associated with each particular force curve results in the force and time axes having the dimensions of newtons and seconds, respectively.

To compare the magnitudes and durations of these force pulses requires that they be normalized to one of the curves. This is achieved by scaling the curves by the ratio \( Fm_\alpha/Fm_1 \) where \( Fm_1 \) is the value of \( Fm \) for the reference curve and \( Fm_\alpha \) is the value of \( Fm \) for the curve being normalized. The same scaling is required for

\[5\] Appendix I discusses the numeric solution for both the normal and dimensionless force model equations.
Figure 3.4: The dimensionless force pulse $f_a / Fm$ as a function of dimensionless time $\tau$ for $\lambda = \{0, 0.1, 0.2, 0.4\}$

Figure 3.5: The dimensionless force pulse $f_a / Fm$ as a function of dimensionless time $\tau$ for $\lambda = \{0.5, 1, 1.5, 2, 3, 4, 6, 8\}$
To obtain this ratio, \( F_m \) and \( T \) should be expressed in terms of one of the sphere or plate parameters. For example, by writing \( F_m \) and \( T \) in terms of \( r_s \), the result

\[
F_m = r_s^2 \left(\frac{4}{3}\right) \left( \frac{E_s'E_p'}{E_s + E_p'} \right)^{\frac{3}{2}} \left( \frac{5}{4} \pi v_s^2 \rho_s \right)^{\frac{1}{2}},
\]

is obtained for the magnitude scaling factor and the result

\[
T = r_s \left(\frac{\pi \rho_s}{\sqrt{v_s}}\right)^{\frac{3}{2}} \left( \frac{E_s'E_p'}{E_s + E_p'} \right)^{\frac{1}{2}},
\]

is obtained for the time scaling factor. For example, assume that the force pulse corresponding to \( \lambda = 0.5 \) is selected as the reference curve so that its \( F_m \) and \( T \) values are denoted \( F_{m1} \) and \( T_1 \). Also assume that for this value of \( \lambda \), \( r_s = r_1 \). Now given three other spheres having radii of \( \sqrt{2}r_1 \), \( \sqrt{3}r_1 \), and \( 2r_1 \), their corresponding values of \( F_m \) and \( T \) are denoted \( F_{m2}, F_{m3}, F_{m4} \) and \( T_2, T_3, T_4 \), respectively. The corresponding values of \( \lambda \) also become \{1.0,1.5,2.0\}, respectively. The ratios needed to scale the magnitudes and times are then \{1,2,3,4\} and \{1,\sqrt{2},\sqrt{3},2\}, for \( F_{m_n}/F_{m1} \) and \( T_n/T_1 \), respectively.

Figure 3.6 shows these force pulses normalized using these values of scaling for both magnitudes and times. The shape of the force pulses have changed only by the scaling factors.

Equation (3.39) indicates that doubling the radius increases the magnitude of the force by the square of the change in radius. Equation (3.41) also shows that the duration of the force pulse is proportional to the change in radius. Unfortunately, this is not true for all values of \( \lambda \). Force pulses associated with \( \lambda \leq 0.5 \), especially as \( \lambda \) approaches zero, do change with radius as indicated. This is due to the force pulses having small changes in its magnitude and duration. Force pulses associated with \( \lambda > 0.5 \), start to have a complex relationship with particle radius due to the non-proportional changes in magnitude and duration.\(^6\)

\(^6\)Similar relationships may be found for \( v_s \) or any other parameter in \( F_m \) and \( T \).
3.4.4 Behaviour of the Force Pulse Spectrum

Referring back to the ideal sensor model and Equation (3.1), it is the product of the spectrum of the force pulse $F_p(\omega)$ and the thin plate transfer function $H_p(\omega)$ that form $Z_p(\omega)$. Therefore, the behaviour of the force pulse spectra as the radius changes is examined in this section.\footnote{Because the force pulse is obtained numerically and is therefore a discrete time signal, the discrete Fourier transform is used to obtain its spectrum. The details of applying a discrete Fourier transform to the discrete time force pulse are discussed in appendix J.}

The force pulse spectra of the dimensionless force pulses are shown in Figure 3.7 for $\lambda = \{0, 0.1, 0.2, 0.4, 0.5, 1, 1.5, 2\}$ and in Figure 3.8 for $\lambda = \{3, 4, 6, 8\}$. The spectra for all values of $\lambda$ are reasonably similar. If spectral magnitudes greater than approximately 10 percent of peak magnitude define the bandwidth of the spectra, than it is seen in Figure 3.7 and Figure 3.8 that the bandwidths are at no greater than 0.5. The bandwidths decrease for larger values of $\lambda$ and becomes about 0.2 for
for $\lambda = 8$. Again, multiplying the dimensionless magnitude and frequency axes by the factors $(Fm)(T)$ and $T^{-1}$ associated with each particular curve, results in the magnitude and frequency axes having the dimensions of newtons per hertz and hertz, respectively.

To examine the change in magnitude and bandwidth as the radius changes, the same normalizing factors $Fm$ and $T$ that were used to examine the force pulse are required. In this case, the ratios needed are $(Fm_n/Fm_1)(T_n/T_1)$ for the magnitude and $(T_n/T_1)^{-1}$ for the frequency. Again, assume that the force pulse spectrum corresponding to $\lambda = 0.5$ is selected as the reference curve so that its $Fm$ and $T$ values are denoted $Fm_1$ and $T_1$. Also assume that for this value of $\lambda$, $r_* = r_1$. Now given three other spheres having radii of $\sqrt{2}r_1$, $\sqrt{3}r_1$, and $2r_1$, their corresponding values of $Fm$ and $T$ are denoted $Fm_2$, $Fm_3$, $Fm_4$ and $T_2$, $T_3$, $T_4$, and the corresponding values of $\lambda$ become 1.0, 1.5, and 2.0, respectively. The ratios needed to scale the magnitudes and frequencies are then $\{1(1), 2(\sqrt{2}), 3(\sqrt{3}), 4(2)\}$ and $\{1, 1/\sqrt{2}, 1/\sqrt{3}, 1/2\}$, for $(Fm_n/Fm_1)(T_n/T_1)$ and $(T_n/T_1)^{-1}$, respectively.

The resulting force pulse spectra are shown in Figure 3.9 using the appropriate ratios of $(Fm_n/Fm_1)(T_n/T_1)$ and $(T_n/T_1)^{-1}$ to scale the magnitude and frequency. The spectra associated with $\lambda = \{0.5, 1, 1.5, 2\}$ have changed only by the appropriate magnitudes and frequency scaling factors. In this particular example, it can be seen from Figure 3.9 that spheres with larger radii have spectra with bandwidths that are narrower and greater in magnitude than spectra associated with the smaller spheres. It is also seen in Figure 3.9 that the magnitudes of the spectra become very similar as the frequency increases. Using the observation that the bandwidths, as defined previously, for all spectra are no greater than 0.5, then using the frequency scaling factors, the bandwidths for the spectra in Figure 3.9 are $0.5$, $0.5/\sqrt{2}$, $0.5/\sqrt{3}$, and $0.5/2$, for $\lambda = \{0.5, 1, 1.5, 2\}$, respectively.

Equation (3.39) and Equation (3.41) can also be used to relate a change in radius to changes in the magnitude and bandwidth of the force spectra. As indi-
Figure 3.7: Dimensionless magnitude spectra $F_{a}/(F_{m}T)$ as a function of dimensionless frequency $fT$.

Figure 3.8: Dimensionless magnitude spectra $F_{a}/(F_{m}T)$ as a function of dimensionless frequency $fT$ for $\lambda = \{3,4,6,8\}$.
Figure 3.9: The shape of the dimensionless force pulse spectra for the imaginary spheres numbered 1 to 4, normalized in both magnitude and frequency by \((Fm_1)(T_1)\) and \(T_1\), respectively.

...forth, with \(\lambda \leq 0.5\) the magnitude and duration of the force pulse are relatively constant. This result in the magnitudes of the spectra being approximately proportional to the square of the change in radius and the bandwidth being inversely proportional to the change in the radius. For \(\lambda > 0.5\) both the magnitude and bandwidth begin to have a complex relationship with the particle radius.

3.4.5 Summary of the Force Pulse Theoretical Analysis

It has been shown that the form of the force pulse is dependent on material parameters of both the sphere and plate. This dependency is explicitly given by \(\lambda\). The sensitivity of \(\lambda\), to changes in the factors forming it, increases as the power of the factors increases. This results in the force pulse being the most sensitive to changes in the sphere radius or plate thickness. In terms of the factors that influence the experimental results, it is the particle radius that is the most significant factor affecting the form of the force.
The change in the magnitude of the force pulse and its bandwidth, given changes in particle radius, exhibited the characteristics discussed in Chapter 1. Increasing the radius increases the magnitude of the spectra and narrows its bandwidth. Specifically, for \( \lambda \leq 0.5 \), the magnitude and duration of the force pulse do not change drastically, so that the magnitudes of both the force pulse and spectra are approximately proportional to the square of the change in radius. The duration of the force pulse and bandwidth of the spectra are approximately proportional and inversely proportional to a change in the radius, respectively. For \( \lambda > 0.5 \) the magnitude and the duration of the force pulse begin to have a complex relationship with the particle radius that requires a numeric evaluation to determine.

3.5 The Transducer Model

The transducer element of the sensor system transforms the plate's transverse motion into a measurable signal. Since real transducers are not ideal elements, they do not provide a faithful reproduction of the plate's motion. The transducer modelled in this thesis is based on the piezo-electric transducer used in the grain loss monitor. The simplicity of the model precludes it from being an accurate representation of the actual transducer but, as shown later in Chapter 4, it exhibits a response that is similar to the response of the actual transducer. The model also provides insight into the factors that influences its response and presents a basis to develop more complex models.

3.5.1 The Mechanical and Electrical Elements

The elements assumed in the mechanical model are shown in Figure 3.10. The transducer is rigidly attached to the plate which moves in the transverse direction.\(^8\) The transducer's motion is reduced that of a simple free beam having a centrally located force applied to it. The beam motion is further reduced to the motion of the spring in a spring-mass-damper system having two degrees of freedom.

\(^8\)The detailed development of the model is covered in appendix G with a description of the grain loss monitor given in appendix L.
Figure 3.10: The mechanical elements assumed in the transducer model. Shown are the free body diagram of the transducer (a), the assumed transducer or beam motion (b), and the analogous spring-mass-damper system (c).

As shown in Figure 3.10(c), the input displacement \( z_i(t) \) represents the transverse motion of the plate and \( z_o(t) \) represents the motion of the mass. The compression of the spring or the analogous beam deflection is then \( z_c(t) = z_o(t) - z_i(t) \).

A spring-mass system will, once disturbed, vibrate indefinitely. This is not realistic, as the vibration or oscillation will eventually dampen out due to a number of factors such as dissipative losses in the piezo-electric transducer and air friction. To account for these losses, a damper, as shown in Figure 3.10(c), is added. Here \( b_t \) is the damping coefficient that accounts for the gross effects of system damping and is assumed to be constant.

The equation of motion for the system shown in Figure 3.10(c), relating the input displacement \( z_i(t) \) to the spring compression \( z_c(t) \), is then

\[
m_i \frac{d^2 z_o(t)}{dt^2} + b_t \frac{dz_c(t)}{dt} + k_i z_c(t) = 0 , \quad (3.42)
\]

\[
m_i \frac{d^2[z_c(t) + z_i(t)]}{dt^2} + b_t \frac{dz_c(t)}{dt} + k_i z_c(t) = 0 , \quad (3.43)
\]

\[
m_i \frac{d^2 z_c(t)}{dt^2} + b_t \frac{dz_c(t)}{dt} + k_i z_c(t) = -m_i \frac{d^2 z_i(t)}{dt^2} , \quad (3.44)
\]
where $k_t$ is the spring constant and $m_t$ is the mass of the mass element. Assuming zero initial conditions, and using the Laplace transform, the deflection $Z_c(s)$ given an input $Z_t(s)$ is

$$Z_c(s) = \frac{-s^2}{s^2 + \frac{k_t}{m_t} s + \frac{k_t}{m_t}} Z_t(s) ,$$

(3.45)

where $Z_c(s)$ and $Z_t(s)$ are the Laplace transforms of $z_c(t)$ and $z_t(t)$, respectively. Equation (3.45) represents the mechanical element of the transducer model.

The electrical element of the transducer model assumes first order piezo-electric effects that result in a linear relationship between the material polarization and the stress, as well as the electric field and strain [32]. The motion of the transducer or analogous spring compression results in a strain within the piezo-electric material that produces an electric field such that the voltage across the electrodes is proportional to the compression. This is expressed as

$$v_t(t) = -K_t z_c(t) ,$$

(3.46)

where $K_t$ is the gain or proportionality constant relating the motion $z_c(t)$ to $v_t(t)$. The complete electrical model is a simple voltage generator in series with the piezo plate capacitance $C$ and a resistive load $R_t$ and is shown in Figure 3.11 [32]. The resistance $R_t$ represents the load of the amplifier measuring the piezo voltage $v_t(t)$.
Using simple voltage division, the output voltage that is measured by the amplifier is given by

\[ V_o(s) = \frac{R_i}{R_i + 1/sC} V_i(s) \]  
\[ = \frac{R_i}{R_i + 1/sC} [-K_t Z_o(s)] \]

where \( V_o(s) \) and \( V_i(s) \) are the Laplace transforms of \( v_o(t) \) and \( v_i(t) \), respectively. Zero initial conditions are also assumed. Equation \( v \) written as

\[ V_o(s) = -R_i C K_i \frac{s}{1 + s R_i C} Z_o(s) \]

and represents the electrical element of the transducer model.

### 3.5.2 The Behaviour of the Transducer Model

The complete piezo-electric transducer model is given using Equation (3.49) and Equation (3.45) and is written

\[ V_o(s) = \left(-R_i C K_i \frac{s}{1 + R_i C s}\right) \left(\frac{-s^2}{s^2 + \frac{K_s}{m_r} s + \frac{K_t}{m_i}}\right) Z_i(s) \]

\[ \frac{V_o(s)}{Z_i(s)} = R_i C K_i \frac{s^3}{(1 + R_i C s) \left(s^2 + \frac{K_s}{m_r} s + \frac{K_t}{m_i}\right)} \]

It is assumed that the break point at \( \omega_e = \frac{1}{R_i C} \), given in the electrical element of the transducer model is much lower in frequency as compared to the resonant point of the mechanical portion, given by \( \omega_m = \sqrt{\frac{K_s}{m_r}} \). Practically, this is a reasonable assumption as \( R_i \) is usually in the order of few megaohms and \( C \) is in the order of a few nanofarads resulting in \( \omega_e \) being below a few hundred hertz and where \( \omega_m \) is typically tens of kilohertz.

With these assumptions the magnitude response of the transducer model, normalized in frequency to \( \omega_m \), is shown in Figure 3.12. Arbitrary values of \( b_t = 1 \), \( R_i C K_i = 1 \), and a ratio of 0.01 for \( \omega_e/\omega_m \) are used to obtain the graph. The frequency axis is in radians per second. The characteristic behaviour of the model is
Figure 3.12: The magnitude response of the complete piezo transducer model

that of a high pass filter with a peak at $\omega_m$. Briefly, from Figure 3.12, the slope of the response between $\omega_m$ and $\omega_c$ is about 40 dB per decade. Below $\omega_c$ the slope of the response is about 60 dB per decade. Above $\omega_m$ the response is flat at 0 dB. The general behaviour predicted by this transducer model is to emphasize the high frequency components of the input displacement, especially at $\omega_m$. 
4. Model Simulations

4.1 Overview

This chapter presents the results of simulating the ideal sensor model that represents the experimental sensor. The objectives of these simulations are to determine the nature of the force pulses and plate displacements generated in the experiments and to compare the behaviour of the simulated and experimental sensor outputs. This is done to provide support for the arguments proposed in Chapter 1 and to generate specific results for the theory given in Chapter 3.

Simulations of the force pulse and its spectra are presented first. Next, the simulations of plate displacements, as measured at the transducer location, are examined as the simulated impact locations move to 0 cm, -1 cm, -2 cm, and -3 cm from impact site B. The sensor outputs are then simulated using the transducer model and compared to the actual sensor outputs. Finally, the spectra of the simulated sensor outputs are examined.

4.2 Force Pulse Simulations

This section presents simulating the force pulses generated in the impacts of the steel spheres with the aluminium plate during the experiments. The spectra of these force pulses are also examined.

The force pulses were simulated by solving the dimensionless form of the force model equations using the values of λ obtained from the material parameters of the sphere and the plate. The modulus of elasticity and Poisson ratio for common steel and aluminium are listed in material property tables [28]. The densities of the steel
<table>
<thead>
<tr>
<th>Item</th>
<th>Material Type</th>
<th>Material Density $\rho$ (kg/m$^3$)</th>
<th>Poisson ratio $\nu$</th>
<th>Modulus of Elasticity $E$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>Steel</td>
<td>7100 to 7900</td>
<td>0.29</td>
<td>200</td>
</tr>
<tr>
<td>Plate</td>
<td>Aluminium</td>
<td>2710</td>
<td>0.33</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 4.1: The sphere and plate material parameters used for the force pulse simulations

<table>
<thead>
<tr>
<th>Particle Radius $r_s$ (mm)</th>
<th>Calculated values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$T$ (usec.)</td>
</tr>
<tr>
<td>1.191</td>
<td>0.781</td>
</tr>
<tr>
<td>1.588</td>
<td>1.480</td>
</tr>
<tr>
<td>1.984</td>
<td>2.31</td>
</tr>
<tr>
<td>2.381</td>
<td>3.25</td>
</tr>
<tr>
<td>2.778</td>
<td>4.53</td>
</tr>
<tr>
<td>3.175</td>
<td>5.87</td>
</tr>
</tbody>
</table>

Table 4.2: The calculated values of $\lambda$, $T$, $0.5T^{-1}$, $F_m$, $f_{a_ph}(t)$ and $t_{end}$ associated with each sphere used in the experiments.

spheres is obtained by averaging the mass of ten spheres and dividing by the sphere's volume. The density of aluminium is given in the material property tables [28]. The resulting values for the material parameters are listed in Table 4.1.

The radii of the spheres and the plate thickness is set to the values used in the experiments or to, 1.191 to 3.175 mm, (3/64 to 8/64 inches) for the sphere radii and 0.7938 mm, (1/32 inches) for the plate's thickness. With a drop height of 10 cm, the particle impact velocity becomes 1.40 m/s for all the spheres.

With these material parameters, the corresponding values of $\lambda$, $T$, $0.5T^{-1}$, $F_m$, $f_{a_ph}(t)$ and $t_{end}$ for each of the six spheres are listed in Table 4.2. The parameters $f_{a_ph}(t)$ and $t_{end}$ indicate the peak force and the duration of the impact, respectively.
Solving the dimensionless force model equations, using the values of \( \lambda \) listed in Table 4.2, and then scaling, in both magnitude and time by the appropriate factors of \( F_m \) and \( T \), results in a force pulse in newtons shown as a function of time in seconds. The results of these simulations and scaling are shown in Figure 4.1 and Figure 4.2.

The shape of the force pulses are as expected given their values of \( \lambda \). The theory in Chapter 3, indicates that with \( \lambda > 0.5 \), the change in the peak magnitudes and durations of the force pulse begins to be a complex function of radius. This is seen by the small increases in the magnitudes and large increases in duration of the force pulses given the changes in the particle radii. The large increase in the force pulse's duration will cause a large change in the bandwidth of the force spectra.

The spectra for the force pulses are shown in Figure 4.3. The expected narrowing of the bandwidths and the increase in the magnitudes as the particle radius increases are present.

## 4.3 Simulated Plate Displacement at the Transducer Location

The simulated force pulses of section 4.2 are used with the ideal thin plate model to simulate plate motion over a duration of 450 \( \mu s \) from the instant of impact. The dimensions of the plate, impact and transducer locations, were set to approximate those in the experiments. The simulated impact locations were chosen to correspond to the impact locations at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B.

Figure 4.4 shows the simulated plate displacements at the transducer due to impacts at the same site. The decrease in the abruptness of the plate motion, as the particle radius is increased, is obvious. Also apparent is the absence of characteristics associated with significant reflections. This indicates that the lateral dimensions of the plate were large enough to prevent reflections of significant magnitude from returning and interfering with the force pulse.
Figure 4.1: Simulated force pulses generated during the impact of steel spheres with a velocity of 1.4 m/s and radii of 1.191 mm, 1.588 mm, and 1.984 mm onto a 0.7938 mm thick aluminium plate.

Figure 4.2: Simulated force pulses generated during the impact of steel spheres with a velocity of 1.4 m/s and radii of 2.381 mm, 2.778 mm, and 3.175 mm onto a 0.7938 mm thick aluminium plate.
Figure 4.3: Comparison of the spectra of simulated force pulses generated during the impact of steel spheres with a velocity of 1.4 m/s and radii of 1.191 mm, 1.588 mm, 1.984 mm, 2.381 mm, 2.778 mm, and 3.175 mm onto a 0.7938 mm thick aluminium plate.

Figure 4.5 through Figure 4.7 shows the simulated plate displacements at the transducer due to impacts at -1 cm, -2 cm, and -3 cm from impact site B, respectively. Further reduction in the abruptness and magnitudes are seen as the impacts move away from the transducer. Also apparent is the delay in the start of the motion and its gradual increase. These characteristics were shown and discussed in the theory presented in Chapter 3.

4.4 Comparison of Actual and Simulated Sensor Signals

This section compares the simulated and actual sensor outputs resulting from the largest and smallest particle impacting at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B.

To simulate the sensor, the transducer model parameters were set to approximate the response of the actual sensor to the impact of the largest particle at the
Figure 4.4: The simulated plate displacements at the transducer location due to particle impacts at approximately 0 cm from impact site B.

Figure 4.5: The simulated plate displacements at the transducer location due to particle impacts at approximately -1 cm from impact site B.
Figure 4.6: The simulated plate displacements at the transducer location due to particle impacts at approximately -2 cm from impact site B.

Figure 4.7: The simulated plate displacements at the transducer location due to particle impacts at approximately -3 cm from impact site B.
Table 4.3: The model parameters used to simulate the transducer location. The simplicity of the model and the differences between it and the actual transducer response results in the selection of the transducer model parameters to be very subjective. These parameters include the resonant frequency, the damping ratio, the gain factor, the plate capacitance, and the load resistance. The resonant frequency $\omega_m$ is approximated by the inverse of twice the interval between zero crossings of the first positive peak of the signal generated in the impact of the largest particle at 0 cm from site B. The mass element $m_t$ of the model is equated to the mass of the transducer that is obtained from its measured volume and given mean density. The model’s spring constant $k_i$ is then found using $\omega_m = \sqrt{\frac{k_i}{m_t}}$. The damping ratio $b_t$ and the gain factor $K_t$ are also selected to approximate the damping and the peak magnitude of the signals generated in the impact of the largest particle at 0 cm from site B. The transducer’s plate capacitance $C$ was measured using a capacitance meter and the load resistance $R_l$ was that of the oscilloscope. The values used for these parameters are listed in Table 4.3.

The discrete time representation of the plate displacement requires determining a discrete time version of the transducer transfer function. This was obtained by performing a bilinear transform on the transfer function to obtain a set of polynomial coefficients that are used to implement a digital filter exhibiting the same magnitude and phase response as the transfer function.

Figure 4.8 shows the actual signals due to three impacts of the largest of the

<table>
<thead>
<tr>
<th>Resonant frequency $\omega_m$ (kHz)</th>
<th>Mass element $m_t$ (g)</th>
<th>Spring constant $k_i$ (N/m)</th>
<th>Damping ratio $b_t$</th>
<th>Gain factor $K_t$ (V/m)</th>
<th>Transducer capacitance $C$ (nF)</th>
<th>Load resistance $R_l$ (MΩ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>44</td>
<td>$7 \times 10^3$</td>
<td>2.2</td>
<td>$1 \times 10^7$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

1. The mean density of the transducer is given by the manufacturer data specification for the particular piezo-electric transducer.
2. Appendix I discusses the details of approximating the transducer transfer function with a discrete time version.
Figure 4.8: Comparison of the experimental and simulated sensor outputs for a 3.175 mm radius particle impacting 0 cm from impact site B. The simulated sensor output is overlayed on these signals for a direct comparison. In spite of the simplicity of the transducer model, the actual and simulated signals are similar in the peak magnitude and the decay rate. However, the simple decay of the simulated sensor signal does not match the complex form of the decay in the actual signal.

