The structure of spaces of valuations and the local uniformization problem

A Thesis Submitted to the
College of Graduate Studies and Research
in Partial Fulfillment of the Requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon

By
Josnei Novacoski

Permission to Use

In presenting this thesis in partial fulfilment of the requirements for a Postgraduate degree from the University of Saskatchewan, I agree that the Libraries of this University may make it freely available for inspection. I further agree that permission for copying of this thesis in any manner, in whole or in part, for scholarly purposes may be granted by the professor or professors who supervised my thesis work or, in their absence, by the Head of the Department or the Dean of the College in which my thesis work was done. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to the University of Saskatchewan in any scholarly use which may be made of any material in my thesis.

Requests for permission to copy or to make other use of material in this thesis in whole or part should be addressed to:

Head of the Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon, Saskatchewan
Canada
S7N 5E6

Abstract

The problem of resolution of singularities is a major problem in algebraic geometry. Local uniformization can be seen as its local version. For varieties over fields of characteristic zero, local uniformization was proved by Zariski in 1940 ([49]) and resolution of singularities was proved by Hironaka in 1964 ([18]). For algebraic varieties over fields of positive characteristic both problems are open in dimension greater than 3. Zariski's idea to solve the resolution of singularities problem for an algebraic variety was to prove local uniformization for all valuations of the associated function field and use the compactness of the Zariski space of valuations to glue the solutions together and construct a global resolution. Hence, his approach deals with two aspects: proving local uniformization and using structural properties of spaces of valuations to glue the local solutions. In this thesis, we present our contribution to both aspects.

In most of the successful cases, local uniformization was first proved for rank one valuations and then extended to the general case. Local uniformization can be stated as a property of a valuation ν centered at a local ring R. One of our contributions to the local uniformization problem (which is a joint work with Spivakovsky) is that in order to prove local uniformization for valuations centered at local rings in a category $\mathcal{M}$ which is closed under taking homomorphic images, finitely generated birational extensions and localizations, it is enough to prove that rank one valuations centered at members of $\mathcal{M}$ admit local uniformization. We also obtain this reduction for different versions of local uniformization, for instance, for embedded local uniformization and inseparable local uniformization. Our proofs are particularly important because they do not depend on the nature of the category $\mathcal{M}$.

We also work with henselian elements. Henselian elements are roots of polynomials appearing in Hensel's Lemma. We summarize unpublished results of Kuhlmann ([24]), van den Dries ([47]) and Roquette ([40]) to obtain that for a finite field extension $(F|L,\nu)$, if F is contained in the absolute inertia field of L, then the valuation ring $\mathcal{O}_F$ of (F,ν) is generated as an $\mathcal{O}_L$-algebra by henselian elements. Moreover, we obtain a list of equivalent conditions

under which $\mathcal{O}_F$ is generated over $\mathcal{O}_L$ by finitely many henselian elements. We prove that if the chain of prime ideals of $\mathcal{O}_L$ is well-ordered by inclusion, then these conditions are satisfied. We give an example of a finite inertial extension $(F|L, \nu)$ for which $\mathcal{O}_F$ is not a finitely generated $\mathcal{O}_L$-algebra. We also present a theorem with a simple proof that relates the problem of local uniformization with the theory of henselian elements. This theorem shows, in particular, that if we obtain elimination of ramification for a function field for a good transcendence basis, then the valuation admits local uniformization.

In our studies of spaces of valuations we defined new topologies on spaces of valuations which extend naturally known topologies. We compare these topologies and show that in general they are not equal. We also obtain criteria under which the space of valuations taking values in a fixed ordered abelian group Γ is a closed subset in the space of all functions taking values in Γ. We study the works of Favre and Jonsson ([14] and [15]) and Granja ([16]) on the valuative tree. Favre and Jonsson prove that the set of all valuations centered at $\mathbb{C}[[x, y]]$ admits a tree structure, which they call the valuative tree. Granja extends this result to any two-dimensional regular local ring. In both works, the definition of non-metric rooted tree is not satisfactory. This is because the definition does not guarantee the existence of an infimum for any non-empty set of valuations. This infimum is necessary in order to define and study many concepts related to such trees. We give a more general definition of a rooted non-metric tree and prove that the set of all valuations satisfies this more general definition, namely, we prove that every non-empty set of valuations centered at a two-dimensional regular domain admits an infimum. We also generalize some topological results related to a non-metric tree, for instance that the weak tree topology is always coarser than the metric topology given by any parametrization.

Acknowledgements

I wish to express my gratitude to my supervisor Dr. Franz-Viktor Kuhlmann for his encouragement, support and patience. His guidance was essential in order to develop my own ideas. I thank my colleagues, friends and collaborators who helped to make this thesis possible. Among them, I want to cite Dr. Hagen Knaf, Dr. Orlando Stanley Juriaans, Dr. Nazar Arakelian and specially Dr. Mark Spivakovsky, who always found time to answer my questions and discuss my ideas. I also would like to thank the University of Saskatchewan for financial support during my program.

Also, I thank the members of my graduate committee Dr. Santiago Encinas, Dr. Murray Marshall, Dr. Murray Bremner and Dr. Mark Keil for their careful reading of this thesis. Their suggestions and corrections made it more accurate and presentable.

To my beloved wife Lucyna and my brother Jonilso.

Contents

Chapter 1

Introduction

Algebraic geometry is the study of systems of polynomial equations. One example is the famous Fermat's Last Theorem which stayed open for three hundred years. Prior to its 1995 proof it was in the Guinness Book of World Records for "most difficult math problems". Algebraic geometry is also applied in the solution of systems of differential equations. These appear in all sorts of problems in science and technology, for instance, weather forecasts and crash-test simulations. One of the main obstacles in handling these systems is the existence of singularities. Singularities are points where the solutions of the system of equations do not depend in a well-behaved way on its parameters. Many excellent mathematicians have worked to achieve "resolution of singularities", that is, the transformation of a given system into a new system that has no singularities. In 1964, Hironaka ([18]) obtained resolution of singularities in an important particular case ("characteristic zero"). For this work, he received a Fields Medal (the equivalent of the Nobel Prize for mathematicians). Since then many attempts (to name a few [1] - [4], [9], [10], [32], [36], [42] and [43]) have been made to settle the remaining case of "positive characteristic", but the problem is still open.

If you cannot solve a problem globally (resolve *all* singularities) you try to solve it locally (resolve one given singularity). Zariski noted in the 1930's that picking out a particular singularity can be done by assigning "valuations", or equivalently "places", to the system of equations. He then achieved in 1940 ([49]) "local uniformization", i.e., the resolution of a single given singularity, in characteristic zero. Again, local uniformization in arbitrary dimension and characteristic remains an open problem.

Zariski's idea was to "glue" all local solutions together to obtain a global solution. For this he worked with the space of all places (or space of all valuations) of a function field and proved that it has a very useful topological property, namely, compactness. When we achieve local uniformization for a valuation, we automatically achieve local uniformization for every valuation in an open neighbourhood (in the Zariski topology) of such a valuation. Using the compactness of the space of valuations to choose a finite subcovering among these open neighbourhoods, we conclude that it is enough to glue finitely many solutions. So far the glueing has only been achieved for small dimensions ([51], [48] and [50]). Therefore, the search for stronger properties and finer structures of spaces of places has begun. A more general kind of spaces of valuations was studied by Knebusch ([19]). Also Berkovich spaces ([7]) appear as an alternative theory. Both Knebusch and Berkovich approaches have been studied as important tools towards local uniformization and resolution of singularities.

An important type of structure in spaces of valuations appears in the interesting work on the "valuative tree" by Favre and Jonsson ([8], [14] and [15]). The aim of this work is to create a stronger structure on spaces of valuations which allows us to handle singularities. They proved that in the dimension two and characteristic zero case, the space of valuations admits a tree-like structure. Granja ([16]) extends their result to more general settings, including the positive characteristic case, but also his studies are restricted to dimension two.

In this thesis, we present our contribution to the local uniformization problem. We divide accordingly to the two aspects previously mentioned: prove local uniformization in more general settings and present and study new structures in spaces of valuations. We present this work in the *Preliminaries* and four chapters: *The valuative tree*, *Topologies on spaces of valuations*, *Henselian elements* and *Reduction of local uniformization to the rank one case.*

1.1 The valuative tree

Favre and Jonsson prove in [14] that the set of normalized valuations centered at $\mathbb{C}[[x,y]]$ admits a tree structure, which they call "the valuative tree". In [16], Granja generalizes this result to the set of normalized valuations centered at any two-dimensional regular local domain. In both works, the definition of "rooted non-metric tree" is not satisfactory (see discussion after Definition 3.2.1). That is because the definition given in the cited papers does not guarantee the existence of the infimum of a non-empty set of valuations. The existence of the infimum (of two valuations) is necessary in order to define some important concepts, such as the "weak tree topology" (see Definition 3.2.6, item **(iv)**) and the metric associated to a "parametrization" (Definition 3.2.6, item **(vi)**).

In [14], it is stated that the existence of the infimum is a consequence of the given definition, which is not true (see Example 3.2.3). In order to make the theory developed there consistent, one needs to prove that there exists the infimum for any given pair of valuations. In the case of $R = \mathbb{C}[[x, y]]$, it is proved in [14], using the sequence of key polynomials associated to a valuation, that the infimum of two valuations exists as long as we can find an element which "minimizes" both valuations. An easy argument (Corollary 3.2.13) shows that one always can obtain such an element (in fact, we can achieve that for any two-dimensional regular domain).

An interesting question which arises naturally is whether we can find the infimum of any non-empty set of valuations centered at a two-dimensional regular local domain. Our next theorem gives an affirmative answer to this question.

Theorem 1.1.1. *Let R be a two-dimensional regular local domain and take any non-empty set $\mathcal{S} = \{\nu_i\}_{i\in I}$ of centered valuations $\nu_i : R \longrightarrow \mathbb{R}_\infty$ normalized by $\nu_i(\mathfrak{m}) = 1$ (see Definition 2.1.15). Then there exists a valuation $\nu : R \longrightarrow \mathbb{R}_\infty$ which is the infimum of $\mathcal{S}$ with respect to the order given by $\nu \leq \mu$ if and only if $\nu(\phi) \leq \mu(\phi)$ for every $\phi \in R$.*

By use of this theorem, it follows from [14] and [16] that the set of all centered normalized valuations on R admits a tree structure and associated to that a weak tree topology (see definitions and discussions in Section 3.2). Every parametrization Ψ of a tree induces a metric d_Ψ on that tree (see Definition 3.2.6). A natural question is whether this metric topology and the weak tree topology are comparable. The next theorem answers this question to the affirmative.

Theorem 1.1.2. *Let $(\mathcal{T}, \leq)$ be a rooted non-metric tree and let $\Psi : \mathcal{T} \longrightarrow [1, \infty]$ be a parametrization of $\mathcal{T}$. The weak tree topology on $\mathcal{T}$ is coarser than or equal to the topology associated with the metric d_Ψ.*

In [14], two parametrizations ("skewness" and "thinness") are presented to prove that the normalized centered valuations on $\mathbb{C}[[x, y]]$ form a parametrized non-metric tree. Also, in [16], Granja presents a new approach and a different parametrization that proves the equivalent result for any two-dimensional regular local domain. In [14], Favre and Jonsson compare the topologies generated by their parametrizations and the weak tree topology. Our theorem above gives a more general comparison, which does not depend on the valuative origin of such a tree.

We also show (Theorem 3.3.4) that if the tree admits a point with uncountably many branches, then these topologies are distinct.

1.2 Topologies on spaces of valuations

The Zariski topology is the best known topology in algebraic geometry. It was introduced by Zariski in the first half of the twentieth century and it has been intensively studied since then. Initially, it was defined as a topology on algebraic varieties, but in a modern language it is defined as a topology on the spectrum of a ring, i.e., the set of all prime ideals of the ring. The space of Krull valuations on a ring (or valuations on a field) admits a natural structure as inverse limit of spectra of rings with their respective Zariski topologies. The corresponding topology is called again the Zariski topology on the space of valuations.

The Zariski topology on the space of valuations is compact but not Hausdorff. The Zariski patch topology is the coarsest compact and Hausdorff topology finer than the Zariski topology. In our studies of spaces of valuations we define three new topologies which are natural extensions of the Zariski and Zariski patch topologies. These topologies can be seen as analogues of spaces of real places. The Harrison topology on spaces of real places is obtained from the natural topology of the residue fields (see for instance [12] and [29]), while the topologies we study here are obtained from the natural topology of the value groups. A surprising result that we obtained (Corollaries 4.3.11 and 4.3.15) is that the three topologies that we introduce are not, in general, equal.

Many questions arose when we started to deal with these topologies. For instance, are they compact? In order to answer this question we reason as follows: let R be a noetherian ring and $\Gamma = \mathbb{R}^d$ with the lexicographic order, where $d = \dim(R)$. We extend the order and addition of Γ to $\Gamma_\infty = \Gamma \cup \{\infty\}$ as usual. Let $\widetilde{\mathcal{W}}_\Gamma^{\geq 0}$ be the subset of $\left(\Gamma_\infty^{\geq 0}\right)^R$ consisting of all non-negative valuations of R taking values in Γ_∞. In view of Proposition 4.3.1, the space $\mathcal{W}^{\geq 0}$ of equivalence classes of non-negative valuations on R can be seen as the quotient of $\widetilde{\mathcal{W}}_\Gamma^{\geq 0}$ under the equivalence presented in Definition 2.1.6. A topology $\mathfrak{A}$ on $\Gamma_\infty^{\geq 0}$ yields a product topology on $\left(\Gamma_\infty^{\geq 0}\right)^R$. This product topology induces the subspace topology on $\widetilde{\mathcal{W}}_\Gamma^{\geq 0}$ and we obtain the quotient topology on $\mathcal{W}^{\geq 0}$. If the topology $\mathfrak{A}$ is compact, then the product topology is also compact. If $\widetilde{\mathcal{W}}_\Gamma^{\geq 0}$ is a closed subset of $\left(\Gamma_\infty^{\geq 0}\right)^R$, then it is compact and so is the quotient topology on $\mathcal{W}^{\geq 0}$. Therefore, we want to know which properties of the topology $\mathfrak{A}$ guarantee that $\widetilde{\mathcal{W}}_\Gamma^{\geq 0}$ is a closed subset of $\left(\Gamma_\infty^{\geq 0}\right)^R$ in the product topology. Our next result gives a sufficient condition.

Theorem 1.2.1. *Let Γ' be any submonoid of Γ_∞ and take a topology $\mathfrak{A}$ on Γ' such that*

(P1) *the addition $+ : \Gamma' \times \Gamma' \longrightarrow \Gamma'$ is continuous, and*

(P2) *for every $\gamma, \gamma' \in \Gamma'$ such that $\gamma < \gamma'$ there exist open sets $U, U' \in \mathfrak{A}$ such that $\gamma \in U, \gamma' \in U'$ and $U < U'$ (i.e., $u < u'$ for every $u \in U$ and $u' \in U'$).*

Then the set $\widetilde{\mathcal{W}}_{\Gamma'}$ of valuations from R to Γ_∞ taking values in Γ' is closed in $(\Gamma')^R$.

Remark 1.2.2. We introduce the submonoid Γ' of Γ_∞, and prove Theorem 1.2.1 for this case, because in many situations we will study valuations which take values in a specific submonoid. For instance, the values of non-negative valuations lie in the submonoid $\left(\Gamma^{\geq 0}\right)_\infty$ of Γ_∞.

We can ask for the converse of Theorem 1.2.1, namely: can we find conditions on $\mathfrak{A}$ which imply that $\widetilde{\mathcal{W}}_{\Gamma'}$ is not closed in $(\Gamma')^R$? The next proposition answers this question for the case $\Gamma' = \Gamma_\infty$.

Proposition 1.2.3. *Take any topology $\mathfrak{A}$ on Γ_∞. If the set of all valuations on R taking values in Γ_∞ is closed in $(\Gamma_\infty)^R$ with respect to the product topology, then $\mathfrak{A}$ is T_1.*

It is easy to show that if **(P2)** holds, then $\mathfrak{A}$ is finer than the order topology and that both Properties **(P1)** and **(P2)** hold for the order topology. On the other hand, if $\mathfrak{A}$ is not

T_1, then the set of all valuations on R taking values in Γ_∞ is not closed in $(\Gamma_\infty)^R$. A natural question is whether the property of being T_1 characterizes the order topology among all the topologies $\mathfrak{A}$ coarser than the order topology. The next proposition (which we prove by giving an example) answers this question to the negative.

Proposition 1.2.4. *There exists a topology $\mathfrak{A}$ on Γ_∞ strictly coarser than the order topology which is T_1.*

Even though it is not clear whether the topologies that we define are compact or not, the criteria that we obtained (Theorem 1.2.1 and Propositions 1.2.3 and 1.2.4) are enlightening. They show how these topologies are dependant of the original topology on Γ_∞.

1.3 Henselian elements

For an extension $(F|L,\nu)$ of valued fields, an element $a \in \mathcal{O}_F$ will be called a **henselian element** (over (L,ν)) if there exists a polynomial $h(x) \in \mathcal{O}_L[x]$ (not necessarily monic) such that $h(a) = 0$ and $\nu(h'(a)) = 0$. In this case, we call h a **henselian polynomial for** a. Henselian elements play an important role implicitly in many problems of valuation theory. The notion of "étale extension" is closely related to the concept of henselian elements. Henselian elements can be used to interpret the property of being étale in more valuation theoretical terms.

The problem of local uniformization for a valued function field $(F|K,\nu)$ turns out to be close to the problem of "elimination of ramification". The valued function field $(F|K,\nu)$ is said to admit **local uniformization** if for every finite set $Z \subseteq \mathcal{O}_F$ there exists an affine model V of $(F|K,\nu)$ such that the center $\mathfrak{p}$ of ν on V is a regular point and $Z \subseteq \mathcal{O}_{V,\mathfrak{p}}$ (see discussion in Section 2.2.5). Elimination of ramification asks whether there exists a transcendence basis T of $F|K$ such that F lies in the "absolute inertia field" L^i of $L = K(T)$ (see Definition 2.2.16).

A possible approach for local uniformization is to prove that the valued function field $(F|K,\nu)$ admits elimination of ramification for a transcendence basis T for which the valued rational function field $(L|K,\nu)$ admits local uniformization. Then to find a model V of

$(F|K,\nu)$ we can find a convenient model V' of $(L|K,\nu)$ and extend it via the inertial extension $F|L$. The set Z appearing in the definition of local uniformization plays an essential role in this task. This is because, when finding the model V', we can require that not only elements obtained from the original set Z, but also elements needed to generate the extension $F|L$ belong to $\mathcal{O}_{V',\mathfrak{p}'}$ ($\mathfrak{p}'$ being the center of ν on V'). Using this approach, Knaf and Kuhlmann proved that every "Abhyankar valuation" admits local uniformization (see [22]) and that every valuation admits local uniformization in a finite separable extension of the function field (see [23]). We also use this approach to prove our Theorem 1.3.8 (and consequently, also Theorem 1.3.9) below.

Since an algebra essentially generated by henselian elements over a regular ring is regular (see Proposition 5.3.1), it is important to answer the following:

Problem 1.3.1. Take a valued field extension $(F|L,\nu)$ such that the field F lies in the absolute inertia field L^i of L and $[F:L]<\infty$ (for short we will call this **a finite inertial extension**). Can we find a generator of F over L which is a henselian element? If that is the case, what can be said about the valuation rings? For instance, is $\mathcal{O}_F$ generated as an $\mathcal{O}_L$-algebra by henselian elements?

In slightly different terms, these questions were posted by Kuhlmann on "The Valuation Theory Home Page" in form of conjectures (see [24]). In the same web page, Roquette and van den Dries (see [40] and [47]) gave interesting answers to these problems. In the chapter on henselian elements we summarize those answers and extend them to more general settings.

The next theorem was proved by Kuhlmann in [24] in a slightly different form. We adapted it to our needs.

Theorem 1.3.2. *Let $(F|L,\nu)$ be a finite inertial extension. Then there is $\eta\in\mathcal{O}_F$ such that $F=L(\eta)$, the (monic) minimal polynomial $h(x)$ of η over L lies in $\mathcal{O}_L[x]$ and η and $h'(\eta)$ are units in $\mathcal{O}_F$. In particular, η is a henselian element over (L,v) and $\mathcal{O}_L[\eta,1/h'(\eta)]$ is a Prüfer domain.*

For applications, for instance to the local uniformization problem, it is important to know whether for a finitely generated $\mathcal{O}_L$-algebra R lying in $\mathcal{O}_F$ there exists a unit u of

$\mathcal{O}_F$ in $\mathcal{O}_L[\eta]$ such that $R \subseteq \mathcal{O}_L[\eta, 1/u]$. The next theorem answers this question to the affirmative. It was proved originally by van den Dries in [47], but our proof is slightly simpler.

Theorem 1.3.3. *Let $(F|L,\nu)$ be a finite inertial extension. Then $\mathcal{O}_F = \mathcal{O}_L[\eta]_{\mathfrak{n}}$ where η is the element appearing in Theorem 1.3.2 and $\mathfrak{n} = \mathfrak{m}_F \cap \mathcal{O}_L[\eta]$. In particular, for every finite set $Z \subseteq \mathcal{O}_F$ there exists a unit u of $\mathcal{O}_F$ in $\mathcal{O}_L[\eta]$ such that $Z \subseteq \mathcal{O}_L[\eta, 1/u]$.*

Another important question is whether we can replace the element $1/u$ obtained in Theorem 1.3.3 by henselian elements. This was answered by Roquette in [40], where he proves the following:

Theorem 1.3.4. *Let $(F|L,\nu)$ be a finite inertial extension. For every element $a \in \mathcal{O}_F$ there exist henselian elements $r, s \in \mathcal{O}_F$ such that $a \in \mathcal{O}_L[\eta, r, s]$, where η is the element obtained in Theorem 1.3.2.*

A natural question is whether the elements η, r, s can be chosen independently of the element $a \in \mathcal{O}_F$, in particular, is $\mathcal{O}_F$ a finitely generated $\mathcal{O}_L$-algebra? We show that this is not always true. Namely, we prove the following:

Theorem 1.3.5. *There exists a finite inertial extension $(F|L,\nu)$ such that $\mathcal{O}_F$ is not a finitely generated $\mathcal{O}_L$-algebra.*

The next theorem gives a list of equivalent conditions under which $\mathcal{O}_F$ is a finitely generated $\mathcal{O}_L$-algebra.

Theorem 1.3.6. *Let $(F|L,\nu)$ be a finite inertial extension. Let η be the element obtained in Theorem 1.3.2. Then the following conditions are equivalent:*

(i) *$\mathcal{O}_F$ is a finitely generated $\mathcal{O}_L$-algebra;*

(ii) *there exists a unit u of $\mathcal{O}_F$ in $\mathcal{O}_L[\eta]$ such that $\mathcal{O}_F = \mathcal{O}_L[\eta, 1/u]$;*

(iii) *there are henselian elements $r, s \in \mathcal{O}_F$ such that $\mathcal{O}_F = \mathcal{O}_L[\eta, r, s]$;*

(iv) *there exists an element $v \notin \mathfrak{n} := \mathfrak{m}_F \cap \mathcal{O}_L[\eta]$ such that v belongs to every prime ideal of $\mathcal{O}_L[\eta]$ not contained in $\mathfrak{n}$, i.e.,*

$$\bigcap_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p} \not\subseteq \mathfrak{n},$$

where $\mathcal{S} = \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_L[\eta]) \mid \mathfrak{p} \not\subseteq \mathfrak{n}\}$;

(v) *for every chain of prime ideals $(\mathfrak{p}_i)_{i \in I}$ of $\mathcal{O}_L[\eta]$, if $\mathfrak{p}_i \not\subseteq \mathfrak{n}$ for every $i \in I$, then*

$$\bigcap_{i \in I} \mathfrak{p}_i \not\subseteq \mathfrak{n}.$$

In the next theorem, we give a valuation theoretical condition on the valued field (L, ν) for which the conditions above are satisfied.

Theorem 1.3.7. *Let $(F|L, \nu)$ be a finite inertial extension. Assume also that the chain of prime ideals of $\mathcal{O}_L$ is well-ordered by inclusion. Then the equivalent Conditions* **(i) - (v)** *of Theorem 1.3.6 are satisfied and in particular, $\mathcal{O}_F$ is a finitely generated $\mathcal{O}_L$-algebra.*

Theorem 1.3.7 is a generalization of part (3) in the main proposition of [47]. There, van den Dries establishes the case when the chain of prime ideals of $\mathcal{O}_L$ has finitely many elements.

The following theorem relates the local uniformization problem with the theory of henselian elements.

Theorem 1.3.8. *Let $(F|K, \nu)$ be a valued function field such that ν is trivial on K. Assume that for every finite set $Z \subseteq \mathcal{O}_F$ there exists a transcendence basis T of $F|K$ and elements $\eta_1, \ldots, \eta_r \in \mathcal{O}_F$ which are henselian over $K(T)$ such that $(K(T)|K, \nu)$ admits local uniformization and $Z \subseteq \mathcal{O}_{K(T)}[\eta_1, \ldots, \eta_r]$. Then $(F|K, \nu)$ admits local uniformization.*

As a consequence of the theorems above we obtain the following:

Theorem 1.3.9. *Let $(F|K, \nu)$ be a valued function field such that ν is trivial on K. Assume that there exists a transcendence basis T of $F|K$ such that $(K(T)|K, \nu)$ admits local uniformization and $F \subseteq K(T)^i$. Then $(F|K, \nu)$ admits local uniformization.*

In [23], Knaf and Kuhlmann proved a version of Theorem 1.3.9 without assuming that ν is trivial on K. They use the theory of local étale extensions and classical results from algebraic geometry. The advantage of our proof is that it is simpler and uses only tools from valuation theory.

1.4 Reduction of local uniformization to the rank one case

To prove local uniformization it is convenient to work first with rank one valuations. This is because, for instance, complete valued fields of rank one are henselian but this is not true, in general, for higher rank valuations. Hence, a natural way to handle local uniformization is to reduce the problem to rank one valuations. This reduction (Theorems 1.4.1, 1.4.2 and 1.4.3), which Spivakovsky and I published in [39], is one of the main results of this thesis.

The notion of local uniformization introduced in the previous section deals with a valued function field $(F|K,\nu)$. However, local uniformization is commonly presented as a property of a valuation centered at a noetherian local ring R. Namely, a valuation ν on $F=\text{Quot}(R)$ centered at R is said to admit **local uniformization** if there exists a local blowing up

$$R\longrightarrow R^{(1)} \tag{1.1}$$

with respect to ν such that $R^{(1)}$ is regular (see definitions of "local blowing up" and "regular local ring" in Section 2.2.5). In Theorem 6.2.1, we show how local uniformization for valued function fields is related to local uniformization for valuations centered at local rings.

Let $\mathcal{N}$ be the category of all noetherian local domains and let $\mathcal{M}\subseteq\mathcal{N}$ be a subcategory of $\mathcal{N}$ which is closed under taking homomorphic images, finitely generated birational extensions and localizations. We want to know for which subcategory $\mathcal{M}$ with these properties, all valuations centered at members of $\mathcal{M}$ admit local uniformization. Grothendieck conjectured that the subcategory which optimizes local uniformization and resolution of singularities is the category of all quasi-excellent local rings. However, this conjecture is widely open.

Our first result on the reduction of local uniformization to the rank one case is the following:

Theorem 1.4.1. *Assume that for every noetherian local domain R in $\mathcal{M}$, every rank one valuation centered at R admits local uniformization. Then all the valuations centered at members of $\mathcal{M}$ admit local uniformization.*

A stronger version of the local uniformization problem, called the **weak embedded local uniformization** problem, asks whether for every given finite subset $\mathcal{F}$ of R we can find a local blowing up as in (1.1) such that $R^{(1)}$ is regular and a regular system of parameters $u = (u_1, \ldots, u_d)$ of $R^{(1)}$ such that all elements of $\mathcal{F}$ are monomials in u.

Theorem 1.4.2. *Assume that for every noetherian local domain R in $\mathcal{M}$, every rank one valuation centered at R admits weak embedded local uniformization. Then all the valuations centered at members of $\mathcal{M}$ admit weak embedded local uniformization.*

We order the elements of the set $\mathcal{F}$ above by their values, i.e., $\mathcal{F} = \{f_1, \ldots, f_q\}$ such that $\nu(f_1) \leq \ldots \leq \nu(f_q)$. An even stronger version of the local uniformization problem asks whether we can find a regular local domain $R^{(1)}$ with regular system of parameters u as before such that the elements f_i are monomials in u and moreover, $f_1 \mid_{R^{(1)}} \ldots \mid_{R^{(1)}} f_q$. This version is called **embedded local uniformization**.

Theorem 1.4.3. *Assume that for every noetherian local domain R in $\mathcal{M}$, every rank one valuation centered at R admits embedded local uniformization. Then all the valuations centered at members of $\mathcal{M}$ admit embedded local uniformization.*

It is important to point out that our proofs for the theorems above do not depend on the nature of the category $\mathcal{M}$.

After de Jong's celebrated work ([20]) on alterations, many mathematicians have been trying to improve his results. A possible extension is to prove that resolution of singularities can be obtained in a purely inseparable extension of the function field. This is a conjecture appearing in [5]. Temkin proved the local form of this extension, i.e., he proved "inseparable local uniformization". To obtain that, he first settled the rank one case and then proved the reduction to the rank one case. We can deduce this reduction from Theorem 1.4.1. Fix a prime number p and consider a subcategory $\mathcal{M}_p$ of $\mathcal{N}$ which is closed under taking homomorphic images, finitely generated birational extensions, localizations and with the additional property:

- *For every* $(R, \mathfrak{m}) \in \mathcal{M}_p$*, char*$(R) = p$ *and if* $a \in \mathfrak{m}$*, then* $(R[a^{1/p}], (\mathfrak{m}, a)) \in \mathcal{M}_p$.

Theorem 1.4.4. *Assume that for every local domain* R *in* $\mathcal{M}_p$*, every rank one valuation centered at* R *admits inseparable local uniformization (see definition of inseparable local uniformization in Section 6.3). Then all the valuations centered at members of* $\mathcal{M}_p$ *admit inseparable local uniformization.*

Our proof of Theorem 1.4.4 is particularly important because it only uses basic tools of valuation theory and commutative algebra.

Chapter 2

Preliminaries

We start by presenting some basic notions related to a valuation.

2.1 Valuations on a ring R

Definition 2.1.1. Take a commutative ring R with unity. A **valuation** on R is a mapping $\nu : R \longrightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$ where Γ is an ordered abelian group (and the extension of addition and order to ∞ is as usual), with the following properties:

(V1) $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ for all $\phi, \psi \in R$.

(V2) $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ for all $\phi, \psi \in R$.

(V3) $\nu(1) = 0$ and $\nu(0) = \infty$.

Remark 2.1.2. Under Conditions **(V1)** and **(V2)** the statement **(V3)** is equivalent to requiring that ν is not constant. Indeed, if ν is not constant, then there exists $\phi \in R$ such that $\nu(\phi) = \alpha \neq \infty$. Then

$$\alpha = \nu(\phi) = \nu(\phi \cdot 1) = \nu(\phi) + \nu(1) = \alpha + \nu(1)$$

which implies that $\nu(1) = 0$. Also, since ν is not constant, there exists $\psi \in R$ such that $\nu(\psi) = \beta \neq 0$. Then we have that

$$\nu(0) = \nu(\psi \cdot 0) = \nu(\psi) + \nu(0) = \beta + \nu(0)$$

which is possible only if $\nu(0) = \infty$. The converse is trivial.

Let $\nu : R \longrightarrow \Gamma_\infty$ be a valuation. The subgroup of Γ generated by

$$\{\nu(\phi) \mid \phi \in R \text{ and } \nu(\phi) \neq \infty\}$$

is called the **value group of** ν and is denoted by νR. The valuation ν is called **trivial** if $\nu R = \{0\}$. The set $\mathfrak{q}_\nu := \nu^{-1}(\infty)$ is a prime ideal of R, called the **support of** ν.

Definition 2.1.3. A valuation $\nu : R \longrightarrow \Gamma_\infty$ is a **Krull valuation** if $\mathfrak{q}_\nu = \{0\}$.

Remark 2.1.4. **(i)** If R admits a Krull valuation ν, then R is a domain and we can extend ν to a valuation on the field $F = \text{Quot}(R)$ by defining

$$\nu\left(\frac{\phi}{\psi}\right) = \nu\phi - \nu\psi.$$

(ii) A non-trivial valuation ν on a field F is automatically a Krull valuation (because the only ideals in a field are the zero ideal and the whole field). Therefore, when we are working with a field, we will not make a distinction between valuations and Krull valuations.

We denote by $\widetilde{\mathcal{W}}$ the class of all valuations on R. Also, we denote by $\widetilde{\mathcal{V}}$ the subset of $\widetilde{\mathcal{W}}$ defined by

$$\widetilde{\mathcal{V}} = \{\nu \in \widetilde{\mathcal{W}} \mid \nu \text{ is a Krull valuation}\}.$$

Given a valuation ν on R, we can define a Krull valuation

$$\overline{\nu} : R/\mathfrak{q}_\nu \longrightarrow \Gamma_\infty$$

by setting $\overline{\nu}(\overline{\phi}) = \nu\phi$, where $\overline{\phi}$ is the reduction of ϕ modulo $\mathfrak{q}_\nu$.

Lemma 2.1.5. *Take two valuations ν and μ of R. Then the following conditions are equivalent*

i) *For all $\phi, \psi \in R$, $\nu(\phi) > \nu(\psi)$ if and only if $\mu(\phi) > \mu(\psi)$.*

ii) *There is an order-preserving isomorphism $f : \nu R \longrightarrow \mu R$ such that $\mu = f \circ \nu$.*

iii) *$\mathfrak{q}_\nu = \mathfrak{q}_\mu$ and for any $\overline{\phi}/\overline{\psi} \in \text{Quot}(R/\mathfrak{q}_\nu) = \text{Quot}(R/\mathfrak{q}_\mu)$ we have that $\overline{\nu}(\overline{\phi}/\overline{\psi}) \geq 0$ if and only if $\overline{\mu}(\overline{\phi}/\overline{\psi}) \geq 0$.*

Proof. Assume first **(i)** and let us prove **(ii)**. We define the mapping $f : \nu R \longrightarrow \mu R$ by setting $f(\nu(\phi)) = \mu(\phi)$ for $\phi \in R$ and extend it to νR by additivity (we can do that because $\{\nu(\phi) \mid \phi \in R\}$ generates νR). Since $\mu = f \circ \nu$, it remains to show that f is an order-preserving group isomorphism. To prove that f is additive we only have to prove that it is additive in the set of generators. Take $\phi, \psi \in R$. Using Property **(V1)** of valuations we obtain that

$$f(\nu(\phi) + \nu(\psi)) = f(\nu(\phi\psi)) = \mu(\phi\psi) = \mu(\phi) + \mu(\psi) = f(\nu(\phi)) + f(\nu(\psi)).$$

Hence, f is a group homomorphism. From the assumption **(i)**, we conclude that f is order preserving and hence injective. The surjectivity follows from the definitions of f and the value group of a valuation. Therefore, f is an order-preserving group isomorphism with $\mu = f \circ \nu$, which is what we wanted to prove.

Now assume that **(ii)** holds. For each $\phi \in R$ we have

$$\phi \notin \mathfrak{q}_\nu \Longleftrightarrow \nu(\phi) \in \nu R \Longleftrightarrow \mu(\phi) = f \circ \nu(\phi) \in \mu R \Longleftrightarrow \phi \notin \mathfrak{q}_\mu.$$

Hence, $\mathfrak{q}_\nu = \mathfrak{q}_\mu$. Take an element $\overline{\phi}/\overline{\psi} \in \operatorname{Quot}(R/\mathfrak{q}_\nu)$ (observe that this implies that $\psi \notin \mathfrak{q}_\nu = \mathfrak{q}_\mu$). Then

$$\overline{\nu}(\overline{\phi}/\overline{\psi}) \geq 0 \Longleftrightarrow \nu(\phi) \geq \nu(\psi) \Longleftrightarrow \mu(\phi) \geq \mu(\psi) \Longleftrightarrow \overline{\mu}(\overline{\phi}/\overline{\psi}) \geq 0,$$

where the second equivalence holds because of our assumption **(ii)**. Therefore, **(iii)** holds.

It remains to prove that **(iii)** implies **(i)**. Assume that **(iii)** holds and take $\phi, \psi \in R$. If $\psi \in \mathfrak{q}_\nu = \mathfrak{q}_\mu$, then neither $\nu(\phi) > \nu(\psi)$ nor $\mu(\phi) > \mu(\psi)$ can be true, so assume that $\psi \notin \mathfrak{q}_\nu = \mathfrak{q}_\mu$. If $\phi \in \mathfrak{q}_\nu = \mathfrak{q}_\mu$, then $\nu(\phi) > \nu(\psi)$ and $\mu(\phi) > \mu(\psi)$. It remains to show that if $\phi, \psi \notin \mathfrak{q}_\nu = \mathfrak{q}_\mu$, then $\nu(\phi) > \nu(\psi)$ if and only if $\mu(\phi) > \mu(\psi)$. For this case we have that

$$\begin{aligned} \nu(\phi) > \nu(\psi) &\Longleftrightarrow \overline{\nu}(\overline{\psi}/\overline{\phi}) < 0 \Longleftrightarrow \overline{\nu}(\overline{\psi}/\overline{\phi}) \ngeq 0 \Longleftrightarrow \overline{\mu}(\overline{\psi}/\overline{\phi}) \ngeq 0 \\ &\Longleftrightarrow \overline{\mu}(\overline{\psi}/\overline{\phi}) < 0 \Longleftrightarrow \mu(\phi) > \mu(\psi) \end{aligned}$$

which concludes our proof.

□

Definition 2.1.6. The valuations ν and μ on R are said to be equivalent (and denote by $\nu \sim \mu$) if one (and hence all) of the conditions above are satisfied.

Remark 2.1.7. If ν and μ are two real valued valuations, i.e., their codomain is the real numbers, then $\nu \sim \mu$ if and only if $\nu = C \cdot \mu$ for some $C \in \mathbb{R}$ and $C > 0$.

We denote by $\mathcal{W}$ (or $\mathcal{V}$) the quotient of $\widetilde{\mathcal{W}}$ (or $\widetilde{\mathcal{V}}$, respectively) by the equivalence relation defined above, i.e., $\mathcal{W} = \widetilde{\mathcal{W}}/\sim$ (or $\mathcal{V} = \widetilde{\mathcal{V}}/\sim$, respectively). Given a valuation $\nu \in \widetilde{\mathcal{W}}$ (or $\widetilde{\mathcal{V}}$, respectively), we denote its equivalence class by $[\nu] \in \mathcal{W}$ (or $\mathcal{V}$, respectively), i.e.,

$$[\nu] = \{\mu \in \widetilde{\mathcal{W}} \text{ (or } \widetilde{\mathcal{V}}, \text{ respectively)} \mid \nu \sim \mu\}.$$

Take a local ring $(R, \mathfrak{m})$ and ν a valuation on R. We will say that ν is **centered** at R if $\nu(\phi) \geq 0$ for all $\phi \in R$ and $\nu(\phi) > 0$ for all $\phi \in \mathfrak{m}$. If in addition R is noetherian, then $\mathfrak{m}$ is finitely generated, so we can define

$$\nu(\mathfrak{m}) := \min\{\nu(\phi) \mid \phi \in \mathfrak{m}\}.$$

If ν is a valuation on a field F and R is a subring of F such that $R \subseteq \mathcal{O}_\nu \subseteq F$ and $F = \operatorname{Quot}(R)$, then the **center** of ν on R is defined to be the prime ideal

$$\mathfrak{p}_\nu = \{\phi \in R \mid \nu(\phi) > 0\} = \mathfrak{m}_\nu \cap R.$$

In this case, ν is centered at $R_{\mathfrak{p}_\nu}$. We denote by $\widetilde{\mathcal{W}}^{\geq 0}$ and $\widetilde{\mathcal{V}}^{\geq 0}$ the subsets of $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{V}}$, respectively, consisting of the valuations on R which take only non-negative values (for short, **non-negative valuations**). We also write $\mathcal{W}^{\geq 0} = \widetilde{\mathcal{W}}^{\geq 0}/\sim$ and $\mathcal{V}^{\geq 0} = \widetilde{\mathcal{V}}^{\geq 0}/\sim$.

