# Compact and weakly Compact Derivations on $\ell^1(\mathbb{Z}_+)$

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# Abstract

In this thesis, we aim to study derivations from $\ell^1(\mathbb{Z}_+)$ to its dual, $\ell^\infty(\mathbb{Z}_+)$. We first characterize them as certain closed subspace of $\ell^\infty(\mathbb{Z}_+)$. Then we present a necessary and sufficient condition, due to M. J. Heath, to make a bounded derivation on $\ell^1(\mathbb{Z}_+)$ into $\ell^\infty(\mathbb{Z}_+)$, a compact linear operator.

After that base on the work in [6], we study weakly compact derivations from $\ell^1(\mathbb{Z}_+)$ to its dual. We introduce T-sets and TF-sets and then state their relation with weakly compact operators on $\ell^1(\mathbb{Z}_+)$. These results are originally due to Y. Choi and M. J. Heath, but we give simpler proofs.

Finally, we will study certain classes of derivations from $L^1(\mathbb{R}_+)$ to $L^\infty(\mathbb{R}_+)$, and give an elementary proof that they are always mapped into $C_0(\mathbb{R}_+)$.

keywords: Banach algebra, Module action, Compact and weakly compact Derivations, T-set, TF-set.

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I lovingly dedicate this thesis to my husband,

Javad

who supported me each step of the way.

# CONTENTS

# LIST OF FIGURES

# INTRODUCTION

One of the very early works on *derivation* was by Kaplansky on Projections in Banach Algebras [18]. Moreover, he studied Modules over Operator Algebras and Inner Derivations on AW-algebras in [17]. Then Singer and Wermer showed in [23] that every bounded derivation of a commutative Banach algebra maps into the radical that is a very important result in derivation study. Since then there have seen various studies of different properties of derivations such as boundedness, compactness, etc.

The problem of determining the weakly compact and compact homomorphisms between various Banach algebras has been much studied. In this thesis we want to intersect these two concepts and study weakly compact and compact derivations.

M. J. Heath in his paper [15] has shown that if there are no (weakly) compact derivations from a commutative Banach algebra, $A$, into its dual module, then there are no (weakly) compact derivations from $A$ into any symmetric $A$-bimodule. He also proved similar results for bounded derivations of finite rank. Then he has described the compact derivations from the convolution algebra $\ell^1(\mathbb{Z}_+)$ to its dual and has given an example of a non-compact, bounded derivation from a uniform algebra $A$ into a symmetric $A$-bimodule.

In [5] Y. Choi and M. J. Heath have shown that all derivations from the disk algebra to its dual are compact. Also in [6] they characterized when derivations from $\ell^1(\mathbb{Z}_+)$ to its dual are weakly compact with examples that are not compact.

Pedersen followed this in [20] for derivations from weighted convolution algebras $L^1(\omega)$ on $\mathbb{R}_+$ to their dual spaces.

In this thesis we will give the background needed to understand these results and then we will present alternative proofs of the main results of Choi and Heath (see Section 2.4).

In the first chapter we will introduce some preliminaries from Banach algebra, derivations and compactness.

In Chapter 2, we characterize derivations on $\ell^1(\mathbb{Z}_+)$ to its dual in Theorem 2.2.6 and compactness and weak compactness of these derivations.

We will study derivations on $L^1(\mathbb{R}_+)$ in Chapter 3.

# CHAPTER 1

# PRELIMINARIES

In this chapter we will review some important concepts and definitions of Banach algebra theory, and their modules and derivations, that we need. The principal references for this chapter are [19], [7], and [21].

## 1.1 Banach Spaces

**Definition 1.1.1.** *Let $X$ be a normed space with the norm $\|\cdot\|$. Then $X$ would be **Banach** **space** when it is complete with its norm. In other word, $X$ is a Banach space when every Cauchy sequence is convergent in $(X, \|\cdot\|)$.*

**Definition 1.1.2.** *Suppose $\mathrm{T} : X \to Y$ is a linear operator between $X, Y$. Then we define **norm** of $T$ by*

$$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| = 1\}.$$

*If $\|T\| < \infty$, then $T$ is a bounded linear operator.*

We can prove that every linear operator $\mathrm{T} : X \to Y$ is bounded if and only if it is continuous. The set of all bounded linear operators from $X$ to $Y$ will be denoted by $\mathcal{B}(X, Y)$, that is a normed space with the norm that we have defined above. In particular, $\mathcal{B}(X, X)$, space of bounded linear operators on normed space $X$, will be shown by $\mathcal{B}(X)$.

**Theorem 1.1.3** (Theorem 1.4, [21])**.** *Let $X, Y$ be normed spaces. If $Y$ is a Banach space, then $\mathcal{B}(X, Y)$ with the norm in Definition 1.1.2 is a Banach space.*

**Example 1.1.4.** *The space $\ell^1 = \{X \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with the norm $\|\cdot\|_1$ on $\ell^1$ that is given by*

$$\|X\|_1 = \sum_{n=1}^{\infty} |x_n| \qquad (X \in \ell^1)$$

*is a Banach space by using Theorem 2.10, [2]. Similarly the space*

$$\ell^2 = \{X \in \mathbb{C}^{\mathbb{N}} : (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}} < \infty\}$$

*is also a Banach space.*

**Definition 1.1.5.** *For a normed space $X$, $\mathcal{B}(X, \mathbb{C})$ is the **dual** of $X$ and we show it with $X^*$.*

If $X$ is a locally compact and Hausdorff space, we define

$$C(X) = \{f : X \to X : f \text{is a continuous function on } X.\},$$

$$BC(X) = \{f \in C(X) : f \text{is a bounded function.}\}$$

and

$$C_c(X) = \{f \in C(X) : Supp(f) \text{ is compact}\},$$

when $Supp(f) = \overline{\{x : f(x) \neq 0\}}$.

We say that $f$ **vanishes at infinity** if for any $\varepsilon > 0$, the set $\{x : |f(x)| \geq \varepsilon\}$ is compact.

Also

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity}\}.$$

We can see that $C_c(X) \subseteq C_0(X) \subseteq BC(X)$.

**Definition 1.1.6.** *Let $X$ be a normed space. The **weak-star topology** on $X^*$ is the weakest topology for which all $x \in X$, the linear functional $x^* \mapsto x^*(x)$ is continuous on $X^*$. Also $X^*$-topology on $X$ is the weakest topology for which all $f \in X^*$ are continuous. We call this topology, **weak topology** on $X$.*

**Definition 1.1.7.** *A net $\{x_\alpha\}$ **converges weakly** in Banach space $X$, if for any $F \in X^*$ the net of complex numbers $\{F(x_\alpha)\}$ converges.( [24]) If $\{x_n\}$ is a sequence in $X$, then $x_n$ **converges weakly** to $x$ if $\varphi(x_n)$ converges to $\varphi(x)$ as $n \to \infty$ for all $\varphi \in X^*$*

**Definition 1.1.8.** *Let $X$ be a locally compact Hausdorff space and $(f_i)$ be a net in $c_0(X)$. $(f_i)$ **converges pointwise** to $f$ if $f_i(x) \to f(x)$ for all $x \in X$.*

**Example 1.1.9.** *Let $(y_i)$ be a bounded net in $c_0(\mathbb{Z}_+)$ and let $y \in c_0(\mathbb{Z}_+)$. Then $(y_i)$ converges weakly to $y$ if and only if it converges pointwise to $y$.*

*Proof.* Let $(y_i)$ converges pointwise to $y$, we want to show $(y_i)$ converges weakly to $y$, so we must show for every $\varphi \in c_0(\mathbb{Z}_+)^* = \ell^1(\mathbb{Z}_+)$,

$$\langle \varphi, y_i \rangle \to \langle \varphi, y \rangle$$

Let $k = \sup_i \|y_i\|$, $\varepsilon > 0$ and $N$ such that $g = \sum_{n=0}^{N} \varphi(n)\delta_n$ satisfies

$$\|\varphi - g\|_1 < \frac{\varepsilon}{3k}.$$

Now

$$\langle g, y_i \rangle \to \langle g, y \rangle$$

since $\langle \delta_n, y_i \rangle \equiv y_i(n)$ that converges to $y(n)$ for every $n$, by assumption and $g$ is finite linear combination of $\delta_n$. So there is an $i_0 > 0$ such that for all $i > i_0$,

$$|\langle g, y_i \rangle - \langle g, y \rangle| < \frac{\varepsilon}{3}.$$

Then

$$
\begin{aligned}
|\langle \varphi, y_i \rangle - \langle \varphi, y \rangle| \;&\leq\; |\langle \varphi - g, y_i \rangle| + |\langle g, y_i - y \rangle| + |\langle \varphi - g, y \rangle| \\
&<\; \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&=\; \varepsilon.
\end{aligned}
$$

Conversely, if $(y_i)$ converges weakly to $y$, so for that every $\varphi \in c_0(\mathbb{Z}_+)^* = \ell^1(\mathbb{Z}_+)$,

$$\langle \varphi, y_i \rangle \to \langle \varphi, y \rangle.$$

Specifically, $\delta_n \in c_0(\mathbb{Z}_+)^*$, for every $n$. So $\langle \delta_n, y_i \rangle \equiv y_i(n)$ converges to $y(n)$ for every $n$. Hence $(y_i)$ converges pointwise to $y$. $\qquad\square$

Now we have a brief review of compact and weakly compact operators.

**Definition 1.1.10.** *A **compact operator** is a linear operator $f$ from a Banach space $X$ to another Banach space $Y$, such that the image of any bounded subset of $X$ under $f$ is a relatively compact subset of $Y$. Such an operator is necessarily a bounded operator, and so continuous.*

**Definition 1.1.11.** *An operator $T : X \to Y$ between Banach spaces is called **weakly compact** if the image of any bounded subset of $X$ under $T$ is a relatively weakly compact subset of $T(X)$.*

**Remark 1.1.12.** *Every compact operator is weakly compact since if $T : X \to Y$ is a compact operator, then $T(B)$ is a compact subset of $T(X)$ when $B$ is a bounded subset of $X$. By the fact that weak topology is weaker than norm topology. $T(B)$ is also compact in weak topology, meaning that $T(B)$ is weakly compact subset of $T(X)$. Hence $T$ is a weakly compact operator.*

Also every weakly compact linear operator between Banach spaces is bounded (Proposition 3.5.3, [19]). In addition, we can show that a bounded operator $T : X \to Y$ between Banach spaces is weakly compact if and only if for every bounded sequence $(x_n)$ of $X$, the sequence $(Tx_n)$ has a weakly convergent subsequence in $Y$.(Proposition 3.5.5, [19])

**Lemma 1.1.13** (Proposition 3.5.9, [19])**.** *Let $(T_m)$ be a bounded sequence in $\mathcal{B}(X,Y)$ $(X,Y$ Banach spaces) such that $\|T_m - T\| \to 0$ as $m \to \infty$. If $T_m$ is weakly compact for all $m$, then $T$ is weakly compact.*

**Definition 1.1.14.** *An operator between Banach spaces is called a **finite-rank operator** when its range is finite-dimensional.*

**Lemma 1.1.15** (Definition 4.1., [1])**.** *Let $S, T$ are finite rank operators between Banach spaces, then $S + T$ is also finite rank.*

**Remark 1.1.16.** *By induction and from the above lemma we conclude that the sum of finite numbers of finite rank operators is also a finite rank operator.*

We denote $\mathcal{K}(A, E)$ as subset of $\mathcal{B}(A, E)$ consisting of operators which are compact maps. It has been shown in (Chapter II, Proposition 4.2, [7]) that $\mathcal{K}(A, E)$ is a closed subspace of $\mathcal{B}(A, E)$. Also we denote the subset of $\mathcal{B}(A, E)$ consisting finite-rank operators by $\mathcal{F}(A, E)$. You can see by (Proposition 3.4.3, [19])

$$\mathcal{F}(A, E) \subseteq \mathcal{K}(A, E)$$

and so

$$\overline{\mathcal{F}(A, E)}^{\|\cdot\|} \subseteq \mathcal{K}(A, E)$$

**Lemma 1.1.17.** *If $\{T_n : n \in \mathbb{N}\}$ is a sequence of finite-rank operators, converging to $T$, Then $T$ is compact.*

