

CONVERGENCE IN THE MEAN-FIELD LIMIT FOR TWO
SPECIES OF BOSONIC PARTICLES

A Thesis Submitted to the
College of Graduate Studies and Research
in Partial Fulfillment of the Requirements
for the degree of Master of Science
in the Department of Mathematics
University of Saskatchewan
Saskatoon

By
Megan Richards

©Megan Richards, August 2014. All rights reserved.

PERMISSION TO USE

In presenting this thesis in partial fulfilment of the requirements for a Postgraduate degree from the University of Saskatchewan, I agree that the Libraries of this University may make it freely available for inspection. I further agree that permission for copying of this thesis in any manner, in whole or in part, for scholarly purposes may be granted by the professor or professors who supervised my thesis work or, in their absence, by the Head of the Department or the Dean of the College in which my thesis work was done. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to the University of Saskatchewan in any scholarly use which may be made of any material in my thesis.

Requests for permission to copy or to make other use of material in this thesis in whole or part should be addressed to:

The Department of Mathematics and Statistics
Room 142 McLean Hall
106 Wiggins Road
University of Saskatchewan
Saskatoon, Saskatchewan
Canada
S7N 5E6

ABSTRACT

The dynamics of a quantum system with a large number N of identical bosonic particles interacting by means of weak two-body potentials can be simplified by using mean-field equations in which all interactions to any one body have been replaced with an average or effective interaction in the mean-field limit $N \rightarrow \infty$. In order to show these mean-field equations are accurate, one needs to show convergence of the quantum N -body dynamics to these equations in the mean-field limit. Previous results on convergence in the mean field limit have been derived for certain initial conditions in the case of one species of bosonic particles, but no results have yet been shown for multi-species.

In this thesis, we look at a quantum bosonic system with two species of particles. For this system, we derive a formula for the rate of convergence in the mean-field limit in the case of an initial coherent state, and we also show convergence in the mean-field limit for the case of an initial factorized state. The analysis for two species can then be extended to multiple species.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. Walid Abou Salem, for all of his support and advice throughout my time at the University of Saskatchewan.

CONTENTS

Permission to Use	i
Abstract	ii
Acknowledgements	iii
Contents	iv
List of Tables	vi
List of Figures	vii
Introduction	1
1 Quantum Bosonic Systems	4
1.1 Introduction to General Fock Spaces	4
1.2 Bosonic Fock space over $L^2(\mathbb{R}^3, dx)$ for one species	8
1.2.1 Creation and annihilation operators	9
1.2.2 The number operator	12
1.2.3 Weyl operators and coherent states	15
1.3 Two species	19
1.3.1 Creation and annihilation operators	19
1.3.2 The number operators	20
1.3.3 Weyl operators and coherent states	23
2 Quantization & Mean-field Limit Through Deformation of Algebras	25
2.1 Introduction to Deformation of Algebras	25
2.2 From the Hartree equation to N body quantum mechanics and back	27
3 Time Evolution of Coherent States for Two Species	32
3.1 Factorized States & Convergence of Marginal Densities	32
3.1.1 One species case	32
3.1.2 Two species case	34
3.2 Previous Results for the Case of an Initial Coherent State (Hepp, Rodnianski & Schlein)	35
3.3 Statement of Main Theorem & Overview of Proof	36
3.4 Control of Growth of Number Operator	43
3.4.1 Lemma (3.4.5)	45
3.4.2 Lemma (3.4.7)	51
3.4.3 Lemma (3.4.9)	58
3.4.4 Proof of Proposition (3.4.1)	66
3.5 Lemma (3.5.1)	67
3.6 Lemma (3.6.1)	70
3.7 Proof of Theorem (3.3.1)	74
4 Time Evolution of Factorized States	78
4.1 Introduction	78
4.2 Preliminary Definitions	79
4.3 Theorem (4.3.1)	81
4.4 Proposition (4.4.1)	92
4.5 Conclusion	94

References	95
A Mathematical & Physical Preliminaries	97
A.1 Operators	97
A.1.1 Operators on a Hilbert space	97
A.1.2 Projections	100
A.1.3 Integral Operators	101
A.1.4 Density Operators	101
A.2 A Few Useful Lemmas & Propositions	102
A.3 *-Algebras and C^* -Algebras	104
A.4 Tensor Products	104
A.5 A Brief Introduction to Classical and Quantum Mechanics	106
A.5.1 Hamiltonian mechanics	106
A.5.2 Lagrangian Mechanics & Noether's Theorem	107
A.5.3 Poisson bracket	109
A.5.4 Phase Space & Symplectic Manifolds	110
A.5.5 The Mathematics of Quantum Mechanics	112
A.5.6 Pictures of Quantum Mechanics	112
A.5.7 Quantum Analogue to the Poisson Bracket	116
A.5.8 The Classical Limit, Quantization, and Phase-Space Formulation	117

LIST OF TABLES

1	A brief summary of each chapter.	3
---	--	---

LIST OF FIGURES

2.1	Classical and quantum mechanics. A right arrow \rightarrow stands for the limit $\hbar \rightarrow 0$. A left arrow \leftarrow stands for quantization with deformation parameter \hbar . A down arrow \downarrow stands for the mean field limit $N \rightarrow \infty$, where N is the number of particles. An up arrow \uparrow stands for second quantization with deformation parameter $1/N$	26
-----	--	----

INTRODUCTION

Suppose we are given a quantum system with a large number N of identical bosonic particles interacting by means of weak two-body potentials. When this happens, one might expect that we can replace all interactions to any one particle with an effective interaction in the mean-field limit $N \rightarrow \infty$. By using such an effective mean-field potential, the dynamics of the N -body problem becomes much simpler.

However, one then needs to justify the validity of these effective or mean-field models. For example, in the case of bosonic systems when kinetic energy scales like potential energy, one wishes to show that in the mean-field limit the quantum N -body dynamics is given by mean-field dynamics described by the nonlinear Hartree equation.

In this thesis, we mainly focus on scalings that produce the Hartree equation. However, the Hartree equation is not the only equation that arises from taking the mean-field limit; the Gross-Pitaevskii equation is another important mean-field equation. Consider a system of N bosonic particles with some initial state ψ_0 . The time evolution ψ_t is given by the N -body Schrödinger equation

$$i \frac{\partial}{\partial t} \psi_t = H_N \psi_t$$

where H_N is the mean-field Hamiltonian given by

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \sum_{j \neq k=1}^N v_{\beta}^N(x_j - x_k)$$

defined on the Hilbert space $L_s^2(\mathbb{R}^{3N})$, which is the subspace of $L^2(\mathbb{R}^{3N})$ that contains all functions symmetric with respect to arbitrary permutations of the N particles. Here v_{β}^N is a symmetric interaction potential, where $\beta \in \mathbb{R}$ stands for the scaling behavior. As an example, we can take v_{β}^N to be

$$v_{\beta}^N = N^{-1+3\beta} v(N^{\beta} x)$$

for a compactly supported, spherically symmetric, positive potential $v \in L^{\infty}$.

Now, suppose $\psi_0 = \prod_{j=1}^N \varphi(x_j)$. One expects that $\psi_t \approx \prod_{j=1}^N \varphi_t(x_j)$, where φ_t solves the non-linear Schrödinger equation

$$i \frac{\partial}{\partial t} \varphi_t = -(\Delta + V_{\varphi_t}) \varphi_t. \tag{1}$$

Here V_{φ_t} is the mean-field potential and depends on φ_t , which makes the equation (1) non-linear. Now, for different regimes of β we get different mean-field potentials and hence different non-linear equations. For $\beta = 0$, the mean-field potential is given by the convolution $V_{\varphi_t} = v * |\varphi_t|^2$, which produces the non-linear

Hartree equation. For $0 < \beta < 1$, we obtain the mean-field potential $V_{\varphi_t} = \|v\|_{L^1} |\varphi_t|^2$, which produces the Gross-Pitaevskii equation. See [12] for a longer discussion of the above, and see, for example, [6], [5], and [4] for information concerning derivation of the Gross-Pitaevskii equation.

The study of convergence in the mean-field limit is not new. In 1974 Hepp was the first to rigorously discuss the mean-field limit of quantum Bose gases in his paper [9], in which he showed convergence in the mean field limit. Recently, in 2009, Rodnianski and Schlein extended the analysis of Hepp by deriving an explicit formula for the rate of convergence in the mean-field limit, for both an initial coherent state as well as an initial factorized state [19]. Meanwhile, Fröhlich, Pizzo, and Knowles studied convergence in the mean field limit at the level of algebra of observables in [8].

In these papers, it was assumed that only one species of bosonic particles was present. Rates of convergence have not yet been shown for the case of multi-species. In this thesis, we look at the case of two species of bosonic particles and extend the work of Rodnianski and Schlein by finding the rate of convergence for an initial coherent state in such a system. We also look at an initial factorized state. Rather than continuing the method of Rodnianski and Schlein to show the rate of convergence, however, we instead look at a method developed in [17, 12] to show convergence. The extension of the two-species case to multi-species is straightforward.

Our work shows convergence to Hartree dynamics in the mean-field limit. Future work includes calculating soliton solutions of the coupled non-linear Hartree equations. Given that like species of bosonic particles attract while opposite species repel, for example, we would be interested in finding the ground state of the system and seeing what patterns emerge in the long term effective dynamics. This is relevant to pattern formation in chemical solutions with multi-species of nanoparticles, see [15, 2, 10, 13, 14].

We begin this thesis by discussing Fock spaces and in particular the bosonic Fock space. Next we discuss how quantization and the mean-field limit can be understood as deformation of algebras, as in [8]. We then consider a system of two species of bosonic particles, and we derive estimates on the rate of convergence in the mean field limit for an initial coherent state of this system, which is a new result. Then, using the method of Pickl, we show convergence in the case of an initial factorized state in a system of two species of bosonic particles, which has also not been shown before.

This thesis also contains an appendix, which gives some basic preliminary concepts on operators, *-algebras, tensor products, and classical and quantum mechanics. A few useful propositions and lemmas pertaining to operators that we use in the thesis are also included in the appendix. A reader may wish to start this thesis by reading through the appendix.

To summarize, this thesis is laid out as follows. Note new results occur in Chapters 3 and 4.

Chapter	Chapter Title	Summary of Chapter
Chapter 1	Quantum Bosonic Systems	Gives an introduction to general Fock spaces and then discusses the bosonic Fock space, first for one species and then for two species.
Chapter 2	Quantization & Mean-Field Limit Through Deformation of Algebras	A summary of the paper [8] by Fröhlich et al., which explains how deformation of algebras can be used to show second quantization and convergence in the mean-field limit.
Chapter 3	Time Evolution of Coherent States	We derive a formula for the rate of convergence in the mean-field limit for an initial coherent state in the case of two species of bosonic particles, which is a new result.
Chapter 4	Time Evolution of Factorized States	We show, using a method developed by Pickl [17], convergence in the mean-field limit for an initial factorized state in the case of two species of bosonic particles, which is a new result.
Appendix A	Mathematical & Physical Preliminaries	We give some basic preliminary concepts regarding operators on Hilbert spaces, *-algebras, and tensor products, and we give a basic introduction to classical and quantum mechanics.

Table 1: A brief summary of each chapter.

At the beginning of some of the chapters and sections, we give some remarks about notation. In particular, we point out over what space the standard notation for norm and inner product refers to. Since in some chapters or sections we mostly discuss elements over a certain space, this may change from chapter to chapter.

CHAPTER 1

QUANTUM BOSONIC SYSTEMS

In this chapter, we start by discussing general Fock spaces, as in the lecture notes of Merkli [16]. Then we go on to discuss bosonic Fock spaces, first for one species of bosonic particles, and then for two species of bosonic particles. See [1] for more details on bosonic Fock spaces. If the reader is not familiar with tensor products, please refer to (A.4).

1.1 Introduction to General Fock Spaces

Let \mathcal{H} be a complex Hilbert space. Then $\varphi \in \mathcal{H}$ describes the state of a single particle. To describe the state of n distinguishable particles, we use the n -fold tensor product $\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}$. This is a Hilbert space with inner product given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}^{\otimes n}} = \langle \psi_1, \varphi_1 \rangle_{\mathcal{H}} \langle \psi_2, \varphi_2 \rangle_{\mathcal{H}} \cdots \langle \psi_n, \varphi_n \rangle_{\mathcal{H}}$$

for $\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \in \mathcal{H}^{\otimes n}$ and $\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n \in \mathcal{H}^{\otimes n}$. Recall from Definition (A.4.8), if $f \in \mathcal{H}$, we will sometimes write $\underbrace{f \otimes f \otimes \cdots \otimes f}_{n \text{ times}}$ as $f^{\otimes n}$.

Definition 1.1.1. *Fock Space over the Hilbert space \mathcal{H}* is the direct sum space $\mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$ where $\mathcal{H}^{\otimes 0} = \mathbb{C}$. A vector $\varphi \in \mathcal{F}(\mathcal{H})$ is a sequence $\varphi = \{\varphi_n\}_{n \geq 0}$ with $\varphi_n \in \mathcal{H}^{\otimes n}$. The scalar product on $\mathcal{F}(\mathcal{H})$ is given by $\langle \psi, \varphi \rangle = \sum_{n \geq 0} \langle \psi_n, \varphi_n \rangle_{\mathcal{H}^{\otimes n}}$ if $\psi = \{\psi_n\}_{n \geq 0} \in \mathcal{F}$ and $\varphi = \{\varphi_n\}_{n \geq 0} \in \mathcal{F}$.

Fock space is the standard Hilbert space when it comes to describing a system of finitely many identical quantum particles (such as bosons or fermions). States of such a system are vectors in the Fock space; they are sequences where the n th term describes the system having n particles. If $\psi = \{\psi_n\}_{n \geq 0} \in \mathcal{F}$, we sometimes use the notation $[\psi]_N = \psi_N$.

Notation 1.1.2. In this section, any norm and inner product will be the norm and inner product in \mathcal{F} . Any other norm or inner product will be indicated with a subscript.