2.1.1 Topologies on spaces of valuations

We will assume that the reader is familiar with the basic notions of topology. For a reference, we suggest [13]. It is worth to mention that we will treat the properties of being "compact" and "Hausdorff" as two separate properties (hence we are not going to use the term "quasi-compact"), namely:

Definition 2.1.8. A topological space $(X, \mathfrak{A})$ is compact if every open covering $\mathcal{C}$ of X admits a finite subcovering $\mathcal{C}'$. The space $(X, \mathfrak{A})$ is said to be Hausdorff if for every two distinct points $x_1, x_2 \in X$ there exist open sets $U_1, U_2 \in \mathfrak{A}$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Our next goal is to define topologies on $\mathcal{W}$ and $\mathcal{V}$. The best known topology on the space of equivalence classes of Krull valuations is the Zariski topology:

Definition 2.1.9. The **Zariski topology** on $\mathcal{V}$ is the topology having as a subbasis the sets of the form

$$\{[\nu] \in \mathcal{V} \mid \nu(\phi) \geq 0\}$$

where ϕ runs through $F = \mathrm{Quot}(R)$.

Remark 2.1.10. It is proved in [52] that this topology is compact but not Hausdorff.

The coarsest Hausdorff topology on $\mathcal{V}$ which is finer than the Zariski topology is the Zariski patch (also called constructive) topology:

Definition 2.1.11. The **Zariski patch topology** on $\mathcal{V}$ is defined to be the topology having as a subbasis the sets of the form

$$\{[\nu] \in \mathcal{V} \mid \nu(\phi) \geq 0\} \text{ and } \{[\nu] \in \mathcal{V} \mid \nu(\psi) > 0\}$$

where ϕ and ψ run through F.

For a discussion about these topologies, see the appendix of [28].

Definition 2.1.12. Take a set X and a family $\mathcal{F} = \{(X_i, \mathfrak{A}_i, \Phi_i) \mid i \in I\}$ where for every $i \in I$, $(X_i, \mathfrak{A}_i)$ is a topological space and $\Phi_i : X \longrightarrow X_i$ is a function. We define the **weak topology on X associated to $\mathcal{F}$** to be the coarsest topology which makes all the Φ_i continuous. It is equivalent to say that this topology is the topology having as a subbasis all sets of the form $\Phi_i^{-1}(U_i)$ with $U_i \in \mathfrak{A}_i$ and $i \in I$.

Remark 2.1.13. For every $\phi \in F = \mathrm{Quot}(R)$ let

$$\phi^* : \mathcal{V} \longrightarrow \{0, -, +\}$$

be the function given by

$$\phi^*([\nu]) = \begin{cases} 0 & \text{if } \nu(\phi) = 0, \\ - & \text{if } \nu(\phi) < 0, \\ + & \text{if } \nu(\phi) > 0. \end{cases}$$

Endow $X := \{0, -, +\}$ with the topologies

$$\mathfrak{A}_1 := \{\emptyset, \{0, +\}, X\} \text{ and } \mathfrak{A}_2 := \{\emptyset, \{+\}, \{0, +\}, X\}.$$

Then the Zariski topology is the weak topology on $\mathcal{V}$ induced by $\{(X, \mathfrak{A}_1, \phi^*) \mid \phi \in F\}$ and the Zariski patch topology is the weak topology on $\mathcal{V}$ induced by $\{(X, \mathfrak{A}_2, \phi^*) \mid \phi \in F\}$.

The way that the Zariski topology is constructed on $\mathcal{V}$ (Definition 2.1.9) cannot be applied to the set $\mathcal{W}$ of all equivalence classes of valuations. That is because R does not need to be a domain. Even if R is a domain we cannot guarantee that every valuation on R can be extended to $F = \operatorname{Quot}(R)$. To overcome this problem, Huber and Knebusch introduced the valuation spectrum topology (see [19]).

Definition 2.1.14. We define the **valuation spectrum topology** on $\mathcal{W}$ as the topology having as a subbasis the sets

$$\{[\nu] \in \mathcal{W} \mid \nu\phi \geq \nu\psi \neq \infty\}$$

where ϕ and ψ run through R.

One can see that the restriction of the valuation spectrum topology from $\mathcal{W}$ to $\mathcal{V}$ is the Zariski topology.

We will describe below an approach used by Berkovich in [7] and by Favre and Jonsson in [14] to define topologies on sets of valuations. Take a noetherian local ring R with maximal ideal $\mathfrak{m}$ and an ordered abelian group Γ.

Definition 2.1.15. For each positive element $\gamma \in \Gamma$ we say that a centered valuation $\nu : R \longrightarrow \Gamma_\infty$ is **normalized by** γ if $\nu(\mathfrak{m}) = \gamma$.

We denote by $\widetilde{\mathcal{W}}_\Gamma$ (or $\widetilde{\mathcal{V}}_\Gamma$) the subset of $\widetilde{\mathcal{W}}$ (or $\widetilde{\mathcal{V}}$, respectively) consisting of all centered valuations having Γ_∞ as their codomain, i.e.,

$$\widetilde{\mathcal{W}}_\Gamma = \{\nu \in \widetilde{\mathcal{W}} \mid \nu : R \longrightarrow \Gamma_\infty\}.$$

Definition 2.1.16. Consider the subset $\widetilde{\mathcal{W}}_{\mathfrak{m}}$ of $\widetilde{\mathcal{W}}_{\mathbb{R}}$ consisting of all centered valuations normalized by 1. We define the weak topology on $\widetilde{\mathcal{W}}_{\mathfrak{m}}$ to be the topology having as a subbasis the sets of the form

$$\{\nu \in \widetilde{\mathcal{W}}_{\mathfrak{m}} \mid \nu(\phi) > \alpha\} \text{ and } \{\nu \in \widetilde{\mathcal{W}}_{\mathfrak{m}} \mid \nu(\phi) < \alpha\}$$

where α runs through $\mathbb{R}_\infty$ and ϕ runs through R.

Remark 2.1.17. If R is a two-dimensional regular local ring, then $\widetilde{\mathcal{W}}_{\mathfrak{m}}$ admits a tree structure, which will be called the valuative tree of R. See Chapter 3 for a detailed discussion.

Remark 2.1.18. The sets of the form

$$\{r \in \mathbb{R}_\infty \mid r > \alpha\} \text{ and } \{r \in \mathbb{R}_\infty \mid r < \alpha\} \text{ with } \alpha \text{ running through } \mathbb{R}_\infty$$

form a subbasis of open sets for the order topology on $\mathbb{R}_\infty$. The product topology on $(\mathbb{R}_\infty)^R$ is the weak topology associated to the projections into $\mathbb{R}_\infty$. Hence, the topology defined in 2.1.16 is the topology on $\widetilde{\mathcal{W}}_{\mathfrak{m}} \subseteq (\mathbb{R}_\infty)^R$ induced by the product topology on $(\mathbb{R}_\infty)^R$.

2.2 Valuations on a field F

From remark 2.1.4 we know that there is a bijection between the set of valuations on a field F and the set of Krull valuations on a fixed ring R with $F = \mathrm{Quot}(R)$.

2.2.1 Valuation vs. valuation ring

Definition 2.2.1. A **valuation ring** is a domain $\mathcal{O}$ such that for every non-zero element $r \in \mathrm{Quot}(\mathcal{O})$, if $r \notin \mathcal{O}$, then $r^{-1} \in \mathcal{O}$. If F is a field, then a valuation ring of F is a valuation ring $\mathcal{O}$ for which $F = \mathrm{Quot}(\mathcal{O})$.

Lemma 2.2.2. *Take a valuation ν on a field F. The ring*

$$\mathcal{O}_\nu := \{a \in F \mid \nu(a) \geq 0\}$$

is a valuation ring, which will be called the **valuation ring of** ν. *Moreover, if $r \notin \mathcal{O}_\nu$, then $r^{-1} \in \mathfrak{m}_\nu = \{a \in F \mid \nu(a) > 0\}$.*

Proof. If $r \notin \mathcal{O}_\nu$, then $\nu(r) < 0$. Consequently, $\nu(r^{-1}) = -\nu(r) > 0$ and so $r^{-1} \in \mathfrak{m}_\nu$. $\square$

Remark 2.2.3. From the definition of equivalences of valuations, we see that two valuations on a field are equivalent if and only if they have the same valuation ring.

Lemma 2.2.4. *Every valuation ring is a local ring.*

Proof. We have to prove that the set $\mathfrak{m} = \mathcal{O} \setminus \mathcal{O}^\times$ is an ideal of $\mathcal{O}$, where $\mathcal{O}^\times$ denotes the set of units of $\mathcal{O}$. If $r \in \mathfrak{m}$, then $r \notin \mathcal{O}^\times$. Hence, $rs \notin \mathcal{O}^\times$ and consequently $rs \in \mathfrak{m}$ for every $s \in \mathcal{O}$. Take $r, s \in \mathfrak{m}$. If $r = 0$ or $s = 0$, then $r + s \in \mathfrak{m}$. Since $\mathcal{O}$ is a valuation ring, if $r \neq 0$ and $s \neq 0$, then $s/r \in \mathcal{O}$ or $r/s \in \mathcal{O}$. Assume, without loss of generality that $s/r \in \mathcal{O}$. Then $r + s = r(1 + s/r) \in \mathfrak{m}$ because $r \in \mathfrak{m}$ and $1 + s/r \in \mathcal{O}$. Therefore, $\mathfrak{m}$ is an ideal of $\mathcal{O}$. $\square$

As a corollary of the lemmas above, we obtain that $\mathcal{O}_\nu$ is a local ring and one can easily check that its unique maximal ideal is $\mathfrak{m}_\nu$.

Lemma 2.2.5. *Take a valuation ring $\mathcal{O}$. Then there exists a valuation ν on $F = \mathrm{Quot}(\mathcal{O})$ such that $\mathcal{O} = \mathcal{O}_\nu$. This valuation will be called the* **valuation associated to** $\mathcal{O}$.

Proof. Given the valuation ring $\mathcal{O}$ we set $\Gamma = F^\times/\mathcal{O}^\times$, where $F^\times$ is seen as the multiplicative group of $F = \mathrm{Quot}(\mathcal{O})$ and $\mathcal{O}^\times$ as a (normal) subgroup of $F^\times$. Since this group is abelian we write it additively, i.e.,

$$r\mathcal{O}^\times + s\mathcal{O}^\times := rs\mathcal{O}^\times.$$

We define an order on Γ by setting $r\mathcal{O}^\times \geq s\mathcal{O}^\times$ if and only if $r/s \in \mathcal{O}$. We have to prove that:

- The order $\geq$ is well-defined.

  Take elements $r, s, r', s' \in F$ such that $r'\mathcal{O}^\times = r\mathcal{O}^\times$ and $s'\mathcal{O}^\times = s\mathcal{O}^\times$. This implies that r'/r and s'/s belong to $\mathcal{O}$. If $r/s \in \mathcal{O}$, then

  $$r'/s' = r'/r \cdot r/s \cdot s/s' \in \mathcal{O}.$$

  Therefore, the order $\geq$ is well-defined.

- The order $\geq$ is a group order.

  Take any elements $r, s \in F^\times$ and suppose that $r\mathcal{O}^\times \geq s\mathcal{O}^\times$. For every $t \in F^\times$ we have that $tr/ts = r/s \in \mathcal{O}$. Hence, $t\mathcal{O}^\times + r\mathcal{O}^\times \geq t\mathcal{O}^\times + s\mathcal{O}^\times$, which is what we wanted to prove.

Define now the mapping $\nu : F \longrightarrow \Gamma_\infty$ given by $\nu(0) = \infty$ and $\nu(r) = r\mathcal{O}^\times$ if $r \neq 0$. We have to prove that ν is a valuation. From the definition, we have that

$$\nu(rs) = \nu(r) + \nu(s),\ \nu(0) = \infty \text{ and } \nu(1) = 0 \text{ (because } 1 \in \mathcal{O}^\times).$$

It remains to prove that $\nu(r + s) \geq \min\{\nu(r), \nu(s)\}$. If $r = 0$ or $s = 0$ the assertion is trivial, so assume that $r \neq 0$ and $s \neq 0$. Assume without loss of generality that $\nu(r) \geq \nu(s)$. This means that $r/s \in \mathcal{O}$. Then also $(r + s)/s = r/s + 1$ belongs to $\mathcal{O}$. Therefore, $\nu(r + s) \geq \nu(s) = \min\{\nu(r), \nu(s)\}$.

To conclude our proof we have to show that $\mathcal{O} = \mathcal{O}_\nu$. Take $r \in F$. If $r = 0$, then $r \in \mathcal{O}_\nu$ and $r \in \mathcal{O}$ because these are both subrings of F. If $r \neq 0$, then

$$r \in \mathcal{O}_\nu \Longleftrightarrow \nu(r) \geq 0 = \nu(1) \Longleftrightarrow r = r/1 \in \mathcal{O}.$$

Therefore, $\mathcal{O} = \mathcal{O}_\nu$. □

A **valued field** is a pair (F, ν) where F is a field and ν is a valuation on F. For any subfield L of F we denote by

$$\mathcal{O}_L := \mathcal{O}_\nu \cap L \text{ and } \mathfrak{m}_L := \mathfrak{m}_\nu \cap L.$$

A **valued field extension** $(F|L, \nu)$ is a field extension $F|L$ together with a valuation ν on F. If the extension $F|K$ is an algebraic function field, then $(F|K, \nu)$ is called a **valued function field**.

2.2.2 Valuation ring vs. place

Definition 2.2.6. A **place on the field** F is a homomorphic mapping P of a subring $\mathcal{O}_P$ of F into a field FP, such that the following conditions are satisfied:

(P1) if $r \in F$ and $r \notin \mathcal{O}_P$, then $1/r \in \mathcal{O}_P$ and $(1/r)P = 0$;

(P2) $rP \neq 0$ for some $r \in \mathcal{O}_P$.

In this case, the field FP is called the **residue field of** P.

We introduce the symbol ∞ and agree to write $rP = \infty$ if $r \notin \mathcal{O}_P$. From the definitions of place and valuation ring, we see that for every place P of F the ring $\mathcal{O}_P$ is a valuation ring, which will be called the **valuation ring of the place** P.

Lemma 2.2.7. *Given a valuation ring $\mathcal{O}$ there exists a place P such that $\mathcal{O}_P = \mathcal{O}$.*

Proof. Let $\mathcal{O}$ be any valuation ring of F. From Lemma 2.2.4 we have that $\mathcal{O}$ is a local ring with unique maximal ideal $\mathfrak{m} = \mathcal{O} \setminus \mathcal{O}^{\times}$. We define the mapping P by

$$\begin{aligned} P: \ \mathcal{O} &\longrightarrow \mathcal{O}/\mathfrak{m} \\ r &\longmapsto r + \mathfrak{m} \end{aligned}$$

Since $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}$ we have that $\mathcal{O}/\mathfrak{m}$ is a field. If $r \in F$ and $r \notin \mathcal{O}$, then from the definition of $\mathcal{O}$ we have that $1/r \in \mathcal{O}$. Moreover, $1/r$ is not a unit of $\mathcal{O}$ which implies that $1/r \in \mathfrak{m}$. Hence, $(1/r)P = 0$ and Property **(P1)** is satisfied. Since $\mathfrak{m}$ is the set of non-units of $\mathcal{O}$ we have that $1 \in \mathcal{O} \setminus \mathfrak{m}$, so $1P \neq 0$. Consequently, Property **(P2)** is satisfied and P is a place of F with $\mathcal{O}_P = \mathcal{O}$. □

The place P for which $\mathcal{O}_P = \mathcal{O}$ is called the **place associated to** $\mathcal{O}$ and if $\mathcal{O}$ is the valuation ring of a valuation ν, then it is called the **place associated to** ν. For any place P, the valuation associated to $\mathcal{O}_P$ is also called the **valuation associated to** P.

2.2.3 The value group of a valuation

Let Γ be an ordered abelian group. We denote by $\mathcal{S}_\Gamma$ the set of convex subgroups of Γ.

Lemma 2.2.8. *The set $\mathcal{S}_\Gamma$ is totally ordered by subset inclusion.*

Proof. Subset inclusion is always a partial order, so we just have to prove that for every pair of convex subgroups Δ, Δ' of Γ, either $\Delta \subseteq \Delta'$ or $\Delta' \subseteq \Delta$.

Assume that $\Delta \not\subseteq \Delta'$. Then there exists $\gamma \in \Gamma$ such that $\gamma \in \Delta \setminus \Delta'$. If there would exist $\gamma_1, \gamma_2 \in \Delta'$ such that $\gamma_1 \leq \gamma \leq \gamma_2$, then γ would belong to Δ' which is a contradiction. Therefore, either $\gamma > \Delta'$ or $\gamma < \Delta'$. This means that either $-\gamma < \Delta' < \gamma$ or $\gamma < \Delta' < -\gamma$. In either case we have that $\Delta' \subseteq \Delta$ because Δ is a convex subset of Γ and $\gamma, -\gamma \in \Delta$. □

Definition 2.2.9. From Lemma 2.2.8 we obtain that the set $\mathcal{S}_{\nu F}$ is totally ordered, so we define the **rank of** ν to be the order type of $\mathcal{S}_{\nu F}$.

Remark 2.2.10. Observe that if $\mathfrak{p}$ is an ideal of $\mathcal{O}_\nu$, then

$$\nu(\mathfrak{p}) = \{\nu\phi \mid \phi \in \mathfrak{p}\}$$

is a final segment of $\nu F \cup \{\infty\}$. Indeed, take $\phi \in \mathfrak{p}$ and $\gamma = \nu\psi \in \nu F \cup \{\infty\}$ such that $\gamma = \nu\psi > \nu\phi$. From the definition of $\mathcal{O}_\nu$ we have that $\dfrac{\psi}{\phi} \in \mathcal{O}_\nu$. Consequently,

$$\psi = \phi \cdot \frac{\psi}{\phi} \in \mathfrak{p},$$

and hence $\gamma \in \nu(\mathfrak{p})$.

Lemma 2.2.11. *There exists an order-reversing bijection between* $\mathrm{Spec}(\mathcal{O}_\nu)$ *and* $\mathcal{S}_{\nu F}$.

Proof. Define

$$\Phi : \mathrm{Spec}(\mathcal{O}_\nu) \longrightarrow \mathcal{S}_{\nu F} \text{ and } \Psi : \mathcal{S}_{\nu F} \longrightarrow \mathrm{Spec}(\mathcal{O}_\nu)$$

by

$$\Phi(\mathfrak{p}) = \Delta_\mathfrak{p} := \{\gamma \in \nu F \mid -\nu(\mathfrak{p}) < \gamma < \nu(\mathfrak{p})\} \text{ and } \Psi(\Delta) = \mathfrak{p}_\Delta := \{\phi \in \mathcal{O}_\nu \mid \nu\phi > \Delta\}.$$

We have to prove:

- If $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_\nu)$, then $\Delta_\mathfrak{p} \in \mathcal{S}_{\nu F}$.

  From the definition, we have that $\Delta_\mathfrak{p}$ is convex and that $0 \in \Delta_\mathfrak{p}$. It remains to prove that it is closed under addition. Take any elements $\gamma, \gamma' \in \Delta_\mathfrak{p}$, say, $\gamma = \nu(\phi)$ and $\gamma' = \nu(\phi')$. If $\gamma, \gamma' \geq 0$, then $\phi, \phi' \in \mathcal{O}_\nu \setminus \mathfrak{p}$ and since $\mathfrak{p}$ is a prime ideal we have that $\phi \cdot \phi' \notin \mathfrak{p}$. Since $\nu(\mathfrak{p})$ is a final segment in νF we conclude that $\nu(\phi \cdot \phi') < \nu(\mathfrak{p})$. Thus

$$-\nu(\mathfrak{p}) < 0 \leq \gamma + \gamma' = \nu(\phi) + \nu(\phi') = \nu(\phi \cdot \phi') < \nu(\mathfrak{p}),$$

and hence $\gamma+\gamma' \in \Delta_{\mathfrak{p}}$. If $\gamma,\gamma' \leq 0$, then $-\gamma,-\gamma' \geq 0$ and we obtain that $0 \leq -(\gamma+\gamma') < \nu(\mathfrak{p})$. Therefore, $-\nu(\mathfrak{p}) < \gamma+\gamma' \leq 0$ and again $\gamma+\gamma' \in \Delta_{\mathfrak{p}}$. For the remaining case, assume without loss of generality that $\gamma' \leq 0 \leq \gamma$. Then

$$\gamma' \leq \gamma+\gamma' \leq \gamma$$

and by the convexity of $\Delta_{\mathfrak{p}}$ we obtain that $\gamma+\gamma' \in \Delta_{\mathfrak{p}}$.

- If $\Delta \in \mathcal{S}_{\nu F}$, then $\mathfrak{p}_\Delta \in \operatorname{Spec}(\mathcal{O}_\nu)$.

  From the properties of valuations we have that $\mathfrak{p}_\Delta$ is an ideal of $\mathcal{O}_\nu$. Take $\phi \in \mathcal{O}_\nu \setminus \mathfrak{p}_\Delta$. Since $\nu(\phi) \not> \Delta$, there exists $\gamma \in \Delta$ such that $\nu(\phi) \leq \gamma$. By the convexity of Δ (and from the fact that $\nu(\phi) \geq 0$) we obtain that $\nu(\phi) \in \Delta$. Take $\phi,\phi' \in \mathcal{O}_\nu \setminus \mathfrak{p}_\Delta$. Then $\nu(\phi),\nu(\phi') \in \Delta$, and since Δ is a group we have

  $$\nu(\phi\cdot\phi') = \nu(\phi)+\nu(\phi') \in \Delta.$$

  Therefore, $\phi\cdot\phi' \notin \mathfrak{p}_\Delta$ and consequently $\mathfrak{p}_\Delta \in \operatorname{Spec}(\mathcal{O}_\nu)$.

- Φ is a bijection.

  We are going to prove that $\Psi\circ\Phi(\mathfrak{p}) = \mathfrak{p}$ and $\Phi\circ\Psi(\Delta) = \Delta$ for every $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_\nu)$ and every $\Delta \in \mathcal{S}_{\nu F}$.

  If $\phi \in \mathfrak{p}$, then for every $\gamma \in \Delta_{\mathfrak{p}}$ we have that $\gamma < \nu\phi$. Thus $\nu\phi > \Delta_{\mathfrak{p}}$ and consequently $\phi \in \mathfrak{p}_{\Delta_{\mathfrak{p}}} = \Psi\circ\Phi(\mathfrak{p})$. On the other hand, if $\phi \in \mathcal{O}_\nu \setminus \mathfrak{p}$, then $0 \leq \nu\phi < \nu(\mathfrak{p})$ (because $\nu(\mathfrak{p})$ is a final segment of νF) which means that $\nu\phi \in \Delta_{\mathfrak{p}}$. Consequently, $\phi \notin \mathfrak{p}_{\Delta_{\mathfrak{p}}} = \Psi\circ\Phi(\mathfrak{p})$. Therefore, $\Psi\circ\Phi(\mathfrak{p}) = \mathfrak{p}$.

  If $\gamma \in \Delta$, then for every $\phi \in \mathfrak{p}_\Delta$ we have $-\nu\phi < \gamma < \nu\phi$. Thus $-\nu(\mathfrak{p}) < \gamma < \nu(\mathfrak{p})$ and consequently $\gamma \in \Phi\circ\Psi(\Delta)$. On the other hand, take $\gamma = \nu\phi \in \nu F$ and assume that $\gamma \notin \Delta$. Since Δ is a convex subset of νF we have that either $\gamma > \Delta$ or $\gamma < \Delta$. In this case, if $\gamma > \Delta$, then $\phi \in \mathfrak{p}_\Delta$ and if $\gamma < \Delta$, then $\phi^{-1} \in \mathfrak{p}_\Delta$. In either case we conclude that $\gamma \notin \Phi\circ\Psi(\Delta)$. Therefore, $\Phi\circ\Psi(\Delta) = \Delta$.

- Φ is order reversing.

  Suppose that $\mathfrak{p} \subseteq \mathfrak{q}$. Then $\nu(\mathfrak{p}) \subseteq \nu(\mathfrak{q})$ and therefore, $\Delta_{\mathfrak{q}} \subseteq \Delta_{\mathfrak{p}}$.

□

Corollary 2.2.12. *The set of prime ideals of a valuation ring is totally ordered with respect to set inclusion.*

Proof. Take any valuation ring $\mathcal{O}$. From Lemma 2.2.5 there exists a valuation ν on $F = \mathrm{Quot}(\mathcal{O})$ such that $\mathcal{O} = \mathcal{O}_\nu$. Take two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathcal{O}$. Lemma 2.2.8 gives us that either $\Delta_\mathfrak{p} \subseteq \Delta_\mathfrak{q}$ or $\Delta_\mathfrak{q} \subseteq \Delta_\mathfrak{p}$. Applying Lemma 2.2.11, we obtain that either $\mathfrak{q} \subseteq \mathfrak{p}$ or $\mathfrak{p} \subseteq \mathfrak{q}$. □

2.2.4 The decomposition of a valuation

Take a valued field (F, ν) and a convex subgroup Δ of νF. Let

$$\pi_\Delta : \nu F \longrightarrow \nu F/\Delta$$

be the canonical epimorphism of νF onto the quotient group $\nu F/\Delta$. Then the function

$$\nu_\Delta = \pi_\Delta \circ \nu$$

is a valuation on F whose valuation ring $\mathcal{O}_{\nu_\Delta}$ contains $\mathcal{O}_\nu$. The valuation ν induces a valuation $\overline{\nu}_\Delta : F\nu_\Delta \longrightarrow \Delta_\infty$ by setting $\nu(0) = \infty$ and

$$\overline{\nu}_\Delta(a + \mathfrak{m}_{\nu_\Delta}) = \nu(a)$$

for $a \in \mathcal{O}_\nu \setminus \mathfrak{m}_\nu$.

Consider the places P_Δ and $\overline{P}_\Delta$ associated to ν_Δ and $\overline{\nu}_\Delta$, respectively, and let $P_\Delta\overline{P}_\Delta$ be the composition of P_Δ and $\overline{P}_\Delta$ as functions, i.e.,

$$\begin{array}{rccl} P_\Delta\overline{P}_\Delta : & F & \longrightarrow & F\overline{\nu}_\Delta \cup \{\infty\} \\ & a & \longmapsto & \begin{cases} (aP_\Delta)\,\overline{P}_\Delta & \text{, if } a \in \mathcal{O}_{\nu_\Delta} \\ \infty & \text{, otherwise.} \end{cases} \end{array}$$

Then $P_\Delta\overline{P}_\Delta$ is a place of F and we can consider the valuation $\nu_\Delta \circ \overline{\nu}_\Delta$ in F as the valuation associated to $P_\Delta\overline{P}_\Delta$. It is easy to see that the valuation rings of ν and $\nu_\Delta \circ \overline{\nu}_\Delta$ are equal, and consequently these valuations are equivalent. When we will be considering equivalence classes of valuations instead of distinguishing equivalent valuations, we will say that $\nu = \nu_\Delta \circ \overline{\nu}_\Delta$ is **the decomposition of ν associated to Δ**.

Let ν be a valuation on $F = \mathrm{Quot}(R)$ centered at the local ring $(R, \mathfrak{m})$. If $\nu = \nu_1 \circ \nu_2$, then ν_1 has a center $\mathfrak{p} = \mathfrak{m}_{\nu_1} \cap R$ which is a subset of $\mathfrak{m}$. If we consider the decomposition $\nu = \nu_\Delta \circ \overline{\nu}_\Delta$ associated to the group $\Delta = \Delta_{\mathfrak{p}}$, then the valuations ν_1 and ν_2 are equivalent to ν_Δ and $\overline{\nu}_\Delta$ respectively. In particular, for any element $r \in \mathfrak{m} \setminus \mathfrak{p}$ we have $\nu(s) > \nu(r)$ for all $s \in \mathfrak{p}$.

2.2.5 Local uniformization

We will give now the main definitions associated to local uniformization. Given a local noetherian ring $(R, \mathfrak{m})$ (not necessarily equicharacteristic) we define the **dimension of R** (denoted by $\dim(R)$) as the Krull dimension of R, i.e., the maximum length of chains of prime ideals in R. If $(R, \mathfrak{m})$ and $(R^{(1)}, \mathfrak{m}^{(1)})$ are local rings, a **local ring homomorphism** is a ring homomorphism

$$\Phi : R \longrightarrow R^{(1)}$$

such that $\Phi^{-1}\left(\mathfrak{m}^{(1)}\right) = \mathfrak{m}$.

An ideal $\mathfrak{p}$ of R is said to be **$\mathfrak{m}$-primary** if for every element $\phi \in \mathfrak{m}$ there exists $n \in \mathbb{N}$ such that $\phi^n \in \mathfrak{p}$.

Remark 2.2.13 (Theorem 11.14 of [6])**.** For any noetherian local ring $(R, \mathfrak{m})$, the dimension of R is equal to the least number of generators of an $\mathfrak{m}$-primary ideal.

Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d and let $u_1, \dots, u_d \in \mathfrak{m}$. We say that

$$u = (u_1, \dots, u_d)$$

is **a system of parameters** of R (or of $\mathfrak{m}$) if $u_1, \dots, u_d$ generate an $\mathfrak{m}$-primary ideal. An element $f \in R$ is said to be a **monomial in** u if there exists

$$\gamma := \left(\gamma^{(1)}, \dots, \gamma^{(d)}\right) \in (\mathbb{N} \cup \{0\})^d$$

and $c \in R^\times$ such that

$$f = cu^\gamma := c \prod_{i=1}^{d} u_i^{\gamma^{(i)}}.$$

Remark 2.2.14. We will sometimes say "the local ring (R,u)" meaning that R is a local ring with maximal ideal $\mathfrak{m}$ such that $u=(u_1,\ldots,u_d)$ is a fixed system of parameters of R.

The local ring R is said to be **regular** if $\mathfrak{m}$ can be generated by $\dim(R)$ many elements. In this case, a set of generators $(u_1,\ldots,u_d)$ of $\mathfrak{m}$ such that $d=\dim(R)$ is called a regular system of parameters of R.

Definition 2.2.15. Let $(R,\mathfrak{m})$ be a noetherian local domain with quotient field F and ν a valuation on F centered at R. A **local blowing up** of $(R,\mathfrak{m})$ with respect to ν is a local ring homomorphism

$$\pi:(R,\mathfrak{m})\longrightarrow\left(R^{(1)},\mathfrak{m}^{(1)}\right)$$

of the following form: take elements $r_i,s_i\in R$, $i=1,\ldots,l$ such that $\nu\left(s_i\right)\leq\nu\left(r_i\right)$ for all $i=1,\ldots,l$ and set

$$R'=R\left[\frac{r_1}{s_1},\ldots,\frac{r_l}{s_l}\right]\text{ and }\mathfrak{m}'=\mathfrak{m}_\nu\cap R'.$$

The local ring $\left(R^{(1)},\mathfrak{m}^{(1)}\right)$ is the localization of R' with respect to the prime ideal $\mathfrak{m}'$, that is,

$$R^{(1)}=R'_{\mathfrak{m}'}=\left\{\frac{a}{b}\in K\;\middle|\;b\notin\mathfrak{m}'\right\}\text{ and }\mathfrak{m}^{(1)}=\mathfrak{m}'R'_{\mathfrak{m}'}.$$

The local blowing up now is the natural inclusion $\pi:R\longrightarrow R^{(1)}$. We will say that the local blowing up π is **simple** if $l=1$, i.e., $R^{(1)}=R\left[\frac{r}{s}\right]_{\mathfrak{m}'}$.

Classically, the problems of local uniformization and resolution of singularities are presented as follows: An algebraic variety V over K is said to admit **resolution of singularities** if there exists a "proper birational morphism" $V'\longrightarrow V$ from a "regular variety" V' to V. If ν is a valuation on $K(V)$ having a center on V, then the pair (V,ν) is said to admit **local uniformization** if there exists a proper birational morphism $V'\longrightarrow V$ from a variety V' to V such that the center of ν on V' is a "regular point".

In the modern language, an algebraic variety over K means an "integral separated scheme of finite type over K". A scheme is the analogue in algebraic geometry of a manifold. A manifold is a topological space which is the union of open sets which are homeomorphic to $\mathbb{R}^n$. Analogously, a scheme of finite type over K is the union of objects which are isomorphic (in the category of schemes) to affine schemes of the form Spec(A) where

$$A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_l),\ f_1, \ldots, f_l \in K[x_1, \ldots, x_n].$$

Since we are working with valuations, the problem of local uniformization is a local problem, so instead of talking about a scheme, we can talk just about affine schemes. Therefore, we can think of V as $V = \mathrm{Spec}(A)$.

We write $A = K[a_1, \ldots, a_n]$ where $a_i = x_i + (f_1, \ldots, f_l)$, $1 \leq i \leq n$. Take a valuation ν on $K(V) = K(a_1, \ldots, a_n)$. We say that the valuation ν has a center on V if $A \subseteq \mathcal{O}_\nu$ (in which case we say that the center is $\mathfrak{p} = \mathfrak{m}_\nu \cap A$). Then ν is centered at $\mathcal{O}_{V,\mathfrak{p}} := A_{\mathfrak{m}_\nu \cap A}$ (see last paragraph of Section 2.1). Hence, we are studying valuations of the field $K(V)$ which are centered at a noetherian local domain $\mathcal{O}_{V,\mathfrak{p}}$. This leads us to the definition of local uniformization for a valuation centered at a local domain given in Section 1.4, where local blowing ups play the role of proper birational morphisms appearing in the classical formulation.

Take an algebraic variety V over K and a valuation ν on $K(V)$ having a center on V. A variety V' is birationally equivalent to V if and only if $K(V) = K(V')$. Therefore, we can fix a valued function field $(F|K, \nu)$ and try to find a **model** V **for** $(F|K, \nu)$ (i.e., $V = \mathrm{Spec}(A)$, with $A = K[a_1, \ldots, a_r] \subseteq \mathcal{O}_F$ and $F = K(a_1, \ldots, a_r)$) such that $\mathcal{O}_{V,\mathfrak{p}} = A_{\mathfrak{m}_\nu \cap A}$ is regular. Since the original problem of local uniformization as presented earlier in this section starts from a given model, one wishes to guarantee that this new model is related in a strong way with the original one. This can be done by fixing a finite subset Z of $\mathcal{O}_L$ and requiring that with respect to the new model, Z is contained in $\mathcal{O}_{V,\mathfrak{p}}$. This motivates the definition of local uniformization for valued function fields presented in Section 1.3. In this definition, one could ask for $V = \mathrm{Spec}(A)$ with the elements of Z appearing among the generators $a_1, ..., a_r$ of A. Then one could for instance take Z to consist of the generators of the ring appearing in the original model and so achieve that this ring will be contained in the new ring A. However, this seemingly stronger condition is equivalent to the one in the definition we presented. This fact is a consequence of Lemma 6.1.4.

2.2.6 Henselization and absolute inertia field

Let $(F|L,\nu)$ be a normal algebraic extension of valued fields. Then we define the **decomposition group** $G^d(F|L,\nu)$ and the **inertia group** $G^i(F|L,\nu)$ of $(F|L,\nu)$ by

$$G^d(F|L,\nu) = \{\sigma \in \mathrm{Gal}(F|L) \mid \nu \circ \sigma = \nu \text{ on } F\}$$

and

$$G^i(F|L,\nu) = \{\sigma \in \mathrm{Gal}(F|L) \mid \forall x \in \mathcal{O}_F : x - \sigma(x) \in \mathfrak{m}_F\},$$

respectively.

Definition 2.2.16. We define the **decomposition field** $(F|L,\nu)^d$ and **inertia field** $(F|L,\nu)^i$ of $(F|L,\nu)$ to be the fixed field of $G^d(F|L,\nu)$ and $G^i(F|L,\nu)$, respectively. The **absolute inertia field** L^i of L is defined to be $(L^{\mathrm{sep}}|L,\nu)^i$. The **henselization** L^h of L is defined to be $(L^{\mathrm{sep}}|L,\nu)^d$.

Chapter 3

The valuative tree

For this chapter we take a two-dimensional regular local domain R and ordered abelian group Γ. Consider the set

$$\widetilde{\mathcal{W}}_\Gamma := \{\nu \in \widetilde{\mathcal{W}} \mid \nu : R \longrightarrow \Gamma_\infty\}.$$

The family

$$\Big(\{\nu \in \widetilde{\mathcal{W}}_\Gamma \mid \nu \text{ is centered and normalized by } \gamma \ \}\Big)_{\gamma \in \Gamma_\infty^{\geq 0}}$$

forms a partition of $\widetilde{\mathcal{W}}_\Gamma$. If we consider the particular case of $\Gamma = \mathbb{R}$, then every non-trivial valuation on $\widetilde{\mathcal{W}}_\mathbb{R}$ is equivalent to a unique valuation normalized by 1.

3.1 Valuation vs. Krull valuation on R

Since $\widetilde{\mathcal{V}} \subseteq \widetilde{\mathcal{W}}$, we can ask whether there exists a natural subset of $\widetilde{\mathcal{W}}$ which can be identified with $\widetilde{\mathcal{V}}$. Would $\widetilde{\mathcal{W}}_\mathbb{R}$ work? We are looking here for a mapping

$$\widetilde{\mathcal{W}}_\mathbb{R} \longrightarrow \widetilde{\mathcal{V}}$$

which is surjective and injective. Moreover, would this map respect equivalence classes? We present below a mapping (denoted by $.^{\mathrm{kr}}$) which respects equivalence classes, is injective, but not surjective.

Take a non-trivial valuation $\nu \in \widetilde{\mathcal{W}}_\mathbb{R}$ (we set the image under $.^{\mathrm{kr}}$ of the trivial valuation of $\widetilde{\mathcal{W}}_\mathbb{R}$ to be the trivial valuation of $\widetilde{\mathcal{V}}$). If $\mathfrak{p}_\nu = (0)$, then ν is a Krull valuation and we define $\nu^{\mathrm{kr}} := \nu$. If $\mathfrak{p}_\nu \neq (0)$, then $(0) \subsetneq \mathfrak{p}_\nu \subsetneq \mathfrak{m}$ which means that $\mathfrak{p}_\nu = (\phi)$ where $\phi \in R$ is an irreducible element. Indeed, $\mathfrak{p}_\nu \neq \mathfrak{m}$ by assumption and if we take any irreducible element

$\phi \in \mathfrak{p}_\nu$, then

$$(0) \subsetneq (\phi) \subseteq \mathfrak{p}_\nu \subsetneq \mathfrak{m}$$

and hence $(\phi) = \mathfrak{p}_\nu$ because (ϕ) is a prime ideal and $\dim(R) = 2$. Define now the Krull valuation

$$\nu^{\mathrm{kr}} : R \longrightarrow \mathbb{Z} \times \mathbb{R}$$

given by $\nu^{\mathrm{kr}}(\psi) = (r, \nu(\psi'))$ where $\psi = \phi^r \psi'$ and $(\phi, \psi') = 1$, where the order on $\mathbb{Z} \times \mathbb{R}$ is the lexicographic order.

Observe that this definition does not depend on the choice of ϕ. Indeed, since R is an unique factorization domain, any irreducible element $\psi \in \mathfrak{p}_\nu = (\phi)$, would be of the form $u \cdot \phi$ where $u \in R^\times$.

It is easy to see that given two valuations $\nu, \mu \in \widetilde{\mathcal{W}}_{\mathbb{R}}$, then

$$\nu \sim \mu \Longleftrightarrow \nu^{\mathrm{kr}} \sim \mu^{\mathrm{kr}}.$$

Therefore, the mapping $\nu \mapsto \nu^{\mathrm{kr}}$ induces an injective mapping $\mathcal{W}_{\mathbb{R}} \longrightarrow \mathcal{V}$. We want to study the properties of this mapping.

Take any Krull valuation $\nu : R \longrightarrow \Gamma_\infty$. If $\mathrm{rk}(\Gamma) = 1$, then we can embed Γ in $\mathbb{R}$, hence there exists a valuation $\nu' \in \widetilde{\mathcal{W}}_{\mathbb{R}}$ equivalent to ν. If $\mathrm{rk}(\Gamma) = 2$, then we can embed Γ in $\mathbb{R}^2$ with the lexicographic order. Consider the mapping

$$\nu'(\phi) := \begin{cases} \pi_2(\nu(\phi)) & , \text{ if } \pi_1(\nu(\phi)) = 0 \\ \infty & , \text{ otherwise.} \end{cases}$$

If this mapping assumes a value different from 0 and ∞, then it is a valuation on R and $\nu'^{\mathrm{kr}} \sim \nu$. If the only values of ν' are 0 and ∞, then there is no valuation ν' on R such that $\nu'^{\mathrm{kr}} = \nu$. Therefore, the mapping $\cdot^{\mathrm{kr}} : \mathcal{W}_{\mathbb{R}} \longrightarrow \mathcal{V}$ is not surjective.

