THE SIMPLEX ALGORITHM FOR
UNDERWATER SOURCE LOCALIZATION

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University of Saskatchewan

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ABSTRACT

The objective of this thesis is to find an effective method for underwater acoustic source localization in a multipath deep-ocean environment. The model discussed in this thesis consists of a submerged three-sensor receiver system and a broad-band noise source. Only two propagation paths (direct path and surface reflection path) from the source to the receiver array are considered, and the time difference of arrival (TDOA) between every two paths are used as the raw data for source localization.

After a brief description of the multipath environment and the system geometry, three analytical methods are discussed in this thesis: linear approximation method, spherical interpolation (SI) method and hybrid method. These methods can give a quasi-optimum least-squares estimate of the source location with a relatively low computational cost. The accuracy of all these analytical methods decreases rapidly with the increasing source-receiver distance. This limits the application of these methods in a very small area around the receiver array.

A numerical method, called the simplex method, is then discussed. As an iterative algorithm, the simplex method can get an excellent accuracy of localization with a relatively high computational cost. Moreover, as a numerical algorithm, the simplex method can use the absolute function (L1 norm) as error function, so as to make the estimator more robust than the least-squares estimator, especially when there are “bad points” in the raw data.

To reduce the computational time of the simplex method, the estimation result of the analytical method can be used as the initial simplex. The simulation results prove that this improvement can greatly reduce the iterative time of the simplex algorithm and make the algorithm more efficient.

The simulation results in this thesis show that the performance of the simplex method is much better than that of the analytical methods. Therefore the conclusion of this thesis is that the simplex method can be used as an effective algorithm in the area of passive localization.
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Contents

1 INTRODUCTION ........................................ 1
  1.1 BACKGROUND .................................... 1
      1.1.1 Application Background of Passive Localization 1
      1.1.2 Multipath Environment of the Underwater Acoustic Channel 4
  1.2 PROBLEM DEFINITION ............................. 6
      1.2.1 Localization in Multipath Environment 6
      1.2.2 Different Methods for Localization 7
      1.2.3 Summary .................................. 8

2 SYSTEM MODEL FOR MULTIPATH LOCALIZATION 10
  2.1 SYSTEM MODEL .................................. 11
      2.1.1 Source .................................. 11
      2.1.2 Ocean Channel ............................ 14
      2.1.3 The Receiver Array ........................ 16
      2.1.4 Summary of the System Model ............ 20
  2.2 TIME DIFFERENCE OF ARRIVAL (TDOA) .......... 20
2.2.1 Expression of TDOA ............................................. 20
2.2.2 Extraction of TDOA ............................................. 23
2.2.3 Variance of TDOA ............................................. 28

3 ANALYTICAL METHODS FOR PASSIVE LOCALIZATION ........ 29
3.1 LINEAR LEAST-SQUARES ESTIMATION .......................... 30
  3.1.1 Definition of Linear Least-Squares Estimation ............. 31
  3.1.2 Estimation Using the TDOA's ................................. 33
3.2 LINEAR APPROXIMATION METHOD ............................... 34
3.3 SPHERICAL INTERPOLATION METHOD FOR LOCALIZATION ..... 38
3.4 HYBRID METHOD OF LOCALIZATION ............................... 43
  3.4.1 Least Squares Time Delay Equations ......................... 44
  3.4.2 Least Squares Estimation of the Source Location ........... 48

4 SIMPLEX ALGORITHM FOR LOCALIZATION ......................... 52
4.1 NUMERICAL ESTIMATION ......................................... 52
4.2 THE SIMPLEX METHOD ............................................ 53
  4.2.1 The Basic Idea of the Simplex Method ....................... 54
  4.2.2 Basic Operations of the Simplex Algorithm ................. 55
4.3 DISCUSSION OF SIMPLEX ALGORITHM ............................ 62
  4.3.1 The Error Function ........................................ 62
  4.3.2 The Strategy ............................................... 63
  4.3.3 The Initial Simplex ....................................... 64
  4.3.4 The Stop Criterion ....................................... 66
4.4 SUMMERY OF THE SIMPLEX METHOD .......................... 67

5 THE SIMULATION CONFIGURATION AND RESULTS 68

5.1 THE GENERATION OF THE TDOA ............................. 68

5.1.1 Geometry of the Experiment System ......................... 69

5.1.2 TDOA as a Random Variable ............................... 70

5.1.3 The Generation of TDOA ................................. 72

5.2 THE SIMULATION OF LOCALIZATION ......................... 72

6 SUMMARY AND CONCLUSIONS 78

A PERFORMANCE OF DIFFERENT LOCALIZATION METHODS 85
List of Figures

1.1 Illustration of the anti-submarine helicopter .............................. 2
1.2 Illustration of the multipath environment ................................... 5

2.1 Illustration of the source generator. ............................................. 12
2.2 An example of the source signal in frequency domain ...................... 13
2.3 Multipath ocean channel model .................................................. 16
2.4 Geometry of the source-sensor configuration .................................. 17
2.5 The signal received by actual sensor and virtual sensor .................... 18
2.6 Auto-correlation of the received signal ........................................ 25
2.7 Cross-correlation of two received signals, four peaks can be observed 26
2.8 TDOA Extraction from Cross-Correlation ..................................... 27
2.9 TDOA Extraction from Auto-Correlation. ..................................... 27

4.1 Three-dimensional simplex: illustrating reflection. ......................... 56
4.2 Three-dimensional simplex: illustrating expansion. ......................... 57
4.3 Three-dimensional simplex: illustrating contraction ........................ 58
4.4 Three-dimensional simplex: illustrating Shrinkage ........................ 59
4.5 The flowchart of the simplex algorithm. .................................... 60
4.6 An example of the simplex moving in the error space ........................................... 61
4.7 Different initial simplexes ......................................................................................... 65
5.1 The variance of TDOA ............................................................................................. 71
5.2 The generation of the TDOA. .................................................................................. 73
5.3 Performance comparison of different localization methods ...................................... 75
5.4 Performance comparison of different localization methods ...................................... 76
A.1 Variance of the location estimate x̂, Linear Approximation method,
source depth=100m ................................................................................................. 86
A.2 Variance of the location estimate x̂, Linear Approximation method,
source depth=200m ................................................................................................. 87
A.3 Variance of the location estimate x̂, Linear Approximation method,
source depth=300m ................................................................................................. 88
A.4 Variance of the location estimate x̂, SI method, source depth=100m .................... 89
A.5 Variance of the location estimate x̂, SI method, source depth=200m .................... 90
A.6 Variance of the location estimate x̂, SI method, source depth=300m .................... 91
A.7 Variance of the location estimate x̂, hybrid method, source depth=100m .............. 92
A.8 Variance of the location estimate x̂, hybrid method, source depth=200m .............. 93
A.9 Variance of the location estimate x̂, hybrid method, source depth=300m .............. 94
A.10 Variance of the location estimate $\hat{x}_s$, Simplex method (L1 norm),
source depth=100m ......................................................... 95

A.11 Variance of the location estimate $\hat{x}_s$, Simplex method (L1 norm),
source depth=200m ......................................................... 96

A.12 Variance of the location estimate $\hat{x}_s$, Simplex method (L1 norm),
source depth=300m ......................................................... 97

A.13 Variance of the location estimate $\hat{x}_s$, Simplex method (L2 norm),
source depth=100m ......................................................... 98

A.14 Variance of the location estimate $\hat{x}_s$, Simplex method (L2 norm),
source depth=200m ......................................................... 99

A.15 Variance of the location estimate $\hat{x}_s$, Simplex method (L2 norm),
source depth=300m ......................................................... 100
List of Tables

2.1 Relation between $k$ and $i,j$ ............................................. 23

4.1 The variance of location estimator under L1 and L2 norm (no bad point in the TDOA measurements) (Unit: $m^2$) ............... 62

4.2 The variance of location estimator under L1 and L2 norm (there is one bad point in the TDOA measurements) (Unit: $m^2$) ............... 63

4.3 Mean Numbers of Iterations Under Different Strategy .................. 64

4.4 Mean number of iterations under three different initial simplexes shown in Figure 4.7 (L1 norm is used), the configuration of the entire system will be shown in Chapter 5. ............................................. 65

4.5 Mean number of iterations under three different initial simplexes shown in Figure 4.7 (L2 norm is used), the configuration of the entire system will be shown in Chapter 5. ............................................. 66

5.1 Mean number of iteration under different initial simplexes (using L1 and L2 norm) ....................................................... 77
Chapter 1

INTRODUCTION

1.1 BACKGROUND

1.1.1 Application Background of Passive Localization

As a research topic, the passive localization of a radiating source has received much attention, mainly for its potential application in anti-submarine systems. Over the years, various methods, including acoustic and non-acoustic, have been studied and proposed for underwater localization. Most non-acoustic methods, such as MAD (Magnetic Anomaly Detections) and infrared sensing, offer little promise for future application, and it seems that only acoustic methods have an appreciable search rate for the localization of a submerged source.[1, 2]

Passive acoustic localization relies on the detection of target-generated noise, in-
cluding machinery noise and wave noise, the target submarine is therefore regarded as an underwater noise-like sound source, and the word "localization" here means obtaining the radiated acoustic signals using a set of transducers and getting the 3-dimension coordinate of the source from the obtained signals[3]. Actually the application of passive localization is not only restricted to underwater acoustics, this technique can also be used in other areas such as navigation, surveillance and geophysics.

Figure 1.1: Illustration of the anti-submarine helicopter

Surface ships are the oldest and most-used platform for passive localization systems, but the vessel itself is also a noise source, which degrades the system performance to a certain degree. Modern air-borne localization systems, developed in the postwar years, are considered to be more efficient, since the aircraft hovers in a different medium above the submerged transducer and its own machinery noises
are thus effectively isolated[4]. Usually localization starts when a suspected source is first detected by a surface ship-mounted passive sonar system or a towed array sonar system. After a coarse localization process, the source is located in a certain region (target region). Aircrafts are then sent for more accurate localization and tracking. Passive sonobuoy is used by fixed wing airplanes and dipped sonar system is used by helicopters [5].

As shown in Figure 1.1, when the helicopter gets to the target region (usually an area of less than one hundred square miles), it will place its sonar transducer into a depth of several hundred meters and “listen” for a while, then haul up the transducer and fly to next sampling point. Repeating this procedure at several different sampling points, the source can be located from the received signals. For fixed-wing planes, the “listening” is carried out by thrown sonobuoys, which are float mechanisms with a transducer extending to a depth of several hundred meters, and every sonobuoy will forward the received acoustic signals to the aircraft using its RF (radio frequency) transmitter.[5]

Therefore, locating a radiating acoustic source using a few submerged omnidirectional sensors is a practical problem in the area of passive sonar localization. For the air-borne anti-submarine application, the technical question can be summarized as follows: considering an underwater sound source, radiating broad-band noise-like signals, which are received at several sampling points in the vicinity of the source (the received signal may be distorted by the channel and corrupted with additive noise during propagation). Using these signals, we must locate the source in 3-dimensional
1.1.2 Multipath Environment of the Underwater Acoustic Channel

It has been reported that received signals can be directly used to locate the target source[6]. But considering the multipath nature of the underwater acoustic channel, a better method can be chosen, in which the different arrival times of different multipath signals will be used.

Underwater acoustic propagation can be described using a multipath model, which means the source signal travels from the source to the receiver through different paths. For a multipath ocean environment, the received signal is the sum of a set of shifted replicas of the source signal, each one corresponds to one of the paths between the source and the receiver.[7]

A simplified diagram of the multipath environment is shown in Figure 1.2, which is a two dimensional diagram of the three dimensional environment. Shown in the figure are the direct path, surface bounce path, and the bottom bounce path. There may exist other paths: surface-bottom bounce, surface-bottom-surface bounce...etc.