*Proof.* See the proof of $(c \to a)$, Theorem 4.4, Chapter II, [7]. $\qquad\square$

**Example 1.1.18** (Example 3.4.5., [19])**.** *Define the linear operator $T$ from $\ell^2$ into itself by the formula $T(\delta_n) = (\frac{\delta_n}{n})$. So $T = diag(1, \frac{1}{2}, \frac{1}{3}, \dots)$. If we consider the sequence $(T_k)$ such that*

$$T_k(\delta_n) = \begin{cases} T(\delta_n) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$$

*Then $T_k$ is a finite rank operator for each $k$. Now for each $a = \sum_n a_n \delta_n \in \ell^2$,*

$$(T - T_k)(a) = \sum_{n>k} (a_n \delta_n) \frac{1}{n},$$

*means that*

$$T - T_k = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \frac{1}{k+1} & & \\ & & & & \frac{1}{k+2} & \\ & & & & & \ddots \end{bmatrix}$$

*So*

$$
\begin{aligned}
\|(T - T_k)(a)\|_2^2 &= \sum_{n>k} |a_n|^2 \frac{1}{n^2} \\
&\leq \sum_{n>k} |a_n|^2 \frac{1}{(k+1)^2} \\
&\leq \|a\|_2^2 \frac{1}{(k+1)^2}
\end{aligned}
$$

*then $\|T - T_k\|_2 \leq \frac{1}{k+1}$. So $(T_k)$ converges to $T$ and by Lemma 1.1.17, $T$ is compact. However, it does not have finite rank, since if it has finite rank $\dim(Im(T)) = d < \infty$ for some $d$. But $\delta_1, \dots, \delta_{d+1} \in Im(T)$, that is a contradiction. Hence $T$ is an example of a compact operator that does not have finite rank.*

## 1.2  Banach algebras

**Definition 1.2.1.** *A **Banach algebra**, A, is a complex Banach space with a product $(x, y) \mapsto xy$ mapping $A \times A \mapsto A$ so that :*
*(i) The product is associative, and distributive laws relate addition and product. Moreover, for $x, y \in A$ and $\lambda \in \mathbb{C}$, $(\lambda x)y = x(\lambda y) = \lambda(xy)$. Thus the product is complex bilinear.*

*(ii) A is **unital**, if it has an identity, meaning there is an $e \in A$ (necessarily unique) and $\|e\| = 1$, such that for all $x \in A$ :*

$$xe = ex = x$$

*(iii) For all $x, y \in A$:*

$$\|xy\| \leq \|x\| \, \|y\|$$

*If the product is commutative, we call A **Abelian**.*

**Example 1.2.2.** *a) Let $G$ be a locally compact group and $f, g$ be integrable functions on $G$. The **convolution** of $f$ and $g$ on $s \in G$ will be defined with:*

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)dm(t)$$

*when $m$ is a left Haar measure on $G$. For every $f, g \in L^1(G)$, $f * g \in L^1(G)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. So $(L^1(G), *, \|\cdot\|_1)$ is a Banach algebra.*

*b) By the above example $(L^1(\mathbb{R}), \|\cdot\|, *)$ is a commutative Banach algebra. Now consider the Banach space $L^1(\mathbb{R}_+)$, this space would be regarded as a closed subspace of $L^1(\mathbb{R})$ by extending each $f \in L^1(\mathbb{R}_+)$ to be equal to $0$ on the negative half-line $\mathbb{R}^{-\bullet} = (-\infty, 0)$. In this case, the product of two elements $f$ and $g$ of $L^1(\mathbb{R}_+)$ is given by the formula*

$$(f * g)(x) = \int_0^x f(x - t)g(t)dt \qquad (x \in \mathbb{R}_+);$$

*clearly $L^1(\mathbb{R}_+)$ is a closed subalgebra of $L^1(\mathbb{R})$.*

*c) Let $S$ be a semigroup, that is, $S$ is a non-empty set together with $(s, t) \mapsto st$, $S \times S \to S$ such that $(rs)t = r(st)$ $(r, s, t \in S)$. We have defined the Banach Space $(\ell^1(S), \|\cdot\|_1)$ in example 1.1.4. Let $f, g \in \ell^1(S)$. Then we set*

$$(f * g)(t) = \sum \{f(r)g(s) : r, s \in S, rs = t\} \qquad (t \in S)$$

*where we take $(f * g)(t) = 0$ when there are no elements $r, s \in S$ with $rs = t$. It is easy to verify that $(\ell^1(S), \|\cdot\|_1, *)$ is a Banach algebra; it is called the **semigroup algebra** of $S$, and $*$ is the convolution product. A semigroup algebra $\ell^1(S)$ is commutative if and only if $S$ is Abelian.*

*d) Let $\mathcal{F} = \mathbb{C}[[X]]$ be the algebra of all formal sums of the form*

$$\sum_{n=0}^{\infty} \alpha_n X^n,$$

where $\alpha_0, \alpha_1, \cdots \in \mathbb{C}$ and where the product is determined by the rule that $X^m \cdot X^n = X^{m+n}$ for all $m, n \in \mathbb{Z}_+$. Then $\mathcal{F}$ is a commutative algebra with an identity. Indeed, the definition for the sum and product of $a = \sum_{n=0}^{\infty} \alpha_n X^n$ and $b = \sum_{n=0}^{\infty} \beta_n X^n$ in $\mathcal{F}$ are

$$a + b = \sum_{n=0}^{\infty} (\alpha_n + \beta_n) X^n, \qquad a * b = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} \alpha_k \beta_{n-k}) X^n;$$

note that the inner sum in the formula for the product is finite sum, despite the fact that elements of $\mathcal{F}$ are infinite sums, and so the product is well defined. When we set

$$\ell^1(\mathbb{Z}_+) = \{a = \sum_{n=0}^{\infty} \alpha_n X^n \in \mathcal{F} : \|a\|_1 = \sum_{n=0}^{\infty} |\alpha_n| < \infty\}$$

It can easily be checked that $(\ell^1(\mathbb{Z}_+), \|\cdot\|_1)$ is a Banach algebra of power series. It is a semigroup algebra on $\mathbb{Z}_+$ by example (c).

**Remark 1.2.3.** Not that for a set $S$ with discrete topology, $c_0 0(S) = \{x = (x_n) \in S : x_n \neq 0$ for finite number of $n\}$.

**Theorem 1.2.4.** Let $S$ be a set with discrete topology, and let $\varphi \in c_0(S)$. Then there is a sequence $(\varphi^{(m)})_{m \in \mathbb{N}}$ that for every $m$, $supp(\varphi^{(m)})$ is finite and $\|\varphi^{(m)} - \varphi\|_{\infty} \to 0$ as $m \to \infty$. (In other word $\overline{c_{00}(S)} = c_0(S)$)

*Proof.* Let $\varphi \in c_0(S)$. Hence, for every $m \in \mathbb{N}$, $S_m = \{n \in \mathbb{N} : |\varphi_n| \geq \frac{1}{m}\}$ is finite. Get $g_m = 1$ on $S_m$ and $g_m = 0$ outside of $S_m$ and define $\varphi^{(m)} = g_m \varphi$. So for every $m$, $\varphi^{(m)} \in c_{00}(S)$ and $\|\varphi^{(m)} - \varphi\|_{\infty} \leq \frac{1}{m}$. Hence $\|\varphi^{(m)} - \varphi\|_{\infty} \to 0$ as $m \to \infty$. $\square$

## 1.3 Modules on Banach Algebras and Derivations

**Definition 1.3.1.** Let $A$ be an algebra on field $\mathbb{F}$ and $M$ be a linear space on $\mathbb{F}$. $M$ is a **left** $A$-**module**, if modular product, $A \times M \longrightarrow M$ with $(a, m) \mapsto am$ satisfies:
a) For every constant $a \in A$, mapping $\alpha : M \to M$ with $\alpha(m) = am$ be linear on $M$.
b) For every constant $m \in M$, mapping $\beta : A \to M$ with $\beta(a) = am$ be linear on $A$.
c) For every $a_1, a_2 \in A$ and every $m \in M$,

$$a_1(a_2 m) = (a_1 a_2) m.$$

Similarly, we can define **right** $A$-**module**. $M$ is an $A$-**bimodule**, when it is right $A$-module and left $A$-module and $a(mb) = (am)b$ for all $a, b \in A, m \in M$.

When $A$ is a Banach algebra and $X$ is a Banach space that is also left $A$-module, we call $X$ a **left Banach** $A$**-module**, when

$$\|am\| \leq \|a\| \, \|m\| \qquad (a \in A, m \in X).$$

Similarly we define right Banach $A$-module and Banach $A$-bimodule.

In continuance, we will mention some concepts about derivations, specially bounded derivations on Banach algebras and their modules.

**Definition 1.3.2.** *If $A$ is a Banach algebra and $X$ is a Banach $A$-bimodule, a (bounded)* **derivation** *from $A$ to $X$, is a (bounded) linear map $D : A \to X$, such that*

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A). \tag{1.3.1}$$

**Definition 1.3.3.** *Let $A$ be a Banach algebra Then:*

*(i) If $X$ is a left Banach $A$-module, then $X^*$ becomes a right Banach $A$-module through*

$$\langle x, \varphi \cdot a \rangle := \langle a \cdot x, \varphi \rangle \qquad (a \in A, x \in X, \varphi \in X^*)$$

*(ii) If $X$ is a right Banach $A$-module, then $X^*$ becomes a left Banach $A$-module through*

$$\langle x, a \cdot \varphi \rangle := \langle x \cdot a, \varphi \rangle \qquad (a \in A, x \in X, \varphi \in X^*)$$

*(iii) If $X$ is a Banach $A$-bimodule, then $X^*$ equipped with the left and right module actions of $A$ from (i) and (ii), respectively, is a Banach $A$-bimodule.*

**Definition 1.3.4.** *The Derivation $D$ is called* **compact** *if $D$ is a compact operator between the Banach spaces $A$ and $X$, and* **weakly compact** *if $D$ is a weakly compact operator from $A$ to $X$ (i.e. $D(B_1)$ is relatively weakly compact in $X$, where $B_1$ is the unit ball of $A$. [11])*

# DERIVATIONS ON $\ell^1(\mathbb{Z}_+)$

## 2.1 Introduction

In this chapter we want to study derivations from a specific Banach algebra to its dual space. We study the Banach algebra

$$\ell^1(\mathbb{Z}_+) := \{\sum_{n \geq 0} a_n t^n : \quad \sum_{n \geq 0} |a_n| < \infty\}$$

that has been introduced at Example 1.2.2 part (d) and derivations on it. In the following, compact and weakly compact derivations on $\ell^1(\mathbb{Z}_+)$ will be discussed. Then the concept of *Translation finite sets* will be introduced in Definition 2.4.4 and study its relation with weakly compact operators on $\ell^1(\mathbb{Z}_+)$ will be studied in Theorem 2.4.13, Corollary 2.4.17 and Theorem 2.4.19.

## 2.2 Derivations on $\ell^1(\mathbb{Z}_+)$

For a Banach algebra $A$, derivations on $A$ are mapped into some Banach $A$-bimodule. (See Definition 1.3.2). One natural example of a Banach $A$-bimodule is $A$ itself, where the module actions are just the multiplication. Hence one could study derivations from $A$ into $A$. However, as we see in the following proposition, this becomes trivial when $A = \ell^1(\mathbb{Z}_+)$.