Example 3.1.1. Let us give a few examples for the case of $R = \mathbb{C}[[x, y]]$. The monomial valuation on R defined by $\nu(x) = \alpha$ and $\nu(y) = \beta$ is given by

$$\nu\left(\sum a_{ij} x^i y^j\right) = \min\{i\alpha + j\beta \mid a_{ij} \neq 0\}.$$

(i) Take the monomial valuation defined by $\nu(x) = \nu(y) = 1$. Then ν is a rank one Krull valuation and $\nu = \nu^{\text{kr}}$.

(ii) Let ν be the monomial valuation defined by $\nu(x) = 1$ and $\nu(y) = \infty$. Then $\mathfrak{p}_\nu = (y)$ and ν^{kr} is the monomial Krull valuation defined by $\nu(x) = (0, 1)$ and $\nu(y) = (1, 0)$.

(iii) Consider the monomial Krull valuation $\mu : R \longrightarrow (\mathbb{Z} \times \mathbb{Z})_\infty$ (with lexicographic order) given by $\mu(x) = (1, 0)$ and $\mu(y) = (1, 1)$. This is a Krull valuation on R such that there is no valuation ν on R with $\nu^{\text{kr}} = \mu$. This shows that $\cdot\ ^{\text{kr}} : \mathcal{W}_{\mathbb{R}} \longrightarrow \mathcal{V}$ is not surjective.

3.2 The existence of the infimum of a set of valuations

We will now define rooted non-metric trees and discuss the difference between our way of defining it and the definition given in [14] and [16].

Definition 3.2.1. A **rooted non-metric tree** is a poset $(\mathcal{T}, \leq)$ such that:

(T1) There exists a (unique) smallest element $\tau_0 \in \mathcal{T}$.

(T2) Every set of the form $I_\tau = \{\sigma \in \mathcal{T} \mid \sigma \leq \tau\}$ is isomorphic (as ordered sets) to a real interval.

(T3) Every totally ordered convex subset of $\mathcal{T}$ is isomorphic to a real interval.

(T4) Every non-empty subset $\mathcal{S}$ of $\mathcal{T}$ admits an infimum in $\mathcal{T}$.

Remark 3.2.2. In [14] and [16], rooted non-metric tree is defined without Condition **(T4)**. In [14], the authors state (page 40 after Remark 3.4) that Condition **(T4)** follows from the completeness of the real numbers and the previous conditions, which is not true, as the following example shows.

Example 3.2.3. Take $X = [0, 1) \cup \{x, y\}$ and extend the natural order on $[0, 1)$ to X by setting $x, y > [0, 1)$ and stating that x and y are incomparable. Then **(T1)**, **(T2)** and **(T3)** hold for $(X, \leq)$, but the set $\{x, y\}$ does not admit an infimum.

We prove the following interesting fact:

Lemma 3.2.4. *Under Conditions* **(T1)** *and* **(T2)**, *the following conditions are equivalent:*

(T4) *Every non-empty subset* $\mathcal{S} \subseteq \mathcal{T}$ *admits an infimum.*

(T4') *Given two elements* $\tau, \sigma \in \mathcal{T}$, *the set* $\{\tau, \sigma\}$ *admits an infimum* $\tau \wedge \sigma$.

Proof. **(T4')** is a particular case of **(T4)**. Assume now that **(T4')** holds and take a non-empty subset $\mathcal{S} \subseteq \mathcal{T}$. We have to prove that $\mathcal{S}$ admits an infimum. Fix an element $\tau \in \mathcal{S}$ and let

$$\Phi_\tau : [\tau_0, \tau] \longrightarrow [a, b] \subseteq \mathbb{R}$$

be the isomorphism given by Condition **(T2)**. For each $\sigma \in \mathcal{S}$, by Condition **(T4')** there exists an element $\tau \wedge \sigma \in \mathcal{T}$ which is the infimum of $\{\tau, \sigma\}$ in $\mathcal{T}$. Define the element

$$a_\sigma = \Phi_\tau(\tau \wedge \sigma) \in [a, b].$$

Since $\mathbb{R}$ is complete, we have that $\{a_\sigma \mid \sigma \in \mathcal{S}\}$ admits an infimum $a_0 \in [a, b]$. Define the element $\sigma_0 = \Phi_\tau^{-1}(a_0) \in \mathcal{T}$. Let us prove that $\sigma_0 = \inf \mathcal{S}$. If not, there would exist an element $\sigma_0' \in \mathcal{T}$ such that $\sigma_0 < \sigma_0' \leq \sigma$ for all $\sigma \in \mathcal{S}$. Then $\sigma_0' \leq \tau \wedge \sigma$ and hence $a_0 = \Phi_\tau(\sigma_0) < \Phi_\tau(\sigma_0') \leq a_\sigma$ for all $\sigma \in \mathcal{S}$, which shows that $a_0 < \inf\{a_\sigma \mid \sigma \in \mathcal{S}\}$, a contradiction. □

Remark 3.2.5. The lemma above shows that if a partially ordered set with Conditions **(T1)** and **(T2)** is directed (with respect to reverse set inclusion), then its order is a directed complete partial order (with respect to reverse set inclusion).

We will now define some properties associated to a non-metric tree.

Definition 3.2.6. **(i)** Given a non-empty subset $\mathcal{S} \subseteq \mathcal{T}$ we define the **join** $\bigwedge_{\tau \in \mathcal{S}} \tau$ of $\mathcal{S}$ to be the infimum of $\mathcal{S}$.

(ii) Given two elements $\tau, \sigma \in \mathcal{T}$ we define the **closed segment** connecting them by

$$[\tau, \sigma] := \{\alpha \in \mathcal{T} \mid (\tau \wedge \sigma \leq \alpha \leq \tau) \vee (\tau \wedge \sigma \leq \alpha \leq \sigma)\}.$$

We define $]\tau, \sigma]$ and $[\tau, \sigma[$ similarly.

(iii) For a point $\tau \in \mathcal{T}$, define an equivalence relation on $\mathcal{T} \backslash \{\tau\}$ by setting

$$\sigma \sim_\tau \alpha \Longleftrightarrow]\tau, \sigma] \cap]\tau, \alpha] \neq \emptyset.$$

The **tangent space of $\mathcal{T}$ at** τ is the set of equivalence classes of $\mathcal{T} \backslash \{\tau\}$. An equivalence class $[\sigma]_\tau \in \mathcal{T} \backslash \{\tau\} / \sim_\tau$ is called a **tangent vector at** τ.

(iv) The **weak tree topology** on $\mathcal{T}$ is the topology having as a subbasis the tangent vectors at the points of $\mathcal{T}$, i.e., the open sets are unions of finite intersections of sets of the form $[\sigma]_\tau$.

(v) A **parametrization** of a rooted non-metric tree is an increasing (or decreasing) mapping $\Psi : \mathcal{T} \longrightarrow [-\infty, \infty]$ such that its restriction to every totally ordered convex subset of $\mathcal{T}$ is an isomorphism (of ordered sets) onto a real interval.

(vi) Given an increasing parametrization $\Psi : \mathcal{T} \longrightarrow [1, \infty]$ we define a metric on $\mathcal{T}$ by setting

$$d_\Psi(\tau, \sigma) = \left(\frac{1}{\Psi(\tau \wedge \sigma)} - \frac{1}{\Psi(\tau)} \right) + \left(\frac{1}{\Psi(\tau \wedge \sigma)} - \frac{1}{\Psi(\sigma)} \right).$$

Remark 3.2.7. Observe that the definitions above depend strongly on the existence of an infimum for any two given elements.

We will start to prove now that every pair of valuations centered at a two-dimensional regular local domain $(R, \mathfrak{m})$ admits an infimum.

Remark 3.2.8. We point out that valuations on any ring R have the following good property:

Lemma 3.2.9. (i) *Every totally ordered set $\mathcal{S} \subseteq \widetilde{\mathcal{W}_\mathbb{R}}$ admits a supremum.*

(ii) *Take two valuations $\nu, \nu' \in \widetilde{\mathcal{W}_\mathbb{R}}$ and assume that there exists a valuation $\nu_0 : R \longrightarrow \mathbb{R}_\infty$ such that one of the sets $\{\mu \in \widetilde{\mathcal{W}_\mathbb{R}} \mid \nu_0 \leq \mu \leq \nu\}$ or $\{\mu \in \widetilde{\mathcal{W}_\mathbb{R}} \mid \nu_0 \leq \mu \leq \nu'\}$ is totally ordered and both are non-empty. Then there exists an infimum for $\{\nu, \nu'\}$ in $\{\mu \in \widetilde{\mathcal{W}_\mathbb{R}} \mid \nu_0 \leq \mu\}$.*

Hence, if we have proven that for the valuative tree, Conditions **(T1)** and **(T2)** are satisfied, then (by use of Lemma 3.2.9) we have the desired existence of the infimum. The advantage of our proof of Theorem 1.1.1 is that we do not assume that **(T1)** and **(T2)** hold.

Proof of Lemma 3.2.9. **(i)** Define the function $\nu_{\mathcal{S}}(\phi) = \sup\{\nu(\phi) \mid \nu \in \mathcal{S}\}$, which is well-defined because $\mathbb{R}$ is complete. We have to prove that $\nu_{\mathcal{S}}$ is a valuation. The assertion that $\nu_{\mathcal{S}}(0) = \infty$ and $\nu_{\mathcal{S}}(1) = 0$ are trivial.

For every valuation $\nu \in \mathcal{S}$ we have that

$$\nu(\phi\psi) = \nu(\phi) + \nu(\psi) \leq \nu_{\mathcal{S}}(\phi) + \nu_{\mathcal{S}}(\psi) \text{ for every } \phi, \psi \in R.$$

Hence, $\nu_{\mathcal{S}}(\phi\psi) \leq \nu_{\mathcal{S}}(\phi) + \nu_{\mathcal{S}}(\psi)$. Suppose that there exist elements $\phi, \psi \in R$ such that $\nu_{\mathcal{S}}(\phi\psi) < \nu_{\mathcal{S}}(\phi) + \nu_{\mathcal{S}}(\psi)$. Assume that

$$\epsilon := \nu_{\mathcal{S}}(\phi) + \nu_{\mathcal{S}}(\psi) - \nu_{\mathcal{S}}(\phi\psi) \neq \infty.$$

From the definition of $\nu_{\mathcal{S}}$, there exist valuations $\nu, \nu' \in \mathcal{S}$ such that

$$\nu(\phi) > \nu_{\mathcal{S}}(\phi) - \epsilon/2 \text{ and } \nu'(\psi) > \nu_{\mathcal{S}}(\psi) - \epsilon/2.$$

Since $\mathcal{S}$ is totally ordered, the valuations ν and ν' are comparable, say, $\nu \geq \nu'$. Hence, $\nu(\psi) \geq \nu'(\psi) > \nu_{\mathcal{S}}(\psi) - \epsilon/2$. Then

$$\nu(\phi) + \nu(\psi) > \nu_{\mathcal{S}}(\phi) + \nu_{\mathcal{S}}(\psi) - \epsilon = \nu_{\mathcal{S}}(\phi\psi) \geq \nu(\phi\psi),$$

which is a contradiction to the fact that ν is a valuation. If $\epsilon = \infty$, a similar argument gives us the desired contradiction.

Suppose towards a contradiction that $\nu_{\mathcal{S}}(\phi+\psi) < \min\{\nu_{\mathcal{S}}(\phi), \nu_{\mathcal{S}}(\psi)\}$ for some $\phi, \psi \in \mathcal{S}$. Then there exist valuations $\nu, \nu' \in \mathcal{S}$ such that

$$\nu_{\mathcal{S}}(\phi + \psi) < \nu(\phi) \text{ and } \nu_{\mathcal{S}}(\phi + \psi) < \nu'(\psi).$$

Again, we assume that $\nu \geq \nu'$, which implies that $\nu_{\mathcal{S}}(\phi + \psi) < \nu'(\psi) \leq \nu(\psi)$. Therefore,

$$\nu(\phi + \psi) \leq \nu_{\mathcal{S}}(\phi + \psi) < \min\{\nu(\phi), \nu(\psi)\},$$

which is the desired contradiction because ν is a valuation.

(ii) Assume without loss of generality that $I_\nu := \{\mu \in \widetilde{\mathcal{W}}_\mathbb{R} \mid \nu_0 \leq \mu \leq \nu\}$ is totally ordered. Take the set $I_{\nu'\wedge\nu} := \{\mu \in I_\nu \mid \mu \leq \nu'\}$. Since $I_{\nu'\wedge\nu}$ is a subset of a totally ordered set, it is also a totally ordered set. By part **(i)**, there exists a supremum $\nu \wedge \nu'$ in $\widetilde{\mathcal{W}}_\mathbb{R}$ for $I_{\nu'\wedge\nu}$. Take a valuation μ such that $\nu_0 \leq \mu \leq \nu$ and $\nu_0 \leq \mu \leq \nu'$. Then $\mu \in I_{\nu'\wedge\nu}$ and since $\nu' \wedge \nu$ is the supremum of $I_{\nu'\wedge\nu}$, we have that $\mu \leq \nu \wedge \nu'$. Also, if $\nu \wedge \nu'(\phi) > \nu(\phi)$ (or $\nu \wedge \nu'(\phi) > \nu'(\phi)$) for some $\phi \in R$, then there would exist a valuation $\mu \in I_{\nu'\wedge\nu}$ with $\mu(\phi) > \nu(\phi)$ (or $\mu(\phi) > \nu'(\phi)$) which is a contradiction. Hence, $\nu \wedge \nu' \leq \nu$ and $\nu \wedge \nu' \leq \nu'$. Therefore, $\nu \wedge \nu' = \inf\{\nu, \nu'\}$. □

Let $(R, \mathfrak{m})$ be a local domain and consider the set

$$\mathcal{S} = \left(R^\times \cup \{0\}\right)^2 \setminus \{(0,0)\}.$$

Define a relation on $\mathcal{S}$ by setting

$$(a_1, b_1) \sim (a_2, b_2) \iff a_1 b_2 - a_2 b_1 \in \mathfrak{m}.$$

Lemma 3.2.10. *This relation is an equivalence relation.*

Proof. This relation is clearly reflexive and symmetric, so it remains to show that it is transitive. Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(a_2, b_2) \sim (a_3, b_3)$. By definition, we have that $a_1b_2 - a_2b_1, a_2b_3 - a_3b_2 \in \mathfrak{m}$. If $a_2 \neq 0$, then we have that

$$a_2(a_3b_1 - a_1b_3) = a_3a_2b_1 - a_3a_1b_2 + a_1a_3b_2 - a_1a_2b_3 = a_3(a_2b_1 - a_1b_2) + a_1(a_3b_2 - a_2b_3) \in \mathfrak{m}$$

and since $a_2 \in R^\times$ we have that $a_3b_1 - a_1b_3 \in \mathfrak{m}$. If $a_2 = 0$, then $b_2 \neq 0$ and we have

$$b_2(a_3b_1 - a_1b_3) = b_1a_3b_2 - b_1a_2b_3 + b_3a_2b_1 - b_3a_1b_2 = b_1(a_3b_2 - a_2b_3) + b_3(a_2b_1 - a_1b_2) \in \mathfrak{m}$$

and again $a_3b_1 - a_1b_3 \in \mathfrak{m}$. □

Suppose that $(R, \mathfrak{m})$ is a two-dimensional regular local domain and let (x, y) be a regular system of parameters of $\mathfrak{m}$. Take a valuation ν centered at R.

Lemma 3.2.11. *Take $(a_1, b_1), (a_2, b_2) \in \mathcal{S}$ with $(a_1, b_1) \sim (a_2, b_2)$. Then*

$$\nu(a_1x + b_1y) > \nu(\mathfrak{m}) \iff \nu(a_2x + b_2y) > \nu(\mathfrak{m}).$$

Also, if there exist $(a_1, b_1), (a_2, b_2) \in \mathcal{S}$ such that $\nu(a_1x + b_1y) > \nu(\mathfrak{m})$ and $\nu(a_2x + b_2y) > \nu(\mathfrak{m})$, then $(a_1, b_1) \sim (a_2, b_2)$.

Proof. For the first statement, suppose that $a_1 \neq 0$ (and so $a_2 \neq 0$). Then

$$\begin{aligned} a_2x + b_2y &= \frac{a_2}{a_1}\left(a_1x + \frac{a_1b_2}{a_2}y\right) \\ &= \frac{a_2}{a_1}\left(a_1x + b_1y - b_1y + \frac{a_1b_2}{a_2}y\right) \\ &= \frac{a_2}{a_1}(a_1x + b_1y) + \frac{a_1b_2 - a_2b_1}{a_1}y \end{aligned}$$

Since $(a_1, b_1) \sim (a_2, b_2)$ we have that

$$\nu\left(\frac{a_1b_2 - a_2b_1}{a_1}y\right) = \nu(a_1b_2 - a_2b_1) - \nu(a_1) + \nu(y) > \nu(y) \geq \nu(\mathfrak{m}).$$

If $\nu(a_1x + b_1y) > \nu(\mathfrak{m})$, then

$$\nu(a_2x + b_2y) \geq \min\left\{\nu\left(\frac{a_2}{a_1}(a_1x + b_1y)\right), \nu\left(\frac{a_1b_2 - a_2b_1}{a_1}y\right)\right\} > \nu(\mathfrak{m}),$$

and if $\nu(a_1x + b_1y) = \nu(\mathfrak{m}) < \nu\left(\frac{a_1b_2 - a_2b_1}{a_1}y\right)$, then

$$\nu(a_2x + b_2y) = \min\left\{\nu\left(\frac{a_2}{a_1}(a_1x + b_1y)\right), \nu\left(\frac{a_1b_2 - a_2b_1}{a_1}y\right)\right\} = \nu(\mathfrak{m}).$$

If $a_1 = 0$, then $b_1 \neq 0 \neq b_2$ and we proceed similarly to obtain

$$a_2x + b_2y = \frac{b_2}{b_1}(a_1x + b_1y) + \frac{a_2b_1 - a_1b_2}{b_1}x.$$

For the second statement, suppose that

$$\nu(a_1x + b_1y) > \nu(\mathfrak{m}) \text{ and } \nu(a_2x + b_2y) > \nu(\mathfrak{m})$$

and that $(a_1, b_1) \nsim (a_2, b_2)$. This would mean that $a_1b_2 - a_2b_1 \notin \mathfrak{m}$, so $\nu(a_2b_1 - a_1b_2) = 0$. Then we would have that

$$\begin{aligned} \nu(x) &= \nu(a_2b_1x - a_1b_2x) \\ &= \nu(a_2b_1x + b_1b_2y - b_1b_2y - a_1b_2x) \\ &= \nu(b_1(a_2x + b_2y) - b_2(a_1x + b_1y)) \\ &> \nu(\mathfrak{m}) \end{aligned}$$

and

$$\begin{aligned} \nu(y) &= \nu(a_2b_1y - a_1b_2y) \\ &= \nu(a_2b_1y + a_1a_2x - a_1a_2x - a_1b_2y) \\ &= \nu(a_2(a_1x + b_1y) - a_1(a_2x + b_2y)) \\ &> \nu(\mathfrak{m}) \end{aligned}$$

which is a contradiction to $\nu(\mathfrak{m}) = \min\{\nu(x), \nu(y)\}$. □

Definition 3.2.12. Take an element $\lambda \in \mathcal{S}/\sim$. We say that

$$\nu(x+\lambda y) > \nu(\mathfrak{m})$$

if $\nu(a_1x+b_1y) > \nu(\mathfrak{m})$ for some (and hence for every) $(a_1, b_1) \in \lambda$. Analogously, we say that

$$\nu(x+\lambda y) = \nu(\mathfrak{m})$$

if $\nu(a_1x+b_1y) = \nu(\mathfrak{m})$ for some (and hence for every) $(a_1, b_1) \in \lambda$.

Corollary 3.2.13. *Given two centered valuations $\nu, \mu : R \longrightarrow \mathbb{R}_\infty$, there exist elements $a, b \in R^\times \cup \{0\}$ such that $\nu(\mathfrak{m}) = \nu(ax+by)$ and $\mu(\mathfrak{m}) = \mu(ax+by)$.*

Proof. By the second part of Lemma 3.2.11, there exist at most one $\lambda_\nu \in \mathcal{S}/\sim$ and at most one $\lambda_\mu \in \mathcal{S}/\sim$ such that $\nu(x+\lambda_\nu y) > \nu(\mathfrak{m})$ and $\mu(x+\lambda_\mu y) > \mu(\mathfrak{m})$. Since $|\mathcal{S}/\sim| \geq 3$ for any domain R there exists an element $\lambda \in \mathcal{S}/\sim$ with $\lambda_\nu \neq \lambda \neq \lambda_\mu$. For any $(a,b) \in \lambda$, we have that $\nu(ax+by) = \nu(\mathfrak{m})$ and $\mu(ax+by) = \mu(\mathfrak{m})$. □

Corollary 3.2.14. *If $R = k[[x,y]]$ for an algebraically closed field k and $\nu \neq \nu(\mathfrak{m}) \cdot \nu_\mathfrak{m}$, then there exists $\lambda \in \mathbb{P}^1(k)$ such that $\nu(x+\lambda y) > \nu(\mathfrak{m})$.*

Proof. We will prove that if $R = k[[x,y]]$ and $\nu \neq \nu(\mathfrak{m}) \cdot \nu_\mathfrak{m}$ where k is any field, then there exists a homogeneous polynomial m in (x,y) such that $\nu(m) > \nu_\mathfrak{m}(m) \cdot \nu(\mathfrak{m})$. Consequently, if k is algebraically closed, then we can find a homogeneous polynomial m of degree 1 such that $\nu(m) > \nu_\mathfrak{m}(m) \cdot \nu(\mathfrak{m}) = \deg(m) \cdot \nu(\mathfrak{m}) = \nu(\mathfrak{m})$.

Since $\nu \neq \nu_\mathfrak{m} \cdot \nu(\mathfrak{m})$ there exists a power series

$$p(x,y) = \sum_{i=1}^{\infty} m_i \in k[[x,y]] \text{ where } m_i \text{ are monomials in } x \text{ and } y,$$

such that $\nu(p) > \nu_\mathfrak{m}(p) \cdot \nu(\mathfrak{m}) = k \cdot \nu(\mathfrak{m})$ where $k := \mathrm{ord}_\mathfrak{m}(p)$. Write p as sum of homogeneous polynomials, i.e.,

$$p(x,y) = \sum_{j \geq k} p_j, \text{ where } p_j = \sum_{\deg(m_i)=j} m_i.$$

Since $\nu(p_j) \geq \min\{\nu(m_i) \mid \deg(m_i) = j\} \geq j \cdot \nu(\mathfrak{m})$ and $\nu(p - p_k) > k \cdot \nu(\mathfrak{m})$ we have that $\nu(p_k) > k \cdot \nu(\mathfrak{m})$. Indeed, if $\nu(p_k) = k \cdot \nu(\mathfrak{m})$ we would have that

$$\nu(p) = \min\{\nu(p-p_k), \nu(p_k)\} = \nu(p_k) = k \cdot \nu(\mathfrak{m}) = \nu_\mathfrak{m}(p) \cdot \nu(\mathfrak{m}).$$

Therefore, there exists a homogeneous polynomial $m \in k[[x,y]]$ (namely $m = p_k$) such that $\nu(m) > \nu_{\mathfrak{m}}(m) \cdot \nu(\mathfrak{m})$. If k is algebraically closed, then m can be chosen to be of degree one. □

Remark 3.2.15. In [14] Corollary 3.19, page 48, it is proved that if two valuations ν and μ on $\mathbb{C}[[x,y]]$ are given where (x,y) are coordinates such that $\nu(x) = \mu(x) = 1 \leq \min\{\nu(y), \mu(y)\}$, then there exists the infimum for ν and μ. By the corollary above, we conclude that every pair of valuations on $\mathbb{C}[[x,y]]$ have an infimum.

For each valuation ν centered at R take a regular system of parameters (x,y) such that $\nu(x) \leq \nu(y)$. Let ν' be the unique extension of ν to $R\left[\frac{y}{x}\right]$ with $\nu'\left(\frac{y}{x}\right) = \nu(y) - \nu(x)$ and let

$$\mathfrak{q}_\nu^{(1)} = \left\{r \in R\left[\frac{y}{x}\right] \mid \nu'(r) > 0\right\}.$$

Then the local domain

$$R_\nu^{(1)} = R\left[\frac{y}{x}\right]_{\mathfrak{q}_\nu^{(1)}}$$

is called the **quadratic dilatation of R with respect to ν**. If $\dim\left(R_\nu^{(1)}\right) = 1$, then $\nu = a \cdot \nu_{\mathfrak{m}}$ for some $a > 0$. If $\dim\left(R_\nu^{(1)}\right) = 2$, then we proceed as before to obtain a new local domain $R_\nu^{(2)}$ which is the quadratic dilatation of $R_\nu^{(1)}$ with respect to $\nu^{(1)}$. We can construct inductively a sequence (finite or infinite)

$$R \subseteq R_\nu^{(1)} \subseteq R_\nu^{(2)} \subseteq \ldots \subseteq R_\nu^{(n)} \subseteq \ldots$$

of regular local domains such that $R_\nu^{(i)}$ is the quadratic dilatation of $R_\nu^{(i-1)}$ with respect to $\nu^{(i-1)}$ (here $R_\nu^{(0)} := R$). Let $\lambda(\nu)$ be the length of the sequence above, i.e.,

$$\lambda(\nu) = \begin{cases} n+1, \text{ if } \dim\left(R_\nu^{(n)}\right) = 2 \text{ and } \dim\left(R_\nu^{(n+1)}\right) = 1 \\ \infty \quad, \text{ if } \dim\left(R_\nu^{(n)}\right) = 2 \text{ for every } n \in \mathbb{N}. \end{cases}$$

It is proved in [2] that

$$\mathcal{O}_{\nu^{\mathrm{kr}}} = \bigcup_{i=0}^{\lambda(\nu)} R_\nu^{(i)}$$

where $\mathcal{O}_{\nu^{\mathrm{kr}}}$ is the valuation domain of ν^{kr} in $\mathrm{Quot}(R)$. The sequence $\left\{R_\nu^{(i)}\right\}_{i=0}^{\lambda(\nu)}$ is called the sequence of quadratic dilatations of ν and the sequence $\left\{m_\nu^{(i)}\right\}_{i=0}^{\lambda(\nu)}$, where $m_\nu^{(i)} = \nu^{(i)}\left(\mathfrak{m}_\nu^{(i)}\right)$, is called the **multiplicity sequence of ν**.

Fix a regular system of parameters (x,y) of $\mathfrak{m}$. For each $\phi \in R\setminus\{0\}$ there exists a unique decomposition $\phi = a_1M_1 + \ldots + a_nM_n$, where $a_i \in R\setminus\mathfrak{m}$ and $M_i = x^{r_i}y^{s_i}$ is a pure monomial in (x,y), $0 \leq i \leq n$. Take $\gamma_1, \gamma_2 \in \mathbb{R}_\infty$ not both equal to ∞. Then we define the valuation

$$\nu(\phi) = \min_{1\leq i\leq n}\{r_i\gamma_1 + s_i\gamma_2\}.$$

This is indeed a valuation (see Lemma 7 of [16]) and it is called a **monomial valuation in** (x,y). It is a Krull valuation if $\gamma_1 \neq \infty \neq \gamma_2$ and it is centered if $\gamma_1 > 0$ and $\gamma_2 > 0$ (see Lemma 8 of [16]).

To prove Theorem 1.1.1 we will use the following Theorem (Theorem 18 of [16]):

Theorem 3.2.16. *Let ν and μ be two centered valuations of R and suppose that $\mu \leq \nu$. Assume that there exists $s \geq 0$ such that $\dim\left(R_\mu^{(i)}\right) = 2$, $m_\nu^{(i)} = m_\mu^{(i)}$ for $0 \leq i \leq s$ and either $\dim\left(R_\nu^{(s+1)}\right) = 1$ or $\dim\left(R_\nu^{(s+1)}\right) = 2$ and $m_\nu^{(s+1)} > m_\mu^{(s+1)}$. Then $R_\nu^{(i)} = R_\mu^{(i)}$ and $0 \leq \mu^{(i)}(\phi) \leq \nu^{(i)}(\phi)$ for each $\phi \in R^{(i)}$, $0 \leq i \leq s$. Moreover, we have the following possibilities:*

(a) *If $\dim\left(R_\nu^{(s+1)}\right) = 1$, then $\lambda(\nu) = \lambda(\mu) = s+1$ and $\nu^{(s)} = \mu^{(s)} = m_\nu^{(s)} \cdot \nu_{\mathfrak{m}^{(s)}}$.*

(b) *If $\dim\left(R_\nu^{(s+1)}\right) = 2$ and $\dim\left(R_\mu^{(s+1)}\right) = 1$, then $s+1 = \lambda(\mu) < \lambda(\nu)$ and there exists a monomial valuation $\overline{\mu}^{(s+1)}$ on $R_\nu^{(s+1)}$ such that $\mu^{(s)}$ is the restriction of $\overline{\mu}^{(s+1)}$ to $R_\nu^{(s)}$.*

(c) *If $\dim\left(R_\nu^{(s+1)}\right) = 2$ and $\dim\left(R_\mu^{(s+1)}\right) = 2$, then $s+1 < \min\{\lambda(\nu), \lambda(\mu)\}$, $R_\nu^{(s+1)} = R_\mu^{(s+1)}$, $0 \leq \mu^{(s+1)}(\phi) \leq \nu^{(s+1)}(\phi)$ for all $\phi \in R^{(s+1)}$ and $\mu^{(s+1)}$ is a monomial valuation on $R^{(s+1)}$.*

Proof of Theorem 1.1.1. Take two centered valuations $\nu, \mu : R \longrightarrow \mathbb{R}_\infty$ such that $\nu(\mathfrak{m}) = \mu(\mathfrak{m}) = 1$. Since $R_\nu^{(0)} = R = R_\mu^{(0)}$ and $1 = \nu(\mathfrak{m}) = m_\nu^{(0)} = m_\mu^{(0)}$, we can define

$$s = \max\{i \mid R_\nu^{(i)} = R_\mu^{(i)} \text{ and } m_\nu^{(i)} = m_\mu^{(i)}\}.$$

If $s = \infty$, then $\mathcal{O}_{\nu^{\text{kr}}} = \mathcal{O}_{\mu^{\text{kr}}}$ and consequently $\nu \sim \mu$. Since these valuations are normalized by $\nu(\mathfrak{m}) = 1 = \mu(\mathfrak{m})$, we must have that $\nu = \mu$ and there is nothing to prove. Therefore, assume that $s < \infty$. We define $R^{(i)} := R_\nu^{(i)} = R_\mu^{(i)}$ and $m^{(i)} := m_\nu^{(i)} = m_\mu^{(i)}$ for $0 \leq i \leq s$.

We will divide our proof in cases, starting with the case where $R_{\nu}^{(s+1)} \neq R_{\mu}^{(s+1)}$. By Corollary 3.2.13 there exists $x^{(s)} \in \mathfrak{m}^{(s)}$ such that

$$\nu^{(s)}\left(x^{(s)}\right) = \nu^{(s)}\left(\mathfrak{m}^{(s)}\right) = \mu^{(s)}\left(\mathfrak{m}^{(s)}\right) = \mu^{(s)}\left(x^{(s)}\right).$$

Take any $y^{(s)} \in \mathfrak{m}^{(s)}$ such that $\left(x^{(s)}, y^{(s)}\right)$ is a regular system of parameters for $\mathfrak{m}^{(s)}$. Define $\omega^{(s)}$ to be the monomial valuation on $R^{(s)}$ defined by

$$\omega^{(s)}\left(x^{(s)}\right) = \nu^{(s)}\left(x^{(s)}\right) = \mu^{(s)}\left(x^{(s)}\right) \text{ and } \omega^{(s)}\left(y^{(s)}\right) = \min\{\nu\left(y^{(s)}\right), \mu\left(y^{(s)}\right)\}.$$

Let ω be the restriction of $\omega^{(s)}$ to R. From the definition of monomial valuation, we conclude that $\omega \leq \nu$ and $\omega \leq \mu$. We want to prove that if ω' is a valuation on R such that $\omega \leq \omega' \leq \nu$ and $\omega \leq \omega' \leq \mu$, then $\omega = \omega'$.

If $\dim\left(R_{\omega'}^{(s+1)}\right) = 1$, applying Theorem 3.2.16 **(a)** for $\omega \leq \omega'$ we have that $\omega = \omega'$. If $\dim\left(R_{\omega'}^{(s+1)}\right) = 2$, then $\dim\left(R_{\nu}^{(s+1)}\right) = 2$ and $\dim\left(R_{\mu}^{(s+1)}\right) = 2$. Moreover, applying Theorem 3.2.16 **(c)** for $\omega' \leq \nu$ and $\omega' \leq \mu$ we have that $R_{\mu}^{(s+1)} = R_{\omega'}^{(s+1)}$ and $R_{\nu}^{(s+1)} = R_{\omega'}^{(s+1)}$. Consequently, $R_{\nu}^{(s+1)} = R_{\mu}^{(s+1)}$, which is a contradiction to our assumption. Therefore, $\dim\left(R_{\omega'}^{(s+1)}\right) = 1$ and $\omega = \omega'$.

The remaining case that of $R_{\nu}^{(s+1)} = R_{\mu}^{(s+1)} =: R^{(s+1)}$ and $m_{\nu}^{(s+1)} \neq m_{\mu}^{(s+1)}$, say $m_{\nu}^{(s+1)} < m_{\mu}^{(s+1)}$. Define the valuation $\omega^{(s+1)}$ in $R_{\nu}^{(s+1)}$ to be the monomial valuation given by

$$\omega^{(s+1)}\left(x^{(s+1)}\right) = \min\left\{\nu^{(s+1)}\left(x^{(s+1)}\right), \mu^{(s+1)}\left(x^{(s+1)}\right)\right\}$$

and

$$\omega^{(s+1)}\left(y^{(s+1)}\right) = \min\left\{\nu^{(s+1)}\left(y^{(s+1)}\right), \mu^{(s+1)}\left(y^{(s+1)}\right)\right\},$$

where $\left(x^{(s+1)}, y^{(s+1)}\right)$ is a regular system of parameters for $R^{(s+1)}$ with the property that $\nu^{(s+1)}\left(x^{(s+1)}\right) = \nu^{(s+1)}\left(\mathfrak{m}^{(s+1)}\right)$ and $\mu^{(s+1)}\left(x^{(s+1)}\right) = \mu^{(s+1)}\left(\mathfrak{m}^{(s+1)}\right)$ (such $x^{(s+1)}$ exists by Corollary 3.2.13). Let ω be the restriction of $\omega^{(s+1)}$ to R. Take a valuation ω' on R such that $\omega \leq \omega' \leq \nu$ and $\omega \leq \omega' \leq \mu$. We want to prove that $\omega = \omega'$.

From our definition, we obtain that $m_{\omega}^{(s+1)} = m_{\nu}^{(s+1)} < m_{\mu}^{(s+1)}$. If $m_{\omega'}^{(s+1)} > m_{\omega}^{(s+1)}$, then we would have that $\omega' \not\leq \nu$ which is a contradiction. Thus $m_{\omega'}^{(s+1)} = m_{\omega}^{(s+1)}$. Since $\omega' \leq \mu$ and $m_{\mu}^{(s+1)} > m_{\omega'}^{(s+1)}$ we are in the situation of Theorem 3.2.16, so by **(c)** we have that $\omega'^{(s+1)}$ is monomial (on $\left(x^{(s+1)}, y^{(s+1)}\right)$). Therefore, $\omega'^{(s+1)} = \omega^{(s+1)}$ and consequently $\omega = \omega'$. $\square$

Remark 3.2.17. In the proof, we used the fact that in the situation above, the valuation $\omega'^{(s+1)}$ is a monomial valuation on the coordinates $\left(x^{(s+1)}, y^{(s+1)}\right)$. This fact was not explicitly stated but appears in the proof of Theorem 3.2.16 in [16].

3.3 Comparison of topologies

In this section we compare the topologies defined above with classical topologies. Also, we compare the weak tree topology and the metric topology given by a parametrization of a rooted non-metric tree.

An interesting fact, proved in [14] (Theorem 5.1), is the following:

Proposition 3.3.1. *The weak tree topology and the weak topology on $\widetilde{\mathcal{W}}_{\mathfrak{m}}$ are equal.*

We will proceed with the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. It is enough to show that every subbasic set $[\sigma]_\tau$ in the weak tree topology is open in the metric topology, i.e., for every $\gamma \in [\sigma]_\tau$ there exist $\epsilon > 0$ such that

$$B_\epsilon(\gamma) = \{\alpha \in \mathcal{T} \mid d_\Psi(\gamma, \alpha) < \epsilon\} \subseteq [\sigma]_\tau.$$

By definition $\gamma \neq \tau$, so $\epsilon := d_\Psi(\gamma, \tau) > 0$. Let us prove that $B_\epsilon(\gamma) \subseteq [\sigma]_\tau$.

Claim 3.3.2. $\alpha \notin [\sigma]_\tau \Longleftrightarrow \tau \in [\alpha, \sigma]$.

Proof. Suppose first that $\alpha \notin [\sigma]_\tau$. This means that $]\tau, \alpha] \cap]\tau, \sigma] = \emptyset$. Suppose towards a contradiction that $\tau > \alpha$ and $\tau > \sigma$. By Condition **(T2)**, we would have that α and σ are comparable, say $\alpha \leq \sigma$. This means that $]\tau, \sigma] \subseteq]\tau, \alpha]$ which is a contradiction. Consequently, $\tau < \sigma$ or $\tau < \alpha$. It remains to show that $\tau \geq \alpha \wedge \sigma$. Suppose not, i.e., that $\tau < \alpha \wedge \sigma$. Then we would have that $\alpha \wedge \sigma \in]\tau, \alpha] \cap]\tau, \sigma]$ which is again a contradiction.

For the converse, assume that $\tau \in [\alpha, \sigma]$. If $\tau = \alpha \wedge \sigma$, then we obtain by definition of $\alpha \wedge \sigma$ that $]\tau, \alpha] \cap]\tau, \sigma] = \emptyset$. Otherwise, assume without loss of generality that $\alpha \wedge \sigma < \tau < \alpha$. Then

$$]\tau, \alpha] = \{\gamma \in \mathcal{T} \mid \tau < \gamma \leq \alpha\}$$

and

$$]\tau, \sigma] = \{\gamma \in \mathcal{T} \mid (\alpha \wedge \sigma \leq \gamma < \tau) \vee (\alpha \wedge \sigma \leq \gamma \leq \sigma)\}$$

which are disjoint sets. Therefore, $\alpha \nsim_\tau \sigma$, so $\alpha \notin [\sigma]_\tau$. □

Claim 3.3.3. *If $\tau \in [\alpha, \sigma]$, then $d_\Psi(\alpha, \sigma) = d_\Psi(\alpha, \tau) + d_\Psi(\tau, \sigma)$*

Proof. Suppose without loss of generality that $\alpha \wedge \sigma \leq \tau \leq \sigma$. Then we have that $\alpha \wedge \tau = \alpha \wedge \sigma$ and that $\tau \wedge \sigma = \tau$. Therefore,

$$\begin{aligned} d_\Psi(\alpha, \sigma) &= \left(\frac{1}{\Psi(\alpha \wedge \sigma)} - \frac{1}{\Psi(\alpha)}\right) + \left(\frac{1}{\Psi(\alpha \wedge \sigma)} - \frac{1}{\Psi(\sigma)}\right) \\ &= \left(\frac{1}{\Psi(\alpha \wedge \tau)} - \frac{1}{\Psi(\alpha)}\right) + \left(\frac{1}{\Psi(\alpha \wedge \tau)} - \frac{1}{\Psi(\sigma)}\right) + \left(\frac{2}{\Psi(\tau \wedge \sigma)} - \frac{2}{\Psi(\tau)}\right) \\ &= \left(\frac{1}{\Psi(\alpha \wedge \tau)} - \frac{1}{\Psi(\alpha)}\right) + \left(\frac{1}{\Psi(\alpha \wedge \tau)} - \frac{1}{\Psi(\tau)}\right) \\ &\quad + \left(\frac{1}{\Psi(\tau \wedge \sigma)} - \frac{1}{\Psi(\sigma)}\right) + \left(\frac{1}{\Psi(\tau \wedge \sigma)} - \frac{1}{\Psi(\tau)}\right) \\ &= d_\Psi(\alpha, \tau) + d_\Psi(\tau, \sigma). \end{aligned}$$

□

Take an element $\alpha \notin [\sigma]_\tau = [\gamma]_\tau$. By Claim 3.3.2 we have that $\tau \in [\alpha, \gamma]$. By Claim 3.3.3 we have that

$$d_\Psi(\alpha, \gamma) = d_\Psi(\alpha, \tau) + d_\Psi(\tau, \gamma) = d_\Psi(\alpha, \tau) + \epsilon \geq \epsilon.$$

Therefore, $\alpha \notin B_\epsilon(\gamma)$ and consequently, $B_\epsilon(\gamma) \subseteq [\sigma]_\tau$. □

We now analyse whether these topologies are equal:

Theorem 3.3.4. *If there is an element $\sigma \in \mathcal{T}$ with uncountably many branches ($|\mathcal{T}_\sigma| > |\mathbb{N}|$), then the weak tree topology is not first countable. In particular, the metric topology given by any parametrization is strictly coarser than the weak tree topology.*

Proof. Take an element $\sigma \in \mathcal{T}$ which has uncountably many branches. Observe that $[\sigma]_\tau$ contains all branches emanating from σ except for the one on which τ lies. That means that any basic open set (therefore any open set) that contains σ contains uncountably many branches emanating from σ. Take now any family $\{V_n\}_{n\in\mathbb{N}}$ of open sets containing σ. Since each V_n contains uncountably many branches, their intersection contains uncountably many branches. Take one of these branches and choose an element α on it. Take now the subbasic open set $[\sigma]_\alpha$. Then $\alpha \in V_n$ for all $n \in \mathbb{N}$ and $\alpha \notin [\sigma]_\alpha$, hence $V_n \nsubseteq [\sigma]_\alpha$. Therefore, there is no countable system of neighbourhoods for the element σ. □

Corollary 3.3.5. *In the valuative tree, the weak tree topology is strictly coarser than the topology generated by a parametrization.*

Proof. It is proved in [14] that divisorial valuations have uncountably many branches. By the theorem above we obtain that the weak tree topology and the metric topology defined by a parametrization are different. □

Remark 3.3.6. As a criterion for the topologies to be equal or different, the fact that there exists a point with uncountably many branches is the best sufficient condition that we can obtain. We can present examples of trees in which every point has finitely (or countably) many branches and the topologies are equal and examples where they are different.