In a deep ocean environment, the signals that reflect off of the bottom are significantly attenuated relative to the direct and surface bounce paths. Therefore only the signals through direct paths and surface paths are considered in this thesis. The propagation time of the signals from the source to the receivers for each path is the ideal information needed for localization, but using a passive system we cannot get
the exact value of the time delay. What we can get is the difference in propagation time between various paths, which is referred to as the Time Difference of Arrival (TDOA). The TDOA, usually extracted from the received signals by auto-correlation and cross-correlation algorithms, can be used to locate the source.

Figure 1.2 shows the various paths as straight lines, actually this will only occur if the ocean has a constant sound speed from the surface to the bottom. Generally there is a change in sound speed with depth, due to variation in temperature and pressure with depth. This sound speed variation causes the various source-receiver paths to curve, which changes the path lengths and complicates the procedure of locating the source.

The sound-speed profile is not the only factor that makes source localization a complicated problem. Other complicating factors include the surface and bottom conditions which will affect the phase and amplitude of the reflected wave. For exam-
ple, if the surface is rough, with wave action, the direction of the reflected sound will vary, which is referred to as scattering. The composition and terrain of the bottom also has a significant effect on the reflections. Some assumptions for the environmental conditions are described in Chapter 2.

1.2 PROBLEM DEFINITION

1.2.1 Localization in Multipath Environment

Locating a radiating acoustic source using a few submerged omnidirectional sensors is a practical problem in the area of passive sonar localization. Much work has been done on this subject [8, 9, 10]. Most authors use the time difference of arrival (TDOA), which are the differences in propagation time among various paths, to locate the source[11]. In this case, the source must be relatively close to the sensors. With the application of multipath information, a better quality of localization can be obtained.

The source can be located by finding the intersection of the surfaces defined by TDOA measurements (for constant sound speed, these surfaces are hyperbolas of one sheet)[12]. Considering the multipath nature of the underwater acoustic channel, in most cases there exist redundant TDOA measurements. Localization can therefore be regarded as a question of optimum estimation: all the TDOA, which are measured from several position-known sensors, are input data of the estimation system, and
the desired output is the source coordinate (2-dimension or 3-dimension).

1.2.2 Different Methods for Localization

Passive localization using TDOA's can be described as a question of non-linear estimation. If the nonlinearity of this question can be eliminated, then the question becomes a linear question and a closed-form expression of the source location may be obtained. Therefore, how to eliminate the non-linear term in the TDOA equation set is the focus of many papers [8, 9, 10, 11, 13, 14]. It has been shown that after selecting a reference sensor, the nonlinearity of the TDOA equation set can be eliminated by matrix operations to yield a closed-form expression for the source location[14, 13]. Another approach, which is discussed in this thesis, solves the problem in two steps: first we estimate the source-sensor distances from the TDOA measurements (this is a linear estimation question); then since every distance is a function of the source location, in the second step we can estimate the source location from the obtained “distance equations”.

Since the above analytical methods are based on approximations, they can not get the actual optimum estimate. Considering the complicated nature of the system model, it's very difficult to get a closed-form expression for the actual optimum estimate of the source location. Therefore, another approach based on numerical algorithms is studied as an attractive solution. The simplex algorithm, is a numerical method which has been used in the area of geology localization for many years[15].
This method is simple, it doesn’t require derivative calculations and avoids matrix inversions. It’s shown in this thesis that the simplex method is also effective in the area of underwater acoustic localization.

1.2.3 Summary

The objective of this thesis is to find an effective passive localization method of an stationary underwater acoustic source in a multipath environment. Analytical methods and numerical methods for passive localization are both described in this thesis, comparison of their performance, their advantages and drawbacks are also discussed. To obtain a source location estimate with both high precision and low computational burden, the best choice is to combine these two methods: using the result of an analytical method as the initial value of the numerical iteration. In this manner an optimum estimation of the source location is obtained.

The structure of the thesis is as follows:

1. The environment and system analysis, including the description of the deep ocean multipath channel, the noise spectrum of the target source, the extraction of TDOA’s from the received multipath signal, and the expression of the variance of the TDOA are presented in Chapter 2.

2. The definition of optimum estimation, especially the least-squares estimation are presented in Chapter 3. Three analytical methods for obtaining the closed-form expression of quasi-optimum estimate of the source location, are also discussed in this chapter.
3. The simplex algorithm, a numerical method for directly searching for the optimum estimation of the source location, are introduced in Chapter 4. A brief discussion of its technical details, are also included in this chapter.

4. The computer simulation results of all the localization methods (analytical, numerical and combined methods) discussed in chapter 3 and 4, are presented and compared in Chapter 5.
Chapter 2

SYSTEM MODEL FOR MULTIPATH LOCALIZATION

As mentioned in Chapter 1, time difference of arrivals (TDOA's), which are used for localization, can be obtained from the auto-correlation of the signal received at each individual receiver and from the cross-correlation of the received signals from two receivers. Three sensors are required for locating a single point source, so the environment under consideration is a submerged horizontal array consisting of three omnidirectional sensors, receiving signals from a single point source, which is in the near vicinity of the sensors.
2.1 SYSTEM MODEL

In this section, the physical system model of the multipath environment is described. The model consists of three parts: a point source (which is the target for localization), the multipath ocean channel, and the receiver array. For each part, its physical description, underlying assumptions and model are presented.

2.1.1 Source

The underwater acoustic source under consideration is a submarine or remotely operated vehicle (ROV) in deep water, radiating noise in every direction. Its noise can be grouped into three classes[16, 3]: Machinery noise comprises the noise caused by the machinery within the vessel; Propeller noise is a hybrid form of noise having features common to both machinery and hydrodynamic noise; and hydrodynamic noise is radiated noise caused by the irregular flow of water past the surface of vessel.

Machinery noise, which is due to the mechanical vibration, is radiated to the ocean channel via the hull of the target. Its spectrum consists of several line component (caused by rotating mechanical parts) and a low-level continuous component (caused by cavitation and turbulence in the fluid flow such as pumps, pipes and valves). Propeller noise originates outside of the hull as a consequence of the rotating propeller, it has a different frequency spectrum from machinery noise. Because the cavitation noise mainly consists of random small bursts caused by bubble collapse, it has a continuous spectrum with peaks occurring between 100 and 1000 Hz, with increased speed this peak shifts to lower frequencies. In old textbooks the hydrodynamic noise
was regarded as a minor contributor to radiated noise and thought to be masked by machinery and propeller noise. But for a modern target source, since the machinery and propeller noise have been greatly reduced, the hydrodynamic noise becomes more important than ever.

In present day sources, the tonal components have been significantly attenuated, so it's reasonable to assume the noise source has a continuous broadband spectrum. And it's also assumed that the radial distance is much larger than the size of the source, so the source can be modelled as a point source with a uniform omnidirectional radiation pattern.

From unclassified materials, we can only find noise spectrum measured from very old sources. One example is given by Urick in Figure 10.19 of [16]. It's the noise spectrum measured from HMS "Graph", a Nazi Germany submarine captured by British navy in WWII. The source levels vary from 170 to 100 dB/$\mu$Pa/1Hz @ 1 yd. Though "Gragh" is almost the best designed submarine in 1940's, its data has only theoretical value and is almost useless for today's practical work.

![Block diagram](image)

**Figure 2.1:** Illustration of the source generator.

A block diagram for the physical source used in this thesis is shown in Figure 2.1.
Figure 2.2: An example of the source signal in frequency domain

The source of interest can be regarded as a low-pass white Gaussian noise source. A time based signal $W(t)$ produced by the white noise generator is filtered by a low-pass filter $H_s(f)$ to produce the source signal $S(t)$. The spectrum of the source signal is shown in Figure 2.2.
2.1.2 Ocean Channel

The ocean channel is generally known as a very complex acoustic medium. Some important factors, such as the sound speed profile, the reflections, and the transmission loss must be considered in modelling the ocean channel. It can be shown that with some restrictions and assumptions, the model can be simplified.

The sound speed in the sea varies with many factors, such as temperature, salinity and pressure. Because all these factors vary with the depth, the speed profile is generally used to represent the variation of the sound speed with depth. The variation of the sound speed with depth will change the direction of the desired signal during its propagation, which will make the localization process very complicated. For the problem under consideration, it was assumed that the source was in deep water and near the vicinity of the sensors. In this case there's a nearly straight line path between the source and the receiver[16], implying that the sound speed is constant with depth.

In the problem under study, because the source is located in deep water, only signals due to the direct path and surface reflected path are considered. Other multipath signals, including the bottom reflected path, surface-bottom path, bottom-surface path are ignored because in this case paths involving a bottom bounce have a greater attenuation.

For short range propagation, the transmission loss is determined by spherical spreading, plus a loss due to absorption. Spreading loss is due to the geometrical effect of the weakening of a sound signal as it spreads outward from the source. For the single point source under consideration, the loss, given in dB, is $10 \log(l^2)$, where $l$
is the distance from the source to the point of measurement. Since the absorption loss is relatively small compared to the spreading loss, it can be assumed to be frequency independent and a constant $\alpha_c$ can be chosen as the absorption factor.

The above assumptions are used in generating the ocean channel model which is defined as follow. The source radiates the signal, $S(t)$, which is a low-pass Gaussian random process, and the signal is transmitted through two paths, a surface reflected path (designated with subscript S) and a direct path (designated with subscript D). The signal is attenuated by path loss and absorption during the propagation, the attenuation coefficients for both paths are:

$$g_s = \frac{g_r}{l_s} 10^{-\alpha_c l_s/20} \tag{2.1}$$

and,

$$g_d = \frac{1}{l_d} 10^{-\alpha_c l_d/20} \tag{2.2}$$

Where $l_s$ and $l_d$ are the length of the two paths, $g_r$ in equation (2.1) is the surface reflection coefficient, so $g_s$ also includes the surface reflection attenuation. If the ocean surface is perfectly smooth, $g_r$ is nearly minus one.

Assuming constant sound speed profile, we can get the time delay of the paths:

$$D_s = \frac{l_s}{c} \quad D_d = \frac{l_d}{c} \tag{2.3}$$

The signal $S_r(t)$ received at the sensor is the sum of two attenuated and time-shifted source signal and the ambient noise $n(t)$, as shown in Figure 2.3, the ambient noise

15
can be modelled as a broadband random process.

2.1.3 The Receiver Array

The sensors which make up the receiver array are submerged hydrophones. A hydrophone is a transducer which converts the acoustical energy to electrical energy. For our application we assume the hydrophones to be omnidirectional, which means they have the same response magnitude in all directions.

In our application, three sensors are used for receiving. A Cartesian coordinate system is set in Figure 2.4 to describe the configuration of the receiver array: the x-y plane is the ocean surface and z direction is downward.

The source signal travels to the sensor through two paths, the direct path and the
Figure 2.4: Geometry of the source-sensor configuration
surface reflection path. Because the source signal is low-pass Gaussian noise and we only care about the time delay of the signal, if the observation time is long enough, we can distinguish the signals through different paths by using time average correlation. In this case, just as shown in Figure 2.5, every signal can be “decomposed” into two signals: one actual signal corresponding to the direct path and one virtual signal corresponding to the surface reflection path. For a M-sensor system, the receiver can be regarded as an array composed of 2M elements, M of them are under the water surface and the rest are above the water surface (as shown in Figure 2.5). The signal received by every element is a delayed and attenuated replica of the source signal. In the following discussions we will use this model as the base of the multipath localization question, it will help us to understand the basic idea of localization through

Figure 2.5: The signal received by actual sensor and virtual sensor
TDOA's.