**Proposition 2.2.1.** *Let $D : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^1(\mathbb{Z}_+)$ be a bounded derivation. Then $D = 0$.*

*Proof.* Let $h = D(t)$, then by (Equation 1.3.1) and induction we have

$$
\begin{aligned}
D(t^n) = D(t^{n-1} \cdot t) &= D(t^{n-1}) \cdot t + t^{n-1} \cdot D(t) \\
&= D(t^{n-2}) \cdot t^2 + 2t^{n-1} \cdot D(t) \\
&= \ldots \\
&= D(t^0) \cdot t^n + nt^{n-1} \cdot D(t) \\
&= nt^{n-1} D(t) = nt^{n-1} h.
\end{aligned}
$$

So if $h = \sum_{j=0}^{\infty} b_j t^j$ as an element of $\ell^1(\mathbb{Z}_+)$, then

$$
\begin{aligned}
\|D(t^n)\|_1 = \left\| nt^{n-1} h \right\|_1 &= \left\| nt^{n-1} \sum_{j=0}^{\infty} b_j t^j \right\|_1 \\
&= \left\| n \sum_{j=0}^{\infty} b_j t^{n-1+j} \right\|_1 \\
&= n \left\| \sum_{j=0}^{\infty} b_j t^{n-1+j} \right\|_1 \\
&= n \sum_{j \in \mathbb{Z}_+} |b_j| = n \|h\|_1.
\end{aligned}
$$

So $\|h\|_1 = \frac{1}{n} \|D(t^n)\|_1 \le \frac{1}{n} \|D\|_1$, for all $n$. Hence $\|h\|_1 = 0$. Then $h = 0$, means $D(t^n) = 0$, for all $n$. Now since the span of all $t^n$ is dense in $\ell^1(\mathbb{Z}_+)$ and $D$ is continuous, we have $D = 0$. $\qquad \square$

Proposition 2.2.1 is a special case of a more general result proved by Singer and Wermer [23].

Another natural example of a Banach bimodule is the dual of the algebra, which in the case of $\ell^1(\mathbb{Z}_+)$, is $\ell^\infty(\mathbb{Z}_+)$ as we show below:

**Definition 2.2.2.** $\ell^\infty(\mathbb{Z}_+)$ *becomes a Banach* $\ell^1(\mathbb{Z}_+)$*-module by Definition 1.3.3 through:*

$$
\begin{aligned}
\langle f, \psi \cdot h \rangle &:= \langle h * f, \psi \rangle \\
&= \langle \sum_{n=0}^{\infty} (\sum_{i=0}^{n} h_i f_{n-i}) t^n, \psi \rangle \\
&= \sum_{n=0}^{\infty} (\sum_{i=0}^{n} h_i f_{n-i}) \psi_n \qquad (f, h \in \ell^1(\mathbb{Z}_+), \psi \in \ell^\infty(\mathbb{Z}_+))
\end{aligned}
$$

when $\cdot$ is the module product of $\ell^1$ and $\ell^\infty$ but $*$ is the product of $\ell^1$ as a Banach algebra. Also for each $k \in \mathbb{Z}_+$ and $\sum f_i t^i = t^k$, $f_i = 1$ when $i = k$ and elsewhere $f_i = 0$. Then we have

$$
\begin{aligned}
(\psi \cdot h)_k = \langle t^k, \psi \cdot h \rangle &= \langle h * t^k, \psi \rangle \\
&= \langle \sum_{n=0}^{\infty} (\sum_{i=0}^{n} h_i f_{n-i}) t^n, \psi \rangle \\
&= \sum_{n=k}^{\infty} h_{n-k} \psi_n \\
&= \sum_{n=0}^{\infty} h_n \psi_{n+k}
\end{aligned}
$$

for $h \in \ell^1(\mathbb{Z}_+), \psi \in \ell^\infty(\mathbb{Z}_+)$.

**Remark 2.2.3.** *For a function $D : A \to A^*$, and arbitrary $f, g, h \in A$ we say that*

$$
D(f \cdot g)(h) = D(f)(g \cdot h) + D(g)(h \cdot f)
$$

*is **Derivation Identity**.*

**Lemma 2.2.4.** *Every bounded linear map $T : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ can be identified with a bounded bilinear form $\ell^1(\mathbb{Z}_+) \times \ell^1(\mathbb{Z}_+) \longrightarrow \mathbb{C}$ which sends $(f, g)$ to $T(f)(g)$ and $\|T\| = \sup\{|T(f)(g)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\}$.*

*Proof.* Define the linear mapping $\Omega : B(\ell^1(\mathbb{Z}_+), \ell^\infty(\mathbb{Z}_+)) \longrightarrow Bil(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+); \mathbb{C})$ with $\Omega(T) = \varphi_T$, for all $T \in B(\ell^1(\mathbb{Z}_+), \ell^\infty(\mathbb{Z}_+))$ such that $\varphi_T(f, g) = T(f)(g)$ for all $f, g \in \ell^1(\mathbb{Z}_+)$. It is isometry since

$$
\begin{aligned}
\|T\| &= \sup\{\|T(f)\| : f \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1\} \\
&\leq \sup\{|T(f)(g)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \sup\{|\varphi_T(f, g)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \|\varphi_T\|_\infty.
\end{aligned}
$$

On the other hand, for $f, g \in \ell^1(\mathbb{Z}_+)$ with $\|f\|_1, \|g\|_1 \leq 1$, we have $|\varphi_T(f, g)| = |T(f)(g)| \leq \sup |T(f)| \leq \|T\|$. So $\sup |\varphi_T(f, g)| \leq \|T\|$. Then we have $\|\varphi_T\|_\infty \leq \|T\|$. And hence $\|\varphi_T\|_\infty = \|T\|$.

Also $\Omega$ is surjective since if you let $\varphi \in Bil(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+); \mathbb{C})$, then define $T_\varphi : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ such that for all $f \in \ell^1(\mathbb{Z}_+)$,

$$
T_\varphi(f) = \varphi(f, -)
$$

that is a bounded linear function on $\ell^1(\mathbb{Z}_+)$ and so it is in $\ell^\infty(\mathbb{Z}_+)$. So for all $g \in \ell^1(\mathbb{Z}_+)$ we have :

$$T_\varphi(f)(g) = \varphi(f, -)(g) = \varphi(f, g)$$

and now we show that $\Omega(T_\varphi) = \varphi$:

$$\Omega(T_\varphi)(f, g) = T_\varphi(f)(g) = \varphi(f, g).$$

So $\Omega$ is surjective. In addition, $\Omega$ is injective, since it is an isometry between normed vector spaces.

$\square$

**Theorem 2.2.5.** *For every bounded derivation* $D : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ *there is a* $\psi \in \ell^\infty(\mathbb{Z}_+)$, *such that*

$$\psi_0 = 0, \quad D(t^j)(t^k) = \frac{j}{j+k}\psi_{j+k} \qquad (j, k \in \mathbb{Z}_+, j \neq 0)$$

*Proof.* By following the definition of a derivation we have $D(1) = D(t^0) = 0$ and

$$
\begin{aligned}
D(t^n) &= D(t * t^{n-1}) \\
&= t \cdot D(t^{n-1}) + t^{n-1} \cdot D(t) \\
&= t^2 \cdot D(t^{n-2}) + 2t^{n-1} \cdot D(t) \\
&= \dots \\
&= nt^{n-1} \cdot D(t).
\end{aligned}
$$

So by Lemma 2.2.4

$$D(t^j)(t^k) = (jt^{j-1} \cdot D(t))(t^k) = D(t)(jt^{k+j-1}).$$

Hence

$$D(t)(t^{k+j-1}) = \frac{1}{j}D(t^j)(t^k) \tag{2.2.1}$$

Now let $\psi : \ell^1(\mathbb{Z}_+) \longrightarrow \mathbb{C}$ be the linear functional defined by $\psi(f) = D(f)(t^0)$, so $\psi_0 = D(t^0)(t^0) = 0$ and

$$D(t^{j+k})(t^0) = (j+k)D(t)(t^{j+k-1}) = \frac{j+k}{j}D(t^j)(t^k) \quad (j \neq 0).$$

We used ( 2.2.1) in last equation. Hence

$$D(t^j)(t^k) = \frac{j}{j+k}D(t^{j+k})(t^0) = \frac{j}{j+k}\psi(t^{j+k}).$$

Finally, $|\psi(f)| \leq \|D\| \|f\|_1$, for all $f \in \ell^1(\mathbb{Z}_+)$. Hence, $\psi \in \ell^\infty(\mathbb{Z}_+)$. $\square$

14

The set of all derivations from a Banach algebra, $A$, to its dual, $A^*$, is noted by $Der(A, A^*)$ and we name the set of all members of $\ell^\infty(\mathbb{Z}_+)$, like $\psi$, that $\psi(t^0) = 0$, set $E = \{\psi \in \ell^\infty(\mathbb{Z}_+) : \psi(t^0) = 0\}$.

**Theorem 2.2.6.** *There is a bijective linear isometry*

$$\text{B} : Der(\ell^1(\mathbb{Z}_+), \ell^\infty(\mathbb{Z}_+)) \longrightarrow E$$

*which defines with $B(D) = \psi_D$, where $\psi_D \in \ell^\infty(\mathbb{Z}_+)$ is the associated element defined in Theorem 2.2.5.*

*Proof.* By the previous theorem this map is well-defined and linear. It is isometry since for every $f, g \in \ell^1(\mathbb{Z}_+), f = \sum_j f_j t^j, g = \sum_k g_k t^k, j, k \in \mathbb{Z}_+$, we have

$$
\begin{aligned}
\|D\| &= \sup\{\|D(f)\| : f \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1\} \\
&\leq \sup\{|D(f)(g)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&= \sup\{|\sum_{j,k=0}^{\infty} f_j g_k D(t^j)(t^k)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&= \sup\{|\sum_{j,k\in\mathbb{Z}_+, j,k\neq 0} f_j g_k \frac{j}{j+k} \psi_D(t^{j+k})| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \sup\{\sum_{j,k\in\mathbb{Z}_+, j,k\neq 0} |f_j g_k| |\frac{j}{j+k} \psi_D(t^{j+k})| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \sup\{\sum_{j,k=0}^{\infty} |f_j g_k| \|\psi_D\|_\infty : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \|\psi_D\|_\infty \sup\{\sum_{j,k=0}^{\infty} |f_j g_k| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \|\psi_D\|_\infty \|f\|_1 \|g\|_1 \\
&\leq \|\psi_D\|_\infty.
\end{aligned}
$$

On the other hand,
$$|\psi_D(t^n)| = |D(t^n)(t^0)| \leq \sup |D(t^n)| \leq \|D\|$$

so
$$\sup_{n\in\mathbb{Z}_+} |\psi_D(t^n)| \leq \|D\|$$

then $\|\psi_D\|_\infty \leq \|D\|$. Hence we can conclude $\|\psi_D\|_\infty = \|D\|$.
Also $B$ is injective, since if $B(D) = 0$, then since $\|D\| = \|\psi_D\|_\infty = 0$, we have $\|D\| = 0$ and

it means $D = 0$.