Definition 1.1.3. The self-adjoint *number operator* \mathcal{N} on $\mathcal{F}(\mathcal{H})$ is defined by

$$\mathcal{N}\varphi = \{n\varphi_n\}_{n \geq 0}$$

for $\varphi \in D(\mathcal{N})$, where $D(\mathcal{N})$ denotes the domain of \mathcal{N} .

Remark 1.1.4. We can easily calculate $D(\mathcal{N})$ as follows:

$$\begin{aligned} D(\mathcal{N}) &= \left\{ \varphi \in \mathcal{F}(\mathcal{H}) \mid \|\mathcal{N}\varphi\| < \infty \right\} = \left\{ \varphi \in \mathcal{F}(\mathcal{H}) \mid \sqrt{\sum_{n \geq 0} \|n\varphi_n\|_{\mathcal{H}^{\otimes n}}^2} < \infty \right\} \\ &= \left\{ \varphi \in \mathcal{F}(\mathcal{H}) \mid \sum_{n \geq 0} n^2 \|\varphi_n\|_{\mathcal{H}^{\otimes n}}^2 < \infty \right\}. \end{aligned}$$

Definition 1.1.5. The *vacuum vector* $\Omega \in \mathcal{F}(\mathcal{H})$ is given by $[\Omega]_0 = 1 \in \mathbb{C}$ and $[\Omega]_n = 0 \in \mathcal{H}^{\otimes n}$ for all $n > 0$, i.e. $\Omega = \{1, 0, 0, \dots\}$.

Remark 1.1.6. The norm of the vacuum vector is 1. To see this, note that

$$\|\Omega\| = \left(\sum_{n=0}^{\infty} \|[\Omega]_n\|_{\mathcal{H}^{\otimes n}}^2 \right)^{1/2} = (\|[\Omega]_0\|_{\mathcal{H}^{\otimes 0}}^2)^{1/2} = 1.$$

Remark 1.1.7. Ω spans the 1-dimensional kernel of \mathcal{N} . To see this, note if $\varphi = \{\varphi_n\}_{n \geq 0} \in \mathcal{F}(\mathcal{H})$, $\mathcal{N}\varphi = \{n\varphi_n\}_{n \geq 0} = \{0, \varphi_1, 2\varphi_2, 3\varphi_3, \dots\}$. So $\mathcal{N}\varphi = 0$ if and only if $\varphi = \{\varphi_0, 0, 0, \dots\} = \varphi_0\Omega$ for some $\varphi_0 \in \mathbb{C}$.

Next, we want to describe the state of n bosons or n fermions. If $\{f_k\}_{k=1}^n \subseteq \mathcal{H}$ are n state vectors of a single particle, the vector $f_1 \otimes \dots \otimes f_n \in \mathcal{H}^{\otimes n}$ is the state of an n -particle system where the k^{th} particle is in the state f_k . Bosons are those in which their state vector is symmetric under interchange of each pair of coordinates; fermions, on the other hand, are those in which their state vector is antisymmetric under interchange of each pair of coordinates.

If we have n bosons, each of which is in a different state, we cannot tell which is in which state. Hence, the state describing n bosons, where one is in the state f_1 , one is in state f_2 , and so on, is given by

$$\frac{1}{n!} \sum_{\pi \in S_n} f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)} \in \mathcal{H}^{\otimes n}$$

where S_n is the group of all permutations π of n objects.

The state describing n fermions is given by

$$\frac{1}{n!} \sum_{\pi \in S_n} \epsilon(\pi) f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)} \in \mathcal{H}^{\otimes n},$$

where $\epsilon(\pi)$ is 1 if π is an even permutation and -1 if π is an odd permutation.

Definition 1.1.8. The symmetrization operator P_+ and the antisymmetrization operator P_- are defined on $\mathcal{F}(\mathcal{H})$ by $P_{\pm}\Omega = \Omega$ and if $\varphi \in \mathcal{F}(\mathcal{H})$, $\varphi = \{\varphi_n\}_{n \geq 0}$, then $P_{\pm}\varphi = \{P_{\pm}\varphi_n\}_{n \geq 0}$, where for $\{f_k\}_{k=1}^n \subseteq \mathcal{H}$ ($n \geq 1$)

$$\begin{aligned} P_+ f_1 \otimes \dots \otimes f_n &= \frac{1}{n!} \sum_{\pi \in S_n} f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)} \\ P_- f_1 \otimes \dots \otimes f_n &= \frac{1}{n!} \sum_{\pi \in S_n} \epsilon(\pi) f_{\pi(1)} \otimes \dots \otimes f_{\pi(n)} \end{aligned}$$

Remark 1.1.9. P_{\pm} are actually self-adjoint projections. That is, we have that $P_{\pm}^2 = P_{\pm}$, $P_{\pm}^* = P_{\pm}$ and $\|P_{\pm}\| = 1$.

Example 1.1.10. Let $\varphi \in \mathcal{F}(\mathcal{H})$, $\varphi = \{\varphi_n\}_{n \geq 0}$. Then $P_+\varphi = \{P_+\varphi_n\}_{n \geq 0}$. If $\varphi_3 = f_1 \otimes f_2 \otimes f_3$,

$$\begin{aligned} [P_+\varphi]_3 &= P_+\varphi_3 = P_+f_1 \otimes f_2 \otimes f_3 \\ &= \frac{1}{3!} \sum_{\pi \in S_3} f_{\pi(1)} \otimes f_{\pi(2)} \otimes f_{\pi(3)} \\ &= \frac{1}{3!} \left(f_1 \otimes f_2 \otimes f_3 + f_1 \otimes f_3 \otimes f_2 + f_2 \otimes f_1 \otimes f_3 + \right. \\ &\quad \left. + f_2 \otimes f_3 \otimes f_1 + f_3 \otimes f_1 \otimes f_2 + f_3 \otimes f_2 \otimes f_1 \right) \end{aligned}$$

Note if we take P_+ on the right hand side it changes nothing.

Definition 1.1.11. The *Bosonic Fock Space* $\mathcal{F}_+(\mathcal{H})$ and the *Fermionic Fock Space* $\mathcal{F}_-(\mathcal{H})$ are given by

$$\mathcal{F}_{\pm}(\mathcal{H}) = P_{\pm}\mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} P_{\pm}\mathcal{H}^{\otimes n}.$$

Definition 1.1.12. Let $f \in \mathcal{H}$ and let $\varphi \in \mathcal{F}(\mathcal{H})$, $\varphi = \{\varphi_n\}_{n \geq 0}$, where $\varphi_n = f_1^{(n)} \otimes f_2^{(n)} \otimes \cdots \otimes f_n^{(n)}$ for some $\{f_k^{(n)}\}_{k=1}^n \subseteq \mathcal{H}$ ($n \geq 1$). Then the *annihilation operator* $a(f)$ defined on $\mathcal{F}(\mathcal{H})$ is given by

$$[a(f)\varphi]_{n-1} = a(f)\varphi_n = a(f)f_1^{(n)} \otimes \cdots \otimes f_n^{(n)} \stackrel{\text{def}}{=} \sqrt{n} \langle f, f_1^{(n)} \rangle_{\mathcal{H}} f_2^{(n)} \otimes \cdots \otimes f_n^{(n)}, \quad n \geq 1.$$

The *creation operator* $a^*(f)$ defined on $\mathcal{F}(\mathcal{H})$ is given by $[a^*(f)\varphi]_0 = 0$, $[a^*(f)\varphi]_1 = f$ and

$$[a^*(f)\varphi]_{n+1} = a^*(f)\varphi_n = a^*(f)f_1^{(n)} \otimes \cdots \otimes f_n^{(n)} \stackrel{\text{def}}{=} \sqrt{n+1} f \otimes f_1^{(n)} \otimes \cdots \otimes f_n^{(n)}, \quad n \geq 1.$$

Remark 1.1.13. Note the map $f \mapsto a(f)$ is antilinear, while the map $f \mapsto a^*(f)$ is linear.

Example 1.1.14. Let $f \in \mathcal{H}$. Then $a(f)\Omega = \{0, 0, \dots\}$ and $a^*(f)\Omega = \{0, f, 0, 0, \dots\}$.

Proposition 1.1.15. Let $\varphi_n \in \mathcal{H}^{\otimes n}$, $f \in \mathcal{H}$. Then

- (1) $\|a(f)\varphi_n\|_{\mathcal{H}^{\otimes(n-1)}} \leq \sqrt{n} \|f\|_{\mathcal{H}} \|\varphi_n\|_{\mathcal{H}^{\otimes n}}$
- (2) $\|a^*(f)\varphi_n\|_{\mathcal{H}^{\otimes(n+1)}} = \sqrt{n+1} \|f\|_{\mathcal{H}} \|\varphi_n\|_{\mathcal{H}^{\otimes n}}$

Proof. (1) Let $\varphi_n \in \mathcal{H}^{\otimes n}$. Then $\varphi_n = \varphi_n^{(1)} \otimes \varphi_n^{(2)} \otimes \cdots \otimes \varphi_n^{(n)}$, where $\varphi_n^{(k)} \in \mathcal{H}^{\otimes k}$ for $k = 1, 2, \dots, n$. Note

$$\begin{aligned} \|a(f)\varphi_n\|_{\mathcal{H}^{\otimes(n-1)}} &= \sup_{h \in \mathcal{H}^{\otimes(n-1)}, \|h\|=1} |\langle h, a(f)\varphi_n \rangle_{\mathcal{H}^{\otimes(n-1)}}| \\ &= \sup_{\|h\|=1} |\langle h, \sqrt{n} \langle f, \varphi_n^{(1)} \rangle_{\mathcal{H}} \varphi_n^{(2)} \otimes \cdots \otimes \varphi_n^{(n)} \rangle_{\mathcal{H}^{\otimes(n-1)}}| \\ &= \sup_{\|h\|=1} |\sqrt{n} \langle f, \varphi_n^{(1)} \rangle_{\mathcal{H}} \langle h, \varphi_n^{(2)} \otimes \cdots \otimes \varphi_n^{(n)} \rangle_{\mathcal{H}^{\otimes(n-1)}}| \\ &= \sup_{\|h\|=1} |\sqrt{n} \langle f \otimes h, \varphi_n^{(1)} \otimes \varphi_n^{(2)} \otimes \cdots \otimes \varphi_n^{(n)} \rangle_{\mathcal{H}^{\otimes n}}| \\ &\leq \sqrt{n} \sup_{\|h\|=1} \|f \otimes h\|_{\mathcal{H}^{\otimes n}} \|\varphi_n\|_{\mathcal{H}^{\otimes n}} \\ &= \sqrt{n} \sup_{\|h\|=1} \|f\|_{\mathcal{H}} \|h\|_{\mathcal{H}^{\otimes(n-1)}} \|\varphi_n\|_{\mathcal{H}^{\otimes n}} \end{aligned}$$

$$= \sqrt{n} \|f\|_{\mathcal{H}} \|\varphi_n\|_{\mathcal{H}^{\otimes n}}$$

$$(2) \quad \begin{aligned} \|a^*(f)\varphi_n\|_{\mathcal{H}^{\otimes(n+1)}} &= \sqrt{n+1} \|f \otimes \varphi_n^{(1)} \otimes \cdots \otimes \varphi_n^{(n)}\|_{\mathcal{H}^{\otimes(n+1)}} \\ &= \sqrt{n+1} \|f\|_{\mathcal{H}} \|\varphi_n\|_{\mathcal{H}^{\otimes n}} \end{aligned}$$

□

Proposition 1.1.16. *Let $f \in \mathcal{H}$. The following inequality*

$$\|a^\#(f)\varphi\| \leq \|f\|_{\mathcal{H}} \|(\mathcal{N} + 1)^{1/2}\varphi\|$$

holds for $\varphi \in D(\mathcal{N}^{1/2}) \subset \mathcal{F}(\mathcal{H})$, where $a^\#(f)$ means either $a^*(f)$ or $a(f)$.

Proof. Let $\varphi = \{\varphi_n\}_{n \geq 0} \in D(\mathcal{N}^{1/2})$. Recall that $\mathcal{N}\varphi = \{n\varphi_n\}_{n \geq 0}$. Thus, $\mathcal{N}^2\varphi = \mathcal{N}\mathcal{N}\varphi = \{n^2\varphi_n\}_{n \geq 0}$, $\mathcal{N}^3\varphi = \{n^3\varphi_n\}_{n \geq 0}$, and so on. In general, $\mathcal{N}^k\varphi = \{n^k\varphi_n\}_{n \geq 0}$ ($k \geq 1$). Now, one can show by expanding $(\mathcal{N} + 1)^{1/2}$ as a Taylor series about $\mathcal{N} = 0$ that $(\mathcal{N} + 1)^{1/2}\varphi = \{\sqrt{n+1}\varphi_n\}_{n \geq 0}$. So

$$\|(\mathcal{N} + 1)^{1/2}\varphi\|^2 = \sum_{n \geq 0} \|\sqrt{n+1}\varphi_n\|_{\mathcal{H}^{\otimes n}}^2 = \sum_{n \geq 0} (n+1) \|\varphi_n\|_{\mathcal{H}^{\otimes n}}^2.$$

Now, note

$$\|a(f)\varphi\|^2 = \sum_{n \geq 1} \|a(f)\varphi_n\|_{\mathcal{H}^{\otimes(n-1)}}^2 \leq \sum_{n \geq 1} n \|f\|_{\mathcal{H}}^2 \|\varphi_n\|_{\mathcal{H}^{\otimes n}}^2$$

using (1) of Proposition (1.1.15). So

$$\|a(f)\varphi\|^2 \leq \sum_{n \geq 1} (n+1) \|f\|_{\mathcal{H}}^2 \|\varphi_n\|_{\mathcal{H}^{\otimes n}}^2 \leq \|(\mathcal{N} + 1)^{1/2}\varphi\|^2 \|f\|_{\mathcal{H}}^2$$

Also,

$$\|a^*(f)\varphi\|^2 = \sum_{n \geq 0} \|a^*(f)\varphi_n\|_{\mathcal{H}^{\otimes(n+1)}}^2 = \sum_{n \geq 0} (n+1) \|f\|_{\mathcal{H}}^2 \|\varphi_n\|_{\mathcal{H}^{\otimes n}}^2$$

using (2) of Proposition (1.1.15). So

$$\|a^*(f)\varphi\| = \|f\|_{\mathcal{H}} \|(\mathcal{N} + 1)^{1/2}\varphi\|.$$

□

Remark 1.1.17. The adjoint of $a(f)$ is actually $a^*(f)$. That is, $a(f)^* = a^*(f)$.