For a three-sensor system, the three actual sensors, referred to as sensor 1, 3, 5, are located at a depth of $z_0$ and make up an equilateral triangle horizontal array. Expressed as vectors in Euclidean space, the location of the three sensors are:

$$x_1 = [x_1, y_1, z_1] = [-x_0, 0, z_0] \quad (2.4)$$

$$x_3 = [x_3, y_3, z_3] = [x_0, 0, z_0] \quad (2.5)$$

$$x_5 = [x_5, y_5, z_5] = [0, y_0, z_0] \quad (2.6)$$

The three virtual sensors just located directly above the actual sensors, at a height of $z_0$:

$$x_2 = [x_2, y_2, z_2] = [x_1, y_1, -z_1] = [-x_0, 0, -z_0] \quad (2.7)$$

$$x_4 = [x_4, y_4, z_4] = [x_3, y_3, -z_3] = [x_0, 0, -z_0] \quad (2.8)$$

$$x_6 = [x_6, y_6, z_6] = [x_5, y_5, -z_5] = [0, y_0, -z_0] \quad (2.9)$$

considering the single point source $S$, located at $x_S = [x_s, y_s, z_s]$, the distance from sensor $i$ to the source is:

$$l_i = |x_i - x_S| \quad (2.10)$$

$l_1, l_3, l_5$ are length of the direct paths, $l_2, l_4, l_6$ are length of the corresponding surface reflection paths.
2.1.4 Summary of the System Model

The system model discussed in this thesis can be summarized with the following assumptions:

1. The sound speed is constant with depth (c=1465 m/s).
2. Only direct and surface-reflected paths exist and the reflection from the ocean surface is assumed to be specular, which causes a phase reversal at the surface boundary.
3. The bottom-bounce paths are assumed to be highly attenuated or absorbed.
4. Spreading loss and absorption loss are not a function of signal frequency.
5. The spreading is spherical. For a uniform radiating point source, the spreading power loss is inversely proportional to the square of the distance.
6. The absorption, which is due to the conversion of acoustic energy into heat, is taken to have the value $a_c = 0.2187 \times 10^{-3}$ dB/meter.[7]

2.2 TIME DIFFERENCE OF ARRIVAL (TDOA)

2.2.1 Expression of TDOA

Considering an M-sensor system, including 2M actual and virtual sensors. The signal received by sensor $i$ (actual or virtual) has a time delay $D_i$, which is the propagation
The vector $D = (D_1, ..., D_i, ..., D_{2M})$ is defined as the time delay vector, $x_s = (x_s, y_s, z_s)$ is the source location, and $c$ is the sound speed.

For the localization system discussed in this thesis, the propagation time of the signal through every path is the ideal information for localization. However, in a passive system we cannot directly measure the propagation time. The information used for localization are the time difference of arrival (TDOA) of the source signals.

For an $M$-sensor system, because only direct path and surface-reflection path are considered, we can get $N$ different path pairs, where:

$$N = \binom{2M}{2} = M(2M - 1)$$

Because the TDOA's are actually differences between the propagation time of different
paths, we can get one TDOA from every path pair:

\[
\begin{align*}
D_{12} &= D_1 - D_2 \\
\vdots \\
D_{ij} &= D_i - D_j \\
\vdots \\
D_{(2M-1)(2M)} &= D_{2M-1} - D_{2M}
\end{align*}
\]  

(2.13)

where \( i = 1 \) to \( 2M - 1 \), \( j = i + 1 \) to \( 2M \), \( x_i = (x_i, y_i, z_i) \) and \( x_j = (x_j, y_j, z_j) \) are known locations of the actual or virtual sensors.

\( D_{ij} \) is not the only way to express the TDOA, another equivalent expression is the vector \( r = (r_1, \ldots r_k, \ldots r_N) \), which is defined as

\[
\begin{align*}
\begin{cases}
r_1 &= D_{12} = D_1 - D_2 \\
\vdots \\
r_k &= D_{ij} = D_i - D_j \\
\vdots \\
r_N &= D_{(2M-1)(2M)} = D_{2M-1} - D_{2M}
\end{cases}
\end{align*}
\]  

(2.14)

where \( N = M(2M - 1) \) is the greatest value of \( k \), and \( k \) is the function of \( i \) and \( j \). For the three-sensor system discussed in this thesis, \( i = 1, \ldots 5 \), \( j = i + 1, \ldots 6 \), therefore in total 15 TDOA's can be obtained. The relation between \( k \) and \( i, j \) is presented in Table 2.1.

the vector \( r = (r_1, \ldots r_k, \ldots r_N) \) is defined as the TDOA set and the equation set
Table 2.1: Relation between $k$ and $i, j$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$i=1$</th>
<th>$i=2$</th>
<th>$i=3$</th>
<th>$i=4$</th>
<th>$i=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>8</td>
<td>11</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>9</td>
<td>12</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

(2.14) is defined as the TDOA equations. With known sound speed $c$ and sensor location $x_i$, $r$ is only the function of location vector $x_s$, so it is possible to determine $x_s$ from the obtained TDOA's.

2.2.2 Extraction of TDOA

The received signal $S_r(t)$ for the three actual sensors is the sum of two delayed source signals. From Figure 2.4, we know that the three actual sensors are sensor 1, 3 and 5, so the received signals are:

$$S_{r1}(t) = g_{d1}S(t - D_1) + g_{s1}S(t - D_2) + n_1$$
$$S_{r3}(t) = g_{d3}S(t - D_3) + g_{s3}S(t - D_4) + n_3$$
$$S_{r5}(t) = g_{d5}S(t - D_5) + g_{s5}S(t - D_6) + n_5$$

(2.15)

where $S_{ri}(t)$ is the signal received by the $i$-th sensor, $g_{di}$ and $g_{si}$ are the corresponding attenuation coefficients for direct and surface reflection path, as expressed in equation (2.1) and (2.2), and $n_i$ is the background noise received by the $i$-th sensor.
The time difference of arrival is extracted from either the auto-correlation or the cross-correlation of these received signals. For example, the cross-correlation of \( S_{r1}(t) \) and \( S_{r3}(t) \) is:

\[
R_{13}(\tau) = E[S_{r1}(t)S_{r3}(t - \tau)]
= gd_1g_3R_{ss}(\tau - D_{13}) + gd_1g_3R_{ss}(\tau - D_{14})
+gd_1g_3R_{ss}(\tau - D_{23}) + g_3g_3R_{ss}(\tau - D_{24}) + R_{n13}(\tau) \tag{2.16}
\]

where \( R_{ss}(\tau) \) is the auto-correlation of the source signal \( s(t) \), \( E[] \) represents the expected value, and \( R_n(\tau) \) is the auto-correlation of the noise.

We can also get the auto-correlation of \( S_{r1}(t) \) and \( S_{r3}(t) \):

\[
R_{11}(\tau) = E[S_{r1}(t)S_{r1}(t - \tau)]
= (gd_1 + g_3^2 + g_{ss})R_{ss}(\tau) + gd_1g_3R_{ss}(\tau + D_{12})
+gd_1g_2R_{ss}(\tau - D_{12}) + R_{n11}(\tau), \tag{2.17}
\]

and,

\[
R_{33}(\tau) = E[S_{r3}(t)S_{r3}(t - \tau)]
= (gd_3 + g_3^2)R_{ss}(\tau) + gd_3g_3R_{ss}(\tau + D_{34})
+gd_3g_3R_{ss}(\tau - D_{34}) + R_{n33}(\tau), \tag{2.18}
\]

These equations indicate that \( R_{13}(\tau) \) should have 4 distinct peaks, at shift of \( D_{13} \),
$D_{14}$, $D_{23}$ and $D_{24}$, and from $R_{11}(\tau)$ and $R_{33}(\tau)$ the shifts of $D_{12}$ and $D_{34}$ can be obtained. These peaks can also be observed in Figures 2.6 and 2.7.

![Figure 2.6: Auto-correlation of the received signal](image)

Therefore, using similar procedures, all the different TDOA's can be obtained from either the auto-correlation or the cross-correlation. In practical applications the auto-correlation and cross-correlation functions of the received signals are estimated
Figure 2.7: Cross-correlation of two received signals, four peaks can be observed
with the time average:

\[
\hat{R}_{ij}(\tau) = \frac{1}{T} \int_0^T S_{ri}(t)S_{rj}(t-\tau) dt \\
\hat{R}_{ii}(\tau) = \frac{1}{T} \int_0^T S_{ri}(t)S_{ri}(t-\tau) dt
\]

(2.19)

where \(i\) and \(j\) can be 1, 3 or 5 and \(i \neq j\). \(T\) is assumed sufficiently large to satisfy \(TB \gg 1\), and \(B\) is the bandwidth of the signal \(S(t)\).

The extraction processing is shown by Figure 2.8 and 2.9, where \(H_0(f)\) is the optimizing filter, which is required for signals with arbitrary spectra to ensure good time delay resolution. The optimizing filters are discussed in detail in [17].

---

![Figure 2.8: TDOA Extraction from Cross-Correlation](image)

![Figure 2.9: TDOA Extraction from Auto-Correlation.](image)
In Figure 2.8, after filtering, $S_{rj}(t)$ is delayed by $\tau$ and multiplied by the filtered $S_{ri}(t)$. The time average of this product gives the estimated cross-correlation $\hat{R}_{ij}(\tau)$. A peak locator is used to locate the peaks in $\hat{R}_{ij}(\tau)$, The time delay of these peaks are the estimated TDOA’s. For example, $\hat{D}_{13}$, $\hat{D}_{14}$, $\hat{D}_{23}$ and $\hat{D}_{24}$ can be extracted from $\hat{R}_{13}(\tau)$. The corresponding model for the auto-correlation is shown in Figure 2.9, where $i = 1, 3, 5$.

### 2.2.3 Variance of TDOA

Considering the error in the estimation caused by background noise, all the TDOA values extracted from the received signals can be regarded as random variables. An expression for the variance of error of the TDOA’s estimated from time average correlations of length $T$ is given in [7]:

$$\text{var}[\hat{D}_{ij}] \approx \frac{1}{2\pi T} \int_{-\infty}^{+\infty} \omega^2 |H_0(\omega)|^4 S_{ii}(\omega) S_{jj}(\omega) d\omega - \frac{1}{2\pi T} \int_{-\infty}^{+\infty} \omega^2 |H_0(\omega)|^4 S_{ij}(-\omega) S_{ij}(\omega) e^{-j\omega D_{ij}} d\omega$$

$$- \frac{1}{2\pi T} \int_{-\infty}^{+\infty} \omega^2 |H_0(\omega)|^2 S_{ij}(\omega) e^{j\omega D_{ij}} d\omega$$

(2.20)

Where $i \neq j$ and $\hat{D}_{ij}$ is the corresponding TDOA estimate. The function $|H_0(\omega)|$ is the magnitude of the frequency response of the optimizing filters. $S_{ii}(\omega)$ and $S_{jj}(\omega)$ are the Fourier transform of $R_{ii}$ and $R_{jj}$ from equation (2.19), and $S_{ij}$ is the cross-power spectral density of the two multipath signals received by the two sensors.

This expression will be used in the simulation experiments of the source localization, which will be described in detail in Chapter 5.
Chapter 3

ANALYTICAL METHODS FOR PASSIVE LOCALIZATION

The TDOA's are defined by the source location. If we can get three independent noise-free TDOA values (from actual or virtual sensors), the 3-dimension coordinate of the point source can be obtained from the nonlinear hyperbolic equations. In a noisy environment, obtaining the source location with only three noise-corrupted TDOA equations may cause a large error; on the other hand, in most cases we can get redundant TDOA equations. To take the advantage of these redundant TDOA equations, estimating the source location from all the available TDOA’s is a better choice. Therefore, passive localization using the TDOA’s is a problem of nonlinear estimation. If the nonlinearity of the TDOA equations can be eliminated, we can get a set of linear equations and obtain a linear least-squares estimation of the source
location from these equations.