$B$ is surjective since if we let $\varphi \in E$ arbitrary, define a function $D_\varphi$ by $D_\varphi(t^j)(t^k) := \frac{j}{j+k}\varphi_{j+k}$, when $j, k \in \mathbb{Z}_+, j + k \neq 0$ and define $D_\varphi(t^0)(t^0) = 0$. For general $f, g \in \ell^1(\mathbb{Z}_+)$ we have $f = \sum_{j=0}^\infty f_j t^j$ and $g = \sum_{k=0}^\infty g_k t^k$. So

$$D_\varphi(f)(g) = \sum_{j,k=0}^\infty f_j g_k D_\varphi(t^j)(t^k).$$

This is well-defined since $(D_\varphi(t^j))_{j \in \mathbb{Z}_+}$ is a bounded sequence in $\ell^\infty(\mathbb{Z}_+)$ :

$$\begin{aligned}
\left\|D_\varphi(t^j)\right\| &= \sup\{|D_\varphi(t^j)(t^k)| : j, k \in \mathbb{Z}_+, j + k \neq 0\} \\
&= \sup\{|\frac{j}{j+k}\varphi_{j+k}| : j, k \in \mathbb{Z}_+, j + k \neq 0\} \\
&\leq \sup\{|\varphi_{j+k}| : j, k \in \mathbb{Z}_+, j + k \neq 0\} \\
&\leq \|\varphi\|_\infty < \infty
\end{aligned}$$

That argument is for $j \geq 1$ and note that by definition of $D_\varphi$, $D_\varphi(t^0)(t^k) = 0$ for all $k \in \mathbb{Z}_+$. We now show that $D_\varphi$ is bounded. Consider every $f, g \in \ell^1(\mathbb{Z}_+), f = \sum_j f_j t^j, g = \sum_k g_k t^k, j, k \in \mathbb{Z}_+$, then:

$$\begin{aligned}
\|D_\varphi\| &= \sup\{\|D_\varphi(f)\| : f \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1\} \\
&\leq \sup\{|D_\varphi(f)(g)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&= \sup\{|\sum_{j,k=0}^\infty f_j g_k D_\varphi(t^j)(t^k)| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&= \sup\{|\sum_{j,k\in\mathbb{Z}_+,j,k\neq 0} f_j g_k \frac{j}{j+k}\varphi_{j+k}| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \sup\{\sum_{j,k\in\mathbb{Z}_+,j,k\neq 0} |f_j g_k| |\frac{j}{j+k}\varphi_{j+k}| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \sup\{\sum_{j,k=0}^\infty |f_j g_k| \|\varphi\|_\infty : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \|\varphi\|_\infty \sup\{\sum_{j,k=0}^\infty |f_j g_k| : f, g \in \ell^1(\mathbb{Z}_+), \|f\|_1 \leq 1, \|g\|_1 \leq 1\} \\
&\leq \|\varphi\|_\infty \|f\|_1 \|g\|_1 \\
&\leq \|\varphi\|_\infty
\end{aligned}$$

Now we show $D_\varphi$ is a derivation, suppose $j, k, \ell \in \mathbb{N}, j + k + \ell \neq 0$. Then

$$D_\varphi(t^j * t^k)(t^\ell) = D_\varphi(t^{j+k})(t^\ell) = \frac{j+k}{j+k+\ell}\varphi_{j+k+\ell}$$

16

and

$$D_\varphi(t^j)(t^k * t^\ell) = D_\varphi(t^j)(t^{k+\ell}) = \frac{j}{j+k+\ell}\varphi_{j+k+\ell}$$

and

$$D_\varphi(t^k)(t^\ell * t^j) = D_\varphi(t^k)(t^{j+\ell}) = \frac{k}{j+k+\ell}\varphi_{j+k+\ell}$$

So

$$(D_\varphi t^j)(t^k * t^\ell) + (D_\varphi t^k)(t^\ell * t^j) = \frac{j+k}{j+k+\ell}\varphi_{j+k+\ell} = D_\varphi(t^j * t^k)(t^\ell).$$

Also when $j + k + \ell = 0$,

$$D_\varphi(t^0 * t^0)(t^0) = 0$$

and

$$D_\varphi(t^0)(t^0 * t^0) = 0$$

So this holds for $j, k, \ell \in \mathbb{Z}_+$. Since $\ell$ is arbitrary, we have

$$D_\varphi(t^j * t^k) = (D_\varphi t^j) \cdot t^k + t^j \cdot (D_\varphi t^k) \qquad (2.2.2)$$

Now for arbitrary $a, b \in \ell^1(\mathbb{Z}_+)$ we have $a = \sum_{j=0}^{\infty} a_j t^j$ and $b = \sum_{k=0}^{\infty} b_k t^k$. So

$$D_\varphi(a * b) = \sum_{j,k=0}^{\infty} a_j b_k D_\varphi(t^j * t^k)$$

and

$$(D_\varphi a) \cdot b = (\sum_{j=0}^{\infty} a_j D_\varphi(t^j)) \cdot \sum_{k=0}^{\infty} b_k t^k = \sum_{j,k=0}^{\infty} a_j b_k D_\varphi(t^j) \cdot t^k$$

and

$$a \cdot (D_\varphi b) = (\sum_{j=0}^{\infty} a_j t^j) \cdot (\sum_{k=0}^{\infty} b_k D_\varphi(t^k)) = \sum_{j,k=0}^{\infty} a_j b_k t^j \cdot D_\varphi(t^k).$$

So by ( 2.2.2)

$$\sum_{j,k=0}^{\infty} a_j b_k D_\varphi(t^j * t^k) = \sum_{j,k=0}^{\infty} a_j b_k (D_\varphi(t^j) \cdot t^k + t^j \cdot D_\varphi(t^k))$$

and it means for arbitrary $a, b \in \ell^1(\mathbb{Z}_+)$

$$D_\varphi(a * b) = (D_\varphi a) \cdot b + a \cdot D_\varphi b$$

So $D_\varphi$ is a derivation from $\ell^1(\mathbb{Z}_+)$ to $\ell^\infty(\mathbb{Z}_+)$. The last thing is to show $B(D_\varphi) = \varphi$. Let $f = t^m$ for $m \in \mathbb{Z}_+$ and $m \neq 0$,

$$B(D_\varphi)(t^m) = D_\varphi(t^m)(t^0) = \frac{m}{m+0}\varphi(t^{m+0}) = \varphi(t^m)$$

17

and for $m = 0$ we have
$$B(D_\varphi)(t^0) = D_\varphi(t^0)(t^0) = 0 = \varphi(t^0).$$

This means that $B(D_\varphi)$ and $\varphi$ has the same coefficient as the elements of $\ell^\infty(\mathbb{Z}_+)$, so $B(D_\varphi) = \varphi$ and by proving this, B is an isometric isomorphism. $\qquad \square$

## 2.3 Compact derivations from $\ell^1(\mathbb{Z}_+)$ to its dual

Now we are ready to study compact and weakly compact derivations, $D : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$. For convenience we write $c_0 := c_0(\mathbb{N})$ and $\ell^\infty := \ell^\infty(\mathbb{N})$. We also consider $c_0$ and $\ell^\infty$ as closed linear subspaces of $c_0(\mathbb{Z}_+)$ and $\ell^\infty(\mathbb{Z}_+)$, respectively, consisting of the sequences $\{a_n\}_{n=0}^\infty$ with $a_0 = 0$.

By the following theorems we conclude that the space of compact derivations from $\ell^1(\mathbb{Z}_+)$ to $\ell^\infty(\mathbb{Z}_+)$ is linearly isomorphic to $c_0$.

**Lemma 2.3.1.** *If $\psi = \delta_n$ $(n \geq 1)$, then $D_\psi$ is a finite-rank linear map.*

*Proof.* We need to show that the linear span of $\{D_\psi(t^j) : j \in \mathbb{Z}_+\}$ has a finite dimension. Let $x_j := D_\psi(t^j)$. Also we know $(D_\psi(t^j))_k = D_\psi(t^j)(t^k) = \frac{j}{j+k}\psi_{j+k}$ when $j + k \neq 0$ and $D_\psi(t^0)(t^0) = 0$, so
$$(x_j)_k = \begin{cases} \frac{j}{j+k} = \frac{j}{n} & \text{if } j + k = n \\ 0 & \text{if } j + k \neq n \end{cases}$$

Hence, if $j > n$, $x_j = 0$. So just for finite numbers of $j$, $x_j$ is nonzero and it means that the linear span of $\{D_\psi(t^j) : j \in \mathbb{Z}_+\}$ has finite dimension. Hence $D_\psi$ is a finite rank operator. $\quad \square$

**Theorem 2.3.2.** *Let $\psi \in c_0$. Then the bounded derivation $D_\psi : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ is compact.*

*Proof.* Let $\psi \in c_0$, since $c_0 = \overline{c_{00}}^{\|\cdot\|_\infty}$, there exists a sequence $(\psi_k) \subset c_{00}$ such that $\|\psi_k - \psi\|_\infty \to 0$ when $k \to \infty$. Now for each $k \in \mathbb{N}$, since $\psi_k \in c_{00}$, we have
$$\psi_k = \sum_{n=1}^m a_n \delta_n$$

where $a_1, a_2, \ldots, a_m \in \mathbb{C}$ for some $m$. So by Theorem 2.2.6, $B^{-1}(\psi_k) = \sum_{n=1}^m a_n D_{\delta_n}$, where $B$ is the linear map defined in Theorem 2.2.6. Hence by Lemma 2.3.1 and Remark 1.1.16, $D_{\psi_k}$ is finite rank for all $k \in \mathbb{N}$. Also by Theorem 2.2.5, $\|D_{\psi_k} - D_\psi\| \to 0$ as $k \to \infty$, and

because the limit of every sequences of finite rank operators is compact by Lemma 1.1.17, $D_\psi$ is a compact derivation. $\qquad\square$

By the next theorem we prove converse of the above theorem:

**Theorem 2.3.3.** *If the bounded derivation $D_\psi : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ is compact, then $\psi \in c_0$.*

*Proof.* This argument is based on Heath's thesis (Proposition 2.6., [15]). Let $D_\psi : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ be a compact derivation such that $\psi \in \ell^\infty \setminus c_0$. We should show that the sequence $(D_\psi(t^k))_{k\in\mathbb{N}}$ has a subsequence with no convergent sub-subsequence. Without loss of generality, we assume that $\|D_\psi\| = 1$. Since $\psi \notin c_0$, so there exists $\varepsilon > 0$ such that $\{n \in \mathbb{N} : |\psi_n| \geq \varepsilon\}$ is infinite. Hence there exist an infinite set $S \subset \mathbb{N}$ such that for all $n \in S$, $|D_\psi(t^n)(1)| = |\frac{n}{n+0}\psi_n| = |\psi_n| > \varepsilon$. Let $k, l \in \mathbb{N}$, then by Theorem 2.2.5

$$|D_\psi(t^k)(t^l)| = \frac{k}{k+l}|D_\psi(t^{k+l})(1)| \leq \frac{k}{k+l} \qquad (2.3.3)$$

Now suppose that $k + l \in S$. Then

$$\begin{aligned}
|D_\psi(t^k)(t^l)| &= |kt^{k+l-1} \cdot D_\psi(t)(1)| \\
&= \frac{k}{k+l}|D_\psi(t^{k+l})(1)| \geq \frac{\varepsilon k}{k+l} \qquad (2.3.4)
\end{aligned}$$

Let $j_1 = 1$. Suppose $j_1 < \cdots < j_{k-1}$ such that for all $i, i' \in \mathbb{N}$ with $i < i' \leq k - 1$, $\|D_\psi(t^{j_i}) - D_\psi(t^{j_{i'}})\| > \frac{\varepsilon}{4}$. Now choose $N \in S$ with $N > 10\varepsilon^{-1}j_{k-1}$, and let $l_k = \lfloor\frac{N}{2}\rfloor$ and $j_k = N - l_k$. Then, by ( 2.3.4),

$$\begin{aligned}
|D_\psi(t^{j_k})(t^{l_k})| &\geq \frac{\varepsilon j_k}{j_k + l_k} \\
&= \varepsilon(\frac{N - l_k}{N}) \\
&= \varepsilon(1 - \frac{l_k}{N}) \\
&= \varepsilon(1 - \frac{\lfloor\frac{N}{2}\rfloor}{N}) \\
&\geq \varepsilon(1 - \frac{1}{2}) \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