This differences becomes more pronounced as the impact site is moved away from the transducer, as shown in Figure 4.9 through to Figure 4.11 comparing the actual and simulated sensor signals due to the largest particle impacting at -1 cm, -2 cm, and -3 cm from impact site B, respectively.

Another similarity between the signals. During the interval of time between 100 μs and 200 μs, both signals appear to be shifted in the negative direction and then, over the remaining interval of time, they both slowly rise back to zero volts. This effect also becomes more pronounced as the impact sites move away from the transducer. If the duration, of about 200 μs, represents four time constants for this behaviour, then this characteristic response occurs below about 3 kHz.
Figure 4.9: Comparison of the experimental and simulated sensor outputs for a 3.175 mm radius particle impacting -1 cm from impact site B.

Figure 4.10: Comparison of the experimental and simulated sensor outputs for a 3.175 mm radius particle impacting -2 cm from impact site B.
Figure 4.11: Comparison of the experimental and simulated sensor outputs for a 3.175 mm radius particle impacting -3 cm from impact site B.

Figure 4.12 through to Figure 4.15 compares the actual and simulated sensor signal due to the smallest particle impacting at 0 cm, -1 cm, -2 cm, and -3 cm from impact site B, respectively. The simulated signals, especially for the impact further from the transducer, are subjectively more similar to the actual signals. This similarity is based on the observation that both signals are of the same magnitudes and moving in the same directions at about the same time more often than the signals due to the impacts of the largest particle.

The increase in similarity may be explained by proposing that the model’s accuracy increases at higher frequencies and that its linearity increases for small movements. Evidence for these characteristics is provided in Figure 4.14 and Figure 4.15. In these figures, the actual and simulated signals are almost identical after about 400 µs and both exhibit an oscillation at about 40 kHz.\(^3\) It is not surprising to see

\(^3\)Due to the fixed trigger level of the digital oscilloscope, the actual signal required a small shift or delay in plotting in order to align the signal in the time domain so that a comparison to the simulated output could be made. This was done by either aligning the first peak of both signals or moving the initial rise of the actual signal to the zero time point of the simulated data.
Figure 4.12: Comparison of the experimental and simulated sensor outputs for a 1.191 mm radius particle impacting 0 cm from impact site B.

Figure 4.13: Comparison of the experimental and simulated sensor outputs for a 1.191 mm radius particle impacting -1 cm from impact site B.
Figure 4.14: Comparison of the experimental and simulated sensor outputs for a 1.191 mm radius particle impacting -2 cm from impact site B.

Figure 4.15: Comparison of the experimental and simulated sensor outputs for a 1.191 mm radius particle impacting -3 cm from impact site B.
these high frequencies since the force pulse simulations show that the bandwidths, associated with the impacts of the smaller particles, are well above 40 kHz. In addition, the plate displacement simulations for the smallest particles also reveals that the change in the motion is very small after 400 μ. Regardless, of the transducer’s response, given a small enough motion, the transducer’s response will appear linear over the range of motion. Therefore, in spite of the simplicity of the transducer model, the simulated sensor response indicates that the low and high frequency response of the transducer model is reasonably accurate. This is especially true for small signal levels where the actual transducer appears to have a more linear response.

The discrepancies between the two signals are seen in the peak magnitudes and in the form of the signals during the decay interval. The differences during the decay interval could be due to the complex boundary conditions or mounting of the transducer which must include the effects of the connecting wires. This could add extra resonant frequencies that would affect the signal. The differences in the peak magnitudes can be explained by proposing that the transducer’s response is not linear for large excitations. The differences in the simulated and actual peak magnitudes also indicates that comparing the simulated and actual peak signal as a function of particle radius or distance will not provide any meaningful results.

4.5 The Spectral Content of the Simulated Sensor Output

This section presents a set of power spectral densities calculated from the simulated sensor output signals. It is shown that the changes in the ratio of power densities are similar to that seen in the actual ratios.

The simulated power spectra were obtained using the same procedure as that used to obtain the actual spectra given in Chapter 3. Figure 4.16 and Figure 4.17 show the simulated spectra of the sensor signal due to impacts of all the particles at 0 cm from impact site B. It is apparent from Table 4.4, that as the particle radius
Table 4.4: The simulated power spectral densities near the frequencies at 10 kHz and 20 kHz for the signals produced by the particles at 0 cm from impact site B.

<table>
<thead>
<tr>
<th>Particle Radius $r_*$ (mm)</th>
<th>Power spectral densities</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>at $\approx$ 10 kHz</td>
<td>$P_1$ ($\mu W/Hz$)</td>
<td>$P_2$ ($\mu W/Hz$)</td>
</tr>
<tr>
<td>3.175</td>
<td>5.2</td>
<td>31</td>
<td>6.0</td>
</tr>
<tr>
<td>2.778</td>
<td>4.2</td>
<td>26</td>
<td>6.2</td>
</tr>
<tr>
<td>2.381</td>
<td>2.8</td>
<td>21</td>
<td>7.5</td>
</tr>
<tr>
<td>1.984</td>
<td>1.49</td>
<td>16.3</td>
<td>11.0</td>
</tr>
<tr>
<td>1.588</td>
<td>0.46</td>
<td>7.8</td>
<td>17.0</td>
</tr>
<tr>
<td>1.191</td>
<td>0.087</td>
<td>1.57</td>
<td>18.0</td>
</tr>
</tbody>
</table>

increases the ratio of the power densities at 20 kHz to that at 10 kHz decreases. This suggests the bandwidth of $Z_t(\omega)$ is decreasing with particle radius. This behaviour was also seen in the actual spectra.

Figure 4.18 and Figure 4.19 shows the spectra of the simulated transducer signal due to all the particles impacting at -3 cm from impact site B, respectively. The large emphasis of the frequencies below approximately 10 kHz over that seen in Figure 4.16 is apparent, especially for the largest particles. A reduction in the magnitudes is also present. This same behaviour was seen in the actual spectra.

4.6 Summary of the Simulations

This chapter has shown that varying the simulated particle radius and impact location results in the simulated sensor output behaving in a manner that is consistent with the discussion in Chapter 1. Specifically, the impact of large particles produces force pulses with greater magnitudes and narrower bandwidths than the impact of smaller particles. The resulting plate motion, as measured by the transducer, is also less abrupt than the motion due to the impact of smaller particles. As the impact location is moved away from the transducer, the sensor signals are reduced.

*These particular frequencies were selected for ease of measuring power densities.*
Figure 4.16: The power spectral density of the simulated sensor output due to impacts of particles with radii of 1.984 mm, 1.588 mm, and 1.191 mm at the location 0 cm from impact site B.

Figure 4.17: The power spectral density of the simulated sensor output due to impacts of particles with radii 3.175 mm, 2.778 mm, and 2.381 mm at the location 0 cm from impact site B.
Figure 4.18: The power spectral density of the simulated sensor output due to impacts of particles with radii 1.984 mm, 1.588 mm, and 1.191 mm at the location -3 cm from impact site B.

Figure 4.19: The power spectral density of the simulated sensor output due to impacts of particles with radii 3.175 mm, 2.778 mm, and 2.381 mm at the location -3 cm from impact site B.
in magnitude and abruptness. The characteristics of the simulated sensor signal is similar to the actual sensor output for small signals below 3 kHz and possibly near 40 kHz. The simplicity of the transducer model results in the response of the simulated and actual sensor output being very different for large signals. This prevents using the simulated sensor to gain additional insight into the peak sensor signal. In spite of this difficulty, the spectra of the sensor signals do show changes, due to changes in particle radius and impact location, that are similar to the changes observed in the actual spectra of the sensor signals.
5. Discussion

5.1 Overview

This chapter opens with a brief discussion summarizing the significance of the experimental and simulated results. Next the possible sources of error and their influence on results is presented. Then using the theoretical model and the simulation results, the behaviour of the the grain loss monitor sensor is hypothesized. It is argued that changes in the impact position of less than ±0.25 mm and ±0.5 mm in the longest and shortest lateral dimensions, respectively, were responsible for the reported peak sensor signal variation of 15% of the mean. Finally, additional research is suggested for future work.

5.2 Significance of the Experimental and Simulation Results

The experimental and simulation results showed that the variation in the magnitude of the peak sensor signal is primarily due to changes in the particle impact position. The results also showed that the sensitivity of the peak sensor signal to changes in impact location varies with the impact location and to some degree with the radius of the impacting particle. Specifically from the impact locations tested, the experiments indicated that the peak sensor signal is the most sensitive to changes in impact position at the transducer site and the least sensitive at approximately 2 cm from the transducer. The experimental results also indicated that the peak sensor signal had almost a linear relationship to particle radius at all the tested impact locations. These results are not obvious from theoretical expressions but were observed through simulations based on the theory. Though differences between the
peak signals of the simulation and experimental results were seen, the results for small signals at low and high frequencies are reasonably similar. This suggests that the theory provides a some basis to describe the processes that occur in the operation of the sensor.

The ease of obtaining peak sensor signals from experiments is also significant and suggests that a series of tests could be used to characterize the sensor so that its performance in measuring particle radius can be determined. These tests require obtaining data to graph $v_{pk}$ as a function of radius at fixed distances from the transducer and graphs showing $v_{pk}$ as a function of distance from the transducer for fixed radii. In addition, graphs of $dv_{pk}/dr_s$ at fixed distances and curves of $dv_{pk}/dr_i$ at fixed radii, where $r_i = \sqrt{(x_i - x_t)^2 + (y_i - y_t)^2}$ represents the distance between the impact site and the transducer, will indicate the sensitivity of the peak sensor output to changes in radius and distance. These graphs may also indicate locations that are the least sensitive to changes in impact position or the most sensitive to radii changes.

The ease of performing these tests also hides the complexity of the processes that generate these signals. This complexity is exemplified in the theory where numerical simulation are required to predict the sensor signal for a particular change in the impact location and particle radius. Regardless of the complexity of the theory, it provides a qualitative means to assess the effects a change in sphere and plate parameters has on the sensor output. Detailed determinations can be obtained using numerical simulations. For example, determining how the plate displacement spectrum $Z_t(\omega_{mn})$ is affected by changes in the lateral dimensions of the plate can initially be considered qualitatively by examining Equation (3.17) and then in detail by numeric simulations. This process avoids testing the characteristics of a large numbers of different sized plates and spheres and allows a directed approach in selecting a suitable combination of plate and sphere parameters.
5.3 Experimental and Modelling Concerns

5.3.1 Experimental Measurement Errors

Impact Location and Peak Sensor Voltage

The errors in measuring the sensor voltage are not significant and are due to the errors introduced by the digital storage oscilloscope. It is assumed that negligible error is introduced by the accompanying measuring probe which consists of a coaxial cable with connecting clips. These errors are separated into fixed and random errors. The fixed errors include the vertical gain errors of the oscilloscope and are assumed to be constant over the oscilloscope's bandwidth of 0 Hz to 100 MHz. The vertical gain errors, as per the instrument data specifications, are no greater than ±1.5%. This fixed error only affects the accuracy of the absolute value of the results. The relationship between the peak sensor voltage and the impact location is not affected. The random errors include the quantization errors of the oscilloscope and the influence of any external noise sources. The quantization error of the oscilloscope is given as ±0.4% and the external noise levels were measured to be less than 20 mV peak to peak over the bandwidth of the oscilloscope. This 20 mV noise signal represents an error of approximately ±1% of the smallest peak sensor signal. Therefore, the worst case random errors in measuring the peak sensor signal is approximately ±1.4%, which is the sum of both errors. The small random errors clearly do not significantly change the experimental results.

In the particle impact experiments at site B, the particles did not fall at precisely the same point. For these experiments the variation of the impact sites was measured by first noting where, relative to the reference axis, the particles impacted. This was done by marking the impact location and then estimating the change in the subsequent impact positions. These changes were estimated to be about ±0.5 mm. Exact measurements could not be made because of the suddenness of the impact. Therefore, a best estimate of each impact site was made. This estimate was aided
by the particles exhibiting almost no rebounding thereby remaining near the impact site. By observing the complete set of impact sites, a location representing the most common or mean impact position was selected. This location obviously has an error, which is difficult to determine, but by noting the locations of all the observed impact sites a worst case error of ±0.5 mm can be assumed since almost all the observed impacts occurred within this distance of the mean location. This error in the mean impact location translates into a 5%, 2.4%, and 1.6% error in the impact locations at 1 cm, 2.1 cm, and 3.15 cm from impact site B.

Finally, the estimate of the mean location of impact site A was selected using the impact locations of the largest particle since it had the smallest observed variation. The error in estimating the mean location is determined in the same manner as described previously and is given as ±1 mm. It should be noted that the smallest particle had a variation of approximately ±2 about the mean location, whereas the other particles all had variations of about ±1 mm.

**Dimensional Measurements**

These measurements include determining the dimensions of the plate, the radii of the spheres, and the location of both the reference axis and the transducer. The lateral dimensions of the plate and the location of the reference axis and the transducer, were measured to within ±0.5 mm or about 0.1% of the plate’s lateral dimensions. The thickness of the plate was measured to within ±0.0002 inches or approximately ±0.7% of its thickness. The radii of the sphere had a tolerance that was listed by the distributor to be +0%, -1%. In the experiments, the same particle was dropped onto the plate so that small changes in the radius of the particles would not influence the results. The remaining errors do not affect the experimental results but they must be considered in the simulation results.
5.3.2 Error Considerations in the Simulations

Dimensional Measurements

These measurements also include the dimensions of the plate, the radii of the spheres, and the location of the reference axis, the transducer, and the impact sites. Section 5.3.1 indicated the magnitude of these errors. The significance of these errors on the simulation results is examined by using the appropriate equations describing the model. For example, the thin plate transfer function \( H_p(\omega_{mn}) \) given by Equation (3.17) will indicate the influence these dimensional errors have on the thin plate behaviour. This is seen by considering the terms where the lateral dimensions are used and include \( K_p, A_{mn}, \) and \( K_{mn} \). These are listed below for convenience.

\[
K_p = \frac{2}{\rho_p l_x l_y},
\]

\[
K_{mn} = \frac{1}{\omega_{mn}} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right),
\]

and

\[
A_{mn} = \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right).
\]

Given the stated dimensional errors, \( K_p \) has an error of approximately \( \pm0.9\% \) and results in \( H_p(\omega_{mn}) \) being scaled by a small amount. This does not affect the simulation results significantly.

More importantly are the effects of the dimensional errors on \( A_{mn} \) and \( K_{mn} \). These errors are frequency dependent in a manner that is similar to the frequency dependency seen in the \( z_i(t) \) or \( Z_i(\omega_{mn}) \), as described in section 3.3.3. The influence of the dimensional errors on \( A_{mn} \), and \( K_{mn} \) is understood by noting that \( A_{mn} \), and \( K_{mn} \) describe the vibrational mode shapes of the thin plate. At any particular frequency a number of half sinusoids are formed over the surface of the plate. A dimensional change results in changing \( A_{mn} \) and \( K_{mn} \). The greater the distance between nodes or zero points of any half sinusoids formed over the surface, the smaller the influence a dimensional change has on \( A_{mn} \) and \( K_{mn} \). If it is assumed that changes in dimension
of less than 10% of the distance between nodes produce negligible effects in the results of $H_p(\omega_{mn})$, then frequencies above 40 kHz, associated with inter-node distances of less than 10 mm, may begin to introduce errors.\textsuperscript{1} With most of the spectral content of the sensor output confined to less than 40 kHz, these errors do not affect the simulation results significantly.

Sphere and Plate Material Parameters

The material parameters for the plate and sphere were either assigned typical values or were equated to a measured quantity. In particular for the sphere and the plate, the values for the modulus of elasticity, Poisson ratio, and material density of common aluminium and steel were obtained from material data tables [28]. The density of the steel spheres was measured by using the mass of ten spheres and then dividing the average weight by the respective volume.

The concern in using typical values for some of the parameters is the possibility of the actual plate and sphere parameters being significantly different from the assumed values. This was addressed by learning that the parameters for most aluminium and steels typically vary less than ±10% amongst themselves [28]. These variations can alter the simulation results, especially if the variations from each parameter produces changes in the results that are cumulative. This is not the case and is shown by considering the transfer function of the thin plate, given by Equation (3.17), the resonant frequency of the plate, given by Equation (3.6), and the inelasticity parameter, given by Equation (3.35). For example, given a 10% increase in the plate’s density $\rho_p$, a decrease in the resonant frequencies of approximately 5% results. A 10% increase in $K_p$ also occurs. The net effect is to decrease the magnitude of $H_p(\omega_{mn})$ by 5%. In addition, the inelasticity parameter decreases by 6%.

Now consider the case where not only is the plate’s density $\rho_p$ increased by 10% but

\textsuperscript{1}This result are obtained by noting that the distance between the nodes of the first sinusoid mode is equal to the dimensions of the plate or about 0.5 m. A distance of 10 times 1 mm or 10 mm requires 50 half sinusoids across the length of the plate. Using Equations (3.6) with $m = n = 50$, a frequency of approximately 40 kHz results.
its modulus of elasticity $E_p$ is also decreased by 10%. This causes the resonant frequencies to decrease by 10% and $K_p$ to increase by 10%. The net effect is to keep the magnitude of $H_p(\omega_{mn})$ constant. The inelasticity parameter also decreases by about 7%. These simple examples show that increasing the error in one term can reduce the error in another.

Changes in the magnitude of $H_p(\omega_{mn})$ on the order of ±10% or small changes in the resonant frequencies $\omega_{mn}$ do not influence the results of the simulation significantly since only fractional changes in the magnitude of $Z_1(\omega_{mn})$ or $z_1(t)$ are expected. Errors on the order of ±10% in the inelasticity parameter also do not significantly affect the force pulse and its associated spectrum.

5.3.3 Simulation Assumptions

The influence of all the previous errors was taken into account by using the appropriate equations and assuming specific errors for the parameters. Other source of errors, due to the assumptions made in the theory, may compounded the effect of these previous errors. Since it is difficult to assess the consequence of these assumptions, they are discussed qualitatively in terms of how they influence the results.

Boundary Conditions

Changes to the boundary conditions affect the mode shape and resonant frequency of the plate and therefore change the summation in the solution of the plate displacement and its sensitivity to changes in impact location. It is difficult to assess the influence of these changes analytically since it involves finding the solution to the classical thin plate equation for complex boundary conditions. The simulations assumed simply supported edges along all sides of the plate. The approximation used in the actual plate had two round edged clamps holding the plate within 2 mm of every side except for areas about 3 cm from the corners. Since the simulation and experimental results were reasonably similar it is likely that the experimental boundary condition were also a good approximation of a simply supported boundary.
Plate Vibrations

The simulations assumed that once excited the plate vibrates indefinitely. In reality this is not the case since there are factors, such as air friction and the mass of the actual transducer that can cause these vibration to dampen out. More importantly, the attenuation of these vibrations occur in the plate material itself. This attenuation factor is function of frequency that typically increases with increasing frequency [24, 26]. Again, since the simulation and experimental results were reasonably similar it is likely that attenuation was not a significant factor to consider, at least for the small distances and time intervals simulated.

The Force of Impact

There are two concerns that could influence the force of impact, they are the return of reflections and the assumption of perfectly elastic impacts. Since both the simulations and experimental results showed no influence of reflections on the resulting sensor signals, it is assumed that reflections are not affecting the force pulse significantly.

The second concern is the assumption of perfectly elastic impacts. It was observed during the experiments that very small indentations in the plate were being made after a particle impact. This suggests that a portion of the impact energy is being used to deform the plate and not to set up vibrations. Therefore, the force pulse will be different from that modelled. Since the indentations were very minute they may not be a significant influence but may be of concern when they are directly above the transducer. In this case, the transducer could experience additional forces or shock that were not considered in the simulations. This could have been factor in producing output signals larger than simulated.

Truncation of the Force Pulse Spectra

The simulations of the plate displacements did not use an infinite number of frequencies to determine the solution. Instead, a finite number of frequencies were
used that corresponded to force pulse spectral components having magnitudes greater than approximately 10% of the maximum magnitude. As shown in Appendix I, this produces results that are well within ±1.5% of the final result.

It should be noted that all numeric simulations used double precision, 16 digit, floating point numbers with a range of approximately $10^{-308}$ to $10^{308}$. Therefore round off errors in the numeric calculations are not expected to be a major concern.

5.4 Extending Results to the Grain Loss Monitor Sensor

It is reasonable to ask how the peak sensor signal changes given changes in the sphere and plate parameters, especially for parameters that attempted to approximate the grain loss monitor sensor. As discussed in Chapter 1, the grain loss monitor sensor is different from the thin plate impact sensor in many aspects and it was initially proposed that the theory and experiments could be of questionable use on the grain loss monitor sensor. Using the experimental results and the developed theory, it is shown that if any concerns in extending the results to the grain loss monitor sensor exist, they are not significant.

5.4.1 The Changes in the Behaviour of the Thin Plate

Reducing the lateral dimensions of the plate to be equal to the dimensions of the grain loss monitor sensor results in $l_x = 12$ cm and $l_y = 6$ cm. This change reduces the size of the mode shapes for the vibrating plate. For example, instead of mode $\{m, n\} = \{1, 1\}$ being formed over approximately 0.5 m in the experimental sensor, it is formed over a $12 \times 6$ cm surface. Similarly, the higher order modes are formed over a smaller distance. This results in a given change in impact position producing more of a change in $H_p(\omega_{mn})$ in the smaller plate than the larger plate. For example, given a change of impact position of $\Delta x_i = 0.5$ mm and $\Delta y_i = 0$ mm from an initial position of $x_i = l_x/2$, $y_i = l_y/2$, the change in $H_p(\omega_{mn})$ for $\{m, n\} = \{11, 11\}$
for this small plate is expressed as the ratio

\[
\Delta H_p(\omega_{11}) = \frac{H_p(\omega_{11})|_{x_i, y_i}}{H_p(\omega_{11})|_{x_i+\Delta x_i, y_i+\Delta y_i}} = \frac{\pi K_p K_{11} A_{11}|_{x_i, y_i}}{\pi K_p K_{11} A_{11}|_{x_i+\Delta x_i, y_i+\Delta y_i}} \frac{(\sin \frac{\pi}{l_x} x_i \sin \frac{\pi}{l_y} y_i)}{(\sin \frac{\pi}{l_x} (x_i + \Delta x_i) \sin \frac{\pi}{l_y} (y_i + \Delta y_i))} = \frac{(\sin \frac{11\pi}{2} \sin \frac{11\pi}{2})}{(\sin \frac{11\pi}{12} (6.05) \sin \frac{11\pi}{2})} = \frac{1}{0.9897} = 1.010.
\]

The change in \(H_p(\omega_{mn})\) due to this small change in impact position is given by a factor of approximately 1.01. For this same change in impact position, the large plate has a change in \(H_p(\omega_{mn})\) given by a factor of approximately 1.0006. This suggests that a smaller plate has a greater sensitivity to changes in impact position that could consequently cause the peak sensor signal to be more sensitive to changes in impact position.

If \(\Delta H_p(\omega_{mn})\) provides a measure of the tolerable error given a change in impact position, then \(\Delta x_i\) for the small plate must be reduced by approximately 4 times or to 0.125 mm to correspond to the values of \(\Delta H_p(\omega_{mn})\) for the large plate. This result is seen by noting that the distance between nodes for any mode has decreased by the ratio of the change in dimensions or in this case a decrease of approximately 4 times for \(l_x\) and 8 times for \(l_y\). Therefore, to obtain an equivalent change in \(\Delta H_p(\omega_{mn})\) for the smaller plate requires a reduction in \(\Delta x_i\) or \(\Delta y_i\) equivalent to the reduction in \(l_x\) or \(l_y\), respectively. In this example, the changes in the \(y\) direction then needs to be reduced by a factor of 8 or to a distance of about 0.063 mm.

These results are used to estimate the variation in the impact position that resulted in a peak signal variation reported for the grain loss monitor sensor. This estimate is also based on the observation that the peak signal variation appears to be proportional to the change in impact location for small changes in position.
Consider the variation of approximately 4% of the mean in the peak signal seen in the experiments for all particles dropped at locations referenced to site B. Assuming that the variation in the impact position is reasonably proportional to the peak signal variation, then the ±0.5 mm changes in impact position needs to be increased by a factor of 4 or to about ±2 mm to produce a peak signal variation of approximately 16% of the mean. Therefore, reducing the size of the plate to approximate the size of the grain loss monitor requires the ±2 mm change in impact position to be reduced by a factor of 4 and 8 in the x and y directions, respectively. This results in variation in impact positions of $\Delta x_i \approx 0.5$ mm and $\Delta y_i \approx 0.25$ mm. These small changes in the impact position would be difficult to detected without careful observation or a measurement device. Since changes in the impact position were not reported in the experiments using the grain loss monitor to size potash particles, it is likely that small undetected changes in the impact position were responsible for the variation of the peak signal.