Chapter 4

Topologies

In order to introduce our topologies on spaces of valuations, we will need some basic facts about ordered abelian groups.

4.1 On the Hahn product

Definition 4.1.1. Let $(G_i, <_i)_{i\in I}$ be a family of ordered abelian groups where I is a totally ordered set.

(i) The **Hahn product** of $\{G_i\}_{i\in I}$ is defined as

$$\mathop{\mathbf{H}}_{i\in I} G_i := \{(a^i)_{i\in I} \in \prod_{i\in I} G_i \mid \mathrm{supp}((a^i)_{i\in I}) \text{ is well-ordered } \},$$

where $\mathrm{supp}((a^i)_{i\in I}) = \{i \in I \mid a^i \neq 0\}$ is the support of $(a^i)_{i\in I}$. The Hahn product is a group with respect to componentwise addition.

(ii) We define the group valuation $v_h : \mathop{\mathbf{H}}_{i\in I} G_i \longrightarrow I \cup \{\infty\}$ by

$$v_h((a^i)_{i\in I}) := \min \mathrm{supp}((a^i)_{i\in I}) \text{ with } v_h(0) = \infty.$$

(iii) The lexicographic order $<_{\text{lex}}$ on $\mathop{\mathbf{H}}_{i\in I} G_i$ is defined by

$$a < b \Longleftrightarrow (b-a)^{v_h(b-a)} > 0.$$

With this order, the Hahn product is an ordered abelian group.

(iv) For an ordered abelian group Γ (in particular, for $\Gamma = \mathop{\mathbf{H}}_{i\in I} G_i$) and an element $\gamma \in \Gamma$ we define the **sign of** γ (and denote it by $\mathrm{sign}(\gamma)$) by $-$ or $+$ according to $\gamma < 0$ or $\gamma > 0$.

Definition 4.1.2. Take a totally ordered set I. A square matrix $A = (a^i_j)_{i,j\in I} \in M_{I\times I}(\mathbb{R})$ is in **special lower echelon form** if:

(1) For each $i \in I$ the support of $a^i := (a^i_j)_{j\in I}$ is finite.

(2) For each $j \in I$, the support of $a_j := (a^i_j)_{i\in I}$ is well-ordered (i.e., $a_j \in \mathop{\mathbf{H}}_{i\in I}\mathbb{R}$) and $a_j > 0$.

(3) If $j < j'$, then $v_h(a^i_j)_{i\in I} < v_h(a^i_{j'})_{i\in I}$.

Remark 4.1.3. Each square matrix $A = (a^i_j)_{i,j\in I} \in M_{I\times I}(\mathbb{R})$ having Property **(1)** induces a group homomorphism $A : \mathop{\mathbf{H}}_{i\in I}\mathbb{R} \longrightarrow \mathop{\mathbf{H}}_{i\in I}\mathbb{R}$ given by

$$A((b^i)_{i\in I}) = (c^i)_{i\in I} \text{ with } c^i := \sum_{j\in I} a^i_j b^j.$$

Proposition 4.1.4. *If the matrix $A = (a^i_j)_{i,j\in I} \in M_{I\times I}(\mathbb{R})$ is in special lower echelon form, then the induced mapping $A : \mathop{\mathbf{H}}_{i\in I}\mathbb{R} \longrightarrow \mathop{\mathbf{H}}_{i\in I}\mathbb{R}$ is an (injective) order-preserving group homomorphism.*

Proof. We take a non-zero element $b := (b^i)_{i\in I} \in \mathop{\mathbf{H}}_{i\in I}\mathbb{R}$ and set $A(b) = (c^i)_{i\in I}$ as before. Since $b \neq 0$ we have that $j_b := v_h(b) \in I$. From Condition **(2)** we obtain that $(a^i_{j_b})_{i\in I} \in \mathop{\mathbf{H}}_{i\in I}\mathbb{R}$ and we set $i_b := v_h\left((a^i_{j_b})_{i\in I}\right) \in I$. Since $b^j = 0$ for every $j < j_b$ we have

$$c^{i_b} = \sum_{j\in I} a^{i_b}_j b^j = \sum_{j\geq j_b} a^{i_b}_j b^j.$$

On the other hand, if $j_b < j$, then Condition **(3)** gives us that $i_b = v_h(a^i_{j_b})_{i\in I} < v_h(a^i_j)_{i\in I}$ and hence $a^{i_b}_j = 0$. Therefore, $c^{i_b} = a^{i_b}_{j_b} b^{j_b}$.

If $b > 0$, then we obtain that $A(b) = c = (c^i)_{i\in I}$ with $v_h(c) = i_b$. Also, $c^{i_b} = a^{i_b}_{j_b} b^{j_b} > 0$ because $a^{i_b}_{j_b} > 0$ (A is in special lower echelon form) and $b^{j_b} > 0$ (because $b > 0$). Therefore, A is order preserving (hence injective). □

Corollary 4.1.5. *Take $\Gamma = \mathop{\mathbf{H}}_{i=1}^{n}\mathbb{R}$ and $a, b \in \Gamma$ both non-zero. Assume that $v_h a = v_h b$ and* $\operatorname{sign}(a) = \operatorname{sign}(b)$. *Then there exists an order-preserving isomorphism $A : \Gamma \longrightarrow \Gamma$ such that $A(a) = b$.*

Proof. Write $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Let $i_0 = v_h(a) = v_h(b)$ and consider the matrix

$$A = \left(\begin{array}{ccc|cccc} 1 & \cdots & 0 & & & & \\ \vdots & \ddots & \vdots & & & 0 & \\ 0 & \cdots & 1 & & & & \\ \hline & & & \frac{b_{i_0}}{a_{i_0}} & 0 & \dots & 0 \\ & 0 & & \frac{b_{i_0+1}-a_{i_0+1}}{a_{i_0}} & 1 & \dots & 0 \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & \frac{b_n - a_n}{a_{i_0}} & 0 & \cdots & 1 \end{array}\right) \in M_n(\mathbb{R}).$$

It is easy to see that the matrix A is in special lower echelon form (because $\operatorname{sign}(a) = \operatorname{sign}(b)$) and so, by Proposition 4.1.4, the mapping A is an injective order-preserving group homomorphism. Also, one can check that $A(a) = b$. On the other hand, the mapping A is an $\mathbb{R}$-linear mapping (if we see Γ as an $\mathbb{R}$-vector space) and so it is also surjective. □

4.2 Topologies on $\mathcal{W}_\Gamma$ derived from topologies on Γ_∞

Our goal is to define new topologies on the set of equivalence classes of valuations $\mathcal{W}$ of a noetherian domain R (see Section 2.1.1). If we do not assume anything about the domain R, then we can run into the problem that $\mathcal{W}$ consists only of the trivial valuation (for instance if $R = \mathbb{F}_p$). Hence, we will assume that R is a domain such that there exists an element in R which can have as its value any element of any ordered abelian group.

A natural way to define topologies on $\mathcal{W}$ is to define a topology on $\widetilde{\mathcal{W}}$ and consider the quotient topology on $\mathcal{W} = \widetilde{\mathcal{W}}/\sim$. For every ordered abelian group Γ there are many different topologies on Γ_∞. Even if we fix a topology on Γ there are different ways of extending this topology to Γ_∞. If $\mathfrak{A}$ is a topology on Γ_∞, then we consider the product topology on $(\Gamma_\infty)^R$ associated to $\mathfrak{A}$. Since the set $\widetilde{\mathcal{W}_\Gamma}$ of valuations on R which take values in Γ_∞ is a subset of $(\Gamma_\infty)^R$, every topology on $(\Gamma_\infty)^R$ induces a topology on $\widetilde{\mathcal{W}_\Gamma}$. Therefore, every topology on Γ_∞ induces naturally a topology on $\widetilde{\mathcal{W}_\Gamma}$, which will be denoted by $\widetilde{\mathfrak{B}}_\mathfrak{A}$. We want to study the relation between the topology $\mathfrak{A}$ and the corresponding topology $\widetilde{\mathfrak{B}}_\mathfrak{A}$.

We start by presenting some natural topologies on Γ_∞. Since Γ is a totally ordered set, it carries a topology induced by the order. There are (at least) three natural ways to extend this topology to Γ_∞.

Remark 4.2.1. For a totally ordered set X and elements $x_0, x_1 \in X$ we will denote by $]x_0, x_1[$ the set $\{x \in X \mid x_0 < x < x_1\}$ (and analogously, for $[x_0, x_1[$, $]x_0, x_1]$ and $[x_0, x_1]$). However, since we will often use the symbol ∞ to represent an element (of Γ_∞ for instance) we will not use the notation $]x_0, \infty[$ to represent the set $\{x \in X \mid x > x_0\}$.

Definition 4.2.2. (i) The **order topology**.

For a totally ordered set X, the order topology is defined as the topology having as a subbasis the sets of the form

$$\{x \in X \mid x > x_0\} \text{ and } \{x \in X \mid x < x_0\}$$

where x_0 runs through X. We denote by X_∞ the set $X \cup \{\infty\}$ where ∞ is an element not belonging to X and extend the order from X to X_∞ by setting $\infty > x$ for every $x \in X$. In this manner, X_∞ is a totally ordered set and hence we can talk about the order topology on X_∞. A system of open neighbourhoods of ∞ in this topology is given by the sets

$$\{x \in X_\infty \mid x > x_0\},$$

with x_0 running through X.

(ii) The **circle topology**.

Take a totally ordered set X and an element $y \notin X$. Define the **circle topology** on $X' = X \cup \{y\}$ as follows: consider the order topology on X and extend it to X' by taking

$$\{y\} \cup \{x \in X \mid (x < x_0) \vee (x > x_1)\} = \{y\} \cup X \setminus [x_0, x_1], \tag{4.1}$$

as a system of open neighbourhoods of y, where x_0 and x_1 run through X.

(iii) The one **point compactification**.

Take any topological space $(X, \mathfrak{A})$. The one point compactification of $(X, \mathfrak{A})$ is the topological space given by $(X', \mathfrak{A}')$ where $X' = X \cup \{y\}$ with $y \notin X$ and

$$\mathfrak{A}' = \mathfrak{A} \cup \{\{y\} \cup (X \setminus K) \mid K \text{ is closed and compact in } (X, \mathfrak{A})\}.$$

Definition 4.2.3. We define the topologies $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ on Γ_∞ as the order, the circle and the one point compactification topologies, respectively, where the topology on Γ is the order topology and we set ∞ to be the element denoted by y in the previous definition.

Remark 4.2.4. Since the order topology on Γ is Hausdorff, the open sets of $\mathfrak{A}_1$ which contain ∞ are $\{\infty\} \cup (X \setminus K)$ where K is any compact subset of Γ (in a Hausdorff space every compact set is closed).

Remark 4.2.5. It is well known that the one point compactification is compact. To see that, take any open covering $\{U_i\}_{i\in I}$ of X'. There exists $i_0 \in I$ such that $y \in U_{i_0}$. By definition of the topology, the set $K = X \setminus U_{i_0}$ is compact and since $\{U_i \setminus \{y\}\}_{i\in I\setminus\{i_0\}}$ covers K there exists a finite open subcovering $\{U_i \setminus \{y\}\}_{i\in J}$ of K. Then $\{U_i\}_{i\in J\cup\{i_0\}}$ is a finite subcovering of X'.

Before proceeding with the relations between the topology $\mathfrak{A}$ on Γ_∞ and the corresponding topology $\widetilde{\mathfrak{B}}_{\mathfrak{A}}$ on $\widetilde{\mathcal{W}}_\Gamma$, we prove a few properties of the topologies defined above.

Lemma 4.2.6. *Take any ordered abelian group* Γ. *Then the order topology on* Γ *is discrete if and only if* Γ *has a smallest positive element.*

Proof. If Γ has a smallest positive element α, then $]0,\alpha[\, = \emptyset$. Then $]\alpha, 2\alpha[\, = \emptyset$, because if $\alpha_1 \in\,]\alpha, 2\alpha[$ we would have $\alpha_1 - \alpha \in\,]0,\alpha[$. Therefore, $\{\alpha\} = \,]0, 2\alpha[$ is open in the order topology. For any element $\beta \in \Gamma$ we have that $\{\beta\} = \,]\beta - \alpha, \beta + \alpha[$ is open in the order topology. Indeed, if

$$\beta' \in\,]\beta - \alpha, \beta + \alpha[\text{ and } \beta \neq \beta'$$

we would have that

$$\beta' - \beta + \alpha \in\,]0, 2\alpha[\text{ and } \beta' - \beta + \alpha \neq \alpha$$

which is a contradiction. Therefore, the order topology is discrete.

Suppose that the order topology is discrete. This implies that $\{0\}$ is an open set, hence there exist $\alpha > 0$ and $\beta < 0$ such that

$$\{\gamma \in \Gamma \mid \beta < \gamma < \alpha\} \subseteq \{0\}.$$

Therefore, α is the smallest positive element for Γ. □

Definition 4.2.7. A totally ordered set X is said to be **complete** if every non-empty set bounded from above admits a supremum.

Proposition 4.2.8. *Take a totally ordered set X. We have the following:*

(i) *If the order topology on X_∞ is compact, then X has a smallest element.*

(ii) *The order topology on X_∞ is finer than or equal to the circle topology on X_∞. Moreover, these topologies are equal if and only if X has a smallest element.*

(iii) *The circle topology on X_∞ is finer than or equal to the one point compactification on X_∞. Moreover, they are equal if and only if X is complete.*

Proof. **(i)** Suppose that X does not have smallest element. Then the family $\{U_x\}_{x\in X}$, with $U_x =]x, \infty]$, is an open covering of X_∞ in the order topology. Also, for every finite subfamily $\{U_{x_i}\}_{1\le i\le n}$ of $\{U_x\}_{x\in X}$, we have that

$$X_\infty \neq U_{x_0} = \bigcup_{i=1}^{n} U_{x_i},$$

where $x_0 = \min\limits_{1\le i\le n} x_i$. Therefore, the order topology is not compact.

(ii) Since both topologies extend the order topology of X, we just have to consider neighbourhoods of y. A subbasic open neighbourhood of ∞ in the circle topology is of the form $U_1 \cup U_2$ where

$$U_1 = \{x \in X \mid x < x_0\} \text{ and } U_2 = \{x \in X_\infty \mid x > x_1\},$$

for some $x_0, x_1 \in X$. Since both U_1 and U_2 are open sets in the order topology, so is $U_1 \cup U_2$.

Assume now that X has smallest element x'. Then every subbasic open neighbourhood of ∞ in the order topology is of the form

$$\{x \in X_\infty \mid x > x_0\} = \{x \in X \mid x < x'\} \cup \{x \in X_\infty \mid x > x_0\},$$

which is open in the circle topology. On the other hand, if X does not have a smallest element, then for $x_0 \in X$ the set $U = \{x \in X_\infty \mid x > x_0\}$ is open in the order topology, but not in the circle topology.

(iii) Take any open subset U of X_∞ in the one point compactification. For every $x \in U$ we have to show that there exists an open set V in the circle topology such that $x \in V \subseteq U$. Since the case is trivial for $x \neq \infty$ we suppose that $x = \infty$ and $U = \{\infty\} \cup \Gamma \setminus K$ such that K is compact in X. Since K is compact, it must be bounded, i.e., $K \subseteq [x_0, x_1]$ for some $x_0, x_1 \in X$. Therefore,

$$\infty \in V := \{\infty\} \cup X \setminus [x_0, x_1] \subseteq \{\infty\} \cup X \setminus K = U.$$

Assume now that X is not complete. Then there exists a non-empty subset S of X, bounded from above but without a supremum. Hence, for every x such that $x \geq S$ there exists $x_1 \in X$ such that $S \leq x_1 < x$. Define the following family of open sets in the circle topology:

$$\mathcal{F} = \{U_{x_0}^{x_1} \mid x_0 \in S, S \leq x_1\}, \text{ where } U_{x_0}^{x_1} = \{\infty\} \cup X \setminus [x_0, x_1].$$

Take an element $x \in X$. If $x \geq S$, then there exists $x_1 \in X$ such that $S \leq x_1 < x$. Then $x \in U_{x_0}^{x_1}$ for any $x_0 \in S$. If $x \not\geq S$ there exists $x_0 \in S$ such that $x < x_0$. Again, $x \in U_{x_0}^{x_1}$ where $x_1 \geq S$ is any element. Therefore,

$$X_\infty = \bigcup_{x_0 \in S, x_1 \geq S} U_{x_0}^{x_1} = \bigcup \mathcal{F}.$$

It is easy to see that for every finite subfamily $\mathcal{F}'$ of $\mathcal{F}$ we have that $X_\infty \neq \bigcup \mathcal{F}'$. Therefore, the circle topology is not compact and hence distinct from the one point compactification.

It remains to show that if X is complete, then the one point compactification and the circle topologies are equal. In view of (4.1), it is enough to show that every subset of the form $[x_0, x_1]$ is compact in X with the order topology. Take any open covering $\{U_i\}_{i \in I}$ of $[x_0, x_1]$ and consider the set

$$\mathcal{S} = \{x \in [x_0, x_1] \mid \exists i_1, \cdots, i_n \in I \text{ such that } [x_0, x] \subseteq \bigcup_{j=1}^{n} U_{i_j}\}.$$

This set is non-empty and bounded, so it admits a supremum x'. Since $\mathcal{S} \leq x_1$ we must have $x' \leq x_1$. Suppose towards a contradiction that $x' < x_1$. If x' has an immediate successor x'', then we take any $i_{n+1} \in I$ such that $x'' \in U_{i_{n+1}}$. Consequently,

$$[x_0, x''] = [x_0, x'] \cup \{x''\} \subseteq \bigcup_{j=1}^{n+1} U_{i_j}$$

which implies that $x'' \in \mathcal{S}$. This is a contradiction with $x' = \sup \mathcal{S}$. If x' does not have an immediate successor we take any $i_{n+1} \in I$ such that $x' \in U_{i_{n+1}}$. Since $U_{i_{n+1}}$ is open in the order topology there exists $x'' > x'$ such that

$$[x_0, x''] \subseteq \bigcup_{j=1}^{n+1} U_{i_j},$$

which gives the desired contradiction. □

Using the proposition above and the well-known fact that every complete ordered abelian group is isomorphic to the real numbers we obtain the following:

Corollary 4.2.9. $\mathfrak{A}_3 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}_1$. *Also,* Γ *does not have smallest element, hence* $\mathfrak{A}_2 \subsetneq \mathfrak{A}_1$. *Moreover,* $\mathfrak{A}_3 = \mathfrak{A}_2$ *if and only if* Γ *is isomorphic to the real numbers.*

Fix a topology $\mathfrak{A}$ on Γ_∞ (for instance one as defined above). If $(\Gamma_\infty, \mathfrak{A})$ is compact, then also $(\Gamma_\infty)^R$ is compact with respect to the product topology. If $\widetilde{\mathcal{W}_\Gamma}$ is closed in $(\Gamma_\infty)^R$, then it is also compact. It is natural to ask which properties of a given topology on Γ_∞ guarantee that $\widetilde{\mathcal{W}_\Gamma}$ is closed in $(\Gamma_\infty)^R$. The next result gives us a sufficient condition.

Theorem 4.2.10. *Let* Γ' *be a submonoid of* Γ_∞ *and take a topology* $\mathfrak{A}$ *on* Γ' *such that*

(P1) *the addition* $+ : \Gamma' \times \Gamma' \longrightarrow \Gamma'$ *is continuous, and*

(P2) *for every* $\gamma, \gamma' \in \Gamma'$ *such that* $\gamma < \gamma'$ *there exist open sets* $U, U' \in \mathfrak{A}$ *such that* $\gamma \in U, \gamma' \in U'$ *and* $U < U'$ *(i.e.,* $u < u'$ *for every* $u \in U$ *and* $u' \in U'$*).*

Then $\widetilde{\mathcal{W}_{\Gamma'}} := \{\nu \in \widetilde{\mathcal{W}_\Gamma} \mid \nu(\phi) \in \Gamma'$ *for every* $\phi \in R\}$ *is closed in* $(\Gamma')^R$.

Proof. We will prove that $(\Gamma')^R \setminus \widetilde{\mathcal{W}_{\Gamma'}}$ is an open set in the product topology. Take a function $f : R \longrightarrow \Gamma'$ which is not a valuation. Then one of the three axioms **(V1)**, **(V2)** or **(V3)** does not hold for f. We will divide the proof in cases:

- $f(\phi\psi) \neq f(\phi) + f(\psi)$ for some $\phi, \psi \in R$.

  Property **(P2)** implies that $\mathfrak{A}$ is Hausdorff, so there exist $U, W \in \mathfrak{A}$ such that $f(\phi) + f(\psi) \in U$, $f(\phi\psi) \in W$ and $U \cap W = \emptyset$. By **(P1)** there exist $V, V' \in \mathfrak{A}$ with $f(\phi) \in V$

and $f(\psi) \in V'$ such that $V + V' \subseteq U$. For an element $\phi \in R$ we define the map $\phi^* : (\Gamma')^R \longrightarrow \Gamma'$ by $\phi^*(f) := f(\phi)$. Take the open set given by

$$O = \phi^{*\,-1}(V) \cap \psi^{*\,-1}(V') \cap (\phi\psi)^{*\,-1}(W).$$

It is easy to see that $f \in O$. Take any element $g \in O$ and let us prove that g is not a valuation. Since $g(\phi) \in V$ and $g(\psi) \in V'$ we must have $g(\phi) + g(\psi) \in V + V' \subseteq U$. Also, $g(\phi\psi) \in W$ which means that $g(\phi\psi) \neq g(\phi) + g(\psi)$ because $U \cap W = \emptyset$.

- $f(\phi + \psi) < \min\{f(\phi), f(\psi)\}$ for some $\phi, \psi \in R$.

  In this case we have that $f(\phi+\psi) < f(\phi)$ and $f(\phi+\psi) < f(\psi)$. By Property **(P2)** we have that there exist open sets $U, U', W, W' \in \mathfrak{A}$ such that $f(\phi+\psi) \in U < W \ni f(\phi)$ and $f(\phi+\psi) \in U' < W' \ni f(\psi)$. Take now

  $$O = \phi^{*\,-1}(W) \cap \psi^{*\,-1}(W') \cap (\phi+\psi)^{*\,-1}(U \cap U').$$

  Again we have that $f \in O$. If $g \in O$ we have that $g(\phi+\psi) < \min\{g(\phi), g(\psi)\}$ which means that g is not a valuation.

- $f(1) \neq 0$.

  Since $\mathfrak{A}$ is Hausdorff the set $\Gamma_\infty \setminus \{0\}$ is open. Take the set $O = 1^{*\,-1}(\Gamma_\infty \setminus \{0\})$. Then $f \in O$ but $O \cap \widetilde{\mathcal{W}}_{\Gamma'} = \emptyset$.

  The case of $f(0) \neq \infty$ is treated analogously.

□

Remark 4.2.11. If **(P2)** holds, then $\mathfrak{A}$ is finer than the order topology. Indeed, take any $\gamma \in \Gamma'$. For every $\gamma' < \gamma$ (or $\gamma' > \gamma$) there exists an open set $U_{\gamma'} \in \mathfrak{A}$ (or $V_{\gamma'} \in \mathfrak{A}$, respectively) such that $U_{\gamma'} < \gamma$ (or $V_{\gamma'} > \gamma$, respectively). Therefore,

$$\{\gamma' \in \Gamma' \mid \gamma' < \gamma\} = \bigcup_{\gamma'<\gamma} U_{\gamma'} \in \mathfrak{A} \qquad (\text{or } \{\gamma' \in \Gamma' \mid \gamma' > \gamma\} = \bigcup_{\gamma'>\gamma} V_{\gamma'} \in \mathfrak{A}, \text{ respectively}).$$

Since every subbasic open set of the order topology belongs to $\mathfrak{A}$, this topology is finer than the order topology.

Lemma 4.2.12. *We have the following:*

(a) *Properties* **(P1)** *and* **(P2)** *hold for* $\mathfrak{A}_1$*;*

(b) *Properties* **(P1)** *and* **(P2)** *are satisfied neither by* $\mathfrak{A}_2$ *nor by* $\mathfrak{A}_3$*.*

As a consequence of Theorem 4.2.10 and Lemma 4.2.12 we obtain:

Corollary 4.2.13. *The set* $\widetilde{\mathcal{W}_\Gamma}$ *is closed in* $(\Gamma_\infty)^R$ *if we take the order topology* $\mathfrak{A}_1$ *on* Γ_∞*.*

Proof of Lemma 4.2.12. **(a)** Take $\gamma, \gamma' \in \Gamma$ such that $\gamma < \gamma'$. If there is an element $\alpha \in]\gamma, \gamma'[$ we take

$$U =]-\infty, \alpha[\text{ and } U' =]\alpha, \infty[\,.$$

If $]\gamma, \gamma'[= \emptyset$ we take

$$U =]-\infty, \gamma'[\text{ and } U' =]\gamma, \infty[\,.$$

In each case we have that $\gamma \in U < U' \ni \gamma'$. Therefore, **(P2)** holds for $\mathfrak{A}_1$.

In order to show that **(P1)** holds we must show that for any $\gamma, \gamma' \in \Gamma_\infty$ and $U \in \mathfrak{A}_1$, if $\gamma + \gamma' \in U$, then there exist $V, V' \in \mathfrak{A}_1$ with $\gamma \in V$ and $\gamma' \in V'$ such that $V + V' \subseteq U$.

First, consider the case where $\gamma \neq \infty \neq \gamma'$. If the order topology is discrete we just take $V = \{\gamma\}$ and $V' = \{\gamma'\}$. In the other case, take $\alpha, \beta \in \Gamma$ with $\alpha, \beta > 0$ such that

$$\gamma + \gamma' \in]\gamma + \gamma' - \alpha, \gamma + \gamma' + \beta[\subseteq U.$$

There exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$ such that

$$\alpha_1, \alpha_2, \beta_1, \beta_2 > 0 \text{ and } \alpha_1 + \alpha_2 = \alpha \text{ and } \beta_1 + \beta_2 = \beta.$$

Consider now the open sets

$$V =]\gamma - \alpha_1, \gamma + \beta_1[\text{ and } V' =]\gamma' - \alpha_2, \gamma' + \beta_2[\,.$$

Then,

$$V + V' \subseteq]\gamma + \gamma' - \alpha_1 - \alpha_2, \gamma + \gamma' + \beta_1 + \beta_2[\subseteq U.$$

It remains to prove that given any open neighbourhood U of ∞ and any $\gamma \in \Gamma_\infty$ there exist $V, V' \in \mathfrak{A}_1$ with $\infty \in V$ and $\gamma \in V'$ such that $V + V' \in U$. Since U is a neighbourhood of ∞ there exists $\alpha \in \Gamma$ such that $\{\alpha' \in \Gamma_\infty \mid \alpha' > \alpha\} \subseteq U$. If $\gamma = \infty$ we just take

$$V = \{\alpha' \in \Gamma_\infty \mid \alpha' > \alpha\} \text{ and } V' = \{\alpha' \in \Gamma_\infty \mid \alpha' > 0\}$$

and if $\gamma \neq \infty$ we just take any $\beta > 0$ and define $V = \{\alpha' \in \Gamma_\infty \mid \alpha' > \alpha - \gamma + \beta\}$ and $V' = \{\alpha' \in \Gamma_\infty \mid \alpha' > \gamma - \beta\}$. In any case we have that $V + V' \subseteq U$.

(b) To prove that **(P1)** does not hold for $\mathfrak{A}_2$ and $\mathfrak{A}_3$ we just have to observe that in each case the set

$$U = \{\gamma \in \Gamma_\infty \mid \gamma \neq 0\}$$

is an open neighbourhood of ∞. On the other hand, if V, V' are open neighbourhoods of ∞, then there exists $\gamma \in \Gamma$ such that $\gamma \in V$ and $-\gamma \in V'$. Therefore, $V + V' \nsubseteq U$.

Take an element $\gamma \in \Gamma$ (hence $\gamma < \infty$ in Γ_∞) and an open neighbourhood U of ∞ in either $\mathfrak{A}_2$ or $\mathfrak{A}_3$. From the definition of the topologies $\mathfrak{A}_2$ and $\mathfrak{A}_3$ we have that there exists an element $\gamma' \in U$ such that $\gamma' < \gamma$. Therefore, Property **(P2)** cannot hold for $\mathfrak{A}_2$ or $\mathfrak{A}_3$. □

We can ask for the converse of Theorem 4.2.10, namely: can we find conditions on $\mathfrak{A}$ which imply that $\widetilde{\mathcal{W}_\Gamma}$ is not closed in $(\Gamma_\infty)^R$? The next proposition answers this question.

Proposition 4.2.14. *Take any topology $\mathfrak{A}$ on Γ_∞. If $\widetilde{\mathcal{W}_\Gamma}$ is closed in $(\Gamma_\infty)^R$ with respect to the product topology, then $\mathfrak{A}$ is T_1.*

Proof. Suppose $\mathfrak{A}$ is not T_1 and let us prove that there exists $f \in (\Gamma_\infty)^R$ such that if f belongs to an open set in the product topology, then such set contains a valuation. Since $\mathfrak{A}$ is not T_1 there exist elements $\gamma, \gamma' \in \Gamma_\infty, \gamma \neq \gamma'$ such that for every $U \in \mathfrak{A}$, if $\gamma' \in U$, then $\gamma \in U$. If $\gamma = 0$ (or $\gamma = \infty$), then we just take a valuation ν and define the function

$$f(\phi) = \begin{cases} \nu(\phi) \ , & \text{if } \phi \neq 1 \text{ (or } \phi \neq 0 \text{ resp.)} \\ \gamma' \ , & \text{if } \phi = 1 \text{ (or } \phi = 0 \text{ resp.).} \end{cases}$$

It is easy to see that f is not a valuation and that every open set that contains f contains ν.

If $\gamma \neq 0$ and $\gamma \neq \infty$ we take a valuation $\nu : R \longrightarrow \Gamma_\infty$ such that $\nu(\phi_0) = \gamma$ for some $\phi_0 \in R$ (such ϕ_0 exists because of our assumption on R). Define now the function

$$f(\phi) = \begin{cases} \nu(\phi) \ , \text{ if } \phi \neq \phi_0 \\ \gamma' \quad \ , \text{ if } \phi = \phi_0. \end{cases}$$

Since $\phi_0^2 \neq \phi_0$,

$$f(\phi_0^2) = \nu(\phi_0^2) = 2\nu(\phi_0) = 2\gamma \neq 2\gamma' = 2\nu(\phi_0) = 2f(\phi_0).$$

Hence, f is not a valuation. Take any open set U' in the product topology such that $f \in U'$. We want to prove that $\nu \in U'$. Take any $\phi \in R$. If $f(\phi) \neq \nu(\phi)$, then $f(\phi) = \gamma$ and $\nu(\phi) = \gamma'$, hence $\nu \in U'$. Therefore, $\Gamma_\infty^R \setminus \widetilde{\mathcal{W}}_\Gamma$ is not open in the product topology. □

Take a topology $\mathfrak{A}$ on Γ_∞ such that $\mathfrak{A} \subseteq \mathfrak{A}_1$. Lemma 4.2.14 states that if $\mathfrak{A}$ is not T_1, then $\widetilde{\mathcal{W}}_\Gamma$ is not closed in $(\Gamma_\infty)^R$. On the other hand, if $\mathfrak{A} = \mathfrak{A}_1$, then Corollary 4.2.13 tells us that $\widetilde{\mathcal{W}}_\Gamma$ is closed in $(\Gamma_\infty)^R$. A natural question is whether the property of being T_1 characterizes the order topology among the topologies $\mathfrak{A}$ such that $\mathfrak{A} \subseteq \mathfrak{A}_1$. The next lemma (which we prove by giving an example) answers this question to the negative.

Lemma 4.2.15. *There exists a topology $\mathfrak{A} \subsetneq \mathfrak{A}_1$ on Γ_∞ which is T_1.*

Proof. Take any element $\gamma \in \Gamma$. Define the topology $\mathfrak{A} \subsetneq \mathfrak{A}_1$ on Γ_∞ by

$$\mathfrak{A} = \{U \in \mathfrak{A}_1 \mid (\gamma \notin U) \vee (\exists \gamma_1, \gamma_2 \in \Gamma \text{ with } \ ]-\infty, \gamma_1[\cup]\gamma_2, \infty[\ \subseteq U)\}.$$

It is easy to check that this topology is T_1 (it can be even proved that $\mathfrak{A}$ is normal). □

For some applications, for instance in algebraic geometry, the interesting valuations are those which are centered. This implies that such valuations take values in $\left(\Gamma^{\geq 0}\right)_\infty$. We define the topologies $\mathfrak{A}'_1$, $\mathfrak{A}'_2$ and $\mathfrak{A}'_3$ on Γ'_∞ analogously to the topologies $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ on Γ_∞.

As a consequence of Lemma 4.2.8 we obtain the following:

Proposition 4.2.16. *The topologies $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$ of $\left(\Gamma^{\geq 0}\right)_\infty$ are equal. Also, $\mathfrak{A}'_3 \subseteq \mathfrak{A}'_2$, and $\mathfrak{A}'_3 = \mathfrak{A}'_2$ if and only if $\Gamma \simeq \mathbb{R}$.*

Corollary 4.2.17. *The set of all non-negative valuations of any ring R with values in $\mathbb{R}_\infty$ is compact with respect to the product topology of $[0,\infty]^R$. (Here, $[0,\infty] = \{x \in \mathbb{R}_\infty \mid 0 \leq x \leq \infty\}$ is endowed with the order topology).*

Proof. Theorem 4.2.10 gives us that the set of valuations on R with values in the monoid $\mathbb{R}'_\infty$ is closed in $(\mathbb{R}'_\infty)^R = [0,\infty]^R$ with respect to the product topology. Also, Lemma 4.2.8 guarantees that $[0,\infty]$ is compact, hence $[0,\infty]^R$ is compact. Since $\widetilde{\mathcal{W}}_\mathbb{R}$ is a closed subset of a compact space it is compact. □

We do not know if the same result as in Corollary 4.2.17 is true for $\Gamma = \mathbb{R}^n$ where $n > 1$, i.e., we do not have an answer for the following problem:

Problem 4.2.18. Take a topology $\mathfrak{A}$ on Γ_∞ where $\Gamma = \mathbb{R}^n$. Is the corresponding topology on the set of all non-negative valuations taking values in $(\mathbb{R}^n)_\infty$ compact?

Our criteria developed so far cannot fully answer this question. Since Properties **(P1)** and **(P2)** hold for $\mathfrak{A}_1$ they also hold for $\mathfrak{A}'_2 = \mathfrak{A}'_1$. Hence, $\widetilde{\mathcal{W}}_{\Gamma'}$ is closed in $(\Gamma')^R$ for the topology $\mathfrak{A}'_2 = \mathfrak{A}'_1$ on $\Gamma' = \left((\mathbb{R}^n)^{\geq 0}\right)_\infty$. However, since $\mathfrak{A}'_2 = \mathfrak{A}'_1$ is not compact (for $n > 1$) we cannot conclude whether $\widetilde{\mathcal{W}}_{\Gamma'}$ is compact or not.

On the other hand, if we consider the compact topology $\mathfrak{A}'_3$ of Γ', then Properties **(P1)** and **(P2)** do not hold for $\mathfrak{A}'_3$ and again we cannot conclude whether $\widetilde{\mathcal{W}}_{\Gamma'}$ is compact or not.

4.3 New topologies on spaces of equivalence classes of valuations

Our goal in this section is to define topologies on the set of equivalence classes of non-negative valuations $\mathcal{W}^{\geq 0} = \widetilde{\mathcal{W}}^{\geq 0}/\sim$. In order to introduce new topologies, we will need the following:

Proposition 4.3.1. *Let R be a commutative ring of dimension d. If ν is a non-negative valuation on R, then νR admits an order-preserving embedding in $\mathop{\mathbf{H}}_{i=1}^{d}\mathbb{R}$ (i.e., $\mathbb{R}^d$ with the lexicographic order).*

To prove Proposition 4.3.1, we will use a theorem by Abhyankar (the so called **Abhyankar inequality**) proved in [2].

Theorem 4.3.2 (Abhyankar inequality). *Let R be a local domain of dimension d with maximal ideal $\mathfrak{m}$ and quotient field F. Let ν be an arbitrary valuation of F having center $\mathfrak{m}$ in R. Let r be the rational rank of ν and let s be the transcendence degree of the residue field $F\nu$ of ν over $R/\mathfrak{m}$. Then $r+s\leq d$.*

Proof of Proposition 4.3.1. If ν is a Krull valuation, then R is a domain (see item **(i)** of Remark 2.1.4). If ν is not a Krull valuation, then it induces a Krull valuation $\overline{\nu}:R/\mathfrak{q}_\nu\longrightarrow \nu R$ by setting $\overline{\nu}(\phi+\mathfrak{q}_\nu):=\nu(\phi)$. Observe that, in this case, $\overline{\nu}\left(R/\mathfrak{q}_\nu\right)=\nu R$ and $R/\mathfrak{q}_\nu$ is a domain (because $\mathfrak{q}_\nu$ is a prime ideal of R). Since $\dim(R/\mathfrak{q}_\nu)\leq\dim(R)$, we reduce our proof to the case that ν is a Krull valuation on a domain R taking only non-negative values.

Since the valuation ν takes only non-negative values in R, it must have a center $\mathfrak{p}=R\cap\mathfrak{m}_\nu$ in R. Also, since ν is a Krull valuation it can be extended to $F=\mathrm{Quot}(R)=\mathrm{Quot}(R_\mathfrak{p})$. Hence, ν can be seen as a valuation on F centered at the local domain $R_\mathfrak{p}$. Since $\dim(R_\mathfrak{p})=\mathrm{ht}(\mathfrak{p})\leq\dim R$ and $\nu R=\nu R_\mathfrak{p}$, we reduce our problem to the case that R is a local domain and ν is a valuation on F centered at R.

