A SMALL-SIGNAL HIGH FREQUENCY EQUIVALENT CIRCUIT FOR THE FIELD-EFFECT TRANSISTOR

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by
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ABSTRACT

The field effect transistor is considered as an active, distributed non-uniform transmission line and a differential equation for the small-signal a.c. case is derived. The short-circuit admittance parameters of the device are determined from the solution of the differential equation. A high frequency equivalent circuit for the intrinsic device is then obtained from the first-order approximation of the analysis and the expressions for the elements of this circuit are derived for both the saturated and the non-saturated conditions. The normalized values of these elements are computed as functions of the gate and drain bias voltages and the results of these computations are presented graphically.
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LIST OF PRINCIPAL SYMBOLS

2a = the distance between the two gates
2b = the channel height
E_x = x-component of electric field in the channel
S_m = the transconductance of the F.E.T.
S_m0 = d.c. or low frequency value of S_m
S_o = output conductance of the F.E.T.
I_d' = total value of channel current
I_d = d.c. value of channel current
i = a.c. value of channel current
L = length of the channel
q = electronic charge
V_g = d.c. potential of the gate referred to the source
V_d = d.c. potential of the drain referred to the source
v = a.c. value of the channel bias
V_g = a.c. potential of the gate referred to the source
V_d = a.c. potential of the drain referred to the source
W = the d.c. value of the channel bias
W' = the total channel bias
W_p = the value of channel bias at pinch-off
y = the ratio (b/a)
y_s = the value of y at the source end of the channel
y_d = the value of y at the drain end of the channel
y_{1,2,0} = the admittances of the equivalent circuit
y_m = the transadmittance of the F.E.T.
\( Z \) = the width of the F.E.T.
\( \varepsilon \) = permittivity of the material of the F.E.T.
\( \mu \) = mobility of the current carriers
\( \rho_0 \) = charge density of the space-charge regions
\( \rho_g \) = charge density of the gate regions
\( \sigma_0 \) = conductivity of the channel
\( \Psi \) = equilibrium barrier potential of the p-n junction

The subscripts \( s \), \( g \), and \( d \) refer to the source, gate and drain respectively.
1. INTRODUCTION

1.1 General

Recent advances in the construction techniques of field-effect transistors have enabled devices of greatly improved high frequency performance to be produced. Application of such devices in an optimal fashion to the design of high frequency circuits is facilitated by the use of an equivalent circuit. This thesis is concerned with the theoretical derivation of a high frequency equivalent circuit for the field-effect transistor from a consideration of the basic physical principles involved in the operation of the device.

1.2 Review of the Literature

The field-effect transistor (F.E.T.) is a semiconductor device in which the conductance of the current path is modulated by the application of a transverse electric field. It was first described by Shockley\(^1\) who presented the basic theory of the device. This theory assumed an abrupt p-n junction and a uniformly graded channel along which the potential varied in a gradual way. A brief account of his analysis is given below.

Consider an F.E.T. with an n-type channel and two p-type gates as shown in Fig. 1.1. Suppose that the two gates are shorted to the source and a positive potential \(V_d\) is applied to the drain. A current \(I_d\) will flow between the source and
Fig. 1.1 Schematic diagram of an F.E.T.

Fig. 1.2 Common source drain voltage-current characteristic of the F.E.T.
the drain causing a voltage drop along the channel. Since the drain is more positive than the source which is directly connected to the gates, the p-n junctions are reverse biased. This reverse bias increases in the positive x direction along the channel and, hence, the space-charge regions will be wider near the drain. As the drain to source voltage is increased the channel becomes narrower until, at a value of channel bias equal to $W_p$, the two space-charge regions from the opposite gates meet at the drain end. This is called the 'pinch-off' condition and $W_p$ is called the 'pinch-off potential'. The drain current remains essentially saturated for bias voltages higher than $W_p$ as most of the increased voltage appears across the space-charge regions near the drain.

If a negative potential $V_g$ is applied between the gate and the source, the magnitude of the drain voltage $V_d$ required to cut off the channel will be reduced by the amount of gate bias. The drain current will therefore saturate at lower values of drain voltage and current. Thus the common source characteristic ($I_d$ versus $V_d$) with gate voltage as a parameter is of the form shown in Fig. 1.2.

If $W$ is the potential of the gate with respect to the channel, then according to Shockley,\textsuperscript{1}

$$W = W_p(1 - \frac{b}{a})^2 \quad (1.1)$$
where \( W_p \) = pinch off potential
2\( \delta \) = channel thickness (Fig. 1)
2\( \alpha \) = distance between the two gates

The value of \( W \) at the source is given by

\[ W = W_s = V_g + \psi \]

and at the drain

\[ W = W_d = (V_g - V_d) + \psi. \]

\( V_g \) and \( V_d \) are the gate and drain voltages referred to the source, respectively. In the above equation, \( \psi \) is the equilibrium barrier potential and is a negative quantity for an F.E.T. with an n-type channel.

The \( x \)-component of the electric field, \( E_x \), at a distance \( x \) from the source along the channel is

\[ E_x = \frac{dW}{dx}. \]  \hspace{1cm} (1.2)

Hence, the channel current is given by

\[ I_d = -2\pi b \cdot E_x \]

\[ = -2\pi \sigma a \left( \frac{b}{a} \right) \frac{dW}{dx}. \]  \hspace{1cm} (1.3)

where \( \sigma \) = conductivity of the channel

\( Z \) = width of the channel.

Shockley has shown that the channel current is given by
\[ I_d = -\frac{2Z\alpha \sigma_w W_p}{L} \left[ \frac{W_d}{W_p} - \frac{W_s}{W_p} - \frac{2}{3} \left( \frac{W_d^3}{W_p^3} \right) - \left( \frac{W_s}{W_p} \right)^2 \right] \]  

(1.4)

where \( L \) is the length of the device. The transconductance, \( g_{mo} \), of the device is therefore given by

\[ g_{mo} = \frac{dI_d}{dV_g} \bigg|_{V_d=\text{const}} = \frac{2Z\alpha \sigma_w}{L} \left[ \left( \frac{W_d}{W_p} \right)^2 - \left( \frac{W_s}{W_p} \right)^2 \right] \]  

(1.5)

The output conductance, \( \frac{1}{R_o} \), of the device becomes

\[ \frac{1}{R_o} = \frac{dI_d}{dV_d} \bigg|_{V_g=\text{const}} = \frac{2Z\alpha \sigma_w}{L} \left[ 1 - \left( \frac{W_d}{W_p} \right)^2 \right] \]  

(1.6)

The equivalent circuit of the F.E.T. for the low frequency case takes the form shown in Fig. 1.3.

---

**Fig. 1.3** Small-signal low frequency equivalent circuit of the F.E.T.
Shockley's theory has been extended\textsuperscript{2-6} to include the effects of changes in the mobility of the carriers with the electric field and non-uniform channel doping.

A number of high frequency equivalent circuits have been proposed\textsuperscript{7,8,9,10,11,12} for the F.E.T. In general, these take the form shown in Fig. 1.4. Note that the equivalent circuit of Fig. 1.3 is of this form ($y_1,y_2 = 0$, $y_m = g_{m0}$ and $y_0 = \frac{1}{R_0}$). An improvement over the equivalent circuit of Fig. 1.3 has been made by van der Ziel\textsuperscript{7} and others\textsuperscript{8} by considering charge storage in the space-charge regions of the p-n junction of the device. Analyses of this type lead to the conclusion that $y_1$ consists of a capacitance $C_1$ and $y_2$ consists of a capacitance $C_2$ and expressions for $C_1$ and $C_2$ as functions of the bias voltages have been presented. Olsen\textsuperscript{9} has suggested that $y_0$ consists of $\frac{1}{R_0}$ in parallel with a capacitance $C_0$.

The solid-state devices group at Texas Instruments Inc.\textsuperscript{10} has suggested the equivalent circuit shown in Fig. 1.5. This circuit is based on a qualitative analysis of the field-effect transistor operation. The resistors $R_3$ and $R_4$ are the d.c. leakage resistances of the reverse biased p-n junctions while $R_5$ and $R_6$ are parasitic resistances associated with the source and drain contacts.

Silverthorn's\textsuperscript{11,12} equivalent circuit (Fig. 1.6) is a simplification of the Texas Instruments Inc. equivalent circuit valid for the frequency range in which $R_3$ and $R_4$ can be
Fig. 1.4 General form of the F.E.T. equivalent circuit.

Fig. 1.5 An equivalent circuit for the F.E.T. (suggested by Texas Instruments Inc.)

Fig. 1.6 Silverthorn's equivalent circuit for the F.E.T.
neglected. He has shown that \( R_5 \) and \( R_6 \) can be absorbed into \( R_1 \) and \( R_2 \) over a wide frequency range and has presented a method for obtaining the equivalent circuit elements from measurements of the short-circuit admittance parameters.

Ero and van der Ziel have presented an improved small-signal high-frequency analysis of the F.E.T. In this theory, the device is considered to be an active, distributed, non-uniform transmission line and the wave equation for this structure has been solved by an approximate method. The expressions for the short-circuit admittance parameters have been derived for the saturation (pinch-off) case. Expressions for \( C_1 \) and \( R_1 \) have been obtained and it has been demonstrated that the value of \( C_1 \) so computed is in agreement with that obtained from a charge storage analysis. These computations show that \( y_2 = 0 \) for the pinch-off case in agreement with the previous computations showing that \( C_2 = 0 \) at pinch-off.

1.3 An Outline of the Scope of This Work

The purpose of this work is to derive an intrinsic high frequency equivalent circuit of the F.E.T. A differential equation for the a.c. case is derived from a knowledge of the physics of the device and a series solution to this equation is obtained. The expressions for short-circuit admittance parameters of the device are derived from the solution. A hybrid-\( \pi \)-network of a general form is assumed to represent the equivalent circuit and the expressions for the circuit
elements are determined by comparing the admittance parameters of the hybrid-w-circuit with the derived expressions.

According to van der Ziel's terminology, the d.c. theory of the device, given in Section 1.2, is called the 'zero-order approximation', which means that this theory is for the case of (jω) approaching zero. The resulting equivalent circuit, shown in Fig. 1.3, has therefore no reactive elements. The 'first-order approximation', then, is taken to mean that only the d.c. and (jω) terms are retained in the theory and the higher order terms in (jω) are neglected. The equivalent circuit for this case will have one reactive element in each branch. In this thesis, the equivalent circuit is derived from the 'first-order approximation' of the theory of the device.