Definition 1.1.18. The bosonic (+) and fermionic (-) creation and annihilation operators are defined to be

$$a_\pm^*(f) = P_\pm a^*(f) P_\pm \quad \text{and} \quad a_\pm(f) = P_\pm a(f) P_\pm,$$

respectively. We can also write them as $a_\pm(f) = a(f) P_\pm$ and $a_\pm^*(f) = P_\pm a^*(f)$.

The bosonic creation and annihilation operators satisfy the canonical commutation relations (CCR):

$$\begin{aligned} [a_+(f), a_+^*(g)] &= \langle f, g \rangle \mathbb{1}_{\mathcal{F}_+(\mathcal{H})} \\ [a_+(f), a_+(g)] &= [a_+^*(f), a_+^*(g)] = 0 \end{aligned}$$

The fermionic creation and annihilation operators satisfy the canonical anti-commutation relations (CAR) which is the same as above with the commutators $[\]$ replaced by anticommutators $\{ \ }$ (where the anticommutator $\{x, y\} = xy + yx$.)

Remark 1.1.19. There is no Pauli Principle for bosons. That is, there is no bound on the number of particles which can occupy a given state. This corresponds to the unboundedness of bosonic creation and annihilation operators. Fermions obey the Pauli Principle and it follows that the fermionic creation and annihilation operators are bounded. However, because the bosonic creation and annihilation operators are unbounded, we basically need to “replace” them with bounded operators called Weyl operators.

1.2 Bosonic Fock space over $L^2(\mathbb{R}^3, dx)$ for one species

Now we consider a system of N bosonic particles of one species which are described on the Hilbert space $L_s^2(\mathbb{R}^{3N})$, which is the subspace of $L^2(\mathbb{R}^{3N})$ that contains all functions symmetric with respect to arbitrary permutations of the N particles. We have the following definition.

Definition 1.2.1. The *bosonic Fock space over $L^2(\mathbb{R}^3, dx)$* can be written as the Hilbert space

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2(\mathbb{R}^3, dx)^{\otimes n} \cong \mathbb{C} \oplus \bigoplus_{n \geq 1} L_s^2(\mathbb{R}^{3n}, dx_1 \dots dx_n). \quad (1.1)$$

Remark 1.2.2. If the system has an initial wave function ψ_N , then the time evolution $\psi_{N,t}$ is given by the Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \quad \psi_{N,0} = \psi_N$$

where we will consider the mean-field Hamiltonian H_N to be

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N} \sum_{i < j}^N V(x_i - x_j).$$

Remark 1.2.3. Let $\psi \in \mathcal{F}$. Consider ψ_N for some $N \in \mathbb{N}$. Note ψ_N can be written as a symmetrized tensor product

$$\psi_N = \varphi_1 \otimes_s \varphi_2 \otimes_s \dots \otimes_s \varphi_N = \frac{1}{N!} \sum_{\pi \in S_N} \varphi_{\pi(1)} \otimes \dots \otimes \varphi_{\pi(N)} \in L^2(\mathbb{R}^3)^{\otimes_s N}$$

where $\varphi_k \in L^2(\mathbb{R}^3)$. Equivalently, because of the isomorphism in (1.1), we can use Definition (A.4.10) to write

$$\psi_N = \psi_N(x_1, \dots, x_N) = \frac{1}{N!} \sum_{\pi \in S_N} \varphi_1(x_{\pi(1)}) \varphi_2(x_{\pi(2)}) \dots \varphi_N(x_{\pi(N)}) \in L_s^2(\mathbb{R}^{3N})$$

where $x_k \in \mathbb{R}^3$. Thus, if ψ is an element of \mathcal{F} it can be represented as a sequence $\{\psi_n\}_{n \geq 0}$ where $\psi_n \in L_s^2(\mathbb{R}^{3n})$ is a symmetric n -particle wave function. Alternatively we can represent ψ_n as a symmetrized tensor product. We will use these two representations interchangeably.

Notation 1.2.4. The above remark means that if $\psi, \varphi \in \mathcal{F}$, where $\psi = \{\psi_n\}_{n \geq 0}$ and $\varphi = \{\varphi_n\}_{n \geq 0}$, we can write either

$$\langle \psi, \varphi \rangle = \sum_{n \geq 0} \langle \psi_n, \varphi_n \rangle_{L^2(\mathbb{R}^3)^{\otimes_s n}}$$

when $\psi_n, \varphi_n \in L^2(\mathbb{R}^3)^{\otimes_s n}$, or we can write

$$\langle \psi, \varphi \rangle = \sum_{n \geq 0} \langle \psi_n, \varphi_n \rangle_{L^2(\mathbb{R}^{3n})}$$

when $\psi_n, \varphi_n \in L_s^2(\mathbb{R}^{3n})$. Since the two representations are equivalent, we use the notation

$$\langle \psi, \varphi \rangle = \sum_{n \geq 0} \langle \psi_n, \varphi_n \rangle_{L^2(\mathbb{R}^3)^{\otimes_s n}} = \sum_{n \geq 0} \langle \psi_n, \varphi_n \rangle_{L^2(\mathbb{R}^{3n})}.$$

Again, all norms and inner products used without subscripts in this section will be a norm or inner product over the Fock space.

Again \mathcal{F} has a vacuum vector Ω given by $\Omega = \{1, 0, 0, \dots\}$. We also have the number operator \mathcal{N} as well as the bosonic creation and annihilation operators $a_+^*(f)$ and $a_+(f)$, as before, except now we can choose to define them slightly differently.

1.2.1 Creation and annihilation operators

Let us now use the previous definition of the bosonic creation and annihilation operators given in Definition (1.1.18) to derive a new expression for our context where $\mathcal{H} = L^2(\mathbb{R}^3, dx)$. Let $\psi = \{\psi_n\}_{n \geq 0}$ be in our bosonic Fock space over $L^2(\mathbb{R}^3, dx)$. Then $\psi_n = P_+ \varphi_1^{(n)} \otimes \dots \otimes \varphi_n^{(n)}$, where $\varphi_k^{(n)} \in L^2(\mathbb{R}^3)$. Note $[a_+^*(f)\psi]_0 = 0$ and if $n \geq 1$, we have

$$\begin{aligned} [a_+^*(f)\psi]_n &= [P_+ a^*(f)\psi]_n = P_+ a^*(f) \frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \varphi_1^{(n-1)}(x_{\pi(1)}) \varphi_2^{(n-1)}(x_{\pi(2)}) \dots \varphi_{n-1}^{(n-1)}(x_{\pi(n-1)}) \\ &= \sqrt{n} P_+ f \otimes \left(\frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \varphi_1^{(n-1)}(x_{\pi(1)}) \varphi_2^{(n-1)}(x_{\pi(2)}) \dots \varphi_{n-1}^{(n-1)}(x_{\pi(n-1)}) \right) \\ &= \frac{\sqrt{n}}{(n-1)!} \frac{(n-1)!}{n!} \sum_{\pi \in S_n} f(x_{\pi(1)}) \varphi_1^{(n-1)}(x_{\pi(2)}) \dots \varphi_{n-1}^{(n-1)}(x_{\pi(n)}) \\ &= \frac{\sqrt{n}}{n} \sum_{j=1}^n f(x_j) \left(\frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}(\{1, \dots, j-1, j+1, \dots, n\})} \varphi_1^{(n-1)}(x_{\pi(1)}) \dots \varphi_{n-1}^{(n-1)}(x_{\pi(n-1)}) \right) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n f(x_j) \psi_{n-1}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n). \end{aligned}$$

To derive our new expression for $a_+(f)$, note for $n \geq 0$ we have

$$\begin{aligned}
[a_+(f)\psi]_n &= [a(f)P_+\psi]_n = a(f)P_+\psi_{n+1} \\
&= a(f)\frac{1}{(n+1)!} \sum_{\pi \in S_{n+1}} \varphi_{\pi(1)}^{(n+1)} \otimes \cdots \otimes \varphi_{\pi(n+1)}^{(n+1)} \\
&= \frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi \in S_{n+1}} \langle f, \varphi_{\pi(1)}^{(n+1)} \rangle \varphi_{\pi(2)}^{(n+1)} \otimes \cdots \otimes \varphi_{\pi(n+1)}^{(n+1)} \\
&= \sqrt{n+1} \int \overline{f(x)} \frac{1}{(n+1)!} \sum_{\pi \in S_{n+1}} \varphi_{\pi(1)}^{(n+1)}(x) \varphi_{\pi(2)}^{(n+1)} \otimes \cdots \otimes \varphi_{\pi(n+1)}^{(n+1)} dx \\
&= \sqrt{n+1} \int \overline{f(x)} \psi_{n+1}(x, x_1, \dots, x_n) dx.
\end{aligned}$$

This gives rise to new definitions for the bosonic creation and annihilation operators. To simplify notation, from now on we will refer to the bosonic creation and annihilation operators as $a^*(f)$ and $a(f)$, respectively.

Definition 1.2.5. The *bosonic creation and annihilation operators* $a^*(f)$ and $a(f)$, respectively, are defined on \mathcal{F} by

$$\begin{aligned}
[a^*(f)\psi]_0 &= 0, \quad [a^*(f)\psi]_n(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n f(x_j) \psi_{n-1}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \quad n \geq 1 \\
[a(f)\psi]_n(x_1, \dots, x_n) &= \sqrt{n+1} \int dx \overline{f(x)} \psi_{n+1}(x, x_1, x_2, \dots, x_n), \quad n \geq 0
\end{aligned} \tag{1.2}$$

As before, the bosonic creation and annihilation operators satisfy the CCR

$$[a(f), a^*(g)] = \langle f, g \rangle_{L^2(\mathbb{R}^3)}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0. \tag{1.3}$$

Sometimes it is also useful to use the following notation.

Definition 1.2.6. For every $f \in L^2(\mathbb{R}^3)$, we define $\Phi(f)$ to be the self-adjoint operator given by

$$\Phi(f) = a^*(f) + a(f)$$

where $a^*(f)$ and $a(f)$ are the bosonic creation and annihilation operators given in (1.2).

In what follows below we use the notation that if $[\psi]_n = 0$ for all $n \neq N$ and $[\psi]_N = f^{\otimes N}$, then $\psi = f^{\otimes N}$.

Proposition 1.2.7. *We have that*

$$a^*(f)^n \Omega = \sqrt{n!} \{0, \dots, 0, f^{\otimes n}, 0, \dots\} = \sqrt{n!} f^{\otimes n}, \quad n \geq 1.$$

A proof can be given using induction. Here we will simply calculate $a^*(f)^n \Omega$ for $n = 1, 2, 3$ so that the reader may see the general pattern.

Note we have that $a^*(f)\Omega = (0, f, 0, \dots) = f = f^{\otimes 1}$. Then

$$\begin{aligned}
a^*(f)a^*(f)\Omega &= a^*(f)f \\
&= \frac{1}{\sqrt{2}} \sum_{j=1}^2 f(x_j) f(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_2) \\
&= \frac{1}{\sqrt{2}} (f(x_1)f(x_2) + f(x_2)f(x_1)) = \sqrt{2}f^{\otimes 2}.
\end{aligned}$$

So

$$\begin{aligned}
a^*(f)^3\Omega &= a^*(f)\sqrt{2}f^{\otimes 2} \\
&= \frac{1}{\sqrt{3}}\sqrt{2}\sum_{j=1}^3 f(x_j)f^{\otimes 2}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_3) \\
&= \frac{1}{\sqrt{3}}\sqrt{2}(f(x_1)f^{\otimes 2}(x_2, x_3) + f(x_2)f^{\otimes 2}(x_1, x_3) + f(x_3)f^{\otimes 2}(x_1, x_2)) \\
&= \sqrt{3!}f^{\otimes 3}.
\end{aligned}$$

We also have the following relationship.

Definition 1.2.8. The operator valued distributions a_x^* and a_x (where $x \in \mathbb{R}^3$) are defined so that

$$a^*(f) = \int dx f(x)a_x^*, \quad (1.4)$$

$$a(f) = \int dx \overline{f(x)}a_x \quad (1.5)$$

for every $f \in L^2(\mathbb{R}^3)$.

Using the CCR (1.3) we can calculate the following CCR for a_x^* and a_x :

$$[a_x, a_y^*] = \delta(x - y), \quad [a_x, a_y] = [a_x^*, a_y^*] = 0 \quad (1.6)$$

Remark 1.2.9. We also often refer to a_x^* and a_x as creation and annihilation operators, respectively.

Remark 1.2.10. We can also find explicit formulas for a_x^* and a_x . Let $\psi = \{\psi_n\}_{n \geq 0} \in \mathcal{F}$. First, note

$$0 = [a^*(f)\psi]_0 = \int dx f(x)[a_x^*\psi]_0 \quad \Rightarrow \quad [a_x^*\psi]_0 = 0.$$

Then, for $n \geq 0$, we have

$$\begin{aligned}
\frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1} f(x_j)\psi_n(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) &= [a^*(f)\psi]_{n+1}(x_1, \dots, x_{n+1}) = a^*(f)\psi_n(x_1, \dots, x_n) \\
&= \int dx f(x)a_x^*\psi_n(x_1, \dots, x_n) \\
\Rightarrow a_x^*\psi_n(x_1, \dots, x_n) &= \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1} \delta(x - x_j)\psi_n(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}).
\end{aligned}$$

Also, note, for $n \geq 1$,

$$\begin{aligned}
\sqrt{n}\int dx \overline{f(x)}\psi_n(x, x_1, \dots, x_{n-1}) &= [a(f)\psi]_{n-1}(x_1, \dots, x_{n-1}) = a(f)\psi_n(x_1, \dots, x_n) \\
&= \int dx \overline{f(x)}a_x\psi_n(x_1, \dots, x_n) \\
\Rightarrow a_x\psi_n(x_1, \dots, x_n) &= \sqrt{n}\psi_n(x, x_1, \dots, x_{n-1}).
\end{aligned}$$

Definition 1.2.11. The Hamiltonian operator \mathcal{H}_N on \mathcal{F} is defined by $[\mathcal{H}_N\psi]_n = \mathcal{H}_N^{(n)}[\psi]_n$, where

$$\mathcal{H}_N^{(n)} = -\sum_{j=1}^n \Delta_{x_j} + \frac{1}{N} \sum_{i<j}^n V(x_i - x_j).$$

Using the distributions a_x and a_x^* , \mathcal{H}_N can be rewritten as

$$\mathcal{H}_N = \int dx \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N} \int dx dy V(x-y) a_x^* a_y^* a_y a_x. \quad (1.7)$$

Remark 1.2.12. Note that the N-particle sector $\mathcal{H}_N^{(N)}$ is the same as the Hamiltonian H_N .