Many authors have studied this idea and they have obtained some closed-form expressions for the estimator. The estimate of the source location can be obtained directly by these analytical methods with a relative low computational burden. In practical applications, if the accuracy requirements can be satisfied, the analytical methods will be very effective because of the low computational burden. Even if the accuracy requirements can be satisfied, the estimate obtained by analytical methods can be used as the initial value of a more accurate iterative estimation (In Chapter 5 we will find that the iteration time will be considerably reduced with the analytical initial values). Because of these reasons, it's meaningful to study these analytical localization methods.

In this chapter, the basic ideas of the linear least-squares estimation are first discussed. Then, considering a 3-sensor receiver array, three methods are derived for closed-form localization of the point source. The simulation results of all these methods will be included in Chapter 5.

3.1 LINEAR LEAST-SQUARES ESTIMATION

Considering an unknown parameter (usually a vector) and a set of noise-corrupted observations or measurements, which are functions of the parameter, estimation is defined as the operation of assigning a value to this unknown parameter based on the observation set. The value assigned is called the estimate.
In many applications it's meaningful to assign a cost to an estimate representing a quantitative measure of how "good" an estimate is. This cost should then be a function of the estimation error, i.e. the observed value and the estimated value of the desired parameter. An optimum estimate is the value to minimize the cost function.

3.1.1 Definition of Linear Least-Squares Estimation

For least-squares estimation, which is most often used for localization, the cost function is defined as:

\[ J = \sum_{i=1}^{m} ||y(\hat{x})_i - z_i||^2 \]  

(3.1)

where m-dimension vector \( x \) is the parameter for estimation, \( z_i \) is the measurement of \( y(x) \), \( y(\hat{x})_i \) is the value generated by the estimate \( \hat{x} \). If \( y() \) is a linear function of the parameter \( x \), the estimation is defined as linear least-squares.

We assume that there are N measurements available:

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_N
\end{pmatrix}, \quad
\begin{pmatrix}
  n_1 \\
  n_2 \\
  \vdots \\
  n_N
\end{pmatrix}
\]  

(3.2)
The measurement vector can be expressed as:

\[ z = Hx + n \]  

(3.3)

Where \( H \) is a \( m \times N \) matrix, and \( n \) is the background noise in measurement.

The problem is, then, to select an estimate \( \hat{x} \) of \( x \) such that the cost function

\[ J(\hat{x}) = (z - H\hat{x})^TW^{-1}(z - H\hat{x}) \]  

(3.4)

is minimized. The optimum estimate which minimizes the foregoing expression will be called the least-squares estimate \( \hat{x}_{LS} \), that is \( J(\hat{x}) \geq J(\hat{x}_{LS}) \). The weighting matrix, \( W^{-1} \), is related to the statistics of the measurement noise \( v \) and assumed to be positive definite and symmetric. In our application, we assume that the noise in every measurement is independent to each other, and all the noise have the same statistics, so as to make \( W = I \) and

\[ J(\hat{x}) = (z - H\hat{x})^T(z - H\hat{x}) \]  

(3.5)

Because the minimization of \( J(\hat{x}) \) is an ordinary deterministic minimization problem, \( \hat{x}_{LS} \) (if it exists) is obtained by setting

\[ \frac{\partial J(\hat{x})}{\partial \hat{x}} |_{\hat{x}=\hat{x}_{LS}} = 0 \]  

(3.6)

By using (3.5) for \( J(\hat{x}) \) and carrying out the indicated partial differentiation, we
obtain:

\[
\frac{\partial J(\hat{x})}{\partial \hat{x}} |_{z} = H^T (z - H \hat{x}_{LS}) = 0
\] (3.7)

or

\[
\hat{x}_{LS} = (H^T H)^{-1} (H^T z)
\] (3.8)

which is the desired least-squares estimate of \( x \).

### 3.1.2 Estimation Using the TDOA's

Considering \( \hat{x}_s \) to be the estimate of vector \( x_s \), a set of corresponding TDOAs can be calculated from it:

\[
\begin{align*}
\hat{r}_1 &= \frac{1}{c} \sqrt{(\hat{x}_e - x_1)^2 + (\hat{y}_e - y_1)^2 + (\hat{z}_e - z_1)^2} - \sqrt{(\hat{x}_e - x_2)^2 + (\hat{y}_e - y_2)^2 + (\hat{z}_e - z_2)^2} \\
&\quad \text{(3.9)} \\
\hat{r}_k &= \frac{1}{c} \sqrt{(\hat{x}_e - x_k)^2 + (\hat{y}_e - y_k)^2 + (\hat{z}_e - z_k)^2} - \sqrt{(\hat{x}_e - x_{2M-1})^2 + (\hat{y}_e - y_{2M-1})^2 + (\hat{z}_e - z_{2M-1})^2} \\
&\quad \text{...} \\
\hat{r}_N &= \frac{1}{c} \sqrt{(\hat{x}_e - x_{2M-1})^2 + (\hat{y}_e - y_{2M-1})^2 + (\hat{z}_e - z_{2M-1})^2} - \sqrt{(\hat{x}_e - x_{2M})^2 + (\hat{y}_e - y_{2M})^2 + (\hat{z}_e - z_{2M})^2}
\end{align*}
\]

For \( r = (r_1, \ldots, r_k, \ldots, r_N) \) and \( \hat{r} = (\hat{r}_1, \ldots, \hat{r}_k, \ldots, \hat{r}_N) \), we can define the cost function by

\[
J = ||r - \hat{r}||^2,
\] (3.10)
Because \( \hat{r} \) is the function of \( \hat{x}_s \), the cost function can also be defined as:

\[
J = ||r - r(\hat{x}_s)||^2, \tag{3.11}
\]

From (3.9) we know that \( r(\hat{x}_s) \) is a non-linear function of \( \hat{x}_s \), so the expression in (3.8) cannot be applied directly. In the following sections three methods are discussed, in which the nonlinearity of the TDOA equations is eliminated, the TDOA equation set (or part of it) is transformed into a new linear equation set, and the closed-form expression of the linear least-squares estimator of the source location can be obtained.

### 3.2 LINEAR APPROXIMATION METHOD

In a conventional sonar system for bearing estimation, if the target range is far enough, the transition time from the source to the sensor (real or virtual) can be approximated by a linear function of the source location \( x_s \) (derived from equation (1a) of [11]):

\[
D_i = \frac{1}{c} \sqrt{(x_s - x_i)^2 + (y_s - y_i)^2 + (z_s - z_i)^2} \\
= \frac{1}{c} \sqrt{R_s^2 + R_i^2 - 2x_i x_s - 2y_i y_s - 2z_i z_s} 
\tag{3.12}
\]

where, \( R_s = \sqrt{x_s^2 + y_s^2 + z_s^2} \) is the distance from the target source to the origin;

\( R_i = \sqrt{x_i^2 + y_i^2 + z_i^2} \) is the distance from the \( i \)-th sensor to the origin;
If $R_i^2 \gg |R_i^2 - 2x_i x_s - 2y_i y_s - 2z_i z_s|^1$ we can get a linear approximation using the linear expansion of the square root function (where $y \gg \Delta y$):

$$\sqrt{y + \Delta y} \approx \sqrt{y} + \frac{\Delta y}{2\sqrt{y}}$$

(3.13)

that is:

$$D_i = \frac{1}{c} \sqrt{(x_s - x_i)^2 + (y_s - y_i)^2 + (z_s - z_i)^2}$$

$$\approx \frac{1}{c} \left(R_i + \frac{1}{2R_i}(R_i^2 - 2x_i x_s - 2y_i y_s - 2z_i z_s)\right)$$

(3.14)

Therefore, using equation (3.14), the TDOA in equation (2.14) can be approximated by linear functions:

$$r_k = D_i - D_j + n_k$$

$$= \frac{1}{2cR_i} \left[(R_i^2 - R_j^2) - 2(x_i - x_j)x_s - 2(y_i - y_j)y_s - 2(z_i - z_j)z_s\right] + n_k,$$

(3.15)

Expressed in matrix form, the equation set is:

$$A x_s = \Delta - R_i r + n_k,$$

(3.16)

---

1This condition can be easily satisfied in underwater acoustic application. For example, let $x_s = (2000m, 2000m, 500m), x_i = (1000m, 0, 2000m)$, then we have $R_i^2 = 8.25 \times 10^6 m^2$, and $|R_i^2 - 2x_i x_s - 2y_i y_s - 2z_i z_s| = 0.25 \times 10^6 m^2$, which is $\frac{1}{40}$ of $R_i^2$. 
where for the 3-sensor system, we have:

\[
\mathbf{A} = \frac{1}{c} \begin{pmatrix}
  x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\
  \vdots & \vdots & \vdots \\
  x_i - x_j & y_i - y_j & z_i - z_j \\
  \vdots & \vdots & \vdots \\
  x_5 - x_6 & y_5 - y_6 & z_5 - z_6
\end{pmatrix}
\]

\[
\Delta = \frac{1}{2c} \begin{pmatrix}
  R_1^2 - R_2^2 \\
  \vdots \\
  R_i^2 - R_j^2 \\
  \vdots \\
  R_5^2 - R_6^2
\end{pmatrix}
\]

(3.17)

(3.18)

This is a standard linear equation set and the least squares estimation of the source location can be obtained:

\[
\hat{x}_s = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T(\Delta - R_s r)
\]

(3.19)

If we set every sensor at a proper location so as to let the distance from every
sensor to the origin be equal, then we have:

\[
\Delta = \frac{1}{2c} \begin{pmatrix}
R_1^2 - R_2^2 \\
... \\
R_i^2 - R_j^2 \\
... \\
R_5^2 - R_6^2 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
... \\
0 \\
\end{pmatrix},
\]  

(3.20)

\( \Delta \) becomes a zero vector, and the expression in (3.19) can be simplified to:

\[
\hat{x}_s = -R_s (A^T A)^{-1} A^T r,
\]

(3.21)

So we can get the bearing estimation of the source:

\[
\frac{\hat{x}_s}{R_s} = -(A^T A)^{-1} A^T r
\]

(3.22)

As a bearing estimator, this method has a good performance. If the estimate of \( R_s \) can be provided by another system (usually a sonar system on the surface ship), the source location can be obtained.

This method is considered because it is simple in concept and easy for implementation. As a localization algorithm, its performance highly depends on the accuracy of \( \hat{R}_s \). For more accurate localization, some more complicated and more effective methods are needed.
3.3 SPHERICAL INTERPOLATION METHOD FOR LOCALIZATION

Another more effective localization method was developed in recent years. This closed-form localization technique, termed the spherical interpolation (SI) method, was first described by J. S. Able in [13], and discussed by many other authors in different formats.[14, 18, 19]. The key point of this method is that the nonlinearity of the TDOA equations can be eliminated by a least square procedure to yield a closed-form estimate for the source location. Because the equations are derived without any approximation, the accuracy of the estimation is better than the linear approximation method.