The analysis presented in this work is of a more general nature than van der Ziel's analysis because the expressions for the short-circuit admittance parameters and the circuit elements derived here are applicable to both the saturated and the non-saturated conditions.
2. THE DIFFERENTIAL EQUATION AND ITS SOLUTION

2.1 Introduction

The F.E.T. is considered as an active, distributed and non-uniform transmission line and a differential equation of this structure is derived for the small-signal case. It is assumed that the channel is uniformly doped and that the p-n junctions are of the abrupt type. A solution of the differential equation is obtained in the form of a power series which gives the channel current in terms of the channel thickness.

2.2 The Differential Equation

In the small signal case, the d.c. values of channel bias and current have superimposed upon them small sinusoidal a.c. voltage and current variations; hence the expressions for the total channel bias and current can be written as follows:

\[ W' = W + v e^{j\omega t} \]  \hspace{1cm} (2.1)
\[ I_d' = I_d + i e^{j\omega t} \]  \hspace{1cm} (2.2)

where \( v \) and \( i \) are the a.c. components of channel bias and current, respectively.

The statement of Ohm's law as given by equation (1.3) must be augmented by the charge continuity equation in order to derive the differential equation.
Fig. 2.1 Current flow in a section \( x \) of the device.

Consider a volume element of width \( \Delta x \) at a distance \( x \) from the source of an F.E.T. with an n-type channel. Under operating conditions, the channel current will flow as shown in Fig. 2.1. The volume of this element is \( 2aZ\Delta x \) and the net charge \( \Delta Q \) contained in it is given by

\[
\Delta Q = 2(1 - \frac{b'}{a})aZ\rho_0\Delta x
\]

where \( 2b' \) is the instantaneous value of the channel height and \( \rho_0 \) is the charge density of the space charge regions. From equation (1.1) one obtains

\[
1 - \frac{b}{a} = 1 - y = \left(\frac{W}{W_p}\right)^\frac{1}{2}
\]

(2.1)

where \( y = \frac{b}{a} \). Similarly, for the a.c. case,

\[
1 - \frac{b'}{a} = 1 - y' = \left(\frac{W'}{W_p}\right)^\frac{1}{2}
\]

(2.2)

where \( y' \) is the instantaneous value of \( y \). Therefore the expression for \( \Delta Q \) becomes
\[ \Delta Q = 2az \rho_o (\frac{W'}{W_P})^\frac{1}{3} \Delta x \quad (2.5) \]

Considerations of charge continuity for this element require that:

\[ I_d' = (I_d' + \frac{\partial I_d'}{\partial x} \Delta x) = -\frac{\partial}{\partial t} (\Delta Q) \]

or

\[ \frac{\partial I_d'}{\partial x} = \frac{\partial}{\partial t} \left[ 2az \rho_o (\frac{W'}{W_P})^\frac{1}{3} \right] \]

where \( t \) is the time.

Since \( \frac{\partial I_d}{\partial x} = 0 \), use of equations (2.1) and (2.2) yields

\[ \frac{\partial I}{\partial x} e^{j\omega t} = 2az \rho_o \frac{\partial}{\partial t} \left[ (\frac{W}{W_P})^\frac{1}{3} (1 + \frac{v}{W} e^{j\omega t})^\frac{1}{3} \right] \]

For the small signal case \((\frac{v}{W})\) is a small quantity and hence the second and higher order terms in \((\frac{v}{W})\) can be neglected.

Thus the above equation reduces to

\[ \frac{\partial I}{\partial x} e^{j\omega t} = 2az \rho_o \left[ (\frac{W}{W_P})^\frac{1}{3} \cdot \frac{\frac{v}{W}}{3} \omega e^{j\omega t} \right] \]

With the aid of (2.3), this can be written as

\[ \frac{\partial I}{\partial x} = \frac{az \rho_o}{W_P (1-y)} j\omega v \quad (2.6) \]

Equation (1.3) can be written for the a.c. case as follows:

\[ I_d' = -A \cdot y' \cdot \frac{dW'}{dx} \quad (2.7) \]

where \( A = 2az \sigma_o \).
Use of equations (2.1), (2.2), and (2.4) in (2.7) yields

$$I_d + i e^{j \omega t} = -A \left[ 1 - \left( \frac{W'}{W_p} \right)^{\frac{3}{2}} \right] \left( \frac{dW}{dx} + \frac{dv}{dx} e^{j \omega t} \right)$$

$$= -A \left[ 1 - \left( \frac{W}{W_p} \right)^{\frac{3}{2}} \left( 1 + \frac{v}{W} e^{j \omega t} \right)^{\frac{3}{2}} \right] \left( \frac{dW}{dx} + \frac{dv}{dx} e^{j \omega t} \right)$$

On neglecting the second and higher order terms in \( \left( \frac{V}{W} \right) \), this reduces, with the aid of (2.3), to

$$I_d + i e^{j \omega t} = -A \cdot y \cdot \frac{dW}{dx} + A \cdot \frac{A}{\xi} (1 - y) \frac{v}{W} \frac{dW}{dx} e^{j \omega t} - A \cdot y \cdot \frac{dv}{dx} e^{j \omega t} \tag{2.8}$$

The time-independent part of (2.8) along with (2.3) gives

$$\frac{dy}{dx} = \frac{I_d/I_o}{a(1-y)y} \tag{2.9}$$

where \( I_o = 4Z \sigma_o W_p \).

Equations (2.3), (2.9), and the time-dependent part of (2.8) can be manipulated to yield

$$\frac{dv}{dy} = - \frac{1(1-y)a}{2Z \sigma_o a(I_d/I_o)} - \frac{v}{y} \tag{2.10}$$

Dividing (2.6) by (2.9), one obtains

$$\frac{d1}{dy} = \frac{\xi^2 \rho_o}{W_p} \cdot \frac{i \omega y v}{(I_d/I_o)} \tag{2.11}$$

The pinch-off potential \( W_p \) is given by

$$W_p = - \frac{\xi^2}{2 \xi} \rho_o \left[ 1 - \frac{\rho_o}{\rho_g} \right] \tag{2.12}$$
where $\varepsilon$ is the permittivity of the channel material and $\rho_g$ is the charge density of the gate regions. Use of (2.12) in (2.11) gives

$$\frac{dl}{dy} = -\frac{2\varepsilon \omega \omega y}{(I_d/I_0)_D}$$

Equations (2.10) and (2.13) can be combined to yield

$$\frac{d^2 l}{dy^2} + 2k^2 y(1-y) l = 0$$

where

$$k^2 = -\frac{\omega}{2\sigma (I_d/I_0)^2} \frac{\varepsilon}{(I_d/I_0)^2}$$

and

$$\sigma = \sigma_c D = \sigma_c (1 - \frac{\rho_o}{\rho_g})$$

Equation (2.14) is the required differential equation which expresses the channel current $l$ in terms of the variable parameter $y$ for the a.c. case.

2.3 Solution of the Differential Equation

The differential equation (2.14) is assumed to have a series solution of the following type.

$$i(y) = \sum_{n=0}^{\infty} C_n y^n$$
Hence
\[ i''(y) = \sum_{\gamma=2}^{\infty} \gamma(\gamma-1)C_\gamma y^{\gamma-2} \]
\[ = 2C_2 + 6C_3 y + \sum_{\gamma=4}^{\infty} \gamma(\gamma-1)C_\gamma y^{\gamma-2} \]
and
\[ 2k^2y(1-y) \sum_{\gamma=0}^{\infty} C_\gamma y^\gamma = 2k^2C_0 y - 2k^2C_0 y^2 + 2k^2C_1 y^2 \\
- 2k^2C_1 y^3 + 2k^2C_2 y^3 \ldots \ldots . \]
\[ = 2k^2C_0 y + \sum_{\gamma=2}^{\infty} 2k^2(C_\gamma - C_{\gamma-2}) y^\gamma \]
so that the equation (2.14) becomes
\[ 2C_2 + 6C_3 y + \sum_{\gamma=4}^{\infty} \gamma(\gamma-1)C_\gamma y^{\gamma-2} + 2k^2C_0 y \\
+ \sum_{\gamma=2}^{\infty} 2k^2(C_\gamma - C_{\gamma-2}) y^\gamma = 0 \]
This can be written as
\[ 2C_2 + 6C_3 y + 2k^2C_0 y + \sum_{\gamma=4}^{\infty} \gamma(\gamma-1)C_\gamma y^{\gamma-2} \\
+ \sum_{\gamma=4}^{\infty} 2k^2(C_\gamma - C_{\gamma-4}) y^{\gamma-2} = 0 \quad (2.17) \]
It is clear that the coefficient of each term should be separately equal to zero.
\[ 2C_2 = 0 \quad \text{or} \quad C_2 = 0 \]
\[ 6C_3 + 2k^2C_0 = 0 \quad \text{or} \quad C_3 = -\frac{k^2}{3} C_0 \]
The last two terms of the equation (2.17) give the recursion formula
\[ c_y = \frac{2k^2(c_y - 4c_y - 3)}{y(y-1)} \text{ for } y > 4. \] (2.18)

Hence
\[ c_4 = \frac{k^2}{6} (c_0 - c_1) \]
\[ c_5 = \frac{k^2}{10} (c_1 - c_2) = \frac{k^2}{10} c_1 \text{ and so on.} \]

It is therefore obvious that \( C_y \) can be expressed as follows:
\[ C_y = \alpha_y c_0 + \beta_y c_1, \quad 0 \leq y \leq \infty \] (2.19)

where \( C_0 \) and \( C_1 \) are arbitrary quantities which can be determined by means of boundary conditions.

The most general solution of the equation (2.14) is therefore given by
\[ i(y) = c_0 \sum_{y=0}^{\infty} \alpha_y y^y + c_1 \sum_{y=0}^{\infty} \beta_y y^y \] (2.20)

where \( \alpha_y \) and \( \beta_y \) can be determined by the use of the relations (2.18) and (2.19).