19

Also, if $m \leq j_{k-1}$, then by ( 2.3.3)

$$
\begin{aligned}
|D_\psi(t^m)(t^{l_k})| &\leq \frac{m}{m + l_k} \leq \frac{j_{k-1}}{j_{k-1} + l_k} \\
&= \frac{j_{k-1}}{j_{k-1} + \lfloor \frac{N}{2} \rfloor} \\
&\leq \frac{j_{k-1}}{j_{k-1} + \lfloor \frac{10\varepsilon^{-1} j_{k-1}}{2} \rfloor} \\
&= \frac{1}{1 + \lfloor 5\varepsilon^{-1} \rfloor} \\
&\leq \frac{1}{1 + (5\varepsilon^{-1} - 1)} \\
&\leq \frac{1}{5\varepsilon^{-1}} \\
&= \frac{\varepsilon}{5}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|D_\psi(t^m)(t^{l_k}) - D_\psi(t^{j_k})(t^{l_k})| &> ||D_\psi(t^m)(t^{l_k})| - |D_\psi(t^{j_k})(t^{l_k})|| \\
&= ||D_\psi(t^{j_k})(t^{l_k})| - |D_\psi(t^m)(t^{l_k})|| \\
&> |\frac{\varepsilon}{2} - \frac{\varepsilon}{5}| \\
&= |\frac{3\varepsilon}{10}| \\
&> \frac{\varepsilon}{4}.
\end{aligned}
$$

In particular, if $i < k$, then $\|D_\psi(t^{j_i}) - D_\psi(t^{j_k})\| > \frac{\varepsilon}{4}$. Hence, by induction, we obtain a sequence, $(j_i)_{i \in \mathbb{N}}$, such that, if $i, k \in \mathbb{N}$ and $i \neq k$ then $\|D_\psi(t^{j_i}) - D_\psi(t^{j_k})\| > \frac{\varepsilon}{4}$. Thus $(D_\psi(t^{j_i}))_{i \in \mathbb{N}}$ has no convergent subsequence, and so, $D_\psi$ is not compact. □

Therefor, by Theorem 2.3.2 and Theorem 2.3.3 we conclude that $D_\psi$ is compact if and only if $\psi \in c_0$.

## 2.4 TF-sets and weakly compact derivations

In this section we introduce concepts "$T$-set, $TF$-set and $TFc_0$" and study their relation with weakly compact operators. Note that, the results of this section were proved by Choi and Heath [6], but we present different proofs that are based on suggestions of M. Daws (personal communication with Y. Choi).

**Definition 2.4.1.** *Let $S \subseteq \mathbb{Z}_+$. We say that $S$ is **translation set** (T-set for short) if, for all $k \in \mathbb{Z}_+$, $S \cap (S - k)$ is finite or empty.*

**Theorem 2.4.2.** *Let $x_1 < x_2 < x_3 < \ldots$ is a sequence in $\mathbb{N}$, and $x_{n+1} - x_n \to \infty$ as $n \to \infty$. Then $S = \{x_n : n \in \mathbb{N}\}$ is a T-set.*

*Proof.* If $S$ is not a T-set, then there is a $k \in \mathbb{N}$ such that $S \cap (S - k)$ is infinite. Choose an increasing subsequence $n(1) < n(2) < n(3) < \ldots$ such that $\{x_{n(j)} : j \in \mathbb{N}\} = S \cap (S - k)$. For each $j \in \mathbb{N}$,

$$x_{n(j)} + k = x_{m(j)} \qquad \text{for some } m(j).$$

Then $x_{m(j)} - x_{n(j)} = k$ and $1 + n(j) \leq m(j)$, so

$$x_{n(j)+1} - x_{n(j)} \leq x_{m(j)} - x_{n(j)} = k$$

and this contradicts with the fact that $x_{n(j)+1} - x_{n(j)} \to \infty$ for large $j$. Hence $S$ is a T-set. $\square$

Converse of the above theorem is also true:

**Theorem 2.4.3.** *Let $x_1 < x_2 < x_3 < \ldots$ be a sequence in $\mathbb{N}$, and $S = \{x_n : n \in \mathbb{N}\}$ a T-set. Then $x_{n+1} - x_n \to \infty$ as $n \to \infty$.*

*Proof.* Suppose that $\{x_{n+1} - x_n\}_{n \in \mathbb{N}}$ does not converge to infinity as $n$ grows. Hence there is an integer $k \in \mathbb{N}$ and an infinite subsequence $\{x_{n_i}\}_{i=1}^{\infty} \subseteq S$ such that $x_{n_i+1} - x_{n_i} \leq k$ for all $i \in \mathbb{N}$. Moreover, if we let $k$ to be the minimum positive integer with the above property, then we can assume that $x_{n_i+1} - x_{n_i} = k$. In particular, $x_{n_i} = x_{n_i+1} - k \in S \cap (S - k)$ for all $i \in \mathbb{N}$. Which is impossible since $S$ is a T-set. Thus $x_{n+1} - x_n \to \infty$ as $n \to \infty$. $\square$

**Definition 2.4.4.** *Let $S \subseteq \mathbb{Z}_+$. We say that $S$ is **translation-finite** (TF for short) if, for every sequence $n_1 < n_2 < \ldots$ in $\mathbb{Z}_+$, there exists $k$ such that $\bigcap_{i=1}^{k}(S - n_i)$ is finite or empty.*

**Remark 2.4.5.** *Note that every T-sets are TF-set. But the converse is not always true (see Example 2.4.23).*

**Example 2.4.6.** *By Theorem 2.4.2 we can see that $S = \{2^n : n \in \mathbb{N}\}$ is a T-set and so it is a TF-set.*

**Example 2.4.7.** *$\mathbb{Z}_+$ is not a TF-set. Since for the sequence $1 < 2 < 3 < \ldots$,*

$$\mathbb{Z}_+ - 1 = \{x : x + 1 \in \mathbb{Z}_+\} = \mathbb{Z}_+$$

*and similarly for every $k \in \mathbb{Z}_+$, $\mathbb{Z}_+ - k = \mathbb{Z}_+$. So $\bigcap_{i=1}^{k}(\mathbb{Z}_+ - i) = \mathbb{Z}_+$ and this is an infinite set.*

**Definition 2.4.8.** *Let $X$ and $Y$ be non-empty sets, and let*

$$f : X \times Y \longrightarrow \mathbb{C}$$

*be a function. Then*
*i) $f$ **clusters** on $X \times Y$ if*

$$\lim_m \lim_n f(x_m, y_n) = \lim_n \lim_m f(x_m, y_n)$$

*whenever $(x_m)$ and $(y_n)$ are sequences in $X$ and $Y$, respectively, each consisting of distinct points, and both repeated limits exist;*
*ii) $f$ **0-clusters** on $X \times Y$ if*

$$\lim_m \lim_n f(x_m, y_n) = \lim_n \lim_m f(x_m, y_n) = 0$$

*whenever $(x_m)$ and $(y_n)$ are sequences in $X$ and $Y$, respectively, each consisting of distinct points, and both repeated limits exist.( See Definition 3.2, [9])*

The following proposition is a famous condition that was originally proved by Grothendieck in 1952 [14].

**Proposition 2.4.9.** *Let $E$, $F$ be Banach spaces, and suppose $T \in B(E, F)$. Then $T$ is weakly compact if and only if the function*

$$E \times F^* \longrightarrow \mathbb{C} \quad , \quad (x, \lambda) \mapsto \langle Tx, \lambda \rangle$$

*clusters on $E_{(1)} \times F^*_{(1)}$, when $E_{(1)}$ and $F^*_{(1)}$ are respectively closed unit balls of $E$ and $F^*$.*

*Proof.* See proposition 3.4, [9]. □

Now let $X$ and $Y$ be non-empty, locally compact sets and $f : X \times Y \to \mathbb{C}$ be bounded , separately continuous function. For $y \in Y$, we set

$$f_y : x \mapsto f(x, y) \quad , \quad X \longrightarrow \mathbb{C}$$

and we regard $f_y$ as an element of $CB(X) = C(\beta X)$; when $\beta X$ is the Stone-Čech compactification of $X$. We then set

$$f(x, y) := f_y(x) \qquad (x \in \beta X, y \in Y).$$

Also we set $\mathcal{F} = \{f_y : y \in Y\}$.

**Theorem 2.4.10** (Proposition 3.3, [9]). *Let $X$ and $Y$ be non-empty, locally compact spaces and $f : X \times Y \to \mathbb{C}$ be a bounded, separately continuous function. Then $\mathcal{F}$ is relatively weakly compact in $C(\beta X)$ if and only if $f$ clusters on $X \times Y$.*

**Proposition 2.4.11.** *Let $T : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ be a bounded linear operator. Then $T$ is weakly compact if and only if the function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{C}$ with $f(x, y) = \langle T(t^y), t^x \rangle$ clusters on $\mathbb{Z}_+ \times \mathbb{Z}_+$.*

*Proof.* By Daws remarks in [10] after Proposition 5.4, $T$ is weakly compact if and only if the set $\{T(t^j) : j \in \mathbb{Z}_+\}$ is relatively weakly compact, and we know $f_y(x) := f(x, y) = \langle T(t^y), t^x \rangle$. So $\mathcal{F} = \{f_y : y \in \mathbb{Z}_+\} = \{T(t^j) : j \in \mathbb{Z}_+\}$. Then by Theorem 2.4.10, $T$ is weakly compact if and only if $f$ clusters on $\mathbb{Z}_+ \times \mathbb{Z}_+$. $\qquad\square$

**Example 2.4.12.** *Let $D : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ be a derivation associated to $1_\mathbb{N} \in \ell^\infty(\mathbb{Z}_+)$, where $1_\mathbb{N}$ is a characteristic function on $\mathbb{N}$ (See Theorem 2.2.6). Then $D$ is not weakly compact.*

*Proof.* By Theorem 2.2.5 and Theorem 2.2.6, $D(\delta_j)_k = \frac{j}{j+k}$ for $j, k \in \mathbb{Z}_+$ and $j + k \neq 0$ and $D(\delta_0) = 0$. Now let $(j_n), (k_m)$ be increasing sequences in $\mathbb{N}$. Then

$$\lim_m \lim_n D(\delta_{j_n})_{k_m} = \lim_m \lim_n \frac{j_n}{j_n + k_m} = \lim_m 1 = 1$$

and

$$\lim_n \lim_m D(\delta_{j_n})_{k_m} = \lim_n \lim_m \frac{j_n}{j_n + k_m} = \lim_n 0 = 0$$

So $\lim_m \lim_n D(\delta_{j_n})_{k_m}$ and $\lim_n \lim_m D(\delta_{j_n})_{k_m}$ both exist but are not equal. Hence by Definition 2.4.8 and Proposition 2.4.11, $D$ is not weakly compact. $\qquad\square$

Example 2.4.12 shows that not every derivation from $\ell^1(\mathbb{Z}_+)$ to $\ell^\infty(\mathbb{Z}_+)$ is weakly compact.