The effect of reducing only the lateral dimensions also increases both the resonant frequencies and the frequency spacing between the resonant frequencies, as indicated by Equation (3.6). Specifically using Equation (3.6), this reduction in $l_x$ and $l_y$ of approximately 4 and 8 times, respectively, causes the frequencies to increase by approximately $4^2 + 8^2 = 80$ times. Other parameters also affect these frequencies, in particular, the modulus of elasticity and Poisson ratio. It is difficult to select precise values for these parameters given an unknown type of plastic but, using simple assumptions, estimates are possible.

It is reasonable to expect the modulus of elasticity of the plastic to be less than that of aluminium and the Poisson ratio to possibly be equal to or greater than that of aluminium [33]. It is also reasonable to assume that the decrease in the modulus of elasticity for the plastic plate is relatively large and that it could approach a factor 100 or more [33]. The Poisson ratio, if it increases, can not exceed 0.5 so that it is a factor of 1 to 1.5 times larger than aluminium. In addition, the density of the plastic is expected to be at least one half to one third of that of
aluminium [33]. Therefore using Equation (3.6), the combination of these parameters tends to lower the resonant frequencies by a factor of 5 to 9. Finally, increasing the thickness of the plate from approximately 0.8 mm to 2 mm also tends to increase the resonant frequencies by a factor of about 2.5 times. The total expected increase in the resonant frequencies, due to the reduction in the lateral dimensions and increase in plate thickness, is approximately \((2.5)(80) = 280\) times. The tendency to reduce the resonant frequencies, due to material parameter changes, is expected to be less than a factor of 10. The final result is a net increase in the frequencies of at least 28 times.

For any set of adjacent vibrational modes, the plate displacement spectrum \(Z_t(\omega_{mn})\) of the smaller plastic plate, associates these modes with frequencies that are higher and spread further apart than those associated with the larger plate. According to the model, this does not cause any difficulty but in reality the higher frequency component of \(Z_t(\omega_{mn})\), associated with any mode \(\{m, n\}\) of the smaller plate, may be subjected to greater attenuation relative to that found in a larger plate.

Another concern in using a thicker plate is the violation of the assumption used in the thin plate equation. This assumption requires the wavelength of the plate vibrations to be less than 10 times the plate thickness. This assumption is related to the plate vibration frequency using [24, 26]

\[
\omega_{mn_{max}} \leq \frac{E_p}{3 \rho_p (1 - \nu^2_p)} \left[ \frac{(m\pi)}{l_x} \right]^2 + \left[ \frac{(n\pi)}{l_y} \right]^2 ,
\]

\[
\leq \frac{E_p}{3 \rho_p l_x} \left( \frac{2\pi}{l_x} \right)^2 ,
\]

\[
\leq \frac{E_p}{3 \rho_p l_x} \left( \frac{2\pi}{10(2l_x)} \right)^2 ,
\]

\[
\leq \frac{E_p}{3 \rho_p l_x} \left( \frac{\pi^2}{100l_x} \right) ,
\]

where \(\omega_{mn_{max}}\) represents the frequency above which a violation of the previous assumption results. Equation (5.4) indicates that as the plate thickness increases, the
maximum frequency, that satisfies the previous assumption, decreases. Therefore, the theory is applicable for a smaller range frequencies.

Another concern affecting the sensitivity of the peak sensor signal are the boundary conditions. It was shown that the boundary conditions affect the shape of the modes formed over the surface of the plate and that the shape of the modes determines the sensitivity of the peak sensor signal to changes in impact position at any one frequency. It is reasonable to expect that the sensitivity of the peak sensor signal to changes in impact position increases with the plate resonant frequency. This is argued by noting that an increase in the plate resonant frequency always decreases inter-node spacing regardless of the mode shape.

5.4.2 The Changes in the Behaviour of the Dimensionless Force

The change in the force pulse is estimated by applying the changes in the plate parameters to Equation (3.3S). This equation expresses $\lambda$ in terms of the sphere and plate parameters, which allows changes in $\lambda$ to be estimated. For convenience, the change in the plate parameters that have been previously given are listed as

$$
\begin{align*}
\Delta \rho_p & \approx 0.5 \text{ to } 0.3, \\
\Delta E_p & \approx 0.01, \\
\Delta k_p & \approx 1 \text{ to } 1.5, \\
\Delta u_p & \approx 0.14 \text{ to } 0.21, \\
\Delta 2l_s & \approx 2.5,
\end{align*}
$$

(5.5)

where $\Delta$ represents the ratio of a plastic parameter to aluminium parameter. Therefore from Equation (3.35), these changes in the plate parameters result in $\lambda$ for the plastic plate decreasing by a factor of 2 to 3 relative to that of the aluminium plate. This decrease in $\lambda$ causes the force pulse to become more half sinusoidal pulse-like with a corresponding change in the shape of the force pulse spectra. It should be noted that in spite of the decrease in most of the plate parameters, the increase in the plate thickness is the dominant factor that determines the behaviour of $\lambda$.

The duration of the dimensionless force pulse and the bandwidth of the dimensionless force pulse spectra are proportional to $T$ and $T^{-1}$, respectively. The factor
<table>
<thead>
<tr>
<th>Particle Radius $r_p$ (mm)</th>
<th>Calculated values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda$ (µsec.)</td>
</tr>
<tr>
<td>1.191</td>
<td>0.312</td>
</tr>
<tr>
<td>1.588</td>
<td>0.592</td>
</tr>
<tr>
<td>1.984</td>
<td>0.924</td>
</tr>
<tr>
<td>2.381</td>
<td>1.300</td>
</tr>
<tr>
<td>2.778</td>
<td>1.812</td>
</tr>
<tr>
<td>3.175</td>
<td>2.35</td>
</tr>
</tbody>
</table>

Table 5.1: The values of $\lambda$, $T$, $0.5T^{-1}$, and $F_m$, $f_{ap}(t)$ and $t_{end}$ associated with each sphere assuming that plate parameters are changed to approximate the plastic plate of the grain loss monitor.

$T$ is given by Equation (3.33) and with the assumed change in plate parameters, it increases by a factor of 5 to 6 over that associated with the aluminium plate. This reduces the bandwidth of the force pulse spectra by the same factor. Finally, the normalizing factor $F_m$ used to obtain the dimensionless force pulse decreases by a factor of about 5.

### 5.4.3 The Changes in the Behaviour of the Simulated Force

The force pulses simulations, described in Chapter 4, listed the associated values of $T$, and $F_m$ in Table 4.2. The values of $T$, and $F_m$ for the plastic plate are calculated by applying the appropriate change in factors. In addition, changes in $\lambda$ change the shape of force pulse and, therefore, change the values for $f_{ap}(t)$ and $t_{end}$. The change in these values are obtained by numerical evaluation. To calculate the changes in these values, it is assumed that $\Delta \lambda = 1/2.5 = 0.4$, $\Delta T = 5.5$, and $\Delta F_m = 1/5 = 0.2$. The resulting change in $T$, $F_m$, $f_{ap}(t)$, and $t_{end}$ have been calculated and are listed in Table 5.1.

As seen in Table 5.1 the duration of the force pulse has increased and the magnitude of the force pulses and the bandwidths of the force pulse spectra have decreased as compared to the values in Table 4.2. The increase in the duration of the
force pulse raises the concern that reflections could return before the force pulse is completed, especially when the dimensions of the plate have been reduced. But if the decrease in $\Delta v_p$ can be assumed to be proportional to the corresponding decrease in propagation speed for the plastic plate, then with $l_p$ decreasing by a factor of 8 and $\Delta v_p$ decreasing by a factor of 5 to 7, reflections are not a problem since they were not a problem in the aluminium plate. The decrease in the bandwidth of the force spectra also indicates that a significant portion of the spectra is below the major resonant frequencies of the transducer. This results in the sensor output spectra having a large e

5.5 Future Research and Suggestions

5.5.1 Grain Loss Monitor Sensor Investigation

The results of the experiment and simulations showed that impact location was a major factor affecting the peak transducer signal. It would be interesting to determine the actual relationship between the grain loss monitor sensor output and the impact location of spherical particles, as discussed in section 5.4. It is expected that the smaller dimensions of the grain loss monitor sensor will result in the output signal being very sensitive to changes in impact location. This requires a mechanism to determine the impact location to an accuracy that is much better than $\pm 0.5$ mm. As a start, the error in locating the impact site should be less than $\pm 0.063$ mm, which corresponds to a proposed peak signal variation of approximately 4% of the mean.

To characterize the sensor, data are required to obtain graphs showing $v_{pk}$ as a function of particle radius at fixed distances from the transducer and graphs showing $v_{pk}$ as a function of distance from the transducer for fixed radii. This graphs are used to evaluate the performance of the sensor in measuring particle radius using the peak sensor signal. In addition graphs are needed to show $dv_{pk}/dr$, at fixed distances and $dv_{pk}/dr$, at fixed radii, where $r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ represents the distance between the impact site and the transducer. These graphs indicate the sensitivity of
the peak sensor output to changes in radius and distance and could reveal locations that have reduced sensitivity to changes in impact position.

These investigations are expected to reveal that changes in the impact position sensor signal.

5.5.2 Model Verification

In the experiments of this thesis, many of the estimates for the errors and possible influence of the theoretical assumptions centre around the force of impact and plate displacements. These quantities were not measured and therefore, were not compared to the simulated results. Establishing the validity of the model requires measuring these quantities.

The difficulty in measuring the force pulse is due not only to the rapid rise and small duration of the force pulses but also because measurement methods may be too complex and not economical. Measuring the plate displacements also presents problems and requires a transducer to accurately measure plate displacements on the order of fractions of a micrometre with bandwidths on the order of hundreds of kilohertz. The difficulties in obtaining or designing such a transducer are significant.

If these measurements are possible, particular questions that can be investigated include:

1. What are the practical limits of the model? For example, what range of particle impact velocities or particle radii result in accurate or inaccurate force pulse simulations?

2. What is the influence of small indentations on the force pulse simulations?

3. What influence does boundary conditions have on the measured plate displacements?

4. Having measured the force pulses and spectra, do the simulated plate displacement accurately represent the actual measured plate displacements at all tested impact sites?
5.5.3 Investigating Other Measures of Particle Radius

It became clear in the experiments that the use of the peak sensor signal was not the only possible measure that can be used for the particle radius. Other measures could prove more effective. Investigating possible measures should begin by examining the plate displacements directly. This includes examining the peak displacements over some defined interval, the peak velocities or the accelerations, and the spectral content of these quantities. Having found a measure, the next step is to determine the influence of the transducer so that appropriate processing can be used to extract the information.

5.5.4 Model Extensions

Having established the model, extensions are possible. In a practical sense, model extensions that include more real world conditions are needed. This includes investigating the use of particles that are not spheres and the effect of simultaneous impact of particles at different locations. Other extensions include the use of non-rectangular plates, the use of different boundary conditions, and plates and spheres made of material other than a metal.

Another less physical extension is the development of statistical models for the variation in the impact location and the plate and sphere parameters. This would result in having an estimate of the particle radius that has statistics associated with it. An optimum radius estimator could then be designed.

This and the previous sections provide a few suggestions that could lead to the development of a novel particle analyser.
6. Conclusions

This thesis shows that a change in the impact position is the primary source of variation in the peak sensor output of the thin plate impact sensor. Specifically from the experimental results, using a hypothetical change in the impact position of approximately 0.1% of the lateral dimensions of the plate or about 0.5 mm, results in a standard deviation of the peak signal that averages 4.7%, 3.9%, 1.9%, and 4.5% of the mean for impact locations 0 cm, 1 cm, 2 cm, and 3 cm from the transducer, respectively. Extending these results to the grain loss monitor sensor, suggests that unperceived changes in impact position of approximately 0.4% of the sensor's lateral dimension, which correspond to changes in distance of about 0.25 mm and 0.5 mm in the shortest and longest lateral plate dimensions, respectively, could possibly result in a standard deviation of 16% of the mean peak sensor signal.

Experiments using six steel spheres, ranging in radius from 3/32 inches to 8/32 inches, impacting onto a 1/32 inch thick aluminium plate, shows that the peak sensor signal varies with the impact position and that the sensitivity of the peak signal to changes in impact position depends on the location of the impact site. Specifically, from the impact locations tested, this sensitivity exhibits a maximum sensitivity at the transducer site. The peak sensor signal also shows an almost linear response to changes in particle radius at all the impact locations tested. This indicates that the peak signal of this particular sensor configuration provides a measure of particle radius. The small changes in particle impact velocity were shown to have a negligible influence on the experimental results.

The theoretical analysis develops equations that described the generation of the force pulse, the propagation of the disturbance, and an approximation of the
transducer response. The impact force equations show that the sphere’s radius, density, modulus of elasticity, Poisson ratio, and impact velocity, as well as the plate thickness, density, modulus of elasticity, and Poisson ratio influence the shape of the resulting force pulse generated during the impact. Impacts of small particles produce small magnitude, wide bandwidth force pulse spectra as opposed to the large magnitude, narrow bandwidth spectra produced by the impacts of large particles. The solution to the thin plate equation indicates that changes in the impact location and transducer position influence the measured plate motion. The transducer model uses simplifying assumptions to describe it mechanically as a spring-mass-damper system and electrically as a voltage source loaded by series plate capacitance and resistance.

Simulations, using the theoretical results to approximate the experimental system, indicated the possible forms of the force pulses and measured plate motions that existed in the experiments. The simplicity of the transducer model did not allow the peak sensor signal to be reproduced exactly. The sensor signal and its spectrum did show similar characteristics for small signals and for frequencies below approximately 3 kHz and near 40 kHz.

These results suggest that the equations describing the force pulse and plate propagation are representative of the behaviour of the experimental system and that the transducer model can provide a basis to develop more accurate and complex transducer models.

These results also suggest that it is possible to measure particle radius using a thin plate sensor system. Appropriate selection of the materials, the plate dimensions, and the transducer along with controlled impact positions will result in a functional sensor system.
Bibliography


A. Supporting Experiments

A.1 The Piezo-electric Transducer

A.1.1 Experimental Setup

Figure A.1 shows a simplified diagram of the setup used to measure the magnitude and phase response of the piezo-electric transducer. A vibration analysis system consisting of a controller, an electro-magnetic shaker, and a calibrated force transducer are configured to measure the unknown magnitude and phase response of the piezo-electric transducer. The controller is set to excite the electro-magnetic shaker with random signals having a bandwidth of approximately 25 kHz. The response of the piezo-electric transducer is obtained using the known response of the calibrated force transducer. This measurement is handled by the controller that is also set to average the results of 1000 fast Fourier transforms of the piezo-electric transducer’s response. The final result is plotted onto graph paper.

The piezo-electric transducer is mounted in a manner, as similar as possible, to that used in the particle impact experiments. This included wire placements and torque values used to tighten the holding nut. The measurements were terminated after three consistent response were measured. In every trial the holding nut was loosened and tightened to the appropriate torque.

One hundred equally spaced points from the final plotted response curves were measured from these plots and entered into a data file for processing on the computer workstation.
A.1.2 The Transducer Magnitude and Phase Response

Using the previous setup, the magnitude and phase response of the piezo-electric transducer is measured. A number of factors made it difficult to obtain repeatable results. It was discovered that the response was very sensitive to changes in the tightness of the holding nut and to the position that the connecting wires were held in. After a number of trials, with the tightness of the holding nut and position of the connecting wires approximating that used in the particle impact experiments, three reasonably consistent magnitude and phase responses were measured. These results are shown in Figure A.2 giving the magnitude response and Figure A.3 illustrating the phase response.

The important features of the transducer's response are the resonant and anti-resonant regions. Briefly, from Figure A.2, these include the resonances at approximately 19 kHz and at 24 kHz. Additional less defined resonant peaks occur at approximately 13 kHz and 10 kHz. Anti-resonant regions are seen at approximately 21 kHz and 17 kHz. A erratic type response asserts itself below 1 kHz.

If the thin plate behaviour is used as an analogy for the behaviour of the transducer, then the transducer's response is the result of a particular combination of the transducer characteristics, its shape, and its boundary conditions. The slight changes
Figure A.2: The transducer's magnitude response for three separate trials.

Figure A.3: The transducer's phase response for three separate trials.
in the torque used to tighten the holding nut and the small changes in the position of the connecting wires causes the boundary conditions to vary that directly affect the transducer's response. In particular, the low frequency response, below 5 kHz, may be the result of the connecting wires exhibiting resonant behaviour with the transducer at a number of frequencies. The response between approximately 5 kHz and 22 kHz could be the complex interaction of the higher modes of vibration of connecting wires and the transducer since this region exhibited the greatest degree of change. The region above 22 kHz, though showing changes in magnitude, has the peak at 24 kHz showing small changes in frequency. This suggests that changes in the torque and position of the connecting wires has less of an influence on this region of the response. Without extensive testing it is difficult to assert with certainty the exact reasons for the measured response.

The sensitivity of the transducer's response to changes in tightness of the holding nut and position of the connecting wire raises the possibility that a different response exists in the particle impact experiments. Though care was taken to mount the transducer in the same manner for both these tests and the particle impact experiments, differences in the transducer mounting condition exist, resulting in a change in the response, but it is expected not to be significant.

A.2 Particle Impact velocities

A.2.1 Experimental Setup

Under simple free fall conditions the variation of the particle velocity is expected to be small but the magnetic holding device could change these condition and possibly influence the release velocity of the steel particles. These experiments were conducted to establish the degree of variation that exists in the particle velocities.

Figure A.4 shows a simplified diagram of the setup used to measure the particle velocities after being released from the magnetic holding mechanism. The velocity measurement system measures the duration an infrared light source is interrupted by
Figure A.4: A simplified diagram of the experimental setup used to measure the velocities of the impacting particles.

A falling particle. The system places an infrared light emitting diode approximately 1 cm from an infrared receiver. A mask with a 1.5 mm circular aperture is placed in front of the receiver allowing only light directly incident from the transmitter to be detected. The transmitter and receiver were placed approximately 7.4 cm below the particle dropping mechanism and positioned so that the central axis of the particle would pass directly in front of the opening of the aperture.

In operation, the signals from the receiver are captured by a digital storage oscilloscope. This allows the rise and fall times and pulse width of the interrupted light source to be measured. By carefully positioning the transmitter and receiver, a maximum pulse width having symmetrical rise and fall times, can be obtained. In these instances the particles were observed to pass directly in front of the opening of the aperture. Each of the six different sized particles were dropped nine to ten times to obtain a set of pulse width, rise and fall time measurements. These experiments were done independently from those involving the particle impacts at site A and B.

A.2.2 Velocity Measurements

To determine the particle velocities the configuration in Figure A.5 is used. At point A the particle is held by the holding device. After releasing the particle
Figure A.5: The configuration and nomenclature used to measure the particle velocities

it falls through a distance of \( z_h - 2r_s \) to arrive at point B with a velocity of \( v_{sB} \). At this point the particle partially blocks the opening of the aperture and prevents approximately 50% of the light from entering, thereby resulting in the receiver signal being reduced to 50%. At point C the particle has now moved a distance of \( 2r_s \) and is travelling at a velocity of \( v_{sC} \). The particle is again partially blocking the opening of the aperture and is preventing approximately 50% of the light from entering. In this case, the receiver signal has now risen from having no signal to being at 50% of the maximum level. The interval of time between the signal falling and rising to 50% of the difference between the maximum and minimum signal levels, is defined to be the interrupt duration \( t_D \) and is used to find the velocity \( v_{sB} \).

The measured velocity \( v_{sB} \) is determined from \( t_D \) by using

\[
2r_s = \frac{1}{2}gt_D^2 + v_{sB} t_D
\]

(A.1)

and writing it as

\[
v_{sB} = \frac{2r_s}{t_D} - g \frac{t_D}{2}
\]

(A.2)

where \( g = 9.81 \) metres per second squared.

Knowing the distance the particle falls, it is also possible to calculate the expected value of \( t_D \) and \( v_{sB} \) by assuming simple free fall conditions. These values can then be compared to the measured values to determine the degree of influence the
magnetic holding device has on the particle velocities. The expected velocity $v_{eB*}$ is found by using

$$v_{eB*} = \sqrt{2g(z_h - 2r_s)} \quad (A.3)$$

The expected duration $t_{De}$ can then be found by using

$$2r_s = \frac{1}{2}gt_{De}^2 + v_{eB*}t_{De} \quad (A.4)$$

and substituting Equation (A.3) into Equation (A.4) and using the positive root, the result is written as

$$t_{De} = \sqrt{\frac{2z_h}{g}} - \sqrt{\frac{2z_h}{g} - \frac{4r_s}{g}} \quad (A.5)$$

The values and maximum errors of the expected interrupt durations and particle velocities are recorded in Table A.1 and Table A.2, respectively. These table also show the mean and error, given as three standard deviations, resulting from measuring the interrupt duration of nine to ten drops of each particle and the corresponding calculated velocities. The error in the interrupt durations and velocities is also listed as a percentage of the mean. The variation in the expected values assumes that the radius $r_s$, the drop height $z_h$, and $g$ have a maximum error of approximately $\pm 0\% - 1\%$, $\pm 1.4\%$, and $\pm 0.2\%$, respectively.\(^1\) This results in the maximum error in the expected interrupt durations and velocities being approximately $0.9\%$.

It is clear from Table A.2 that the velocities of any one particle is consistent from drop to drop and has a variation, using three standard deviations expressed as a percentage of the mean, that is less than two percent for all but the smallest particle. The increase in the variation for the smaller particles is the result of the holding device having less control on releasing the particles resulting in the central axis of the particle not passing directly in front of the aperture. The smaller particles have a slight increase in velocities due to the small increase in the distance they fall. It is also apparent that influences due to the magnetic holding device, if present, can not be measured since the error in the expected particle velocities places them

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\(^1\)The errors in the radius are given in Appendix K. The error in $z_h$ was $\pm 1$ mm or $\pm 1.4\%$, and the error in $g$ is given as $\pm 0.2\%$ [34]
Within the range of the measured velocities. In spite of this, the measured velocities are consistently less than the expected velocities. This may suggest that the holding device effectively slows the velocities by gradually, instead of abruptly, releasing the particle. Another possibility is aerodynamic drag slowing the particles down. This effect should be more pronounced for the smaller particles but this result is not seen.

Finally, using Equation (1.4), a maximum change in velocity of approximately 2.2%, is equivalent to a 0.2% change in particle radius. With the error in the radius of the spheres given as ±0% -1%, the change in the form of the force pulse, due to the variation in the velocities, is not expected to be a significant influence in changing the sensor output.

Table A.1: The expected and measured interrupted pulse duration

Table A.2: The expected and measured particle velocities
B. The Taut String

To understand the mechanisms governing the behaviour of the thin plate, this appendix examines a simple physical system, that of a taut string. The taut string system is useful for illustrating the fundamental behaviour of a thin plate to impacts, including the nature of the initial epi-centre impact deformation, its outward propagation, and reflections from boundaries.

B.1 String Motion

The system shown in Figure B.1 consists of a string of length, \( l \) m, that is assumed to be infinite. The string is also under a tension \( T \) N. The string is assumed to be stationary prior to application of the force \( f_s(x, t) \). This sets the initial string displacement and velocity, \( F_i(x) \) and \( G_i(x) \), respectively, to zero. The governing equation of motion for the string is given by [25]

\[
\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = f_s(x, t) \quad ,
\]

(B.1)

![Figure B.1: The simple taut string system.](image)
where the constant $c$ is the phase speed in m/s given by [25]

$$c = \sqrt{\frac{T}{\rho_i}},$$  \hspace{1cm} (B.2)

and where $\rho_i$ is the mass per unit length of the string in kg/m, and $f_s(x,t)$ is the linear force density function in N/m. The general solution to the non-homogeneous string wave Equation (B.1) is [31]

$$y_s(x,t) = \frac{1}{2} [F_i(x + ct) + F_i(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} G_i(\eta) d\eta$$

$$+ \frac{1}{2c\rho_i} \int_0^t \int_{x-ct(\tau-\delta)}^{x+ct(\tau-\delta)} f_s(\eta, \delta) d\eta d\delta. \hspace{1cm} (B.3)$$

The terms $F_i(x \pm ct)$ and $G_i(\eta)$ represent the influence of the initial displacement $F_i(x)$ and velocity $G_i(x)$. Since the string is stationary prior to the impact, both these terms go to zero and the general solution becomes

$$y_s(x,t) = \frac{1}{2c\rho_i} \int_0^t \int_{x-ct(\tau-\delta)}^{x+ct(\tau-\delta)} f_s(\eta, \delta) d\eta d\delta. \hspace{1cm} (B.4)$$

Though both $F_i(x)$ and $G_i(x)$ do not play a part in the general solution to this impact problem, it is instructive to review their individual influence on the general solution and also to examine their propagation characteristics.