The basic idea of the SI method can be described as following:

Considering the i-th sensor of the receiver array, the distance from the sensor to the source is:

\[ c^2 D_i^2 = R_i^2 + R_s^2 - 2x_i^T x_s, \]  

(3.23)

where,

\[ R_s = \sqrt{x_s^2 + y_s^2 + z_s^2} \] is the distance from the target source to the origin;

\[ R_i = \sqrt{x_i^2 + y_i^2 + z_i^2} \] is the distance from the i-th sensor to the origin;

Considering two sensors i and j, we have:

\[ c^2 (D_i^2 - D_j^2) = R_i^2 - R_j^2 - 2(x_i^T - x_j^T) x_s, \]  

(3.24)
From the TDOA equation: \( D_{ij} = D_i - D_j \), we also have:

\[
D_i^2 = (D_{ij} + D_j)^2 = D_{ij}^2 + D_j^2 + 2D_{ij}D_j
\]  

(3.25)

and,

\[
D_i^2 - D_j^2 = D_{ij}^2 + 2D_{ij}D_j,
\]  

(3.26)

Substituting (3.26) into (3.24) yields,

\[
2(x_i^T - x_j^T)x_s = (R_i^2 - R_j^2) - c^2(D_{ij}^2 + 2D_{ij}D_j),
\]  

(3.27)

The receiver array discussed in this thesis consists of six sensors (including the three virtual sensors). If we take sensor \( j \) as the reference sensor, using the relations expressed in equation (3.23) to (3.27), 5 equations can be obtained, the equation set written in matrix is:

\[
A_jx_s = u_j - D_jr_j.
\]  

(3.28)

The reference sensor \( j \) can be any one of the six real or virtual sensors, here we just choose sensor 1, which defines the matrices in (3.28) as:
\[ A_1 = \frac{2}{c^2} \begin{pmatrix}
  x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\
  x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\
  x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\
  x_5 - x_1 & y_5 - y_1 & z_5 - z_1 \\
  x_6 - x_1 & y_6 - y_1 & z_6 - z_1 
\end{pmatrix} \quad (3.29) \]

\[ u_1 = \frac{1}{c^2} \begin{pmatrix}
  R_2^2 - R_1^2 \\
  R_3^2 - R_1^2 \\
  R_4^2 - R_1^2 \\
  R_5^2 - R_1^2 \\
  R_6^2 - R_1^2 
\end{pmatrix} - \begin{pmatrix}
  D_{21}^2 \\
  D_{31}^2 \\
  D_{41}^2 \\
  D_{51}^2 \\
  D_{61}^2 
\end{pmatrix} \quad (3.30) \]

\[ r_1 = 2 \times \begin{pmatrix}
  D_{21} \\
  D_{31} \\
  D_{41} \\
  D_{51} \\
  D_{61} 
\end{pmatrix} \quad (3.31) \]

Note that the measurements vector \( r_j \) is a subset of the TDOA set \( r \) defined in equation (2.14), the vector \( u_j \) and the matrix \( A \) are decided by the known sensor locations. \( D_j \) is the only nonlinear term contained in equation (3.28). If \( D_j \) can be removed, we can get a linear estimate of \( x \), from these equations.

The approach is based on the idea of premultiplying \( r_j \) by a matrix \( M_j \) which has
\( r_j \) in its null space, i.e., \( M_j r_j = 0 \). The matrix is:

\[
M_j = (I - Z)P_j, \tag{3.32}
\]

where

\[
P_1 = \begin{pmatrix}
D_{21} & 0 & 0 & 0 & 0 \\
0 & D_{31} & 0 & 0 & 0 \\
0 & 0 & D_{41} & 0 & 0 \\
0 & 0 & 0 & D_{51} & 0 \\
0 & 0 & 0 & 0 & D_{61}
\end{pmatrix}
\]

\[
Z = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{3.34}
\]

Because \( M_j r_j = 0 \), after multiplying both sides of equation (3.28) by \( M_j \), we have:

\[
M_j A_j x_s = M_j u_j, \tag{3.35}
\]

and from equation (3.35) we can get the least-squares estimate of source location, with a closed-form expression:

\[
\hat{x}_s = (A_j^T M_j^T M_j A_j)^{-1} A_j^T M_j^T M_j u_j. \tag{3.36}
\]
In [13] the SI method is described in a little different way, which is based on a two-step procedure. If we assume that the nonlinear term $D_j$ is known, from (3.28) we can get a linear least-squares estimate of $x_s$:

$$x_s = (A_j^T A_j)^{-1} A_j^T (u_j - D_j r_j) \quad (3.37)$$

Then the cost value $J$ becomes a function of $D_j$:

$$J = ||A_j x_s - (u_j - D_j r_j)||^2$$
$$= ||(A_j (A_j^T A_j)^{-1} A_j^T - I)(u_j - D_j r_j)||^2 \quad (3.38)$$

Let:

$$T = ||A_j (A_j^T A_j)^{-1} A_j^T - I||^2 \quad (3.39)$$

and then $J$ can be expressed as:

$$J = (u_j - D_j r_j)^T T (u_j - D_j r_j). \quad (3.40)$$

The second step is minimizing $J$ with respect to $D_j$, so as to get the least-squares estimate of $D_j$:

$$\hat{D}_j = \frac{r_j^T T r_j}{r_j^T T u_j} \quad (3.41)$$
Thus the source location estimate is:

\[ x_0 = (A_j^T A_j)^{-1} A_j^T (u_j - \frac{r_j^T r_j}{r_j^T T r_j_j}) \]  

(3.42)

It has been shown by B. Friedlander in [14] that equations (3.36) and (3.42) are mathematically equivalent. Equation (3.36) is somewhat simpler to understand and equation (3.42) reveals the underlying structure of the method.

Note the the SI estimate given in equations (3.36) or (3.42) is based on TDOA's relative to a single reference sensor. To use all the available TDOA's, the SI estimate can be obtained using different reference sensors, and the final estimate can be computed as the average of all the SI estimates with different reference sensors.

3.4 HYBRID METHOD OF LOCALIZATION

The hybrid method consisting of two steps: the first step is to get the least-squares estimate of the time delay vector \( D \) (defined in equation (2.11)) from the TDOA equations (this is a linear estimation question), which will generate a set of least squares time delay equations. Because \( D \) is the function of \( x_0 \), as the second step, we can obtain the least-squares estimate of \( x_0 \) from the time delay equations.
3.4.1 Least Squares Time Delay Equations

This method stems from the least squares estimation of vector $D$ [19]. Recalling the TDOA equations, if these equations are expressed by the time delay of the signal from the source to every sensor, it should be:

$$ r = HD $$ (3.43)

where $D$ is the time delay vector. For an $M$-sensor system in a multipath environment, $D = [D_1, ..., D_{2M}]^T$ and $H$ is a $N \times 2M$ matrix. $D_i$ and $N$ are defined in equation (2.11) and (2.12), and

$$ H = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \vdots \\ 0 & \ldots & 0 & 1 & -1 \end{pmatrix} $$ (3.44)

Equation (3.43) is a set of linear equations of $D$, so we can get the least-squares estimation of vector $D$ from the following equation set:

$$ PD = HT r, $$ (3.45)

where $P = HT H$. 

44
For the three-sensor system, \( M=3, N=15 \), so \( P \) is a \( 6 \times 6 \) matrix:

\[
P = H^T H = \begin{pmatrix}
5 & -1 & -1 & -1 & -1 & -1 \\
-1 & 5 & -1 & -1 & -1 & -1 \\
-1 & -1 & 5 & -1 & -1 & -1 \\
-1 & -1 & -1 & 5 & -1 & -1 \\
-1 & -1 & -1 & -1 & 5 & -1 \\
-1 & -1 & -1 & -1 & -1 & 5
\end{pmatrix}
\]

(3.46)

\( P \) isn't a full rank matrix, but it can be expressed as the product of two essentially triangular matrices, one of them is a permutation of a lower triangular matrix and the other is an upper triangular matrix:

\[
P = LU = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{5} & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{5} & -\frac{1}{4} & 1 & 0 & 0 & 0 \\
-\frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 & 0 \\
-\frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 & 0 \\
-\frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 & 1
\end{pmatrix} \begin{pmatrix}
5 & -1 & -1 & -1 & -1 & -1 \\
0 & 4.8 & -1.2 & -1.2 & -1.2 & -1.2 \\
0 & 0 & 4.5 & -1.5 & -1.5 & -1.5 \\
0 & 0 & 0 & 4.0 & -2 & -2 \\
0 & 0 & 0 & 0 & 3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(3.47)

So we have:

\[
LU \hat{D} = H^T r
\]

(3.48)

and

\[
U \hat{D} = L^{-1} H^T r,
\]

(3.49)
Since \( L \) is a full rank matrix.

Evaluating \( L^{-1}H^T \) gives,

\[
L^{-1}H^T = \begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
-\frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\
0 & 0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(3.50)

and the right side of the equation can expressed as:

\[
L^{-1}H^Tr = \begin{pmatrix}
L_n \\
-\_\_ \\
0 \\
\end{pmatrix}
\]

(3.51)
Matrix $U$ can also be divided into 4 parts: $U_n, K, 0$ and a single element $0$:

$$U = \begin{pmatrix} U_n & K \\ - & + \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -1 & -1 & | & -1 \\ 0 & 4.8 & -1.2 & -1.2 & -1.2 & | & -1.2 \\ 0 & 0 & 4.5 & -1.5 & -1.5 & | & -1.5 \\ 0 & 0 & 0 & 4.0 & -2 & | & -2 \\ 0 & 0 & 0 & 0 & 3 & | & -3 \\ - & - & - & - & - \end{pmatrix}$$

(3.52)

Where $K$ is a 5-element column vector and $0$ is a 5-element row vector, and $U_n$ is a $5 \times 5$ nonsingular matrix. Substituting (3.51) and (3.52) into (3.49) we can get 5 independent equations:

$$U_n \hat{D}_n + \hat{D}_n K = L_n,$$

(3.53)

where,

$$\hat{D}_n = \begin{pmatrix} \hat{D}_1 \\ \hat{D}_2 \\ \hat{D}_3 \\ \hat{D}_4 \\ \hat{D}_5 \end{pmatrix}$$

(3.54)
Since

\[
U_n^{-1}K = -\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1
\]  

(3.55)

thus we have:

\[
\hat{D}_n = U_n^{-1}L_n - \hat{D}_\delta U_n^{-1}K \\
= v + \hat{D}_\delta 1
\]  

(3.56)

where \( v = U_n^{-1}L_n \).

Equation (3.56) is defined as the least squares time delay equation set. Where \( v \) is a 5-element column vector, the TDOA vector \( \vec{r} \) is included in \( v \).

3.4.2 Least Squares Estimation of the Source Location

Every equation in (3.56) can be expressed as:

\[
\hat{D}_i = v_i + \hat{D}_\delta
\]  

(3.57)
Since $\hat{D}_t$ is defined by the estimate $\hat{x}_a$, (3.57) can also be expressed by $\hat{x}_a$:

$$\sqrt{(\hat{x}_a - x_i)^2 + (\hat{y}_a - y_i)^2 + (\hat{z}_a - z_i)^2} = c(v_i + \hat{D}_a)$$

(3.58)

So we also have:

$$(\hat{x}_a - x_i)^2 + (\hat{y}_a - y_i)^2 + (\hat{z}_a - z_i)^2$$

$$= c^2(v_i + \hat{D}_a)^2$$

$$= c^2(v_i^2 + \hat{D}_a^2 + 2v_i\hat{D}_a)$$

$$= c^2v_i^2 + ((\hat{x}_a - x_a)^2 + (\hat{y}_a - y_a)^2 + (\hat{z}_a - z_a)^2) + 2c^2v_i\hat{D}_a$$

(3.59)

From (3.59) we can get:

$$x_i^2 + y_i^2 + z_i^2 - 2\hat{x}_a x_i - 2\hat{y}_a y_i - 2\hat{z}_a z_i = c^2v_i^2 + x_a^2 + y_a^2 + z_a^2 - 2\hat{x}_a x_a - 2\hat{y}_a y_a - 2\hat{z}_a z_a + 2c^2v_i\hat{D}_a$$

(3.60)