Remark 1.2.13. Let us show, using the explicit formulas for a_x^* and a_x that we derived in Remark (1.2.10), that \mathcal{H}_N can indeed be written as in (1.7). First, note

$$\begin{aligned} & \int dx \nabla_x a_x^* \nabla_x a_x \psi_n(x_1, \dots, x_n) \\ &= \sqrt{n} \int dx \nabla_x a_x^* \nabla_x \psi_n(x, x_1, \dots, x_{n-1}) \\ &= \sqrt{n} \int dx \nabla_x \frac{1}{\sqrt{n}} \sum_{j=1}^n \delta(x - x_j) \nabla_x \psi_n(x, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ &= -\sum_{j=1}^n \Delta_{x_j} \psi_n(x_1, \dots, x_n). \end{aligned}$$

Also,

$$\begin{aligned} & \frac{1}{2N} \int dx dy V(x-y) a_x^* a_y^* a_y a_x \psi_n(x_1, \dots, x_n) \\ &= \frac{1}{2N} \sqrt{n(n-1)} \int dx dy V(x-y) a_x^* a_y^* \psi_n(x, y, x_1, \dots, x_{n-2}) \\ &= \frac{\sqrt{n(n-1)}}{2N} \int dx dy V(x-y) a_x^* \frac{1}{\sqrt{n-1}} \sum_{j=1}^{n-1} \delta(y - x_j) \psi_n(x, y, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}) \\ &= \frac{\sqrt{n}}{2N} \int dx dy V(x-x_2) a_x^* (n-1) \psi_n(x, x_1, x_2, \dots, x_{n-1}) \\ &= \frac{1}{2N} \int dx V(x-x_2) (n-1) \sum_{j=1}^n \psi_n(x, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ &= \frac{n(n-1)}{2N} V(x_1 - x_2) \psi_n(x_1, \dots, x_n) \\ &= \frac{1}{N} \sum_{i<j}^n V(x_i - x_j) \psi_n(x_1, \dots, x_n). \end{aligned}$$

1.2.2 The number operator

Now we can express the number particle operator \mathcal{N} as

$$\mathcal{N} = \int dx a_x^* a_x. \quad (1.8)$$

Remark 1.2.14. Let us show that, using (1.8), we obtain $\mathcal{N}\psi_n(x_1, \dots, x_n) = n\psi_n(x_1, \dots, x_n)$. Note, using the explicit formulas for a_x^* and a_x in Remark (1.2.10),

$$\begin{aligned} \int dx a_x^* a_x \psi_n(x_1, \dots, x_n) &= \sqrt{n} \int dx a_x^* \psi_n(x, x_1, \dots, x_{n-1}) \\ &= \int dx \sum_{j=1}^n \delta(x - x_j) \psi_n(x, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ &= n\psi_n(x_1, \dots, x_n). \end{aligned}$$

Let us look at example which does not use the explicit formulas for a_x and a_x^* , but rather only (1.8) and Definition (1.2.8).

Example 1.2.15. Let $\psi \in \mathcal{F}$ be given by $\psi = \{0, f, 0, \dots\} = a^*(f)\Omega$. We will show that $\mathcal{N}\psi = \psi$. Note

$$\mathcal{N}\psi = \int dx a_x^* a_x a^*(f)\Omega = \iint dx dy f(y) a_x^* a_x a_y^* \Omega.$$

Now we use the CCR (1.6) to get

$$\begin{aligned} \mathcal{N}\psi &= \iint dx dy f(y) a_x^* a_y^* a_x \Omega + \int dx f(x) a_x^* \Omega \\ &= \int dy f(y) a_y^* \int dx a_x^* a_x \Omega + a^*(f)\Omega \\ &= a^*(f)\mathcal{N}\Omega + \psi = 0 + \psi = \psi. \end{aligned}$$

Proposition 1.2.16. *The number operator satisfies*

$$\begin{aligned} (i) \quad [\mathcal{N}, a_x^*] &= a_x^* \quad \text{and} \quad [\mathcal{N}, a_x] = -a_x, \\ (ii) \quad [\mathcal{N}, a^*(\varphi)] &= a^*(\varphi) \quad \text{and} \quad [\mathcal{N}, a(\varphi)] = -a(\varphi). \end{aligned}$$

Proof. To show (i), note

$$\begin{aligned} \mathcal{N}a_x^* &= \int dy a_y^* a_y a_x^* = \int dy a_y^* (a_x^* a_y + \delta(y-x)) = a_x^*(\mathcal{N}+1), \\ \mathcal{N}a_x &= \int dy a_y^* a_x a_y = \int dy (a_x a_y a_y^* - \delta(y-x)) = a_x(\mathcal{N}-1). \end{aligned}$$

Now (ii) follows from (i) and Definition (1.2.8). □

We also have the following useful propositions.

Proposition 1.2.17. *An integral containing both an annihilation and creation operator commutes with $(\mathcal{N}+1)^j$; that is*

$$\left[\int dx f(x) a_x^* a_x, (\mathcal{N}+1)^j \right] = 0, \quad \text{and} \tag{1.9}$$

$$\left[\int dx dy f(x, y) a_x^* a_y, (\mathcal{N}+1)^j \right] = 0 \tag{1.10}$$

and the same would hold if we switched the order of the creation and annihilation operators.

Proof. To show (1.9), it suffices to show $[\int dx f(x)a_x^*a_x, \mathcal{N}] = 0$. Note by (1.2.16)(ii), we have

$$\begin{aligned} \int dx f(x)a_x^*a_x\mathcal{N} &= \int dx f(x)a_x(\mathcal{N} - 1)a_x^* \\ &= \int dx f(x)(\mathcal{N} + 1)a_xa_x^* - \int dx f(x)a_xa_x^* \\ &= \mathcal{N} \int dx f(x)a_x^*a_x. \end{aligned}$$

It is easy to see (1.10) follows similarly. □

Proposition 1.2.18. *Let $\psi \in \mathcal{F}$. Then*

$$\int dx \|a_x\psi\|^2 = \|\mathcal{N}^{1/2}\psi\|^2. \quad (1.11)$$

Proof. Note

$$\begin{aligned} \|\mathcal{N}^{1/2}\psi\|^2 &= \langle \mathcal{N}^{1/2}\psi, \mathcal{N}^{1/2}\psi \rangle = \langle \psi, \mathcal{N}\psi \rangle = \left\langle \psi, \int dx a_x^*a_x\psi \right\rangle \\ &= \sum_{n \geq 1} \int dx_1 \cdots dx_n \bar{\psi}_n(x_1, \dots, x_n) \int dx a_x^*a_x\psi_n(x_1, \dots, x_n) \\ &= \sum_{n \geq 1} \int dx_1 \cdots dx_n \int dx |a_x\psi_n(x_1, \dots, x_n)|^2 \\ &= \int dx \langle a_x\psi, a_x\psi \rangle = \int dx \|a_x\psi\|^2. \end{aligned}$$

□

Proposition 1.2.19. *Let $\psi \in \mathcal{F}$ and let $\varphi \in L^2$ such that $\|\varphi\|_{L^2} = 1$. Then*

$$\int dx |\varphi(x)| \|a_x\psi\| \leq \|\mathcal{N}^{1/2}\psi\|. \quad (1.12)$$

Proof. Let $f(x) = \|a_x\psi\|$. Then

$$\begin{aligned} \int dx |\varphi(x)| \|a_x\psi\| &= \int dx |\varphi(x)|f(x) = \langle \varphi, f \rangle_{L^2} \\ &\leq \|\varphi\|_{L^2} \|f\|_{L^2} \\ &= \left(\int dx |f(x)|^2 \right)^{1/2} \\ &= \left(\int dx \|a_x\psi\|^2 \right)^{1/2} \\ &= \|\mathcal{N}^{1/2}\psi\| \end{aligned}$$

where in the last line we used (1.11). □

We also have the following important inequalities.

Proposition 1.2.20. *Let $f \in L^2(\mathbb{R}^3)$. Then*

$$\|a(f)\psi\| \leq \|f\|_{L^2} \|\mathcal{N}^{1/2}\psi\|, \quad (1.13)$$

$$\|a^*(f)\psi\| \leq \|f\|_{L^2} \|(\mathcal{N} + 1)^{1/2}\psi\|, \quad (1.14)$$

$$\|\Phi(f)\psi\| \leq 2\|f\|_{L^2} \|(\mathcal{N} + 1)^{1/2}\psi\|. \quad (1.15)$$

Proof. To prove (1.13), note

$$\begin{aligned} \|a(f)\psi\| &= \left\| \int dx \overline{f(x)} a_x \psi \right\| \\ &\leq \int dx |f(x)| \|a_x \psi\| \\ &\leq \left(\int dx |f(x)|^2 \right)^{1/2} \left(\int dx \|a_x \psi\|^2 \right)^{1/2} \\ &= \|f\|_{L^2} \|\mathcal{N}^{1/2}\psi\| \end{aligned}$$

where in the last line we used (1.11). To prove (1.14), first note, using the CCR in (1.3) and the inequality (1.13), we have

$$\begin{aligned} \|a^*(f)\psi\|^2 &= \langle \psi, a(f)a^*(f)\psi \rangle = \langle \psi, (a^*(f)a(f) + \|f\|_{L^2}^2) \psi \rangle \\ &= \langle a(f)\psi, a(f)\psi \rangle + \|f\|_{L^2}^2 \langle \psi, \psi \rangle = \|a(f)\psi\|^2 + \|f\|_{L^2}^2 \|\psi\|^2 \\ &\leq \|f\|_{L^2}^2 (\|\mathcal{N}^{1/2}\psi\|^2 + \|\psi\|^2). \end{aligned}$$

Next, note, again using (1.11),

$$\begin{aligned} \|\mathcal{N}^{1/2}\psi\|^2 + \|\psi\|^2 &= \int dx \|a_x \psi\|^2 + \|\psi\|^2 \\ &= \int dx \langle \psi, a_x^* a_x \psi \rangle + \|\psi\|^2 \\ &= \left\langle \psi, \int dx a_x^* a_x \psi \right\rangle + \langle \psi, \psi \rangle \\ &= \langle \psi, (\mathcal{N} + 1) \psi \rangle \\ &= \|(\mathcal{N} + 1)^{1/2} \psi\|^2, \end{aligned}$$

which gives us the desired inequality (1.14). Finally, (1.15) follows from (1.13) and (1.14). \square

1.2.3 Weyl operators and coherent states

Definition 1.2.21. Let $f \in L^2(\mathbb{R}^3)$. Then the *Weyl operator* is given by

$$W(f) = e^{a^*(f) - a(f)} = e^{\int dx (f(x)a_x^* - \overline{f(x)}a_x)}.$$

Now we state and prove several properties of the Weyl operator.

Proposition 1.2.22. *Let $f, g \in L^2(\mathbb{R}^3)$. Then (i) the Weyl operator satisfies the relations*

$$W(f)W(g) = W(g)W(f)e^{-2i\text{Im}\langle f, g \rangle_{L^2}} = W(f+g)e^{-i\text{Im}\langle f, g \rangle_{L^2}},$$

known as the Weyl form of the CCR. In addition, $W(f)$ has the properties

- (ii) $W(f)^* = W(f)^{-1} = W(-f)$,
- (iii) $W^*(f)a_x W(f) = a_x + f(x)$, and
- (iv) $W^*(f)a(f)W(f) = a(f) + \|f\|_{L^2}^2$.

Proof. (i) First we show that $W(f)W(g) = W(g)W(f)e^{-2i\text{Im}\langle f, g \rangle_{L^2}}$. Note

$$W(f)W(g) = e^{a^*(f)-a(f)}e^{a^*(g)-a(g)} = e^{-\frac{1}{2}\|f\|_{L^2}^2}e^{a^*(f)}e^{-a(f)}e^{a^*(g)-a(g)}$$

using Lemma (A.2.1). Now using Proposition (A.2.2),

$$e^{a^*(f)}e^{-a(f)}e^{a^*(g)-a(g)} = e^{[-a(f), a^*(g)-a(g)]}e^{a^*(g)-a(g)}e^{-a(f)} = e^{-\langle f, g \rangle_{L^2}}e^{a^*(g)-a(g)}e^{-a(f)}.$$

So

$$W(f)W(g) = e^{-\frac{1}{2}\|f\|_{L^2}^2}e^{-\langle f, g \rangle_{L^2}}e^{a^*(f)}e^{a^*(g)-a(g)}e^{-a(f)}$$

We use Proposition (A.2.2) again to get $e^{a^*(f)}e^{a^*(g)-a(g)} = e^{\langle g, f \rangle_{L^2}}e^{a^*(g)-a(g)}e^{a^*(f)}$. So

$$W(f)W(g) = e^{-\frac{1}{2}\|f\|_{L^2}^2}e^{-\langle f, g \rangle_{L^2}}e^{\langle g, f \rangle_{L^2}}e^{a^*(g)-a(g)}e^{a^*(f)}e^{-a(f)}$$

Finally we use Lemma (A.2.1) to get $e^{a^*(f)}e^{-a(f)} = e^{\frac{1}{2}\|f\|_{L^2}^2}e^{a^*(f)-a(f)}$. So

$$\begin{aligned} W(f)W(g) &= e^{-\langle f, g \rangle_{L^2} + \overline{\langle f, g \rangle_{L^2}}}e^{a^*(g)-a(g)}e^{a^*(f)-a(f)} \\ &= e^{-2i\text{Im}\langle f, g \rangle_{L^2}}W(g)W(f). \end{aligned}$$

Thus we have shown that $W(f)W(g) = W(g)W(f)e^{-2i\text{Im}\langle f, g \rangle_{L^2}}$. Now we show that

$W(g)W(f)e^{-2i\text{Im}\langle f, g \rangle_{L^2}} = W(f+g)e^{-i\text{Im}\langle f, g \rangle_{L^2}}$. First, note that

$$[a^*(g) - a(g), a^*(f) - a(f)] = [a(f), a^*(g)] - [a(g), a^*(f)] = \langle f, g \rangle_{L^2} - \overline{\langle f, g \rangle_{L^2}}.$$

So, using Lemma (A.2.1),

$$\begin{aligned} W(g)W(f)e^{-2i\text{Im}\langle f, g \rangle_{L^2}} &= e^{\frac{1}{2}(\langle f, g \rangle_{L^2} - \overline{\langle f, g \rangle_{L^2}})}e^{a^*(g)-a(g)+a^*(f)-a(f)}e^{-2i\text{Im}\langle f, g \rangle_{L^2}} \\ &= e^{i\text{Im}\langle f, g \rangle_{L^2}}e^{a^*(f+g)-a(f+g)}e^{-2i\text{Im}\langle f, g \rangle_{L^2}} \\ &= e^{-i\text{Im}\langle f, g \rangle_{L^2}}W(f+g). \end{aligned}$$

(ii) Note

$$W(f)^* = e^{(a^*(f)-a(f))^*} = e^{a(f)-a^*(f)} = e^{a^*(-f)-a(-f)} = W(-f).$$

Also using (i) we see that $W(f)W(-f) = W(0)e^{-i\text{Im}\langle f, -f \rangle_{L^2}} = \mathbf{1} = W(-f)W(f)$, and so $W(f)^* = W(f)^{-1}$.