Consider now the case described with

$$y_s(x,0) = F_i(x), \hspace{1cm} (B.5)$$

$$G_i(x) = 0, \hspace{1cm} (B.6)$$

$$f_s(x,t) = 0, \hspace{1cm} (B.7)$$

representing a string with no applied force or initial velocity but with some arbitrary initial displacement. The general solution is then written

$$y_s(x,t) = \frac{1}{2} [F_i(x + ct) + F_i(x - ct)]. \hspace{1cm} (B.8)$$

This solution describes two disturbances propagating in opposite direction with the shape of $\frac{1}{2}F_i(x)$. This is shown in Figure B.2. Given a string of infinite length, these waves will propagate indefinitely in their respective direction.
Consider now the second case described with

\[
F_1(x) = 0 , \quad (B.9)
\]
\[
\frac{dy_a(x,0)}{dx} = G_1(x) , \quad (B.10)
\]
\[
f_a(x,t) = 0 , \quad (B.11)
\]
representing a string with no applied force or initial displacement but with some arbitrary initial velocity. This case is used in many instances to describe the effect of impulsively striking a string but is in fact an approximation of an object applying a force over a very short duration to a stationary string \[25\]. A general solution for this case is

\[
y_a(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G_1(\eta) d\eta . \quad (B.12)
\]
The solution to this case is illustrated by defining the initial velocity \( G_1(x) \) to be applied at the point \( x_i \). This particular function for \( G_1(x) \) results in a solution that is similar to that obtained for the problem of a point mass impacting a stationary string. Specifically \( G_1(x) \) is defined as

\[
\frac{dy_a(x)}{dx} = G_1(x) = \delta(x - x_i)v . \quad (B.13)
\]
The solution to Equation \((B.12)\) then becomes

\[
u_a(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G_1(\eta) d\eta . \quad (B.14)
\]
Figure B.3: Propagation of a disturbance on a string given an initial velocity:

\[
G_l(x) = \begin{cases} 
  v; & x = x_l \\
  0; & \text{elsewhere}
\end{cases}
\]

\[t_0 = 0\]

\[+x_1\]

\[t_1 = \Delta x_1/c\]

\[\Delta x_1\]

\[\Delta x_2\]

\[t_2 = \Delta x_2/c\]

\[x_l\]

where \(u(x)\) is the unit step function.

This result is shown in Figure B.3 at various instances of time. At time \(t_0 = 0\) the initial velocity, \(G_l(x)\) is imposed at the point \(x_l\). At time \(t_1\), two disturbances have moved out in opposite directions with speed \(c\). The shape of the disturbance resembles that of a widening rectangular pulse of amplitude \(\frac{v}{2c}\). It can be seen from Figure B.3, that the string reaches a maximum displacement given by \(\frac{1}{2c} \int_{x_1}^{x_2} G_l(\eta)d\eta\), where \(x_1\) and \(x_2\) encompass, inclusively, the region of \(G_l(x)\). In addition, as the disturbance passes a point on the string, that point on the string remains stationary.

Returning to the original problem of a point force being applied to the string, the general solution given by Equation (B.4), has an unspecified term given by \(f_a(x,t)\). Before this solution can be determined, the forcing function \(f_a(x,t)\) needs to be specified. Instead of considering a point mass impact, it is more convenient to apply a point force that is equivalent to an impact of a mass \(m\) having a velocity \(v\). From section B.5, this force is expressed as

\[f_a(x) = \delta x_1(x) - \frac{2v \Delta x_1}{c}\]  \(\text{(B.15)}\)
Figure B.4: Epi-centre displacement of a string due to a point force.

so that the motion of the string at the impact point, from section B.5, is

\[ y_a(x_i, t) = \frac{mv}{2c\rho_l} \left[ 1 - e^{-\frac{2c\rho_l t}{m}} \right]. \]  \hspace{1cm} (B.16)

The expression, given by Equation (B.16), describes the displacement of the string at the point of impact, \( x_i \), due to a point force that is equivalent to a point mass \( m \) having a velocity \( v \). This is referred to as the epi-centre displacement and represents the initial deformation of the string due to the applied force. Figure B.4 shows the general form of this deformation as a function of time.

Examining Equation (B.16) and Figure B.4, shows that the initial displacement rises exponentially from zero to the limit \( \frac{mv}{2c\rho_l} \) with a time constant of \( \frac{m}{2c\rho_l} \). This indicates that the magnitude of the displacement is directly proportional to the \( mv \) and the time constant is proportional to \( m \).
The propagation of the disturbance, shown in Figure B.4 can be expressed as [31]

\[
y_a(x, t) = \begin{cases} 
\frac{1}{2c\delta t} \int_{t_0 - \Delta x/c}^{t} f_a(\delta) d\delta, & t > \Delta x/c \\
0, & t \leq \Delta x/c
\end{cases},
\]  

(B.17)

or as

\[
y_a(x, t) = \frac{mv}{2c\delta t} \left[ 1 - e^{-\frac{2\pi m t'}{x}} \right] u(t'), \quad \text{for } t > 0
\]

(B.18)

where \(\Delta x\) is distance between the point of impact \(x_i\) and the point of interest \(x\) and \(t' = \Delta x/c\). These expressions given by Equation (B.17) or Equation (B.18), indicates that given a point \(x\) on the string, the displacement, beginning at time \(t - (\Delta x/c)\), has the same form as that found at the epi-centre. The propagation characteristic of this disturbance is similar to that given by Equation (B.12) for the case of having only an initial velocity \(G_i(x)\) imposed on the string. Two disturbances, of the form given by Equation (B.16), propagate in opposite direction with speed \(c\) as shown in Figure B.5.

### B.2 Boundary Condition Influences: Reflections

The previous section showed that changes to the initial conditions cause different forms of a disturbance to propagate along a string. In all cases the string
Figure B.6: General string boundary conditions for a finite length string. In (a) the end of a string is fastened to a stationary support. The fastener is shown in (b) and consists of a mass-spring-damper system.

was of infinite length and any disturbance, once begun, propagated indefinitely in their respective direction at the phase speed $c$. If instead $l$ is finite, any disturbances will, in time, encounter the end of the string. The behaviour of the disturbance at this point is examined in this section.

A string of finite length has its ends fastened in some manner to an object that may be stationary or moving. The method used to fasten a string to this object can vary from a firmly held end, allowing no motion, to a free end, allowing motion without restriction. These fastening methods are known as the string's boundary condition. For simplicity, the boundaries are assumed stationary. To precisely represent this range of fastening methods or boundary conditions the system shown in Figure B.6 is used [25].

Figure B.6 shows the end of a string attached to a spring-mass-damper system that has its motion confined to the the y-axis. By varying the coefficients $M_l$, $B_l$, or $K_l$ of this system, very complex and realistic boundary conditions can be approximated. For the present discussion only the simplest of boundary conditions are examined.
A string with an end fixed to a boundary is a simple fastening method that allows no string motion. This boundary condition is obtained by setting any one of \( M_i, B_i, \) or \( K_i \) to infinity. More precisely this boundary condition is expressed as,

\[
y_a(0, t) = 0 \quad ,
\]

(B.19)
where the end of the string is defined to be at \( x = 0 \). To see the effect this boundary condition has on a propagating disturbance, consider a disturbance \( F_i (x + ct) \) travelling to the left and about to encounter this boundary. The general solution for this is,

\[
y_a(x, t) = F_i (x + ct) + F_{ii} (x - ct) \quad .
\]

(B.20)
Applying the boundary condition at \( x = 0 \), Equation (B.20) becomes

\[
y_a(0, t) = F_i (0 + ct) + F_{ii} (0 - ct) = 0 \quad ,
\]

(B.21)
resulting in [25]

\[
F_i (ct) = -F_{ii} (-ct) \quad .
\]

(B.22)
So the general solution becomes

\[
y_a(x, t) = F_i (x + ct) - F_{ii} (x - ct) \quad .
\]

(B.23)
Figure B.7(a) illustrates an interpretation of the sequence of events given by Equation (B.23). A real arbitrary disturbance moving to the left and a phantom mirrored inverted disturbance moving to the right, meet at the fixed boundary. The speed and distance from the boundary of both disturbances are identical. As the disturbances pass through the boundary the real disturbance is transformed into the phantom disturbance and the phantom disturbance is transformed into the real disturbance. The resulting disturbance at the boundary is the sum of real parts of either disturbance.

As a contrast to the fixed boundary, consider the same disturbance encountering a free end boundary condition at \( x = 0 \). A free end implies no \( M_i, B_i, \) and
Figure B.7: Reflections on a string due to fixed and free boundary conditions shown in (a) and (b), respectively.

As in the fixed end, this indicates that at a free boundary the disturbance travelling to the left, given by \( F_1(x + ct) \), remains unchanged and begins to travel to the right. Alternately Figure B.7(b) shows the sequence of events as an arbitrary disturbance encounters a free boundary. Changing \( M_1 \), \( B_1 \), and \( K_1 \) to values other then zero or infinity will result in different and more complex forms of the reflected disturbance.
B.3 Boundary Condition Influences: Vibration Modes

In the previous section only the disturbance at one end of the string was considered, showing that a disturbance propagating on a string is reflected after encountering a boundary and that the form of the reflection is dependent on the boundary conditions. In this section the reflections from both ends are considered together resulting in another representation of string motion.

Consider the function $F_i(x \pm ct)$ representing the propagating disturbances in the general solution given by Equation (B.20). These functions are completely arbitrary and some examples include $\log(x \pm ct)$, $\sin(\omega(x \pm t))$, and $\exp(j\omega(x \pm t))$ that could result by suitable application of the initial and boundary conditions [25].

If the system shown in Figure B.1 has the conditions

\[
\begin{align*}
    f_a(x, t) & = 0 , \\
    y_a(x, 0) & = F_i(x) , \\
    \frac{dy_a(x, 0)}{dt} & = G_i(x) , \\
    y_a(0, t) & = y_a(l, t) = 0 ,
\end{align*}
\]

then this describes a string of length $l$ fixed at both ends, having no applied force, and with some arbitrary initial displacement and velocity defined over its length. A general solution of the form [25]

\[
y(x, t) = Ae^{j(k_n x + \omega t)} + Be^{-j(k_n x - \omega t)} ,
\]

(B.26)

satisfies the homogeneous string wave equation. The boldface notation indicates complex quantities. This solution represents two sinusoidal waves propagating in opposite directions. These waves can be viewed as oscillating in time at frequency $\omega$ or having $\frac{k_n}{2}$ periods over the length of the string, where $k_n$ is the wave number. The coefficients $A$ and $B$ specify the complex amplitude of these waves. Applying the boundary conditions to this solution at $x = 0$ and $x = l$ respectively, results in [25]

\[
0 = A + B ,
\]

(B.27)
Solving for \( A \) gives

\[
0 = Ae^{jknl} + Be^{-jknl} .
\]  

(B.28)

Not including the trivial solution of \( A = 0 \), only the discrete values, \( k_n l = nx \), for \( n = 1, 2, 3, \ldots \), result in solutions to Equation \((B.29)\). Since \( \omega / k_n = c \), only discrete frequencies, given by, \( \omega = \frac{\pi}{l} n \), are allowed. The solution of Equation \((B.26)\) is then written as a series of solutions

\[
y_{an}(x, t) = A_n 2j \sin(k_n x) e^{j\omega_n t} .
\]  

(B.30)

Either the real or imaginary part can considered a solution [25]. By using \( A_n = a_n + j b_n \), the real component of the \( n^{th} \) solution is written

\[
y_{an}(x, t) = (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin k_n x .
\]  

(B.31)

The complete solution is a sum of all the individual solutions or [25]

\[
y_a(x, t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin k_n x ,
\]  

(B.32)

where the coefficients \( a_n \) and \( b_n \) are determined by applying the initial conditions given by \( F_l(x) \) and \( G_l(x) \). Equation \((B.32)\) is also called the normal mode solution [25].

Application of the boundary conditions limits the solution to a discrete set of functions called eigenfunctions or normal modes. Associated with each eigenfunction are normal mode frequencies or eigenvalues given by \( \omega_n = \frac{\pi}{l} n \). Each of these normal modes represents the string vibrating at a frequency \( \omega_n \). Figure B.8 shows the first few modes on a string with fixed boundary conditions. In this view, each wave does not move along the string but vibrates, in time, at the normal mode frequency. For this reason these waves are also called standing waves. Figure B.8 shows the string vibrating with one mode at any particular instant, but there is nothing to prevent any number of modes, each with a different amplitude, to be vibrating on the string at the same time. If this occurs the resulting string motion becomes the sum of each
of the individual wave motion. Though difficult to visualise, this results in portions of the string having transverse displacements ranging from zero to some maximum as the individual wave motions tend to cancel or add in various regions. As time progresses these regions sum differently and the string displacement changes. Though this appears to be a more complex view, it must still be compatible with the earlier general solution of propagating disturbances.

The constraint of having a discrete set of sinusoidal waves vibrating on the string, can be viewed with the additional fact that a disturbance also propagate along a string. By selecting particular modes or sine waves and setting their amplitudes appropriately, the sum of these discrete waves, at any instant, can result in a disturbance to appear within a certain region of the string. At all other points the waves sum to zero and there appears to be no displacement. At a later instant in time, this summation changes such that the same disturbance appears over a different string region. In this way, it appears that the disturbance moves down the string with phase speed $c$.

It is apparent that the coefficients of the solution, given by Equation (B.31), indicate the magnitude of these sinusoidal waves or normal modes and are such that
their sum over the length of the string results in a particular disturbance to appear. These coefficients, in this case of fixed ends, result from the Fourier sine transform of the initial conditions \( F_i(x) \) and \( G_i(x) \) or
\[
\begin{align*}
a_n &= \frac{2}{l} \int_0^l F_i(x) \sin(k_n x) \, dx, \\
b_n &= \frac{2}{\omega_n l} \int_0^l G_i(x) \sin(k_n x) \, dx.
\end{align*}
\] (B.33) and

The result of this section show that a disturbance on a string can be represented as a series of normal mode solutions vibrating at a unique normal mode frequency. In the simple case of a string with both ends fixed, these are sine waves vibrating at integer multiple values of frequencies \( \omega_n \). The normal mode solutions and frequencies are determined by the boundary conditions. The relative amplitude distribution of these modes are determined using the initial conditions. For more complex boundary conditions, the normal mode solution and frequency relationship becomes more difficult to obtain.

B.4 The Behaviour of a Fixed Finite Length Taut String to a Point Force

Consider the system shown in Figure B.1 having a string with a linear density \( \rho_l \), length \( l \), under tension \( T_l \) that is assumed to be initially at rest and fixed at the both ends. A point force \( f_a(x, t) \) is now applied that is equivalent to a point mass \( m \) with velocity \( v \) impacting onto the string at the point \( x_i \). From section B.5 the normal mode solution can be expressed as
\[
y(x_i, t) = \frac{2}{l} \sum_{k=1}^{\infty} \left( \frac{B}{A^2 + \omega_k^2} \right) \left( e^{-A t} - \cos \omega_k t + \frac{A}{\omega_k} \sin \omega_k t \right) \left( \sin \frac{k \pi}{l} x_i \right) \left( \sin \frac{k \pi}{l} x \right),
\] (B.35) where
\[
\begin{align*}
A &= \frac{2c \rho_l}{m}, \\
B &= 2c v \rho_l.
\end{align*}
\] (B.36) (B.37)
Equation (B.35) indicates that the measured motion at \( x = x_i \) depends on the impact and measurement locations and a number of other factors. Insight into the meaning of these other factors can be gained by writing Equation (B.35) in the form

\[
y_a(x_i, t) = \sqrt{\frac{2}{l}} \sum_{k=1}^{\infty} N_k(t) \sin \frac{k\pi}{l} x_i ,
\]

where

\[
N_k(t) = \sqrt{\frac{2}{l}} \left( \sin \frac{k\pi}{l} x_i \right) \left( \frac{B}{a^2 + \omega_k^2} \right) \left( e^{-\lambda t} - \cos \omega_k t + \frac{A}{\omega_k} \sin \omega_k t \right) ,
\]

are the Fourier coefficients of \( y(x, t) \) that are time dependent [31]. These coefficients are now written as

\[
N_k(t) = \left( \sin \frac{k\pi}{l} x_i \right) \left[ \frac{e^{-\lambda t}}{\omega_k} \left( \frac{B}{A^2 + \omega_k^2} \right) + \frac{1}{\omega_k} L_k \right] ,
\]

where the constant \( L_k \) is defined as

\[
L_k = \left( \frac{B}{A^2 + \omega_k^2} \right) \left( A \sin \omega_k t - \omega_k \cos \omega_k t \right)
\]

The magnitude of \( L_k \) as a function of \( \omega_k \) is

\[
|L_k(\omega_k)| = \left( \frac{B}{A^2 + \omega_k^2} \right) \sqrt{A^2 + \omega_k^2}
\]

Now consider the Fourier transform of an applied impact force \( f_a(t) \), given by

\[
F_a(\omega) = \int_{-\infty}^{\infty} f_a(t) e^{-j\omega t} dt
\]

where \( f_a(t) = Be^{-\lambda t} \). The magnitude of \( F_a(\omega) \) is

\[
|F_a(\omega)| = \left( \frac{B}{A^2 + \omega^2} \right) \sqrt{A^2 + \omega^2}
\]

which is the same form as that given by Equation (B.42). The difference is that Equation (B.42) is a sampled version of the continuous transform of Equation (B.44). This shows that the modes excited in the impact are related to the spectral content of the forcing function and, in fact, a sampled version of the spectrum of the forcing function. This statement is not completely accurate as the set of modes that are
excited also depend on the location of the impact site. Specifically, the magnitude of
the modes vibrating on the plate is given by three factors, the spectral magnitude of
the applied force at the sampled point $\omega_k$, the impact location, and the $\frac{1}{\omega_k}$ term. This
last term indicates that the magnitude of the vibrations is inversely proportional to
frequency of oscillation.

B.5 Taut String Solution to a Point Force

For a stationary taut string of infinite length the general solution to the
non-homogeneous wave equation, given by Equation (B.1), was given in section B.1
and is re\(\overline{\text{f}}\)

$$y_a(x, t) = \frac{1}{2c\rho I} \int_0^t \int_{x-c(t-\zeta)}^{x+(t-\zeta)} f_a(\zeta, \eta) d\zeta d\eta \quad \text{(B.45)}$$

If the force is applied at the point $x = x_i$, it can be expressed as

$$f_a(x, t) = \delta(x - x_i)f_a(t), \quad t \geq 0, \quad \text{(B.46)}$$

where $\delta(x)$ is the Dirac delta function that results in

$$\int_{-\infty}^{\infty} f_a(x, t) dx = \int_{-\infty}^{\infty} \delta(x - x_i)f_a(t) dx = f_a(t) \quad \text{(B.47)}$$

Equation (B.45) then becomes at $x = x_i$

$$y_a(t) = \begin{cases} \frac{1}{2c\rho I} \int_0^t \int_{x-c(t-\zeta)}^{x+(t-\zeta)} f_a(\zeta, \eta) d\zeta d\eta, & t > 0, \\ 0, & t = 0 \end{cases} \quad \text{(B.48)}$$

where the notation $y_a(t)$ is used with the understanding that it represents $y_a(x_i, t)$.

Differentiating Equation (B.48) with respect to time results in

$$\frac{dy_a(t)}{dt} = \frac{1}{2c\rho I} f_a(t) \quad \text{(B.49)}$$

or

$$f_a(t) = 2c\rho I \frac{dy_a(t)}{dt} \quad \text{(B.50)}$$

Assume that the force at $x = x_i$ is expressed using

$$f_a(t) = -ma = -m \frac{d^2y_a(t)}{dt^2} \quad \text{(B.51)}$$
Though Equation (B.51) is the equation of motion of a particle of mass $m$, in the context of this problem it is simply a force. Now equating Equation (B.50) to Equation (B.51) results in

$$2cpl \frac{dy_x(t)}{dt} = u_x,$$  \hspace{1cm} (B.52)

or

$$\frac{d^2y_x(t)}{dt^2} + \frac{2cpl}{m} \frac{dy_x(t)}{dt} = 0.$$  \hspace{1cm} (B.53)

Given the initial conditions

$$y_x(x,0) = 0,$$  \hspace{1cm} (B.54)

$$\frac{dy_x(x,0)}{dx} = v.$$  \hspace{1cm} (B.55)

The Laplace transform of Equation (B.53) is [23]

$$s^2Y_x(s) - sy_x(0^+) - \frac{dy_x(0^+)}{dt} + \frac{2cpl}{m} \left[sY_x(s) - y_x(0^+)\right] = 0,$$  \hspace{1cm} (B.56)

$$Y_x(s) \left[s^2 + \frac{2cpl}{m} s\right] = v,$$  \hspace{1cm} (B.57)

or

$$Y_x(s) = \frac{v}{s(s + \frac{2cpl}{m})},$$  \hspace{1cm} (B.58)

$$= \frac{A_1}{s} + \frac{A_2}{s + \frac{2cpl}{m}},$$  \hspace{1cm} (B.59)

where $A_1$ and $A_2$ are found using partial fraction expansion to yield

$$\frac{v}{s(s + \frac{2cpl}{m})} = \frac{A_1(s + \frac{2cpl}{m}) + A_2 s}{s(s + \frac{2cpl}{m})},$$  \hspace{1cm} (B.60)

$$= \frac{s(A_1 + A_2) + 2cpl/m(A_1)}{s(s + \frac{2cpl}{m})}.$$  \hspace{1cm} (B.61)

Therefore, $A_1 + A_2 = 0$ and $v = 2cpl/m(A_1)$, or

$$A_1 = \frac{mv}{2cpl},$$  \hspace{1cm} (B.62)

$$A_2 = -A_1.$$  \hspace{1cm} (B.63)

Then

$$Y_x(s) = \frac{mv}{2cpl} \left[\frac{1}{s} - \frac{1}{s + \frac{2cpl}{m}}\right].$$  \hspace{1cm} (B.65)
The inverse Laplace transform of Equation (B.65), from tables [23], is

\[ y_a(t) = \frac{mv}{2cp} \left[ 1 - e^{-\frac{2cp_f}{m}t} \right], \quad (B.66) \]

and represents the solution for the motion of the taut string at the point \( x = x_i \) due to the point force, given by Equation (B.51), being applied at \( x = x_i \). Differentiating Equation (B.66) twice with respect to time yields

\[ \frac{d^2y_a(t)}{dt^2} = -\frac{2cp_v}{m} e^{-\frac{2cp_f}{m}t}. \quad (B.67) \]

Therefore, the force at \( x = x_i \) can be explicitly expressed as

\[ f_a(t) = 2cp_v e^{-\frac{2cp_f}{m}t}, \quad (B.68) \]

using either Equation (B.50) or Equation (B.51). From Equation (B.46), the general expression for the force is then

\[ f_a(x, t) = \delta(x - x_i)f_a(t), \quad t \geq 0, \quad (B.69) \]

\[ = \delta(x - x_i)2cp_v e^{-\frac{2cp_f}{m}t}, \quad (B.70) \]

\[ = \delta(x - x_i)Be^{-At}, \quad (B.71) \]

where

\[ A = \frac{2cp_f}{m}, \quad (B.72) \]

\[ B = 2cp_v. \quad (B.73) \]