So finally we can get a new equation set of $\hat{x}_a$ from these derivation. It can be expressed in matrix form:

$$B\hat{x}_a = W + \hat{D}_a v$$

(3.61)
where

\[
B = -2 \begin{pmatrix}
  x_1 - x_6 & y_1 - y_6 & z_1 - z_6 \\
  x_2 - x_6 & y_2 - y_6 & z_2 - z_6 \\
  x_3 - x_6 & y_3 - y_6 & z_3 - z_6 \\
  x_4 - x_6 & y_4 - y_6 & z_4 - z_6 \\
  x_5 - x_6 & y_5 - y_6 & z_5 - z_6
\end{pmatrix}, \quad W = c^2 \begin{pmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \\ v_4^2 \\ v_5^2 \end{pmatrix}, \quad V = 2c^2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}
\] (3.62)

Then the question is similar to that in section 3.2: if the nonlinear term \( D_6 \) can be removed, we can get the linear least squares estimate of \( x_6 \). Here \( v \) should be premultiplyed by a matrix \( M_6 \) so as to let \( M_6 v = 0 \). The matrix is:

\[
M_6 = (I - Z)P_6
\] (3.63)

Where

\[
P_6 = \begin{pmatrix}
  v_1 & 0 & 0 & 0 & 0 \\
  0 & v_2 & 0 & 0 & 0 \\
  0 & 0 & v_3 & 0 & 0 \\
  0 & 0 & 0 & v_4 & 0 \\
  0 & 0 & 0 & 0 & v_5
\end{pmatrix}
\] (3.64)
\[ Z = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \] (3.65)

Because \( M_\delta v = 0 \), after multiplying both side of equation (3.61), we have:

\[ M_\delta B x_* = M_\delta w \] (3.66)

from equation (3.66) we can get the least square estimation of source location, with a closed-form expression:

\[ \hat{x}_* = (B^T M_\delta^T M_\delta B)^{-1} B^T M_\delta^T M_\delta w \] (3.67)

The simulation results in chapter 5 show that this method has a better performance than that of the SI method.
Chapter 4

SIMPLEX ALGORITHM FOR LOCALIZATION

4.1 NUMERICAL ESTIMATION

As mentioned in Chapter 3, passive localization means optimum estimation: searching for the minimum point of the cost function $J(\hat{x})$, to get the corresponding best-fit location $\hat{x}$. Obtaining this minimum point by derivation is not the only method available. Another class of methods are numerical estimation methods, which directly search for the minimum of the cost function, obtaining the desired value by a series of trials. Once the minimum error value is obtained, the corresponding point in 3-dimension space is the estimate of the source location.

Actually the value $J$ is a statistic of the errors between the TDOA measurements $r_1,...,r_N$ and the function of the estimate: $r(\hat{x})_1,...,r(\hat{x})_N$ (see equation(3.11)).
Least-squares estimation means to select the least-squares norm (L2 norm) for the error statistic. Because in numerical optimization the concept of "error function" is also used, in the following discussion we will use the term "error function" $E(\hat{x}_*)$ to replace the term "cost function" $J(\hat{x}_*)$, and take them to be equivalent in this thesis.

For a given set of TDOA measurements, we know that there is an error $E$ associated with every point in 3-dimensional space. The entire region of values $E$ in space is defined as the error space. The numerical methods directly search for the minimum in this error space.

The simplex method is one of such numerical methods that has been used to solve a variety of non-linear optimization problems. Its successful application in geology localization questions inspires us to apply this algorithm to the area of underwater acoustics. This use of the word "simplex" is not to be confused with the simplex method of linear programming.[20]

It must be emphasized that least-squares norm (L2 norm) is not the only error function that can be used by the simplex method, another error function (L1 norm) will be discussed in section 4.3.

4.2 THE SIMPLEX METHOD

The simplex method was first discussed as a method for minimization by Spendley and Hext in 1962 [21]. A much more efficient simplex method was presented by
Nelder and Mead in 1965 [22], and an excellent discussion of the simplex method and its application to nonlinear questions is given by Caceci and Cacheris in 1984[23].

Since the simplex method is based on the concept of the simplex in n-dimensional space, we must start with the general principles of the simplex.

4.2.1 The Basic Idea of the Simplex Method

A simplex in n-dimensional space may be thought of as a polyhedron with n+1 distinct vertexes. In passive localization questions, the simplex needed is a tetrahedron in 3-dimensional space, the four vertexes of the given simplex are denoted by \( V_i(i = 1, \ldots, 4) \), with the location vectors \( v_i(i = 1, \ldots, 4) \). Let \( V' \) be any point in the space distinct from \( v_i(i = 1, \ldots, 4) \). A new simplex can be obtained by replacing any one of the \( V_i \) with \( V' \). Hence by changing one vertex of a simplex at a time, we can generate a sequence of simplexes.

Every point in the three-dimensional space corresponds to an error value \( E \), localization means finding the minimum \( E \) value and the corresponding location vector in 3-dimensional space, which is the estimate of source location.

The basic idea of the simplex algorithm for finding this minimum is to replace the vertex corresponding to the highest error with another point corresponding to a lower error, so as to get a new simplex, repeat this operation again and again, the simplex moves and distorts to converge to the minimum error point, which is the best
4.2.2 Basic Operations of the Simplex Algorithm

At the beginning of the iteration, an initial simplex is set, then the error value $E$ corresponding to each vertex is calculated. For vertex $V_i$, we can use one of the two different error functions. If L1 norm is used, we have,

$$E_i = \sum_{k=1}^{N} |r_{vik} - r_k|, \quad (4.1)$$

if L2 norm is used, we have,

$$E_i = \sum_{k=1}^{N} (r_{vik} - r_k)^2, \quad (4.2)$$

where $r = (r_1, r_2, ... r_N)$ is the measured TDOA vector and $r_{vi} = (r_{vi1}, r_{vi2}, ... r_{vin})$ is the corresponding TDOA vector assuming that the source is located at position $V_i$. More discussion about L1 and L2 norm is presented in next section.

Comparing these error values, we can get the vertex $V_h$ with the highest error value of the four vertexes, the next step is to determine the point $V_c$, which is the centroid of the remaining vertices with lower error values:

$$v_c = (\sum_{i=1}^{4} v_i - v_h)/3 \quad (4.3)$$

Where $v_c$ is the location vector of the point $V_c$. 

fit solution.
The direction $v_h - v_c$ is likely to be a direction along which error value decreases, and we have three ways to deform the present simplex along this direction.

1. Reflection:

As shown in Figure 4.1, the reflection point of $v_h$ through $v_c$ is:

$$v_h^R = v_c + \alpha(v_c - v_h) \tag{4.4}$$

Where $\alpha$ is the reflection coefficient, $0 < \alpha < 1$.

Let $E_h^R$ be the $E$ value associated with $v_h^R$ and $E_l$ be the lowest $E$ value of the vertexes, if $E_l < E_h^R < E_h$, then we replace $v_h$ with $v_h^R$ to obtain the new simplex $(v_1, \ldots, v_{h-1}, v_h^R, v_{h+1}, \ldots, v_4)$.

2. Expansion:

If $E_h^R < E_l$, then reflection has produced a new minimum $E$ value. Clearly in this
Figure 4.2: Three-dimensional simplex: illustrating expansion.

case the direction $v_h - v_c$ is worthy of further exploration, so we expand $v_h^R$ to $v_h^E$:

$$v_h^E = v_c + \gamma (v_h^R - v_h) \quad (4.5)$$

where $\gamma$ is the expansion coefficient, $\gamma > 1$.

Let $E_h^E$ be the $E$ value associated with $v_h^E$, if $E_h^E < E_l$, we replace $v_h$ with $v_h^E$ to obtain the new simplex $(v_1, \ldots, v_{h-1}, v_h^E, v_{h+1}, \ldots v_d)$. If $E_h^E \geq E_l$, we have a failed expansion. The simplex has moved too far along the direction $v_h - v_c$, in this case we choose $v_h^R$ to replace $v_h$ since we know that $E_h^R < E_l$. The expansion is shown in Figure 4.2.

3. Contraction:
If on reflecting $v_h$ to $v^R_h$ we find that $E^R_h > E_h$, a contraction is needed:

$$v^C_h = v_c + \beta(v_h - v_c) \quad (4.6)$$

Where $\beta$ is the contraction coefficient, $0 < \beta < 1$. Let $E^C_h$ be the $E$ value associated with $v^C_h$, if $E^C_h < \min(E_h, E^R_h)$, we replace $v_h$ with $v^C_h$ to obtain the new simplex $(v_1, \ldots, v_{h-1}, v^C_h, v_{h+1}, \ldots v_4)$. The contraction is shown in Figure 4.3.

4. **Shrinkage:**

If $E^C_h \geq \min(E_h, E^R_h)$, $v^C_h$ is a worse replacement for $v_h$, then we construct a new simplex by replacing every $v_i$ with $v'_i$, $(i=1,\ldots,4)$:

$$v'_i = (v_i + v_i)/2, \quad (4.7)$$

That is to say, we reduce the size of the simplex to one half of its previous size.
The new simplex

The old simplex

Worst Error vertex

Shrunken vertexes

Figure 4.4: Three-dimensional simplex: illustrating Shrinkage

but retain the vertex \( v_i \), as shown in Figure 4.4.

Having constructed the new simplex, we repeat the procedure to obtain a sequence of simplexes. It should be clear that the iteration procedure changes the shape and size of the current simplex, and the error values at the vertexes of the next simplex won't be greater than that of the current ones. Expressed geometrically, the current simplex tends to move downhill towards the minimum of error space, changing its shape and size in order to negotiate curving valleys, until the minimum is enclosed by it.

The flowchart of the simplex algorithm is shown in Figure 4.5. An example of a simplex iteration is shown in Figure 4.6, which is a two dimensional (x-y plane) diagram of the three dimensional localization. The source is located at the position of \((5000m, 5000m, 500m)\). What shown by the dot-line contour is a \(x - y\) plane slice of the error space at \(z = 500m\), which has a minimum at \((5000m, 5000m)\). It can be
Figure 4.5: The flowchart of the simplex algorithm.
Figure 4.6: An example of the simplex moving in the error space

observed that the iterations start from the initial simplex (the triangle at the lower-left corner of the figure). The simplex, during the iterations, moves downhill towards the minimum of error space, becomes smaller and smaller, and finally converges to the minimum.
<table>
<thead>
<tr>
<th>$x_s$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_s = 1.4km, y_s = 1.4km, z_s = 300m$</td>
<td>3.11</td>
<td>3.63</td>
<td>0.05</td>
<td>1.34</td>
<td>1.54</td>
<td>0.01</td>
</tr>
<tr>
<td>$x_s = 2.1km, y_s = 2.1km, z_s = 300m$</td>
<td>3.37</td>
<td>3.81</td>
<td>0.05</td>
<td>0.71</td>
<td>0.80</td>
<td>0.01</td>
</tr>
<tr>
<td>$x_s = 2.8km, y_s = 2.8km, z_s = 300m$</td>
<td>3.87</td>
<td>4.11</td>
<td>0.08</td>
<td>2.45</td>
<td>2.69</td>
<td>0.05</td>
</tr>
<tr>
<td>$x_s = 3.5km, y_s = 3.5km, z_s = 300m$</td>
<td>25.57</td>
<td>26.54</td>
<td>0.18</td>
<td>10.53</td>
<td>11.23</td>
<td>0.24</td>
</tr>
<tr>
<td>$x_s = 4.2km, y_s = 4.2km, z_s = 300m$</td>
<td>42.24</td>
<td>45.50</td>
<td>0.82</td>
<td>55.06</td>
<td>57.84</td>
<td>0.39</td>
</tr>
<tr>
<td>$x_s = 4.9km, y_s = 4.9km, z_s = 300m$</td>
<td>155.89</td>
<td>161.75</td>
<td>1.63</td>
<td>169.15</td>
<td>177.17</td>
<td>2.11</td>
</tr>
</tbody>
</table>

Table 4.1: The variance of location estimator under L1 and L2 norm (no bad point in the TDOA measurements) (Unit: m²)

### 4.3 DISCUSSION OF SIMPLEX ALGORITHM

#### 4.3.1 The Error Function

As we have shown in equation 4.1 and 4.2, L1 norm and L2 norm are two different error functions used in the numerical localization. L2 norm, or least-squares error function, has been discussed in Section 3.1. The L1 norm is also popularly used in estimation questions [24].