Hence, we have all the assumptions of Theorem 4.3.2 and consequently we have that $\mathrm{rk}(\nu)=\mathrm{rk}(\nu R)\leq\mathrm{rr}(\nu)\leq d$. It is a well-known fact that every ordered abelian group of rank smaller or equal to d admits an order-preserving embedding in $\mathbb{R}^d$ ordered lexicographically. This concludes our proof. $\square$

If $d=\dim(R)$, then by use of Proposition 4.3.1 for every non-negative valuation $\nu\in\widetilde{\mathcal{W}}$ there exists a (non-negative) valuation $\mu\in\widetilde{\mathcal{W}}_\Gamma$ such that $\nu\sim\mu$. Therefore,

$$\mathcal{W}^{\geq0}=\widetilde{\mathcal{W}}^{\geq0}/\sim\;=\widetilde{\mathcal{W}}_\Gamma^{\geq0}/\sim.$$

For the remainder of this chapter we fix $\Gamma=\prod_{i=1}^{d}\mathbb{R}$. Let $\mathfrak{A}$ be a topology on Γ_∞.

Definition 4.3.3. As described in the previous section, the topology $\mathfrak{A}$ induces a topology $\widetilde{\mathfrak{B}}_\mathfrak{A}$ on $\widetilde{\mathcal{W}}_\Gamma$ (and hence on $\widetilde{\mathcal{W}}_\Gamma^{\geq0}$). We define the topology $\mathfrak{B}_\mathfrak{A}$ on $\mathcal{W}^{\geq0}$ to be the quotient topology on $\mathcal{W}^{\geq0}=\widetilde{\mathcal{W}}_\Gamma^{\geq0}/\sim$ associated to $\widetilde{\mathfrak{B}}_\mathfrak{A}$.

If $\widetilde{\mathcal{W}}_\Gamma^{\geq 0}$ is compact, then so is $\mathcal{W}^{\geq 0}$ because the quotient map is continuous when we take on $\mathcal{W}^{\geq 0}$ the quotient topology. However, it may happen that the quotient topology is compact but the original topology is not. Hence, it is interesting to study topologies which are defined directly on $\mathcal{W}^{\geq 0}$.

Definition 4.3.4. Define the topology $\mathfrak{C}_\mathfrak{A}$ on $\mathcal{W}^{\geq 0}$ to be the topology having as a subbasis the sets of the form

$$O_U^\phi = \{[\nu] \in \mathcal{W}^{\geq 0} \mid \exists \mu \in [\nu] \text{ with } \mu(\phi) \in U\}$$

where ϕ runs through R and U runs through $\mathfrak{A}$.

Remark 4.3.5. The subbasic open sets in the topology $\mathfrak{C}_\mathfrak{A}$ are images under the quotient map of subbasic open sets in the topology $\widetilde{\mathfrak{B}}_\mathfrak{A}$. Indeed, given a subbasic open set O_U^ϕ in $\mathfrak{C}_\mathfrak{A}$ we consider the subbasic open set in $\widetilde{\mathfrak{B}}_\mathfrak{A}$ given by

$$\widetilde{O}_U^\phi = \{\nu \in \widetilde{\mathcal{W}}_\Gamma \mid \nu(\phi) \in U\}.$$

From the definition of the quotient map we obtain that $q\left(\widetilde{O}_U^\phi\right) = O_U^\phi$.

From now on, we fix the topology on Γ_∞ to be the order topology, i.e., $\mathfrak{A} = \mathfrak{A}_1$.

Definition 4.3.6. Take any element $\phi \in R$ and $1 \leq i \leq d$. We define the sets associated to ϕ and i as

$$O_{(\phi,i,+)} = \{[\nu] \in \mathcal{W} \mid \exists \mu \in [\nu] \text{ with } v_h(\mu(\phi)) = i \text{ and } \mu(\phi) > 0\},$$

$$O_{(\phi,i)} = \{[\nu] \in \mathcal{W} \mid \exists \mu \in [\nu] \text{ with } v_h(\mu(\phi)) \geq i\}$$

and

$$O_{(\phi,\infty)} = \{[\nu] \in \mathcal{W} \mid \exists \mu \in [\nu] \text{ such that either } \mu(\phi) = \infty \text{ or } v_h(\mu(\phi)) = 1 \text{ and } \mu(\phi) > 0\}.$$

Proposition 4.3.7. *The sets defined in 4.3.6 are open in the topology $\mathfrak{C}_\mathfrak{A}$.*

Proof. Consider the element $e_i \in \mathbb{R}^d$ having 1 in the i-th coordinate and 0 in the other coordinates. For each $n \in \mathbb{N}$ we define the sets

$$U_n^+ = \left]\frac{1}{n}e_i, ne_i\right[, \; U_n =]-ne_i, ne_i[\text{ and } U_n^\infty = \left\{\gamma \in \Gamma_\infty \mid \gamma > \frac{1}{n}e_1\right\}.$$

Then we have that

$$O_{(\phi,i,+)} = \bigcup_{n\in\mathbb{N}} O^{\phi}_{U_n^+},\ O_{(\phi,i)} = \bigcup_{n\in\mathbb{N}} O^{\phi}_{U_n} \text{ and } O_{(\phi,\infty)} = \bigcup_{n\in\mathbb{N}} O^{\phi}_{U_n^\infty},$$

and hence these sets are open in the topology $\mathfrak{C}_{\mathfrak{A}}$. □

Lemma 4.3.8. *Take any valuation $\nu : R \longrightarrow \Gamma_\infty$ and any element $\gamma \in \Gamma$. If $v_h(\nu(\phi)) = v_h\gamma$ and $\mathrm{sign}(\nu(\phi)) = \mathrm{sign}(\gamma)$, then there exists $\mu \in [\nu]$ such that $\mu(\phi) = \gamma$.*

Proof. From Corollary 4.1.5 there exists an order-preserving isomorphism $\varphi : \Gamma \longrightarrow \Gamma$ such that $\varphi(\nu(\phi)) = \gamma$. Take now the valuation $\mu := \varphi \circ \mu$. Since φ is order preserving and injective we have that $\mu \in [\nu]$ and $\mu(\phi) = \varphi(\nu(\phi)) = \gamma$ by definition. □

Theorem 4.3.9. *The sets of the form defined in 4.3.6 form a subbasis for the topology $\mathfrak{C}_{\mathfrak{A}}$.*

Proof. In view of Proposition 4.3.7 it remains to prove that every subbasic open set in the topology $\mathfrak{C}_{\mathfrak{A}}$ can be written as the union of sets which are the intersection of finitely many sets as in Definition 4.3.6.

Take any open set of the form $U =]\alpha, \beta[\in \mathfrak{A}_1$ in Γ_∞. Let us prove first that if $\infty > \beta > \alpha \geq 0$, then

$$O^{\phi}_U = \bigcup_{i=v_h\beta}^{v_h\alpha} O_{(\phi,i,+)} \ .$$

Take $[\nu] \in O^{\phi}_U$. From the definition of O^{ϕ}_U, we know that there exists a non-negative valuation $\mu \in [\nu]$ such that $\alpha < \mu(\phi) < \beta$. Hence, $v_h(\beta) \leq v_h(\mu(\phi)) \leq v_h(\alpha)$, which implies (since $\mu\phi > 0$) that

$$[\nu] \in \bigcup_{i=v_h\beta}^{v_h\alpha} O_{(\phi,i,+)} \ .$$

For the converse, assume that $[\nu] \in O_{(\phi,i,+)}$ for some i with $v_h(\beta) \leq i \leq v_h(\alpha)$. Take an element $\gamma \in]\alpha, \beta[$ such that $v_h(\gamma) = i$. Since $\gamma > \alpha \geq 0$ we can apply Lemma 4.3.8 to conclude that there exists a valuation $\mu' \in [\nu]$ such that $\mu'(\phi) = \gamma$. Therefore, $[\nu] \in O^{\phi}_U$. If $\beta = \infty$, then a similar argument shows that

$$O^{\phi}_U = \bigcup_{i=1}^{v_h\alpha} O_{(\phi,i,+)} \ .$$

If $\alpha < \beta \leq 0$, then $O_U^\phi = \emptyset$ because we are dealing with non-negative valuations. The case $\alpha < 0 < \beta < \infty$ must be divided into two distinct cases, namely when $v_h\alpha \leq v_h\beta$ or $v_h\beta \leq v_h\alpha$. One can use arguments as above to show that for the first case we have

$$O_U^\phi = O_{(\phi, v_h\beta)}$$

and for the second

$$O_U^\phi = O_{(\phi, v_h\alpha)} \cup \bigcup_{i=v_h\beta}^{v_h\alpha} O_{(\phi,i,+)} \ .$$

The case when $\beta = \infty$ is treated as before.

It remains to show that if $U = \{\gamma \mid \gamma > \alpha\}$ for some $\alpha \in \Gamma_\infty$, then O_U^ϕ can be written as union of sets which are the intersection of finitely many sets as in Definition 4.3.6. A similar argument as before can be used to show that, for this case,

$$O_U^\phi = O_{(\phi,\infty)} \cup \bigcup_{i=1}^{v_h\alpha} O_{(\phi,i,+)} \ .$$

□

Theorem 4.3.10. *Take a Krull valuation ν and a non-zero element $\phi \in R$. If $\mathrm{rk}(\nu) \leq d-i$, then $[\nu] \in O_{(\phi,i)}$. Also, if $\nu(\phi) > 0$, then $[\nu] \in O_{(\phi,i',+)}$ for some $i' \geq i$. Moreover, every non-empty open set of $\mathfrak{C}_{\mathfrak{A}}$ contains the equivalence class of a rank one Krull valuation.*

Proof. Take any valuation ν of rank smaller than or equal to $d - i$. Then we can find a Krull valuation

$$\mu : R \longrightarrow \left(\mathop{\mathbf{H}}_{j=1}^{d-i} \mathbb{R} \right)_\infty$$

equivalent to ν. If $k = \min \mathrm{supp}\,(\mu(\phi)) \geq i$, then the assertions about ν are trivial, so assume that $k < i$.

Consider the embedding

$$\Psi : \mathop{\mathbf{H}}_{j=1}^{d-i} \mathbb{R} \longrightarrow \mathop{\mathbf{H}}_{j=1}^{d} \mathbb{R}$$

defined by $\Psi(e_j) = e_{j+i-k}$. The valuation $\mu' : R \longrightarrow \left(\mathop{\mathbf{H}}_{j=1}^{d} \mathbb{R} \right)_\infty$ given by $\mu' = \Psi \circ \mu$ is equivalent to ν and $v_h(\mu'\phi) = i$. Hence, $[\nu] \in O_{(\phi,i)}$. If $\nu(\phi) > 0$, then $[\nu] \in O_{(\phi,i,+)}$ from the definition of these sets.

To prove the last assertion we just take a non-empty subbasic open set O of $\mathfrak{C}_{\mathfrak{A}}$. If $O = O_{(\phi,i)}$, then from what we proved above, all equivalence classes of rank one valuations belong to O. If $O = O_{(\phi,i,+)}$, then there exists a rank one valuation ν such that $\nu(\phi) > 0$. Since ν is of rank one, νR can be embedded in $\mathbb{R}$. Considering the embedding $\Phi : \mathbb{R} \longrightarrow \mathbf{H}_{j=1}^{d} \mathbb{R}$ given by $\Phi(a) = ae_i$ we proceed as before to show that $[\nu] \in O$. □

Corollary 4.3.11. *In general, the topologies $\mathfrak{B}_{\mathfrak{A}}$ and $\mathfrak{C}_{\mathfrak{A}}$ are not equal.*

Proof. The proof is given by the following counterexample:

Example 4.3.12. Take $R = k[x, y]$ where x and y are algebraically independent over $k = \mathbb{F}_p$ and consider $\Gamma = \mathbb{R} \times \mathbb{R}$ ordered lexicographically. Consider

$$U_1 = \bigcup_{n\in\mathbb{N}} \left] \frac{1}{n} e_1, ne_1 \right[\text{ and } U_2 = \bigcup_{n\in\mathbb{N}} \left] \frac{1}{n} e_2, ne_2 \right[,$$

which are both open sets in the order topology of Γ. Consider the subset of $\widetilde{\mathcal{W}}_{\Gamma}^{\geq 0}$ given by

$$V = \{\nu \in \widetilde{\mathcal{W}}_{\Gamma}^{\geq 0} \mid \nu x \in U_1 \text{ and } \nu y \in U_2\}, \tag{4.2}$$

which is open in the product topology. Let $q : \widetilde{\mathcal{W}}_{\Gamma}^{\geq 0} \longrightarrow \mathcal{W}^{\geq 0}$ be the natural projection. We want to prove that $q(V) \in \mathfrak{B}_{\mathfrak{A}}$ but $q(V) \notin \mathfrak{C}_{\mathfrak{A}}$.

Take a valuation $\nu \in V$ and a valuation $\mu : R \longrightarrow (\mathbb{R} \times \mathbb{R})_{\infty}$ such that $\nu \sim \mu$. Let $\mu(x) = (a, b)$ and $\mu(y) = (c, d)$. Since $\nu(y) > 0$ we have that $c \geq 0$. If $c > 0$, then there exists $n \in \mathbb{N}$ such that $\mu(y^n) > \mu(x)$ and then $\nu(y^n) > \nu(x)$ which is a contradiction. Hence, $c = 0$ and $d > 0$ which implies that $\mu(y) \in U_2$. Since $\nu(x) > 0$ we have that $a \geq 0$. If $a = 0$, then there exists $n \in \mathbb{N}$ such that $\mu(y^n) > \mu(x)$ which is a contradiction. Hence, $a > 0$ and consequently $\mu(x) \in U_1$. Thus $[\nu] \subseteq V$ which implies that $q^{-1}(q(V)) = V$. Therefore, $q(V) \in \mathfrak{B}_{\mathfrak{A}}$.

To prove that $q(V) \notin \mathfrak{C}_{\mathfrak{A}}$ it is enough to present an equivalence class $[\nu] \in q(V)$ such that for every $V' \in \mathfrak{C}_{\mathfrak{A}}$ with $[\nu] \in V'$ we have that $V' \nsubseteq q(V)$. Take an equivalence class $[\nu]$ in $q(V)$ and $V' \in \mathfrak{C}_{\mathfrak{A}}$ such that $[\nu] \in V'$. Thus $V' \neq \emptyset$ and by Theorem 4.3.10 we obtain that V' contains the equivalence class of a rank one Krull valuation μ. In view of (4.2), we obtain that $\mu \notin V$ and hence $V' \nsubseteq q(V)$. Therefore, $q(V) \notin \mathfrak{C}_{\mathfrak{A}}$.

□

We want to introduce topologies on $\mathcal{W}$ which do not depend on the chosen topology $\mathfrak{A}$ on Γ_∞ and are similar to the topologies defined above. Take $\gamma, \gamma' \in \Gamma$, $\gamma, \gamma' \geq 0$. We say that γ is infinitely bigger than γ' and write $\gamma \gg \gamma'$ if $\gamma > n\gamma'$ for every $n \in \mathbb{N}$.

Take a valuation ν such that $[\nu] \in O_{(\phi,i,+)}$ for some i, $1 \leq i \leq d$. Then there exists a valuation $\mu : R \longrightarrow \Gamma_\infty$ equivalent to ν such that $v_h(\mu(\phi)) = i$ and $\mu(\phi) > 0$. This means that $\mu(\phi) \gg e_{i+1} \gg \ldots \gg e_d \geq 0$. Assume that there exist elements $\phi_1, \ldots, \phi_{d-i} \in R$ such that $\mu(\phi_j) = e_{i+j}$ for every j, $1 \leq j \leq d-i$. Then $\mu(\phi) \gg \mu(\phi_1) \gg \ldots \gg \mu(\phi_{d-i}) \geq 0$ and since $\nu \sim \mu$ we obtain that $\nu(\phi) \gg \nu(\phi_1) \gg \ldots \gg \nu(\phi_{d-i}) \geq 0 = \nu(1)$. This motivates the following definition:

Definition 4.3.13. We define the topology $\mathfrak{D}$ on $\mathcal{W}$ to be the topology having as a subbasis the sets of the form

$$O'_{(\phi,\psi,k)} = \{[\nu] \in \mathcal{W} \mid \exists \phi_0, \cdots, \phi_k \in R \text{ such that } \nu(\phi) = \nu(\phi_0) \gg \cdots \gg \nu(\phi_k) \geq \nu(\psi) \neq \infty\},$$

where ϕ and ψ run through R and k runs through $\mathbb{N} \cup \{0\}$.

Remark 4.3.14. This topology is particularly interesting because it is a "natural" generalization of the valuation spectrum topology introduced by Huber and Knebusch in [19] (see Definition 2.1.14). A subbasic open set in the valuation spectrum topology is of the form

$$\{[\nu] \in \mathcal{W} \mid \nu(\phi) \geq \nu(\psi) \neq \infty\},$$

for some $\phi, \psi \in R$. This set is exactly the set $O'_{(\phi,\psi,0)}$ in our definition. Therefore, the topology $\mathfrak{D}$ is finer than the valuation spectrum topology on $\mathcal{W}$. Also, it is easy to see that

$$O'_{(\phi,\psi,n)} \subseteq O'_{(\phi,\psi,n-1)} \subseteq \cdots \subseteq O'_{(\phi,\psi,0)}$$

and that $O'_{(\phi,\psi,m)} = \emptyset$ if $m > d = \dim(R)$. Hence, the subbasic open sets in $\mathfrak{D}$ consist of chains of finite sets lying in the subbasic open sets of the valuation spectrum topology. We can also see that if $\dim(R) = 1$, then these two topologies are equal.

Corollary 4.3.15. *Let R be a ring containing a field K and two elements which are algebraically independent over K. Then the topology $\mathfrak{C}_{\mathfrak{A}}$ and the topology on $\mathcal{W}^{\geq 0}$ induced by $\mathfrak{D}$ are not equal.*

Proof. From our assumptions on R there exists a set of the form $O'_{(\phi,\psi,2)} \in \mathfrak{D}$ which is not empty. From the definition of $O'_{(\phi,\psi,2)}$ we conclude that every valuation in $O'_{(\phi,\psi,2)}$ has rank greater than one. Therefore, from Theorem 4.3.10 we obtain that $O'_{(\phi,\psi,2)} \notin \mathfrak{C}_{\mathfrak{A}}$. □

We do not know much about the topologies that we defined above. We still cannot answer whether they are compact or not. However, the fact that they are not in general equal shows that these structures are worth to be studied.

Chapter 5

Henselian elements

Before we start to prove our results related to henselian elements we present and prove a few lemmas which we will need later.

5.1 Basic facts associated to henselian elements

Lemma 5.1.1. *Let R be a domain, $\mathfrak{p}$ a prime ideal and ϕ an element of R such that $\phi \notin \mathfrak{p}$. Then $R_\phi = R_\mathfrak{p}$ (as subsets of* $\operatorname{Quot}(R)$*) if and only if $\phi \in \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of R such that $\mathfrak{q} \not\subseteq \mathfrak{p}$.*

Proof. First observe that if ϕ does not belong to a prime ideal $\mathfrak{q}$, then $R_\phi \subseteq R_\mathfrak{q}$. Indeed, take an element $r \in R_\phi$. Then $r = f/\phi^n$ for some $f \in R$ and $n \in \mathbb{N}$. Since $\phi \notin \mathfrak{q}$ and $\mathfrak{q}$ is a prime ideal we have that $\phi^n \notin \mathfrak{q}$. Therefore, $r \in R_\mathfrak{q}$.

Assume that $R_\phi = R_\mathfrak{p}$ and take a prime ideal $\mathfrak{q}$ such that $\mathfrak{q} \not\subseteq \mathfrak{p}$. This implies that $R_\phi = R_\mathfrak{p} \not\subseteq R_\mathfrak{q}$. Therefore, by the previous paragraph, we must have that $\phi \in \mathfrak{q}$.

For the converse, assume that ϕ belongs to every prime ideal of R not contained in $\mathfrak{p}$. By the first paragraph and our assumption that $\phi \notin \mathfrak{p}$ we have that $R_\phi \subseteq R_\mathfrak{p}$. Now take an $r \in R_\mathfrak{p}$. This means that $r = f/g$ for some $f, g \in R$ and $g \notin \mathfrak{p}$. If there exists $\psi \in R$ and $n \in \mathbb{N}$ such that $g \cdot \psi = \phi^n$, then

$$r = \frac{f}{g} = \frac{f \cdot \psi}{\phi^n} \in R_\phi.$$

Suppose that such ψ and n do not exist. Then $gR \cap \{\phi^n \mid n \in \mathbb{N}\} = \emptyset$. Define the set

$$\mathcal{S} = \{\mathfrak{q} \subseteq R \mid \mathfrak{q} \text{ is an ideal of } R,\ g \in \mathfrak{q} \text{ and } \mathfrak{q} \cap \{\phi^n \mid n \in \mathbb{N}\} = \emptyset\}.$$

By the assumption on g we have that $gR \in \mathcal{S}$. It is easy to see that every chain of ideals in $\mathcal{S}$ admits an upper bound in $\mathcal{S}$, thus by Zorn's Lemma $\mathcal{S}$ has a maximal element $\mathfrak{q}$. It is also easy to see that $\mathfrak{q}$ is prime. Indeed, if not there would exist elements $\alpha, \beta \in R \setminus \mathfrak{q}$ such that $\alpha \cdot \beta \in \mathfrak{q}$. By the maximality of $\mathfrak{q}$ we have that

$$(\mathfrak{q} + \alpha R) \cap \{\phi^n \mid n \in \mathbb{N}\} \neq \emptyset \neq (\mathfrak{q} + \beta R) \cap \{\phi^n \mid n \in \mathbb{N}\}.$$

Thus $\phi^n = p + \alpha s$ and $\phi^m = q + \beta s'$ for some $p, q \in \mathfrak{q}$, $s, s' \in R$ and $m, n \in \mathbb{N}$. This means that

$$\phi^{m+n} = (p + \alpha s) \cdot (q + \beta s') = pq + p\beta s' + q\alpha s + \alpha\beta s s' \in \mathfrak{q},$$

which is a contradiction.

Since $g \in \mathfrak{q} \setminus \mathfrak{p}$ and $\mathfrak{q}$ is a prime ideal, by our assumption on ϕ we have that $\phi \in \mathfrak{q}$, which is a contradiction to the construction of $\mathfrak{q}$. □

Lemma 5.1.2. *Let $\mathfrak{m}$ be a maximal ideal of a domain R. If S is a local ring such that*

$$R \subseteq S \subseteq R_{\mathfrak{m}},$$

then $S = R_{\mathfrak{m}}$.

Proof. Take an element $r \in \mathrm{Quot}(R)$ such that $r \in R_{\mathfrak{m}}$. Then $r = f/g$ with $f, g \in R$ and $g \notin \mathfrak{m}$. Since $\mathfrak{m}$ is maximal and $g \notin \mathfrak{m}$ we have that the ideal generated by g and $\mathfrak{m}$ is the whole of R. Thus there exist $a \in R$ and $b \in \mathfrak{m}$ such that $1 = ag + b$. Let $\mathfrak{m}_S$ be the unique maximal ideal of S. Since $\mathfrak{m}R_{\mathfrak{m}} \cap S$ is an ideal of S we must have that $\mathfrak{m}R_{\mathfrak{m}} \cap S \subseteq \mathfrak{m}_S$, hence $b \in \mathfrak{m}_S$. This implies that $1 - b$ is a unit of S and since $ag = 1 - b$ we have that $1/g = a/(1-b) \in S$. Therefore, $r \in S$ and $S = R_{\mathfrak{m}}$. □

Lemma 5.1.3. *Let R be an integrally closed domain with fraction field L. Let $F|L$ be a finite separable extension and set R^* as the integral closure of R in F. Take $a \in R^*$ such that $F = L(a)$. Then the minimal polynomial $f(x)$ of a over L lies in $R[x]$ and $R^* \subseteq f'(a)^{-1}R[a]$.*

Proof. Let $k = [F : L]$ and $\sigma_1, \ldots, \sigma_k$ be all the L-embeddings of F in $\widetilde{L} = \widetilde{F}$. Then we have that

$$f(x) = \prod_{i=1}^{k}(x - \sigma_i(a)),$$

where we assume that $\sigma_1 = \text{id}$. Then $\sigma_i(a)$ belongs to the integral closure of R in $\widetilde{L}$ for $1 \leq i \leq k$, because $a \in R^*$. This means that every coefficient of f is integral over R and since R is integrally closed we have that $f(x) \in R[x]$.

Define now

$$g_i(x) = \frac{f(x)}{x - \sigma_i(a)} = \prod_{j \neq i} (x - \sigma_j(a)) \in \widetilde{L}[x].$$

We shall prove that

$$g_1(x) = c_0 + \ldots + c_{k-2}x^{k-2} + x^{k-1} \in R^*[x].$$

Observe first that R^* is the intersection of all valuation rings $\mathcal{O}$ of F containing R. Since $f(x) \in R[x]$ and $f(x) = g_1(x) \cdot (x - a)$ (in $F[x]$), using Gauss Lemma we obtain that all coefficients of g_1 must belong to $\mathcal{O}$. Hence, every coefficient of g_1 belongs to R^*.

If $i > 1$, then $g_i(a) = 0$ because $x - a$ divides g_i. Also, since $f(x) = g_1(x) \cdot (x - a)$ we have

$$f'(x) = g_1'(x) \cdot (x - a) + g_1(x)$$

hence $f'(a) = g_1(a)$. Observe that

$$g_i(x) = \sigma_i(c_0) + \ldots + \sigma_i(c_{k-2})x^{k-2} + x^{k-1}.$$

Take $b \in R^*$. Then

$$\begin{aligned} b \cdot f'(a) &= b \cdot g_1(a) \\ &= \sum_{i=1}^{k} \sigma_i(b) \cdot g_i(a) \\ &= \sum_{i=1}^{k} \sigma_i(b) \cdot \big(\sigma_i(c_0) + \ldots + \sigma_i(c_{k-2})a^{k-2} + a^{k-1}\big) \\ &= \sum_{i=1}^{k} \sigma_i(bc_0) + \ldots + \left(\sum_{i=1}^{k} \sigma_i(bc_{k-2})\right) a^{k-2} + \left(\sum_{i=1}^{k} \sigma_i(b)\right) a^{k-1}. \end{aligned}$$

Since each coefficient of $b \cdot f'(a)$ is the trace of some element in F they all belong to L. On the other hand, they are integral over R, hence they belong to R. Thus $b \cdot f'(a) \in R[a]$ and consequently, $b \in f'(a)^{-1}R[a]$. Therefore, $R^* \subseteq f'(a)^{-1}R[a]$. $\square$

Lemma 5.1.4 (Prime Avoidance Lemma). *Let R be a ring and $J, I_1, \dots, I_r$, $r \in \mathbb{N}$ be ideals of R such that at most two of them are not prime. If*

$$J \subseteq \bigcup_{i=1}^{r} I_i$$

then $J \subseteq I_i$ for some $0 \leq i \leq r$.

Proof. We will prove the lemma by induction on r. If $r = 1$, then the lemma is trivial. Assume that the result is true for every $r < k$ and let us prove it is true for $r = k$.

If $J \subseteq \bigcup_{i \neq j} I_i$ for some $0 \leq j \leq r$, then by the induction hypothesis we have that $J \subseteq I_i$ for some $i \neq j$. Assume towards a contradiction that $J \not\subseteq \bigcup_{i \neq j} I_i$ for every i, $0 \leq j \leq r$. For each j, $0 \leq j \leq r$, we pick an element $a_j \in J$ such that $a_j \notin \bigcup_{i \neq j} I_i$ and consequently $a_j \notin I_i$ if $i \neq j$. Since $J \subseteq \bigcup_{i=1}^{r} I_i$ we must have that $a_j \in I_j$.

We want to prove that there exists an element $a \in J$ such that $a \notin I_i$ for every $0 \leq i \leq r$ which is the desired contradiction since we assumed that $J \subseteq \bigcup_{i=1}^{r} I_i$. If $r = 2$, then $a = a_1 + a_2$ cannot belong to either I_1 or I_2. Indeed, assume without loss of generality that $a_1 + a_2 \in I_1$ then

$$a_2 = (a_1 + a_2) - a_1 \in I_1$$

which is a contradiction. If $r > 2$, then there exists $j \leq r$ such that I_j is prime. Let

$$a = a_j + \prod_{i \neq j} a_i.$$

If $l \neq j$ and $a \in I_l$, then

$$a_j = a - \prod_{i \neq j} a_i \in I_l$$

which is a contradiction. If $a \in I_j$, then

$$\prod_{i \neq j} a_i = a - a_j \in I_j$$

which is a contradiction because I_j is prime and $a_i \notin I_j$ if $i \neq j$.

□

Lemma 5.1.5. *Let R be a ring and G a finite group of automorphisms of R. Put*

$$A = \{r \in R \mid \sigma r = r \text{ for every } \sigma \in G\}.$$

If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of R such that $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$, then there exists $\sigma \in G$ such that $\mathfrak{q}_2 = \sigma\mathfrak{q}_1$.

Proof. Let us prove first that there exists $\sigma \in G$ such that $\mathfrak{q}_2 \subseteq \sigma\mathfrak{q}_1$. Suppose towards a contradiction that $\mathfrak{q}_2 \nsubseteq \sigma\mathfrak{q}_1$ for every $\sigma \in G$. From Lemma 5.1.4 there exists $a \in R$ such that $a \in \mathfrak{q}_2$ but $a \notin \sigma\mathfrak{q}_1$ for every $\sigma \in G$ (observe that since $\mathfrak{q}_1$ is prime and σ is an automorphism of R the ideal $\sigma\mathfrak{q}_1$ is also prime). Put

$$r = \prod_{\sigma \in G} \sigma a \in R.$$

Since $\sigma r = r$ for every $\sigma \in G$ we have that $r \in A$. Moreover, since $a \in \mathfrak{q}_2$ $\sigma \in G$ we have that $r \in \mathfrak{q}_2$. Hence,

$$r \in \mathfrak{q}_2 \cap A = \mathfrak{q}_1 \cap A \subseteq \mathfrak{q}_1.$$

This is a contradiction to the fact that $a \notin \sigma\mathfrak{q}_1$ for every $\sigma \in G$.

Since $\mathfrak{q}_2 \subseteq \sigma\mathfrak{q}_1$ for some $\sigma \in G$, if the extension $A \subseteq R$ is integral, then $\mathfrak{q}_2 = \sigma\mathfrak{q}_1$ because integral extensions have the incomparability property (see [35] Theorem 5 (ii) of chapter 2, for instance). Hence, it remains to prove that $A \subseteq R$ is integral. Take any element $a \in R$. Then the monic polynomial

$$f(x) = \prod_{\sigma \in G} (x - \sigma a) \in R[x]$$

has a as a root. If we extend σ to $R[x]$ by setting $\sigma(x) = x$, then a polynomial is fixed by σ if and only if all of its coefficients are fixed by σ. It is easy to see that $\sigma(f) = f$ for every $\sigma \in G$. Hence, every coefficient of f belongs to A. Therefore, a is integral over A, which is what we wanted to prove. □

Proposition 5.1.6. *Let $F|L$ be a finite separable extension of fields and $\mathcal{O}_L$ a valuation ring of L. Take $\mathcal{O}_L^*$ to be the integral closure of $\mathcal{O}_L$ in the normal hull F' of $F|L$. Let $(\mathfrak{p}_i)_{i\in I}$ be the chain of prime ideals of $\mathcal{O}_L$ and $(\mathfrak{q}_i)_{i\in I}$ a chain of prime ideals of $\mathcal{O}_L^*$ such* *that* $\mathfrak{p}_i = \mathcal{O}_L \cap \mathfrak{q}_i$ *for every* $i \in I$. *For every ring* S *such that* $\mathcal{O}_L \subseteq S \subseteq \mathcal{O}_L^*$ *we have that the prime ideals of* S *are given by* $\sigma(\mathfrak{q}_i) \cap S$ *for some* $\sigma \in \mathrm{Gal}(F|L)$ *and some* $i \in I$. *In particular,* $\mathrm{Spec}(S)$ *is a union of finitely many chains of prime ideals of* S.

Proof. Since $\sigma(\mathcal{O}_L^*) = \mathcal{O}_L^*$ for every $\sigma \in \mathrm{Gal}(F'|L)$ and

$$\mathcal{O}_L = \{r \in \mathcal{O}_L^* \mid \sigma r = r \text{ for every } \sigma \in \mathrm{Gal}(F'|L)\}$$

we use Lemma 5.1.5 to conclude that every prime ideal of $\mathcal{O}_L^*$ is of the form $\sigma(\mathfrak{q}_i)$ for some $\sigma \in \mathrm{Gal}(F'|L)$ and for some $i \in I$. Since the ring extension $S \subseteq \mathcal{O}_L^*$ is integral, for every prime ideal $\mathfrak{p}$ of S there exists a prime ideal $\mathfrak{q}$ of $\mathcal{O}_L^*$ such that $\mathfrak{p} = \mathfrak{q} \cap S$. Therefore,

$$\mathfrak{p} = \mathfrak{q} \cap S = \sigma(\mathfrak{q}_i) \cap S$$

for some $\sigma \in \mathrm{Gal}(F'|L)$ and for some $i \in I$. □

Lemma 5.1.7. *Let* R *be a domain and* $\mathfrak{p}$ *a prime ideal of* R. *Then every monic polynomial* $\overline{f} \in R/\mathfrak{p}[x]$ *admits a factorization into irreducible elements of* $R/\mathfrak{p}[x]$.

Proof. If $\overline{f}$ is irreducible we are done. If $\overline{f}$ is reducible, then $\overline{f} = \overline{f}_1\overline{f}_2$ with $\overline{f}_1$ and $\overline{f}_2$ non-units of $R/\mathfrak{p}[x]$. If $\overline{f}_1$ and $\overline{f}_2$ are irreducible we are done and if not we have a new decomposition of $\overline{f}$ into more than two factors which are non-units in $R/\mathfrak{p}[x]$. We have to prove that if we proceed like that we will obtain a factorization in which every factor is irreducible. Since f is monic, for each decomposition

$$\overline{f} = \prod_{i=1}^{r} \overline{f}_i \text{ with } \overline{f}_i \text{ non-unit in } R/\mathfrak{p}[x] \text{ for } 1 \leq i \leq r,$$

the leading coefficients of each $\overline{f}_i$ is a unit in $R/\mathfrak{p}$. Hence, the degree of each $\overline{f}_i$ is greater than zero and consequently, r cannot exceed $\deg \overline{f}$. Therefore, the desired decomposition exists. □

Lemma 5.1.8. *Take a valuation ring* R *and a prime ideal* $\mathfrak{p}$ *of* R. *Then the polynomial ring* $R/\mathfrak{p}[x]$ *is a GCD domain.*

Proof. We observe first that since R is a valuation ring so it is $R/\mathfrak{p}$. Take two elements a, b in a valuation ring. Then either a/b or b/a belongs to that valuation ring, which means that either a divides b or b divides a. Hence, every valuation ring is a GCD domain. It is a well-known fact that the polynomial ring over a GCD domain is a GCD domain. Therefore, $R/\mathfrak{p}[x]$ is a GCD domain. $\square$

Proposition 5.1.9. *Take R a valuation ring, $f(x) \in R[x]$ a monic polynomial and $\mathfrak{p}$ a prime ideal of R. Set*

$$R[a] = R[x]/(f), \text{ for } a = x + (f).$$

For a polynomial $g \in R[x]$, we denote by $\overline{g}$ the polynomial obtained from g by the reduction modulo $\mathfrak{p}$ of its coefficients. Let

$$\overline{f} = \prod \overline{f_i}^{e_i}, \text{ with } e_i > 0$$

be the factorization (which exists by Lemma 5.1.7) of $\overline{f}(x) \in R/\mathfrak{p}[x]$ into powers of distinct irreducible polynomials $\overline{f_i}$. Then the prime ideals of $R[a]$ lying over $\mathfrak{p}$ are precisely

$$\mathfrak{q}_i = (\mathfrak{p}, f_i(a))R[a].$$

Moreover, $\mathfrak{q}_i \neq \mathfrak{q}_j$ if $i \neq j$, and $R[a]/\mathfrak{q}_i \cong (R/\mathfrak{p}[x])/(\overline{f_i})$.

Proof. Consider the map

$$\Phi_i : R[a] = R[x]/(f) \longrightarrow (R/\mathfrak{p}[x])/(\overline{f}_i),$$

given by $\Phi_i(g(x) + (f)) = \overline{g}(x) + (\overline{f}_i)$. We will prove that $\ker(\Phi_i) = \mathfrak{q}_i$. It is easy to see that

$$\mathfrak{q}_i = (\mathfrak{p}, f_i(a))R[a] \subseteq \ker(\Phi_i).$$

On the other hand, if $\Phi_i(\phi(a)) = 0$, then $g(x) := \phi(x) - f_i(x)h(x) \in \mathfrak{p}R[x]$ for some $h(x) \in R[x]$. Therefore,

$$\phi(a) = g(a) + f_i(a)h(a) \in (\mathfrak{p}, f_i(a))R[a].$$

Since $R/\mathfrak{p}[x]$ is a GCD domain (Lemma 5.1.8) and $\overline{f}_i$ is irreducible over $R/\mathfrak{p}[x]$ we have that $R/\mathfrak{p}[x]/(\overline{f}_i)$ is a domain. Also, since Φ_i is surjective we conclude that $\mathfrak{q}_i$ is a prime ideal.

If $j \neq i$, then the image of $f_j(x)$ is not zero in $R/\mathfrak{p}[x]/(\overline{f}_i)$, which implies that $f_j(a) \notin \mathfrak{q}_i$. Therefore, $\mathfrak{q}_i \neq \mathfrak{q}_j$.

It remains to prove that every prime ideal $\mathfrak{q}'$ of $R[a]$ lying over $\mathfrak{p}$ is of the form $\mathfrak{q}_i$ for some i. Since

$$\prod f_i(x)^{e_i} - f(x) \in \mathfrak{p}R[x],$$

and $f(a) = 0$ we have that $\prod f_i(a)^{e_i}$ belongs to $\mathfrak{q}'$. Consequently, for some i we have that $f_i(a) \in \mathfrak{q}'$ and hence $\mathfrak{q}_i = (\mathfrak{p}, f_i(a))R[a] \subseteq \mathfrak{q}'$. Since both $\mathfrak{q}'$ and $\mathfrak{q}_i$ are prime ideals lying over $\mathfrak{p}$ and $R[a]$ is integral over R, from the incomparability property we conclude that $\mathfrak{q}_i = \mathfrak{q}'$. □

5.2 Henselian elements

Let (L, ν) be a valued field and $(F|L, \nu)$ a finite valued field extension such that $F \subseteq L^i$.

Lemma 5.2.1. *Let $a \in \mathcal{O}_F$ be a henselian element such that $\nu(a) = 0$. Then a^{-1} is also henselian.*

Proof. Let $h(x) \in \mathcal{O}_L[x]$ be a henselian polynomial for a. Define the polynomial

$$g(x) = x^n \cdot h(x^{-1}) \in \mathcal{O}_L[x],$$

where $n = \deg h$. Then $g\left(a^{-1}\right) = 0$. Moreover,

$$g'(x) = nx^{n-1}h(x^{-1}) - x^{n-2}h'(x^{-1})$$

and hence $g'(a^{-1}) = -a^{-(n-2)}h'(a)$. Therefore, $\nu(g'(a^{-1})) = \nu(h'(a)) = 0$. □

Let W be the set of all valuations μ of F such that $\nu|_L = \mu|_L$ and $\nu \neq \mu$.

Lemma 5.2.2. *Let $\vartheta \in F\nu$ be a non-zero generator of $F\nu$ over $L\nu$ and take $z \in F$ such that $z\nu = \vartheta$. If $\mu(z) \neq 0$ for every valuation $\mu \in W$, then z is a henselian element.*

Proof. We assume ν to be extended to the algebraic closure $\widetilde{L} = \widetilde{F}$. Let (L^h, ν) and (F^h, ν) denote the henselizations of (L, ν) and (F, ν) inside of (L^i, ν). Then $F^h = F.L^h$. This lies in L^i since $L^h \subseteq L^i$ and by assumption, $F \subseteq L^i$. Since the henselization is an immediate extension, we have that $L^h\nu = L\nu$ and $F^h\nu = F\nu$. As $F.L^h|L^h$ is a finite subextension of $L^i|L^h$, $[F.L^h : L^h] = [F^h\nu : L^h\nu] = [F\nu : L\nu]$. Therefore, there are $k := [F\nu : L\nu]$ many automorphisms $\sigma_1, \ldots, \sigma_k \in \mathrm{Aut}(L^{\mathrm{sep}}|L^h)$ which induce distinct embeddings $\overline{\sigma}_1, \ldots, \overline{\sigma}_k$ of $F\nu$ in $(L\nu)^{\mathrm{sep}}$ over $L\nu$ (note that $F\nu|L\nu$ is separable since $F \subseteq L^i$). Their restriction to F are k distinct embeddings of F over L in L^{sep}.