Equations (2.18) and (2.19) yield the following values for the coefficients \( \alpha \)'s and \( \beta \)'s.

\[
\begin{align*}
\alpha_0 &= 1 & \beta_0 &= 0 \\
\alpha_1 &= 0 & \beta_1 &= 1 \\
\alpha_2 &= 0 & \beta_2 &= 0 \\
\alpha_3 &= -\frac{k^2}{3} & \beta_3 &= 0 \\
\alpha_4 &= \frac{k^2}{6} & \beta_4 &= -\frac{k^2}{6}
\end{align*}
\]
\[ \alpha_5 = 0 \quad \beta_5 = \frac{k^2}{10} \]
\[ \alpha_6 = \frac{k^4}{45} \quad \beta_6 = 0 \]
\[ \alpha_7 = -\frac{k^4}{42} \quad \beta_7 = \frac{k^4}{126} \]
\[ \alpha_8 = \frac{k^4}{168} \quad \beta_8 = \frac{k^4}{105} \]
\[ \alpha_9 = \frac{k^6}{1620} \quad \beta_9 = \frac{k^4}{360} \]
\[ \alpha_{10} = \frac{29}{28350} k^6 \quad \vdots \]
\[ \alpha_{11} = \frac{k^6}{1848} \quad \vdots \]
\[ \alpha_{12} = \frac{1149}{124740} k^6 \quad \vdots \]

It will be useful to recall the definition of \( k^2 \) as given by the relation (2.15)

\[ k^2 = -\omega \left[ \frac{\epsilon}{2\sigma(I_d/I_0)^2} \right] \]

It was mentioned in section 1.3 that only the first-order approximation of the theory was going to be used in deriving the equivalent circuit. After carrying out the calculations, it was found that the second order terms in (j\omega) had to be retained in the numerator of the expression for the channel current as they give rise to an additional (j\omega) term. This
fact will become clear in chapter 4. Therefore, those values of the coefficients \( \alpha \) and \( \beta \) which give rise to \( (j\omega)^2 \) terms will also be retained in the expression for the channel current. Higher order terms in \( (j\omega) \) will be neglected.

When the above values of the coefficients are substituted, equation (2.20) becomes

\[
I(y) = c_0 \cdot T(y) + c_1 \cdot R(y)
\]  

(2.21)

where

\[
T(y) = 1 - \frac{1}{3}k^2y^3 + \frac{1}{6}k^2y^4 + \frac{1}{45}k^4y^6 - \frac{1}{12}k^4y^7 + \frac{1}{16}k^4y^8
- \frac{1}{1620}k^6y^9 + \frac{29}{26350}k^6y^{10} - \frac{1}{1848}k^6y^{11} + \frac{1}{121740}k^6y^{12}
\]

(2.22)

\[
R(y) = y - \frac{1}{6}k^2y^4 + \frac{1}{10}k^2y^5 + \frac{1}{126}k^4y^7 - \frac{1}{105}k^4y^8 + \frac{1}{360}k^4y^9
\]

(2.23)

Equation (2.21) is the second-order approximation of the general solution (2.20).
3. EXPRESSIONS FOR ADMITTANCE PARAMETERS

3.1 Introduction

The short-circuit admittance parameters of the F.E.T. are determined in this chapter from the expression for the channel current which was obtained in the previous section. The coefficients $C_0$ and $C_1$ contained in that expression are evaluated by means of the boundary conditions in both the short-circuited output and the short-circuited input cases.

3.2 Evaluation of $C_0$ and $C_1$ for the Short-Circuited Output Case.

The channel current is given by equation (2.21)

$$i(y) = C_0 \cdot T(y) + C_1 \cdot R(y)$$

Hence

$$\frac{di}{dy} = C_0 \cdot T'(y) + C_1 \cdot R'(y)$$

This expression is substituted in equation (2.13) to give

$$v = -\left(\frac{I_d}{I_o}\right)\frac{1}{(j\omega)2\pi\xi}\left[C_0 \cdot \frac{T'(y)}{y} + C_1 \cdot \frac{R'(y)}{y}\right]$$

(3.1)

In order to determine the input admittance of the device, the output has to be short-circuited. Therefore, the drain is a.c. short-circuited to the source and this boundary condition gives
\[ y = y_s \] at the source \hspace{1cm} (3.2a)

\[ v = v_g \]

and

\[ y = y_d \] at the drain \hspace{1cm} (3.2b)

where \( v_g \) = a.c. potential of the gate with respect to the source.

When the values (3.2a) are substituted, equation (3.1) becomes (at the source)

\[ -\frac{v_g}{B} = c_0 \left[ \frac{T'(y_s)}{y_s} \right] + c_1 \left[ \frac{R'(y_s)}{y_s} \right] \] \hspace{1cm} (3.3)

where \( B = \frac{I_d}{I_0} \omega \omega \),

\[ \frac{T'(y_s)}{y_s} = -k^2 \left[ y_s - \frac{2}{3} y_s^2 - \frac{2}{15} k^2 y_s^4 + \frac{1}{6} k^2 y_s^5 - \frac{1}{2} k^2 y_s^6 
+ \frac{1}{18} k^4 y_s^7 - \frac{29}{2835} k^4 y_s^8 + \frac{1}{168} k^4 y_s^9 - \frac{149}{124770} y_s^{10} \right] \]

\[ = -k^2 \cdot Y_{s1} \]

and

\[ \frac{R'(y_s)}{y_s} = \frac{1}{y_s} - \frac{2}{3} k^2 y_s^2 + \frac{1}{6} k^2 y_s^3 + \frac{1}{10} k^4 y_s^5 - \frac{3}{105} k^4 y_s^6 + \frac{1}{40} k^4 y_s^7 \]

\[ = Y_{s2} \]

Equation (3.3) can therefore be written as

\[ -\frac{v_g}{B} = -c_0 k^2 \cdot Y_{s1} + c_1 \cdot Y_{s2} \] \hspace{1cm} (3.4)
Similarly, use of relations (3.2b) in equation (3.1) gives (at the drain)

\[- \frac{V^E_B}{B} = - c_0 k^2 Y_{d1} + c_1 Y_{d2} \]  \hspace{1cm} (3.5)

where \( Y_{d1} = \frac{T'(Y_d)}{k^2 Y_d} \)

and \( Y_{d2} = \frac{R'(Y_d)}{Y_d} \)

Equations (3.4) and (3.5) yield

\[- \frac{V^E_B}{B} Y_{d2} = - c_0 k^2 Y_{s1} Y_{d2} + c_1 Y_{s2} Y_{d2} \]  \hspace{1cm} (3.6a)

\[- \frac{V^E_B}{B} Y_{s2} = - c_0 k^2 Y_{d1} Y_{s2} + c_1 Y_{d2} Y_{s2} \]  \hspace{1cm} (3.6b)

\( c_0 \) is given by subtracting equation (3.6b) from (3.6a).

\[ c_0 = \frac{\frac{V^E_B}{B} (Y_{s2} - Y_{d2})}{k^2 (Y_{d1} Y_{s2} - Y_{s1} Y_{d2})} \]

As explained in section 2.3, \((j\omega)^2\) terms are retained in the numerator of the expression for current, whereas in the denominator they will be neglected as they are not necessary for the first-order approximation of the theory. After simplification, \( c_0 \) reduces to

\[ c_0 = \frac{\frac{V^E_B}{B} Y_1}{k^2 \Delta} \]  \hspace{1cm} (3.7)
where
\[
Y_1 = \frac{1}{y_s y_d} (u-1) - k^2 y_s \left\{ \frac{2}{3} (1-u^2) - \frac{2}{3} y_s (1-u^3) \right\}
\]
\[
+ k^4 y_s \left\{ \frac{1}{10} (1-u^5) - \frac{8}{105} y_s (1-u^6) + \frac{1}{40} y_s^2 (1-u^7) \right\}
\]
\[
\Delta = \frac{2}{3} \frac{1}{u} (1-u^3) - \frac{1}{y_d} (1-u^2) - \frac{2}{3} k^2 y_s^2 u \left\{ 1 - u - \frac{3}{4} y_s (1-u^2) \right\}
\]
\[
+ \frac{1}{2} y_s^2 u (1-u) - \frac{1}{5} \frac{1}{u^2} (1-u^5) + \frac{1}{y_s u^2} (1-u^6)
\]
\[
- \frac{1}{10} y_s^2 \frac{1}{u} (1-u^7) \}
\]
and
\[
u = \frac{y_d}{y_s} \]  

The coefficient $C_1$ is determined in the same way. Equations (3.4) and (3.5) can be written as
\[
- \frac{v}{B} \cdot Y_{dl} = -C_0 k^2 \cdot Y_s1 \cdot Y_{dl} + C_1 \cdot Y_s2 \cdot Y_{dl}
\]
\[
- \frac{v}{B} \cdot Y_{s1} = -C_0 k^2 \cdot Y_{dl} \cdot Y_{s1} + C_1 \cdot Y_{d2} \cdot Y_{s1}
\]
These two relations are solved for $C_1$.
\[
C_1 = \frac{\frac{v}{B} (Y_{s1} - Y_{dl})}{Y_{s2} \cdot Y_{dl} - Y_{d2} \cdot Y_{s1}}
\]
The terms containing $k^4$ and $k^6$ are again neglected in the denominator. The expression for $C_1$ is then simplified and rearranged to give
\[
C_1 = \frac{\frac{v}{B} \cdot Y_2}{\Delta} \]
where
\[ Y_2 = 1 - u - \frac{2}{3} y_s (1-u^2) - \frac{2}{15} k^2 y_s^3 (1-u^4) + \frac{1}{6} k^2 y_s^4 (1-u^5) \]
\[ - \frac{1}{21} k^2 y_s^5 (1-u^6) + \frac{1}{180} k^4 y_s^6 (1-u^7) - \frac{29}{2835} k^4 y_s^7 (1-u^8) \]
\[ + \frac{1}{168} k^4 y_s^8 (1-u^9) - \frac{1149}{124740} k^4 y_s^9 (1-u^{10}) \]

\( \Delta \) and \( u \) are given by the equations (3.8) and (3.9), respectively.