Equation (B.71) represents the force applied at the point \( x = x_i \). The motion of the string at that same point is given by Equation (B.66). The motion of the string at any arbitrary point \( x \) must be determined from the non-homogeneous wave equation for a taut string, repeated here as

\[ \frac{\partial^2y(x, t)}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2y(x, t)}{\partial x^2} = f_a(x, t), \quad (B.74) \]

where for the problem in this case, the taut string has a finite length \( l \) with both ends fixed. These initial and boundary conditions can be given as

\[ y_a(x, 0) = F_l = 0, \quad (B.75) \]
\[
\frac{dy_a(x,0)}{dt} = G_t = 0, \quad (B.76)
\]
\[
y_a(0,t) = y_a(l,t) = 0, \quad (B.77)
\]

where \( f_a(x,t) \) is given by Equation (B.71). The solution to Equation (B.74) can be expressed as [31]
\[
y_a(x,t) = \sum_{k=1}^{\infty} N_k(t) M_k(x), \quad (B.78)
\]
where \( N_k(t) \) are the time dependent Fourier coefficients that will need to be determined and where \( M_k(x) \) represent a series of eigenfunctions that can be obtained from the eigenproblem [31]
\[
-\frac{c^2 M_k}{dx^2} = \lambda_k M_k(x), \quad (B.79)
\]
where \( \lambda_k \) are the eigenvalues. With fixed boundaries, the normalized eigenfunctions can be found to be [31]
\[
M_k(x) = \sqrt{\frac{2}{l}} \sin \frac{k\pi}{l} x, \quad (B.80)
\]
and the eigenvalues or eigenfrequencies can be found from Equation (B.79) to be
\[
\lambda_k = \omega^2_k = \left( \frac{\pi c}{l} k \right)^2. \quad (B.81)
\]

Consider a single \( k \) term of Equation (B.78), substitute this into Equation (B.74), multiply the result by \( M_k(x) \), then substitute in Equation (B.79), and integrate over \( 0 \leq x \leq l \) to obtain
\[
\int_{0}^{l} \frac{d^2 [N_k(t) M_k(x)]}{dt^2} M_k(x) dx + \int_{0}^{l} [N_k(t) M_k(x)] \lambda_k M_k(x) dx = \int_{0}^{l} f_a(x,t) M_k(x) dx. \quad (B.82)
\]

Using the property of eigenfunctions that [31]
\[
\int_{0}^{l} M_k(x) M_k(x) dx = 1, \quad \text{for all } k, \quad (B.83)
\]
reduces Equation (B.82) to
\[
\frac{d^2 N_k(t)}{dt^2} + \lambda_k N_k(t) = F_k(t), \quad (B.84)
\]
where

\[ F_k(t) = \int_0^t f_a(x, t) M_k(x) \, dx \quad \text{(B.85)} \]

Equation (B.85) can be expressed explicitly by substituting in Equation (B.71) for \( f_a(x, t) \) and Equation (B.80) for \( M_k(x) \) to obtain

\[ F_k(t) = B \sqrt{\frac{2}{l}} \int_0^t \delta(x - x_i) e^{-At} \sin \left( \frac{k\pi}{l} x \right) \, dx \quad \text{, (B.86)} \]

\[ = B \sqrt{\frac{2}{l}} e^{-At} \sin \frac{k\pi}{l} x_i \quad \text{, (B.87)} \]

Now assuming zero initial conditions and taking the Laplace transform of Equation (B.84) results in

\[ s^2 N_k(s) + \lambda_k N_k(s) = F_k(s) \quad \text{, (B.88)} \]

or

\[ N_k(s) = F_k(s) \frac{1}{s^2 + \lambda_k} \quad \text{, (B.89)} \]

Applying an inverse Laplace transform to Equation (B.89) and using the convolution theorem [23], results in

\[ N_k(t) = \frac{1}{\sqrt{\lambda_k}} \int_0^t F_k(\eta) \sin \left[ \sqrt{\lambda_k}(t - \eta) \right] \, d\eta \quad \text{. (B.90)} \]

Substitute into Equation (B.90), Equation (B.87) for \( F_k(t) \) and use the relationship \( \omega_k = \sqrt{\lambda_k} \) from Equation (B.81) to obtain

\[ N_k(t) = \frac{B}{\omega_k} \sqrt{\frac{2}{l}} \sin \frac{k\pi}{l} x_i \int_0^t e^{\lambda_k \eta} \sin [\omega_k(t - \eta)] \, d\eta \quad \text{. (B.91)} \]

Equation (B.91) can be solved by integrating by parts twice. From

\[ \int udw = uw - \int wdu \quad \text{, (B.92)} \]

let

\[ u = \sin [\omega_k(t - \eta)], \quad dw = e^{-\lambda_k \eta} d\eta, \]

\[ du = -\omega_k \cos [\omega_k(t - \eta)] d\eta, \quad w = -\frac{1}{\lambda_k} e^{-\lambda_k \eta}. \quad \text{(B.93)} \]
Now working only with the integral term of Equation (B.91), it becomes

\[
\int_0^t e^{-\lambda \eta} \sin [\omega_k(t - \eta)] d\eta
\]

\[
= -\frac{1}{A} e^{-\lambda \eta} \sin [\omega_k(t - \eta)] \int_0^t - \frac{\omega_k}{A} \int_0^t e^{-\lambda \eta} \cos [\omega_k(t - \eta)] d\eta,
\]

\[
= \frac{1}{A} \sin(\omega_k t) - \frac{\omega_k}{A} \int_0^t e^{-\lambda \eta} \cos [\omega_k(t - \eta)] d\eta.
\]  

(B.94)

Now integrts again by letting

\[
\begin{align*}
    u &= \cos[\omega_k(t - \eta)], & dw &= e^{-\lambda \eta} d\eta, \\
    du &= -\omega_k \sin[\omega_k(t - \eta)] d\eta, & w &= -\frac{1}{A} e^{-\lambda \eta},
\end{align*}
\]

so that the integral term in Equation (B.94) becomes

\[
-\frac{\omega_k}{A} \int_0^t e^{-\lambda \eta} \cos [\omega_k(t - \eta)] d\eta
\]

\[
= \frac{\omega_k}{A^2} e^{-\lambda \eta} \cos [\omega_k(t - \eta)] \int_0^t - \frac{\omega_k^2}{A^2} \int_0^t e^{-\lambda \eta} \sin [\omega_k(t - \eta)] d\eta,
\]

(B.96)

\[
= \frac{\omega_k}{A^2} e^{-\lambda t} - \frac{\omega_k}{A^2} \cos(\omega_k t) - \frac{\omega_k^2}{A^2} \int_0^t e^{-\lambda \eta} \sin [\omega_k(t - \eta)] d\eta.
\]  

(B.97)

Now the integral of Equation (B.91) can be written as

\[
\int_0^t e^{-\lambda \eta} \sin [\omega_k(t - \eta)] d\eta \left[ 1 + \frac{\omega_k^2}{A^2} \right] A^2
\]

\[= \omega_k e^{-\lambda t} + A \sin(\omega_k t) - \omega_k \cos(\omega_k t), \]

(B.98)

or as

\[
\int_0^t e^{-\lambda \eta} \sin [\omega_k(t - \eta)] d\eta
\]

\[
= \frac{1}{\omega_k^2 + A^2} [\omega_k e^{-\lambda t} + A \sin(\omega_k t) - \omega_k \cos(\omega_k t)].
\]  

(B.99)

By substituting Equation (B.99) into Equation (B.91) it becomes

\[
N_k(t)
\]

\[
= \frac{B}{\omega_k} \sqrt{2 \sin \frac{k\pi}{l} x_i} \frac{1}{\omega_k^2 + A^2} [\omega_k e^{-\lambda t} + A \sin(\omega_k t) - \omega_k \cos(\omega_k t)].
\]  

(B.100)

\[
= \sqrt{\frac{2}{l}} \left[ \sin \frac{k\pi}{l} x_i \right] \left( \frac{B}{\omega_k} \sqrt{\frac{1}{42}} \right) \left[ e^{-\lambda t} - \cos(\omega_k t) + \frac{A}{\omega_k} \sin(\omega_k t) \right].
\]  

(B.101)
Final from Equation (B.78) and Equation (B.80), the solution is expressed as

$$y(x, t) = \sum_{k=1}^{\infty} N_k(t) \sqrt{\frac{2}{l}} \sin \frac{k\pi}{l} x ,$$

(B.102)

or

$$y_a(x, t) = \frac{2}{l} \sum_{k=1}^{\infty} \left( \frac{B}{\omega_k^2 + A^2} \right) \times$$

$$\left[ e^{-At} - \cos(\omega_k t) + \frac{A}{\omega_k} \sin(\omega_k t) \right] \left( \sin \frac{k\pi}{l} x_i \right) \left( \sin \frac{k\pi}{l} x_t \right) ,$$

(B.103)

where the solution is evaluated at the point $x = x_t$.

The numerical evaluation of Equation (B.103) requires that all the time functions be replaced by their discrete equivalents. In addition, the summation must be truncated at a value $K$ such that the error due to this truncation is acceptable to the application of interest. The discrete versions of the time functions also require that they be sampled at least twice the highest frequency given by $\omega_k = \omega_K$ or $\omega_s = \omega_K N$, where $\omega_s$ is the sampling frequency and $N \geq 2$. The sampling interval is then $T_s = 1/(2\pi \omega_s)$. The discrete version of Equation (B.103) can then be expressed as

$$y_a(x, nT_s) = w(x, n), \quad n = 0, 1, 2, \ldots ,$$

(B.104)

or more explicitly as

$$w(x, n) = \frac{2}{l} \sum_{k=1}^{K} \left( \frac{B}{\omega_k^2 + A^2} \right) \times$$

$$\left[ e^{-K\pi nT_s} - \cos(\omega_k nT_s) + \frac{A}{\omega_k} \sin(\omega_k nT_s) \right] \left( \sin \frac{k\pi}{l} x_i \right) \left( \sin \frac{k\pi}{l} x_t \right) ,$$

(B.105)

and represents a form that can be numerically evaluated.
C. Thin Plate Propagation

C.1 Dispersion Relationships

A taut string is a non-dispersive medium, allowing a disturbance to retain its shape as it propagates. This is the result of the propagation speed of every frequency component of the disturbance travelling at the same phase speed. The phase speed of each frequency component is determined by relating a quantity known as the frequency component's wave number to the speed of propagation using a dispersion relationship [31].

The dispersion relationship requires using a solution of the wave equation for the system in question. Using the simple taut string system as an example, the wave equation, given by Equation (B.1), has a normal mode solution of the form [25]

\[ y(x, t) = Ae^{i(kx-\omega t)} \]  \hspace{1cm} (C.1)

where \( A \) is the magnitude of complex exponential, \( k \) is the wave number, and \( \omega \) the frequency. Substituted Equation (C.1) into the string wave equation results in the dispersion relationship for a taut string given as

\[ \omega = ck \]
\[ \omega(k) = c \]  \hspace{1cm} (C.2)

The left hand term of Equation (C.2) is generally written as \( \frac{\omega(k)}{k} \), emphasising that it is a function of the wave number \( k \). The left hand term of Equation (C.2) is also the phase speed of any component at the frequency \( \omega \). Every frequency component is also associated with a wave number \( k \) as indicated by Equation (C.2). It is obvious
from Equation (C.2) that the phase speed of every component travels at the same rate resulting in a disturbance propagating without changing shape.

Insight into the meaning of the phase speed can be understood by considering the given as

\[ \theta = kx - \omega(k)t \]  

(C.3)

This term is called the phase of the normal mode and values of constant phase represent the same point on a waveform \[31\]. Holding the phase constant and differentiating Equation (C.3) with respect to time, results in the expression

\[ \frac{dx}{dt} = \frac{\omega(k)}{k} \]  

(C.4)

that gives the speed of propagation of a constant phase point along the \( x \) axis. It can be seen that the right hand side of Equation (C.4) is identical to the left hand side of Equation (C.2) and shows why the term \( \frac{\omega(k)}{k} \) is referred to as the phase speed. A Constant phase speed for all \( k \) implies \( \frac{d\omega(k)}{dt} = 0 \). If this is not true the phase speed depends on \( k \) and indicates that the phase speed of each normal mode is different. In this case a disturbance will change shape as it propagates.

To find the dispersion relationship for the classical thin plate equation, a normal mode solution, of the form of, Equation (C.1), is substituted into thin plate equation, given by Equation (3.2), to obtain the dispersion relationship \[24, 28\]

\[ \frac{\omega}{k} = kl_c \sqrt{\frac{E_p}{3\rho_r(1 - \nu^2)}} \]  

(C.5)

which shows that the phase speed is dependent on the wave number \( k \). The propagation of a disturbance on a thin plate disperses since the phase speed of the components increases with increasing wave number.

The wave number for a thin plate is dependent on the shape and boundary conditions. For a rectangular plate, simply supported on all sides, the wave number is given by \[24, 28, 26\]

\[ k_{mn} = \sqrt{\frac{m\pi^2}{l_x} + \frac{n\pi^2}{l_y}} \]  

(C.6)
Where \( l_x \) and \( l_y \) are the lateral dimensions of the plate, and \( m \), and \( n \) are integers. The frequency associated with the wave number \( k_{mn} \) is given by [24, 28, 26]

\[
\omega_{mn} = l_x \sqrt{\frac{E_p}{3\rho_p(1-\nu^2)}} \left[ \left( \frac{m\pi}{l_x} \right)^2 + \left( \frac{n\pi}{l_y} \right)^2 \right],
\]

\[
= l_x \left( \sqrt{\frac{E_p}{3\rho_p(1-\nu^2)}} \right) k_{mn}^2.
\]  

(C.7)

Since the classical thin plate equation is an approximate expression describing the behaviour of plates with relatively large lateral dimensions as compared to their thickness, the validity of these equations decreases as the wavelength, \( \Lambda_{mn} = \frac{2\pi}{k_{mn}} \), approaches the plate thickness.

### C.2 Validity of The Thin Plate Expressions

Section 3.3.1 indicated that the classic thin plate equation is an approximations of the expressions that describe the propagation of waves through a solid and results when the thickness is small relative to the solid's other lateral dimensions [24, 28]. As an approximation it provides an accurate description of plate behaviour when the propagating waves, with wavelengths \( \Lambda \), are long relative to \( 2l_x \), the thickness of the plate [24, 28, 26]. The range over which the frequency and phase speed expressions are valid can be quantified by using a dimensionless wave number given as [24]

\[
\xi = \frac{2k l_x}{\pi},
\]

and a dimensionless frequency variable given as [24]

\[
\Omega = \frac{2l_x\omega}{\pi \sqrt{\frac{E_p}{\rho_p 2(1+\nu)}}},
\]

\[
= \frac{2l_x\omega}{\pi c_tr}
\]

where \( c_tr \) is referred to as the speed of propagation of the transverse waves [24]. It can be shown that case \( \xi > \omega \) and \( \xi \ll 1 \) corresponds to the thin plate approximations [24]. In general values of \( \xi \) less than 0.1 and possibly 0.2 can be taken to satisfy \( \xi \ll 1 \).
Using the thickness of the experimental plate, given as approximately 0.0008 m, the result \( \xi = 2.546 \times 10^{-4}k \) is obtained. Setting \( \xi < 0.2 \) results in \( k < 786 \). These wave numbers are associated with a wavelengths \( \Lambda > \frac{2\pi}{k} \approx 0.008 \) m, that are ten times the thickness of the thin plate. The corresponding frequencies are then \( \omega < 7.6 \times 10^5 \) rad/s or \( f < 121 \) kHz, using Equation (C.7) and the material parameters of the experimental plate, given in Appendix K. These range of frequencies and wavelengths, for the experimental thin plate, meet the assumption of using components having wavelengths less than ten times the thickness of the plate. As a note regarding the experimental plate, the values of \( K \), and \( \omega_{mn} \) are approximately \( 1.940 \text{kH} \) and \( 1.094 \times 10^5 \) rad/s, respectively.

### C.3 Propagation Velocities

The phase speed of these frequency components can be obtained using Equation (C.5) for values of \( \xi \ll 1 \). As \( \xi \) increases, the speed of propagation approaches that of Rayleigh waves given by [24]

\[
\mathbf{c}_r \approx \frac{(0.862 + 1.14\nu_p)}{(1 + \nu_p)}c_{tr} \quad ,
\]

which for the experimental aluminium plate is approximately

\[
\mathbf{c}_r \approx 0.932c_{tr} = 0.932(3122) = 2910 \text{ m/s} \quad .\]

The phase speed of a component associated with a frequency of approximately 121 kHz is, from Equation (C.5)

\[
\frac{\omega}{k} = k\sqrt{\frac{E_p}{3\rho_p(1 - \nu^2)}} \quad ,
\]

\[
= 786(1.234) = 970 \text{ m/s} \quad ,
\]

that is less than the speed \( \mathbf{c}_r \).

The bulk propagation velocity of aluminium, at 25°C, is at approximately 6300 m/s [25].
C.4 Impact duration for steel spheres and infinitely thick plates

The duration of an impact of a steel sphere unto a infinitely thick plate can be approximated using [26]

\[ t_{\text{end}} = 4.53 \left[ \frac{E'^{\prime} + E^{\prime}}{\pi E_{\text{p}}^2 m_{\text{s}}} \right]^{\frac{1}{2}} \left[ \frac{16 \sqrt{r_{\text{s}} v_{\text{s}}}}{} \right] \]  

which for the experimental plate and the steel sphere having \( r_{\text{s}} = 3.175 \text{ mm} \) and \( v_{\text{s}} = 1.4 \text{ m/s} \) evaluates to approxi
D. General Solution of the Thin Plate Equation

This appendix reviews the general solution of the non-homogeneous classical thin plate equation using a rectangle thin plate simply supported on all sides with a point applied force. A detailed treatment is given by a number of sources [26, 18, 31]. The non-homogeneous classical thin plate equation is given by [26]

\[ D \nabla^4 z(x, y, t) + (2 \rho_y l_x) \frac{\partial^2 z(x, y, t)}{\partial t^2} = f_s(x, y, t) \quad \text{(D.1)} \]

where for notational brevity \( z \) represents \( z(x, y, t) \) and the squared Laplacian operator is

\[ \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^2}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad \text{(D.2)} \]

and \( D \), the flexural rigidity is given by

\[ D = \frac{6E_p l_x^3}{12(1 - \nu^2)} \quad \text{(D.3)} \]

A general solution for Equation (D.1) can be obtained by separation of variables using [26, 18, 31]

\[ z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}(t) \overline{U}_{mn}(x, y) \quad \text{(D.4)} \]

where \( \overline{U}_{mn}(x, y) \) are the normalized eigenfunctions found by using a rectangular plate defined to be confined to the region

\[ 0 < x < l_x \quad , \quad 0 < y < l_y \quad \text{(D.5)} \]
with simply supported boundary conditions expressed by

\[ z(0, y) = \frac{\partial^2 z(0, y)}{\partial x^2} = z(l_x, y) = \frac{\partial^2 z(l_x, y)}{\partial x^2} = 0, \quad 0 < y < l_y, \quad (D.6) \]

along the \( x \) axis and

\[ z(x, 0) = \frac{\partial^2 z(x, 0)}{\partial y^2} = z(x, l_y) = \frac{\partial^2 z(x, l_y)}{\partial y^2} = 0, \quad 0 < x < l_x, \quad (D.7) \]

along the \( y \) axis. The simply supported boundary condition allows no transverse motion of the plate but allows rotation so that there is no moment about the boundary. These conditions are applied to the expression or eigenproblem [26, 31]

\[ \nabla^4 U_{mn}(x, y) - \left( \frac{2\rho_l l_x}{D} \right) \omega_{mn}^2 U_{mn}(x, y) = 0, \quad (D.8) \]

where \( \omega_{mn} \) are the eigenfrequencies. From tables this results in [26]

\[ \omega_{mn} = l_x \sqrt{\frac{E_p}{3\rho_p(1 - \nu_p^2)}} \left[ \left( \frac{m\pi}{l_x} \right)^2 + \left( \frac{n\pi}{l_y} \right)^2 \right], \quad (D.9) \]

for the eigenfrequencies and

\[ U_{mn}(x, y) = \frac{4}{l_x l_y} \left( \sin \frac{m\pi}{l_x} x \sin \frac{n\pi}{l_y} y \right), \quad (D.10) \]

for the normalized eigenfunctions.

Now substitute Equation \((D.4)\) into Equation \((D.1)\) to obtain

\[ D \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \nabla^4 C_{mn}(t) U_{mn}(x, y) + (2\rho_l l_x) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^2 C_{mn}(t) U_{mn}(x, y)}{\partial t^2} = f_a(x, y, t) \quad . \quad (D.11) \]

Using Equation \((D.8)\), Equation \((D.11)\) becomes

\[ \omega_{mn}^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}(t) U_{mn}(x, y) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^2 C_{mn}(t) U_{mn}(x, y)}{\partial t^2} = \frac{1}{2\rho_l l_x} f_a(x, y, t) \quad . \quad (D.12) \]
Multiply both sides of Equation (D.11) by \( \bar{u}_{mn}(x,y) \), and integrate over the surface \( S \) of the plate to obtain

\[
\omega_{mn}^2 \int_S \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}(t) \bar{U}_{mn}(x,y) \bar{U}_{mn}(x,y) dS \\
+ \int_S \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^2 C_{mn}(t)}{\partial t^2} \bar{U}_{mn}(x,y) \bar{U}_{mn}(x,y) dS \\
= \frac{1}{2\rho_pl_s} \int_S f_a(x,y,t) \bar{U}_{mn}(x,y) . \tag{D.13}
\]

Using the property of eigenfunctions that

\[
\int_S \bar{U}_{mn}(x,y) \bar{U}_{mn}(x,y) dS = 1 \tag{D.14}
\]

Equation (D.13) becomes a series of equations of the form

\[
\frac{d^2 C_{mn}(t)}{dt^2} + \omega_{mn}^2 C_{mn}(t) = \frac{1}{2\rho_pl_s} \int_S \bar{U}_{mn}(x,y) f_a(x,y,t) dS . \tag{D.15}
\]

The solution to Equation (D.15) using a point applied force at \( (x_i,y_i) \), is given by [18]

\[
C_{mn}(t) = \frac{1}{2\rho_pl_s\omega_{mn}} \bar{U}_{mn}(x_i,y_i) \int_0^t f_a(\tau) \sin \omega_{mn}(t-\tau) d\tau . \tag{D.16}
\]

Equation (D.16) is substituted back into Equation (D.4) to obtain [26, 31]

\[
x(x,y,t) = \frac{2}{\rho_pl_s l_x l_y} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right) \\
\times \left( \sin \frac{m\pi}{l_x} x \sin \frac{n\pi}{l_y} y \right) \int_0^t f_a(\tau) \sin \omega_{mn}(t-\tau) d\tau . \tag{D.17}
\]

the general solution to the non-homogeneous wave equation.
E. Epi-centre Displacement

The epi-centre displacement of a thin plate due to a point force was developed and treated in detail in 1941 by Zener.

Using the results of appendix D with the general solution for \( z(x, y, t) \) evaluated at \( (x = x_i, y = y_i) \) becomes [26, 31]

\[
Z(x_i, y_i, t) = \frac{2}{\rho_p l_x l_y} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right)^2 \int_0^t f_a(\tau) \sin \omega_{mn}(t - \tau) d\tau ,
\]  

(E.1)

To simplify Equation (E.1), Zener [18] makes the assumption that the impact, or impulse, is completed before the reflections return. Physically, this requires large lateral plate dimensions or slow propagating disturbances. This assumption allows the evaluation of the term [18]

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}} \frac{4}{l_x l_y} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right)^2 \sin \omega_{mn}(t - \tau) ,
\]

(E.2)

before integration with respect to \( \tau \). As Zener indicates, the expression given by Equation (E.2) is independent of the shape of the plate or its boundary conditions as long as \( t \) is so small that the impact is completed before the disturbance is reflected by the boundaries. This approximation then leads to the expression, for the epi-centre displacement [18]

\[
z(x_i, y_i, t) = B_p \int_0^t f_a(\tau) d\tau ,
\]

(E.3)

where

\[
B_p = \sqrt{\frac{3\rho_p}{E_p}} \left( \frac{1}{16\rho_p l_x^2} \right) .
\]

(E.4)
F. Force of Impact of a Sphere With a Plate

F.1 The Normal Force of Impact

In the impact of a solid sphere with a solid plate, both objects deform. If the deformation does not exceed the elastic limits of either solid, both will return to their original shape [26]. For two solids pressed together, the deformation is not a simple linear relationship to force. It is dependent on the shape of the solids in the region of contact and is expressed using the force-deformation law [26]

\[ f_a(t) = K_I z_d(t)^\frac{3}{2} , \quad (F.1) \]

where for a solid sphere and a plane solid surface

\[ K_I = \frac{4}{3} \sqrt{\frac{E_s'}{E_p'}} \left( E_s + E_p' \right) . \quad (F.2) \]

If both the sphere and plane move while being compressed \( z_d(t) \) becomes the relative distance between the two or

\[ z_d(t) = z_s(t) - z_p(t) \quad , \quad (F.3) \]

where \( z_s(t) \) and \( z_p(t) \) represents the relative displacement of the sphere and the plate, respectively, and, as shown in Figure F.1, is measured from the point of impact. If Equation (F.3) is differentiated twice with respect to time the expression [18]

\[ \frac{d^2 z_d(t)}{dt^2} = \frac{d^2 z_s(t)}{dt^2} - \frac{d^2 z_p(t)}{dt^2} \quad , \quad (F.4) \]

is obtained. The expressions relating the motion of the plate \( z_p \) and that of the sphere \( z_s(t) \) is required to determine there relative motion. The sphere's motion is simply expressed as

\[ \frac{d^2 z_s(t)}{dt^2} = -\frac{f_s(t)}{m_s} . \quad (F.5) \]
Figure F.1: The relative position of a sphere and thin plate during impact.