(iii) First note that

$$\begin{aligned} a^*(f)a_x &= \int dy f(y)a_y^*a_x = \int dy f(y)(a_x a_y^* - \delta(x-y)) = a_x a^*(f) - f(x), \text{ and} \\ a(f)a_x &= \int dy \overline{f(y)}a_y a_x = a_x a(f). \end{aligned}$$

So $[-a^*(f) + a(f), a_x] = -[a^*(f), a_x] = f(x)$. Thus, using Lemma (A.4),

$$W^*(f)a_x W(f) = e^{-a^*(f)+a(f)}a_x e^{a^*(f)-a(f)} = a_x + [-a^*(f) + a(f), a_x] = a_x + f(x).$$

(iv) This follows from (iii). (It can also easily be proved using Lemma (A.4).) Note

$$W^*(f)a(f)W(f) = \int dx \overline{f(x)}W^*(f)a_x W(f) = \int dx \overline{f(x)}(a_x + f(x)) = a(f) + \|f\|_{L^2}^2.$$

□

Definition 1.2.23. Given $f \in L^2(\mathbb{R}^3)$, the *coherent state* $\psi(f) \in \mathcal{F}$ is defined by

$$\psi(f) = W(f)\Omega.$$

Coherent states have the following properties.

Proposition 1.2.24. Let ψ be a coherent state, $f, g \in L^2(\mathbb{R}^3)$. Then

(i) We have that

$$\psi(f) = e^{-\frac{1}{2}\|f\|_{L^2}^2} e^{a^*(f)}\Omega = e^{-\frac{1}{2}\|f\|_{L^2}^2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

(ii) The norm of ψ is equal to 1; that is, $\|\psi\| = 1$.

(iii) Coherent states are eigenvectors of the annihilation operators. In particular,

$$a_x \psi(f) = f(x)\psi(f), \text{ and it follows that } a(g)\psi(f) = \langle g, f \rangle_{L^2} \psi(f).$$

(iv) The expectation of the number of particles in the coherent state $\psi(f)$ is equal to $\|f\|_{L^2}^2$; that is,

$$\langle \psi(f), \mathcal{N}\psi(f) \rangle = \|f\|_{L^2}^2.$$

(v) Coherent states are not orthogonal to each other but rather

$$\langle \psi(f), \psi(g) \rangle = e^{-1/2(\|f\|_{L^2}^2 + \|g\|_{L^2}^2 - 2\langle f, g \rangle_{L^2})}.$$

Proof. (i) First, note using Lemma (A.2.1),

$$W(f)\Omega = e^{a^*(f)-a(f)}\Omega = e^{-\frac{1}{2}\|f\|_{L^2}^2} e^{a^*(f)} e^{-a(f)}\Omega.$$

But $a(f)\Omega = 0$, so $e^{-a(f)}\Omega = \left(\mathbf{1} - a(f) + \frac{a(f)^2}{2!} - \frac{a(f)^3}{3!} + \dots \right) \Omega = \Omega$. So $W(f)\Omega = e^{-\frac{1}{2}\|f\|_{L^2}^2} e^{a^*(f)}\Omega$. Now, expanding $e^{a^*(f)}$ as a series and using Proposition (1.2.7), we see that

$$W(f)\Omega = e^{-\frac{1}{2}\|f\|_{L^2}^2} \sum_{n=0}^{\infty} \frac{a^*(f)^n}{n!} \Omega = e^{-\frac{1}{2}\|f\|_{L^2}^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

(ii) Note

$$\begin{aligned}\|\psi(f)\|^2 &= \langle W(f)\Omega, W(f)\Omega \rangle = \langle W(-f)W(f)\Omega, \Omega \rangle = \langle \Omega, \Omega \rangle = 1. \\ \Rightarrow \|\psi(f)\| &= 1.\end{aligned}$$

(iii) Note from Proposition (1.2.22) (iii),

$$a_x\psi(f) = a_xW(f)\Omega = W(f)(a_x + f(x))\Omega.$$

Then since $a_x\Omega = 0$, we get that

$$a_x\psi(f) = f(x)W(f)\Omega = f(x)\psi(f).$$

Hence it follows that

$$a(g)\psi(f) = \int dx \overline{g(x)} a_x\psi(f) = \int dx \overline{g(x)} f(x)\psi(f) = \langle g, f \rangle_{L^2} \psi(f).$$

(iv) Note

$$\begin{aligned}\mathcal{N}\psi(f) &= \int dx a_x^* a_x W(f)\Omega \\ &= \int dx W(f) (a_x^* + \overline{f(x)}) (a_x + f(x)) \Omega\end{aligned}$$

using (iii). Then using the fact that $\mathcal{N}\Omega = 0$ and $a(f)\Omega = 0$, as well as Proposition (1.2.22)(iv), we have

$$\mathcal{N}\psi(f) = W(f)a^*(f)\Omega + \|f\|_{L^2}^2 W(f)\Omega = (a^*(f) - \|f\|_{L^2}^2) W(f)\Omega + \|f\|_{L^2}^2 W(f)\Omega = a^*(f)\psi(f).$$

It follows that

$$\langle \psi(f), \mathcal{N}\psi(f) \rangle = \langle \psi(f), a^*(f)\psi(f) \rangle = \langle a(f)\psi(f), \psi(f) \rangle.$$

Now $a(f)\psi(f) = a(f)W(f)\Omega = W(f)(a(f) + \|f\|_{L^2}^2)\Omega = \|f\|_{L^2}^2 \psi(f)$. So, using (ii), we get that

$$\langle \psi(f), \mathcal{N}\psi(f) \rangle = \langle \|f\|_{L^2}^2 \psi(f), \psi(f) \rangle = \|f\|_{L^2}^2 \|\psi(f)\|^2 = \|f\|_{L^2}^2.$$

(v) Note, using (i) and Proposition (1.2.22),

$$\begin{aligned}\langle \psi(f), \psi(g) \rangle &= \langle \Omega, W(-f)W(g)\Omega \rangle \\ &= \left\langle \Omega, W(-f+g)e^{-i\text{Im}\langle -f, g \rangle_{L^2}} \Omega \right\rangle \\ &= e^{-1/2(\langle -f, g \rangle_{L^2} - \overline{\langle -f, g \rangle_{L^2}})} \langle \Omega, \psi(g-f) \rangle \\ &= e^{-1/2(\langle g, f \rangle_{L^2} - \langle f, g \rangle_{L^2})} \left\langle \Omega, e^{-1/2\|g-f\|_{L^2}^2} e^{a^*(g-f)} \Omega \right\rangle \\ &= e^{-1/2(\langle g, f \rangle_{L^2} - \langle f, g \rangle_{L^2})} e^{-1/2(\|g\|_{L^2}^2 - \langle g, f \rangle_{L^2} - \langle f, g \rangle_{L^2} + \|f\|_{L^2}^2)} \left\langle \Omega, e^{a^*(g-f)} \Omega \right\rangle \\ &= e^{-1/2(\|f\|_{L^2}^2 + \|g\|_{L^2}^2 - 2\langle f, g \rangle_{L^2})}.\end{aligned}$$

□

1.3 Two species

Now we wish to consider a system of N_1 bosonic particles of one species and N_2 bosonic particles of another species. A state of this system will be an element of the two-fold tensor product of the Fock space \mathcal{F} given in (1.1). In other words, if ψ describes the state of a system containing bosonic particles of two species, then $\psi \in \mathcal{F} \otimes \mathcal{F}$.

Remark 1.3.1. Now if the system has an initial wave function ψ_{N_1, N_2} , then the time evolution $\psi_{N_1, N_2, t}$ is given by the Schrödinger equation

$$i\partial_t \psi_{N_1, N_2, t} = H_{N_1, N_2} \psi_{N_1, N_2, t}, \quad \psi_{N_1, N_2, 0} = \psi_{N_1, N_2}$$

where our Hamiltonian H_{N_1, N_2} will look like

$$\begin{aligned} H_{N_1, N_2} = & \sum_{j=1}^{N_1} -\Delta_{x_j} + \sum_{k=1}^{N_2} -\Delta_{y_k} + \frac{1}{N_1} \sum_{1 \leq r < s \leq N_1} V_1(x_s - x_r) + \frac{1}{N_2} \sum_{1 \leq p < q \leq N_2} V_2(y_q - y_p) \\ & + \frac{1}{N_1 + N_2} \sum_{\ell=1}^{N_1} \sum_{m=1}^{N_2} W(x_\ell - y_m). \end{aligned} \quad (1.16)$$

Remark 1.3.2. Consider $\psi_{N_1} = \{0, \dots, 0, \psi_{N_1}, 0, \dots\} \in \mathcal{F}$ and $\psi_{N_2} = \{0, \dots, 0, \psi_{N_2}, 0, \dots\} \in \mathcal{F}$ for some $N_1, N_2 \in \mathbb{N}$. Note by Remark (1.2.3) and Definition (A.4.5) we can choose to write $\psi_{N_1} \otimes \psi_{N_2}$ as a function of $L^2(\mathbb{R}^{3(N_1+N_2)})$. That is, we can write

$$\psi_{N_1} \otimes \psi_{N_2} = \psi_{N_1}(x_1, x_2, \dots, x_{N_1}) \psi_{N_2}(y_1, y_2, \dots, y_{N_2})$$

where $\psi_{N_1} \in L_s^2(\mathbb{R}^{3N_1})$ and $\psi_{N_2} \in L_s^2(\mathbb{R}^{3N_2})$.

Definition 1.3.3. The vacuum vector of $\mathcal{F} \otimes \mathcal{F}$ is given by $\Omega_{\mathcal{F} \otimes \mathcal{F}} = \Omega \otimes \Omega$, where Ω is the vacuum vector of \mathcal{F} . Thus,

$$\Omega_{\mathcal{F} \otimes \mathcal{F}} = \{1, 0, \dots\} \otimes \{1, 0, \dots\}$$

Remark 1.3.4. Note that we still have that the norm of the vacuum vector in $\mathcal{F} \otimes \mathcal{F}$ is equal to 1, since

$$\|\Omega_{\mathcal{F} \otimes \mathcal{F}}\|_{\mathcal{F} \otimes \mathcal{F}} = \|\Omega \otimes \Omega\|_{\mathcal{F} \otimes \mathcal{F}} = \|\Omega\|_{\mathcal{F}} \|\Omega\|_{\mathcal{F}} = 1.$$

1.3.1 Creation and annihilation operators

Let $a^*(f)$ and $a(f)$ denote the creation and annihilation operators, respectively, on \mathcal{F} , as in Definition (1.2.5).

Now there are also two new sets of creation and annihilation operators.

Definition 1.3.5. Let $\psi = \psi_1 \otimes \psi_2 \in \mathcal{F} \otimes \mathcal{F}$. Then the creation and annihilation operators $a_j^*(f)$ and $a_j(f)$, where $j = 1, 2$, are defined on $\mathcal{F} \otimes \mathcal{F}$ by

$$\begin{aligned} a_1^\#(f)\psi &= a_1^\#(f)(\psi_1 \otimes \psi_2) = (a^\#(f)\psi_1) \otimes \psi_2 \quad \text{and} \\ a_2^\#(f)\psi &= \psi_1 \otimes (a^\#(f)\psi_2), \end{aligned}$$

where $a_j^\#(f)$ means either $a_j^*(f)$ or $a_j(f)$.

We can also define $a_{j,x}^*$ and $a_{j,x}$, where $j = 1, 2$, as follows.

Definition 1.3.6. The distributions $a_{j,x}^*$ and $a_{j,x}$, where $j = 1, 2$, are defined so that

$$\begin{aligned} a_j^*(f) &= \int dx f(x) a_{j,x}^* \quad \text{and} \\ a_j(f) &= \int dx \overline{f(x)} a_{j,x}, \quad \text{where } j = 1, 2. \end{aligned}$$

Let us look at an example to see how our new creation operators $a_1^*(f_1)$ and $a_2^*(f_2)$ act on the vacuum vector of $\mathcal{F} \otimes \mathcal{F}$.