The performance of the two error functions are measured using the variance of the source location estimator in three dimension. Different source locations are chosen for estimation, and the results are shown in Table 4.1 and 4.2, the configuration of the entire system and the method for obtaining the variance will be presented in Chapter 5.

Under ideal situations, the performance of L1 and L2 will have no obvious differences, (Table 4.1). If there are single, large errors in the measured TDOA’s which are known as bad points, the performance of L1 will be better than that of L2. That
Table 4.2: The variance of location estimator under L1 and L2 norm (there is one bad point in the TDOA measurements) (Unit: m²)

<table>
<thead>
<tr>
<th>x_2</th>
<th>v_x</th>
<th>v_y</th>
<th>v_z</th>
<th>v_x</th>
<th>v_y</th>
<th>v_z</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.41</td>
<td>3.99</td>
<td>0.06</td>
<td>4.36E+4</td>
<td>5.25E+4</td>
<td>461</td>
<td></td>
</tr>
<tr>
<td>2.13</td>
<td>2.39</td>
<td>0.07</td>
<td>1.89E+4</td>
<td>2.23E+4</td>
<td>70.6</td>
<td></td>
</tr>
<tr>
<td>2.56</td>
<td>2.79</td>
<td>0.17</td>
<td>4.30E+4</td>
<td>4.97E+4</td>
<td>99.3</td>
<td></td>
</tr>
<tr>
<td>17.40</td>
<td>19.28</td>
<td>0.17</td>
<td>1.25E+5</td>
<td>1.42E+5</td>
<td>193.9</td>
<td></td>
</tr>
<tr>
<td>15.49</td>
<td>16.82</td>
<td>0.52</td>
<td>2.49E+5</td>
<td>2.77E+5</td>
<td>272.6</td>
<td></td>
</tr>
<tr>
<td>624.32</td>
<td>662.17</td>
<td>3.74</td>
<td>1.88E+5</td>
<td>2.07E+5</td>
<td>274.8</td>
<td></td>
</tr>
</tbody>
</table>

is because the L2 norm weights each input TDOA by the square of its error, so a bad point will tend to bias the final result. On the other hand, the L1 norm tends to deemphasize the effect of single bad point [24], so the final result won’t be greatly influenced by single bad point. Table 4.2 shows the comparison of the performance under L1 and L2 norm with one bad point in the measured TDOAs. Because of the application environment for the localization system, the appearance of “bad points” in the measured TDOA’s is possible and must be considered. Therefore, L1 norm is a better choice for underwater acoustic questions.

4.3.2 The Strategy

The strategy involves selecting optimum coefficient values, i.e. the value of α, β and γ. A better strategy can give a faster convergence and the best values can only be extracted from experiments.

From Table 4.3, we can select α = 1.0, β = 0.25 and γ = 2.0 to be the best strategy. This may be only effective under the experiment configuration presented in
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x, & \alpha = 0.5 & \alpha = 0.5 & \alpha = 1.0 & \alpha = 1.0 & \alpha = 1.0 \\
\gamma = 2.0 & \beta = 0.5 & \beta = 0.25 & \beta = 0.25 & \beta = 0.5 & \beta = 0.5 \\
\hline
x = 1.4km, y = 1.4km, z = 300m & 24 & 27 & 20 & 39 & 45 \\
x = 2.1km, y = 2.1km, z = 300m & 28 & 26 & 17 & 41 & 43 \\
x = 2.8km, y = 2.8km, z = 300m & 29 & 32 & 23 & 48 & 46 \\
x = 3.5km, y = 3.5km, z = 300m & 34 & 30 & 22 & 45 & 47 \\
x = 4.2km, y = 4.2km, z = 300m & 27 & 25 & 21 & 49 & 46 \\
x = 4.9km, y = 4.9km, z = 300m & 28 & 28 & 23 & 47 & 53 \\
\hline
\end{array}
\]

Table 4.3: Mean Numbers of Iterations Under Different Strategy

this thesis.

4.3.3 The Initial Simplex

In the simplex method, the initial value will also influence the convergence speed of the estimation. In Figure 4.7 three different initial simplexes are set, the first one is far away from the source location, the second has a relatively large size and wraps around the source location, the third is just in the vicinity of the source. We can find from Table 4.4 and 4.5 that if the initial simplex is located too far away from the desired minimum (as Simplex 1 in Figure 4.7 is selected), the simplex may "stick" to a local minimum point or navigate in a wrong direction, which will cause a long convergence time. On the other hand, if the initial simplex locates near the source location or "wraps" the source location, as the simplex 2 or simplex 3 shown in Figure 4.7, the convergence will be much faster.

If the analytical and numerical methods are combined, the estimate from the
Table 4.4: Mean number of iterations under three different initial simplexes shown in Figure 4.7 (L1 norm is used), the configuration of the entire system will be shown in Chapter 5.

<table>
<thead>
<tr>
<th>$x_s$</th>
<th>simplex 1</th>
<th>simplex 2</th>
<th>simplex 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_s = 1.4\text{km}, y_s = 1.4\text{km}, z_s = 300$</td>
<td>81</td>
<td>32</td>
<td>21</td>
</tr>
<tr>
<td>$x_s = 2.1\text{km}, y_s = 2.1\text{km}, z_s = 300$</td>
<td>87</td>
<td>34</td>
<td>26</td>
</tr>
<tr>
<td>$x_s = 2.8\text{km}, y_s = 2.8\text{km}, z_s = 300$</td>
<td>84</td>
<td>44</td>
<td>27</td>
</tr>
<tr>
<td>$x_s = 3.5\text{km}, y_s = 3.5\text{km}, z_s = 300$</td>
<td>97</td>
<td>47</td>
<td>34</td>
</tr>
<tr>
<td>$x_s = 4.2\text{km}, y_s = 4.2\text{km}, z_s = 300$</td>
<td>105</td>
<td>42</td>
<td>38</td>
</tr>
<tr>
<td>$x_s = 4.9\text{km}, y_s = 4.9\text{km}, z_s = 300$</td>
<td>110</td>
<td>63</td>
<td>44</td>
</tr>
</tbody>
</table>

analytical method can be used as the initial value of the simplex method, so as to start the iteration from a reasonable initial simplex. With this improvement, the convergence time can be greatly reduced. The simulation results of this idea are shown in Chapter 5.

Figure 4.7: Different initial simplexes
<table>
<thead>
<tr>
<th>$x_s$</th>
<th>simplex 1</th>
<th>simplex 2</th>
<th>simplex 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_s = 1.4\text{km}, y_s = 1.4\text{km}, z_s = 300$</td>
<td>74</td>
<td>38</td>
<td>22</td>
</tr>
<tr>
<td>$x_s = 2.1\text{km}, y_s = 2.1\text{km}, z_s = 300$</td>
<td>82</td>
<td>40</td>
<td>27</td>
</tr>
<tr>
<td>$x_s = 2.8\text{km}, y_s = 2.8\text{km}, z_s = 300$</td>
<td>88</td>
<td>40</td>
<td>25</td>
</tr>
<tr>
<td>$x_s = 3.5\text{km}, y_s = 3.5\text{km}, z_s = 300$</td>
<td>95</td>
<td>71</td>
<td>36</td>
</tr>
<tr>
<td>$x_s = 4.2\text{km}, y_s = 4.2\text{km}, z_s = 300$</td>
<td>101</td>
<td>78</td>
<td>44</td>
</tr>
<tr>
<td>$x_s = 4.9\text{km}, y_s = 4.9\text{km}, z_s = 300$</td>
<td>102</td>
<td>79</td>
<td>53</td>
</tr>
</tbody>
</table>

Table 4.5: Mean number of iterations under three different initial simplexes shown in Figure 4.7 (L2 norm is used), the configuration of the entire system will be shown in Chapter 5.

### 4.3.4 The Stop Criterion

It has been suggested that a pre-established threshold of the error value $E$ could be used to stop the iteration when the $E$ value falls below it [22, 23]. But in practice, the minimum $E$ value may not be zero or very small. Here we use the value $C_a$, which is the average of the relative distance between every two vertices as the criterion

$$C_a = \frac{1}{6} \sum_{i=1}^{3} \sum_{j=i+1}^{4} |v_i - v_j|$$  \hspace{1cm} (4.8)

A value $C_a^t = 1$ meter is set as the threshold of $C_a$, the search will stop when $C_a \leq C_a^t$, which means the simplex figure has contracted such that all its vertices are physically close to one another. As an insurance, a maximum number of iterations is also kept as a criterion. That is: if $C_a$ is still greater than $C_a^t$ after 300 iterations, the program will stop automatically.
4.4 SUMMERY OF THE SIMPLEX METHOD

The simplex algorithm directly searches for the minimum in the error space, the result of simplex iterations is the optimum estimate of the source location. The simulation results in Chapter 5 will show that the simplex method has a much better performance than any of the analytical methods discussed in chapter 3, because these analytical methods can only give quasi-optimum estimates. Moreover, as a numerical method, the simplex method can take the advantage of another error function (L1 norm). This makes the method more robust than the analytical least-squares methods, especially when some single, great errors (bad points) appear in the measurements.

The drawback of the simplex method is its computational burden, it may take more than 100 iterations to obtain the desired estimate. So it's difficult to use the simplex method directly in some real-time applications. However, it is noted that the time of convergence depends on the initial simplex. If a reasonable initial simplex is selected, the convergence time will be greatly reduced.

Therefore, the best choice is to combine the analytical and numerical methods together, take the estimate from the analytical method as the initial value of the simplex method, so as to start the iteration from a reasonable initial simplex. With this improvement, the simplex method can be applied to the underwater acoustic passive localization question as a more effective method, with an acceptable computational burden.
Chapter 5

THE SIMULATION CONFIGURATION AND RESULTS

5.1 THE GENERATION OF THE TDOA

In this chapter, a computer simulation of the localization methods discussed in previous chapters are presented. Because all of these methods take the TDOA measurements as input, a TDOA generator is needed for simulation. Since the TDOA measurements are decided by the system geometry and corrupted by environment noise and path attenuation, as a start we should consider these questions.
5.1.1 Geometry of the Experiment System

As mentioned in Chapter 2, the system under consideration is a 3-sensor system. According to the definition in Section 2.1, we define \( x_0 = 500m, y_0 = 866m, \) \( z_0 = 200m \). In terms of an \((x, y, z)\) coordinate system the sensors are located at \((-500, 0, 200)\), \((500, 0, 200)\) and \((0, 866, 200)\) where \( z \) is the depth. This places the three sensors on the corners of an equilateral triangle with 1000 meters sides. For the deep ocean multipath model, the three actual sensors corresponding to the direct paths are:

\[
\begin{align*}
  x_1 &= (-500, 0, 200) \\
  x_2 &= (500, 0, 200) \\
  x_3 &= (0, 866, 200) \\
  x_4 &= (-500, 0, -200) \\
  x_5 &= (500, 0, -200) \\
  x_6 &= (0, 866, -200)
\end{align*}
\] (5.1)

and the three virtual sensors corresponding to the surface reflecting paths are:

\[
\begin{align*}
  x_2 &= (-500, 0, -200) \\
  x_4 &= (500, 0, -200) \\
  x_6 &= (0, 866, -200)
\end{align*}
\] (5.2)

Another question is the location of the source, for fixed sensors, a single source at different locations will generate different TDOA measurements. In our simulation experiment, the location of the single target source is limited within the following range: \( x_s = -5000m \sim 5000m, y_s = -5000m \sim 5000m, z_s = 100m \sim 300m \). In the simulation experiment the source placed at different locations within this region are tested by all of the localization methods.
5.1.2 TDOA as a Random Variable

Every TDOA measurement can be regarded as the sum of the true TDOA value (calculated from (2.14)) and a random variable (caused by the background noise)[12]. The random variable can be assumed to be a zero mean and has a normal distribution function. Therefore we can take every TDOA measurement as a random variable (RV). For a fixed receiver array, the location of every sensor is known, so the mean value of this RV, which is the true value of the TDOA, can be calculated using (2.14), and its variance can be estimated by (2.20). A simplified expression of the variance is also given in the appendix of [7]:

\[
\text{var}[\hat{D}_{ij}] \approx \frac{(2\pi B)^2 (u S_i^2 + v S_{ni} S_s + w S_{nj} S_s + S_{ni} S_{nj})}{(2\pi T g_i g_j S_s)^2},
\]

where \( u, v, w \) are factors defined by the corresponding path attenuation, \( S_s \) is the spectrum level of the source signal \( S(t) \) (assumed to be band-pass white noise), \( S_n \) is the spectrum level of the ambient noise in the paths i and j. We note that the variance is mainly decided by the source and ocean noise power spectral densities and the path attenuation of the source signal.