**Theorem 2.4.13.** *Let $\psi \in \ell^\infty(\mathbb{N})$. Then $D_\psi$ is weakly compact if $Supp(\psi)$ is a TF-set.*

*Proof.* Define $D_\psi : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ with $D_\psi(t^j)(t^k) := \frac{j}{j+k}\psi_{j+k}$ for $j, k \in \mathbb{Z}_+$ and $j + k \neq 0$, and $D_\psi(t^0)(t^0) = 0$. Let $\{x_n\}, \{y_m\} \subseteq \mathbb{Z}_+$ are sequences of distinct points and both repeated limits

$$\lim_n \lim_m D_\psi(t^{x_n})(t^{y_m}), \quad \lim_m \lim_n D_\psi(t^{x_n})(t^{y_m})$$

exist. Note that

$$\lim_n \lim_m D_\psi(t^{x_n})(t^{y_m}) = \lim_n \lim_m \frac{x_n}{x_n + y_m}\psi_{x_n+y_m} = 0.$$

23

So by Proposition 2.4.11, $D_\psi$ is weakly compact if

$$\lim_m \lim_n \frac{x_n}{x_n + y_m} \psi_{x_n+y_m} = \lim_m \lim_n \psi_{x_n+y_m} = 0.$$

By contradiction suppose the limit is not zero, so there is increasing subsequences $\{x_{n(j)}\}$ of $\{x_n\}$ and $\{y_{m(k)}\}$ of $\{y_m\}$ in $\mathbb{Z}_+$ that

$$\lim_m \lim_n \psi_{x_n+y_m} = \lim_k (\lim_j \psi_{x_{n(j)}+y_{m(k)}}) = a \neq 0.$$

So there is an $N \in \mathbb{N}$, such that for all $k \in \mathbb{N}$, $k \geq N$

$$|\lim_j \psi_{x_{n(j)}+y_{m(k)}} - a| < \frac{|a|}{2}.$$

By triangular inequality $|\lim_j \psi_{x_{n(j)}+y_{m(k)}}| > \frac{|a|}{2}$. If we fix $k \geq N$, then there is a $J(k) \in \mathbb{N}$ that for all $j \geq J(k)$

$$|\psi_{x_{n(j)}+y_{m(k)}}| > \frac{|a|}{4}$$

Since $Supp(\psi) = \{p \in \mathbb{N} : |\psi_p| > 0\}$, so for all $j \geq J(k)$, $x_{n(j)} + y_{m(k)} \in Supp(\psi)$. Hence for all $j \geq J(k)$, $x_{n(j)} \in Supp(\psi) - y_{m(k)}$. Given $M$, consider $Supp(\psi) - y_{m(k+N)}$ for $1 \leq k \leq M$ and let $R = \max(j(N+1), j(N+2), \ldots, j(N+M))$. Then for every $j \geq R$ and $1 \leq k \leq M$, $x_{n(j)} \in Supp(\psi) - y_{m(k+N)}$ (See Figure 2.1). Hence $\bigcap_{k=1}^M (Supp(\psi) - y_{m(k+N)})$ contains $\{x_{n(j)} : j \geq R\}$ and this contradicts the fact that $\bigcap_{k=1}^M (Supp(\psi) - y_{m(k+N)})$ is finite or empty for some $M$. So $D_\psi$ is weakly compact.
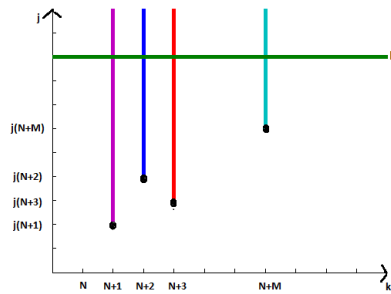


**Figure 2.1:** *For all $k \geq N$, then there is a $J(k) \in \mathbb{N}$ that for all $j \geq J(k)$, $x_{n(j)} \in Supp(\psi) - y_{m(k)}$*

$\square$

In Section 2.3, we showed that compact $D_\psi$ is equivalent with $\psi \in c_0$ means that for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |\psi_n| \geq \varepsilon\}$ is finite and so is a TF-set. Now we want to show that if we have only $D_\psi$ weakly compact, what is it equivalent with? (See Figure 2.2)

**Definition 2.4.14.** *We say $\psi \in TFc_0$ if*

*(i) $\psi \in \ell^\infty(\mathbb{N})$;*

*(ii) for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |\psi_n| \geq \varepsilon\}$ is a TF-set.*

**Example 2.4.15.** *Let $\psi(n) = \frac{1}{n}$, so $\psi \in c_0$, and by Theorem 2.3.2, $D_\psi$ is compact but $Supp(\psi) = \mathbb{N}$ and by Example 2.4.7 it is not TF.*
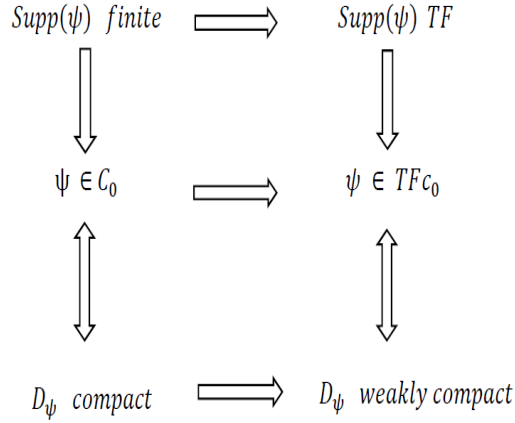
$$
\begin{array}{ccc}
\text{Supp}(\psi) \ \text{finite} & \Longrightarrow & \text{Supp}(\psi) \ \text{TF} \\
\Big\Updownarrow & & \Big\Updownarrow \\
\psi \in C_0 & \Longrightarrow & \psi \in TFc_0 \\
\Big\Updownarrow & & \Big\Updownarrow \\
D_\psi \ \text{compact} & \Longrightarrow & D_\psi \ \text{weakly compact}
\end{array}
$$

**Figure 2.2:** *Weakening compactness of $D_\psi$ to weak compactness, corresponds to changing from $\psi \in c_0$ to only $\psi \in TFc_0$.*

**Lemma 2.4.16.** *Let $\psi \in TFc_0$. Then there is a sequence $(\psi^{(m)})_{m \in \mathbb{N}}$, $\psi^{(m)} \in \ell^\infty$ for every $m$, $Supp(\psi^{(m)})$ is TF and $\left\| \psi^{(m)} - \psi \right\|_\infty \to 0$ as $m \to \infty$.*

*Proof.* Let $\psi \in TFc_0$, so for every $m \in \mathbb{N}$, $S_m = \{n \in \mathbb{N} : |\psi_n| \geq \frac{1}{m}\}$ is TF. Let $g_m = 1$ on $S_m$ and $g_m = 0$ outside of $S_m$. Define $\psi^{(m)} = g_m \psi$, so $\psi^{(m)} \in \ell^\infty$ and $Supp(\psi^{(m)}) = S_m$ and so it is TF. Also $\left\| \psi^{(m)} - \psi \right\|_\infty < \frac{1}{m} \to 0$ as $m \to \infty$. $\qquad\square$

**Corollary 2.4.17.** *Let $\psi \in TFc_0$. Then $D_\psi$ is weakly compact.*

*Proof.* Let $\psi \in TFc_0$, by Lemma 2.4.16 there is a sequence $(\psi^{(m)}) \subseteq \ell^\infty$, such that $Supp(\psi^{(m)})$ is TF for each $m$. By Theorem 2.4.13 $D_{\psi^{(m)}}$ is weakly compact for each $m$. By Theorem 2.2.6 since B $: Der(\ell^1(\mathbb{Z}_+), \ell^\infty(\mathbb{Z}_+)) \longrightarrow E$ is bijective and $B(D_\psi) = \psi$, $(D_\psi(t^j)(t^0) = \psi(t^j) = \psi_j)$, so

$$
\begin{aligned}
\left\| D_{\psi^{(m)}} - D_\psi \right\| &= \left\| B(D_{\psi^{(m)}} - D_\psi) \right\|_\infty \\
&= \left\| B(D_{\psi^{(m)}}) - B(D_\psi) \right\|_\infty \\
&= \left\| \psi^{(m)} - \psi \right\|_\infty \to 0 \quad \text{as} \quad m \to \infty.
\end{aligned}
$$

25

So by Lemma 1.1.13, $D_\psi$ is weakly compact. $\qquad\square$

**Lemma 2.4.18.** *Let $F : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be a bounded function. Then there are increasing sequences $n(1) < n(2) < n(3) < \ldots$ and $m(1) < m(2) < m(3) < \ldots$ such that*

$$\lim_k(\lim_j F(n(j), m(k)))$$

*exists.*

*Proof.* Start with $m = 1$, $(F(n, 1))_{n \geq 1}$ is a bounded sequence, so by Bolzano-Weierstrass Theorem there is a subsequence $p_1(1) < p_1(2) < p_1(3) < \ldots$ such that $\lim_j F(p_1(j), 1)$ exists, call it $\mathcal{F}(1)$.

Consider $m = 2$, $(F(p_1(j), 2))_{j \geq 1}$ is a bounded sequence, so by Bolzano-Weierstrass Theorem there is a subsequence $p_2(1) < p_2(2) < p_2(3) < \ldots$ such that $\lim_j F(p_1(p_2(j)), 2)$ exists, call it $\mathcal{F}(2)$. Note $\lim_j F(p_1(p_2(j)), 1)$ exists and equals $\mathcal{F}(1)$.

By induction let $\lim_j F(p_1(p_2(\ldots (p_{m-1}(j)) \ldots)), m-1)$ exists and equals $\mathcal{F}(m-1)$. Then consider $m$, $(F(p_1(p_2(\ldots (p_{m-1}(j)) \ldots)), m))_{j \geq 1}$ is a bounded sequence, so by Bolzano-Weierstrass Theorem there is a subsequence $p_m(1) < p_m(2) < p_m(3) < \ldots$ such that

$$\lim_j F(p_1(p_2(\ldots (p_m(j)) \ldots)), m)$$

exists, call it $\mathcal{F}(m)$. Now let

$$
\begin{aligned}
n(1) &= p_1(1) \\
n(2) &= p_1(p_2(2)) \\
&\vdots \\
n(m) &= p_1(p_2(\ldots (p_m(m)) \ldots))
\end{aligned}
$$

then the sequence

$$F(n(m), m), F(n(m+1), m), F(n(m+2), m), \ldots$$

is a subsequence of $(F(p_1(p_2(\ldots (p_m(j)) \ldots)), m))_{j \geq 1}$ which converges for all $m$. So $n(1) < n(2) < n(3) < \ldots$ is an increasing sequence (since $p_1(p_2(1)) \geq p_1(1)$ and so $p_1(p_2(2)) > p_1(1)$, means that $n(2) > n(1)$) that $F(n(j), m)$ converges for all $m$ to $\mathcal{F}(m)$. (See Figure 2.3)

So for every $m$, $\lim_j F(n(j), m) = \mathcal{F}(m)$, and $(\mathcal{F}(m))_{m \geq 1}$ is a bounded sequence, so by Bolzano-Weierstrass Theorem there is a subsequence $m(1) < m(2) < m(3) < \ldots$ such that $\lim_k \mathcal{F}(m(k))$ exists and

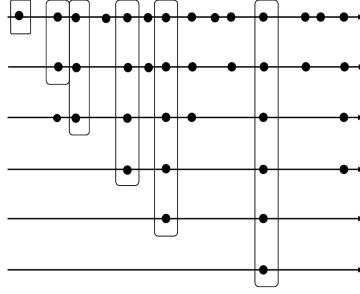$$\lim_k \mathcal{F}(m(k)) = \lim_k(\lim_j F(n(j), m(k))).$$

$\qquad\square$

**Figure 2.3:** *The Diagonal Argument on $p_m(j)$ to get an increasing sequence $n(j)$ that converges for all $m$.*

Next theorem that is the converse of Corollary 2.4.17 was originally proved by Choi and Heath using a direct but complicated induction [6]. We give a simple proof using repeated limits (as suggested by M. Daws, see the remarks at the start of Section 2.4).