### 3.3 Expression For Channel Current

According to equation (2.15),
\[ \frac{-j \omega \xi}{I_d} = k^2 \]
\[ \frac{2 \sigma(I_d)}{I_0} \]

or
\[ \frac{-j \omega \xi}{I_d} = k^2 \cdot 4 \pi \sigma \frac{I_d}{I_0} \]
\[ \frac{2 \pi \sigma(I_d)}{I_0} \]

Therefore,
\[ \frac{j \omega \cdot 2 \pi E}{I_d} = -k^2 \cdot 4 \pi \sigma \frac{I_d}{I_0} \]
\[ \frac{1}{B} = -k^2 \cdot 4 \pi \sigma \frac{I_d}{I_0} \] (3.11)

Equations (3.7), (3.10) and (3.11) are used in (2.25) to get the general expression for the channel current.
\[ i(y) = C_0 \cdot T(y) + C_1 \cdot R(y) \]

\[ = \frac{V_g}{E} \cdot \frac{Y_1}{k^2} \cdot T(y) + \frac{V_g}{\Delta} \cdot \frac{Y_2}{k^2} \cdot R(y) \]

\[ = -k^2 (\frac{1}{Z_s} I_d) \frac{V_g}{\Delta} \left\{ \frac{Y_1}{k^2} \cdot T(y) + \frac{Y_2}{k^2} \cdot R(y) \right\} \]

Third and higher order terms in \((j\omega)\) are again neglected. It can be shown that when simplified and rearranged, the expression for current reduces to

\[ i(y) = \frac{I_d}{\Delta} \frac{V_g}{\sum} \left[ \frac{1}{Y_s Y_d} (1-u) + k^2 Y_s \left\{ \frac{2}{3} (1-u^2) - \frac{1}{3} Y_s (1-u^3) \right\} \right. \]

\[ - \frac{1}{3} \frac{Y^3}{Y_s^2 Y_d} (1-u) + \frac{1}{6} \frac{Y^4}{Y_s^2 Y_d} (1-u) - \frac{Y}{Y_s} (1-u) + \frac{2}{3} Y (1-u^2) \]

\[ + k^4 Y_s \left\{ - \frac{1}{10} (1-u^5) + \frac{1}{105} Y_s (1-u^6) - \frac{1}{40} Y_s^2 (1-u^7) \right\} \]

\[ - \frac{2}{9} \frac{Y^3}{Y_s^3} (1-u^2) + \frac{1}{6} \frac{Y^3}{Y_s^2} (1-u^3) + \frac{1}{9} \frac{Y^4}{Y_s^3} (1-u^2) \]

\[ - \frac{1}{12} \frac{Y^4}{Y_s^2} (1-u^3) + \frac{1}{45} \frac{Y^6}{Y_s^2 Y_d} (1-u) - \frac{1}{42} \frac{Y^7}{Y_s^2 Y_d} (1-u) \]

\[ + \frac{1}{168} \frac{Y^8}{Y_s^5 Y_d} (1-u) + \frac{2}{15} \frac{Y_s (1-u^4) - \frac{1}{6} Y_s (1-u^5) + \frac{1}{27} Y_s y (1-u^6)}{Y_s} \]

\[ + \frac{1}{6} \frac{Y^4}{Y_s^2} (1-u) - \frac{1}{9} \frac{Y^4}{Y_s^3} (1-u^2) + \frac{1}{10} \frac{Y^5}{Y_s^4} (1-u) + \frac{1}{15} \frac{Y^5}{Y_s^3} (1-u^2) \]

\[ (3.12) \]
This is the expression for current at any point along the channel for the case when the drain is a.c. short-circuited to the source.

3.4 The D.C. Case

It is of interest to see what the expression (3.12) for the channel current yields in the d.c. case. Before considering this case, it is necessary to determine \( \frac{I_d}{I_0} \) in terms of \( y \).

Equation (2.8) gives

\[
\frac{1}{a} \int_0^a \frac{I_d}{I_0} \, dx = y(1-y) \, dy
\]

which becomes, on integration

\[
\int_0^L \frac{1}{a} \frac{I_d}{I_0} \, dx = \int_y^0 y(1-y) \, dy
\]

or,

\[
\frac{I_d}{I_0} \left( \frac{L}{a} \right) = \frac{y_d^2}{2} - \frac{y_d^3}{3} - \frac{y_s^2}{2} + \frac{y_s^3}{3}
\]

\[
= - \frac{1}{3} (y_s^2 - y_d^2) + \frac{1}{3} (y_s^3 - y_d^3)
\]

\[
= - \frac{y_s^2}{2} \left[ 1 - u^2 - \frac{2}{3} y_s (1-u^3) \right]
\]

Hence

\[
\frac{I_d}{I_0} = - \frac{a}{2\pi} \cdot y_s^2 \left[ 1 - u^2 - \frac{2}{3} y_s (1-u^3) \right]
\]

(3.13a)

As \( \omega \to 0 \) (d.c. case) equation (3.12) reduces to
On substituting the value of \( \frac{I_d}{I_o} \) from equation (3.13), this expression becomes

\[
\frac{I_d}{V_g} = \frac{1}{v_g} \left( \frac{4Z\sigma \cdot I_o \cdot \frac{1}{y_s} y_d(1-u)}{2 \frac{1}{3} u (1-u^3) - \frac{1}{y_d} (1-u^2)} \right)
\]

\[
= \frac{I_d}{I_o} \cdot \frac{1}{y_s} (1-u) \left( \frac{1}{y_d} \right)
\]

\[
= \left\{ \frac{1}{1 - u^2 - \frac{2}{3} y_s (1-u^3)} \right\} \left( \frac{1}{y_d} \right)
\]

which agrees with the equation (1.5)

3.5 Expression for \( \Delta \)

It will be useful to reduce the expression for \( \Delta \) to a convenient form. Equation (3.8) can be written as

\[
\Delta = \frac{2}{3u} (1-u^3) - \frac{1}{y_d} (1-u^2) - \frac{2}{3} k^2 y_s^2 u (Y_3)
\]

where

\[
Y_3 = 1 - u - \frac{3}{4} y_s (1-u^2) + \frac{1}{2} y_s^2 u (1-u) - \frac{1}{5u^2} (1-u^5)
\]

\[
+ \frac{1}{4u^2} y_s (1-u^6) - \frac{1}{14u^2} y_s^2 (1-u^7)
\]
and \( u = \frac{Y_d}{Y_s} \)

Hence
\[
\Delta = \left[ \frac{2}{3u} (1-u^3) - \frac{1}{Y_d} (1-u^2) \right] \left[ 1 - \frac{2}{3u} k \frac{k^2 Y_s^2 u Y_3}{(1-u^3) - \frac{1}{Y_d} (1-u^2)} \right]
\]
\[
= \left[ \frac{2}{3u} (1-u^3) - \frac{1}{Y_d} (1-u^2) \right] \left[ 1 + j\omega \frac{\xi}{2\sigma} \frac{(1-d)}{(1-d)^2} \frac{1}{\frac{1}{Y_s}} \left[ 1 - \frac{2}{3u} \frac{Y_s^2 u Y_3}{(1-u^3) - \frac{1}{Y_d} (1-u^2)} \right] \right]
\]

It can be shown that this expression reduces, with the aid of (3.13a), to
\[
\Delta = \left[ \frac{2}{3u} (1-u^3) - \frac{1}{Y_d} (1-u^2) \right] \left[ 1 + j\omega \tau_1 \right]
\]

(3.14a)

where
\[
\tau_1 = \frac{\xi^2}{15\sigma^2}, \quad \frac{1}{\frac{Y_s}{Y_0}} \frac{Y_4}{Y_0^3}
\]

\[
Y_0 = 1 - u^2 - \frac{2}{3} Y_s (1-u^3)
\]

(3.14b)

and
\[
Y_4 = 1 - 5u^2 + 5u^3 - u^5 - \frac{5}{4} Y_s (1-3u^2+3u^4-u^6)
\]
\[
+ \frac{5}{14} Y_s^2 (1-7u^3+7u^4-u^7)
\]

3.6 Expression for \( Y_{21} \)

When \( y = Y_d \), equation (3.12) yields
\[
\frac{i_d}{v_g} = \frac{I_d}{v_d} \left[ \frac{1}{\Delta} \cdot \frac{1}{y_s y_d} \right] \\
= \frac{4Z_{10} I_d}{\Delta} \left[ \frac{1}{y_s y_d} \left( 1 - u \right) + k^2 y_s \left\{ \frac{2}{3} (1-u^2) - \frac{2}{3} y_s (1-u^3) - \frac{1}{3} u^2 (1-u) \right\} \\
+ \frac{1}{6} y_s u^3 (1-u) - u(1-u) + \frac{2}{3} y_s u (1-u^2) \right\} + k^4 y_s \left\{ - \frac{1}{18} (1-u^5) \right\} \\
+ \frac{\delta}{10^5} y_s (1-u^6) - \frac{1}{40} y_s^2 (1-u^7) - \frac{2}{5} y^3 u^2 (1-u^2) + \frac{1}{6} y_s u^3 (1-u^3) \\
+ \frac{1}{8} y_s u^4 (1-u^2) - \frac{1}{12} y_s^2 u^4 (1-u^3) + \frac{1}{15} u^5 (1-u) \\
- \frac{1}{42} y_s u^6 (1-u) + \frac{1}{180} y_s^2 u^7 (1-u) + \frac{2}{15} u (1-u^4) - \frac{1}{6} y_s u (1-u^5) \\
+ \frac{1}{21} y_s^2 u (1-u^6) + \frac{1}{6} y_s u^4 (1-u) - \frac{1}{6} y_s u^4 (1-u^2) - \frac{1}{10} y_s^5 (1-u) \\
+ \frac{1}{15} y_s^2 u^5 (1-u^2) \right\} \\
\right]
\]

which becomes, on simplification,

\[
\frac{i_d}{v_g} = \frac{I_d}{v_d} \left[ \frac{1}{\Delta} \cdot \frac{1}{y_s y_d} \right] \left\{ \frac{1}{3} + \frac{2}{3} k^2 y_s^2 y_d \left\{ 1 + \frac{2}{3} u^2 - \frac{2}{3} y_s (1 - \frac{1}{3} u - \frac{1}{3} u^2 \right\} \\
- \frac{1}{3} y^3 \right\} - \frac{1}{15} k^4 y_s^5 y_d \left\{ 1 - \frac{7}{5} u - \frac{7}{5} u^2 + \frac{13}{5} u^3 - \frac{2}{5} y_s u^4 \\
- \frac{2}{5} y_s^2 \left( 1 - \frac{19}{21} u - \frac{19}{21} u^2 + \frac{19}{21} u^3 + u^4 - \frac{5}{16} u^5 - \frac{5}{16} u^6 \right) \\
+ \frac{9}{20} y_s^2 \left( 1 - \frac{19}{21} u - \frac{19}{21} u^2 - \frac{19}{21} u^3 + \frac{51}{21} u^4 - \frac{5}{21} u^5 - \frac{5}{21} u^6 - \frac{5}{21} u^7 \right) \right\} \\
\right]
\]