Now we choose further automorphisms $\sigma_{k+1}, \ldots, \sigma_{k+l} \in \mathrm{Aut}(L^{\mathrm{sep}}|L)$ such that the restrictions of $\sigma_1, \ldots, \sigma_{k+l}$ to F are precisely all distinct embeddings of F over L in L^{sep}. As a subextension of $L^i|L$, also $F|L$ is separable; so $k + l = n = [F : L]$.

We claim that $\nu\sigma_i \neq \nu$ on F for $k < i \leq n$. Assume that $\nu\sigma = \nu$ on F for some $\sigma \in \mathrm{Aut}(L^{\mathrm{sep}}|L)$. As all extensions of a valuation to an algebraic extension are conjugate, there is some $\sigma' \in \mathrm{Aut}(L^{\mathrm{sep}}|L)$ such that $\nu\sigma\sigma' = \nu$ on L^{sep}. By definition, $\sigma\sigma'$ is thus an element of the decomposition group of the extension $(L^{\mathrm{sep}}|L, \nu)$. On the other hand, this decomposition group is the Galois group of $L^{\mathrm{sep}}|L^h$, where L^h is the henselization of L in (L^{sep}, ν). Therefore, $\sigma\sigma'$ is trivial on L^h and must consequently coincide on $F.L^h$ with σ_j for some $j \in \{1, \ldots, k\}$. It follows that σ coincides with σ_j on L. This proves our claim.

Since $\nu_i(z) \neq 0$ for $k < i \leq n$ we assume, without loss of generality, that

$$\nu_j(z) > 0 \text{ for } k < j \leq r$$

and

$$\nu_l(z) < 0 \text{ for } r < l \leq n.$$

Let

$$h_0(x) = \prod_{i=1}^{k}(x - \sigma_i z) \prod_{j=k+1}^{r} (x - \sigma_j z) \prod_{l=r+1}^{n} (x - \sigma_l z).$$

Dividing h_0 by a coefficient c of smallest ν-value, we obtain a polynomial $h(x) \in L[x]$ such that every coefficient of h has nonnegative value and at least one coefficient has value zero (such polynomial will be called **ν-primitive**). Consider

$$h_1(x) = \prod_{i=1}^{k}(x - \sigma_i z) \prod_{j=k+1}^{r} (x - \sigma_j z) \prod_{l=r+1}^{n} ((\sigma_l z)^{-1}x - 1),$$

which is obtained by dividing $h_0(x)$ by the factor $\prod_l \sigma_l z \in \widetilde{L}$. The polynomials h and h_1

differ by a constant factor $d \in \widetilde{L}$ and since both are ν-primitive we have $\nu d = 0$. Thus

$$h(x) = d \cdot h_1(x) \text{ with } d \in \widetilde{L} \text{ and } d\nu \neq 0, \infty.$$

Consequently,

$$h\nu(x) = d\nu \cdot g(x) \cdot x^{r-k} \cdot (-1)^{n-r}$$

where $g(x)$ is the minimal polynomial of ϑ over $L\nu$. Since ϑ is non-zero, the polynomial $g(x)$ is different of x and hence $z\nu = \vartheta$ is a simple root of $h\nu(x)$. This means that $h'(z)\nu \neq 0$ and consequently z is a henselian element. □

We will prove now our main theorems.

Proof of Theorem 1.3.2. Let ϑ be a generator of the separable extension $F\nu|L\nu$. Employing the Chinese Remainder Theorem (see Lemma 6.60 of [25]) we find an element $\eta \in F$ such that $\eta\nu = \vartheta$ and $\eta\mu = 0$ for every $\mu \in W$. Observe that in this case, the polynomials $h_0(x), h_1(x)$ and $h(x)$ (constructed in the proof of Lemma 5.2.2) are equal and have the form

$$h(x) = \prod_{i=1}^{k}(x - \sigma_i\eta) \prod_{j=k+1}^{n} (x - \sigma_j\eta)$$

where $\nu(\sigma_i\eta) > 0$ for $i > k$. Consequently, $h(x) \in \mathcal{O}_L[x]$. By Lemma 5.2.2 we have that η is henselian with henselian polynomial $h(x)$ (in particular, $h'(\eta)$ is a unit in $\mathcal{O}_F$). It remains to prove that $F = L(\eta)$, i.e., that the polynomial $h(x)$ is irreducible. It is enough to prove that $\sigma_i\eta \neq \eta$ for $1 \leq i \leq n$. From the separability of $F\nu|L\nu$ we have that for $1 \leq i \leq k$, $\sigma_i(\vartheta) \neq \vartheta$, hence $\sigma_i(\eta) \neq \eta$. For $i > k$ we have that

$$\nu(\sigma_i\eta) = \nu_i(\eta) \neq 0 = \nu(\eta),$$

so $\sigma_i\eta \neq \eta$.

Let $\mathcal{O}_L^*$ be the integral closure of $\mathcal{O}_L$ in F. From Lemma 5.1.3 we have that

$$\mathcal{O}_L^* \subseteq \mathcal{O}_L[\eta, 1/h'(\eta)].$$

Since $\mathcal{O}_L^*$ is a Prüfer domain, every ring lying between $\mathcal{O}_L^*$ and its quotient field F is a Prüfer domain. Therefore, $\mathcal{O}_L[\eta, 1/h'(\eta)]$ is a Prüfer domain. □

Proof of Theorem 1.3.3. With the notation of the proof of Theorem 1.3.2 we have that $\mathcal{O}_F = (\mathcal{O}_L^*)_{\mathfrak{m}^*}$ (see Lemma 6.50 of [25]), where $\mathfrak{m}^* = \mathfrak{m}_F \cap \mathcal{O}_L^*$. Set $\mathfrak{n} = \mathfrak{m}_F \cap \mathcal{O}_L[\eta]$ and let $h(X) \in \mathcal{O}_L$ be the monic polynomial constructed in proof of Lemma 5.2.2. Using Lemma 5.1.3 for the first inclusion, we obtain that

$$\mathcal{O}_L^* \subseteq \frac{1}{h'(\eta)}\mathcal{O}_L[\eta] \subseteq \mathcal{O}_L[\eta]_{\mathfrak{n}} \subseteq (\mathcal{O}_L^*)_{\mathfrak{m}^*} = \mathcal{O}_F.$$

Hence, by Lemma 5.1.2 we have that

$$\mathcal{O}_F = \mathcal{O}_L[\eta]_{\mathfrak{n}}.$$

Take any finite set $Z = \{f_1, \ldots, f_r\} \subseteq \mathcal{O}_F$. For every $f_i \in Z \subseteq \mathcal{O}_F = \mathcal{O}_L[\eta]_{\mathfrak{n}}$ we have $f_i = \dfrac{a_i}{b_i}$ for some $a_i, b_i \in \mathcal{O}_L[\eta]$ and $b_i \notin \mathfrak{n}$. Taking

$$u = \prod_{i=1}^{r} b_i \in \mathcal{O}_L[\eta]$$

we obtain that $Z \subseteq \mathcal{O}_L[\eta, 1/u]$. □

Proof of Theorem 1.3.4. Take any element $a \in \mathcal{O}_F$. Since ϑ is a generator of $F\nu$ over $L\nu$, then $a\nu$ can be written as a polynomial on ϑ with coefficients on $L\nu$. Let $f(X) \in \mathcal{O}_L[X]$ be an inverse image of that polynomial and define $b = a - f(\eta) + \eta$. Then $a\nu = f(\eta)\nu$ and $b\nu = \vartheta$. Also, we have that $a \in \mathcal{O}_L[\eta, b]$. Therefore, it is enough to prove that $b \in \mathcal{O}_L[r, s]$ for some henselian elements r and s.

Employing the Chinese Remainder Theorem, we find an element $c \in \mathcal{O}_F$ with following properties:

$$\begin{aligned} \nu(c-1) > 0 &\quad \text{so that } c\nu = 1 \\ \mu(c) > 0 &\quad \text{if } \mu(b) \geq 0 \\ \mu(c) = 0 &\quad \text{if } \mu(b) < 0 \end{aligned}$$

for all $\mu \in W$. Then the element $r = bc$ has the following properties:

$$\begin{array}{rclcl} r\nu & = & b\nu & = & \vartheta \\ \mu(r) & > & 0 & & \text{if } \mu(b) \geq 0 \\ \mu(r) & < & 0 & & \text{if } \mu(b) < 0 \end{array}$$

and hence $\mu(r) \neq 0$ for every $\mu \in W$. According to Lemma 5.2.2, r is a henselian element. On the other hand, the element rc has the same above properties, so it is also henselian. Since $\nu(rc) = 0$ we can apply Lemma 5.2.1 to obtain that $s := (rc)^{-1}$ is also a henselian element. Therefore,

$$b = r \cdot c^{-1} = r^2 \cdot (rc)^{-1} = r^2 \cdot s \in \mathcal{O}_L[r, s]$$

as required. $\square$

Proof of Theorem 1.3.6. **(i)** $\Longrightarrow$ **(ii)**: Since $\mathcal{O}_F$ is a finitely generated $\mathcal{O}_L$-algebra, we have that $\mathcal{O}_F = \mathcal{O}_L[a_1, \ldots, a_r]$ for some $a_1, \ldots, a_r \in \mathcal{O}_F$. Applying Theorem 1.3.3 to the set $Z := \{a_1, \ldots, a_r\}$ we obtain that there exists a unit $u \in \mathcal{O}_F$ such that $Z \subseteq \mathcal{O}_L[\eta, 1/u]$. Therefore,

$$\mathcal{O}_F = \mathcal{O}_L[a_1, \ldots, a_r] \subseteq \mathcal{O}_L[\eta, 1/u] \subseteq \mathcal{O}_F$$

and consequently the equality holds everywhere.

(ii) $\Longrightarrow$ **(iii)**: Just apply Theorem 1.3.4 to the element $a = 1/u$.

(iii) $\Longrightarrow$ **(i)**: This is trivial.

(ii) $\Longleftrightarrow$ **(iv)**: Applying Lemma 5.1.1 with $R = \mathcal{O}_L[\eta]$, $\phi = v$ and $\mathfrak{p} = \mathfrak{n}$ we obtain that $\mathcal{O}_L[\eta]_{\mathfrak{n}} = \mathcal{O}_L[\eta]_v$ if and only if v belongs to every prime ideal of $\mathcal{O}_L[\eta]$ which is not contained in $\mathfrak{n}$. Since $\mathcal{O}_L[\eta]_v = \mathcal{O}_L[\eta, 1/v]$ by definition and $\mathcal{O}_F = \mathcal{O}_L[\eta]_{\mathfrak{n}}$ from Theorem 1.3.3 we have our assertion.

(iv) $\Longrightarrow$ **(v)**: Suppose that **(v)** is not true. This means that there exists a chain of prime ideals $(\mathfrak{p}_i)_{i \in I}$ of $\mathcal{O}_L[\eta]$ such that $\mathfrak{p}_i \not\subseteq \mathfrak{n}$ for every $i \in I$ and

$$\bigcap_{i \in I} \mathfrak{p}_i \subseteq \mathfrak{n}.$$

Since $(\mathfrak{p}_i)_{i \in I} \subseteq \mathcal{S}$, then

$$\bigcap_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p} \subseteq \bigcap_{i \in I} \mathfrak{p}_i$$

and we conclude that **(iv)** does not hold.

(v) $\Longrightarrow$ **(iv)**: From Proposition 5.1.6, $\mathrm{Spec}(\mathcal{O}_L[\eta])$ consists of finitely many chains of prime ideals, say

$$\mathrm{Spec}(\mathcal{O}_L[\eta]) = \bigcup_{j=1}^{r} (\mathfrak{q}_i^j)_{i \in I}.$$

For each chain $(\mathfrak{q}_i^j)_{i\in I}$, if $\mathfrak{q}_i^j \not\subseteq \mathfrak{n}$ for some $i \in I$, then by our assumption

$$\bigcap_{\mathfrak{q}_i^j \not\subseteq \mathfrak{n}} \mathfrak{q}_i^j \not\subseteq \mathfrak{n}.$$

Hence, there exits an element $v_j \in \mathcal{O}_L[\eta] \setminus \mathfrak{n}$ such that $v_j \in \mathfrak{q}_i^j$ for every $i \in I$ with $\mathfrak{q}_i^j \not\subseteq \mathfrak{n}$. Take v to be the product of these v_j's. Then v belongs to every prime ideal of $\mathcal{O}_L[\eta]$ not contained in $\mathfrak{n}$. Therefore, **(iv)** holds. □

In order to prove Theorem 1.3.5 we will need the following result, which is Proposition 4 of [26].

Proposition 5.2.3. *There are valued fields (L, ν) such that νL has no maximal proper convex subgroup, the residue field $(L\mu, \overline{\mu})$ is henselian for every non-trivial coarsening $\mu \neq \nu$ of ν, but (L, ν) is not itself henselian.*

Proof of Theorem 1.3.5. Let (L, ν) be the valued field given in Proposition 5.2.3. Extend ν to the algebraic closure $\widetilde{L}$ of L (denote this extension again by ν). Since (L, ν) is not henselian, there exists a finite extension F of L with $F \subseteq L^h \subseteq L^i$ such that there exists a valuation μ on F with $\mu \neq \nu$ and $\nu\,|_L = \mu\,|_L$, i.e, $W \neq \emptyset$. By Theorem 1.3.2 there exists $\eta \in F$ such that $F = L(\eta)$, the monic minimal polynomial $h(x)$ of η over L lies in $\mathcal{O}_L[x]$ and η and $h'(\eta)$ are units in $\mathcal{O}_F$. Moreover, from the construction of h we have that

$$h\nu(x) = x^l g(x) \in L\nu[x],\ l = |W| \geq 1$$

where $g(x)$ is the separable minimal polynomial of a generator ϑ of $F\nu$ over $L\nu$.

Given any prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$ we consider the coarsening $\nu_\mathfrak{p}$ of ν associated with $\mathfrak{p}$. If $(0) \subsetneq \mathfrak{p} \subsetneq \mathcal{O}_L$, then $\nu_\mathfrak{p}$ is non-trivial and $\nu_\mathfrak{p} \neq \nu$.

Claim 5.2.4. *There exist polynomials $F, G \in \mathcal{O}_L[x]$ such that $h - FG \in \mathfrak{p}\mathcal{O}_L[x]$ with $F\nu(x) = x^l$, $G\nu(x) = g(x)$ and $\deg F = l \geq 1$.*

Proof. By the assumption on the valued field (F, ν) we have that $(L\nu_\mathfrak{p}, \overline{\nu}_\mathfrak{p})$ is henselian. Hence, there exist polynomials $\overline{F}, \overline{G} \in \mathcal{O}_{\overline{\nu}_\mathfrak{p}}[x]$ such that

$$h\nu_\mathfrak{p}(x) = \overline{F}(x)\overline{G}(x),\ \overline{F}\overline{\nu}_\mathfrak{p} = x^l,\ \overline{G}\overline{\nu}_\mathfrak{p} = g(x) \text{ and } \deg \overline{F} = l.$$

Take any polynomials $F(x), G(x) \in \mathcal{O}_L[x]$ such that $F\nu_{\mathfrak{p}} = \overline{F}$, $G\nu_{\mathfrak{p}} = \overline{G}$ and $\deg F = l$. Then $F\nu(x) = x^l$ and $G\nu(x) = g(x)$. Since the residue map associated to $\nu_{\mathfrak{p}}$ in $\mathcal{O}_L$ is the reduction modulo $\mathfrak{p}$ and $h\nu_{\mathfrak{p}}(x) - F\nu_{\mathfrak{p}}(x)G\nu_{\mathfrak{p}}(x) = 0$ we have that every coefficient of $h - FG$ belongs to $\mathfrak{p}$. Therefore, $h - FG \in \mathfrak{p}\mathcal{O}_L[x]$.

$\square$

Take a prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$. We claim that there exists a prime ideal $\mathfrak{q}$ of $\mathcal{O}_L[\eta]$ lying over $\mathfrak{p}$ such that $\mathfrak{q} \not\subseteq \mathfrak{n} = \mathfrak{m}_F \cap \mathcal{O}_L[\eta]$. By Claim 5.2.4, there exists $F(x) \in \mathcal{O}_L[x]$ with $F\nu(x) = x^l$ such that $\overline{h}(x) = \overline{F}(x)\overline{G}(x)$ in $(\mathcal{O}_L/\mathfrak{p})[x]$. Take a monic polynomial $f \in \mathcal{O}_L[x]$ such that its reduction in $(\mathcal{O}_L/\mathfrak{p})[x]$ is irreducible and divides the reduction of F. By Proposition 5.1.9 we achieve that $\mathfrak{q} = (f(\eta), \mathfrak{p})\mathcal{O}[\eta]$ is a prime ideal of $\mathcal{O}_L[\eta]$ lying over $\mathfrak{p}$. Since f divides F modulo $\mathfrak{p}\mathcal{O}_L[x]$ it also divides F modulo $\mathfrak{m}_L\mathcal{O}_L[x]$, so $f\nu(x) = x^r$ for some r, $1 \leq r \leq l$. This means that $f(\eta)\nu = f\nu(\eta\nu) = \vartheta^r \neq 0$ and hence $f(\eta) \notin \mathfrak{n}$. Therefore, $\mathfrak{q} \not\subseteq \mathfrak{n}$.

Suppose towards a contradiction that $\mathcal{O}_F$ is a finitely generated $\mathcal{O}_L$-algebra. From Proposition 5.1.6, $\mathrm{Spec}(\mathcal{O}_L[\eta])$ consists of finitely many chains of prime ideals, say

$$\mathrm{Spec}(\mathcal{O}_L[\eta]) = \bigcup_{j=1}^{r} (\mathfrak{q}_i^j)_{i\in I}.$$

For each chain $(\mathfrak{q}_i^j)_{i\in I}$, if $\mathfrak{q}_i^j \not\subseteq \mathfrak{n}$ for some $i \in I$ we set

$$\mathfrak{q}_j := \bigcap_{\mathfrak{q}_i^j \not\subseteq \mathfrak{n}} \mathfrak{q}_i^j.$$

By Theorem 1.3.6, $\mathfrak{q}_j \not\subseteq \mathfrak{n}$ and in particular, $\mathfrak{q}_j \neq (0)$. Set $\mathfrak{p}_j = \mathfrak{q}_j \cap \mathcal{O}_L$ and let $\mathfrak{p}_{j_0}$ be the intersection of $\mathfrak{p}_j$'s. By the construction of $\mathfrak{p}_{j_0}$, if a prime ideal $\mathfrak{q}$ of $\mathcal{O}_L[\eta]$ is not contained in $\mathfrak{n}$, then $\mathfrak{p}_{j_0} \subseteq \mathfrak{q} \cap \mathcal{O}_L$. Also, $\mathfrak{p}_{j_0}$ is a prime ideal of $\mathcal{O}_L$ lying below the non-zero prime ideal $\mathfrak{q}_{j_0}$ of $\mathcal{O}_L[\eta]$, so $\mathfrak{p}_{j_0} \neq (0)$. Hence, for every prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$ such that $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{p}_{j_0}$ (which exists because of our assumption on the valued group of (L, ν)) and every prime ideal $\mathfrak{q}$ of $\mathcal{O}_L[\eta]$ lying over $\mathfrak{p}$ we have that $\mathfrak{q} \subseteq \mathfrak{n}$. This is a contradiction to conclusion of the previous paragraph.

$\square$

Proof of Theorem 1.3.7. Take any chain $(\mathfrak{q}_i)_{i\in I}$ of prime ideals of $\mathcal{O}_L[\eta]$ such that $\mathfrak{q}_i \not\subseteq \mathfrak{n}$ for every $i \in I$. Observe that I can be seeing as a subset the indexing set J of the prime ideals of $\mathcal{O}_L$. Indeed, for every $i \in I$ the ideal $\mathfrak{p}_i \cap \mathcal{O}_L$ is a prime ideal of $\mathcal{O}_L$. Also, for any two elements $i_1, i_2 \in I$, if $\mathfrak{p}_{i_1} \cap \mathcal{O}_L = \mathfrak{p}_{i_2} \cap \mathcal{O}_L$, then by the incomparability property we have that $\mathfrak{p}_{i_1} = \mathfrak{p}_{i_2}$. Since the prime ideals of $\mathcal{O}_L$ are well-ordered by inclusion I has a minimum i_0. This means that

$$\bigcap_{i \in I} \mathfrak{p}_i = \mathfrak{p}_{i_0} \not\subseteq \mathfrak{n}$$

which proves that Condition **(v)** of Theorem 1.3.6 holds.

□

5.3 Local uniformization

The next proposition is essential for the proof of Theorem 1.3.8.

Proposition 5.3.1. *Let (L, ν) be a valued field and fix an extension of ν to $\widetilde{L}$ (which we call again ν). Let $\eta_1, \ldots, \eta_r \in \widetilde{L}$ be henselian elements over L with henselian polynomials $h_i(x) \in L[x]$. Assume that there exists a regular local ring $R \subseteq L$ dominated by $\mathcal{O}_L$ such that $\operatorname{Quot}(R) = L$ and $h_i(x) \in R[x]$. Then*

$$R[\eta_1, \ldots, \eta_r]_{\mathfrak{m}_\nu \cap R[\eta_1, \ldots, \eta_r]}$$

is also regular.

A ring R is said to be **reduced** if it has no non-zero nilpotent elements.

To prove Proposition 5.3.1 we will need the following Lemma:

Lemma 5.3.2. *Every localization of a zero-dimensional reduced ring R at some prime ideal $\mathfrak{p}$ is a field.*

Proof. Let $\mathfrak{p}R_\mathfrak{p}$ be the maximal ideal of $R_\mathfrak{p}$. It is enough to prove that $\mathfrak{p}R_\mathfrak{p} = 0$. If $\mathfrak{p}R_\mathfrak{p}$ had some non-nilpotent element f, then there would be a prime ideal on $R_\mathfrak{p}$ which did not contain f. Then there would exist a prime ideal $\mathfrak{p}'$ of R such that $(0) \subsetneq \mathfrak{p}' \subsetneq \mathfrak{p}$ which is a contradiction to $\dim(R) = 0$. Hence, every element of $\mathfrak{p}R_\mathfrak{p}$ is nilpotent. On the other hand, since R is reduced and the localization of any reduced ring is reduced, we obtain that $R_\mathfrak{p}$ is reduced. Therefore, $\mathfrak{p}R_\mathfrak{p} = 0$, which is what we wanted to prove. □

Proof of Proposition 5.3.1. Let $R^{(s)} = R[\eta_1, \dots, \eta_s]_{\mathfrak{m}_\nu \cap R[\eta_1,\dots,\eta_s]}$ for $1 \le s \le r$ and $R^{(0)} = R$. Since

$$R^{(s)} = R^{(s-1)}[\eta_s]_{\mathfrak{m}_\nu \cap R^{(s-1)}[\eta_s]} \text{ for each } s, 1 \le s \le r,$$

it is enough to prove our proposition for $r = 1$. For this case we denote $\eta_1 = \eta$, $h_1 = h$ and $R' = R^{(1)} = R[\eta]_{\mathfrak{m}_\nu \cap R[\eta]}$.

We claim that $\dim(R) = \dim(R')$. Indeed, since $R[\eta]$ is integral over R there is a bijection between the chains of prime ideals of R and the chains of prime ideals of $R[\eta]$ contained in $\mathfrak{m}_\nu \cap R[\eta]$. On the other hand, since $R' = R[\eta]_{\mathfrak{m}_\nu \cap R[\eta]}$ there exists a bijection between the chains of prime ideals of R' and the chains of prime ideals of $R[\eta]$ contained in $\mathfrak{m}_\nu \cap R[\eta]$. Therefore, $\dim(R) = \dim(R')$.

Since R is regular, its maximal ideal $\mathfrak{m}$ is generated by $d = \dim(R)$ many elements. We want to prove that also the maximal ideal $\mathfrak{m}'$ of R' is generated by d many elements. To do this, it is sufficient to show that $\mathfrak{m}'$ is generated by the same generators as $\mathfrak{m}$, i.e., that $\mathfrak{m}' = \mathfrak{m}R'$. We have to prove that $\mathfrak{m}R'$ is a maximal ideal, or, equivalently, that $R'/\mathfrak{m}R'$ is a field.

Consider the mapping

$$\begin{array}{rccc} \Psi: & R' & \longrightarrow & \left(R/\mathfrak{m}[x]/(\overline{h}(x))\right)_{\mathfrak{q}} \\ & \dfrac{f}{g} & \longmapsto & \dfrac{\overline{f} + (\overline{h}(x))}{\overline{g} + (\overline{h}(x))} \end{array}$$

where $\mathfrak{q} = \{\overline{g} + (\overline{h}(x)) \mid \nu(g) > 0\}$ is a prime ideal of $(R/\mathfrak{m}[x])/(h(x))$. From the construction of $\mathfrak{q}$ we obtain that Ψ is well-defined and Ψ is surjective because of its construction. Also, for every element $f/g \in \mathfrak{m}R'$ the reduction of f in $R/\mathfrak{m}[x]$ is zero, so $\mathfrak{m}R \subseteq \ker(\Psi)$. We want to prove that $\ker(\Psi) \subseteq \mathfrak{m}R'$. If the element f/g belongs to $\ker(\Psi)$, then $f = a_0 + \ldots + a_n\eta^n \in R[\eta]$ is a polynomial such that all a_i's belong to $\mathfrak{m}$. This means that

$$\frac{f}{g} = a_0 \cdot \frac{1}{g} + \ldots + a_n \cdot \frac{\eta^n}{g} \in \mathfrak{m}R'.$$

Therefore, $R'/\mathfrak{m}R' \cong \left(R/\mathfrak{m}[x]/(\overline{h}(x))\right)_{\mathfrak{q}}$.

Since $\nu(h'(\eta)) = 0$, the polynomial $h(x)$ has no multiple factors modulo $\mathfrak{m}$. Thus $(R/\mathfrak{m}[x])/(\overline{h}(x))$ has no nilpotent elements other than zero, i.e., it is a reduced ring. Also,

$(R/\mathfrak{m}[x])\,/(\overline{h}(x))$ is zero-dimensional because it is an integral extension of a field. Applying Lemma 5.3.2, we obtain that $R'/\mathfrak{m}R'$ is a field, which is what we wanted to prove. □

Proof of Theorem 1.3.8. Take any finite set $Z \subseteq \mathcal{O}_F$. By assumption, there exist a transcendence basis T of $F|K$ and henselian elements $\eta_1, \ldots, \eta_r$ over $L = K(T)$ such that $Z \subseteq \mathcal{O}_L[\eta_1, \ldots, \eta_r]$ and $(L|K, \nu)$ admits local uniformization. Let $Z' \subseteq \mathcal{O}_L$ be the finite set given by all coefficients of all h_i's and the coefficients of all $\zeta \in Z$ (as elements of $\mathcal{O}_L[\eta_1, \ldots, \eta_r]$). Since $(L|K, \nu)$ admits local uniformization, there exists a model $V = \operatorname{Spec}(A)$ of $L|K$ such that $\mathcal{O}_{V,\mathfrak{p}} = A_{\mathfrak{m}_L \cap A}$ is regular (where $\mathfrak{p} = \mathfrak{m}_L \cap A$ is the center on V of the restriction of ν to L) and such that $Z' \subseteq \mathcal{O}_{V,\mathfrak{p}}$. Let $A' = A[\eta_1, \ldots, \eta_r]$ and $V' = \operatorname{Spec}(A')$. Then by Proposition 5.3.1,

$$\mathcal{O}_{V',\mathfrak{p}'} = \mathcal{O}_{V,\mathfrak{p}}[\eta_1, \ldots, \eta_r]_{\mathfrak{m}_F \cap \mathcal{O}_{V,\mathfrak{p}}[\eta_1,\ldots,\eta_r]}$$

is regular, where $\mathfrak{p}' = \mathfrak{m}_F \cap A'$ is the center of ν on V'. Also, since each element $\zeta \in Z$ is a polynomial in $\mathcal{O}_L[\eta_1, \ldots, \eta_r]$ with coefficients in $\mathcal{O}_{V,\mathfrak{p}}$ we have that $\zeta \in \mathcal{O}_{V',\mathfrak{p}'}$. Therefore, $(F|K, \nu)$ admits local uniformization. □

Proof of Theorem 1.3.9. Take a finite subset Z of $\mathcal{O}_L$. By assumption, there exists a transcendence basis T of $F|K$ such that $(K(T)|K, \nu)$ admits local uniformization and $F \subseteq K(T)^i$. By Theorem 1.3.3 and 1.3.4 there exists henselian elements $\eta, r, s \in \mathcal{O}_F$ such that $Z \subseteq \mathcal{O}_{K(T)}[\eta, r, s]$. Hence, we can apply Theorem 1.3.8 to conclude that $(F|K, \nu)$ admits local uniformization. □

Chapter 6

Reduction of local uniformization to the rank one case

We start the chapter by presenting some definitions and results which we will need to prove our theorems on the reduction of local uniformization to the rank one case.

6.1 Basic facts

Definition 6.1.1. Let R be a domain and let $\mathfrak{p}$ be a prime ideal of R. Then the field

$$\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$$

is called **the residue field** of $\mathfrak{p}$.

Lemma 6.1.2. *Let $\nu = \nu_1 \circ \nu_2$ be a valuation on $F = \mathrm{Quot}(R)$ centered at the local domain $(R, \mathfrak{m})$. If $\mathfrak{p} \subseteq \mathfrak{m}$ is the center of ν_1 on R, then $\kappa(\mathfrak{p})$ embeds naturally in $F\nu_1$ and the restriction of ν_2 to $\kappa(\mathfrak{p})$ is centered at $(R/\mathfrak{p}, \mathfrak{m}/\mathfrak{p})$.*

Proof. It is easy to see that $R_{\mathfrak{p}} \subseteq \mathcal{O}_{\nu_1}$ and that $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{m}_{\nu_1} \cap R_{\mathfrak{p}}$. Therefore,

$$\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow \mathcal{O}_{\nu_1}/\mathfrak{m}_{\nu_1} = F\nu_1.$$

On the other hand, $\kappa(\mathfrak{p}) = \mathrm{Quot}(R/\mathfrak{p})$ so it remains to show that $\mathfrak{m}_{\nu_2} \cap R/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$. Take an element $a \in R$. If $a \in \mathfrak{p}$, then $a + \mathfrak{p} = 0 + \mathfrak{p} \in \mathfrak{m}/\mathfrak{p} \cap (\mathfrak{m}_{\nu_2} \cap R/\mathfrak{p})$, so we assume that $a \notin \mathfrak{p}$. In this case, $\nu_2(a + \mathfrak{p}) = \nu(a)$ and we have

$$a + \mathfrak{p} \in \mathfrak{m}/\mathfrak{p} \Longleftrightarrow a \in \mathfrak{m} \Longleftrightarrow \nu(a) > 0 \Longleftrightarrow \nu_2(a + \mathfrak{p}) > 0 \Longleftrightarrow a + \mathfrak{p} \in \mathfrak{m}_{\nu_2}.$$

□

Remark 6.1.3. Let $\mathfrak{p}_\Delta = \mathfrak{m}_{\nu_1} \cap \mathcal{O}_\nu$. Applying the above Lemma to the local domain $R = \mathcal{O}_\nu$, we see that there is a natural surjective homomorphism $\Phi_\Delta : \mathcal{O}_\nu \to \mathcal{O}_{\nu_2}$, whose kernel is $\mathfrak{p}_\Delta$.

Lemma 6.1.4. **(1)** *Let*

$$R \hookrightarrow R' \tag{6.1}$$

be an injective homomorphism of domains, having a common quotient field F*. Let* ν *be a valuation on* F *such that* $R' \subseteq \mathcal{O}_\nu$*. Let* $\mathfrak{m}$ *denote the center of* ν *on* R*, and* $\mathfrak{m}'$ *the center of* ν *in* R'*. Then*

$$R_{\mathfrak{m}} \subseteq R'_{\mathfrak{m}'} \tag{6.2}$$

(viewed as subrings of F*).*

(2) *Let* $S \subseteq R$ *be a multiplicative subset, such that*

$$S \cap \mathfrak{m} = \emptyset. \tag{6.3}$$

Assume that $R' \subseteq R_S$*. Then the inclusion (6.2) is, in fact, an equality.*

Proof. **(1)** The inclusion (6.1) induces the inclusion

$$R \setminus \mathfrak{m} \subseteq R' \setminus \mathfrak{m}'. \tag{6.4}$$

The desired inclusion (6.2) follows immediately from (6.1) and (6.4).

(2) The assumption (6.3) implies that $R' \subseteq R_S \subseteq R_{\mathfrak{m}}$. Then

$$R'_{\mathfrak{m}'} \subseteq R_{\mathfrak{m}} \tag{6.5}$$

by the first part of the lemma, and the result follows.

□

Lemma 6.1.5. *Every local blowing up can be decomposed as a finite sequence of simple local blowing ups, i.e., given a local blowing up*

$$\pi : R \longrightarrow R\left[\frac{a_1}{b_1}, \dots, \frac{a_r}{b_r}\right]_{\mathfrak{m}'}$$

we can find a finite sequence of simple local blowing ups

$$(R,\mathfrak{m}) \longrightarrow \left(R^{(1)},\mathfrak{m}^{(1)}\right) \longrightarrow \cdots \longrightarrow \left(R^{(r)},\mathfrak{m}^{(r)}\right)$$

such that $R^{(r)} = R\left[\frac{a_1}{b_1},\dots,\frac{a_r}{b_r}\right]_{\mathfrak{m}'}$ *and* π *is the composition of the simple local blowing ups* $\pi_i : R^{(i-1)} \longrightarrow R^{(i)}$ *(where we set* $R^{(0)} := R$*).*

Proof. Define the domains

$$R'^{(k)} = R\left[\frac{a_1}{b_1},\dots,\frac{a_k}{b_k}\right]_{\mathfrak{m}_\nu \cap R\left[\frac{a_1}{b_1},\dots,\frac{a_k}{b_k}\right]}, \qquad 1 \le k \le r,$$

and let us define $R^{(k)}$ inductively by setting $R^{(0)} = R$ and

$$R^{(k)} = R^{(k-1)}\left[\frac{a_k}{b_k}\right]_{\mathfrak{m}_\nu \cap R^{(k-1)}\left[\frac{a_k}{b_k}\right]}, \qquad 1 \le k \le r.$$

The inclusions $\pi_k : R^{(k-1)} \longrightarrow R^{(k)}$ are all simple local blowing ups, so we just have to prove that $R^{(r)} = R'^{(r)}$ and we will have $\pi = \pi_r \circ \ldots \circ \pi_1$ because all the π_k are inclusions. We will prove by induction that $R^{(k)} = R'^{(k)}$ for all $k = 1,\dots,r$ and we will be done.

By definition, $R^{(1)} = R'^{(1)}$ so assume that $k > 1$ and that

$$R^{(k-1)} = R'^{(k-1)}. \tag{6.6}$$

Let us prove that $R^{(k)} = R'^{(k)}$. The inclusion $R'^{(k)} \subseteq R^{(k)}$ is trivial so it remains to prove that

$$R^{(k)} \subseteq R'^{(k)}. \tag{6.7}$$

To prove (6.7), first note that $R'^{(k-1)} \subseteq R'^{(k)}$, hence $R^{(k-1)} \subseteq R'^{(k)}$ by (6.6). We have $\frac{a_k}{b_k} \in R'^{(k)}$ by definition, so

$$R^{(k-1)}\left[\frac{a_k}{b_k}\right] \subseteq R'^{(k)}. \tag{6.8}$$

Now (6.7) is given by Lemma 6.1.4 (1). This completes the proof of the lemma. □

Lemma 6.1.6. *Take a domain* R *and a valuation* ν *centered at* R*. For every sequence of local blowing ups*

$$R \longrightarrow R^{(1)} \longrightarrow \ldots \longrightarrow R^{(r)}$$

with respect to ν there exists a local blowing up

$$R \longrightarrow \overline{R}^{(1)}$$

with respect to ν such that $\overline{R}^{(1)} = R^{(r)}$.

Proof. We are going to prove our lemma by induction on r. If $r = 1$, then the statement is trivial. Assume that $r > 1$ an let us prove that if there exists a local blowing up $R \longrightarrow \overline{R}'^{(1)}$ such that $\overline{R}'^{(1)} = R^{(r-1)}$, then there exists a local blowing up $R \longrightarrow \overline{R}^{(1)}$ such that $\overline{R}^{(1)} = R^{(r)}$.

From the definition of local blowing ups we know that there exists $a_1, \ldots, a_s, b_1, \ldots, b_s \in R^{(r-1)}$ such that

$$R^{(r)} = R^{(r-1)} \left[\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right]_{\mathfrak{m}_\nu \cap R^{(r-1)}\left[\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right]}. \tag{6.9}$$

Since $\operatorname{Quot}\left(R^{(r-1)}\right) = \operatorname{Quot}(R)$, for each i, $1 \leq i \leq s$, there exist $a_i', b_i' \in R$ such that $\frac{a_i}{b_i} = \frac{a_i'}{b_i'}$. Hence, we can assume that the elements a_i's and b_i's appearing in (6.9) belong to R. Take $c_1, \ldots, c_{s'}, d_1, \ldots, d_{s'} \in R$ such that

$$R^{(r-1)} = \overline{R}'^{(1)} = R\left[\frac{c_1}{d_1}, \ldots, \frac{c_{s'}}{d_{s'}}\right]_{\mathfrak{m}_\nu \cap R\left[\frac{c_1}{d_1}, \ldots, \frac{c_{s'}}{d_{s'}}\right]}.$$

Then

$$\begin{aligned} R^{(r)} &= \left(\left(R\left[\frac{c_1}{d_1}, \ldots, \frac{c_{s'}}{d_{s'}}\right]_{\mathfrak{m}_\nu \cap R\left[\frac{c_1}{d_1}, \ldots, \frac{c_{s'}}{d_{s'}}\right]}\right)\left[\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right]\right)_{\mathfrak{m}_\nu \cap R^{(r-1)}\left[\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right]} \\ &= R\left[\frac{c_1}{d_1}, \ldots, \frac{c_{s'}}{d_{s'}}, \frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right]_{\mathfrak{m}_\nu \cap R\left[\frac{c_1}{d_1}, \ldots, \frac{c_{s'}}{d_{s'}}, \frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\right]} \end{aligned}$$

where the second equality holds because of Lemma 6.1.4. □

Remark 6.1.7. In view of the lemma above, in order to achieve local uniformization for a valuation ν centered at R, it is enough to find a sequence of local blowing ups

$$R \longrightarrow R^{(1)} \longrightarrow \ldots \longrightarrow R^{(n)}$$

such that $R^{(n)}$ is regular. The equivalent statement is true for weak embedded, embedded and inseparable local uniformization.

Lemma 6.1.8. *Let $(R, \mathfrak{m})$ be a noetherian local domain and ν a valuation on $F = \mathrm{Quot}(R)$ which is centered at $(R, \mathfrak{m})$. Take an ideal $\mathcal{I}$ of R and elements $u_1, \ldots, u_d \in \mathcal{I}$ which generate $\mathcal{I}$. Assume that $\nu(u_{i_1}) = \nu(u_{i_2}) \leq \nu(u_i)$ for $1 \leq i \leq d$ and let*

$$R' = R\left[\frac{u_1}{u_{i_1}}, \ldots, \frac{u_d}{u_{i_1}}\right] \text{ and } R'' = R\left[\frac{u_1}{u_{i_2}}, \ldots, \frac{u_d}{u_{i_2}}\right].$$

Also, consider the prime ideals $\mathfrak{m}' = \mathfrak{m}_\nu \cap R' \subseteq R'$ and $\mathfrak{m}'' = \mathfrak{m}_\nu \cap R'' \subseteq R''$. Then

$$R'_{\mathfrak{m}'} = R''_{\mathfrak{m}''}.$$

Proof. We may assume, without loss of generality, that $i_1 = 1$ and $i_2 = 2$. Since

$$\nu\left(\frac{u_2}{u_1}\right) = \nu(u_1) - \nu(u_2) = 0$$

we have $\frac{u_2}{u_1} \in R''_{\mathfrak{m}''}$. Hence, for every $i \in \{1, \ldots, d\}$ we have $\frac{u_i}{u_1} = \frac{u_i}{u_2}\frac{u_2}{u_1} \in R''_{\mathfrak{m}''}$. Then

$$R' \subseteq R''_{\mathfrak{m}''}. \tag{6.10}$$

From (6.10) and Lemma 6.1.4 (1) we obtain

$$R'_{\mathfrak{m}'} \subseteq R''_{\mathfrak{m}''}.$$

The opposite inclusion is analogous. □

Take now an ideal $\mathcal{I}$ of R and let $u_0 \in \mathcal{I}$ be an element such that $\nu(u_0) \leq \nu(\alpha)$ for all $\alpha \in \mathcal{I}$. Complete u_0 to sets $\{u_0, u_1, \ldots, u_q\}$ and $\{u_0, u'_1, \ldots, u'_{q'}\}$ of generators of $\mathcal{I}$. It is easy to see that

$$R\left[\frac{u_1}{u_0}, \ldots, \frac{u_q}{u_0}\right] = R\left[\frac{u'_1}{u_0}, \ldots, \frac{u'_{q'}}{u_0}\right] =: R'.$$

This, together with Lemma 6.1.8 above, guarantees that given an ideal $\mathcal{I}$ of R the local blowing up

$$R \longrightarrow R'_{\mathfrak{m}_\nu \cap R'}$$

is uniquely determined by $\mathcal{I}$ and is independent of the particular set of generators $\{u_0, u_1, \ldots, u_q\}$.