**Theorem 2.4.19.** *If $\psi \in \ell^\infty$, and $D_\psi$ is weakly compact then $S_\varepsilon := \{n \in \mathbb{N} : |\psi_n| > \varepsilon\}$ is a TF-set for each $\varepsilon > 0$.*

*Proof.* Since $D_\psi$ is weakly compact by Proposition 2.4.11

$$\lim_m \lim_n D_\psi(t^{x_n})(t^{y_m}) = \lim_n \lim_m D_\psi(t^{x_n})(t^{y_m}),$$

where $D_\psi(t^j)(t^k) = \frac{j}{j+k}\psi_{j+k}$ for $j, k \in \mathbb{Z}_+, j + k \neq 0$ and $(x_n), (y_m)$ are sequences in $\mathbb{Z}_+$, consisting distinct points and both repeated limits exist. Also we know

$$\lim_n \lim_m D_\psi(t^{x_n})(t^{y_m}) = \lim_n \lim_m \frac{x_n}{x_n + y_m}\psi_{x_n+y_m} = 0.$$

By contradiction suppose $S_\varepsilon$ is not TF for some $\varepsilon > 0$. Then there is an increasing sequence $(y_m)$ with $X_n := \bigcap_{m=1}^n (S_\varepsilon - y_m)$ is infinite, for all $n \in \mathbb{N}$. So

$$X_1 = S_\varepsilon - y_1$$
$$X_2 = (S_\varepsilon - y_1) \cap (S_\varepsilon - y_2)$$
$$X_3 = (S_\varepsilon - y_1) \cap (S_\varepsilon - y_2) \cap (S_\varepsilon - y_3)$$
$$\vdots$$
$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$$

Pick $x_1 \in X_1$, $x_2 \in X_2$ and $x_2 > x_1$, $x_3 \in X_3$ and $x_3 > x_2$. So for every $n$, $x_n \in X_n$ and $x_n > x_{n-1}$ (See Figure 2.4). Hence $(x_n)$ is an increasing sequence that $x_n + y_m \in S_\varepsilon$ for every $n \geq m$. Now define $F(n, m) := \psi_{x_n+y_m} \in \mathbb{C}$. By Lemma 2.4.18 there are increasing sequences $n(1) < n(2) < n(3) < \dots$ and $m(1) < m(2) < m(3) < \dots$ such that $\lim_k(\lim_j F(n(j), m(k)))$
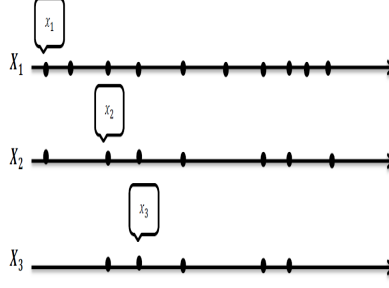
27

*Figure 2.4:* $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ *and pick* $x_n \in X_n$ *such that* $x_n > x_{n-1}$ *for every* $n$.

exists and is non-zero.

(Since $x_{n(j)} + y_{m(k)} \in S_\varepsilon$ for every $n(j) \geq m(k)$ and so $|F(n(j), m(k))| > \varepsilon$ means that $|\psi_{x_{n(j)}+y_{m(k)}}| > \varepsilon$. Then

$$|\lim_j \psi_{x_{n(j)}+y_{m(k)}}| > \frac{\varepsilon}{2}.$$

Hence

$$|\lim_k (\lim_j \psi_{x_{n(j)}+y_{m(k)}})| > \frac{\varepsilon}{4}$$

for every $n(j) \geq m(k)$.)

We know $\lim_j (\lim_k \psi_{x_{n(j)}+y_{m(k)}}) = 0$, so $\lim_j (\lim_k \psi_{x_{n(j)}+y_{m(k)}})$ and $\lim_k (\lim_j \psi_{x_{n(j)}+y_{m(k)}})$ both exist but are not equal. Hence $D_\psi$ is not weakly compact, and it is a contradiction. So we conclude if $D_\psi$ is weakly compact then $S_\varepsilon$ is a TF-set. $\square$

**Corollary 2.4.20.** *If* $\varphi \in TFc_0$ *and* $\psi \in TFc_0$, *then* $\varphi + \psi \in TFc_0$.

*Proof.* Let $\varphi \in TFc_0$, so by Corollary 2.4.17 $D_\varphi$ is weakly compact, also $\psi \in TFc_0$ similarly $D_\psi$ is weakly compact. Then by Theorem 2.2.6

$$D_\varphi + D_\psi = B^{-1}(\varphi) + B^{-1}(\psi) = B^{-1}(\varphi + \psi) = D_{\varphi+\psi}$$

is weakly compact since the set of weakly compact operators is a vector space by Corollary 3.5.10, [19]. So by Theorem 2.4.19 $S_\varepsilon = \{n \in \mathbb{N} : |(\varphi + \psi)_n| > \varepsilon\}$ is TF for every $\varepsilon > 0$ and so $\varphi + \psi \in TFc_0$.

$\square$

**Corollary 2.4.21.** *Let* $X_1, X_2$ *be subsets of* $\mathbb{N}$ *such that* $X_1$ *and* $X_2$ *are TF,* $X_1 \cap X_2 = \varnothing$, *then* $X_1 \cup X_2$ *is TF.*

*Proof.* Define $\varphi$ with $\varphi = 1$ on $X_1$ and $\varphi = 0$ outside $X_1$. So $\varphi \in TFc_0$ by definition. Similarly if we define $\psi = 1$ on $X_2$ and $\psi = 0$ outside $X_2$, then $\psi \in TFc_0$. By Corollary 2.4.20,

$\varphi + \psi \in TFc_0$. So $S_\varepsilon = \{n \in \mathbb{N} : |(\varphi + \psi)_n| > \varepsilon\}$ is TF for every $\varepsilon > 0$ and

$$(\varphi + \psi)_n = \begin{cases} 1 & \text{if } n \in X_1 \cup X_2 \\ 0 & \text{if } n \notin X_1 \cup X_2 \end{cases}$$

then $S_\varepsilon = X_1 \cup X_2$ and hence $X_1 \cup X_2$ is TF. $\qquad\square$

**Remark 2.4.22.** *If $S$ is a T-set then so is $S + m = \{x + m : x \in S\}$ for any $m \in \mathbb{N}$.*

*Proof.* Suppose $S + m$ is not T-set for some $m$, so there is a $k \in \mathbb{N}$ that for it $(S + m) \cap (S + m - k)$ is infinite, so for all $y \in (S + m) \cap (S + m - k)$, $y = x + m$ for some $x \in S$ and $y + k \in S + m$, means $(x + k) + m \in S + m$, so $x + k \in S$. Hence $x \in S - k$. It contradicts that $S$ is T-set. So $S + m$ is T-set. $\qquad\square$

**Example 2.4.23.** *Let $X_1 = \{2^n : n \in \mathbb{N}\}$, we know it is a T-set, also $X_2 = \{2^n + 1 : n \in \mathbb{N}\}$ is a T-set by above remark, and $X_1 \cap X_2 = \varnothing$. So $X_1 \cup X_2$ is a TF-set by Corollary 2.4.21, but it is not a T-set since for $k = 1$,*

$$(X_1 \cup X_2) \cap ((X_1 \cup X_2) - 1) = (X_1 \cup (X_1 + 1)) \cap ((X_1 - 1) \cup X_1) \supseteq X_1$$

*which is infinite. So $X_1 \cup X_2$ is TF but is not T-set.*

# CHAPTER 3

# DERIVATIONS ON $L^1(\mathbb{R}_+)$

## 3.1 Introduction

In this chapter we study derivations, $D : L^1(\mathbb{R}_+) \longrightarrow L^\infty(\mathbb{R}_+)$.

By part (b) of Example 1.2.2, $(L^1(\mathbb{R}_+), \|\cdot\|_1, *)$ is a Banach algebra when for every $f \in L^1(\mathbb{R}_+)$ and $\mu$ Lebesgue measure, we define

$$\|f\|_1 = \int_0^\infty |f| d\mu.$$

Also $(L^\infty(\mathbb{R}_+), \|\cdot\|_\infty)$ is a Banach space when for every $\varphi \in L^\infty(\mathbb{R}_+)$ we have

$$\|\varphi\|_\infty = \operatorname{ess\,sup} |\varphi| = \inf\{C \geq 0 : |\varphi(x)| \leq C \text{ for } \mu\text{-almost every } x\}.$$

Given $\varphi \in L^\infty(\mathbb{R}_+)$, the function $L^1(\mathbb{R}_+) \longrightarrow \mathbb{C}$ defined by $f \mapsto \int_0^\infty f\varphi d\mu$ is a bounded linear map with norm equal to $\|\varphi\|_\infty$. By Example 1.10.2 [19], every element of $L^1(\mathbb{R}_+)^*$ arises in this way.

**Definition 3.1.1.** $L^\infty(\mathbb{R}_+)$ *becomes a Banach $L^1(\mathbb{R}_+)$-module by Definition 1.3.3 through*

$$\langle f, g \cdot \varphi \rangle := \langle f * g, \varphi \rangle$$
$$= \int_0^\infty (f * g)(t)\varphi(t) dt$$
$$= \int_0^\infty \int_0^t f(t - s)g(s)\varphi(t) ds dt \qquad (f, g \in L^1(\mathbb{R}_+), \varphi \in L^\infty(\mathbb{R}_+))$$

*By changing variable $t$ to $t + s$, we get*

$$\langle f, g \cdot \varphi \rangle = \int_0^\infty \int_0^\infty f(t)g(s)\varphi(t + s) ds dt = \int_0^\infty f(t)\left(\int_0^\infty g(s)\varphi(t + s) ds\right) dt$$

*It means that the module action of $L^1(\mathbb{R}_+)$ on $L^\infty(\mathbb{R}_+)$ is*

$$(g \cdot \varphi)(t) = \int_0^\infty g(s)\varphi(t + s) ds.$$

## 3.2 Derivations from $L^1(\mathbb{R}_+)$ to its dual

**Theorem 3.2.1.** *Let $\varphi \in L^\infty(\mathbb{R}_+)$. Then*

$$(D_\varphi f)(t) = \int_0^\infty f(s)\frac{s}{t+s}\varphi(t+s)ds \qquad (t \in \mathbb{R}_+, f \in L^1(\mathbb{R}_+)) \qquad (3.2.1)$$

*defines a continuous derivation from $L^1(\mathbb{R}_+)$ to $L^\infty(\mathbb{R}_+)$.*

*Proof.* Firstly, note that $D_\varphi f$ is a measurable function on $\mathbb{R}_+$. To see this, consider

$$F(t,s) = \frac{s}{s+t}f(s)\varphi(s+t) \qquad (s,t \in \mathbb{R}_+, s+t \neq 0).$$

The function $F$ is measurable,and we can assume that $F$ is non-negative. (We can write $F = F_1 + iF_2$, where $F_1, F_2$ are real and imaginary parts of $F$, respectively. Also $F_k$ ($k = 1, 2$) can be written as difference of two (measurable) non-negative functions.) Then, by Tonelli's Theorem (Theorem 2.37, [12]) the function $t \mapsto \int_{\mathbb{R}_+} F(t,s)ds$ is measurable. Thus $D_\varphi(f)$ is measurable. Also we have $D_\varphi f \in L^\infty(\mathbb{R}_+)$, since

$$\begin{aligned}
|D_\varphi f(t)| &= |\int_0^\infty \frac{s}{s+t}f(s)\varphi(s+t)ds| \\
&\leq \int_0^\infty |f(s)||\frac{s}{s+t}\varphi(s+t)|ds \\
&\leq \int_0^\infty |f(s)|\,\|\varphi\|_\infty\,ds \\
&\leq \|\varphi\|_\infty \int_0^\infty |f(s)|ds \\
&\leq \|\varphi\|_\infty \|f\|_1.
\end{aligned}$$

So

$$\|D_\varphi f\|_\infty = \inf\{C \geq 0 : |D_\varphi f(t)| < C \text{ for a.e. } t\} \leq \|\varphi\|_\infty \|f\|_1 < \infty.$$

So $D_\varphi(f) \in L^\infty(\mathbb{R}_+)$.

Then we prove $D_\varphi$ is a bounded linear map on $L^1(\mathbb{R}_+)$.