Using equations (2.15a), (3.13a), (3.14a) and (3.15b) it can be shown that this expression takes the form

\[
\frac{i_d}{v_g} = Y_{21} = \frac{\delta m_0}{(1 + j \omega T_1)} \left[ 1 - j \omega T_2 (1 + j \omega T_3) \right] \\
(3.15)
\]
where

\[ \tau_2 = \frac{u \xi L^2}{3 \sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{1}{Y_0^2} \cdot u \cdot Y_5 \]

\[ \tau_3 = \frac{\xi L^2}{6 \sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{1}{Y_0^2} \cdot Y_6 \]

and

\[ Y_5 = 1 - 2u - 2u^2 - \frac{3}{4}y_s(1 - \frac{1}{3}u - \frac{1}{3}u^2 - \frac{1}{3}u^3) \]

\[ Y_6 = 1 - \frac{7}{2}u - \frac{7}{2}u^2 + \frac{13}{2}u^3 - \frac{2}{5}u^4 - \frac{2}{5}u^5 - \frac{1}{5}u^6 \]

\[ = \frac{1}{35}y_s(1 - \frac{19}{16}u - \frac{19}{16}u^2 + u^3 + u^4 - \frac{5}{16}u^5 - \frac{5}{16}u^6) \]

\[ + \frac{9}{20}y_s^2(1 - \frac{10}{21}u - \frac{10}{21}u^2 - \frac{10}{21}u^3 + \frac{5}{21}u^4 - \frac{5}{21}u^5 - \frac{5}{21}u^6 - \frac{5}{21}u^7) \]

3.7 Expression for \( Y_{11} \)

When \( y = y_s \) equation (3.12) becomes

\[ \frac{I(y_s)}{y_g} = \frac{4Z_0}{\Delta} \frac{I_0}{y_s Y_d} (1-u) + k^2 y_s \left[ \frac{2}{3}(1-u^2) - \frac{1}{2}y_s(1-u^3) - \frac{1}{3}u(1-u) \right] \]

\[ + \frac{1}{60}y_s(1-u)^2 - (1-u) + \frac{2}{3}y_s(1-u^2) \]

\[ + \frac{1}{40}y_s^2(1-u^6) - \frac{1}{15}y_s^2(1-u^7) - \frac{2}{9}(1-u^2) + \frac{1}{6}y_s(1-u^3) \]

\[ + \frac{1}{9}y_s(1-u^2) - \frac{1}{12}y_s^2(1-u^3) + \frac{1}{45} \frac{1}{u}(1-u) = \frac{1}{42} \frac{1}{u}y_s(1-u) \]

\[ + \frac{1}{168} \frac{1}{u}y_s^2(1-u) + \frac{1}{15}(1-u^4) - \frac{1}{5}y_s(1-u^5) + \frac{1}{21}y_s^2(1-u^6) \]

\[ + \frac{1}{6}(1-u) - \frac{1}{5}y_s(1-u^2) - \frac{1}{10}y_s(1-u) + \frac{1}{15}y_s^2(1-u^2) \]

This is simplified and rearranged to give
\[
\frac{I(y_s)}{V_g} = \frac{4Z_0 I_d (1-u)}{\Delta \cdot y_s y_d} \left[ 1 - \frac{1}{3} x^2 y_s^2 y_d \left\{ \frac{1}{u} + 1 - 2u - \frac{3}{2} y_s \left( \frac{1}{u} + 1 + u - 3u^2 \right) \right\} 
+ \frac{1}{4} y_s^2 y_d \left\{ \frac{1}{y_s^2} + 1 - \frac{13}{2} u + \frac{7}{2} u^2 + \frac{7}{2} u^3 - \frac{5}{3} u^4 \right\} 
- \frac{1}{14} y_s \left( \frac{1}{u} + 1 - \frac{16}{5} u - \frac{16}{5} u^2 + \frac{19}{5} u^3 + \frac{19}{5} u^4 - \frac{16}{5} u^5 \right) 
+ \frac{1}{16} y_s^2 \left( \frac{1}{u} + 1 + u - \frac{51}{5} u^2 + \frac{19}{5} u^3 + \frac{19}{5} u^4 + \frac{19}{5} u^5 - \frac{22}{5} u^6 \right) \right]\]

Use of equations (2.15a), (3.13a), (3.14a) and (3.13b) as in the previous case yields

\[
\frac{I(y_s)}{V_g} = \frac{g_m}{(1 + j \omega \tau_\gamma)} \left[ 1 + j \omega \tau_4 (1 + j \omega \tau_5) \right]
\]

(3.16)

where

\[
\tau_4 = \frac{2 \xi L}{3 \sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{Y_7}{Y_0^2}
\]

\[
\tau_5 = \frac{2 \xi L}{15 \sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{Y_8}{Y_0^2 Y_7}
\]

and

\[
Y_7 = 1 + u - 2u^2 - \frac{3}{2} y_s (1 + u + u^2 - 3u^3)
\]

\[
Y_8 = 1 + u - \frac{13}{2} u^2 + \frac{7}{2} u^3 + \frac{7}{2} u^4 - \frac{5}{2} u^5 - \frac{15}{14} y_s \left\{ 1 + u - \frac{16}{5} u^2 
- \frac{16}{5} u^3 + \frac{19}{5} u^4 + \frac{19}{5} u^5 - \frac{16}{5} u^6 \right\} + \frac{15}{56} y_s^2 \left\{ 1 + u 
+ u^2 - \frac{51}{5} u^3 + \frac{19}{5} u^4 + \frac{19}{5} u^5 + \frac{19}{5} u^6 - \frac{22}{5} u^7 \right\}
\]

\[
Y_0 = 1 - u^2 - \frac{3}{2} y_s (1-u^3)
\]

Since the net charge in the depletion regions is zero
one obtains
\[ i_d + i_s + i_g = 0 \]

\[ \frac{i_g}{V_g} = -\frac{i_s}{V_g} + \frac{i_d}{V_g} \]
\[ = \frac{i(y_s)}{V_g} - \frac{i_d}{V_g} \quad \text{(since } i(y_s) = -i_s) \]
\[ = \frac{\varepsilon_{mo}}{(1 + j\omega \tau_4)} \left[ 1 + j\omega \tau_4 (1 + j\omega \tau_5) - 1 + j\omega \tau_2 (1 + j\omega \tau_3) \right] \]
\[ = \frac{\varepsilon_{mo}}{(1 + j\omega \tau_4)} \left[ j\omega (\tau_4 + \tau_2) + (j\omega)^2 (\tau_4 \tau_5 + \tau_2 \tau_3) \right] \]

which can be written as
\[ \frac{i_g}{V_g} = y_{11} = \frac{\varepsilon_{mo}}{(1 + j\omega \tau_4)} j\omega \tau_6 (1 + j\omega \tau_7) \quad (3.17) \]

where
\[ \tau_6 = \frac{2\pi \ell^2}{3\alpha a^2} \cdot \frac{1}{y_s} \cdot \frac{Y_0}{Y_0^2} \]
\[ T_7 = \frac{2\alpha L^2}{15\sigma a^2} \cdot \frac{1}{Y_s} \cdot \frac{1}{Y_o^2} \cdot \frac{Y_{10}}{Y_9} \]

and

\[ Y_9 = 1 + 3u - 3u^2 - u^3 - \frac{3}{2}Y_s(1 + 4u - 4u^3 - u^4) \]

\[ Y_{10} = 1 + \frac{7}{2}u - 10u^2 + 10u^4 - \frac{7}{2}u^5 - u^6 \]
\[ - \frac{15}{4}Y_s(1 + \frac{21}{5}u - 7u^2 - 7u^3 + 7u^4 + 7u^5 - \frac{21}{5}u^6 - u^7) \]
\[ + \frac{15}{56}Y_s^2(1 + \frac{26}{5}u - \frac{11}{5}u^2 - 11u^3 + 11u^5 + \frac{11}{5}u^6 - \frac{26}{5}u^7 - u^8) \]

The input admittance \( Y_{11} \) of the device is therefore a complex quantity.

3.8 Evaluation of \( C_o \) and \( C_1 \) for the Short-Circuited Input Case

The remaining part of this chapter deals with the derivation of expressions for the short-circuit output admittance \( (Y_{22}) \) and the reverse transfer admittance \( (Y_{12}) \) of the device. In order to determine the output admittance, the input is s.c. short-circuited and, for this case, the boundary conditions are

\[ \begin{align*}
    & Y = Y_s \\
    & v = v_g = 0
\end{align*} \]

at the source \( (3.18a) \)

and \( Y = Y_d \)

\[ \begin{align*}
    & v = -v_d
\end{align*} \]

at the drain \( (3.18b) \)

These relations are substituted in equation \( (3.1) \) to give
\[ 0 = -C_0 k^2 Y_{s1} + C_1 Y_{s2} \quad \text{(at the source)} \quad (3.19) \]

and

\[ \frac{V_d}{B} = -C_0 k^2 Y_{d1} + C_1 Y_{d2} \quad \text{(at the drain)} \quad (3.20) \]

\[ B, Y_{s1}, Y_{s2}, Y_{d1} \] and \[ Y_{d2} \] have been defined in section (3.2).

The coefficients \( C_0 \) and \( C_1 \) are evaluated for this case by using the equations (3.19) and (3.20).

Equations (3.19) and (3.20) yield

\[ 0 = -C_0 k^2 \cdot Y_{s1} \cdot Y_{d2} + C_1 Y_{s2} \cdot Y_{d2} \quad (3.21) \]

\[ \frac{V_d}{B} \cdot Y_{s2} = -C_0 k^2 \cdot Y_{d1} \cdot Y_{s2} + C_1 Y_{d2} \cdot Y_{s2} \quad (3.22) \]

\( C_0 \) is determined by subtracting (3.22) from (3.21).

\[ C_0 = \frac{-\frac{V_d}{B} \cdot Y_{s2}}{k^2 [Y_{d1} \cdot Y_{s2} - Y_{s1} \cdot Y_{d2}]} \]

This can be rearranged and simplified to give

\[ C_0 = \frac{\frac{V_d}{B} \cdot Y_{11}}{k^2 \cdot \Delta} \quad (3.23) \]

where \( Y_{11} = \frac{1}{Y_{s2}^2} - \frac{2}{3} k^2 Y_s + \frac{1}{2} k^2 Y_s^2 + \frac{1}{10} k^4 Y_s^4 - \frac{8}{105} k^4 Y_s^5 + \frac{1}{40} k^4 Y_s^6 \)

and \( \Delta \) is given by equation (3.14a).