Definition 6.1.9. The local blowing up described above is said to be the local blowing up of $(R, \mathfrak{m})$ **with respect to ν along $\mathcal{I}$.**

We will now prove a few Lemmas which will be essential in the proofs of our main results. From here until the end of this section we will assume that $(R, \mathfrak{m})$ is a noetherian local domain and ν a valuation on $F = \mathrm{Quot}(R)$ centered at $(R, \mathfrak{m})$. Also, assume that ν can be decomposed as $\nu = \nu_1 \circ \nu_2$ and write $\mathfrak{p} = \mathfrak{m}_{\nu_1} \cap R$ for the center of ν_1 on R.

Lemma 6.1.10. *Let*

$$\widetilde{\pi} : R_{\mathfrak{p}} \longrightarrow \widetilde{R} = R_{\mathfrak{p}} \left[\frac{a}{b}\right]_{\widetilde{\mathfrak{p}}}$$

be a simple local blowing up with respect to ν_1*, where* $\widetilde{\mathfrak{p}} = \mathfrak{m}_{\nu_1} \cap R_{\mathfrak{p}} \left[\frac{a}{b}\right]$*, and assume that* $\nu(a) \geq \nu(b)$*. Consider the sequence of local blowing ups*

$$R \longrightarrow R^{(1)} = R\left[a, b\right]_{\mathfrak{m}'} \longrightarrow R^{(2)} = R^{(1)} \left[\frac{a}{b}\right]_{\mathfrak{m}''}$$

with respect to ν*, where* $\mathfrak{m}' = \mathfrak{m}_{\nu} \cap R\left[a, b\right]$ *and* $\mathfrak{m}'' = \mathfrak{m}_{\nu} \cap R^{(1)} \left[\frac{a}{b}\right]$*. If* $\mathfrak{p}^{(2)}$ *is the center of* ν_1 *in* $R^{(2)}$*, then* $R^{(2)}_{\mathfrak{p}^{(2)}} = \widetilde{R}$*.*

Proof. Let $\mathfrak{p}^{(1)}$, $\mathfrak{p}_0^{(1)}$ denote the centers of ν_1 in $R^{(1)}$ and in $R[a, b]$, respectively. We have $a, b \in R_{\mathfrak{p}}$ by definition, so $R\left[a, b\right] \subseteq R_{\mathfrak{p}}$ and

$$R[a, b]_{\mathfrak{p}_0^{(1)}} \subseteq R_{\mathfrak{p}} \tag{6.11}$$

by Lemma 6.1.4 (1). Since $\mathfrak{p}_0^{(1)} \subseteq \mathfrak{m}'$, we have

$$R^{(1)}_{\mathfrak{p}^{(1)}} = R[a, b]_{\mathfrak{p}_0^{(1)}} \tag{6.12}$$

by Lemma 6.1.4 (2). Combining (6.11) and (6.12), we obtain

$$R^{(1)}_{\mathfrak{p}^{(1)}} \subseteq R_{\mathfrak{p}}. \tag{6.13}$$

Now, from the natural inclusion $R \subseteq R^{(2)}$ we have $R_{\mathfrak{p}} \subseteq R^{(2)}_{\mathfrak{p}^{(2)}}$ by Lemma 6.1.4 (1). Since $\frac{a}{b} \in R^{(2)}_{\mathfrak{p}^{(2)}}$ by definition, we obtain $R_{\mathfrak{p}} \left[\frac{a}{b}\right] \subseteq R^{(2)}_{\mathfrak{p}^{(2)}}$, so

$$\widetilde{R} \subseteq R^{(2)}_{\mathfrak{p}^{(2)}} \tag{6.14}$$

by Lemma 6.1.4 (1). For the opposite inclusion, Let $\mathfrak{p}_0^{(2)}$ denote the center of ν_1 in $R^{(1)} \left[\frac{a}{b}\right]$. Since $\mathfrak{p}_0^{(2)} \subseteq \mathfrak{m}''$, we have

$$R^{(1)} \left[\frac{a}{b}\right]_{\mathfrak{p}_0^{(2)}} = R^{(2)}_{\mathfrak{p}^{(2)}} \tag{6.15}$$

by Lemma 6.1.4 (2). We have $R^{(1)} \subseteq R_{\mathfrak{p}} \subseteq \widetilde{R}$ by (6.13) and $\frac{a}{b} \in \widetilde{R}$ by definition, so

$$R^{(1)}\left[\frac{a}{b}\right] \subseteq \widetilde{R}.$$

Hence,

$$R^{(1)}\left[\frac{a}{b}\right]_{\mathfrak{p}_0^{(2)}} \subseteq \widetilde{R} \tag{6.16}$$

by Lemma 6.1.4 (1). Combining (6.16) with (6.15), we obtain

$$R^{(2)}_{\mathfrak{p}^{(2)}} \subseteq \widetilde{R}. \tag{6.17}$$

This completes the proof. □

If in the lemma above we had $\nu(a) < \nu(b)$, then we would have $\nu_1(a) = \nu_1(b)$. Indeed, since $\nu(a) < \nu(b)$ we obtain

$$\frac{b}{a} \in \mathfrak{m}_\nu \subseteq \mathcal{O}_\nu \subseteq \mathcal{O}_{\nu_1}$$

which guarantees that $\nu_1(b) \leq \nu_1(a)$. By definition of a local blowing up we have that $\nu_1(a) \leq \nu_1(b)$, so $\nu_1(a) = \nu_1(b)$. From Lemma 6.1.8 we conclude that $\widetilde{R} = R\left[\frac{b}{a}\right]_{\widetilde{\mathfrak{p}'}}$ where $\widetilde{\mathfrak{p}'} = \mathfrak{m}_{\nu_1} \cap R\left[\frac{b}{a}\right]$. Consider now the sequence of local blowing ups

$$R \longrightarrow R^{(1)} = R[a,b]_{\mathfrak{m}'} \longrightarrow R^{(1)}\left[\frac{b}{a}\right]_{\mathfrak{m}''}$$

with respect to ν, where $\mathfrak{m}' = \mathfrak{m}_\nu \cap R[a,b]$ and $\mathfrak{m}'' = \mathfrak{m}_\nu \cap R^{(1)}\left[\frac{b}{a}\right]$ and let $\mathfrak{p}^{(2)} = \mathfrak{m}_{\nu_1} \cap R^{(2)}$. By the previous Lemma we conclude again that $\widetilde{R} = R^{(2)}_{\mathfrak{p}^{(2)}}$. We have then proved the following Corollary.

Corollary 6.1.11. *For every simple local blowing up of* $R_{\mathfrak{p}}$

$$R_{\mathfrak{p}} \longrightarrow \widetilde{R}$$

with respect to ν_1 *there exists a sequence of local blowing ups of* R

$$R \longrightarrow R^{(1)} \longrightarrow R^{(2)}$$

with respect to ν *such that* $\widetilde{R} = R^{(2)}_{\mathfrak{p}^{(2)}}$*, where* $\mathfrak{p}^{(2)}$ *is the center of* ν_1 *in* $R^{(2)}$.

Corollary 6.1.12. *Let*

$$R_{\mathfrak{p}} \longrightarrow \widetilde{R}^{(1)} \longrightarrow \cdots \longrightarrow \widetilde{R}^{(r)} \tag{6.18}$$

be a sequence of local blowing ups with respect to ν_1*. Then there exists a sequence of local blowing ups*

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν *such that* $R^{(n)}_{\mathfrak{p}^{(n)}} = \widetilde{R}^{(r)}$*, where* $\mathfrak{p}^{(n)}$ *is the center* $\mathfrak{m}_{\nu_1} \cap R^{(n)}$ *of* ν_1 *in* $R^{(n)}$*. In particular, if* $\widetilde{R}^{(r)}$ *is regular, then* $R^{(n)}_{\mathfrak{p}^{(n)}}$ *is regular.*

Proof. From Lemma 6.1.5 we may assume that every local blowing up in the sequence (6.18) is simple. Applying Corollary 6.1.11 to each of these simple local blowing ups and using induction on r, we achieve the desired sequence. □

We will now prove some facts about ν_2. Let

$$\phi : R/\mathfrak{p} \longrightarrow \overline{R}$$

be an isomorphism of local domains and denote $\phi(a + \mathfrak{p})$ by $\overline{a}$. Let $\overline{\nu}_2 = \nu_2 \circ \phi^{-1}$ and take elements $a, b \in R \setminus \mathfrak{p}$ such that $\nu(a) = \overline{\nu}_2(\overline{a}) \geq \overline{\nu}_2(\overline{b}) = \nu(b)$. Consider the domains

$$R' = R\left[\frac{a}{b}\right] \text{ and } \overline{R}' = \overline{R}\left[\frac{\overline{a}}{\overline{b}}\right]$$

and the ideals $\mathfrak{m}' = \mathfrak{m}_\nu \cap R'$ and $\overline{\mathfrak{m}}' = \mathfrak{m}_{\overline{\nu}_2} \cap \overline{R}'$. Let

$$R^{(1)} = R'_{\mathfrak{m}'} \text{ and } \overline{R}^{(1)} = \overline{R}'_{\overline{\mathfrak{m}}'}$$

and $\mathfrak{p}^{(1)} = \mathfrak{m}_{\nu_1} \cap R'_{\mathfrak{m}'}$.

Lemma 6.1.13. *In the above situation we have* $\overline{R}^{(1)} \cong R^{(1)}/\mathfrak{p}^{(1)}$ *and* $R_{\mathfrak{p}} = R^{(1)}_{\mathfrak{p}^{(1)}}$*.*

Proof. To prove that $R_{\mathfrak{p}} = R^{(1)}_{\mathfrak{p}^{(1)}}$ it is enough to prove that

$$R_{\mathfrak{p}} \supseteq R^{(1)}_{\mathfrak{p}^{(1)}} \tag{6.19}$$

because $R_{\mathfrak{p}} \subseteq R^{(1)}_{\mathfrak{p}^{(1)}}$ is trivial. To prove (6.19), note that $b \notin \mathfrak{p}$ by definition, hence $\dfrac{a}{b} \in R_{\mathfrak{p}}$, so $R\left[\dfrac{a}{b}\right] \subseteq R_{\mathfrak{p}}$. Now the inclusion (6.19) follows from Lemma 6.1.4 (1).

To prove the first statement of the lemma, first note that we have a natural surjective homomorphism $R \to \overline{R}$. We extend it to a surjective homomorphism $\Phi : R' \to \overline{R}'$ by sending $\dfrac{a}{b}$ to $\dfrac{\overline{a}}{\overline{b}}$. We have $\Phi(\mathfrak{m}') = \overline{\mathfrak{m}}'$ and $\Phi(R' \setminus \mathfrak{m}') = \overline{R}' \setminus \overline{\mathfrak{m}}'$, hence Φ extends to a surjective homomorphism $R^{(1)} \longrightarrow \overline{R}^{(1)}$ of localizations. By abuse of notation, we denote this new homomorphism also by Φ.

It remains to show that $\ker(\Phi) = \mathfrak{p}^{(1)}$. By definitions, we have injective local homomorphisms $R^{(1)} \hookrightarrow \mathcal{O}_\nu$ and $\overline{R}^{(1)} \hookrightarrow \mathcal{O}_{\nu_2}$, and the homomorphism Φ is nothing but the restriction to $R^{(1)}$ of the homomorphism Φ_Δ of Remark 6.1.3. Hence,

$$\ker(\Phi) = \ker(\Phi_\Delta) \cap R^{(1)} = \mathfrak{p}_\Delta \cap R^{(1)} = (\mathfrak{m}_{\nu_1} \cap \mathcal{O}_\nu) \cap R^{(1)} = \mathfrak{m}_{\nu_1} \cap R^{(1)} = \mathfrak{p}^{(1)},$$

as desired. □

Definition 6.1.14. The simple local blowing up

$$\pi : R \longrightarrow R\left[\frac{a}{b}\right]_{\mathfrak{m}'}$$

constructed in the lemma above is called a **lifting** of the simple local blowing up $\overline{\pi}$ from $R/\mathfrak{p}$ to R.

Corollary 6.1.15. *Take a sequence of local blowing ups*

$$R/\mathfrak{p} \longrightarrow \overline{R}^{(1)} \longrightarrow \cdots \longrightarrow \overline{R}^{(r)}$$

with respect to ν_2. Then there exists a sequence of local blowing ups

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}_{\mathfrak{p}^{(n)}} = R_\mathfrak{p}$ and $R^{(n)}/\mathfrak{p}^{(n)} \cong \overline{R}^{(r)}$, where $\mathfrak{p}^{(n)} = \mathfrak{m}_{\nu_1} \cap R^{(n)}$. In particular, if $R_\mathfrak{p}$ and $\overline{R}^{(r)}$ are regular, then so are $R^{(n)}_{\mathfrak{p}^{(n)}}$ and $R^{(n)}/\mathfrak{p}^{(n)}$.

Proof. Since every local blowing up can be decomposed as a finite sequence of simple local blowing ups (see Lemma 6.1.5), we may assume that all local blowing ups in the sequence

$$\overline{R} \longrightarrow \overline{R}^{(1)} \longrightarrow \cdots \longrightarrow \overline{R}^{(r)}$$

are simple. We will prove by induction on k, $1 \leq k \leq r$ that we can lift the simple local blowing up

$$\overline{\pi}_k : \overline{R}^{(k-1)} \longrightarrow \overline{R}^{(k)}$$

$\left(\overline{R}^{(0)} := R/\mathfrak{p}\right)$ to a simple local blowing up

$$\pi_k : R^{(k-1)} \longrightarrow R^{(k)}$$

$\left(R^{(0)} := R\right)$ with respect to ν such that $R^{(k)}_{\mathfrak{p}^{(k)}} = R_{\mathfrak{p}}$ and $\overline{R}^{(k)} \cong R^{(k)}/\mathfrak{p}^{(k)}$. For $k = 1$ we apply Lemma 6.1.10 with $R = R$ and $\overline{R} = R/\mathfrak{p}$. Suppose now that $k > 1$ and that $R^{(k-1)}_{\mathfrak{p}^{(k-1)}} = R_{\mathfrak{p}}$ and $\overline{R}^{(k-1)} \cong R^{(k-1)}/\mathfrak{p}^{(k-1)}$. Applying Lemma 6.1.10 to $R = R^{(k-1)}$ and $\overline{R} = \overline{R}^{(k-1)}$, we obtain $R^{(k)}_{\mathfrak{p}^{(k)}} = R^{(k-1)}_{\mathfrak{p}^{(k-1)}} = R_{\mathfrak{p}}$ and $\overline{R}^{(k)} \cong R^{(k)}/\mathfrak{p}^{(k)}$. Therefore, $R^{(r)}_{\mathfrak{p}^{(r)}} = R_{\mathfrak{p}}$ and $R^{(r)}/\mathfrak{p}^{(r)} \cong \overline{R}^{(r)}$, as desired. □

We will now assume that both $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ are regular and will study the effects of blowing up R with respect to ν.

Lemma 6.1.16. *Let R be a domain and $\mathfrak{p}$ a prime ideal of R such that $R_{\mathfrak{p}}$ is regular. Then there exist $y_1, \ldots, y_r \in \mathfrak{p}$ that form a regular system of parameters for $R_{\mathfrak{p}}$.*

Proof. Since $R_{\mathfrak{p}}$ is regular, there exist $\widetilde{y}_1, \ldots, \widetilde{y}_r \in \mathfrak{p}R_{\mathfrak{p}}$ which form a regular system of parameters for $R_{\mathfrak{p}}$. By definition of $\mathfrak{p}R_{\mathfrak{p}}$, there exist $\beta_i \notin \mathfrak{p}$ and $y_i \in \mathfrak{p}$ such that

$$\widetilde{y}_i = \frac{y_i}{\beta_i}, \qquad 1 \leq i \leq r.$$

Then β_i is a unit in $R_{\mathfrak{p}}$ and therefore $(\widetilde{y}_1, \ldots, \widetilde{y}_r)\, R_{\mathfrak{p}} = (y_1, \ldots, y_r)R_{\mathfrak{p}}$. □

Lemma 6.1.17. *Assume that $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ are regular and take $y_1, \ldots, y_r \in \mathfrak{p}$ which form a regular system of parameters for $R_{\mathfrak{p}}$ and $x_1, \ldots, x_t \in \mathfrak{m} \setminus \mathfrak{p}$ such that $(x_1 + \mathfrak{p}, \ldots, x_t + \mathfrak{p})$ form a regular system of parameters for $R/\mathfrak{p}$. Take $a \in \mathfrak{m} \setminus \mathfrak{p}$, $y_{r+1}, \ldots, y_{r+s} \in \mathfrak{p}$ and let*

$$\pi : R \longrightarrow R^{(1)}$$

be the local blowing up of R with respect to ν along an ideal of the form $(a, y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s'})$ for some s', $0 \leq s' \leq s$. Let

$$y_1^{(1)} = \frac{y_1}{a}, \ldots, y_{r+s'}^{(1)} = \frac{y_{r+s'}}{a}$$

and $\mathfrak{p}^{(1)} = \mathfrak{m}_{\nu_1} \cap R^{(1)}$. Then we have:

(i) *If* $\mathfrak{p} = (y_1, \ldots, y_{r+s})R$*, then* $\mathfrak{p}^{(1)} = \left(y_1^{(1)}, \ldots, y_{r+s'}^{(1)}, y_{r+s'+1}, \ldots, y_{r+s}\right) R^{(1)}$.

(ii) $R^{(1)}/\mathfrak{p}^{(1)} \cong R/\mathfrak{p}$ *(hence it is regular) and* $\left(x_1 + \mathfrak{p}^{(1)}, \ldots, x_t + \mathfrak{p}^{(1)}\right)$ *is a regular system of parameters for* $R^{(1)}/\mathfrak{p}^{(1)}$.

(iii) $R^{(1)}_{\mathfrak{p}^{(1)}} = R_{\mathfrak{p}}$ *(hence it is regular) and* $\left(y_1^{(1)}, \ldots, y_r^{(1)}\right)$ *is a regular system of parameters for* $R^{(1)}_{\mathfrak{p}^{(1)}}$.

Proof. **(i)** Since $y_i \in \mathfrak{p}$ for $1 \leq i \leq r+s'$ and $a \in \mathfrak{m} \setminus \mathfrak{p}$ we have that $\nu_1(y_i) > \nu_1(a)$, hence $y_i^{(1)} \in \mathfrak{p}^{(1)}$ for every i, $1 \leq i \leq r+s'$. Since $y_i \in \mathfrak{p}$ for $r+s' < i \leq r+s$, we obtain that $\left(y_1^{(1)}, \ldots, y_{r+s'}^{(1)}, y_{r+s'+1}, \ldots, y_{r+s}\right) R^{(1)} \subseteq \mathfrak{p}^{(1)}$.

Take an element $r \in \mathfrak{p}^{(1)}$. Then by the definition of local blowing ups,

$$r = \frac{f}{g} \text{ with } \nu(g) = 0, \text{ for some } f, g \in R\left[y_1^{(1)}, \ldots, y_{r+s'}^{(1)}\right].$$

This means that g is a unit in $R^{(1)}$ and consequently $1/g \in R^{(1)}$. Thus,

$$r = \frac{1}{g}\left(f(0) + y_1^{(1)}\overline{f}_1 + \ldots + y_{r+s'}^{(1)}\overline{f}_{r+s'}\right)$$

for some $\overline{f_i} \in R[y_1^{(1)}, \ldots, y_{r+s'}^{(1)}] \subseteq R^{(1)}$. Since $\nu_1(f) > 0$ and $\nu_1(y_i^{(1)}\overline{f}_i) > 0$ for each i, $1 \leq i \leq r+s'$, so $\nu_1(f(0)) > 0$. From this and the fact that $f(0) \in R$ we conclude that $f(0) \in (y_1, \ldots, y_{r+s})R^{(1)}$, which concludes our proof.

(ii) We want to prove first that $R^{(1)}/\mathfrak{p}^{(1)} \cong R/\mathfrak{p}$. Let $i : R \longrightarrow R^{(1)}$ be the natural inclusion and $\pi : R^{(1)} \longrightarrow R^{(1)}/\mathfrak{p}^{(1)}$ be the canonical epimorphism. Consider the mapping $\phi = \pi \circ i$. We have to prove that ϕ is surjective and that $\ker\phi = \mathfrak{p}$. Take an element $\frac{p}{q} \in R^{(1)}$, so $p, q \in R\left[\frac{y_1}{a}, \ldots, \frac{y_{r+s'}}{a}\right]$ and $\nu(q) = 0$. Write

$$p = p_0 + p_1 \text{ and } q = q_0 + q_1$$

where $p_0, q_0 \in R$ and

$$p_1 = \frac{y_1}{a}\overline{p}_1 + \ldots + \frac{y_{r+s'}}{a}\overline{p}_{r+s'} \text{ and } q_1 = \frac{y_1}{a}\overline{q}_1 + \ldots + \frac{y_{r+s'}}{a}\overline{q}_{r+s'}$$

for some $\overline{p}_i, \overline{q}_i \in R\left[\frac{y_1}{a}, \ldots, \frac{y_{r+s'}}{a}\right]$. Since $\nu_1\left(\frac{y_i}{a}\right) = \nu_1(y_i) - \nu_1(a) > 0$ we have $\nu_1(p_1) > 0$ and $\nu_1(q_1) > 0$, in particular $\nu(p_1) > 0$ and $\nu(q_1) > 0$. Since $\nu(q_1) > 0$

and $\nu(q) = 0$ we have

$$\nu(q_0) = \nu(q + q_0 - q) = \nu(q - q_1) = 0$$

and since ν is centered at $(R, \mathfrak{m})$ we have $\dfrac{1}{q_0} \in R$, and consequently $\dfrac{p_0}{q_0} \in R$. Also,

$$\frac{p}{q} = \frac{p_0 + p_1}{q_0 + q_1} = \frac{p_0}{q_0} + \frac{p_1 q_0 - p_0 q_1}{q_0 q} =: \frac{p_0}{q_0} + p^{(1)}.$$

Since $\nu(q) = 0 = \nu(q_0)$ we have $\nu(q_0 q) = 0$ so $p^{(1)} \in R^{(1)}$. On the other hand,

$$\nu_1(p_1 q_0 - p_0 q_1) \geq \min\{\nu_1(p_1 q_0), \nu_1(p_0 q_1)\} > 0$$

and we have $r^{(1)} \in \mathfrak{p}^{(1)}$. Therefore,

$$\frac{p}{q} + \mathfrak{p}^{(1)} = \frac{p_0}{q_0} + \mathfrak{p}^{(1)} = \phi\left(\frac{p_0}{q_0}\right)$$

so ϕ is surjective. The fact that $\ker \phi = \mathfrak{p}$ is trivial because $\mathfrak{p} = R \cap \mathfrak{p}^{(1)}$. Therefore, $R^{(1)}/\mathfrak{p}^{(1)} \cong R/\mathfrak{p}$.

Since $R^{(1)}/\mathfrak{p}^{(1)} \cong R/\mathfrak{p}$, then $R^{(1)}/\mathfrak{p}^{(1)}$ is regular and the images

$$x_1 + \mathfrak{p}^{(1)}, \ldots, x_t + \mathfrak{p}^{(1)} \in R^{(1)}/\mathfrak{p}^{(1)}$$

of $x_1 + \mathfrak{p}, \ldots, x_t + \mathfrak{p} \in R/\mathfrak{p}$ under ϕ form a regular system of parameters for $\mathfrak{m}^{(1)}/\mathfrak{p}^{(1)}$.

(iii) Proving that $R_\mathfrak{p} = R^{(1)}_{\mathfrak{p}^{(1)}}$ is similar to the proof of Lemma 6.1.10. Namely, the inclusion $R_\mathfrak{p} \subseteq R^{(1)}_{\mathfrak{p}^{(1)}}$ is obvious and the opposite inclusion follows from Lemma 6.1.4 (1), since $\dfrac{y_i}{a} \in R_\mathfrak{p}$.

Since $\nu_1(a) = 0$ the element a is a unit in $R^{(1)}_{\mathfrak{p}^{(1)}}$. From this and from the fact that $(y_1, \ldots, y_r)$ is a regular system of parameters for $R_\mathfrak{p}$ we conclude that $\left(y_1^{(1)}, \ldots, y_r^{(1)}\right) = \left(\dfrac{y_1}{a}, \ldots, \dfrac{y_r}{a}\right)$ is a regular system of parameters for $R^{(1)}_{\mathfrak{p}^{(1)}}$.

$\square$

6.2 The local uniformization problem

We defined in Section 1.3 what it means for a valued function field $(F|K, \nu)$ to admit local uniformization. In Section 1.4 we introduced the concept of local uniformization for a valuation ν centered at a local domain R. The main goal of this section is to show the relation between these two concepts.

Theorem 6.2.1. *Take a valued function field $(F|K,\nu)$ such that ν is trivial on K. Then $(F|K,\nu)$ admits local uniformization if and only if for every local domain R essentially of finite type over K with $F = \mathrm{Quot}(R)$ such that ν is centered at R, the pair (R,ν) admits local uniformization.*

Proof. Assume that $(F|K,\nu)$ admits local uniformization. Take a local domain R essentially of finite type over K with $F = \mathrm{Quot}(R)$ such that ν is centered at R. This means that there exist $a_1,\ldots,a_s \in \mathcal{O}_\nu$ such that

$$R = K[\underline{a}]_{\mathfrak{m}_\nu \cap K[\underline{a}]},$$

where $K[\underline{a}] := K[a_1,\ldots,a_s]$. Set $Z := \{a_1,\ldots,a_s\}$. Since $(F|K,\nu)$ admits local uniformization, there exists an affine model $V = \mathrm{Spec}(K[\underline{b}])$ of $F|K$ such that

$$Z \subseteq \mathcal{O}_{V,\mathfrak{p}} = K[\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{b}]}$$

and $\mathcal{O}_{V,\mathfrak{p}}$ is regular, where $K[\underline{b}] := K[b_1,\ldots,b_q]$ and $b_1,\ldots,b_q \in \mathcal{O}_\nu$. Consider the mapping

$$\pi : R \longrightarrow R^{(1)} := R[\underline{b}]_{\mathfrak{m}_\nu \cap R[\underline{b}]}.$$

Since $F = \mathrm{Quot}(R)$, for each i, $1 \leq i \leq q$, the element b_i is of the form r_i/s_i for some $r_i, s_i \in R$. Since $\nu(b_i) \geq 0$ we have that $\nu(s_i) \leq \nu(r_i)$. Hence, π is a local blowing up with respect to ν. If we prove that $R^{(1)} = \mathcal{O}_{V,\mathfrak{p}}$, then (R,ν) admits local uniformization. We will prove that

$$\mathcal{O}_{V,\mathfrak{p}} = K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]} \text{ and } R^{(1)} = K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]}. \tag{6.20}$$

Since $K[\underline{b}] \subseteq K[\underline{a},\underline{b}]$ we apply Lemma 6.1.4 **(1)** to obtain that

$$\mathcal{O}_{V,\mathfrak{p}} = K[\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{b}]} \subseteq K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]}.$$

On the other hand, since $a_1,\ldots,a_s \in \mathcal{O}_{V,\mathfrak{p}}$ we have that

$$K[\underline{a},\underline{b}] \subseteq \mathcal{O}_{V,\mathfrak{p}} = K[\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{b}]}.$$

Applying part **(2)** of Lemma 6.1.4 we obtain that $\mathcal{O}_{V,\mathfrak{p}} = K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]}$. In view of Lemma 6.1.4 **(1)** and $K[\underline{a},\underline{b}] \subseteq R[\underline{b}]$ we have that $K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]} \subseteq R^{(1)}$. Also, since $R[\underline{b}] \subseteq K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]}$ we can apply again part **(2)** of Lemma 6.1.4 to achieve that $K[\underline{a},\underline{b}]_{\mathfrak{m}_\nu \cap K[\underline{a},\underline{b}]} = R^{(1)}$. Therefore, $R^{(1)} = \mathcal{O}_{V,\mathfrak{p}}$.

For the converse, take a finite set $Z = \{a_1, \ldots, a_s\} \subseteq \mathcal{O}_\nu$ and let $F = K(b_1, \ldots, b_q)$ with $b_i \in \mathcal{O}_\nu$. Consider

$$R = K[a_1, \ldots, a_s, b_1, \ldots, b_q]_{\mathfrak{m}_\nu \cap K[a_1,\ldots,a_s,b_1,\ldots,b_q]}.$$

By assumption, there exists a local blowing up

$$R \longrightarrow R^{(1)}$$

with respect to ν such that $R^{(1)}$ is regular. From the definition of local blowing up we obtain that

$$R^{(1)} = [c_1, \ldots, c_l]_{\mathfrak{m}_\nu \cap R[c_1,\ldots,c_l]},$$

for some $c_1, \ldots, c_l \in \mathcal{O}_\nu$. Consider the affine model

$$V = \operatorname{Spec}\left(K[a_1, \ldots, a_s, b_1, \ldots, b_q, c_1, \ldots, c_l]\right)$$

of $F|K$. An argument similar to the one used to prove (6.20) gives us that $\mathcal{O}_{V,\mathfrak{p}} = R^{(1)}$. Hence, $\mathcal{O}_{V,\mathfrak{p}}$ is regular and $Z = \{a_1, \ldots, a_s\} \subseteq \mathcal{O}_{V,\mathfrak{p}}$, which concludes our proof. □

6.3 Inseparable local uniformization

Lemma 6.3.1. *Let $(R, \mathfrak{m})$ be a local domain of characteristic $p > 0$ and take an element a in some extension of* $\operatorname{Quot}(R)$ *such that $a^{p^r} \in R$. If $a^{p^r} \in \mathfrak{m}$, then $R[a]$ is a local domain (with unique maximal ideal being the ideal generated by $\mathfrak{m}$ and a).*

Proof. Take $f(x) \in R[x]$. It is enough to show that $f(a)$ is a unit of $R[a]$ if and only if $f(0)$ is a unit of R. Indeed, if that is true, then one easily shows that $R[a] \setminus (R[a])^\times$ is an ideal of $R[a]$ which means that $R[a]$ is a local domain.

Take any element $b \in R[a]$. Then $b^{p^r} \in R$ and consequently, if b is a unit of $R[a]$, then b^{p^r} is a unit of R. Assume that $f(a)$ is a unit of $R[a]$ (so $f(a)^{p^r}$ is a unit of R) and suppose that

$f(0)$ is not a unit of R (so $f(0)^{p^r} \in \mathfrak{m}$). Since $f(a) = f(0) + a \cdot g(a)$ for some $g(x) \in R[x]$ and $a^{p^r} \in \mathfrak{m}$, we have that

$$f(a)^{p^r} = f(0)^{p^r} + a^{p^r} \cdot g(a)^{p^r} \in \mathfrak{m}$$

which is a contradiction. Therefore, if $f(a)$ is a unit of $R[a]$, then $f(0)$ is a unit of R.

For the converse, assume that $f(0)$ is a unit of R and write $f(a) = f(0) - b$ ($b = -a \cdot g(a)$). Then

$$f(a) \cdot \left(1 + \frac{b}{f(0)} + \ldots + \frac{b^{p^r-1}}{f(0)^{p^r-1}}\right) = f(0) - \frac{b^{p^r}}{f(0)^{p^r-1}} \in R^{\times},$$

because $(R, \mathfrak{m})$ is a local domain. Therefore, $f(a)$ is a unit of $R[a]$. $\square$

Definition 6.3.2 (Inseparable Local Uniformization). Take a local domain R of positive characteristic p and a valuation ν centered at R. Then the pair (R, ν) is said to admit **inseparable local uniformization** if there exists a purely inseparable extension $F' = F(a_1, \ldots, a_s)$ of $F = \text{Quot}(R)$, with $a_i^{p^r} \in \mathfrak{m}$ for some $r \in \mathbb{N}$, such that the the local domain $(R[a_1, \ldots, a_s], \nu)$ admits local uniformization.

Remark 6.3.3. Observe that $R[a_1, \ldots, a_s]$ is local (by induction and Lemma 6.3.1) and that the unique extension of ν from F to F' (which we denote again by ν) is centered at $R[a_1, \ldots, a_s]$.

6.4 Proofs of the main results

We will now prove the main results of this chapter. In order to prove Theorem 1.4.1, we will need the following Theorem:

Theorem 6.4.1. *Take a valuation ν centered at the local domain R, decompose $\nu = \nu_1 \circ \nu_2$ and let $\mathfrak{p} = \mathfrak{m}_{\nu_1} \cap R$ be the center of ν_1 on R. If $(R_\mathfrak{p}, \nu_1)$ and $(R/\mathfrak{p}, \nu_2)$ admit local uniformization, then also (R, ν) admits local uniformization.*

Proof. Since ν_1 admits local uniformization, there exists a local blowing up

$$R_\mathfrak{p} \longrightarrow \widetilde{R}^{(1)}$$

with respect to ν_1 such that $\widetilde{R}^{(1)}$ is regular. From Corollary 6.1.12 we conclude that there exists a sequence of local blowing ups

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}_{\mathfrak{p}^{(n)}}$ is regular, where $\mathfrak{p}^{(n)}$ is the center $\mathfrak{m}_{\nu_1} \cap R^{(n)}$ of ν_1 on $R^{(n)}$. Replacing $R^{(n)}$ by R, we may assume that $R_{\mathfrak{p}}$ is regular.

Since ν_2 admits local uniformization, there exists a local blowing up

$$R/\mathfrak{p} \longrightarrow \overline{R}^{(1)}$$

with respect to ν_2 such that $\overline{R}^{(1)}$ is regular. By use of Corollary 6.1.15, there exists a sequence of local blowing ups

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}_{\mathfrak{p}^{(n)}}$ and $R^{(n)}/\mathfrak{p}^{(n)}$ are regular. Replacing R by $R^{(n)}$, we may assume that both $R_{\mathfrak{p}}$ and $R/\mathfrak{p}$ are regular.

Let $(y_1, \ldots, y_r) \subseteq \mathfrak{p}$ be a regular system of parameters for $\mathfrak{p}R_{\mathfrak{p}}$ (we can take $y_i \in \mathfrak{p}$ by Lemma 6.1.16), and $x_1, \ldots, x_t$ a set of elements of $R \setminus \mathfrak{p}$, whose images modulo $\mathfrak{p}$ form a regular system of parameters of $\mathfrak{m}/\mathfrak{p}$. If $y_1, \ldots, y_r$ generate $\mathfrak{p}$, then R is regular. Indeed, since $y_1, \ldots, y_r, x_1, \ldots, x_t$ generate $\mathfrak{m}$ we have $r + t \geq \dim(R)$. Also, since $r = \dim(R_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p})$ and $t = \dim(R/\mathfrak{p}) = \operatorname{ht}(\mathfrak{m}/\mathfrak{p})$ we have

$$\dim(R) = \operatorname{ht}(\mathfrak{m}) \geq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{m}/\mathfrak{p}) = r + t \geq \dim(R).$$

Therefore, $r + t = \dim(R)$ and $y_1, \ldots, y_r, x_1, \ldots, x_t$ is a minimal set of generators of $\mathfrak{m}$, hence $(R, \mathfrak{m})$ is regular.

If $y_1, \ldots, y_r$ do not generate $\mathfrak{p}$, take $y_{r+1}, \ldots, y_{r+s} \in \mathfrak{p}$ such that $y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s}$ generate $\mathfrak{p}$. Since the residues of $y_1, \ldots, y_r$ modulo $(\mathfrak{p}R_{\mathfrak{p}})^2$ form a $\kappa(\mathfrak{p})$-basis of $\mathfrak{p}R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})^2$, for each k, $1 \leq k \leq s$, we can find an equation

$$a_k y_{r+k} + b_{1k} y_1 + \ldots + b_{rk} y_r - h_k = 0$$

where $a_k \in R \setminus \mathfrak{p}$ and $h_k \in (\mathfrak{p}R_{\mathfrak{p}})^2$. In fact, multiplying the above equations by suitable elements of $R \setminus \mathfrak{p}$, we may assume that

$$h_k \in (y_1, \ldots, y_r)^2, \qquad 1 \leq k \leq s. \tag{6.21}$$

First, let us blow up R with respect to ν along the ideal $(a_1, y_1, \ldots, y_r)$ obtaining a new local domain $R^{(1)}$. Observe that, by part **(i)** of Lemma 6.1.17, the center $\mathfrak{p}^{(1)}$ of ν_1 in $R^{(1)}$ is generated by $y_1^{(1)}, \ldots, y_r^{(1)}, y_{r+1}, \ldots, y_{r+s}$. In $R^{(1)}$ we have $y_1 = a_1 y_1^{(1)}, y_2 = a_1 y_2^{(1)}, \ldots, y_r = a_1 y_r^{(1)}$ and we rewrite the previous relations as

$$a_k y_{r+k} + a_1 b_{1k} y_1^{(1)} + \ldots + a_1 b_{rk} y_r^{(1)} - h_k = 0, \qquad 1 \leq k \leq s$$

Observe that by (6.21) we have $h_k \in a_1^2 \left(y_1^{(1)}, \ldots, y_r^{(1)}\right)^2$. In particular, we have $a_1^2 \mid h_1$ in $R^{(1)}$ and we obtain

$$a_1 \left(y_{r+1} + b_{11} y_1^{(1)} + \ldots + b_{r1} y_r^{(1)} - h_1'\right) = 0$$

and

$$a_k y_{r+k} + a_1 b_{1k} y_1^{(1)} + \ldots + a_1 b_{rk} y_r^{(1)} - h_k = 0$$

for $k > 1$, where $h_1 = a_1 h_1'$ with $h_1', h_2, \ldots, h_s \in \left(y_1^{(1)}, \ldots, y_r^{(1)}\right)^2$. In particular,

$$y_{r+1} + b_{11} y_1^{(1)} + \ldots + b_{r1} y_r^{(1)} - h_1' = 0 \tag{6.22}$$

Since $h_1' \in \left(y_1^{(1)}, \ldots, y_r^{(1)}\right)$ we have $y_{r+1} \in \left(y_1^{(1)}, \ldots, y_r^{(1)}\right)$ and consequently

$$\mathfrak{p}^{(1)} = \left(y_{r+2}, \ldots, y_{r+s}, y_1^{(1)}, \ldots, y_r^{(1)}\right).$$

By parts **(ii)** and **(iii)** of Lemma 6.1.17, $\left(x_1 + \mathfrak{p}^{(1)}, \ldots, x_t + \mathfrak{p}^{(1)}\right)$ is a regular system of parameters for $R^{(1)}/\mathfrak{p}^{(1)}$ and $\left(y_1^{(1)}, \ldots, y_r^{(1)}\right)$ is a regular system of parameters for $R^{(1)}_{\mathfrak{p}^{(1)}}$.

We proceed as before with a_k for all $k = 2, \ldots, s$ until we reach a local domain $R^{(s)}$ for which $\mathfrak{p}^{(s)} = \left(y_1^{(s)}, \ldots, y_r^{(s)}\right) R^{(s)}$, $\left(x_1 + \mathfrak{p}^{(s)}, \ldots, x_t + \mathfrak{p}^{(s)}\right)$ is a regular system of parameters for $R^{(s)}/\mathfrak{p}^{(s)}$ and $\left(y_1^{(s)}, \ldots, y_r^{(s)}\right)$ is a regular system of parameters for $R^{(s)}_{\mathfrak{p}^{(s)}}$. Therefore, $R^{(s)}$ is regular with regular system of parameters $\left(x_1, \ldots, x_t, y_1^{(s)}, \ldots, y_r^{(s)}\right)$. □

6.4.1 Proof of Theorem 1.4.1

We will proceed by induction on the rank of the valuation. Let n be a given natural number and assume that every valuation μ centered at a noetherian local domain

$$(R', \mathfrak{m}') \in \mathcal{M}$$

with $\operatorname{rk}(\mu) < n$ admits local uniformization. Take a valuation ν centered at a noetherian local domain $(R, \mathfrak{m}) \in \mathcal{M}$ such that $\operatorname{rk}(\nu) = n$. We will prove that ν admits local uniformization.