It is linear since if $f, g \in L^1(\mathbb{R}_+)$ and $c$ be an scalar, then

$$D_\varphi(cf+g)(t) = \int_0^\infty (cf+g)(s)\frac{s}{s+t}\varphi(t+s)ds$$

$$= \int_0^\infty (cf(s)+g(s))\frac{s}{s+t}\varphi(t+s)ds$$

$$= \int_0^\infty cf(s)\frac{s}{s+t}\varphi(t+s)ds + \int_0^\infty g(s)\frac{s}{s+t}\varphi(t+s)ds$$

$$= c\int_0^\infty f(s)\frac{s}{s+t}\varphi(t+s)ds + \int_0^\infty g(s)\frac{s}{s+t}\varphi(t+s)ds$$

$$= cD_\varphi(f)(t) + D_\varphi(g)(t).$$

Now, let $f \in L^1(\mathbb{R}_+)$ and $\varphi \in L^\infty(\mathbb{R}_+)$, so

$$\|D_\varphi\| = \sup\{\|D_\varphi f\|_\infty : f \in L^1(\mathbb{R}_+), \|f\|_1 \leq 1\}$$

$$\leq \sup\{\|\varphi\|_\infty \|f\|_1 : f \in L^1(\mathbb{R}_+), \|f\|_1 \leq 1\}$$

$$\leq \|\varphi\|_\infty < \infty.$$

Hence $D_\varphi$ is bounded.

Then we show $D_\varphi$ is a derivation, it means we want to show;

$$\langle D_\varphi(f*g), h\rangle = \langle D_\varphi(f), g*h\rangle + \langle D_\varphi(g), h*f\rangle \qquad (\forall f, g, h \in L^1(\mathbb{R}_+))$$

Let $f, g, h \in L^1(\mathbb{R}_+)$. We have

$$\langle D_\varphi(f*g), h\rangle = \int_0^\infty (D_\varphi(f*g))(t)h(t)dt$$

$$= \int_0^\infty \int_0^\infty \frac{s}{s+t}(f*g)(s)\varphi(t+s)h(t)dsdt$$

$$= \int_0^\infty \int_0^\infty \int_0^s \frac{s}{s+t}f(r)g(s-r)\varphi(s+t)h(t)drdsdt$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \frac{r+u}{r+u+t}f(r)g(u)\varphi(r+u+t)h(t)dudrdt$$

32

and similarly

$$\langle D_\varphi(f), g * h \rangle = \int_0^\infty (D_\varphi(f))(x)(g * h)(x)dx$$

$$= \int_0^\infty \int_0^\infty f(r)\frac{r}{x+r}\varphi(x+r)(g * h)(x)drdx$$

$$= \int_0^\infty \int_0^\infty \int_0^x \frac{r}{r+x}f(r)\varphi(x+r)g(u)h(x-u)dudrdx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \frac{r}{r+u+t}f(r)g(u)h(t)\varphi(r+u+t)dudrdt$$

also,

$$\langle D_\varphi(g), h * f \rangle = \int_0^\infty (D_\varphi(g))(x)(h * f)(x)dx$$

$$= \int_0^\infty \int_0^\infty g(u)\frac{u}{u+x}\varphi(u+x)(h * f)(x)dudx$$

$$= \int_0^\infty \int_0^\infty \int_0^x \frac{u}{u+x}g(u)\varphi(u+x)h(t)f(x-t)dudxdt$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \frac{u}{r+u+t}g(u)\varphi(r+u+t)h(t)f(r)dudrdt.$$

So by adding these two, we have

$$\langle D_\varphi(f), g * h \rangle + \langle D_\varphi(g), h * f \rangle = \int_0^\infty \int_0^\infty \int_0^\infty \frac{r+u}{r+u+t}\varphi(r+u+t)f(r)g(u)h(t)dudrdt$$

$$= \langle D_\varphi(f * g), h \rangle.$$

So $D_\varphi$ is a derivation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the rest of this section, we will show that $D_\varphi(L^1(\mathbb{R}_+)) \subseteq C_0(\mathbb{R}_+)$. This is a special case of some known results, but we give an alternative proof which is more elementary.

**Remark 3.2.2.** *We call* $\mathbb{I}_{[a,b]}$ *the **characteristic function** of the interval* $[a,b]$ *if*

$$\mathbb{I}_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b] \end{cases}$$

**Proposition 3.2.3.** *Let* $\varphi \in L^\infty(\mathbb{R}_+)$ *and* $0 \le a < b$. *Define* $f(x) = D_\varphi(\mathbb{I}_{[a,b]})(x)$. *So* $f : \mathbb{R}_+ \to \mathbb{C}$. *Then* $f$ *is continuous at every point.*

*Proof.* We have $f(t) = \int_a^b \frac{s}{s+t}\varphi(s+t)ds$. We want to show $f$ is continuous. So let

$$f(t_2) - f(t_1) = \int_a^b \frac{s}{s+t_2}\varphi(s+t_2)ds - \int_a^b \frac{s}{s+t_1}\varphi(s+t_1)ds$$

and by changing the variable $s$ to $s-t_2$ in the first integral and changing $s$ to $s-t_1$ in second integral we will have

$$f(t_2) - f(t_1) = \int_{a+t_2}^{b+t_2} \frac{s-t_2}{s}\varphi(s)ds - \int_{a+t_1}^{b+t_1} \frac{s-t_1}{s}\varphi(s)ds$$

Without loss of generality we can assume $0 \le t_1 < t_2$ with $|t_2 - t_1| < |b-a|$, so $a+t_2 < b+t_1$. Then

$$f(t_2) - f(t_1) = \int_{a+t_2}^{b+t_1} \frac{s-t_2}{s}\varphi(s)ds + \int_{b+t_1}^{b+t_2} \frac{s-t_2}{s}\varphi(s)ds$$
$$- \int_{a+t_1}^{a+t_2} \frac{s-t_1}{s}\varphi(s)ds - \int_{a+t_2}^{b+t_1} \frac{s-t_1}{s}\varphi(s)ds$$

If we define

$$g(t_1, t_2) = \int_{b+t_1}^{b+t_2} \frac{s-t_2}{s}\varphi(s)ds - \int_{a+t_1}^{a+t_2} \frac{s-t_1}{s}\varphi(s)ds$$

and

$$h(t_1, t_2) = \int_{a+t_2}^{b+t_1} \frac{s-t_2}{s}\varphi(s)ds - \int_{a+t_2}^{b+t_1} \frac{s-t_1}{s}\varphi(s)ds$$

then $f(t_2) - f(t_1) = g(t_1, t_2) + h(t_1, t_2)$ and so

$$|g(t_1, t_2)| \le \int_{b+t_1}^{b+t_2} \frac{s-t_2}{s}|\varphi(s)|ds + \int_{a+t_1}^{a+t_2} \frac{s-t_1}{s}|\varphi(s)|ds$$
$$\le \int_{b+t_1}^{b+t_2} |\varphi(s)|ds + \int_{a+t_1}^{a+t_2} |\varphi(s)|ds$$
$$\le |t_2 - t_1|\,\|\varphi\|_\infty + |t_2 - t_1|\,\|\varphi\|_\infty$$
$$= 2|t_2 - t_1|\,\|\varphi\|_\infty.$$

For next step we have

$$h(t_1, t_2) = \int_{a+t_2}^{b+t_1} \frac{s-t_2}{s}\varphi(s)ds - \int_{a+t_2}^{b+t_1} \frac{s-t_1}{s}\varphi(s)ds$$
$$= \int_{a+t_2}^{b+t_1} \frac{t_1 - t_2}{s}\varphi(s)ds.$$

34

So

$$|h(t_1, t_2)| \leq (t_2 - t_1) \|\varphi\|_\infty \int_{a+t_2}^{b+t_1} \frac{ds}{s}$$
$$= (t_2 - t_1) \|\varphi\|_\infty \log \frac{b + t_1}{a + t_2}.$$

Now fix $x_0 \geq 0$. Let $\delta > 0$. If $x_0 = 0$, then $|h(0, \delta)| \leq \delta \|\varphi\|_\infty \log \frac{b}{\delta}$ which converges to zero as $\delta \to 0^+$ by the logarithm properties.

If $x_0 > 0$, then provided that $\delta < x_0$ and $\delta < |b - a|$, we get

$$|h(x_0, x_0 + \delta)| \leq \delta \|\varphi\|_\infty \log \frac{b + x_0}{a + x_0}$$

or

$$|h(x_0 - \delta, x_0)| \leq \delta \|\varphi\|_\infty \log \frac{b + x_0 - \delta}{a + x_0}$$

which both converges to zero as $\delta \to 0^+$.

So we showed in all conditions $|h(t_1, t_2)|$ and $|g(t_1, t_2)|$ converges to zero as $|t_2 - t_1| \to 0$. Hence $|f(t_2) - f(t_1)|$ converges to zero as $|t_2 - t_1| \to 0$ and it means $f$ is continuous, and so $D_\varphi(\mathbb{I}_{[a,b]}) \in C_b(\mathbb{R}_+)$. $\qquad \square$

**Lemma 3.2.4.** *Let* $\varphi \in L^\infty(\mathbb{R}_+)$, *then* $D_\varphi(\mathbb{I}_{[a,b]}) \in C_0(\mathbb{R}_+)$.

*Proof.* We know $D_\varphi(\mathbb{I}_{[a,b]})$ is continuous, we need to show it vanishes at infinity. We have $D_\varphi(\mathbb{I}_{[a,b]})(t) = \int_a^b \frac{s}{s+t} \varphi(s + t) ds$. Let $\varepsilon > 0$ and choose $t_0 \in \mathbb{R}_+$ large enough such that $\frac{b}{b+t_0} \|\varphi\|_\infty (b - a) < \varepsilon$. If $t \geq t_0$ and $s \leq b$ then $\frac{s}{s+t} \leq \frac{b}{b+t} \leq \frac{b}{b+t_0}$. So for all $t \geq t_0$ we have

$$|D_\varphi(\mathbb{I}_{[a,b]})(t)| \leq \int_a^b \frac{b}{b + t_0} \|\varphi\|_\infty ds = \frac{b}{b + t_0} \|\varphi\|_\infty (b - a) < \varepsilon$$

So $D_\varphi(\mathbb{I}_{[a,b]}) \in C_0(\mathbb{R}_+)$. $\qquad \square$

**Corollary 3.2.5.** *Let* $\varphi \in L^\infty(\mathbb{R}_+)$ *and* $f \in L^1(\mathbb{R}_+)$. *Then* $D_\varphi(f) \in C_0(\mathbb{R}_+)$.

*Proof.* By (Theorem 2.26, [12]), we know that $V = \mathrm{Span}\{\mathbb{I}_{[a,b]}; 0 \leq a < b < \infty\}$ is dense in the Banach space $(L^1(\mathbb{R}_+), \|\cdot\|_1)$. So there is a sequence $(f_n) \subseteq V$ that converges to $f$ in $L^1(\mathbb{R}_+)$. On the other hand, by Lemma 3.2.4, $D_\varphi(f_n) \in C_0(\mathbb{R}_+)$ for all $n$. Since $C_0(\mathbb{R}_+)$ is closed in $(L^\infty(\mathbb{R}_+), \|\cdot\|_\infty)$, hence $D_\varphi(f) = \lim_n D_\varphi f_n \in C_0(\mathbb{R}_+)$. $\qquad \square$

# CHAPTER 4

# CONCLUSION

In this thesis we characterized derivations on $\ell^1(\mathbb{Z}_+)$ in Theorem 2.2.6, then we found a necessary and sufficient condition to make these kind of derivation compact in Section 2.3. Also we showed in Theorem 2.4.13, Corollary 2.4.17 and Theorem 2.4.19 that all derivations, $D_\psi : \ell^1(\mathbb{Z}_+) \longrightarrow \ell^\infty(\mathbb{Z}_+)$ is weakly compact, if and only if, $\psi \in TFc_0$.

We started study derivations, $D : L^1(\mathbb{R}_+) \longrightarrow L^\infty(\mathbb{R}_+)$. and we showed they are actually map into $C_0(\mathbb{R}_+)$. All these results were already known, but we have given alternative proofs that may be more accessible to future researchers.

One possible future project would be to find the necessary and sufficient condition to make a derivation, $D_\psi : L^1(\mathbb{R}_+) \longrightarrow L^\infty(\mathbb{R}_+)$ compact or weakly compact.

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