In order to determine \( C_1 \), equations (3.19) and (3.20) are written in the following form.
\[ 0 = -C_0 k^2 \cdot Y_{s1} \cdot Y_{d1} + C_1 \cdot Y_{s2} \cdot Y_{d1} \quad (3.24) \]

\[ \frac{v_d}{B} \cdot Y_{s1} = -C_0 k^2 \cdot Y_{d1} \cdot Y_{s1} + C_1 \cdot Y_{d2} \cdot Y_{s1} \quad (3.25) \]

Equations (3.24) and (3.25) are solved for \( C_1 \).

\[ C_1 = \frac{\frac{v_d}{B} \cdot Y_{s1}}{Y_{s2} \cdot Y_{d1} - Y_{d2} \cdot Y_{s1}} \]

It can be shown that this expression reduces to

\[ C_1 = \frac{\frac{v_d}{B} \cdot Y_{12}}{\Delta} \quad (3.26) \]

where \( Y_{12} = 1 - \frac{2}{3} v_s - k^2 (\frac{2}{15} v_s^3 - \frac{1}{6} v_s^4 + \frac{1}{24} v_s^5) \)

\[ + v_s^4 \left( \frac{1}{150} v_s^6 - \frac{29}{2835} v_s^7 + \frac{1}{168} v_s^8 - \frac{149}{124740} v_s^9 \right) \]

Substitution of equations (3.23) and (3.26) in (2.21) gives the expression for channel current in this case.

\[ i(y) = C_0 T(y) + C_1 R(y) \]

\[ = -\frac{v_d}{B \Delta} \left[ \frac{Y_{11}}{k^2} \cdot T(y) + Y_{12} \cdot R(y) \right] \]

Use of equations (3.11), (2.22) and (2.23) yields, after neglecting third and higher order terms in \( j\omega \),
\[ I(y) = \frac{\mu Zv_d I_d}{\Delta} \frac{I_d}{I_0} \left\{ \frac{1}{y_s^2} - \frac{1}{3} y_s \left[ \frac{3}{y_s} - \frac{2}{3} y_s + \frac{2}{y_s^3} \right] - \frac{1}{6} \frac{y^4}{y_s^3} \right. \\
- \frac{1}{y_s} (1 - \frac{2}{3} y_s) \right\} + \mu I_d \frac{y^4}{y_s^4} \left[ \frac{1}{18} - \frac{8}{105} y_s + \frac{1}{15} y_s^2 + \frac{2}{9} \frac{y^3}{y_s^3} \right] \\
- \frac{1}{6} \frac{y^3}{y_s^3} - \frac{1}{9} \frac{y^4}{y_s^3} + \frac{1}{12} \frac{y^3}{y_s^2} + \frac{1}{15} \frac{y^4}{y_s^2} - \frac{1}{12} \frac{y^7}{y_s^6} \\
+ \frac{1}{168} \frac{y^8}{y_s^6} - \frac{2}{15} \frac{y^5}{y_s^3} + \frac{1}{6} - \frac{1}{15} y_s - \frac{1}{6} \frac{y^4}{y_s^3} + \frac{1}{9} \frac{y^4}{y_s^3} \\
+ \frac{1}{10} \frac{y^5}{y_s^4} - \frac{1}{15} \frac{y^5}{y_s^3} \right\} \right) \]  \hspace{1cm} (3.27)

This is the expression for the channel current when the input is a.c. short-circuited.

3.9 Expression for \( y_{22} \)

When \( y = y_d \), equation (3.27) becomes

\[ \frac{I_d}{v_d} = \frac{I(y_d)}{v_d} \]

\[ = \frac{\mu Zv_d I_d}{\Delta} \frac{I_d}{I_0} \left\{ \frac{1}{y_s^2} - \frac{2}{3} y_s \left[ 1 - \frac{3}{4} y_s + \frac{1}{6} y_s^3 - \frac{3}{4} y_s u^4 - \frac{3}{4} u + y_s u \right] \\
+ \frac{1}{18} \frac{y^4}{y_s^4} \left[ 1 - \frac{1}{35} y_s + \frac{9}{20} y_s^2 + 4 u^3 - 3 y_s u^3 + \frac{3}{2} y_s^2 u^4 \right] \\
+ \frac{2}{5} y^6 - \frac{3}{7} y_s u^7 + \frac{3}{28} y_s^2 u^8 - \frac{12}{5} u + 3 y_s u \\
- \frac{6}{7} y_s^2 u - 3 u^4 + \frac{9}{2} y_s u^5 - \frac{6}{5} y_s^2 u^5 \right\} \]

Equations (2.15) and (3.13) are substituted in this expression to give
\[
\frac{1d}{vd} = \frac{4Zp}{\Delta} \frac{1d}{y_s^2} \left\{ 1 + j\omega \left[ \frac{\xi L^2}{36a^2} \cdot \frac{1}{y_s} \cdot \frac{1}{\gamma_0} \left\{ 1 - \frac{3}{2}u + \frac{3}{4}u^3 - \frac{3}{4}y_s (1 - \frac{3}{2}u + \frac{3}{4}u^3) \right\} \right] \right. \\
\left\{ 1 - \frac{3}{2}u + \frac{3}{4}u^4 \right\} \right\} + (j\omega)^2 \left[ \frac{2}{9} \frac{\xi L^4}{\sigma a^4} \cdot \frac{1}{y_s^2} \cdot \frac{1}{\gamma_0^4} \left\{ 1 - \frac{12}{5}u + 4u^3 \right\} \right. \\
- 3u^4 + \frac{2}{5}u^6 - \frac{4}{35}y_s (1 - \frac{35}{16}u + \frac{35}{16}u^3 - \frac{21}{16}u^5 + \frac{5}{8}u^7) \\
\left. + \frac{9}{20}y_s^2 (1 - \frac{40}{21}u + \frac{10}{3}u^4 - \frac{6}{5}u^5 + \frac{5}{21}u^8) \right\} \right\}
\]

which can be written in the form

\[
\frac{1d}{vd} = y_{22} = \frac{6\mu}{(1+j\omega\tau_8)} \left[ \frac{u}{(1-u)} \right] \left\{ 1 + j\omega(1+j\omega\tau_9) \right\}
\]  (3.28)

where

\[
\tau_8 = \frac{\xi L^2}{3\sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{1}{\gamma_0^2}
\]

\[
\tau_9 = \frac{\xi L^2}{6\sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{1}{\gamma_0^2} \cdot \frac{1}{\gamma_1^2}
\]

and

\[
y_{13} = 1 - \frac{3}{2}u + \frac{3}{4}u^3 - \frac{3}{4}y_s (1 - \frac{3}{2}u + \frac{3}{4}u^4)
\]

\[
y_{14} = 1 - \frac{12}{5}u + 4u^3 - 3u^4 + \frac{2}{3}u^6
\]

\[
- \frac{4}{35}y_s (1 - \frac{35}{16}u + \frac{35}{16}u^3 - \frac{21}{16}u^5 + \frac{5}{16}u^7)
\]

\[
+ \frac{9}{20}y_s^2 (1 - \frac{40}{21}u + \frac{10}{3}u^4 - \frac{6}{5}u^5 + \frac{5}{21}u^8)
\]

3.10 Expression for \(y_{12}\)

When \(y = y_s\), equation (3.27) becomes
\[
\frac{i(y_s)}{v_d} = \frac{hZ_0}{\Delta} \cdot \frac{I_d}{I_0} \left\{ \frac{1}{y_s^2} - \frac{2\nu^2y_s}{3} + \frac{1}{2} - \frac{3y_s}{2} + \frac{1}{y_s} \right\}
\]

which reduces, with the aid of (2.15a), (3.13a), (3.13b) and (3.14a), to

\[
\frac{i(y_s)}{v_d} = \frac{Z_{mo}}{(1+j\omega t_1)(1-u)} \quad (3.29)
\]

Since the net charge in the depletion regions is zero,

\[i_d + i_s + i_g = 0\]

Hence

\[i_g = -i_s - i_d = i(y_s) - i_d\]

The expression for \(y_{12}\) is found by using the relations (3.28) and (3.29).

\[y_{12} = \frac{i_g}{v_d} = \frac{i(y_s)}{v_d} - \frac{i_d}{v_d}\]

or \(y_{12} = -\frac{Z_{mo}}{(1+j\omega t_1)(1-u)} \omega t g(1+j\omega t_9) \quad (3.30)\)

The short-circuit admittance parameters of the F.E.T. can therefore be summarized as follows:
\[ y_{11} = \left. \frac{g}{v_g} \right|_{v_d=0} = \frac{\epsilon_{mo}}{(1+j\omega_1)} j\omega_6 (1+j\omega_7) \]
\[ y_{12} = \left. \frac{g}{v_d} \right|_{v_g=0} = \frac{-\epsilon_{mo}}{(1+j\omega_1)} \frac{u}{1-u} j\omega_8 (1+j\omega_9) \]
\[ y_{21} = \left. \frac{d}{v_g} \right|_{v_d=0} = \frac{\epsilon_{mo}}{(1+j\omega_1)} \left[ 1 - j\omega_2 (1+j\omega_3) \right] \]
\[ y_{22} = \left. \frac{d}{v_d} \right|_{v_g=0} = \frac{\epsilon_{mo}}{(1+j\omega_1) (1-u)} \left[ 1 + j\omega_8 (1+j\omega_9) \right] \]

Under saturated (pinch-off) conditions, \( v_d = 0 \) and so \( u = 0 \). Hence

\[ y_{12} = 0 \]

and \( y_{22} = 0 \)

Also, from equation (3.15), \( \tau_2 = 0 \). Therefore, \( y_{21} \) becomes

\[ y_{21} = \frac{\epsilon_{mo}}{(1+j\omega_1)} \]
4. THE EQUIVALENT CIRCUIT

4.1 Introduction

In this chapter, an equivalent circuit of the field-effect transistor is formulated by using the expressions for the short-circuit admittance parameters derived in the previous chapter. The elements of the circuit are determined and graphs are drawn to show how their values vary with the drain and source voltages.

There are two possible approaches to obtaining equivalent circuits for active devices. The 'circuit or black-box approach' is usually employed when only the terminal characteristics of the device are known. In such cases it is not necessary to know the internal physics of the device. The second method, called the 'device approach', is adopted when the physics of the device is sufficiently well understood. Starting from a knowledge of the physical principles involved, the expressions which adequately describe the operation of the device are derived. Then, an equivalent circuit which satisfies these expressions is found. The advantage of this approach is that it enables each component of the equivalent circuit to be expressed in terms of the fundamental device parameters. The device approach is used here to obtain the equivalent circuit.
4.2 Assumed Circuit Model

To start with, a hybrid-$
$ network of a general type is assumed to represent the equivalent circuit of the device and the admittance parameters of this circuit model are determined. By comparing these parameters with those derived in the previous chapter, expressions for the different elements of the circuit are derived.