The power spectrum of the source and ocean noise is assumed to be flat with 400 Hz bandwidth. The ratio of the transmitted source power (at the source location) to the received noise power is assumed to be 80 dB and the observation time is \( T = 2 \) seconds. It is noted that the SNR falls off as a power of range due to spreading loss and exponentially due to absorption. That is:

\[
\text{SNR} = 80 - 20 \log(l_s) - \alpha \times l_s, \text{ where } l_s \text{ is the distance from the source to the}
\]
Figure 5.1: The variance of TDOA
sensor in meters and $\alpha$ is the absorption coefficient ($\alpha = 0.2187 dB/km$). Considering the source with a range from 100 meters to 8000 meters, the SNR at the receiver varies from 40 dB to 0 dB.

In the simulation, after the location of the target source is set, the distance from every sensor to the target source is obtained so as to calculate the SNR. The level of $S_s$ and $S_n$ can be calculated from the SNR and substituted into (5.3) and then the variance can be estimated. From Figure 5.1 we can see that for the target source located from 100m to 8000m the variance of TDOA measurement should be about $10^{-10}$ to $10^{-8}$ sec$^2$.

5.1.3 The Generation of TDOA

The block diagram of the TDOA generator is shown in Figure 5.2: with the input of sensor and target source locations, the distance difference between two paths can be calculated, and the output is divided by sound speed $c$ so as to get the true value of TDOA. Meanwhile a zero mean white noise generator is set, whose variance is defined by the output of the variance calculator. Adding the output of the noise generator to the true value, we get the TDOA measurement output.

5.2 THE SIMULATION OF LOCALIZATION

The simulation results of analytical and numerical localization methods are shown in this section. The sample variance of the source location estimate is used to compare the performance of different methods. Considering the source located at a particular
The sensors' location \[
\rightarrow \text{Calculation of the distance difference} \rightarrow \frac{1}{c} \rightarrow \text{TDOA measurements}
\]

The source location

The final results of different methods can be expressed by the 3-dimensional figures. (Figures A.1-A.15 in Appendix A). From these figures it is observed that in the 10km x 10km vicinity of the receiver, for the estimate of \(x_\ast\) and \(y_\ast\), the variances of the simplex method are almost never greater than 1000m\(^2\) (standard deviation \(\leq 33\) m), and the analytical estimators can get the same performance only within a much smaller and irregular area around the receiver (this is shown by the contour
To show the performance of different methods more clearly, cross-sections are taken from these 3-dimensional figures and two examples are shown in Figures 5.3 and 5.4. Figure 5.3 expresses the cross-section extending in x direction at \( y = 5000m \) and \( z = 300m \); Figure 5.4 expresses the cross-section extending in y direction at \( x = 5000m \) and \( z = 300m \). The top three curves represent the three analytical localization methods (dot line for linear approximation method; solid line for SI method and dash line for hybrid method), and the bottom two represents the simplex method (dash line for L1 norm and solid line for L2 norm).

The simplex method has a much better precision than any of the other methods. This is because the simplex method searches for the minimum in the actual error space. All the other methods work with different approximations of the error space, and obtain the minimum of the approximation space by derivation. With increasing source-receiver distance and decreasing SNR, the difference between the approximation and the actual error space becomes greater and greater, so the performance of analytical methods reduces rapidly with the source-receiver distance.

As an iterative method, the simplex will take a longer computational time, and if the initial simplex is not properly selected, the algorithm may have a convergence problem. To compensate for this drawback, we can choose the result of the analytical method as the initial value of simplex iteration so as to reduce the time of iteration. As a test nine source locations are selected at different range and depth. The mean number of iteration under two different initial simplexes A and B. Initial simplex A is a fixed tetragon with vertexes: (0, 0, 0), (4000, 0, 0), (0, 4000, 0), (0, 0, 0, 200). Initial
Legend:

- : Linear Approximation Method
- (Top): Spherical Interpolation (SI) Method
- (Top): Hybrid Method
- (Bottom): Simplex Method (L2 Norm)
- (Bottom): Simplex Method (L1 Norm)

Figure 5.3: Performance comparison of different localization methods

75
Figure 5.4: Performance comparison of different localization methods
Table 5.1: Mean number of iteration under different initial simplexes (using L1 and L2 norm)

<table>
<thead>
<tr>
<th>$x_s$</th>
<th>init. simplex A</th>
<th></th>
<th>init. simplex B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 1 km, y = 1 km, z = 100 m$</td>
<td>73</td>
<td>72</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$x = 3 km, y = 3 km, z = 100 m$</td>
<td>75</td>
<td>74</td>
<td>21</td>
<td>25</td>
</tr>
<tr>
<td>$x = 5 km, y = 5 km, z = 100 m$</td>
<td>78</td>
<td>75</td>
<td>33</td>
<td>38</td>
</tr>
<tr>
<td>$x = 1 km, y = 1 km, z = 200 m$</td>
<td>70</td>
<td>54</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$x = 3 km, y = 3 km, z = 200 m$</td>
<td>76</td>
<td>58</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td>$x = 5 km, y = 5 km, z = 200 m$</td>
<td>82</td>
<td>63</td>
<td>27</td>
<td>29</td>
</tr>
<tr>
<td>$x = 1 km, y = 1 km, z = 300 m$</td>
<td>87</td>
<td>56</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$x = 3 km, y = 3 km, z = 300 m$</td>
<td>93</td>
<td>66</td>
<td>19</td>
<td>17</td>
</tr>
<tr>
<td>$x = 5 km, y = 5 km, z = 300 m$</td>
<td>99</td>
<td>72</td>
<td>31</td>
<td>30</td>
</tr>
</tbody>
</table>

simplex B is generated using the SI method.

The results of this test are presented in Table 5.1. The results show that using the analytical estimate as the basis for the initial simplex can significantly reduces the number of iterations. This is a very good idea to reduce the computational burden and makes the simplex method a practical efficient algorithm in underwater acoustic passive localization.
Chapter 6

SUMMARY AND CONCLUSIONS

The main application considered in this thesis is the problem of passively locating an underwater acoustic source using multiple omnidirectional receivers in a multipath deep-ocean environment. The model discussed in this thesis consists of a submerged three-sensor receiver system and a broad-band noise source. Only two propagation paths (direct path and surface reflection path) from the source to the receiver array are considered, and the time difference of arrival (TDOA) between every two paths are used as the raw data for source localization.

The objective of this thesis is to find an effective localization method using the TDOA's. Passive localization, which is a question of nonlinear optimum estimation, can be solved by two kinds of methods: analytical methods and numerical methods.

Three analytical methods are discussed in this thesis: linear approximation method,
spherical interpolation (SI) method and hybrid method. They can give a quasi-optimum least-squares estimates of the source location with a relatively low computational cost. The precision of all these analytical methods decreases rapidly with the increasing source-receiver distance, which limits the application of these methods to a very small area around the receiver array.

The numerical method discussed in this thesis is the simplex method, which has been applied in the area of geology localization for several years. As an iterative algorithm, the simplex method produces excellent precision of localization but with a relatively high computational cost. Moreover, as a numerical algorithm, the simplex method can use the absolute function (L1 norm) as an error function, so as to make the estimator more robust than the least-squares estimator, especially when there are "bad points" in the raw data.

To reduce the computational time of the simplex method, the estimation result of the analytical method can be used as the initial simplex. The simulation results prove that this improvement can greatly reduce the iterative time of the simplex algorithm and make the algorithm more efficient. The simulation results in this thesis shows that the performance of the simplex method is much better than that of the analytical methods. Therefore the conclusion of this thesis is that the simplex method can be used as an effective algorithm in the area of passive localization.

Future work on this topic will focus on a more practical question: the tracking of the moving target source. In this case the parameter for estimation is not only the source location vector but also its velocity vector, and the TDOA measurements
may be influenced by the Doppler frequency shift of the source signal. Although this question is complicated and difficult for both analytical and numerical methods, it is believed that with some necessary modifications, the simplex method can solve it successfully.
Bibliography


Appendix A

PERFORMANCE OF DIFFERENT LOCALIZATION METHODS

The following are 15 figures showing the variance of different estimators. Every figure corresponds to a set of source locations (at the same depth) within the vicinity of the receiver array, axis $z$ expresses the logarithm of the variance of the location estimate. It is observed that in positions closer to the receiver the accuracy of the estimate is very good, and it reduces rapidly when the source-receiver distance increases.
Figure A.1: Variance of the location estimate $\hat{x}_s$, Linear Approximation method, source depth=100m
The variance of $X_s$ -5000 -5000

The variance of $Z_s$ 5000 5000

The variance of $V_{x_5}$...

Figure A.2: Variance of the location estimate $x_s$, Linear Approximation method, source depth=200m
Figure A.3: Variance of the location estimate $\hat{x}$, Linear Approximation method, source depth=300m
The variance of $X_s$  

The variance of $Y_s$  

The variance of $Z_s$  

Figure A.4: Variance of the location estimate $\hat{x}$, SI method, source depth=100m
The variance of $X_s$ - $5000 \leq X_s \leq 5000$

The variance of $Y_s$

The variance of $Z_s$ - $5000 \leq Z_s \leq 5000$

Figure A.5: Variance of the location estimate $\hat{x}$, SI method, source depth=200m
The variance of $X_s$. The variance of $Y_s$. The variance of $Z_s$.

Figure A.6: Variance of the location estimate $\hat{x}_s$, SI method, source depth=300m
The variance of $X_s$

The variance of $Y_s$

The variance of $Z_s$

Figure A.7: Variance of the location estimate $\bar{x}_s$, hybrid method, source depth=100m
Figure A.8: Variance of the location estimate $\hat{x}$, hybrid method, source depth=200m
Figure A.9: Variance of the location estimate $\hat{x}_s$, hybrid method, source depth=300m
The variance of $X_s$

The variance of $Y_s$

The variance of $Z_s$
Figure A.11: Variance of the location estimate \( \hat{x}_s \), Simplex method (L1 norm), source depth=200m
Figure A.12: Variance of the location estimate $\hat{x}$, Simplex method (L1 norm), source depth=300m
The variance of $X_s$

Figure A.13: Variance of the location estimate $\hat{x}$, Simplex method (L2 norm), source depth=100m
Figure A.14: Variance of the location estimate $\hat{x}_s$, Simplex method (L2 norm), source depth=200m
The variance of \( X \).
The variance of \( V \).

\[
\text{Figure A.15: Variance of the location estimate } x_s, \text{ Simplex method (L2 norm), source depth=300m}
\]