Example 1.3.7. Note $a_1^*(f_1)$ acting on the vacuum vector of $\mathcal{F} \otimes \mathcal{F}$ is given by

$$a_1^*(f_1)\Omega_{\mathcal{F} \otimes \mathcal{F}} = a_1^*(f_1)(\Omega \otimes \Omega) = (a^*(f_1)\Omega) \otimes \Omega = \{0, f_1, 0, \dots\} \otimes \Omega = f_1 \otimes \Omega$$

while $a_2^*(f_2)$ acting on the vacuum vector of $\mathcal{F} \otimes \mathcal{F}$ is given by

$$a_2^*(f_2)\Omega_{\mathcal{F} \otimes \mathcal{F}} = a_2^*(f_2)(\Omega \otimes \Omega) = \Omega \otimes (a^*(f_2)\Omega) = \Omega \otimes \{0, f_2, 0, \dots\} = \Omega \otimes f_2.$$

We also make the following definition.

Definition 1.3.8. For every $f \in L^2(\mathbb{R}^3)$ we define the self adjoint operators $\Phi_1(f)$ and $\Phi_2(f)$ as

$$\Phi_1(f) = a_1^*(f) + a_1(f) \quad \text{and} \quad \Phi_2(f) = a_2^*(f) + a_2(f).$$

The Hamiltonian operator \mathcal{H} can be written from the definition of the Hamiltonian given in (1.16) in terms of the distributions $a_{1,x}$ and $a_{2,x}$, which leads us to the next definition.

Definition 1.3.9. The Hamiltonian \mathcal{H}_{N_1, N_2} on $\mathcal{F} \otimes \mathcal{F}$ can be written in terms of the distributions $a_{1,x}$ and $a_{2,x}$ as

$$\begin{aligned} \mathcal{H}_{N_1, N_2} &= \int dx \nabla_x a_{1,x}^* \nabla_x a_{1,x} + \int dx \nabla_x a_{2,x}^* \nabla_x a_{2,x} + \\ &+ \frac{1}{2N_1} \int dx dy a_{1,x}^* a_{1,y}^* V_1(x-y) a_{1,y} a_{1,x} + \frac{1}{2N_2} \int dx dy a_{2,x}^* a_{2,y}^* V_2(x-y) a_{2,y} a_{2,x} + \\ &+ \frac{1}{N_1 + N_2} \int dx dy a_{2,x}^* a_{1,y}^* W(x-y) a_{1,y} a_{2,x} \end{aligned}$$

1.3.2 The number operators

This gives rise to two new number operators \mathcal{N}_1 and \mathcal{N}_2 , defined as follows.

Definition 1.3.10. The number operators \mathcal{N}_1 and \mathcal{N}_2 are given by

$$\mathcal{N}_1 = \int dx a_{1,x}^* a_{1,x}, \quad \mathcal{N}_2 = \int dx a_{2,x}^* a_{2,x}.$$

Remark 1.3.11. Note, if $\psi = \psi_1 \otimes \psi_2 \in \mathcal{F} \otimes \mathcal{F}$, then

$$\mathcal{N}_1\psi = (\mathcal{N}\psi_1) \otimes \psi_2 \quad \text{and} \quad \mathcal{N}_2\psi = \psi_1 \otimes (\mathcal{N}\psi_2).$$

We can now give several propositions.

Proposition 1.3.12.

(i) *We have the pull-through formulas*

$$\begin{aligned} a_{l,x}(\mathcal{N}_1 + \mathcal{N}_2) &= (\mathcal{N}_1 + \mathcal{N}_2 + 1) a_{l,x} \quad \text{and} \\ a_{l,x}^*(\mathcal{N}_1 + \mathcal{N}_2) &= (\mathcal{N}_1 + \mathcal{N}_2 - 1) a_{l,x}^*, \quad \text{for } l = 1, 2. \end{aligned}$$

(ii) *It follows that*

$$\begin{aligned} [a_{l,x}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j] &= \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k a_{l,x}^*, \\ [a_{l,x}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j] &= \sum_{k=0}^{j-1} \binom{j}{k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k a_{l,x}, \quad \text{for } l = 1, 2, \end{aligned}$$

(iii) *and*

$$\begin{aligned} [a_{l,x}^* a_{m,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j] &= \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} \left((\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{l,x}^* a_{m,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} + \right. \\ &\quad \left. + (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{l,x}^* a_{m,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \right), \\ [a_{l,x}^* a_{m,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j] &= \sum_{k=0}^{j-1} \binom{j}{k} \left((\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{l,x}^* a_{m,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} + \right. \\ &\quad \left. + (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{l,x}^* a_{m,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \right), \quad \text{for } l, m = 1, 2. \end{aligned}$$

(iv) *Also,*

$$\begin{aligned} [a_{l,x}^* a_{m,y}^* a_{n,z}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j] &= \sum_{k=0}^{j-1} \binom{j}{k} \left((\mathcal{N}_1 + \mathcal{N}_2 - 1)^{k/2} a_{l,x}^* a_{m,y}^* a_{n,z} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} + \right. \\ &\quad \left. + (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{l,x}^* a_{m,y}^* a_{n,z} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} + \right. \\ &\quad \left. + (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{l,x}^* a_{m,y}^* a_{n,z} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \right), \quad \text{for } l, m, n = 1, 2. \end{aligned}$$

Proof. (i) follows from Proposition (1.2.16) (ii). For (ii), note

$$\begin{aligned}
\left[a_{1,x}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] &= a_{1,x}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j a_{1,x}^* \\
&= (\mathcal{N}_1 + \mathcal{N}_2 + 1 - 1)^j a_{1,x}^* - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j a_{1,x}^* \\
&= \left(\sum_{k=0}^j \binom{j}{k} (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right) a_{1,x}^* \\
&= \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k a_{1,x}^*
\end{aligned}$$

and

$$\begin{aligned}
\left[a_{1,x}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] &= a_{1,x} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j a_{1,x} \\
&= (\mathcal{N}_1 + \mathcal{N}_2 + 1 + 1 - 1)^j a_{1,x} - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j a_{1,x} \\
&= \left(\sum_{k=0}^j \binom{j}{k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right) a_{1,x} \\
&= \sum_{k=0}^{j-1} \binom{j}{k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k a_{1,x}.
\end{aligned}$$

To show (iii), note

$$\begin{aligned}
\left[a_{l,x}^* a_{m,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] &= a_{l,x}^* a_{m,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j - (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j a_{l,x}^* a_{m,y}^* \\
&= a_{l,x}^* \left((\mathcal{N}_1 + \mathcal{N}_2 + 1)^j a_{m,y}^* + \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k a_{m,y}^* \right) + \\
&\quad + \left(-a_{l,x}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j + \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^k a_{l,x}^* \right) a_{m,y}^* \\
&= \sum_{k=0}^{j-1} (-1)^{j-k} \left((\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{l,x}^* a_{m,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} + \right. \\
&\quad \left. + (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{l,x}^* a_{m,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \right).
\end{aligned}$$

The expression for $\left[a_{l,x}^* a_{m,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right]$ is shown similarly. Now (iv) follows from (iii). \square

Definition 1.3.13. If A is an operator, we define

$$(-1)^A = e^{\pi i A}.$$

Proposition 1.3.14. For $j = 1, 2$ we have the following relations:

- (i) $a_{j,x} (-1)^{\mathcal{N}_1 + \mathcal{N}_2} = -(-1)^{\mathcal{N}_1 + \mathcal{N}_2} a_{j,x}$
- (ii) $a_{j,x}^* (-1)^{\mathcal{N}_1 + \mathcal{N}_2} = -(-1)^{\mathcal{N}_1 + \mathcal{N}_2} a_{j,x}^*$

Also,

$$(iii) \quad (-1)^{\mathcal{N}_1 + \mathcal{N}_2} \Omega_{\mathcal{F} \otimes \mathcal{F}} = \Omega_{\mathcal{F} \otimes \mathcal{F}}.$$

Proof. Without loss of generality, let $j = 1$. Note both $a_{1,x}$ and $(-1)^{\mathcal{N}_1}$ commute with $(-1)^{\mathcal{N}_2}$ and

$$(-1)^{\mathcal{N}_1} = e^{\pi i \mathcal{N}_1} = \sum_{n=0}^{\infty} \frac{(\pi i \mathcal{N}_1)^n}{n!}.$$

From Proposition (1.2.16)(ii), we have $a_{1,x} \mathcal{N}_1 = (\mathcal{N}_1 + 1) a_{1,x}$. It follows that $a_{1,x} (\mathcal{N}_1)^n = (\mathcal{N}_1 + 1)^n a_{1,x}$.

Hence,

$$\begin{aligned} a_{1,x} (-1)^{\mathcal{N}_1 + \mathcal{N}_2} &= a_{1,x} \sum_{n=0}^{\infty} \frac{(\pi i)^n}{n!} (\mathcal{N}_1)^n (-1)^{\mathcal{N}_2} \\ &= (-1)^{\mathcal{N}_2} \sum_{n=0}^{\infty} \frac{(\pi i)^n}{n!} (\mathcal{N}_1 + 1)^n a_{1,x} \\ &= (-1)^{\mathcal{N}_2} e^{\pi i (\mathcal{N}_1 + 1)} a_{1,x} \\ &= (-1)^{\mathcal{N}_2} \left(- (-1)^{\mathcal{N}_1} \right) a_{1,x} \\ &= - (-1)^{\mathcal{N}_1 + \mathcal{N}_2} a_{1,x}, \end{aligned}$$

which proves (i); (ii) can be shown similarly. To prove (iii), note

$$(-1)^{\mathcal{N}} \Omega = \sum_{n=0}^{\infty} \frac{(\pi i)^n}{n!} (\mathcal{N})^n \Omega = \Omega.$$

Then (iii) follows. □

1.3.3 Weyl operators and coherent states

We also have two new Weyl operators W_1 and W_2 , defined as follows.

Definition 1.3.15. Let $f_1, f_2 \in L^2(\mathbb{R}^3)$. Then the Weyl operators $W_1(f_1)$ and $W_2(f_2)$ are given by

$$\begin{aligned} W_1(f_1) &= e^{a_1^*(f_1) - a_1(f_1)} = e^{\int dx f_1 a_{1,x}^* - \int dx \bar{f}_1 a_{1,x}}, \\ W_2(f_2) &= e^{a_2^*(f_2) - a_2(f_2)} = e^{\int dx f_2 a_{2,x}^* - \int dx \bar{f}_2 a_{2,x}}. \end{aligned}$$

Remark 1.3.16. Note if $\psi = \psi_1 \otimes \psi_2 \in \mathcal{F} \otimes \mathcal{F}$, then $W_1(f_1)$ and $W_2(f_2)$ act on ψ as follows:

$$\begin{aligned} W_1(f_1)\psi &= W_1(f_1)\psi_1 \otimes \psi_2 = (W(f_1)\psi_1) \otimes \psi_2, \text{ and} \\ W_2(f_2)\psi &= W_2(f_2)\psi_1 \otimes \psi_2 = \psi_1 \otimes (W(f_2)\psi_2). \end{aligned}$$

Remark 1.3.17. Note that

$$W_1^*(f_1) W_2^*(f_2) a_{j,x} W_2(f_2) W_1(f_1) = a_{j,x} + f_j(x), \quad \text{for } j = 1, 2.$$

Now we can define a coherent state for two species.

Definition 1.3.18. Given $f_1, f_2 \in L^2(\mathbb{R}^3)$, the coherent state $\psi(f_1, f_2)$ is given by

$$\psi(f_1, f_2) = W_1(f_1) W_2(f_2) \Omega_{\mathcal{F} \otimes \mathcal{F}}$$

Remark 1.3.19. Note we can write the coherent state $\psi(f_1, f_2)$ as

$$\psi(f_1, f_2) = \psi_1(f_1) \otimes \psi_2(f_2),$$

where $\psi_1(f_1) = W(f_1)\Omega \in \mathcal{F}$ and $\psi_2(f_2) = W(f_2)\Omega \in \mathcal{F}$. Thus, a coherent state $\psi(f_1, f_2)$ in $\mathcal{F} \otimes \mathcal{F}$ is made up of two coherent states $\psi_1(f_1)$ and $\psi_2(f_2)$ in \mathcal{F} .

Now we list some properties of coherent states in $\mathcal{F} \otimes \mathcal{F}$.

Proposition 1.3.20. *Let $f_1, f_2 \in L^2(\mathbb{R}^3)$, and let $\psi(f_1, f_2) = \psi_1(f_1) \otimes \psi_2(f_2)$ be a coherent state in $\mathcal{F} \otimes \mathcal{F}$.*

Then

$$(i) \quad \|\psi\|_{\mathcal{F} \otimes \mathcal{F}} = 1$$

$$(ii) \quad a_{1,x}a_{2,x}\psi(f_1, f_2) = f_1(x)f_2(x)\psi(f_1, f_2), \text{ and it follows that}$$

$$a_1(g_1)a_2(g_2)\psi(f_1, f_2) = \langle g_1, f_1 \rangle_{L^2} \langle g_2, f_2 \rangle_{L^2} \psi(f_1, f_2)$$

$$(iii) \quad \langle \psi(f_1, f_2), \mathcal{N}_1 \psi(f_1, f_2) \rangle = \|f_1\|_{L^2}^2 \text{ and } \langle \psi(f_1, f_2), \mathcal{N}_2 \psi(f_1, f_2) \rangle_{L^2} = \|f_2\|_{L^2}^2.$$

Proof. The proposition follows from the properties of coherent states in \mathcal{F} listed in Proposition (1.2.24). \square

CHAPTER 2

QUANTIZATION & MEAN-FIELD LIMIT THROUGH DEFORMATION OF ALGEBRAS

2.1 Introduction to Deformation of Algebras

In this chapter we briefly discuss, for the case of interacting quantum Bose gases, how quantization and the mean-field limit can be understood through deformation theory of algebras, as discussed by Fröhlich et al in their paper [8]. The information in this chapter is not intended to be very detailed; it is just meant as a basic overview and introduction to the subject. Please refer to (A.5.4) for preliminary information on phase space and symplectic manifolds and refer to (A.5.8) for an introduction to the classical limit and quantization.

Characterizing and deforming a physical system

First, we discuss how we can characterize physical systems using $*$ -algebras (usually C^* algebras).