Write $\nu = \nu_1 \circ \nu_2$ with $\operatorname{rk}(\nu_1) < \operatorname{rk}(\nu)$ and $\operatorname{rk}(\nu_2) < \operatorname{rk}(\nu)$. Then ν_1 is a valuation on F with center $\mathfrak{p} \subseteq \mathfrak{m}$ on R (so ν_1 is centered at $R_{\mathfrak{p}}$) and ν_2 is a valuation on $F\nu_1$ whose restriction to $\kappa(\mathfrak{p})$ is centered at $(R/\mathfrak{p}, \mathfrak{m}/\mathfrak{p})$ (see Lemma 6.1.2 above).

Since $\operatorname{rk}(\nu_1) < \operatorname{rk}(\nu)$ and $\operatorname{rk}(\nu_1) < \operatorname{rk}(\nu)$, by the induction hypothesis we obtain that ν_1 and ν_2 admit local uniformization. Theorem 1.4.1 follows now from Theorem 6.4.1.

6.4.2 Proof of Theorem 1.4.2

We will proceed as before. Let ν be a valuation centered at $(R, \mathfrak{m})$ with $\operatorname{rk}(\nu) > 1$, decompose it as $\nu = \nu_1 \circ \nu_2$ and let $\mathfrak{p}$ be the center of ν_1 on R. We want to prove that given $f_1, \ldots, f_q \in R$, there exists a sequence of local blowing ups (and hence by Lemma 6.1.6 a local blowing up)

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}$ is regular and there exists a regular system of parameters $u^{(n)} = \left(u_1^{(n)}, \ldots, u_d^{(n)}\right)$ of $\mathfrak{m}^{(n)}$ such that $f_1, \ldots, f_q$ are monomials in $u^{(n)}$.

By the induction hypothesis, there exists a local blowing up

$$R_{\mathfrak{p}} \longrightarrow \widetilde{R}^{(1)}$$

with respect to ν_1 such that $\widetilde{R}^{(1)}$ is regular and there exists a regular system of parameters $z = (z_1, \ldots, z_r)$ of $\widetilde{R}^{(1)}$ such that $f_i = c_i z^{\gamma_i}$ where c_i is a unit in $\widetilde{R}^{(1)}$. From Corollary 6.1.12 we conclude that there exists a sequence of local blowing ups

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}_{\mathfrak{p}^{(n)}} = \widetilde{R}^{(1)}$, where $\mathfrak{p}^{(n)}$ is the center $\mathfrak{m}_{\nu_1} \cap R^{(n)}$ of ν_1 on $R^{(n)}$. Replacing $R^{(n)}$ by R we may assume that $R_{\mathfrak{p}}$ is regular with regular system of parameters z such that $f_i = c_i z^{\gamma_i}$ with c_i a unit in $R_{\mathfrak{p}}$. Writing $c_i = \dfrac{\alpha_i}{\beta_i}$ with $\alpha_i, \beta_i \in R \setminus \mathfrak{p}$ we obtain

$$f_i = \frac{\alpha_i}{\beta_i} z^{\gamma_i}, \qquad 1 \leq i \leq q.$$

Moreover, we may assume that $z_j \in \mathfrak{p}$. Indeed, since $z_j \in \mathfrak{p}R_{\mathfrak{p}}$ we can write $z_j = \dfrac{1}{a_j} y_j$ with $y_j \in \mathfrak{p}$ and $a_j \in R \setminus \mathfrak{p}$. We have $\mathfrak{p}R_{\mathfrak{p}} = (z_1, \ldots, z_r)\, R_{\mathfrak{p}} = (y_1, \ldots, y_r)\, R_{\mathfrak{p}}$ and defining $\beta_i' = \beta_i \prod_{j=1}^{r} a_j^{\gamma_i^{(j)}}$ we have

$$f_i = \frac{\alpha_i}{\beta_i'} y^{\gamma_i}, \qquad 1 \leq i \leq q.$$

Blowing up R with respect to ν along the ideals $(\beta_i', y_1, \ldots, y_r)$, we may assume that $\beta_i' = 1$.

From the previous paragraph we can assume that $R_{\mathfrak{p}}$ is regular and that there are $y_1, \ldots, y_r \in \mathfrak{p}$ that form a regular system of parameters for $R_{\mathfrak{p}}$ and there exist $\alpha_i \in R \setminus \mathfrak{p}$ such that

$$f_i = \alpha_i y^{\gamma_i}, \qquad 1 \leq i \leq q.$$

Extend now $(y_1, \ldots, y_r)$ to a set of generators of $\mathfrak{p}$, say $(y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s})$. Since the residues of $y_1, \ldots, y_r$ modulo $(\mathfrak{p}R_{\mathfrak{p}})^2$ form a $\kappa(\mathfrak{p})$-basis of $\mathfrak{p}R_{\mathfrak{p}} / (\mathfrak{p}R_{\mathfrak{p}})^2$, for each $k = 1, \ldots, s$ we can find an equation

$$a_k y_{r+k} + b_{1k} y_1 + \ldots + b_{rk} y_r - h_k = 0$$

where $a_k \in R \setminus \mathfrak{p}$ and $h_k \in (\mathfrak{p}R_{\mathfrak{p}})^2$. Multiplying the last equation by a suitable element of $R \setminus \mathfrak{p}$ we can assume that $h_k \in (\mathfrak{p})^2$.

By the induction assumption for ν_2, there exists a local blowing up

$$R/\mathfrak{p} \longrightarrow \overline{R}^{(1)}$$

with respect to ν_2 such that $\overline{R}^{(1)}$ is regular and there exists a regular system of parameters $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_t)$ of $\overline{R}^{(1)}$ such that the residues of α_i and a_k (modulo $\mathfrak{p}$) are monomials in $\overline{x}$, i.e.,

$$\overline{\alpha}_i = \overline{u}_i \overline{x}^{\delta_i}, \qquad 1 \leq i \leq q$$

and

$$\overline{a}_k = \overline{v}_k \overline{x}^{\epsilon_k}, \qquad 1 \leq k \leq s$$

where $\overline{u}_i, \overline{v}_k$ are units in $\overline{R}$. By Corollary 6.1.15 there exists a sequence of local blowing ups

$$R \longrightarrow R^{(1)} \longrightarrow \cdots \longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}_{\mathfrak{p}^{(n)}} = R_{\mathfrak{p}}$ and $R^{(n)}/\mathfrak{p}^{(n)} \cong \overline{R}^{(1)}$ (for $x \in R^{(n)} \setminus \mathfrak{p}^{(n)}$ denote by $\overline{x}$ the element corresponding to $x + \mathfrak{p}^{(n)}$ via this isomorphism and say that x represents $\overline{x}$). Choose elements $x_l, u_i, v_k \in R^{(n)} \setminus \mathfrak{p}^{(n)}$ that represent $\overline{x}_l, \overline{u}_i, \overline{v}_k$ respectively. Then

$$\alpha_i = u_i x^{\delta_i} + r_i, \qquad 1 \leq i \leq q$$

and

$$a_k = v_k x^{\epsilon_k} + s_k, \qquad 1 \leq k \leq s$$

for some $r_i, s_k \in \mathfrak{p}^{(n)}$.

From the last paragraphs we may assume that $R_{\mathfrak{p}}$ is regular with a regular system of parameters $y = (y_1, \ldots, y_r)$ which extends to a set of generators $(y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s})$ of $\mathfrak{p}$ and there exist $x_1, \ldots, x_t \in \mathfrak{m} \setminus \mathfrak{p}$ such that their images in $R/\mathfrak{p}$ form a regular system of parameters of $R/\mathfrak{p}$ such that

$$f_i = \left(u_i x^{\delta_i} + r_i\right) y^{\gamma_i}, \qquad 1 \leq i \leq q \tag{6.23}$$

and

$$v_k x^{\epsilon_k} y_{r+k} + b_{1k} y_1 + \ldots + b_{rk} y_r + h'_k = 0, \qquad 1 \leq k \leq s, \tag{6.24}$$

where u_i, v_k are units in R, and $r_i, s_k \in \mathfrak{p}$ and $h'_k = h_k + s_k y_{r+k} \in (\mathfrak{p})^2$.

From now on we will just blow up R with respect to ν along ideals of the form $(x_l, y_1, \ldots, y_r)$ or $(x_l, y_1, \ldots, y_r, y_{r+s_1}, \ldots, y_{r+s})$ for some $1 \leq s_1 \leq s$. Take $l \in 1, \ldots, t$ such that $x_l \mid x^{\delta_i}$ for some $i = 1, \ldots, q$. Blowing up R with respect to ν along

$$(x_l, y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s})$$

we obtain a system of generators

$$\left(x_1, \ldots, x_t, y_1^{(1)}, \ldots, y_r^{(1)}, y_{r+1}^{(1)}, \ldots, y_{r+s}^{(1)}\right)$$

of $\mathfrak{m}^{(1)}$ such that $y_j = x_l y_j^{(1)}$ for all $j = 1, \ldots, r+s$. Substituting this new system of generators to the equations (6.23) and (6.24), we obtain

$$f_i = \left(u_i x^{\delta_i} + x_l r_i^{(1)}\right) x_l^{|\gamma_i|} \left(y^{(1)}\right)^{\gamma_i} = \left(u_i \frac{x^{\delta_i}}{x_l} + r_i^{(1)}\right) x_l^{|\gamma_i|+1} \left(y^{(1)}\right)^{\gamma_i}, \qquad 1 \leq i \leq q \tag{6.25}$$

and

$$v_k x^{\epsilon_k^{(1)}} y_{r+k}^{(1)} + b_{1k}^{(1)} y_1^{(1)} + \ldots + b_{rk}^{(1)} y_r^{(1)} + h'_k = 0, \qquad 1 \leq k \leq s, \tag{6.26}$$

where $r_i^{(1)} \in \mathfrak{p}^{(1)}$. Observe that $h'_k \in \left(y_1^{(1)}, \ldots, y_r^{(1)}, y_{r+1}^{(1)}, \ldots, y_{r+s}^{(1)}\right)^2$ and that by part **(i)** of Lemma 6.1.17 $\mathfrak{p}^{(1)} = \left(y_1^{(1)}, \ldots, y_r^{(1)}, y_{r+1}^{(1)}, \ldots, y_{r+s}^{(1)}\right)$. After finitely many of these local blowing ups, we obtain a local domain $\left(R^{(n)}, \mathfrak{m}^{(n)}\right)$ such that $\mathfrak{m}^{(n)}$ is generated by

$$\left(x_1, \ldots, x_t, y_1^{(n)}, \ldots, y_r^{(n)}, y_{r+1}^{(n)}, \ldots, y_{r+s}^{(n)}\right)$$

with

$$f_i = \left(u_i + r_i^{(n)}\right) x^{\tau_i} \left(y^{(n)}\right)^{\gamma_i} = u'_i x^{\tau_i} \left(y^{(n)}\right)^{\gamma_i}, \qquad 1 \leq i \leq q$$

and

$$v_k x^{\epsilon_k^{(n)}} y_{r+k}^{(n)} + b_{1k}^{(n)} y_1^{(n)} + \ldots + b_{rk}^{(n)} y_r^{(n)} + h'_k = 0, \qquad 1 \leq k \leq s \tag{6.27}$$

with $h'_k \in \left(y_1^{(n)}, \ldots, y_r^{(n)}, y_{r+1}^{(n)}, \ldots, y_{r+s}^{(n)}\right)^2$ and $u'_i \in \left(R^{(n)}\right)^{\times}$. Therefore, all f_i are monomials in

$$\left(x, y^{(n)}\right) := \left(x_1, \ldots, x_t, y_1^{(n)}, \ldots, y_r^{(n)}\right).$$

Observe that if we blow up $R^{(n)}$ with respect to ν along ideals of the form

$$\left(x_l, y_1^{(n)}, \ldots, y_r^{(n)}\right) \text{ or } \left(x_l, y_1^{(n)}, \ldots, y_r^{(n)}, y_{r+s_1}^{(n)}, \ldots, y_{r+s}^{(n)}\right) \tag{6.28}$$

for some $s_1 \in \{1, \ldots, s\}$ then all $f'_i s$ will be monomials in $\left(x, y^{(n+1)}\right)$.

We still do not have that $R^{(n)}$ is regular. In order to obtain that, we will blow up $R^{(n)}$ with respect to ν along ideals of the form (6.28). Let $x_l \mid x^{\epsilon_1^{(n)}}$ for some $1 \leq l \leq t$ and blow up $R^{(n)}$ with respect to ν along the ideal

$$\left(x_l, y_1^{(n)}, \ldots, y_r^{(n)}\right).$$

In $R^{(n+1)}$ equation (6.27) for $k = 1$ can be rewritten as

$$v_1 x^{\epsilon_1^{(n)}} y_{r+1}^{(n+1)} + x_l \left(b_{11}^{(n)} y_1^{(n+1)} + \ldots + b_{r1}^{(n)} y_r^{(n+1)}\right) + h'_1 = 0. \tag{6.29}$$

with $h'_1 \in \left(y_1^{(n+1)}, \ldots, y_r^{(n+1)}, y_{r+1}^{(n+1)}, \ldots, y_{r+s}^{(n+1)}\right)^2$. Now we blow up $R^{(n+1)}$ with respect to ν along

$$\left(x_l, y_1^{(n+1)}, \ldots, y_r^{(n+1)}, y_{r+1}^{(n+1)}, \ldots, y_{r+s}^{(n+1)}\right)$$

and the equation (6.29) rereads as

$$x_l^2\left(v_1\frac{x^{\epsilon_1^{(n)}}}{x_l}y_{r+1}^{(n+2)}+b_{11}^{(n)}y_1^{(n+2)}+\ldots+b_{r1}^{(n)}y_r^{(n+2)}+h_1''\right)=0$$

and consequently

$$v_1\frac{x^{\epsilon_1^{(n)}}}{x_l}y_{r+1}^{(n+2)}+b_{11}^{(n)}y_1^{(n+2)}+\ldots+b_{r1}^{(n)}y_r^{(n+2)}+h_1''=0,$$

with $h_1''\in\left(y_1^{(n+2)},\ldots,y_r^{(n+2)},y_{r+1}^{(n+2)},\ldots,y_{r+s}^{(n+2)}\right)^2$. After finitely many of these steps we reach a domain $R^{(n+m_1)}$ where

$$v_1y_{r+1}^{(n+m_1)}+b_{11}^{(n)}y_1^{(n+m_1)}+\ldots+b_{r1}^{(n)}y_r^{(n+m_1)}+h_1^{(m_1)}=0,$$

with $h_1^{(m_1)}\in\left(y_1^{(n+m_1)},\ldots,y_r^{(n+m_1)},y_{r+1}^{(n+m_1)},\ldots,y_{r+s}^{(n+m_1)}\right)^2$. It follows now that

$$\mathfrak{p}^{(n+m_1)}=\left(y_1^{(n+m_1)},\ldots,y_r^{(n+m_1)},y_{r+2}^{(n+m_1)},\ldots,y_{r+s}^{(n+m_1)}\right).$$

Repeating the process as above for each $k=2,\ldots,s$, we reach a domain $R^{(n+m)}$ such that $\mathfrak{p}^{(n+m)}$ is generated by

$$\left(y_1^{(n+m)},\ldots,y_r^{(n+m)}\right).$$

Analogously to the proof of Theorem 6.4.1, we note that $\mathfrak{m}^{(n+m)}$ is generated by

$$\left(x,y^{(n+m)}\right)=\left(x_1,\ldots,x_t,y_1^{(n+m)},\ldots,y_r^{(n+m)}\right),$$

so $R^{(n+m)}$ is regular with regular system of parameters $\left(x,y^{(n+m)}\right)$. Moreover, f_i is a monomial on $\left(x,y^{(n+m)}\right)$ for each $i=1,\ldots,q$. Therefore, we have achieved weak embedded local uniformization for ν.

6.4.3 Proof of Theorem 1.4.3

Let ν be a valuation with $\mathrm{rk}\,(\nu)>1$. We want to prove that given $f_1,\ldots,f_q\in R$ such that $\nu\,(f_1)\leq\ldots\leq\nu\,(f_q)$ there exists a sequence of local blowing ups

$$R\longrightarrow R^{(1)}\longrightarrow\cdots\longrightarrow R^{(n)}$$

with respect to ν such that $R^{(n)}$ is regular and there exists a regular system of parameters $u^{(n)} = \left(u_1^{(n)}, \ldots, u_d^{(n)}\right)$ of $R^{(n)}$ such that f_i is a monomial in $u^{(n)}$ for all $i = 1, \ldots, q$ and $f_1 \mid_{R^{(n)}} \ldots \mid_{R^{(n)}} f_q$.

Again, we will proceed by induction on the rank. Write $\nu = \nu_1 \circ \nu_2$ with $\mathrm{rk}\,(\nu_2) = 1$. By induction hypothesis for ν_1 and after changes as in Theorems 6.4.1 and 1.4.2 we can assume that $R_{\mathfrak{p}}$ is regular and there exists $y_1, \ldots, y_r \in \mathfrak{p}$ that form a regular system of parameters for $R_{\mathfrak{p}}$ such that

$$f_i = \alpha_i y^{\gamma_i}, \qquad 1 \le i \le q$$

with $\alpha_i \in R \setminus \mathfrak{p}$ and $y^{\gamma_1} \mid_R \ldots \mid_R y^{\gamma_q}$.

We want to modify α_i in such a way that $\nu_2\left(\alpha_1 + \mathfrak{p}\right) \le \ldots \le \nu_2\left(\alpha_q + \mathfrak{p}\right)$. We will do that by blowing up R with respect to ν along an ideal of the form $(\alpha, y_1, \ldots, y_r)$ for some $\alpha \in R \setminus \mathfrak{p}$. Since $y^{\gamma_1} \mid_R \ldots \mid_R y^{\gamma_q}$ we have that $\gamma_1 \le \ldots \le \gamma_r$ where " $\le$ " is the componentwise partial order of $\left(\mathbb{N} \cup \{0\}\right)^r$. Take $\alpha \in R$ such that

$$\nu_2\left(\alpha + \mathfrak{p}\right) \ge \nu_2(\alpha_i + \mathfrak{p}) - \nu_2(\alpha_{i+1} + \mathfrak{p}), \text{ for every } i = 1, \ldots, q-1,$$

(for instance take α to be the α_i with maximum value). Blowing up R with respect to ν along the ideal $(\alpha, y_1, \ldots, y_r)$ we obtain

$$f_i = \alpha_i' \left(y^{(1)}\right)^{\gamma_i} = \alpha_i \alpha^{|\gamma_i|} \left(y^{(1)}\right)^{\gamma_i}, \qquad 1 \le i \le q.$$

For each i, $1 \le i \le q-1$, either $\gamma_i = \gamma_{i+1}$ or $\gamma_i < \gamma_{i+1}$. If $\gamma_i = \gamma_{i+1}$, then $\nu\left(\alpha_i\right) \le \nu\left(\alpha_{i+1}\right)$ what implies that $\nu_2\left(\alpha_i + \mathfrak{p}\right) \le \nu_2\left(\alpha_{i+1} + \mathfrak{p}\right)$. Consequently

$$\begin{aligned} \nu_2\left(\alpha_i' + \mathfrak{p}^{(1)}\right) &= \nu_2\left(\alpha_i \alpha^{|\gamma_i|} + \mathfrak{p}^{(1)}\right) \\ &= \nu_2\left(\alpha_i \alpha^{|\gamma_{i+1}|} + \mathfrak{p}^{(1)}\right) \\ &\le \nu_2\left(\alpha_{i+1} \alpha^{|\gamma_{i+1}|} + \mathfrak{p}^{(1)}\right) \\ &= \nu_2\left(\alpha_{i+1}' + \mathfrak{p}^{(1)}\right). \end{aligned}$$

If $\gamma_i < \gamma_{i+1}$, then $|\gamma_i| < |\gamma_{i+1}|$ what implies that $1 + |\gamma_i| \leq |\gamma_{i+1}|$. Consequently

$$\begin{aligned}
\nu_2\left(\alpha_i' + \mathfrak{p}^{(1)}\right) &= \nu_2\left(\alpha_i \alpha^{|\gamma_i|} + \mathfrak{p}^{(1)}\right) \\
&= \nu_2\left(\alpha_i + \mathfrak{p}^{(1)}\right) + |\gamma_i| \nu_2\left(\alpha + \mathfrak{p}^{(1)}\right) \\
&\leq \nu_2(\alpha_{i+1} + \mathfrak{p}^{(1)}) + \nu_2\left(\alpha + \mathfrak{p}^{(1)}\right) + |\gamma_i| \nu_2\left(\alpha + \mathfrak{p}^{(1)}\right) \\
&= \nu_2(\alpha_{i+1} + \mathfrak{p}^{(1)}) + (1 + |\gamma_i|)\, \nu_2\left(\alpha + \mathfrak{p}^{(1)}\right) \\
&\leq \nu_2\left(\alpha_{i+1} + \mathfrak{p}^{(1)}\right) + |\gamma_{i+1}| \nu_2\left(\alpha + \mathfrak{p}^{(1)}\right) \\
&= \nu_2\left(\alpha_{i+1}' + \mathfrak{p}^{(1)}\right).
\end{aligned}$$

From the last paragraphs, we can assume that $R_\mathfrak{p}$ is regular and there exist $y_1, \ldots, y_r \in \mathfrak{p}$ that form a regular system of parameters for $R_\mathfrak{p}$ such that

$$f_i = \alpha_i y^{\gamma_i}, \qquad 1 \leq i \leq q, \tag{6.30}$$

where $y^{\gamma_1} \mid \ldots \mid y^{\gamma_q}$ and $\nu_2(\alpha_1 + \mathfrak{p}) \leq \ldots \leq \nu_2(\alpha_q + \mathfrak{p})$.

Extend $(y_1, \ldots, y_r)$ to a set of generators

$$(y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s})$$

of $\mathfrak{p}$. As in the proofs of Theorems 6.4.1 and 1.4.2, we have relations of the form

$$a_k y_{r+k} + b_{1k} y_1 + \ldots + b_{rk} y_r - h_k = 0, \qquad 1 \leq k \leq s, \tag{6.31}$$

where $a_k \in R \setminus \mathfrak{p}$ and $h_k \in \mathfrak{p}^2$.

By the induction hypothesis for $R/\mathfrak{p}$ and after lifting local blowing ups as in Theorems 6.4.1 and 1.4.2 we can assume that there exist $x_1, \ldots, x_t \in \mathfrak{m} \setminus \mathfrak{p}$ such that their images $x_1 + \mathfrak{p}, \ldots, x_t + \mathfrak{p}$ form a regular system of parameters for $R/\mathfrak{p}$ and we have the following relations:

$$\alpha_i = u_i x^{\epsilon_i} + r_i, \qquad 1 \leq i \leq q,$$

and

$$a_k = v_k x^{\delta_k} + s_k, \qquad 1 \leq k \leq s,$$

where u_i, v_k are units in R and $s_k, r_i \in \mathfrak{p}$ for $k = 1, \ldots, s$ and $i = 1, \ldots, r$ with $x^{\epsilon_1} \mid \ldots \mid x^{\epsilon_q}$.

Substituting a_k's and α_i's in equations (6.30) and (6.31) we obtain

$$f_i = \left(u_i x^{\epsilon_i} + r_i\right) y^{\gamma_i}, \qquad 1 \leq i \leq q,$$

and

$$v_k x^{\delta_k} y_{r+k} + b_{1k} y_1 + \ldots + b_{rk} y_r - h'_k = 0, \qquad 1 \leq k \leq s,$$

where $h'_k \in \mathfrak{p}^2$.

Blowing up R with respect to ν along ideals of the form $(x_l, y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+s})$ we have new coordinates $y^{(1)} = \left(y_1^{(1)}, \ldots, y_r^{(1)}\right)$ in which $y_j = x_l y_j^{(1)}$, $j = 1, \ldots, r$. Therefore,

$$f_i = \left(u_i x^{\epsilon_i} + x_l r_i^{(1)}\right) x_l^{|\gamma_i|} \left(y^{(1)}\right)^{\gamma_i}, \qquad 1 \leq i \leq q,$$

where $r'_i \in \mathfrak{p}'$. If $x_l \mid x^{\epsilon_i}$, this equation can be rewritten as

$$f_i = \left(u_i \frac{x^{\epsilon_i}}{x_l} + r_i^{(1)}\right) x_l^{|\gamma_i|+1} \left(y^{(1)}\right)^{\gamma_i}, \qquad 1 \leq i \leq q.$$

After finitely many of these local blowing ups we achieve

$$f_i = \left(u_i + r_i^{(n)}\right) x^{\delta_i} \left(y^{(n)}\right)^{\gamma_i}, \qquad 1 \leq i \leq q$$

where $\delta_i^{(l)} = \epsilon_i^{(q)} |\gamma_i| + \epsilon_i^{(l)}$. Since $\epsilon_1^{(l)} \leq \ldots \leq \epsilon_q^{(l)}$ we have that $x^{\delta_1} \mid \ldots \mid x^{\delta_q}$. Therefore, we achieved that $f_1, \ldots, f_q$ are monomials in $\left(x, y^{(n)}\right)$ and that $f_1 \mid \ldots \mid f_q$.

We still don't have that $\left(R^{(n)}, \mathfrak{m}^{(n)}\right)$ is regular. We can achieve that proceeding as in Theorem 1.4.2. Now $\left(R^{(n+m)}, \mathfrak{m}^{(n+m)}\right)$ is regular with regular system of parameters $\left(x, y^{(n+m)}\right)$ in which f_i are monomials and $f_1 \mid \ldots \mid f_q$. Therefore, we achieved embedded local uniformization for ν.

6.4.4 Proof of Theorem 1.4.4

Decompose $\nu = \nu_1 \circ \nu_2$ and let $\mathfrak{p} = \mathfrak{m}_{\nu_1} \cap R$ be the center of ν_1 on R. Then ν_1 is a valuation on $F = \mathrm{Quot}(R)$ centered at $R_{\mathfrak{p}}$ and ν_2 is a valuation on $F\nu_1$ such that the restriction of ν_2 to $\kappa(\mathfrak{p})$ is centered at $R/\mathfrak{p}$.

We can assume that $\mathrm{rk}(\nu_1) < \mathrm{rk}(\nu)$ and $\mathrm{rk}(\nu_2) < \mathrm{rk}(\nu)$, hence by induction on the rank, both ν_1 and ν_2 admit inseparable local uniformization. Take elements $\overline{a}_1, \ldots, \overline{a}_s$ in some purely inseparable extension of $\kappa(\mathfrak{p})$ such that $(R/\mathfrak{p}[\overline{a}_1, \ldots, \overline{a}_s], \nu_2)$ admits local uniformization and $\overline{a}_i^{p^{r_i}} \in \mathfrak{m}/\mathfrak{p}$. For every $i, 1 \leq i \leq s$, choose $a_i \in F^{\mathrm{ac}}$ to be a root of the polynomial $x^{p^{r_i}} - b_i$, where $b_i + \mathfrak{p} = \overline{a}_i^{p^{r_i}}$ (and $\overline{a}_i^{p^{r'}} \notin \kappa(\mathfrak{p})$ for every $r' < r_i$). Let

$F' := F(a_1, \ldots, a_s)$ be the purely inseparable extension of F generated by $a_1, \ldots, a_s$ and let $R' = R[a_1, \ldots, a_s]$.

Claim 6.4.2. *The pair $(R'/\mathfrak{p}', \nu_2)$ admits local uniformization, where $\mathfrak{p}' = \mathfrak{m}_{\nu_1} \cap R'$ is the center of ν_1 in R'.*

Proof. It is enough to prove that there exists a valuation preserving isomorphism between $R'/\mathfrak{p}'$ and $R/\mathfrak{p}[\overline{a}_1, \ldots, \overline{a}_s]$. Consider the mapping

$$\Phi : R' = R[a_1, \ldots, a_s] \longrightarrow R/\mathfrak{p}[\overline{a}_1, \ldots, \overline{a}_s]$$

given by

$$\Phi\left(\sum_{i=0}^{n} \alpha_i a^{\gamma_i}\right) = \sum_{i=0}^{n} (\alpha_i + \mathfrak{p})\overline{a}^{\gamma_i},$$

where $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{is}) \in \mathbb{N}^s$ and $a^{\gamma_i} = a_1^{\gamma_{i1}} \cdots a_s^{\gamma_{is}}$ (respectively, $\overline{a}^{\gamma_i} = \overline{a}_1^{\gamma_{i1}} \cdots \overline{a}_s^{\gamma_{is}}$).

It is easy to see that Φ is a ring homomorphism and that it is surjective. It remains to prove that $\ker(\Phi) = \mathfrak{p}'$. Take any element $\phi \in R'$. Then $\phi^{p^r} \in R$ for some $r \in \mathbb{N}$ and we have that

$$\phi^{p^r} \in \mathfrak{p} \Longleftrightarrow \phi \in \mathfrak{p}'.$$

Therefore,

$$\phi \in \mathfrak{p}' \Longleftrightarrow \phi^{p^r} \in \mathfrak{p} \Longleftrightarrow \Phi\left(\phi^{p^r}\right) = 0 \Longleftrightarrow \Phi(\phi) = 0.$$

□

Replacing R' by R we can assume that the valuation ν is decomposed as $\nu = \nu_1 \circ \nu_2$ such that $(R/\mathfrak{p}, \nu_2)$ admits local uniformization. Observe that from our extra assumption on the category $\mathcal{M}_p$, the ring R' also belongs to $\mathcal{M}_p$.

Since $\mathrm{rk}(\nu_1) < \mathrm{rk}(\nu)$, by induction hypothesis, $(R_\mathfrak{p}, \nu_1)$ admits inseparable local uniformization, which means that in some extension of $F = \mathrm{Quot}(R)$ there exist elements $b_1, \ldots, b_t$ such that $(R_\mathfrak{p}[b_1, \ldots, b_t], \nu_1)$ with $b_i^{p^r} \in \mathfrak{p}R_\mathfrak{p}$ admits local uniformization. Observe that we can assume that $b_i^{p^r} \in \mathfrak{p}$. Indeed, since $b_i^{p^r} \in \mathfrak{p}R_\mathfrak{p}$ then $b_i^{p^r} = \alpha_i/\beta_i$ for some $\alpha_i \in \mathfrak{p}$ and $\beta_i \in R \setminus \mathfrak{p}$. Taking $c_i = \beta_i \cdot b_i$ we have that $R_\mathfrak{p}[b_1, \ldots, b_t] = R_\mathfrak{p}[c_1, \ldots, c_t]$ and that $c_i^{p^r} \in \mathfrak{p}$.

Claim 6.4.3. *Let* $R' = R[b_1, \ldots, b_t]$ *and* $\mathfrak{p}' = \mathfrak{m}_{\nu_1} \cap R'$ *be the center of* ν_1 *in* R'*. Then* $(R'_{\mathfrak{p}'}, \nu_1)$ *and* $(R'/\mathfrak{p}', \nu_2)$ *admit local uniformization.*

Proof. It is easy to see that

$$R'_{\mathfrak{p}'} = R_{\mathfrak{p}}[b_1, \ldots, b_t].$$

Indeed, take an element $r \in R'_{\mathfrak{p}'}$. Then

$$r = \frac{f}{g}$$

with $f = f(b_1, \ldots, b_t), g = g(b_1, \ldots, b_t) \in R[b_1, \ldots, b_t]$ and $\nu_1(g) = 0$. Choose $s \in \mathbb{N}$ such that $g^{p^s} \in R_{\mathfrak{p}}$. Since $\nu_1\left(g^{p^s}\right) = p^s \nu_1(g) = 0$ we have that g^{p^s} is a unit of $R_{\mathfrak{p}}$. Then

$$r = \frac{f}{g} = \frac{f \cdot g^{p^r - 1}}{g^{p^r}} \in R_{\mathfrak{p}}[b_1, \ldots, b_t].$$

The other inclusion is trivial.

It remains to prove that $R/\mathfrak{p} \simeq R'/\mathfrak{p}'$. Let $\Phi : R \longrightarrow R'/\mathfrak{p}'$ be the composition of the natural inclusion of R into R' with the canonical epimorphism from R' onto $R'/\mathfrak{p}'$, i.e.

$$R \xrightarrow{\ i\ } R' \xrightarrow{\ \pi\ } R'/\mathfrak{p}', \qquad \Phi = \pi \circ i.$$

We have to prove that Φ is surjective and that $\ker(\Phi) = \mathfrak{p}$. Since $i : R \longrightarrow R'$ is an inclusion we have that

$$\ker(\Phi) = \{r \in R \mid r \in \mathfrak{p}'\} = \{r \in R \mid \nu_1(r) > 0\} = \mathfrak{p}.$$

It remains to prove that Φ is surjective. Take any element $f = f(b_1, \ldots, b_t) \in R'$. Since

$$f = f(0) + b_1 \cdot f_1(b_1, \ldots, b_t) + \ldots + b_t \cdot f_t(b_1, \ldots, b_t)$$

and $b_1, \ldots, b_t \in \mathfrak{p}'$ we have that $f(0) + \mathfrak{p}' = f + \mathfrak{p}'$. Therefore,

$$\Phi(f(0)) = f + \mathfrak{p}'$$

and consequently Φ is surjective. □

Theorem 1.4.4 is a direct consequence of Theorem 6.4.1 and Claim 6.4.3.

Bibliography

[1] S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. of Math. **63** (1956), 491 – 526.

[2] S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math. **78** (1956), 321 – 348.

[3] S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces*, Academic Press, New York and London, 1966.

[4] S. Abhyankar, *Simultaneous resolution for algebraic surfaces*. Amer. J. Math. **78** (1956), 761 – 790.

[5] D. Abramovich and F. Oort, *Alterations and resolution of singularities*, In: Resolution of Singularities. A research textbook in tribute to Oscar Zariski, Progress in Math. vol. **181**, Birkhäuser 2000.

[6] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.

[7] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Mathematical Surveys and Monographs **33**, (Amer. Math. Soc., Providence RI 1990).

[8] S. Boucksom, C. Favre and M. Jonsson, *Valuations and Plurisubharmonic Singularities*, Publ. Res. Inst. Math. Sci. **44**, no. 2, (2008), 449 – 494.

[9] V. Cossart, *Desingularisation en dimension 3 et caracteristique p*, Proceedings de La Rabida, Progr. Math. vol. **134**, Birkhäuser, 1996.

[10] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra **320** no. 3 (2008), 1051–1082.

[11] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** no. 7 (2009), 1836–1976.

[12] I. Efrat and K. Osiak, *Topological spaces as spaces of R-places*, J. Pure Appl. Algebra **215** no. 5 (2011), 839 – 846.

[13] R. Engelking, *General topology*, Sigma Ser. Pure Math. **6**, Heldermann Verlag, Berlin, 1989.

[14] C. Favre and M. Jonsson, *The valuative tree*, Lecture Notes in Math. **1853**, Springer-Verlag, Berlin, 2004.

[15] C. Favre and M. Jonsson, *Valuative analysis of planar plurisubharmonic functions*, Invent. Math. **162** no. 2 (2005), 271 – 311.

[16] A. Granja, *The valuative tree of a two-dimensional regular local ring*, Math. Res. Lett. **14** no. 1 (2007), 19 – 34.

[17] F. J. Herrera Govantes, M. A. Olalla Acosta and M. Spivakovsky, *Valuations in algebraic field extensions*, J. Algebra **312** no. 2 (2007), 1033 – 1074.

[18] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), 109 – 326.

[19] R. Huber and M. Knebusch, *On valuation spectra*, Contemp. Math. **155** (1994), 167 – 206.

[20] J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51 – 93.

[21] H. Knaf, *Regular Curves over Prüfer Domains*, Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Commun. **33** (2003), 89–106.

[22] H. Knaf and F.-V. Kuhlmann, *Abhyankar places admit local uniformisation in any characteristic*, Ann. Sci. Éc. Norm. Supér. (4) **38** no. 6 (2005), 833 – 846.

[23] H. Knaf and F.-V. Kuhlmann, *Every place admits local uniformization in a finite extension of the function field*, Adv. Math. **221** no. 2 (2009), 428 –453.

[24] F.-V. Kuhlmann, *A conjecture about extensions within the absolute inertia field of a valued field*, online version at: `http://math.usask.ca/fvk/Locunico.dvi`.

[25] F.-V. Kuhlmann, *Book in preparation.* Preliminary versions of several chapters available at: `http://math.usask.ca/~ fvk/Fvkbook.htm`.

[26] F.-V. Kuhlmann, *Dense subfields of henselian fields, and integer parts*, Logic in Tehran, Proceedings of the Workshop and Conference on Logic, Algebra, and Arithmetic, held October 18–22, 2003, Lecture Notes in Logic **26** (2006), 204–226.

[27] F. -V. Kuhlmann, *Henselian function fields and tame fields*, extended version of Ph.D. thesis, Heidelberg 1990.

[28] F.-V. Kuhlmann, *On places of algebraic function fields in arbitrary characteristic*, Adv. Math. **188** (2004), 399 – 424.

[29] F.-V. Kuhlmann, M. Machura and K. Osiak, *Metrizability of spaces of R-places of function fields of transcendence degree 1 over real closed fields*, Comm. Algebr **39** no. 9 (2011), 3166 – 3177.

[30] F.-V. Kuhlmann and J. Novacoski, *Henselian elements*, in preparation.

[31] F.-V. Kuhlmann and A. Prestel, *On places of algebraic function fields*, J. Reine Angew. Math. **353** (1984), 181 – 195.

[32] J. Lipman, *Desingularization of two-dimensional schemes*, Ann. of Math. **107** (1978), 151 – 207.

[33] S. MacLane, *A construction for absolute values in polynomial rings*, Trans. Amer. Math. Soc. **40** (1936), 363 – 395.

[34] S. MacLane, *A construction for prime ideals as absolute values of an algebraic field*, Duke Math. J. **2** (1936), 492 – 510.

[35] H. Matsumura, *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. **56**, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[36] T. T. Moh, *Desingularization of three-varieties in positive characteristic*, Progr. Math. **134**, 49 – 93.

[37] J. Novacoski, *Topologies on spaces of valuations*, in preparation.

[38] J. Novacoski, *Valuations centered at a two-dimensional regular local ring: Infima and Topologies*, to appear in Proceedings of the Second International Conference and Workshop on Valuation Theory (2012), 14 pages.

[39] J. Novacoski and M. Spivakovsky, *Reduction of Local Uniformization to the rank one case*, submitted (2012), 22 pages.

[40] P. Roquette, *Henselian elements*, online version at: `http://math.usask.ca/fvk/Henselm.dvi`.

[41] M. Spivakovsky, *Valuations in function fields of surfaces*, Amer. J. Math **112**, 1 (1990), 107 – 156.

[42] M. Spivakovsky, *Resolution of singularities I: Local uniformization of an equicaracteristic quasi-excellent local domain whose residue field k satisfies* $[k : k^p] < \infty$, preprint.

[43] B. Teissier, *Valuations, deformations, and toric geometry*. Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), 361 – 459, Fields Inst. Commun. **33**, (2003).

[44] M. Temkin, *Inseparable Local Uniformization*, J. Algebra **373** (2013), 65–119.

[45] M. Vaquie, *Extension d'une valuation*, Trans. Amer. Math. Soc. **359** no. 7 (2007), 3439 – 3481.

[46] M. Vaquie, *Famille admissible de valuations et defaut d'une extension*, J. Algebra **311** no. 2 (2007), 859 – 876.

[47] L. van den Dries, *On finite inertial extensions of a valued field*, online version at: `http://math.usask.ca/fvk/louhens.pdf`.

[48] O. Zariski, *A simplified proof for resolution of singularities of an algebraic surface*, Ann. of Math. **43** (1942), 583 – 593.

[49] O. Zariski, *Local uniformization theorem on algebraic varieties*, Ann. of Math. **41** (1940), 852 – 896.

[50] O. Zariski, *Reduction of singularities of algebraic three dimensional varieties*, Ann. of Math. **45** (1944), 472 – 542.

[51] O. Zariski, *The reduction of singularities of an algebraic surface*, Ann. of Math. **40** (1939), 639 – 689.

[52] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, New York/Heidelberg/Berlin (1960).

Index