![Fig. 4.1 Assumed form of the F.E.T. equivalent circuit.](image)

Fig. 4.1 shows the assumed circuit model of an F.E.T. which is treated as a four terminal device. In this circuit, \( y_1 \), \( y_2 \) and \( y_0 \) are the admittance branches and \( y_m v_g \) is the current generator; \( i_1 \) and \( i_2 \) are the input and output currents, respectively. If \( v_g \) and \( v_d \) are the input and output voltages for this common-source connection, Kirchhoff's current relations
for the two nodes G and D can be written as follows:

\[ i_1 = (y_1 + y_2)v_g - y_2v_d \]
\[ i_2 = -(y_2 - y_m)v_g + (y_2 + y_0)v_d \]

Hence

\[ j_{11} = y_1 + y_2 \]
\[ j_{12} = -y_2 \]
\[ j_{21} = -y_2 + y_m \]
\[ j_{22} = y_2 + y_0 \]

These relations are solved for \( y_1, y_2, y_o \) and \( y_m \).

\[ y_1 = j_{11} + j_{12} \quad (4.1) \]
\[ y_2 = -j_{12} \quad (4.2) \]
\[ y_o = j_{22} + j_{12} \quad (4.3) \]
\[ y_m = j_{21} - j_{12} \quad (4.4) \]

The values of \( j_{11}, j_{12}, j_{21} \) and \( j_{22} \), derived in chapter 3, are now used to determine \( y_1, y_2, y_o \) and \( y_m \).

4.3 Expressions for \( y_1, G_1 \) and \( R_1 \)

By using equations (4.1), (3.17) and (3.30), \( y_1 \) is determined.
\[ y_1 = y_{11} + y_{12} \]
\[ = \frac{\varepsilon_{m0}}{(1 + j\omega \tau_1)} \left[ j\omega \tau_6 (1 + j\omega \tau_7) - \varepsilon \omega \tau_8 (1 + j\omega \tau_9) \right] \]
\[ = \frac{\varepsilon_{m0}}{(1 + j\omega \tau_1)} \left[ \frac{\omega \tau_6}{(1 - j\omega \tau_7)} - \frac{\varepsilon \omega \tau_8}{(1 - j\omega \tau_9)} \right] \]
\[ = \frac{\varepsilon_{m0}}{(1 + j\omega \tau_1)} \left[ \frac{j\omega (\tau_6 - \delta \tau_8) - (j\omega)^2 (\tau_6 \tau_9 - \delta \tau_8 \tau_7)}{(1 - j\omega \tau_7)(1 - j\omega \tau_9)} \right] \]

where \( \delta = \frac{u}{1-u} \) \hspace{1cm} (4.5)

As in the previous cases the second order terms in \( j\omega \) are neglected in the denominator. It can then be shown that the above expression reduces to

\[ y_1 = \frac{\varepsilon_{m0} \cdot j\omega (\tau_6 - \delta \tau_8)}{1 + j\omega \left\{ \tau_1 + \frac{\tau_6 \tau_9 - \delta \tau_8 \tau_7}{\tau_6 - \delta \tau_8} - \tau_7 - \tau_9 \right\}} \]

which can be written in the form

\[ y_1 = \frac{j\omega C_1}{1 + j\omega C_1 R_1} \]

where \( C_1 = \varepsilon_{m0}(\tau_6 - \delta \tau_8) \) \hspace{1cm} (4.6)

and

\[ R_1 = \frac{\tau_1 + \frac{\tau_6 \tau_9 - \delta \tau_8 \tau_7}{\tau_6 - \delta \tau_8} - \tau_7 - \tau_9}{\varepsilon_{m0}(\tau_6 - \delta \tau_8)} \]

\[ \star \] The validity of this approximation will be discussed in Chapter 5.
Equation (4.7) indicates that $y_1$ can be conveniently expressed as the admittance of a resistance $R_1$ in series with a capacitance $C_1$. As the expressions for $g_{mo}$ and $\tau$'s are known, the values of $R_1$ and $C_1$ can be determined.

The capacitance $C_1$ is given by equation (4.8):

$$C_1 = g_{mo}(\tau_6 - \sigma \tau_8)$$

The values of $\tau_6$ and $\tau_8$ are given by (3.17) and (3.28), respectively, and $\sigma$ is given by (4.5)

$$(\tau_6 - \sigma \tau_8) = \frac{2\varepsilon L^2}{3a^2} \cdot \frac{1}{y_s} \cdot \frac{1}{y_0^2} \cdot \frac{1}{(1-u)} \left[ 1 - 3u^2 + 2u^3 - \frac{3}{2} y_s (1 - 4u^3 + 3u^4) \right]$$

Finally, use of equations (3.13b) and (3.14b) yields

$$C_1 = \frac{4\varepsilon L}{3a} \left\{ \frac{1 - 3u^2 + 2u^3 - \frac{3}{2} y_s (1 - 4u^3 + 3u^4)}{[1 - u^2 - \frac{2}{3} y_s (1 - u^3)]^2} \right\}$$

(4.10)

where $u = \frac{y_d}{y_s}$

$C_1$ is the gate-to-source capacitance and it can be shown that (4.10) agrees with van der Ziel's expression for $C_1$.

In the saturation (pinch-off) case, $y_d = 0$ or $u = 0$, so that (4.10) becomes

$$C_1 = C_{1s} = \frac{4\varepsilon L}{3a} \left[ \frac{1 - \frac{3}{2} y_s}{(1 - \frac{2}{3} y_s)^2} \right]$$

(4.11)
The resistance $R_1$ is given by equation (4.9). When the values of $x$'s are substituted in (4.9), it becomes too long and cumbersome to write. Therefore, an expression for $R_1$ in the saturation case is derived.

In the saturation case $y_d = 0$ and $\delta = 0$, so that equation (4.9) becomes

$$R_1 = R_{1S} = \frac{x_1 - x_7}{\gamma_{mo} \cdot \gamma_6}$$

This can be simplified to give

$$R_{1S} = \frac{L}{10c_{a2}} \cdot \left\{ \frac{1}{y_s} \frac{1 - \frac{37}{21} y_s + \frac{55}{56} y_s^2 - \frac{5}{28} y_s^3}{1 - \frac{5}{3} y_s + \frac{11}{12} y_s^2 - \frac{1}{6} y_s^3} \right\} \quad (4.12)$$

Since, for the saturation case,

$$\gamma_{mo} = \frac{298a}{L} y_s$$

$R_{1S}$ can also be written as

$$R_{1S} = \frac{1}{5\gamma_{mo}} \left\{ \frac{1 - \frac{37}{21} y_s + \frac{55}{56} y_s^2 - \frac{5}{28} y_s^3}{1 - \frac{5}{3} y_s + \frac{11}{12} y_s^2 - \frac{1}{6} y_s^3} \right\} \quad (4.13)$$

4.4 Expressions for $y_2$, $C_2$ and $R_2$

Equations (4.2) and (3.30) yield

$$y_2 = -y_{12}$$
When the second order terms in \((j\omega)\) are neglected, this becomes

\[ y_2 = \frac{\varepsilon_{\infty} u}{(1+jo\tau_1) (1-u) j\omega \tau_g (1+jo\tau_9)} \]

which can be written in the form

\[ y_2 = \frac{\varepsilon_{\infty} (\frac{u}{1-u}) j\omega \tau_g}{1+j\omega(\tau_1-\tau_9)} \]

where

\[ c_2 = \varepsilon_{\infty} \frac{u}{1-u} \tau_8 \]  \hspace{1cm} (4.15)

and

\[ R_2 = \frac{\tau_1 - \tau_9}{\varepsilon_{\infty} (\frac{u}{1-u}) \tau_8} \]  \hspace{1cm} (4.16)

Equation (4.14) indicates that \(y_2\) can be considered as the admittance of a resistance \(R_2\) and a capacity \(C_2\) in series. Substitution of equations (3.13b) and (3.28) in (4.15) yields

\[ c_2 = \frac{8ZeL}{3a} \left\{ \frac{u \left[ 1 - \frac{3}{2} u + \frac{1}{2} u^3 - \frac{3}{4} y_s (1+\frac{1}{2} u+\frac{1}{2} u^4) \right]}{1 - u^2 - \frac{2}{3} y_s (1-u^3) \frac{1}{2}} \right\} \]  \hspace{1cm} (4.17)
$C_2$ is the gate-to-drain capacitance. It can be shown that (4.17) agrees with van der Ziel's expression for $C_2$. In the saturation case $u = 0$ so that

$$C_2 = C_{2S} = 0$$

After substituting for $\tau$'s and $s_{mo}$, (4.16) becomes

$$R_2 = \frac{L}{100} \sigma_0 \tau \cdot \frac{1}{y_d} \left[ \frac{1}{y_0 \cdot y_{13}} \right] \left[ y_4 - \frac{5}{8} \frac{y_0}{y_{13}} \cdot y_{14} \right]$$  \hspace{1cm} (4.16)

$y_0$ and $y_4$ are defined in section 3.5, and $y_{13}$ and $y_{14}$ are defined in section 3.8.

Under saturated conditions, $y_d = 0$ and so (4.18) gives

$$R_2 = R_{2S} = \infty$$

4.5 Expressions for $y_0$, $R_0$ and $L_0$

Use of equations (4.3), (3.28) and (3.30) yields the expression for $y_0$.

$$y_0 = y_{22} + y_{12}$$

$$= \frac{s_{mo}}{(1+j\omega T_1)} \frac{u}{(1-u)} \left[ 1 + j\omega \epsilon (1+j\omega \varphi) \right]$$

$$- \frac{s_{mo}}{(1+j\omega T_1)} \frac{u}{(1-u)} j\omega \epsilon (1+j\omega \varphi)$$

$$= \frac{s_{mo} \cdot 6}{(1+j\omega T_1)}$$  \hspace{1cm} (4.19)
where \( \delta = \frac{u}{1-u} \).

Equation (4.19) can be expressed in the form

\[
y_0 = \frac{g_0}{1 + j\omega \delta g_0 L_0}
\]  
(4.20)

where

\[
g_0 = \delta g_{m0}
\]  
(4.21)

and

\[
I_0 = \frac{I_1}{g_{m0} \delta}
\]  
(4.22)

Equation (4.20) shows that the admittance \( y_0 \) consists of a conductance \( g_0 \) in series with an inductance \( I_0 \) whose values are given by the relations (4.21) and (4.22), respectively.

Substituting for \( g_{m0} \) and \( \delta \), equation (4.21) gives

\[
g_0 = \frac{2Z\alpha a}{L} \cdot y_s (1-u) \cdot \frac{u}{(1-u)}
\]

Since \( u = \frac{y_d}{y_s} \), \( g_0 \) becomes

\[
g_0 = \frac{2Z\alpha a}{L} \cdot y_d
\]  
(4.23)

\( g_0 \) is the output conductance of the device. Recalling that

\[
W_d = W_p (1-y_d)^2
\]

equation (4.23) can be written as

\[
g_0 = \frac{2Z\alpha a}{L} \left[ 1 - \left( \frac{W_d}{W_p^2} \right)^{\frac{1}{3}} \right]
\]
which agrees with the expression (1.5) for output conductance given by the d.c. theory.