Definition 2.1.1. A physical system Σ can be characterized by a triple $(\mathcal{A}_\Sigma, \mathcal{C}_\Sigma, \mathcal{G}_\Sigma)$, where: \mathcal{A}_Σ is an associative $*$ -algebra, also called the kinematical algebra of Σ , whose self-adjoint elements are the observables of the system; \mathcal{C}_Σ , a convex set of the states of Σ , is a collection of all real-valued, positive linear functionals on \mathcal{A}_Σ ; \mathcal{G}_Σ are the symmetries of Σ , most often described in terms of $*$ -automorphisms of \mathcal{A}_Σ .

A system Σ characterized by $(\mathcal{A}_\Sigma, \mathcal{C}_\Sigma, \mathcal{G}_\Sigma)$ can be “deformed” to a new system $\hat{\Sigma}$ through deforming the triple $(\mathcal{A}_\Sigma, \mathcal{C}_\Sigma, \mathcal{G}_\Sigma)$ to a new triple $(\mathcal{A}_{\hat{\Sigma}}, \mathcal{C}_{\hat{\Sigma}}, \mathcal{G}_{\hat{\Sigma}})$.

Consider a Hamiltonian system Σ with phase space given by a symplectic manifold Γ_Σ . Then \mathcal{A}_Σ is the abelian algebra of smooth functions on Γ_Σ , equipped with a Poisson bracket. The space \mathcal{C}_Σ of states of Σ is given by the probability measures on Γ_Σ , and the symmetries \mathcal{G}_Σ are described by symplectomorphisms on Γ_Σ . By deforming \mathcal{A}_Σ to a new associative, usually non-abelian, $*$ -algebra $\mathcal{A}_{\hat{\Sigma}}$, replacing the Poisson bracket by a commutator multiplied by i/\hbar , as well as through deforming \mathcal{C}_Σ to a new set $\mathcal{C}_{\hat{\Sigma}}$ of positive linear functionals on $\mathcal{A}_{\hat{\Sigma}}$ and replacing \mathcal{G}_Σ to a group $\mathcal{G}_{\hat{\Sigma}}$ of $*$ -automorphisms on $\mathcal{A}_{\hat{\Sigma}}$, we deform the classical Hamiltonian system Σ to a new system $\hat{\Sigma}$. The parameter \hbar plays the role of the *deformation parameter*.

Mean-field equations & deformation of algebras

Suppose we are given a large N -body problem; that is, suppose we wish to calculate the time evolution of a system containing a large number N of interacting particles. Solving such a system directly would be very difficult. However, if the interactions between the particles are weak, one might expect that we could replace all interactions to any one body with an average or effective interaction. This idea is actually valid, and it gives rise to what is known as *mean-field equations*.

Definition 2.1.2. Given a system of a large number N of interacting particles, the *mean-field limit* is the limit $N \rightarrow \infty$. Taking the mean-field limit in a large N -body problem produces what is known as a *mean-field equation*, a model that describes the evolution of a typical particle subject to an average or effective interaction.

Mean-field models arise in both classical and quantum mechanics. The Vlasov equation arises as a mean-field limit $N \rightarrow \infty$ of a classical Hamiltonian system with N point particles. Similarly, the Hartree equation is known to arise as a mean-field limit of a many-body quantum mechanical system.

Deformation theory of algebras can be used to pass from Vlasov dynamics to Hartree dynamics, thereby undergoing quantization, as well as to go back the other way (classical limit). In addition, it can be used to perform *second quantization*, in which we go from the mean-field limit of N -body dynamics back to the N -body dynamics of a system. Also, it can be used to show that in the mean-field limit, we obtain mean-field equations.

Consider the diagram shown in Figure (2.1). As shown in the figure, we can pass from Vlasov dynamics to Hartree dynamics by *first quantization*, in which we use deformation of algebras, replacing the Poisson bracket by a commutator multiplied by $\frac{i}{\hbar}$. To go from Hartree dynamics back to the many-body quantum dynamics we undergo *second quantization* with deformation parameter $\frac{1}{N}$, which involves deformation of the classical Hartree field theory.



Figure 2.1: Classical and quantum mechanics. A right arrow \rightarrow stands for the limit $\hbar \rightarrow 0$. A left arrow \leftarrow stands for quantization with deformation parameter \hbar . A down arrow \downarrow stands for the mean field limit $N \rightarrow \infty$, where N is the number of particles. An up arrow \uparrow stands for second quantization with deformation parameter $1/N$.

In this chapter we will just focus on the left side of the diagram. We will use certain mathematical concepts, such as Fock space, which are described in Chapter (1). Other mathematical concepts introduced in Appendix (A) will also be used.

2.2 From the Hartree equation to N body quantum mechanics and back

Now we give a brief overview on how the theory of algebra of deformation is used when passing from Hartree dynamics to N body quantum mechanics through second quantization. First we describe the algebra formalism of quantum bose gases and then the algebra formalism of Hartree dynamics. Finally we tie the two together and show how the algebra formalism in Hartree dynamics can be deformed to produce the dynamics in N body quantum mechanics. We also show, through deformation of algebras, how the quantum mechanical evolution and the Hartree evolution are intertwined in the mean-field limit.

Quantum Bose gases

Consider a system of n bosonic particles of one species described on the Hilbert space $L_s^2(\mathbb{R}^{3n})$, the subspace of $L^2(\mathbb{R}^{3n})$ that contains all functions symmetric with respect to arbitrary permutations of the n particles. We consider the mean-field Hamiltonian

$$H_N^{(n)} = H_0^{(n)} + V_N^{(n)} \quad (2.1)$$

where

$$H_0^{(n)} = \sum_{j=1}^n \left(-\frac{1}{2m} \Delta_{x_j} + V_{\text{ext}}(x_j) \right), \quad V_N^{(n)} = \frac{1}{N} \sum_{1 \leq i < j \leq n} V(x_i - x_j).$$

Here $V_{\text{ext}} \geq 0$ is the external potential, V is the interaction potential, $m > 0$ is the mass of a particle, and $1/N$ is a dimensionless coupling constant. The mean-field limit is the limit where $n, N \rightarrow \infty$ in such a way that $n/N = \mathcal{O}(1)$.

If $\varphi_t^{(n)} \in L_s^2(\mathbb{R}^{3n})$, the time evolution of the system is described by the Schrödinger equation

$$i \frac{\partial}{\partial t} \varphi_t^{(n)} = H_N^{(n)} \varphi_t^{(n)}. \quad (2.2)$$

Let $\mathcal{H} = L^2(\mathbb{R}^{3n})^{\otimes_s n} \cong L_s^2(\mathbb{R}^{3n})$. The bosonic Fock space is given by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}.$$

We define $\mathcal{F}^{\leq k}$ to be

$$\mathcal{F}^{\leq k} = \bigoplus_{n=0}^k \mathcal{H}^{(n)}.$$

We rescale the normal creation and annihilation operators a_x and a_x^* , as defined in Definition (1.2.8), by a factor of $N^{-1/2}$, which gives rise to a new set of creation and annihilation operators.

Definition 2.2.1. We define

$$\hat{\psi}_N(x) = N^{-1/2} a_x, \quad \hat{\psi}_N^*(x) = N^{-1/2} a_x^*$$

where a_x and a_x^* are the creation and annihilation operators defined in (1.2.8).

From the CCR (1.6), we have the new CCR

$$\begin{aligned} [\hat{\psi}_N(x), \hat{\psi}_N(y)] &= [\hat{\psi}_N^*(x), \hat{\psi}_N^*(y)] = 0, \\ [\hat{\psi}_N(x), \hat{\psi}_N^*(y)] &= \frac{1}{N} \delta(x - y). \end{aligned}$$

Definition 2.2.2. We define $\hat{\mathcal{H}}$ to be the direct sum of the Hamiltonians in (2.1) rescaled by $1/N$, which we then write in terms of the creation and annihilation operators. That is,

$$\hat{\mathcal{H}} = \frac{1}{N} \bigoplus_{n=0}^{\infty} H_N^{(n)} = \hat{\mathcal{H}}_{0,N} + \hat{\mathcal{V}}_N$$

where

$$\begin{aligned} \hat{\mathcal{H}}_{0,N} &= \int dx \hat{\psi}_N^*(x) \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(x) \right) \hat{\psi}_N(x), \\ \hat{\mathcal{V}}_N &= \frac{1}{2} \int dx dy \hat{\psi}_N^*(x) \hat{\psi}_N^*(y) V(x - y) \hat{\psi}_N(y) \hat{\psi}_N(x). \end{aligned}$$

Now if $\Phi_t \in \mathcal{F}$, the Schrödinger equation (2.2) implies

$$\frac{i}{N} \frac{\partial}{\partial t} \Phi_t = \hat{\mathcal{H}}_N \Phi_t.$$

Definition 2.2.3. Let $a^{(p)}$ be a bounded operator on the p -particle Hilbert space $\mathcal{H}^{(p)}$ whose kernel is given by $a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p)$. Then the operator $\hat{A}_N(a^{(p)})$ is defined on the Fock space by

$$\hat{A}_N(a^{(p)}) = \int \int \prod_{i=1}^p \hat{\psi}_N^*(x_i) dx_i a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p) \prod_{j=1}^p \hat{\psi}_N(y_j) dy_j.$$

Furthermore, we define \hat{A} to be a $*$ -algebra given by the linear span

$$\hat{A} = \text{span}\{\hat{A}_N(a^{(p)}), p = 1, 2, \dots\}.$$

Classical Hartree field theory

The nonlinear Hartree equation has a Hamiltonian nature; that is, although the Hartree equation is a quantum mean-field model, we can use the phase space formulation of quantum mechanics to describe Hartree dynamics in a way that more closely resembles classical mechanics. Thus, we will use Hamilton's equation and the notion of phase space. Therefore, we consider a "classical" Hamiltonian system with phase space $\Gamma = H_{V_{\text{ext}}}^1(\mathbb{R}^3)$, a weighted complex Sobolev space of index 1, defined as the quadratic form domain of the operator $-\frac{\Delta}{2m} + V_{\text{ext}}$, with complex coordinates $\bar{\psi}(x)$ and $\psi(x)$. On Γ we have a Poisson bracket

$$\begin{aligned} \{\psi(x), \bar{\psi}(y)\} &= i\delta(x - y), \\ \{\psi(x), \psi(y)\} &= \{\bar{\psi}(x), \bar{\psi}(y)\} = 0. \end{aligned}$$

We define a Hamilton functional by

$$\mathcal{H}(\bar{\psi}, \psi) = \mathcal{H}_0(\bar{\psi}, \psi) + \mathcal{V}(\bar{\psi}, \psi)$$

where

$$\begin{aligned}\mathcal{H}_0(\bar{\psi}, \psi) &= \int dx \bar{\psi}(x) \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(x) \right) \psi(x) \\ \mathcal{V}(\bar{\psi}, \psi) &= \frac{1}{2} \int dx dy |\psi(x)|^2 V(x-y) |\psi(y)|^2.\end{aligned}$$

Then the Hamiltonian equations of motion corresponding to \mathcal{H} are given by the Hartree equation

$$i \frac{\partial}{\partial t} \psi_t(x) = \{ \mathcal{H}, \psi_t(x) \} = -i \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(x) \right) \psi_t(x) - i (V * |\psi_t|^2)(x) \psi_t(x). \quad (2.3)$$

If ψ_t is the solution to (2.3), then, under sufficient conditions, we can define a global symplectic flow Φ_t on Γ by

$$\Phi_t(\psi) = \psi_t.$$

We define the ‘‘charge’’ \mathcal{N} to be the function

$$\mathcal{N}(\bar{\psi}, \psi) = \int dx |\psi(x)|^2 = \|\psi\|_{L^2}^2.$$

We see that this function \mathcal{N} is invariant under the gauge transformations

$$\psi(x) \mapsto e^{-i\theta} \psi(x), \quad \bar{\psi}(x) \mapsto e^{i\theta} \bar{\psi}(x).$$

This function \mathcal{N} and the energy \mathcal{H} should be conserved. Indeed, we see that through gauge invariance of \mathcal{H} and with the help of Noether’s theorem (which explains how certain symmetries produce conserved quantities) we have that \mathcal{N} is a conserved quantity of the flow Φ_t . (See (A.5.2) for an introduction to Noether’s theorem.) Also, since the system is autonomous (does not depend explicitly on time), we have conservation of \mathcal{H} as well.

Similarly to Definition (2.2.3), we have the following definition.

Definition 2.2.4. Let $a^{(p)}$ be bounded operators on the p -particle Hilbert space $\mathcal{H}^{(p)}$ whose kernel is given by $a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p)$. Then we define the function $A(a^{(p)})$ on Γ by

$$A(a^{(p)}) (\bar{\psi}, \psi) = \int \int \prod_{i=1}^p \bar{\psi}(x_i) dx_i a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p) \prod_{j=1}^p \psi(y_j) dy_j.$$

Furthermore, we define \mathcal{A} to be the abelian $*$ -algebra under pointwise multiplication, equipped with a Poisson bracket, given by

$$\mathcal{A} = \text{span}\{A(a^{(p)}), p = 1, 2, \dots\}.$$

Second Quantization

Now we can perform second quantization to go from the Hartree dynamics to the original quantum-mechanical problem in the case of the quantum bose gases.

Second quantization is the linear map $(\cdot)_N : \mathcal{A} \rightarrow \hat{\mathcal{A}}$ defined by the substitution

$$\begin{aligned}\bar{\psi}(x) &\rightarrow \hat{\psi}_N^*(x), \\ \psi(x) &\rightarrow \hat{\psi}_N(x)\end{aligned}$$

followed by Wick ordering (ordering of the creation and annihilation operators so that all creation operators are to the left of all annihilation operators). Quantization will map the Poisson bracket $\{\psi^{\#1}(x), \psi^{\#2}(y)\}$ to the commutator $iN[\hat{\psi}_N^{\#1}(x), \hat{\psi}_N^{\#2}(y)]$, where $\#_i$ stands for either complex conjugate and adjoint, respectively, or nothing. In addition, under quantization, we have

$$\begin{aligned}\bar{A} &\rightarrow \hat{A}_N^*, \\ A(a^{(p)}) &\rightarrow \hat{A}_N(a^{(p)}), \\ \mathcal{H} &\rightarrow \hat{\mathcal{H}}_N\end{aligned}$$

and so we see that \mathcal{A} is deformed to $\hat{\mathcal{A}}$ with deformation parameter $\frac{1}{N}$.