The motion of the plate at the point of contact is obtained by using the simplified general solution developed by Zener describe in Appendix E. Differentiating this expression twice with respect to time results in

\[
\frac{d^2 z_p(t)}{dt^2} = \frac{d^2 (B_p \int f_a(t) \, dt)}{dt^2} \quad \text{(F.6)}
\]

\[
= B_p \frac{df_a(t)}{dt} \quad \text{(F.7)}
\]

Substituting Equation (F.7) and Equation (F.5) into Equation (F.4), the expression

\[
\frac{d^2 z_d(t)}{dt^2} + B_p \frac{df_a(t)}{dt} + \frac{f_a(t)}{m_s} = 0 \quad \text{(F.8)}
\]

is obtained. With the force-deformation law given by Equation (F.1), the first order term becomes

\[
\frac{df_a(t)}{dt} = \frac{d(K_f z_d(t)^{3/2})}{dt} \quad \text{(F.9)}
\]

\[
= \frac{3}{2} K_f \sqrt{z_d(t)} \frac{dz_d(t)}{dt} \quad \text{(F.10)}
\]

so that Equation (F.8) is written as

\[
\frac{d^2 z_d(t)}{dt^2} + \frac{3}{2} K_f B_p \sqrt{z_d(t)} \frac{dz_d(t)}{dt} + \frac{K_f}{m_s} z_d(t)^{3/2} = 0 \quad \text{(F.11)}
\]

and represents the normal relative plate displacements. The solution to Equation (F.11) is then used in Equation (F.1) to obtain the normal force of impact.
F.2 The Dimensionless Force of Impact

A more convenient form of Equation (F.11) can be obtained by a change of variables [18]

\[ \sigma(\tau) = \frac{zd(\tau)}{Tv_z}, \quad (F.12) \]
\[ \tau = \frac{t}{T}, \quad (F.13) \]

where \( \sigma \) and \( \tau \) represent the dimensionless relative displacement and time, respectively. Note, that \( zd(\tau) \) is shown as a function of \( \tau \). Now using

\[ dzd(\tau) = Tvd(\sigma(\tau)) , \quad dt = Td\tau , \quad (F.14) \]
\[ d^2zd(\tau) = Tvd^2(\sigma(\tau)) , \quad dt^2 = T^2d\tau^2 , \]

in Equation (F.11), it becomes

\[ \frac{Tvd^2(\sigma(\tau))}{T^2} + \frac{3}{2} KfBp \frac{Tvd(\sigma(\tau))}{T} (Tv_z(\sigma(\tau))^\frac{1}{2} \frac{d\sigma(\tau)}{d\tau} + \frac{Kf}{m_v} (Tv_z(\sigma(\tau))^\frac{3}{2} = 0 , (F.15) \]

\[ \frac{d^2(\sigma(\tau))}{d\tau^2} + \frac{3}{2} KfBpT(v_v(\sigma(\tau))^\frac{1}{2} \frac{d\sigma(\tau)}{d\tau} + \frac{Kf}{m_v} (Tv_z(\sigma(\tau))^\frac{3}{2} = 0 , (F.16) \]

\[ \frac{d^2(\sigma(\tau))}{d\tau^2} + \frac{3}{2} KfBpT^\frac{3}{2} (tv_v(\sigma(\tau))^\frac{1}{2} \frac{d\sigma(\tau)}{d\tau} + \frac{Kf}{m_v} T^\frac{1}{2} v_v^\frac{1}{2} (\sigma(\tau))^\frac{3}{2} = 0 . (F.17) \]

Now set \( T \) such that

\[ \frac{Kf}{m_v} T^\frac{3}{2} v_v^\frac{1}{2} = 1 , \quad (F.18) \]
\[ T^\frac{3}{2} = \frac{m_v}{Kf v_v^\frac{1}{2}} , \quad (F.19) \]
\[ T = \left( \frac{m_v}{Kf \sqrt{v_v}} \right)^\frac{4}{3} . \quad (F.20) \]

Equation (F.17) can then be written as

\[ \frac{d^2(\sigma(\tau))}{d\tau^2} + \frac{3}{2} KfBpT^\frac{3}{2} (tv_v(\sigma(\tau))^\frac{1}{2} \frac{d\sigma(\tau)}{d\tau} + \sigma(\tau))^\frac{3}{2} = 0 . \quad (F.21) \]

Now write the first order term as

\[ \frac{3}{2} KfBpT^\frac{3}{2} (tv_v(\sigma(\tau))^\frac{1}{2} \frac{T d\sigma(\tau)}{d\tau} = \frac{3}{2} KfBpT^\frac{3}{2} (tv_v(\sigma(\tau))^\frac{1}{2} \frac{1}{T} \frac{d\sigma(\tau)}{d\tau} , \quad (F.22) \]
and replace the $T^\frac{3}{2}$ factor with the results of Equation (F.20) to obtain for the first order term

$$= \frac{3}{2} K_f B_p \left( \frac{m_s}{K_f \sqrt{v_x}} \right)^{\frac{3}{2}} \left( v_\tau \sigma(\tau) \right)^{\frac{3}{2}} \frac{1}{2} \frac{\dot{\sigma}}{\dot{\tau}} ,$$

$$= \frac{3}{2} B_p m_s \sqrt{\sigma(\tau) \frac{d\sigma}{d\tau}} ,$$

$$= \lambda^\frac{3}{2} \sqrt{\sigma(\tau) \frac{d\sigma}{d\tau}} ,$$

where $\lambda$ is defined as

$$\lambda = \frac{B_p m_s}{T} .$$

so that Equation (F.21) becomes

$$\frac{d^2 \sigma(\tau)}{d\tau^2} + \lambda \left( \frac{3}{2} \sqrt{\sigma(\tau)} \right) \frac{d\sigma(\tau)}{d\tau} + \sigma(\tau)^{\frac{3}{2}} = 0 ,$$

and is the dimensionless relative plate displacements. The solution to Equation (F.25), is then used in Equation (F.1) to obtain the force of impact. The coefficient $\lambda$ is referred to by Zener as the inelasticity parameter.

By substituting in the factors $K_f$, $B_p$, and $T$ forming $\lambda$, the result

$$\lambda = m_s \left( \frac{K_f \sqrt{v_x}}{m_s} \right)^{\frac{3}{2}} B_p ,$$

$$= m_s^\frac{3}{2} K_f^{\frac{1}{2}} v_x^{\frac{1}{2}} B_p ,$$

$$= \left( \frac{4}{3} \pi r_s^3 \rho_s \right)^{\frac{1}{2}} \left( \frac{4}{3} \right)^{\frac{1}{2}} \left( r_s \frac{1}{2} \right) \left( \frac{E'_s E'_p}{E'_s + E'_p} \right)^{\frac{1}{2}} v_x \sqrt{3 \rho_p \left( \frac{1}{16 \rho_p l_s^2} \right)} ,$$

$$= \left( \frac{4}{3} \right) \left( \frac{1}{4} \right) \left( \frac{1}{2a} \right) \sqrt{3 \pi} \left( \frac{1}{4} \rho_s \right) \left( \frac{1}{2a} \right) \left( \frac{E'_s E'_p}{E'_s + E'_p} \right)^{\frac{1}{2}} v_x \rho_p \rho_p^{-1} ,$$

$$= \frac{\pi}{\sqrt{3}} \left( \frac{r_s}{2l_s} \right)^2 \left( \frac{v_x}{v_p} \right) \left( \frac{E'_s E'_p}{E'_s + E'_p} \right)^{\frac{1}{2}} \left( \frac{E'_s}{E'_p} \right) \rho_p \rho_p^{-1} ,$$

$$= \frac{\pi}{\sqrt{3}} \left( \frac{r_s}{2l_s} \right)^2 \left( \frac{v_x}{v_p} \right) \left( \frac{E'_s}{E'_p} \right) \left( \frac{E'_s E'_p}{E'_s + E'_p} \right)^{\frac{1}{2}} \rho_p \rho_p^{-1} ,$$

is obtained showing the influence of the sphere and plate parameters on $\lambda$. 
G. The Transducer Model

G.1 Physical Description

The piezo transducer is shown in Figure G.1(a) and is commercially called a piezo bender that consists of two oppositely polarized layers of a piezo-ceramic made of lead zirconate titanate formed into a thin sheet approximately $8 \times 8 \times 0.6$ millimetres. When the sheet is bend, one side goes into tension while the other is in compression. The oppositely polarized layers then produce voltages that are in the same direction so that resulting output is the sum of the voltage from each layer.

The piezo transducer has a small, approximately two millimetre diameter, hole to allow a small mounting bolt to pass through. A mounting nut on either side of the transducer hold the transducer at some point along the length of the bolt. This bolt is then fixed to the surface of the plate. This configuration is shown in Figure G.1(b).

G.2 Approximating the Piezo Transducer Motion

The piezo transducer appears to be a small thin plate whose motion can be approximated using the classical thin plate equation. This is not a reasonable view as the transducer has piezoelectric coupling that changes the assumptions used in obtaining the classical thin plate equation [35]. This fact and the complex boundary conditions consisting of free boundaries along the outside edges and a centre held by the mounting nuts that is forced into motion by the transverse plate movements, makes finding a solution difficult. The approach taken is to develop a simple one dimensional model that describes the primary motion of the transducer and the resulting voltage generated.
To begin to approximate to the piezo transducer's motion, the effects of the mounting nuts are neglected and only free boundary conditions along the edges are assumed. It can be reasonably expected that the boundary effects of the mounting nuts will tend to complicate the motion and change the resonant frequencies of the piezo and for this reason are ignored. According to manufacturer of the piezo transducer, the primary resonant motion of a free piezo plate is as shown in Figure G.2. This is based on manufacturer's qualitative statements and its sketches of the free piezo motion from the piezo data sheets.

A similar conclusion can be reached by arguing that the application of a voltage across this particular piezo transducer causes the entire surface of the piezo to flex or bend in one direction. If a low frequency alternating sine wave voltage is applied across the electrodes, the flexing direction alternates in accordance with the applied voltage. At the resonant frequency, $\omega$, the motion shown in Figure G.2 results. This argument indicates that areas of the piezo plate, outside the nodal line, are moving in the same direction and that this is opposite to direction the area inside the nodal line is moving. This establishes the form of the motion for this mode given an externally applied voltage.

The piezo transducer, described in the section G.1, is not forced into motion.
by an external applied voltage but by an externally applied force. An additional simplifying assumption is made that the motion due to either cause is reasonably the same and that this is the dominate motion of piezo. The argument supporting this assumption draws an analogy to the motion of a free thin plate and the reduction of the external force to a single point located at the centre of the piezo.

Figure G.3 shows the first few mode shapes of a free plate obtained experimentally [28]. The modes shown in Figure G.3 correspond to modes \( \{mn\} \) of (2, 2), (1, 3), and (3, 1). The modes (1, 1), (1, 2), (2, 1) are rigid body modes and are not shown [28]. The lines on the surface of the plate represent the nodal positions or locations on the plate that are stationary. Each mode vibrates at a particular frequency that increases with the mode number.

A force applied to a nodal point will not excite any modes having nodal lines running through that point. Since modes (2, 2), and (1, 3) have a nodal lines running through the centre, these modes will not be excited and the first mode without a centre nodal location is the (3, 1) mode. The nodal lines for this mode are very similar in shape to those for a piezo in resonant motion.

To reduce this motion to a simple representative system it is also assumed that any cross sectional slice parallel to a side is representative of the general form of the piezo vibration. This cross section is then used to represent the piezo plate's primary resonant motion. Figure G.4(a) illustrates the form of the plate motion at various
The first few vibration modes and nodal lines for a simple free thin plate. The nodal lines shown in (c) resembles the free resonant motion of the transducer.

cross sections. The frequency of vibration of any of these cross sections is at the resonant frequency $\omega_1$ of the piezo transducer.

The nodal points on the piezo plate represent locations that are stationary. For example, the cross section of A-A' has the nodal points at the far ends, whereas section C-C' has a single nodal point mid-way between the ends. This assumption is based on the approximation that plate motion can be reasonably described by the motion of a series of separable beams [28]. In this sense, the cross sections are representative of beams simply supported at the nodal points. This representation is also shown in Figure G.4(b). These cross sections and the beam analogy is then used to further simplify the plate motion to a one dimensional system. This is given in the next section.

**G.3 The Mechanical Model**

In this section a simple one dimensional model of the motion of the piezo cross section is developed. The purpose of this model is to have an expression that relates the flexing motion of the cross section to the motion of a one dimensional system. The final mechanical model is based on a spring-mass-damper system.

Consider the system in Figure G.5 showing a beam of mass $m_b$, modulus of
elasticity $E_b$, and length $l_b$ that is simply supported at both end. This system is analogous to a piezo plate cross section taken through section A-A’ of Figure G.4. This beam bends or deflects an amount due to its own weight. If the deflection is referenced from the initial non bent position, it can be shown that the deflection can be approximated by a mass loaded spring, using [28]

\[
 z_t = \frac{F_b}{k_b} , \\
 = \frac{m_bl_bg}{2.96'\frac{E_b}{l_b}I_b} , \\
 = 2.96'\frac{m_bl_bg}{E_bI_b} , 
\]

where $F_b$ is the force of gravity acting on the beam mass, $k_b$, and $I_b$ are the equivalent spring constant and the moment of inertia for the beam, respectively. The beam is then represented by the equivalent mass-spring system shown in Figure G.5(b).
Figure G.5: A simply supported beam deflects a distance $z_t$ due to the beam mass $m_b$ and gravity $g$. This deflection is approximated using a spring-mass system shown in (b) that is further approximated using the spring-mass-damper system shown in (c).

The resonant frequency of this mass-spring system is

$$\omega_b = \sqrt{\frac{k_b}{m_b}}. \quad (G.4)$$

The resonant frequency of the spring-mass system, given by Equation (G.4) is approximately 11% lower than the exact first resonant frequency of the simply supported beam [28]. Though not exact, the spring-mass system is a reasonable analogous system that represents the motion of the piezo cross section.

Using this analogy, the resonant frequency of the beam cross section, $\omega_b$, is set equal to the resonant frequency of the spring-mass-damper system $\omega_t$, shown in Figure G.5(c) or

$$\omega_b = \omega_t = \sqrt{\frac{k_t}{m_t}}, \quad (G.5)$$

where $k_t$ and $m_t$ represent the appropriate spring constant and mass, respectively, that results in the resonant frequency of $\omega_t$. Then from Equation (G.5), the deflection $z_c(t)$ of the cross section due to an external force $f_c(t)$ is approximated by the spring displacement

$$z_c(t) = \frac{f_c(t)}{k_t}. \quad (G.6)$$
For the sensor configuration under consideration, the piezo plate is forced into motion by the motion of sensor plate. The motion of this sensor plate is denoted \( z_t(t) \). The free body diagram for this configuration is shown in Figure G.5(c) where the displacement \( z_t(t) \) represents the transverse motion of the sensor plate. The spring compression or cross section deflection is then \( z_c(t) = z_o(t) - z_t(t) \) and the system acquires two degrees of freedom. This configuration appears intuitively correct as the piezo plate, mounted as described in section G.1, also has two degrees of freedom.

The spring-mass system shown in Figure G.5(b) will, once disturbed, vibrate indefinitely. This is not realistic, as the vibration or oscillation will eventually dampen out due to a number of factors such as mounting, dissipative losses in the piezo, and air friction. To account for these losses, a damper is added to the spring-mass system as shown in Figure G.5(c). Here \( b_t \) is the damping coefficient that accounts for the gross effects of system damping. From Figure G.5(c), the equation of motion relating the input displacement \( z_t(t) \) to the spring compression \( z_c(t) \), is then written as

\[
m_t \frac{d^2 z_o(t)}{dt^2} + b_t \frac{dz_o(t)}{dt} + k_t z_o(t) = 0 \tag{G.7}
\]

\[
m_t \frac{d^2(z_c(t) + z_t(t))}{dt^2} + b_t \frac{dz_c(t)}{dt} + k_t z_o(t) = 0 \tag{G.8}
\]

\[
m_t \frac{d^2 z_c(t)}{dt^2} + b_t \frac{dz_c(t)}{dt} + k_t z_c(t) = -m_t \frac{d^2 z_t(t)}{dt^2} \tag{G.9}
\]

Assuming zero initial conditions, and taking the Laplace transform, the compression \( z_c(t) \) given an input \( z_t(t) \) is

\[
\frac{Z_c(s)}{Z_t(s)} = \frac{-s^2}{s^2 + \frac{b_t}{m_t} s + \frac{k_t}{m_t}} \tag{G.10}
\]

### G.4 The Electrical Model

In this section a simple electrical model is developed to represent the voltage generated by the piezo transducer. The two dimensional cross sectional representation of the piezo and first order piezoelectric effects are used to develop the
Figure G.6: A cross section of piezo-material with a thickness $2l_t$ is shown in (a) and the internal stresses through the section A-A' are shown in (b).

model. These piezoelectric effects result in a linear relationship between the material polarization and the stress as well as, the electric field and strain [32]. The cross section is assumed to have negligible thin electrodes on the outside surfaces and through the mid-line.

Consider a segment of a cross section that is deflecting as shown in Figure G.6(a). As discussed previously, the concaved surface of the cross section is in compression, while the convexed surface is in tension. It is known that the longitudinal mid-line of the cross section is under no stress and that compression or tension grows linearly as a function of the distance from this mid-line [28]. The maximum stress occurs at the surface $z = l_t$ and is expressed using

$$T_l(z, t) = C_l \frac{z}{l_t} f_c(t) \quad 0 \leq z \leq l_t \quad ,$$

where $T_l$ represent the tension of the material at the point $z$ and $C_l$ represents a proportionality constant. A positive $T_l$ refers to a compression. Equation (G.11) also relates the applied force to the tension through the proportionality constant $C_l$. The tension is also assumed to be only a function of $z$ and $t$. At the surface the stress becomes

$$T_l(l_t, t) = C_l f_c(t) \quad .$$

\hspace{1cm} (G.12)
The tension or stress $T_i$ can also be expressed using Equation (G.6), given as

$$T_i(z, t) = k_d(z)f_c(t) \quad ,$$  \hspace{1cm} (G.13)

$$= k_d(z)(k_i z_c(t)) \quad ,$$  \hspace{1cm} (G.14)

where $k_d(z)$ is defined as

$$k_d(z) = C_i \frac{z}{l_i} \quad .$$  \hspace{1cm} (G.15)

The stress on the piezo material results in strain that produces an electric field. The material of the piezo transducer, from the manufacturer's data sheets, operates in a 31-mode resulting in an electric field $E_i$ that is orthogonal to the stress and strain or, in this case, perpendicular to the surface of the cross section [36]. The direction of the field depends on direction of the material polarization and whether the stress is a compression or a tension. In any case, the opposite polarized layers, with an applied tension and compression, produces a net field that is in the same direction.

To determine the voltage that is generated at the electrodes, the relationship [32]

$$D_t = \epsilon_i E_t + d_{31} T_i \quad ,$$  \hspace{1cm} (G.16)

is used. Where $D_t$ is the electric displacement, $\epsilon_i$, the dielectric constant of the piezo material, $d_{31}$ the piezoelectric constant of the material, and $T_i$ the applied stress. If the electrodes of the piezo transducer are left open circuited, the electric displacement, $D_t = 0$ [32]. Using this, Equation (G.16) is written as

$$E_i(z, t) = -\frac{d_{31} T_i(z, t)}{\epsilon_i} \quad .$$  \hspace{1cm} (G.17)

To obtain the voltage $v_t(t)$ generated across the electrodes, the electric field is integrated over the region between the mid-line and one surface. Noting that the field is symmetric about the mid-line and in same direction, the expression

$$v_t(t) = 2 \int_0^h E_i(z, t)dz \quad ,$$  \hspace{1cm} (G.18)
\[ v(t) = -K_t z_c(t) \]

is obtained with the use of Equation (G.11) and Equation (G.14). Equation (G.21) is written in the form

\[ v(t) = -K_t z_c(t) \]

where \( K_t = \frac{d_{31} k_t C_I}{\epsilon_t} \). Equation (G.22) shows that the voltage is directly proportional to the displacement or compression of the piezo cross section. The electrical model using this result is shown in Figure G.7 [32]. The capacitance \( C \) represents the piezo plate capacitance and the resistance \( R_t \) represents the load of the amplifier measuring the piezo voltage.

Using simple voltage division, the output voltage that is measured by the amplifier is given by

\[
V_o(s) = \frac{R_t}{R_t + 1/sC} V_i(s)
\]

where \( V_o(s) \) and \( V_i(s) \) are the Laplace transforms of \( v_o(t) \) and \( v_i(t) \), respectively.
initial conditions are also assumed. Equation (G.24) is now written as

\[ V_o(s) = -R_t C K_t \frac{s}{1 + s R_t C} Z_t(s), \quad (G.25) \]

and represents the electrical element of the transducer model.

The complete piezo transducer model is given using Equation (G.25) and Equation (G.10) and is written

\[ V_o(s) = \left( -R_t C K_t \frac{s}{1 + R_t C s} \right) \left( \frac{-s^2}{s^2 + \frac{k_i}{m_i} s + \frac{k_i}{m_i}} \right) Z_t(s), \quad (G.26) \]

\[ \frac{V_o(s)}{Z_t(s)} = R_t C K_t \frac{s^3}{(1 + R_t C s) \left( s^2 + \frac{k_i}{m_i} s + \frac{k_i}{m_i} \right)} \quad (G.27) \]

It is assumed that the break point at \( \omega_c = \frac{1}{R_t C} \), given in the electrical element of the transducer model is much lower in frequency as compared to the resonant point of the mechanical portion, given by \( \omega_m = \sqrt{\frac{k_i}{m_i}} \). Practically, this is a reasonable assumption as \( R_t \) is usually in the order of few megohms and \( C \) is in the order of a few nanofarads resulting in a break point below a few hundred Hertz. The resonant frequency of the piezo is in the range of tens of kilohertz.
H. The Frequency Representation of the Thin Plate Solution

The details in transforming the thin plate displacement \( z_i(t) \) into the frequency domain is examined in this section. Consider the thin plate displacement given by

\[
z_i(t) = \frac{2}{\rho_p l_x l_y l_{\omega_{mn}}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right) \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right) \times \int_0^t f_\alpha(\tau) \sin \omega_{mn}(t - \tau) d\tau ,
\]

where all the terms have been previously defined. For brevity Equation (H.1) is written as

\[
z_i(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \int_0^t f_\alpha(\tau) \sin \omega_{mn}(t - \tau) d\tau ,
\]

where

\[
B_{mn} = \frac{2}{\rho_p l_x l_y l_{\omega_{mn}}} \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right) \left( \sin \frac{m\pi}{l_x} x_i \sin \frac{n\pi}{l_y} y_i \right) .
\]

Expanding out each term in Equation (H.2)

\[
z_i(t) = B_{11} \int_0^t f_\alpha(\tau) \sin \omega_{11}(t - \tau) d\tau \\
+ B_{12} \int_0^t f_\alpha(\tau) \sin \omega_{12}(t - \tau) d\tau \\
+ B_{21} \int_0^t f_\alpha(\tau) \sin \omega_{21}(t - \tau) d\tau + \cdots .
\]

Equation (H.5) consists of an infinite summation of weighted convolutions of \( f_\alpha(t) \) with \( \sin \omega_{mn}t \). The frequencies \( \omega_{mn} \) are defined by

\[
\omega_{mn} = l_x \sqrt{\frac{E_p}{3 \rho_p (1 - \nu^2_p)}} \left[ \left( \frac{m\pi}{l_x} \right)^2 + \left( \frac{n\pi}{l_y} \right)^2 \right] .
\]
By setting $\omega_{mn} = \omega_{nm}$ it is possible to determine under what conditions the frequencies $\omega_{mn}$ are unique, from Equation (H.5)

$$\omega_{mn} = \omega_{nm} \quad \text{(H.6)}$$

$$l_x \sqrt{\frac{E_p}{3\rho_p(1 - \nu_p^2)}} \left[ \left( \frac{m\pi}{l_x} \right)^2 + \left( \frac{n\pi}{l_y} \right)^2 \right] = l_x \sqrt{\frac{E_p}{3\rho_p(1 - \nu_p^2)}} \left[ \left( \frac{n\pi}{l_x} \right)^2 + \left( \frac{m\pi}{l_y} \right)^2 \right]$$

$$\left( \frac{m}{l_x} \right)^2 + \left( \frac{n}{l_y} \right)^2 = \left( \frac{n}{l_x} \right)^2 + \left( \frac{m}{l_y} \right)^2$$

$$\frac{1}{l_x^2} (m^2 - n^2) = \frac{1}{l_y^2} (m^2 - n^2)$$

$$\frac{l_x^2}{l_y^2} = \frac{m^2 - n^2}{m^2 - n^2} = 1$$

$$\frac{l_y}{l_x} = 1$$

or only when $l_y = l_x$ will $\omega_{mn} = \omega_{nm}$. In general it is assumed that $l_y \neq l_x$ so that each $\omega_{mn}$ is unique and therefore no two terms in Equation (H.5) can be combined into a single term.