An expression for \( I_o \) can be derived by the use of equations (4.22) and (3.14a),

\[
I_o = \frac{L}{22 \sigma a} \cdot \frac{1}{y_d} \cdot \frac{\varepsilon L^2}{15 \sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{y_b}{y_o^3}
\]

\[
= \frac{2 \varepsilon}{15} \cdot \frac{L^3}{2\sigma^2 a^3} \cdot \frac{1}{y_d y_s} \cdot \frac{y_b}{y_o^3}
\]  

(4.24)

where

\[
y_b = 1 - 5u^2 + 5u^3 - u^5 - \frac{5}{4} y_s (1 - 3u^2 + 3u^4 - u^6)
\]

\[
+ \frac{5}{4} y_s^2 (1 - 7u^3 + 7u^4 - u^7)
\]

and

\[
y_o = 1 - u^2 - \frac{2}{3} y_s (1 - u^3)
\]

Under saturated conditions \((y_d = 0)\) equation (4.23) gives

\[
g_o = g_{os} = 0
\]

or, the output resistance \( R_o \) is

\[
R_o = R_{os} = \frac{1}{g_{os}} = \infty
\]

and the inductance \( L_o \) becomes

\[
L_o = L_{os} = \infty
\]
4.6 Expression for transadmittance $y_m$

The transadmittance '$y_m$' of the device is determined by using equations (4.4), (3.15) and (3.30).

$$y_m = y_{21} - y_{12}$$

$$= \frac{\varepsilon_{mo}}{1 + j\omega \tau_1} \left\{ 1 - j\omega \tau_2 (1 + j\omega \tau_3) \right\} + \frac{\varepsilon_{mo}}{1 + j\omega \tau_1} \cdot \delta \cdot j\omega \tau_3 (1 + j\omega \tau_3)$$

$$= \frac{\varepsilon_{mo}}{1 + j\omega \tau_1} \left\{ 1 - j\omega (\tau_2 - \delta \tau_3) + (j\omega)^2 (\tau_2 \tau_3 + \delta \tau_3 \tau_3) \right\} \quad (4.25)$$

At saturation $\delta = 0$ and $\tau_2 = 0$, so that

$$y_m = \frac{\varepsilon_{mo}}{(1 + j\omega \tau_{1S})} \quad (4.26)$$

where $\tau_{1S}$ is the value of $\tau_1$ at saturation. (see Appendix A)

$$\tau_{1S} = \frac{4\varepsilon I^2}{5\sigma a^2} \cdot \frac{1}{y_s} \left\{ 1 - \frac{5y_s}{12y_s} + \frac{5y_s^2}{12y_s^2} \right\} \left\{ 1 - \frac{5y_s}{12y_s} + \frac{5y_s^2}{12y_s^2} \right\} \left( 1 - \frac{2y_s}{3y_s} \right)^3$$

4.7 The Equivalent Circuit

Having determined all the elements of the assumed circuit model, the equivalent circuit for the F.E.T. can be written as shown in Fig. 4.2.
Fig. 4.2 The derived equivalent circuit for the device.

S, D and G are the source, drain and gate terminals, respectively of the device and $y_m V_g$ is the current generator; $V_g$ being the a.c. gate voltage referred to the source. The expressions for $R_1$, $R_2$, $C_1$, $C_2$ and $L_0$ indicate that the values of these elements vary in a nonlinear manner with the drain and gate voltages.

Normalized values of $R_1$ and $C_1$ as given by equations 4.9 and 4.10 have been computed theoretically and are plotted against $W_s/W_p$ for different values of $W_d/W_p$ in Figs. 4.3 and 4.4. It is apparent that when $W_s = W_p$, $R_1$ goes to infinity.
for all values of $W_d$ and $C_1$ goes to zero for all values of $W_d$ except when $W_d = W_p$ (saturation).

It has been shown in the previous section that under saturated conditions $R_2$ goes to infinity and $C_2$ reduces to zero. In Figs. 4.5 and 4.6, normalized values of $R_2$ and $C_2$ under non-saturated conditions are plotted against $W_d/W_p$. As $W_d/W_p$ decreases, the value of $R_2$ reduces and that of $C_2$ increases.

According to (4.23) and (4.24), both $R_0$ and $L_0$ increase with the drain-to-source voltage and at saturation both go to infinity. It is shown in Chapter 5 that the value of $L_0$ is very small and that it can usually be neglected except at very high frequencies.
Fig. 4.3 Normalized values of $R_1$ and $C_1$ versus normalized gate bias with normalized drain bias as the parameter.

(a) $W_a = W_p$ (saturation); (b) $W_a = (0.9)W_p$

$R_1(\sigma a z) \frac{a}{(2zL)}$

$C_1(\frac{a}{2zL})$
Fig. 4.4 Normalized values of $R_1$ and $C_1$ versus normalized gate bias with normalized drain bias as the parameter. (a) $W_a = (0.75) W_p$; (b) $W_a = (0.5) W_p$. 
Fig. 4.5 Normalized values of \( R_2 \) and \( C_2 \) versus normalized gate bias with normalized drain bias as the parameter. (a) \( W_a = (0.98) \frac{W}{W_p} \); (b) \( W_a = (0.83) \frac{W}{W_p} \).
Fig. 4.6 Normalized values of $R_2$ and $C_2$ versus normalized gate bias with normalized drain bias as the parameter. (a) $W_a = 0.56 \, W_p$; (b) $W_a = 0.35 \, W_p$. 
Fig. 4.7 Normalized value of \((-\tau_1 + \tau_2 - \phi \tau_3)\) versus normalized gate bias with normalized drain bias as the parameter.
5. CONCLUSIONS

Considering the field effect transistor as an active, distributed and non-uniform transmission line, a differential equation for the a.c. case was derived from the physical principles involved in the operation of the device. As these considerations were based on Shockley's basic theory of the device, the analysis presented here is applicable only to those devices for which the assumptions in that theory are valid. In other words, the theory presented in this work is valid for F.E.T.'s with abrupt p-n junctions and uniformly doped channels. Furthermore, since Shockley's theory also assumes a gradual variation of potential along the channel of the device, this analysis is valid as long as the condition \( y_s \geq y_d \) (i.e. \( W_d \geq W_s \)) is satisfied.

The differential equation was solved and the solution was used to derive the expressions for the short-circuit admittance parameters of the device. An equivalent circuit in the form of a hybrid-\( \pi \) network was then obtained from the first order approximation of the theory. The mathematical approximation made in sections 4.3 and 4.4, while deriving the expressions for the elements of the equivalent circuit, is valid as long as the condition

\[
\omega \tau \ll 1
\]

is satisfied. The quantity \( \frac{\epsilon I^2}{\sigma a^2} \) which determines the order
of magnitude of all $\tau$'s, depends upon the material, doping and the physical dimensions of the device. For example, it has been shown\textsuperscript{14} that for the F.E.T. 2N2498 (manufactured by Texas Instruments Inc.) the value of $\frac{\varepsilon L^2}{\sigma a^2}$ is $< 4.0 \times 10^{-9}$. Hence, the approximation is valid up to a frequency of several megacycles.

The $Q$ of the $y_0$ branch can be determined by using this value of $\varepsilon L^2/\sigma a^2$. It is given by

$$Q = \frac{\omega L_0}{R_0}$$

Use of (4.23) and (4.24) yields

$$\frac{L_0}{R_0} = \frac{4}{15} \frac{\varepsilon L^2}{\sigma a^2} \cdot \frac{1}{y_s} \cdot \frac{Y_4}{Y_0}$$

Let $y_s = 0.5$ (i.e., $\frac{W_s}{W_p} = 0.25$). Then

$$\frac{L_0}{R_0} \approx 2.4 \times 10^{-9}$$

Therefore the $Q$ of the $y_0$ branch at a frequency of 5 mc is approximately equal to 0.075 which is a very small quantity. The inductance $L_0$ can therefore be neglected except at very high frequencies. When it is omitted, the equivalent circuit of Fig. 4.2 will have the same configuration as Silverthorn's circuit.

Equation (4.26) shows that the transadmittance $Y_m$ decreases with frequency. Since the order of magnitude of $\tau_1$ is $10^{-9}$
seconds, the variation of $g_m$ with frequency should be very small until a value of about ten megacycles is reached. Measurements made by Silverthorn show this to be correct.

The analysis presented here takes into consideration only the intrinsic device and hence the equivalent circuit obtained is not complete. In an actual F.E.T. structure, there will be some parasitic resistance in series with the source and drain leads, and it has been shown that these parasitic resistances can be absorbed into the equivalent circuit of Fig. 4.2 to yield a circuit having the same configuration. Furthermore, the actual capacitances in a practical device will differ somewhat from the values computed in this idealized model because of the non-ideal conditions at the source and drain ends of the channel due to depletion region "end effects".
6. REFERENCES


APPENDIX A

The values of \( \tau \)'s under saturated condition \( (y_d = 0) \) are given below.

\[
\tau_1 = \frac{4\varepsilon L^2}{15sa^2} \cdot \frac{1}{y_s} \left\{ \frac{1 - \frac{5}{4} y_s + \frac{5}{14} y_s^2}{(1 - \frac{2}{3} y_s)^3} \right\}
\]

\( \tau_2 = 0 \)

\[
\tau_3 = \tau_9 = \frac{\varepsilon L^2}{6sa^2} \cdot \frac{1}{y_s} \left\{ \frac{1 - \frac{48}{35} y_s + \frac{9}{20} y_s^2}{(1 - \frac{2}{3} y_s)^2(1 - \frac{3}{4} y_s)} \right\}
\]

\[
\tau_4 = \tau_6 = \frac{2}{3} \frac{\varepsilon L^2}{sa^2} \cdot \frac{1}{y_s} \left\{ \frac{1 - \frac{3}{2} y_s}{(1 - \frac{2}{3} y_s)^2} \right\}
\]

\[
\tau_5 = \tau_7 = \frac{2\varepsilon L^2}{15sa^2} \cdot \frac{1}{y_s} \left\{ \frac{1 - \frac{15}{14} y_s + \frac{15}{56} y_s^2}{(1 - \frac{2}{3} y_s)^2(1 - \frac{1}{2} y_s)} \right\}
\]

\[
\tau_8 = \frac{4\varepsilon L^2}{3sa^2} \cdot \frac{1}{y_s} \left\{ \frac{1 - \frac{3}{14} y_s}{(1 - \frac{2}{3} y_s)^2} \right\}
\]