Now consider the convolution integral [20, 22]

$$y(t) = \int_{-\infty}^{+\infty} f_a(\tau) g(t - \tau) d\tau \quad \text{(H.7)}$$

$$= f_a(t) * g(t) \quad \text{(H.8)}$$

where $f_a(t) = g(t) = 0$ for $t \leq 0$ and $g(t) = \sin \omega_{mn} t$. Then

$$y(t) = \int_{-\infty}^{+\infty} f_a(\tau) g(t - \tau) d\tau \quad \text{(H.9)}$$

$$= \int_{0}^{t} f_a(\tau) \sin \omega_{mn} (t - \tau) d\tau \quad \text{(H.10)}$$

Using the property that [20, 22]

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{f_a(t) * g(t)\} \quad \text{(H.11)}$$

$$= F_a(\omega)G(\omega) \quad \text{(H.12)}$$
where $\mathcal{F}\{}$ represents the Fourier transform

$$
\mathcal{F}\{g(t)\} = G(\omega) = \int_{-\infty}^{+\infty} g(t)e^{-j\omega t} dt .
$$

(H.13)

From tables [20, 22]

$$
\mathcal{F}\{\sin \omega_m t\} = j\pi \left[ \delta(\omega + \omega_m) - \delta(\omega - \omega_m) \right] ,
$$

(H.14)

so that

$$
\mathcal{F}\{y(t)\} = F_0(\omega)j\pi \left[ \delta(\omega + \omega_m) - \delta(\omega - \omega_m) \right] .
$$

(H.15)

For historical reasons $\delta(x)$ is called the Dirac delta function and is defined with the following properties [37]

$$
\delta(x) = \begin{cases} 
0, & x \neq 0 \\
\infty, & x = 0 
\end{cases} ,
$$

(H.16)

with

$$
\int_{-\infty}^{+\infty} \delta(x) dx = 1 ,
$$

(H.17)

so that

$$
\int_{-\infty}^{+\infty} \delta(x)f_a(x) dx = f_a(0) .
$$

(H.18)

It should be noted that $\delta(x)$ is not really a function but an item called a distribution and should be thought of as a linear operator, for example, $\delta(x - x_0)$ operates on $f_a(x)$ to yield $f_a(x_0)$ [37].

Applying the results of Equation (H.15) to each term in Equation (H.5) to obtain

$$
Z_4(\omega) = B_{11}F_0(\omega)j\pi \left[ \delta(\omega + \omega_{11}) - \delta(\omega - \omega_{11}) \right] + B_{12}F_0(\omega)j\pi \left[ \delta(\omega + \omega_{12}) - \delta(\omega - \omega_{12}) \right] + B_{21}F_0(\omega)j\pi \left[ \delta(\omega + \omega_{21}) - \delta(\omega - \omega_{21}) \right] + \cdots ,
$$

(H.19)
or in more compact form

\[ Z_t(\omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} F_a(\omega) j \pi \left[ \delta(\omega + \omega_{mn}) - \delta(\omega - \omega_{mn}) \right] \]  

(H.20)

The most common interpretation of Equation (H.20) is that it represents a series of unit area impulses centred at \( \omega_{mn} \), each weighted by a complex quantity that includes the value of \( F_a(\omega) \) at \( \omega_{mn} \) [20, 22, 21]. These impulses extend from \(-\infty\) to \(+\infty\).
I. Numerical Solutions

I.1 Overview

This appendix discusses numeric methods to solve the non-linear normal and dimensionless equations describing the relative motion between the sphere and the plate. From these solutions the force of impact can be determined. Three methods are used to calculate the resulting plate displacements at the impact epi-centre, the method given by Zener's approximation, the discrete convolution method, and a method based on linear superposition of sinusoids. Of these three methods, the discrete convolution and linear superposition allow the determination of the plate motion for any arbitrary impact and measurement point. It is assumed that Zener's results are accurate so that the other two methods can be compared to these results. It is shown that both discrete convolution and linear superposition provide results that agree but it is the linear superposition method that provides the most accurate and rapid evaluation of the plate displacements.

This appendix also presents the numeric methods and results used to implement the transducer transfer function and the digital differentiator.

I.2 Solution of the Non-Linear Second Order Differential Equation

The solution to the non-linear normal and dimensionless equations describing the relative motion between the sphere and the plate is achieved by applying the fourth order Runge-Kutta-Nystrom method to both equations [23]. This method
finds the solution to the initial value problem, given \( y_0 = y(x_0), \ y'_0 = y'(x_0) \)

\[
y'' = g(x, y, y')
\]

at the points \( y_n \) and \( x_n = x_0 + nh \) for \( n = 0, 1, \ldots \), where \( h \) is the time increment between points, and where \( y' \) and \( y'' \) represent the first and second derivatives with respect to general variable \( x \). For each time increment the following intermediate values are required

\[
k_1 = \frac{1}{2} h g(x_n, y_n, y'_n)
\]

\[
k_2 = \frac{1}{2} h g(x_n + \frac{1}{2} h, y_n + K_2, y'_n + k_1)
\]

where \( K_2 = \frac{1}{2} h (y'_n + \frac{1}{2} k_1) \),

\[
k_3 = \frac{1}{2} h g(x_n + \frac{1}{2} h, y_n + K_2, y'_n + k_2)
\]

\[
k_4 = \frac{1}{2} h g(x_n + h, y_n + K_4, y'_n + k_3)
\]

where \( K_4 = h (y'_n + k_3) \),

to obtain

\[
y_{n+1} = y_n + h (y'_n + \frac{1}{3} (k_1 + k_2 + k_3))
\]

and

\[
y'_{n+1} = y'_n \frac{1}{3} (k_1 + 2k_2 + 2k_3 + k_4)
\]

For the normal equation, these constants are found by using

\[
z_d(t_n)'' = z_{d_n}'' = g(t_n, z_{d_n}, z'_{d_n}) = -\frac{3}{2} K_f B_p \sqrt{z_{d_n} z'_{d_n}} - \frac{K_f}{m_s} \frac{z''_{d_n}}{2}
\]

where \( t_n = nh \) for \( n = 0, 1, \ldots \) and \( z'_{d_n} \) and \( z''_{d_n} \) represent the first and second derivatives with respect to time, respectively. For the dimensionless equation the constants are found using

\[
\sigma (\tau_n)'' = \sigma_{n}'' = g(\tau_n, \sigma_n, \sigma'_n) = -\lambda \frac{3}{2} \sqrt{\sigma_n} \sigma'_n - \sigma_{n}^\alpha
\]
where \( \tau_n = nh \) for \( n = 0, 1, \ldots \) and \( \sigma'_n \) and \( \sigma''(\tau_n) \) represent the first and second derivatives with respect to \( \tau \), respectively. Numerically solving Equation (1.8) or Equation (1.9) requires an appropriate time increment to allow the solution to converge. As was shown in the development of the force model, the force pulse has a varying duration that is characterised by a rapid rise of the signal followed by a slow exponential decay during its latter half. The time interval must than be set small enough to avoid non-convergence and large enough to prevent large number of iterations to arrive at a solution. Having found \( z_d(t_n) \) or \( \sigma(\tau_n) \), the force deformation law is used to arrive at the discrete force function \( f_a(t_n) \) or \( f_a(\tau_n) \).

The solutions for either Equation (1.8) or Equation (1.9) can be converted to the other by scaling of both the time and magnitude quantities using the previous change of variables

\[
\sigma(\tau) = \frac{z_d(\tau)}{T_{v_s}}, \\
\tau = \frac{t}{T},
\]

where \( T \) must be evaluated for each different set of particle and plate parameters.

Now consider

\[
f_a(\tau) = K_f z_d(\tau)^{\frac{3}{2}}, \tag{1.12}
\]

\[
= K_f (\sigma(\tau)T_{v_s})^{\frac{3}{2}}, \tag{1.13}
\]

where

\[
K_f = \frac{4}{3} \sqrt{\tau} \frac{E'_s E'_p}{E'_s + E'_p}, \tag{1.14}
\]

and

\[
T = \left( \frac{m_s}{K_f \sqrt{v_s}} \right)^{\frac{3}{2}}, \tag{1.15}
\]

so that Equation (1.13) becomes

\[
f_a(\tau) = K_f \sigma(\tau)^{\frac{3}{2}} v_s^{\frac{5}{2}} \left[ \left( \frac{m_s}{K_f \sqrt{v_s}} \right)^{\frac{3}{2}} \right]^{\frac{3}{2}}, \tag{1.16}
\]

\[
= K_f^{\frac{15}{8}} K_f^{\frac{18}{30}} v_s^{\frac{18}{18}} v_s^{-\frac{18}{18}} m_s^{\frac{2}{2}} \sigma(\tau)^{\frac{3}{2}}, \tag{1.17}
\]

\[
= K_f^{\frac{2}{2}} v_s^{\frac{5}{2}} m_s^{\frac{2}{2}} \sigma(\tau)^{\frac{3}{2}}. \tag{1.18}
\]
Now divide Equation (I.18) by \( F_m \) where

\[
F_m = r^2 \left( \frac{4}{3} \right) \left( \frac{E_s' E_p'}{E_s + E_p} \right) \left( \frac{5}{4} \pi v_s^2 \rho_p \right) \frac{r}{r^3} \left( \frac{E_s' E_p'}{E_s + E_p} \right) \left( \frac{5}{4} \pi v_s^2 \rho_p \right) \frac{r}{r^3},
\]

so the result

\[
\frac{f_s(\tau)}{F_m} = \left( \frac{1}{4} \right)^{\frac{3}{2}} r^\frac{3}{2} \left( \frac{E_s' E_p'}{E_s + E_p} \right) \left( \frac{5}{4} \pi v_s^2 \rho_p \right) \frac{r}{r^3} \sigma(\tau)^{\frac{3}{2}} \frac{(\frac{1}{2})^{\frac{3}{2}}}{(\frac{1}{2})^{\frac{3}{2}}},
\]

\[
= \left( \frac{1}{4} \right)^{\frac{3}{2}} r^\frac{3}{2} \left( \frac{E_s' E_p'}{E_s + E_p} \right) \left( \frac{5}{4} \pi v_s^2 \rho_p \right) \sigma(\tau)^{\frac{3}{2}} \frac{(\frac{1}{2})^{\frac{3}{2}}}{(\frac{1}{2})^{\frac{3}{2}}},
\]

\[
= \frac{\sigma(\tau)^{\frac{3}{2}}}{(\frac{1}{2})^{\frac{3}{2}}},
\]

\[
= 0.8^{\frac{3}{2}} \sigma(\tau)^{\frac{3}{2}},
\]

is obtained. Therefore, plotting \((0.8)^{\frac{3}{2}} \sigma^{\frac{3}{2}}\) versus \(\tau\) results in force function that is dimensionless in both time and magnitude. Multiply the time axis by \(T\) and the magnitude by \(F_m\) will result in the normal solution having the units seconds and newtons for the time and magnitude, respectively.

### I.3 Direct Evaluation of \(z_t(t)\)

Consider the general solution of the thin plate equation, having the transducer location \(x_t, y_t\) fixed so that it is written as a function of time, or as

\[
z_t(t) = K_p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn} A_{mn} \int_{0}^{t} f_s(\eta) \sin \omega_{mn}(t - \eta) d\eta,
\]

where

\[
K_p = \frac{2}{\rho_p l_x l_y},
\]

\[
K_{mn} = \frac{1}{\omega_{mn}} \left( \sin \frac{m\pi}{l_x} x_t \sin \frac{n\pi}{l_y} y_t \right),
\]

and

\[
A_{mn} = \left( \sin \frac{m\pi}{l_x} x_t \sin \frac{n\pi}{l_y} y_t \right).
\]
The integral in Equation (1.24) is a convolution of a causal finite duration force pulse \( f_a(t) \) with a sinusoid over the duration \( t \). The infinite summation, over the mode numbers \( \{m, n\} \), of these appropriately weighted convolutions results in the solution \( z_t(t) \).

Consider the discrete time version of \( f_a(t) \) and \( x_n(t) = \sin(\omega_m t) \) expressed as

\[
x_1(l) = \begin{cases} f_a(lT_s), & l = 0, 1, \ldots \\ 0, & l < 0, \text{and} \end{cases}
\]

and

\[
x_2(n) = x_n(nT_s) = \sin(\omega_m nT_s), \quad n = 0, 1, \ldots,
\]

where \( T_s \) is the sampling interval in seconds per sample and where \( L \) represents the total number of sample points over the duration of the force pulse. Using the discrete time functions in Equation (1.24) reduces the convolution integral to a summation so that the discrete time version of Equation (1.24) becomes

\[
w_t(k) = z_t(kT_s), \quad k = 0, 1, 2, \ldots
\]

or

\[
w_t(k) = K_p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn} A_{mn} \sum_{l=0}^{L-1} x_1(l)x_2(k-l) \quad k = 0, 1, 2, \ldots
\]

The plate displacement is now a discrete time function where each time point \( k \) represents the point at \( t = kT_s \).

Equation (1.32) will be a scaled version of Equation (1.24) by virtue of the discrete linear convolution in Equation (1.32) being the sum of points versus the sum of areas calculated by the convolution integral. Scaling Equation (1.32) by \( T_s \) will result in it approximating the values produced by Equation (1.24) since this effectively turns each point into a rectangular area of height \( w_t(k) \) and width \( T_s \). The error in this approximation will approach zero as \( T_s \) approaches zero.
To evaluate Equation (1.32) will require the summation over \( m \) and \( n \) to be truncated to some value \( M \) and \( N \), respectively, such that the resulting error due to insufficient higher order terms is tolerable to the application of interest. This results in

\[
\omega_t(k) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} w_t(k) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{m'} \sum_{n'} w_t(k) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{m'} \sum_{n'} w_t(k) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{m'} \sum_{n'} w_t(k) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{m'} \sum_{n'} w_t(k)
\]

a form that can be numerical evaluated. Though Equation (1.32) can be evaluated numerically, it requires a large number of summation to arrive at a solution. This could take a considerable amount of computational resources.

### I.4 Evaluation of \( z_t(t) \) by Linear Superposition

A method that may be faster to evaluate uses the frequency domain representation of the thin plate solution expressed by

\[
Z_t(\omega_{mn}) = \begin{cases} 
-j \pi K_p K_{mn} A_{mn} F_a(\omega_{mn}), & \omega_{mn} > 0 \\
+j \pi K_p K_{mn} A_{mn} F_a(\omega_{mn}), & \omega_{mn} < 0
\end{cases} \quad (1.33)
\]

One interpretation of Equation (1.33) is that it indicates that the frequency spectrum of \( z_t(t) \) consists of an infinite summation of sinusoids at the frequencies \( \omega_{mn} \) and magnitudes \( |Z_t(\omega_{mn})| \). By calculating the value of each of the sinusoids over a specified duration of time and summing them, \( z_t(t) \), over the given duration, will result.

Since \( F_a(\omega_{mn}) \) will in general be a complex quantity, \( Z_t(\omega_{mn}) \) will consist of complex quantities at both positive and negative frequencies. The continuous time representation of a single sinusoid at a frequency \( \omega_{mn} \) can be expressed as

\[
z_{t_{mn}}(t) = |Z_t(\omega_{mn})| \left[ e^{j(\omega_{mn}t+\int Z_t(\omega_{mn}))} - e^{-j(\omega_{mn}t+\int Z_t(\omega_{mn})} \right] / (2j) \quad (1.34)
\]

or in phaser form

\[
z_{t_{mn}}(t) = |Z_t(\omega_{mn})| \angle (\theta_{\omega_{mn}} - \theta_{-\omega_{mn}}) \quad (1.35)
\]

where

\[
\angle Z_t(\omega_{mn}) = \theta_{\omega_{mn}} \quad (1.36)
\]
and
\[ \angle Z_t(-\omega_{mn}) = \theta_{-\omega_{mn}} . \] (I.37)

The angle \( \angle Z_t(\omega_{mn}) \) is defined as
\[ \angle Z_t(\omega_{mn}) = \tan^{-1} \left[ \frac{\Im\{Z_t(\omega_{mn})\}}{\Re\{Z_t(\omega_{mn})\}} \right] , \] (I.38)
where \( \Im\{\} \) and \( \Re\{\} \) represent the imaginary and real parts of \( Z_t(\omega_{mn}) \) respectively. Equation (I.38) indicates that each sinusoid will be shifted in phase by some absolute angle. Since the plate begins to vibrate at the lowest frequency, the phase of the vibrations of the all the other frequencies can be measured relative to phase of the lowest frequency. This sets the lowest frequency sinusoid to be the reference phase or zero phase frequency so that the relative phase of a sinusoid at \( \omega_{mn} \) can be expressed as
\[ \angle_{rel} Z_t(\omega_{mn}) = \tan^{-1} \left[ \frac{\Im\{Z_t(\omega_{mn})\}}{\Re\{Z_t(\omega_{mn})\}} \right] - \tan^{-1} \left[ \frac{\Im\{Z_t(\omega_{11})\}}{\Re\{Z_t(\omega_{11})\}} \right] . \] (I.39)

In phaser form the sinusoid having this relative phase shift at the frequency \( \omega = \omega_{mn} \) is then
\[ z_{t_{mn}}(t) = |Z_t(\omega_{mn})| \angle_{rel} Z_t(\omega_{mn}) , \] (I.40)

To numerically calculate \( z_t(t) \) using these sinusoids, only a finite number of discrete sinusoids with frequencies between \( \omega_{11} \leq \omega_{mn} \leq \omega_{mn_{max}} \) over a duration \( 0 \leq t \leq t_{end} \) can be used. The discrete version of the single sinusoid \( z_{t_{mn}}(t) \), given by Equation (I.34), is expressed as
\[ \omega_{mn}(n) = z_{t_{mn}}(nT_\omega) , \] (I.41)
where \( T_\omega \leq 1/(2\pi\omega_{mn_{max}}) \) to allow sufficient sampling of the sinusoid. With this sampling interval the total number of points \( K \) that is needed to represent any sinusoid over the interval \( 0 \leq t \leq t_{end} \) will be \( K = 2\pi\omega_{mn_{max}}t_{end} \).

The final solution then consists of the summation of all these individual discrete sinusoids. In the discrete domain, the discrete version of \( z_t(t) \) is expressed as
\[ \omega_t(n) = z_t(nT_\omega) . \] (I.42)
The final solution is then given as

$$w_t(k) = \sum_{\omega_m=\omega_1}^{\omega_{mmmax}} w_m(k) \quad k = 0,1,2,\ldots,K-1$$  \hspace{1cm} (I.43)

and represents the solution using the summation of sinusoids over the duration $0 \leq t \leq t_{end}$.

### I.5 Comparison of numerical methods

This section compares the numerical results obtained from the normal and dimensionless force model equations. It compares the solution provided by Zener's equation for the epi-centre motion to the solution obtained by discrete convolution and linear superposition.

#### I.5.1 Zener's Force Pulse and Plate Displacements

The dimensionless equation describing the relative motion of the sphere and plate expressed using Equation (I.9) was numerically evaluated by a fourth order Runge-Kutta-Nystrom method as discussed in section I.2. The solution for $\sigma_n(\tau_n)$ using values of $\lambda = \{0.0,0.5,1.0,1.5\}$ are shown as $0.8^{0.6}\sigma_n^{1.5}$ versus $\tau_n$ in Figure I.1 and are equivalent to $f_a(\tau_n)/Fm$ versus $\tau_n$. The results shown in Figure I.1 are identical to the results shown by Zener [18].

The forces can also be calculated by first applying the fourth order Runge-Kutta-Nystrom method to the normal force model equations to determine $z_d(t_n)$ using Equation (I.8). The associated force is obtained from

$$f_a(t_n) = K_f z_d(t_n)\frac{3}{\pi}$$\hspace{1cm} (I.44)

The normal force $f_a(t)$ can also be obtained from $f_a(\tau)/Fm$ by multiplying the result $f_a(\tau)/Fm$ by $Fm$ and scaling the time axis by multiplying $\tau$ by $T$. This

---

1Representative points from Zener's force pulse curves were measured from his graph and compared to the values obtained in Figure I.1. The results were well within the error expected in measuring points from a copy of a graph.
Figure I.1: The dimensionless force pulse $f_\phi / F_m$ as a function of dimensionless time $\tau$ for $\lambda = \{0.0, 0.5, 1.0, 1.5\}$

requires the sphere and plate parameters to be known so that the constants $F_m$ and $T$ can be calculated.

The epi-centre displacement of the plate can be determined from either the normal or dimensionless forces. For the normal plate displacements $z_p(t_n)$, the use of Equation (E.3) is required, repeated here as

$$z_p(t_n) = B_p \int_{0}^{t_n} f_\phi(\eta) d\eta \quad , \quad (I.45)$$

The dimensionless epi-centre displacement is determined by using

$$\sigma_p(\tau_n) = \frac{\lambda}{F_m} \int_{0}^{\tau_n} f_\phi(\eta) d\eta \quad , \quad (I.46)$$

where $\sigma_p(t)$ represents the dimensionless plate displacement at the impact site. The normal and dimensionless plate displacements are related by

$$z_p = \frac{\lambda T^2}{m_s} \sigma_p \quad . \quad (I.47)$$

This result is seen by noting that $\lambda = \frac{B_p m_s}{T^2}$, given in section 3.4.2, and that Equation (I.46) requires multiplying by $F_m$ and $T$ to obtain the force as a function of time $t$. 
Since both $f_a(t_n)$ and $f_a(\tau_n)/Fm$ are discrete quantities, the integration in Equation (1.45) or Equation (1.46) requires numerical evaluation.

A simple trapezoidal integration method is chosen to evaluate either equation. A concern in using the trapezoidal integration method is in choosing an interval small enough to insure that the error in the integration is not significant to the application of interest. This interval is determined by the interval used to obtain either $f_a(t_n)$ or $f_a(\tau_n)/Fm$ from the fourth order Runge-Kutta-Nystrom method. To examine the effect of using different interval on the plate displacements, Figure 1.2 shows the normal epi-centre plate displacements $z_p(t)$ due to a 1.4 m/s impact of 3.175 mm radius steel sphere on a 0.7938 mm thick aluminium plate calculated from a discrete time force pulse consisting of 8382 points and 201 points. The steel and aluminium material parameters listed in Chapter 4 were used to calculate the force and displacements. The small difference between the two displacements curves is apparent and represents the error in using different number of points to evaluate the integral in Equation (1.45).

It should be noted that both Figure I.1 and Figure I.2 are not continuous curves but consist of discrete points. They are shown as continuous curves for convenience and actual consist of more than 500 points for the force pulses, and 8382 points and 201 points for the plate displacements.

A comparison of the normal and dimensionless plate displacements, calculated from the normal and dimensionless force pulse, respectively, are shown in Figure I.3. This figure shows the plate displacements over the interval $100\mu s \leq t \leq 450\mu s$ with a number of representative data points. The number of force pulse points used to calculate the normal and dimensionless plate displacements are approximately equal. In addition, the dimensionless plate displacements were scaled by Equation (1.47). It is seen from Figure I.3 that the scaled version of the dimensionless plate displacements are identical to the normal plate displacements for approximately the same number of points.
Figure 1.2: The epi-centre displacements calculated from a discrete time force pulse consisting of 8382 points and 201 points.

Figure 1.3: Comparison of the plate displacements of Figure 1.2 between the normal and scaled dimensionless displacements.
Since the force pulse is a positive function, integration errors accumulate, resulting in the worst case errors occurring at the end of the integration interval. The magnitude of the plate displacement at 450 $\mu$s as a function of the number of points used in the force pulse is shown in Figure I.4. The results from both the normal and scaled dimensionless displacement are shown. It is seen that the final magnitude appears to approach a value that changes very little as the number of points increases. Figure I.5 shows the change in the plate displacement as percent of the magnitude of the plate displacement at 450 $\mu$s as a function of the number of points used in the force pulse. Figure I.5 shows that, even at approximately 200 points for the force pulse, the error in the plate displacements are less than the 1.5%.

I.5.2 Comparing Plate Displacements Calculations

This section compares the results of calculating the epicentre plate displacements using Zener's equation, direct evaluation of the solution of the thin plate equation using discrete convolution and linear superposition.

The plate displacements using discrete convolution, as given by Equation (I.32), requires that the double summation be truncated to $M$ and $N$. Each term in the summation represents a single mode of plate vibration at a particular frequency. The more modes that are included in this summation, the more accurate the results become. The number of modes used depends on the force pulse spectrum $F_a(\omega)$. If the bandwidth of the force pulse spectrum is very narrow, less modes are necessary to evaluate the solution since using modes associated with frequencies outside the bandwidth of the force pulse spectrum contributing very little to the solution.

The results of evaluating the epicentre plate displacement due to a 1.4 m/s central impact of 3.175 mm radius steel sphere on a 0.7938 mm thick aluminium plate\(^2\) using Equation (I.32) is shown in Figure I.6. These results show the use of spectral components having magnitudes at least 50%, 90%, and 95% of the maximum and are compared with the plate displacements calculated using Zener's equation with

\(^2\)The dimensions of the aluminium plate approximate that of the experimental sensor.
Figure 1.4: The magnitude of the plate displacement at 450 $\mu$s as a function of the number of points used in the force pulse

Figure 1.5: The change in the plate displacement as percent of the magnitude of the plate displacement at 450 $\mu$s as a function of the number of points used in the force pulse
a 8382 point force pulse. It is assumed that the displacement calculated by Zener's equation are accurate.

To calculate the plate displacements shown in Figure 1.6 using Equation (I.32), a discrete time force pulse and sinusoid were required. The force pulse has 128 sample points spaced evenly over its 428 µs duration resulting in a sampling interval of approximately 3.34 µs. With the solution required over the interval 0 ≤ t ≤ 450µs, approximately 135 sampling points are needed. The extra sample points are zeros appended to the force pulse. Every discrete sinusoid requires samples at the same interval as the force pulse. From section 4.2, the magnitude of the force pulse spectra is greater than 5% of the maximum below 60 kHz. Therefore, this is the highest frequency convolved with the force pulse. With a sampling interval of approximately 3.34 µs, 5 samples per cycle are used, well above the Nyquist rate.

It can be seen that using only spectral components having magnitudes at least 50% of the maximum results in large differences between the plate displacements solutions. Using component having magnitudes at least 90% and 95% of the maximum, produce, as expected, smaller differences. These differences are examined more closely in Figure 1.7.

The difference between the displacements calculated using Equation (I.32) and Zener's equation is apparent in Figure 1.7. Even the curves were more of the spectrum is used are consistently greater than the results from Zener's equation. Increasing the number of points in the force pulse does not appear to improve the match between the two results and actually causes the differences to increase. Without further investigation it is difficult to explain this but it may be due the accumulation of numeric errors since thousands of summations are needed to arrive at a final result. The results may improve if the number of points in the force pulse increases substantially but this will increase computational times dramatically possibly making the evaluation impractical.

The result of using linear superposition to evaluate the plate displacements is
Figure I.6: The plate displacement evaluated using discrete convolution with spectral components having magnitudes at least 50%, 90%, and 95% of the maximum.

Figure I.7: The plate displacement of Figure I.6 shown over the interval $200 \mu s \leq t \leq 450 \mu s$. 
shown in Figure I.8. Again, the results are shown using spectral components having magnitudes at least 50%, 90%, and 95% of the maximum magnitude of the force pulse spectrum. The plate displacement calculated using Zener's equation for a 8382 point normal force pulse are also shown to allow comparisons to be made.

The difference between the results of using linear superposition and Zener's equation is less than that seen in Figure I.6. All the results of using linear superposition appear very similar to the results of Zener's equation. Figure I.9 shows the curve of Figure I.8 over the interval of $200\mu s \leq t \leq 450\mu s$ It can be seen that the differences are small between the 90%, and 95% displacement curves.

It is apparent that the linear superposition method provides a better agreement to Zener's equation than does the discrete convolution method. Another advantage of the linear superposition method over the discrete convolution method is the reduction in the computational time. Using comparable number of force pulse points and modes, the linear superposition method was a factor of at least 200 times faster. It is for these reasons that the linear superposition method was used to calculate all the plate displacements.

It was not possible to use Zener's equation to calculate plate displacements at points other than the impact site, so that comparisons to the discrete convolution method and linear superposition could not be made. Comparisons between the discrete convolution method and linear superposition could be made to determine if either method produced significantly different results from the other. This was not the case and both methods provided similar results for plate displacements due to impact at other locations.

I.5.3 Transducer response

This section briefly compares the $s$-domain and $z$-domain response of the transducer model used in the simulations.

The $s$-domain transfer function of the transducer was given in section 3.5.2.
Figure I.8: The plate displacement evaluated using linear superposition with spectral components having magnitudes at least 50%, 90%, and 95% of the maximum.

Figure I.9: The plate displacement of Figure I.8 shown over the interval 200µs ≤ t ≤ 450µs.
One method to numerically simulate this transfer function is to perform a bilinear transform on the s-domain transfer function. The bilinear transform maps the s-plane into the z-plane using [27]

\[ H(z) = H(s) \bigg|_{s = \frac{2fs}{1 + fs}} \]  

(1.48)

Consider the transducer model transfer function, repeated here as

\[ V_o(s) = \frac{\left(-R_lCK_l \frac{s}{1 + R_lCs} \right) \left( \frac{-s^2}{s^2 + \frac{b_l}{m_t}s + \frac{k_t}{m_t}} \right)}{Z_t(s)} \]  

(1.49)

\[ \frac{V_o(s)}{Z_t(s)} = \frac{R_lCK_l \frac{s^3}{(1 + R_lCs) \left(s^2 + \frac{b_l}{m_t}s + \frac{k_t}{m_t}\right)}}{Z_t(s)} \]  

(1.50)

Equation (1.50) can be expressed as

\[ \frac{V_o(s)}{Z_t(s)} = \frac{(R_lCK_l)s^3}{(R_lC)s^3 + (R_lCb_l/m_t + 1)s^2 + (R_lCk_l/m_t + b_l/m_t)s + k_l/m_t} \]  

(1.51)

With the values of variables given in Table 4.3 Equation (1.51) becomes

\[ \frac{V_o(s)}{Z_t(s)} = \frac{(2 \times 10^4)s^3}{2 \times 10^{-3}s^3 + 1.01 \times 10^2s^2 + 3.19 \times 10^7s + 1.591 \times 10^{10}} \]  

(1.52)

A bilinear transform of Equation (1.52) using a sampling frequency \( f_s \) of 1 MHz results in

\[ \frac{V_o(z)}{Z_t(z)} = \frac{9.72 \times 10^6 - 2.92 \times 10^7 z^{-1} + 2.92 \times 10^7 z^{-2} - 9.72 \times 10^6 z^{-3}}{1 - 2.94z^{-1} + 2.89z^{-2} - 0.951z^{-3}} \]  

(1.53)

Equation (1.53) can be implemented as a digital filter in any number of ways. A transposed direct form II structure was used to implement Equation (1.53) [27].

A comparison of the magnitude and phase between the s-domain and z-domain response are shown in Figure I.10 and Figure I.11, respectively.

As can be seen from Figure I.10 and Figure I.11 that the z-domain response is identical, over the frequencies of interest, to the s-domain response so that no frequency pre-warping was necessary [27].
Figure I.10: A comparison of the s-domain and z-domain magnitude response of the transducer model.

Figure I.11: A comparison of the s-domain and z-domain phase response of the transducer model.
1.5.4 The digital differentiator

This section briefly reviews the implementation of the digital differentiator used in this thesis.

The ideal differentiator given by

\[ H_d(\omega) = j\omega, \]  \hspace{1cm} (I.54)

or using normalized frequency notation \( \Omega = \omega/\omega_s \),

\[ H_d(\Omega) = j\Omega, \]  \hspace{1cm} (I.55)

The discrete unit sample response is given by the sequence [38]

\[ h_d'(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\Omega) e^{jn\Omega} d\Omega, \] \hspace{1cm} (I.56)

\[ = \frac{\cos(\pi n)}{n}, \quad -\infty < n < \infty, \quad n \neq 0. \] \hspace{1cm} (I.57)

By shifting the impulse response, given by Equation (I.57), \((N - 1)/2 + 1/2\) samples and truncating the sequence to \(N\) points, the resulting new sequence is

\[ h_d(n) = \frac{\cos(\pi(n - N/2))}{(n - N/2)^2} - \frac{\sin(\pi(n - N/2))}{\pi(n - N/2)^2}, \quad 0 \leq n \leq N - 1. \] \hspace{1cm} (I.58)

By limiting \(N\) to be odd, the sequence given by Equation (I.58) will resemble an even sequence so that \(H_d(\pi) \neq 0\) [27]. The resulting sequence can be used as the numerator coefficients for a transposed direct form II digital filter structure [27]. The denominator is simply set to 1.

The magnitude and phase response of shifted sequence of Equation (I.58), using a \(N = 101\) point hamming window, are shown in Figure I.12 and Figure I.13, respectively. The sampling frequency \(f_s\) was set to 1 MHz.

Note that the response \(H_d(\pi) \neq 0\) or \(H_d(500\ \text{kHz}) \neq 0\) and the smooth phase transition at \(\Omega = \pi\) or \(F = 500\ \text{kHz}\). The hamming window provides for additional smoothing of the response.
Figure I.12: The magnitude response of the digital differentiator.

Figure I.13: The phase response of the digital differentiator.
J. Discrete Fourier Transform of the Force Pulse

This appendix discusses application of the discrete Fourier transform on both the discrete normal and discrete dimensionless force pulse, \( f_a(t_n) \) or \( f_a(\tau_n) \), respectively.

J.1 The Discrete Fourier Transform of a Finite Duration Signal

Given a finite duration signal \( x_a(t) \) such that \( x_a(t) = 0 \) for \( t < 0 \) and \( t > t_{\text{end}} \), the discrete time version is then

\[
x(n) = x_a(nT_s), \quad n = 0, 1, \ldots, L - 2, L - 1 ,
\]

where \( T_s \) is the sampling interval in seconds per samples and \( L \) the total number of discrete points or samples over the interval \( 0 \leq nT_s \leq t_{\text{end}} \). These definitions imply the following relationships

\[
t_n = nT_s = n/f_s, \quad T_s = t_{\text{end}}/L, \quad f_s = 1/T_s = L/t_{\text{end}} ,
\]

where \( f_s \) is the sampling frequency in samples per second. The Fourier transform of \( x(n) \) is [38]

\[
X(F) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi F_n} .
\] (J.3)

The Fourier transform of \( x_a(t) \) is [38]

\[
X_a(f) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi f} .
\] (J.4)
The frequency variables $f$ and $F$ are related by $F = f/f_s$ and where

$$-1/2 \leq F \leq 1/2 \quad . \tag{J.5}$$

The frequency $f$ represents the frequencies for the continuous or analog signal $x_a(t)$ and $F$ represents the frequencies for the discrete signal $x(n)$. The frequency variable $F$ can also be used over the interval $0 \leq F \leq 1$ [38]. It can be shown that [38]

$$X(F) = \sum_{k=-\infty}^{\infty} X_a[(F - k)f_s] \quad , \tag{J.6}$$

and is a scaled periodic version of $X_a(f)$ with a period $f_s$. If the bandwidth of $x_a(t)$ is confined to be less than $f_s/2$, the Nyquist rate [38]

$$X(F) = X\left(\frac{f}{f_s}\right) = f_sX_a(f) = \frac{1}{T_s}X_a(f), \quad |f| \leq \frac{f_s}{2} \quad , \tag{J.7}$$

so that the form of $X(F)$ is identical to $X_a(f)$ over, what is referred to as, the fundamental range $|F| \leq \frac{1}{2}$. Equation (J.7) also shows that the spectrum $X(F)$ of $x(n)$ is a scaled version of $X_a(f)$, so that $X(F)$ must be scaled by $T_s$ to obtain an indication of the true magnitude.

Frequency sampling $X(F)$ at the frequencies $F_k$ that are equally spaced over $L-1$ intervals between $0 \leq F \leq 1$ results in the discrete Fourier transform of $x(n)$ [38]

$$X_{DFT}(k) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi F_k n}, \quad k = 0,1,\ldots,L-1 \quad . \tag{J.8}$$

Therefore the discrete Fourier transform of $x(n)$ is related to $X(F)$ and $X_a(f)$ by

$$X_{DFT}(k) = X(F_k) = f_sX_a\left(\frac{f_k}{L}\right)k, \quad k = 0,1,\ldots,L-1 \quad , \tag{J.9}$$

where $F_k = \frac{1}{L}k$ for $k = 0,1,\ldots,L-1$. The frequency samples $F_k$ correspond to the real frequency samples at $f_k = \frac{f_s}{L}k$. Since $X_{DFT}(k) = X(F_k)$ the notation $X(F_k)$ or $X(k)$ will be be used to denote the discrete Fourier transform of $x(n)$.

Given that the number of discrete points for $x(n)$ or $x_a(nT_s)$ is fixed at $L$ points, the discrete Fourier transform will also have frequency samples at $L$ equally spaced
points over the fundamental range. If the spectrum requires sampling at more points
or

\[ X(k) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi F\frac{k}{N}}, \quad k = 0, 1, \ldots, N - 1 \]  \hspace{1cm} (J.10)

With \( x(n) = 0 \) for \( n \geq L \) the summation stops after \( L - 1 \) points but since \( k = 0, 1, \ldots, N - 1 \) the number of frequency sample points has increased. It should be noted that the frequency resolution of \( x(n) \) does not increase with these extra points but that the extra points provide an interpolation of frequency values between the original \( L \) points [38]. Using these extra frequency sample points places the true sampled frequency points at \( f_k = \frac{k}{N} \cdot k \). With both the number of data points for \( x_1(t) \) or \( x(n) \) fixed at \( L \) and the sampling rate at \( f_s = L/t_{\text{end}} \), the true frequencies can also be expressed as \( f_k = \frac{k}{Nt_{\text{end}}} \cdot k \). In terms of the normalized fundamental range this corresponds to \( F_k = \frac{k}{N} \cdot k \). Increasing the number of frequency sample points is equivalent to the signal \( x(n) \) or \( x_1(nT_s) \) having \( N - L \) zero valued points appended or prefixed to the original data sequence [38].

**J.2 The Discrete Fourier Transform of the Force Function**

Consider the discrete version of the normal force function \( f_1(t) \) given as

\[ x_1(n) = f_1(nT_s), \quad n = 0, 1, 2, \ldots \] \hspace{1cm} (J.11)

where \( T_s \) is the sample interval in seconds per sample. Assume also that \( f_s(t) = 0 \) for \( t < 0 \) and \( t > t_{\text{end}} \) and is sampled using \( L \) points over the interval \( 0 \leq t \leq t_{\text{end}} \). This implies that \( x_1(n) = 0 \) for at least for \( n \neq 0, 1, \ldots, L - 1 \) or that \( x_1(n) \) is defined over the interval \( n = 0, 1, \ldots, L - 1 \). The discrete Fourier transform of \( x_1(n) \) is

\[ X_1(k) = \sum_{n=0}^{L-1} x_1(n)e^{-j2\pi F\frac{k}{N}}, \quad k = 0, 1, \ldots, L - 1 \] \hspace{1cm} (J.12)

The sample frequency points over the fundamental range are at \( F_k = k/L \) or \( f_k = \frac{k}{N} \cdot k \) for the true frequencies.
Now consider the dimensionless time force function \( f_a(t/T) = f_a(\tau) \) that is simply a time scaled version of \( f_a(t) \) defined over the scaled interval \( 0 \leq t/T \leq t_{\text{end}}/T \) and with \( f_a(t/T) = 0 \) elsewhere. \( T \) is some quantity with units in seconds. If the discrete version of \( f_a(t/T) \) also consists of \( L \) points sampled evenly over this interval, then the discrete version of the time scaled force function is

\[
x_2(m) = f_a(mT'_s)
\]

where \( T'_s = 1/f'_s = T_s/T \) is the dimensionless sample interval and where \( \tau = mT'_s \). The discrete Fourier transform:

\[
X_2(k) = \sum_{m=0}^{L-1} x_2(m) e^{-j2\pi F_{s} km}, \quad k = 0, 1, \ldots, L - 1
\]  

The sample frequency points are then at \( F_{s} = k/L \) or \( f'_s = \frac{L}{T'}k \).

The relationship between \( x_1(n) \) and \( x_2(m) \) can be examined by noting that they originate from the same function \( f_a \) but are defined over different domains. Since \( \tau = t/T \), the value of \( f_a(t_n) \) at the sample point at \( nT'_s = t_n \) will be equal to the value at \( f_a(\tau_m) \) having its sample point at \( mT'_s = mT_s/T = t_m/T = \tau_m \) for \( m = n \). Therefore \( x_1(n) = x_2(m) \) for \( m = n \). This result can be easily seen graphically by realizing that the sampled values of \( f_a \) will not change value if only the domain or time axis is scaled. With \( x_1(n) = x_2(m) \) for \( m = n \), implies that the discrete Fourier transforms are also equal or

\[
X_2(k) = X_1(k)
\]

In other words, with both sequences \( x_1(n) \) and \( x_2(m) \) being identical, the sequences \( X_1(k) \) and \( X_2(k) \) will be equal. Therefore, a discrete Fourier transform consisting of an \( L \) point sequence for either the normal or dimensionless time force function will produce the same result. The difference shows itself when the value of the frequency sample points are required. Either \( f'_s = \frac{L}{T'}k \) or \( f_k = \frac{L}{T}k \) can be used for these frequency sample points for the force spectra. Using the previous relationships, these frequency sample points are related by
Therefore, given a sequence $X(k)$, that could have resulted from the discrete Fourier transform of either $x_1(n)$ or $x_2(m)$, the true frequency points can be assigned either to $f_k$ or to $f'_k$ and converted to back to the other using Equation (J.16).

If $f_a(mT_a')$ is normalized and made dimensionless in magnitude by the factor $Fm$, the discrete Fourier transform will also be scaled by $1/Fm$. Finally, the discrete Fourier transform of $f_a(mT_a') = x_2(m)$, given by $X_2(k)$, is related to the Fourier transform

$$X_2(k) = f'_aX_a\left(\frac{f'_a}{f_a}k\right), \quad k = 0, 1, \ldots, L - 1 \quad (J.17)$$

where $f'_a = \frac{1}{f_a} = \frac{f_a}{f'_a}$. Similarly, the discrete Fourier transform of $f_a(nT_a) = x_1(n)$, given by $X_1(k)$, is related to the Fourier transform of $f_a(t)$, given by $X_a(f)$, using

$$X_1(k) = f_aX_a\left(\frac{f_a}{f'_a}k\right), \quad k = 0, 1, \ldots, L - 1 \quad (J.18)$$
K. Material Parameters and Measurements

This appendix presents relevant details for the material properties of the steel spheres, aluminium plate and other measurements made for experiments.

K.1 Sensor Plate

The measured dimensions of plate are listed in Table K.1. The assumed material parameters are given Table K.2. The location of the reference axis, the transducer site, and the impact sites are listed in Table K.3. It should be noted that the impact locations are estimates of the mean site.

<table>
<thead>
<tr>
<th>$l_x$ side</th>
<th>$l_y$ side</th>
<th>$2l_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, 0)$ (cm)$\pm0.05$</td>
<td>$(x, l_y)$ (cm)$\pm0.05$</td>
<td>$(0, y)$ (cm)$\pm0.05$</td>
</tr>
<tr>
<td>53.30</td>
<td>53.05</td>
<td>47.35</td>
</tr>
</tbody>
</table>

Table K.1: The measured experimental sensor plate dimensions.

<table>
<thead>
<tr>
<th>Material Type</th>
<th>Material Density $\rho$ (kg/m$^3$)</th>
<th>Poisson ratio $\mu$</th>
<th>Modulus of Elasticity $E$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium</td>
<td>2710</td>
<td>0.33</td>
<td>70</td>
</tr>
</tbody>
</table>

Table K.2: The assumed experimental sensor plate material parameters.
Table K.3: The measured locations for the reference axis, the transducer site, and the estimated locations of the impact sites.

### K.2 Spheres

The radius, the measured mass, and calculated sphere densities are listed in Table K.4. Table K.5 gives the other assumed material parameters.

Table K.4: The assumed experimental sphere radii and measured masses.

<table>
<thead>
<tr>
<th>$r_s$ (mm) +0%-1%</th>
<th>volume $m_s$ (cm$^3$) +0%-3%</th>
<th>$m_s$ (gm)</th>
<th>$\rho_s$ (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.191</td>
<td>7.08x10^{-3}</td>
<td>5.0x10^{-2} ±10%</td>
<td>7100 +10%-13%</td>
</tr>
<tr>
<td>1.588</td>
<td>1.677x10^{-2}</td>
<td>1.33x10^{-1} ±8%</td>
<td>7900 +8%-11%</td>
</tr>
<tr>
<td>1.984</td>
<td>3.27x10^{-2}</td>
<td>2.6x10^{-1} ±4%</td>
<td>8000 +4%-7%</td>
</tr>
<tr>
<td>2.381</td>
<td>5.66x10^{-2}</td>
<td>4.3x10^{-1} ±2%</td>
<td>7600 +2%-5%</td>
</tr>
<tr>
<td>2.778</td>
<td>8.98x10^{-2}</td>
<td>7.1x10^{-1} ±1%</td>
<td>7900 +1%-4%</td>
</tr>
<tr>
<td>3.175</td>
<td>1.341x10^{-1}</td>
<td>1.04x10^{0} ±1%</td>
<td>7800 +1%-4%</td>
</tr>
</tbody>
</table>

Table K.5: The assumed experimental sphere material parameters.

<table>
<thead>
<tr>
<th>Material Type</th>
<th>Poisson ratio $\mu$</th>
<th>Modulus of Elasticity $E$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>0.29</td>
<td>200</td>
</tr>
</tbody>
</table>
L. Description of the Grain Loss Monitor

The grain loss monitor sensor in common use is shown in Figure L.1 (a). It consists of a formed plastic plate approximately two millimetres thick by six centimetres in width and twelve centimetres in length. The plate is mounted on an aluminium base plate by a firm foam gasket that has glue on both faces to fix the plastic plate to the base plate. The foam gasket is approximately five millimetres wide and runs along the edge of both plates. It is also thick enough to have the plastic plate ride above the base plate without direct contact. In the centre of the plastic plate is a small hole meant to hold a tiny brass bolt. Above this bolt is an approximately one millimetre thick rubber pad about three centimetres in diameter. On this tiny bolt, the piezo-electric transducer is mounted and held in position with a nut, as shown in more detail in Figure L.1 (b). Electrical contacts to the transducer are made by soldering small gauge wire to opposite faces of the transducer. The wires are then routed to terminal posts that feed through the base plate to allow connection to the sensor assembly by terminal lugs as shown in the cut-away view of Figure L.2.

The piezo-electric transducer element is also commercially known as a piezo bender and consists of two oppositely polarized layers, as shown in Figure L.1 (b). When the element is bent one layer will be in tension while the other is in compression. The opposite polarized layers will produce an electrical output that will be the sum of the two individual layers.
The physical transducer consists of two oppositely polarized layers of a piezoceramic made of lead zirconate titanate formed into a thin sheet approximately $8 \times 8 \times 0.6$ millimetres. The piezo transducer has a small, approximately two millimetre diameter, hole to allow a small mounting bolt to pass through. A mounting nut on either side of the transducer hold the transducer at some point along the length of the bolt. This bolt is then fixed to the surface of the plate.

The basic operation of the sensor can be deduced from the configuration of the pieces. A particle impacts on the plate at some point and the plate is set into
vibration. This motion is then imparted to the piezo bender whose perimeter begins to move relative to these edges, Figure L.3 shows a simplified version of this motion.