We have the following Lemma.

Lemma 2.2.5. *Let Φ_t^0 denote the classical flow for $\psi = 0$. Also let $\nu > 0$ be an arbitrary constant. Then*

$$(a) \quad A(a^{(p)}) \circ \Phi_t^0 = A\left(e^{itH_0^{(p)}} a^{(p)} e^{-itH_0^{(p)}}\right), \quad (2.4)$$

$$(b) \quad e^{itN\hat{\mathcal{H}}_{0,N}} \hat{A}_N(a^{(p)}) e^{-itN\hat{\mathcal{H}}_{0,N}} = \left(A(a^{(p)}) \circ \Phi_t^0\right)_N \quad (2.5)$$

$$(c) \quad e^{itN\hat{\mathcal{H}}_N} \hat{A}_N(a^{(p)}) e^{-itN\hat{\mathcal{H}}_N} |_{\mathcal{F} \leq \nu N} = \left(A(a^{(p)}) \circ \Phi_t^0\right)_N |_{\mathcal{F} \leq \nu N} + o(1), \quad \text{as } N \rightarrow \infty \quad (2.6)$$

From the Lemma, we see that quantization does not intertwine quantum-mechanical time evolution with the classical (Hartree) time evolution, except when $\psi = 0$. However, as (2.6) shows, quantization does intertwine the full quantum-mechanical time evolution with the Hartree evolution in the mean-field limit.

Remark 2.2.6. Here we remark on the main idea in the proof of (2.6). For more details see [8]. The proof is based on expanding (2.6) and iterating to obtain a Dyson series which can be shown to converge using tree loop diagrams. For example, we can expand the left-hand side of (2.6) as follows:

$$\begin{aligned}e^{itN\hat{\mathcal{H}}_N} \hat{A}_N(a^{(p)}) e^{-itN\hat{\mathcal{H}}_N} &= \hat{A}_N(a_t^{(p)}) \\ &+ \int_0^t ds e^{isN\hat{\mathcal{H}}_N} e^{isN\hat{\mathcal{H}}_{0,N}} \frac{iN}{2} \left[\hat{A}_N(\varphi_s), \hat{A}_N(a_t^{(p)}) \right] e^{isN\hat{\mathcal{H}}_{0,N}} e^{-isN\hat{\mathcal{H}}_N}.\end{aligned} \quad (2.7)$$

where φ is given by

$$\varphi(x_1, x_2; y_1, y_2) = V(x_1 - y_1) \delta(x_2 - y_1) \delta(y_2 - x_1)$$

and $a_t^{(p)}$ is given by

$$a_t^{(p)} = e^{itH_0^{(p)}} a^{(p)} e^{-itH_0^{(p)}}.$$

The commutator in (2.7) can be written as

$$\frac{iN}{2} \left[\hat{A}_N(\varphi_s), \hat{A}_N(a_t^{(p)}) \right] = \sum_{\ell=1}^2 \binom{p}{\ell} \binom{2}{\ell} \frac{i\ell}{2N^{\ell-1}} \hat{A}_N \left(\left[(\varphi_s) \xrightarrow{\ell} (a_t^{(p)}) \right] \right).$$

Here we are using the notation

$$\begin{aligned} & \left(a^{(p)} \xrightarrow{\ell} b^{(q)} \right) (x_1, \dots, x_{p+q-\ell}; y_1, \dots, y_{p+q-\ell}) \\ &= P_s \int \prod_{i=1}^{\ell} du_i a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_{p-\ell}, u_1, \dots, u_{\ell}) \\ & \quad \cdot b^{(q)}(u_1, \dots, u_{\ell}, x_{p+1}, \dots, x_{p+q-\ell}; y_{p-\ell+1}, \dots, y_{p+q-\ell}) \end{aligned}$$

where P_s is the orthogonal projection onto the subspace of vectors symmetric under the exchange of particles.

Iterating (2.7) we obtain a Dyson series. We split the iterated series into two parts, the “tree” terms ($\ell = 1$) and the “loop terms” ($\ell = 2$). When a loop term is generated, we stop expanding and put it into an error term of order $1/N$. We use the bound $\|\hat{A}_N(a^{(p)})|_{\mathcal{F} \leq \nu N}\| \leq \nu^p \|a^{(p)}\|$ to show that all terms then converge absolutely provided that

$$|t| \leq \frac{1}{4\nu \|\varphi\|_{\infty}}.$$

CHAPTER 3

TIME EVOLUTION OF COHERENT STATES FOR TWO SPECIES

In this chapter, we extend the analysis of Rodnianski and Schlein, who in [19] derive an explicit formula for the rate of convergence of the microscopic quantum N -body dynamics to the Hartree dynamics in the case of an initial coherent state in a system of N bosonic particles of one species, to the case of two species. We begin by explaining how we come to look at convergence of marginal densities through discussing initial factorized states. Then we briefly discuss the important result of Hepp in [9], whose approach Rodnianski and Schlein based their work on, and go over Rodnianski and Schlein's result for the case of an initial coherent state. Then we go on to introduce the main theorem of this chapter and give a sketch of the proof with the main ideas. Finally we devote the remaining sections to the full proof of the theorem.

Notation 3.0.7. Throughout this chapter, an inner product or a norm without a subscript will correspond to the norm or inner product in $\mathcal{F} \otimes \mathcal{F}$ where \mathcal{F} is the bosonic Fock space defined in (1.1). All other norms or inner products will be indicated with a subscript. Also, in this chapter, the vacuum vector in $\mathcal{F} \otimes \mathcal{F}$ will be denoted simply as Ω .

3.1 Factorized States & Convergence of Marginal Densities

Although in this chapter we are dealing with initial coherent states, rather than initial factorized states, we give the discussion below so that the reader may get a general idea of why we are looking at convergence of marginal densities, and we give some important definitions.

3.1.1 One species case

As before, consider a system of N bosonic particles of one species. Now, however, consider the initial state of the system to be the factorized wave function given by $\psi_N(x) = \prod_{j=1}^N \varphi(x_j) \in L_s^2(\mathbb{R}^{3N})$, where $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N}$, for some $\varphi \in H^1(\mathbb{R}^3)$ where $\|\varphi\|_{L^2(\mathbb{R}^3)} = 1$.

The time evolution $\psi_{N,t}$ is then given by the Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \quad \psi_{N,0} = \psi_N \tag{3.1}$$

where the mean-field Hamiltonian H_N is given by

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^N V(x_i - x_j).$$

One might expect that $\psi_{N,t}$ should remain approximately factorized, or

$$\psi_{N,t}(x) \approx \prod_{j=1}^N \varphi_t(x_j) \quad \text{for large } N \quad (3.2)$$

in an appropriate sense. It follows from (3.2) that the total potential of a single particle at a position x can be approximated by the convolution $(V * |\varphi_t|^2)(x)$. It follows that the time evolution φ_t of a single particle can be described by the nonlinear Hartree equation

$$i \frac{\partial}{\partial t} \varphi_t = -\Delta \varphi_t + (V * |\varphi_t|^2) \varphi_t. \quad (3.3)$$

It is the main goal to show (3.2). Unfortunately, (3.2) is not true in the sense that

$$\|\psi_{N,t}(x) - \prod_{j=1}^N \varphi_t(x_j)\|_{L^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Rather, it has been found that (3.2) should be thought of in terms of marginal densities.

The solution $\psi_{N,t}$ to (3.1) provides us with the density matrix and the k -particle marginal density.

Definition 3.1.1. The *density matrix* $\gamma_{N,t}$ associated with $\psi_{N,t}$ is given by

$$\gamma_{N,t} = |\psi_{N,t}\rangle\langle\psi_{N,t}|$$

with kernel

$$\gamma_{N,t}(x, x') = \psi_{N,t}(x) \overline{\psi_{N,t}(x')}.$$

Definition 3.1.2. For $k = 1, 2, \dots, N$, the *k -particle marginal density* $\gamma_{N,t}^{(k)}$ associated with $\gamma_{N,t}$ is the positive trace class operator on $L^2(\mathbb{R}^{3k})$ with kernel

$$\gamma_{N,t}^{(k)}(x_k; x'_k) = \int \gamma_{N,t}(x_k, x_{N-k}; x'_k, x_{N-k}) dx_{N-k} \quad (3.4)$$

where $x_k \in \mathbb{R}^{3k}$ and $x_{N-k} \in \mathbb{R}^{3(N-k)}$.

It turns out that (3.2) is true in the sense of marginal densities. What one wants to show then is the following:

$$\gamma_{N,t}^{(k)} \rightarrow |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \quad \text{as } N \rightarrow \infty \quad (3.5)$$

in the trace norm topology, where φ_t is the solution to (3.3).

Remark 3.1.3. It is important to note that (3.5) should not only hold for initial factorized states; it should also hold for other initial states as well, such as initial coherent states.

One normally shows (3.5) for $k = 1$ and then the other cases follow similarly. Let us give another definition, then, which is better suited for our purposes, for the one particle density.

Definition 3.1.4. Let $\psi \in \mathcal{F}$. Then the *one-particle density* $\gamma_\psi^{(1)}$ associated with ψ is the positive trace class operator on $L^2(\mathbb{R}^3)$ with kernel given by

$$\gamma_\psi^{(1)}(x; y) = \frac{1}{\langle \psi, \mathcal{N}\psi \rangle} \langle \psi, a_y^* a_x \psi \rangle. \quad (3.6)$$

Remark 3.1.5. Note that (3.6) is equivalent with (3.4) when $k = 1$. To see this, let $\psi = \{0, \dots, 0, \psi_N, 0, \dots\}$, and let $f \in L^2(\mathbb{R}^3)$. Assuming that the one-particle density $\gamma_\psi^{(1)}$ associated with ψ has kernel given by (3.6), we have

$$\left(\gamma_\psi^{(1)} f \right) (x) = \int \gamma_\psi^{(1)}(x; y) f(y) dy = \int \frac{1}{N} \langle \psi, a_y^* a_x \psi \rangle f(y) dy = \frac{1}{N} \langle a(f) \psi, a_x \psi \rangle.$$

Now note that

$$\begin{aligned} \int dx \bar{f}(x) a_x \psi_N(x_1, x_2, \dots, x_N) &= (a(f) \psi_N)(x_2, \dots, x_N) = \sqrt{N} \int dx \bar{f}(x) \psi_N(x, x_2, \dots, x_N) \\ \Rightarrow a_x \psi_N(x_1, x_2, \dots, x_N) &= \sqrt{N} \psi_N(x, x_2, \dots, x_N). \end{aligned}$$

So

$$\begin{aligned} \left(\gamma_\psi^{(1)} f \right) (x) &= \frac{1}{N} \left\langle \sqrt{N} \int dy \bar{f}(y) \psi_N(y, x_2, \dots, x_N), \sqrt{N} \psi_N(x, x_2, \dots, x_N) \right\rangle \\ &= \int dy \int dx_2 \cdots dx_N \bar{\psi}_N(y, x_2, \dots, x_N) \psi_N(x, x_2, \dots, x_N) f(y) \end{aligned}$$

so the kernel can indeed also be written as

$$\begin{aligned} \gamma_\psi^{(1)}(x; y) &= \int dx_2 \cdots dx_N \psi_N(x, x_2, \dots, x_N) \bar{\psi}_N(y, x_2, \dots, x_N) \\ &= \int \gamma_{N,t}(x, x_2, \dots, x_N; y, x_2, \dots, x_N) dx_2 \cdots dx_N. \end{aligned}$$

3.1.2 Two species case

Let us now consider our system of N_1 bosonic particles of one species and N_2 bosonic particles of another species. We assume N_1 and N_2 scale linearly so that $N_1 = aN_2$. We consider the initial state of this system to be the factorized wave function given by $\psi_{N_1, N_2}(x, y) = \prod_{j=1}^{N_1} \varphi_1(x_j) \prod_{k=1}^{N_2} \varphi_2(y_k) \in L_s^2(\mathbb{R}^{3(N_1+N_2)})$, where $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N}$ and $y = (y_1, y_2, \dots, y_N) \in \mathbb{R}^{3N}$, for some $\varphi_1, \varphi_2 \in H^1(\mathbb{R}^3)$ where $\|\varphi_j\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 1, 2$.

The time evolution $\psi_{N_1, N_2, t}$ is then given by the Schrödinger equation

$$i\partial_t \psi_{N_1, N_2, t} = H_{N_1, N_2} \psi_{N_1, N_2, t}, \quad \psi_{N_1, N_2, 0} = \psi_{N_1, N_2} \quad (3.7)$$

where the mean-field Hamiltonian H_{N_1, N_2} is given by

$$\begin{aligned} H_{N_1, N_2} = & \sum_{j=1}^{N_1} -\Delta_{x_j} + \sum_{k=1}^{N_2} -\Delta_{y_k} + \frac{1}{N_1} \sum_{1 \leq r < s \leq N_1} V_1(x_s - x_r) + \frac{1}{N_2} \sum_{1 \leq p < q \leq N_2} V_2(y_q - y_p) \\ & + \frac{1}{N_1 + N_2} \sum_{\ell=1}^{N_1} \sum_{m=1}^{N_2} Q(x_\ell - y_m). \end{aligned}$$

Similar to as in the one-species case, we wish to show that the evolving wave function $\psi_{N_1, N_2, t}$ remains approximately factorized as $N_1, N_2 \rightarrow \infty$; that is we wish to show that

$$\psi_{N_1, N_2, t}(x, y) \approx \prod_{j=1}^{N_1} \varphi_{1, t}(x_j) \prod_{k=1}^{N_2} \varphi_{2, t}(y_k) \quad \text{for large } N_1, N_2 \quad (3.8)$$

in an appropriate sense. As in the one-species case, (3.8) is understood in terms of marginal densities. We wish to show that

$$\Gamma_{N_1, N_2, t}^{(k)} \rightarrow |\varphi_{1, t}\rangle \langle \varphi_{1, t}|^{\otimes k} \otimes |\varphi_{2, t}\rangle \langle \varphi_{2, t}|^{\otimes k} \quad \text{as } N_1, N_2 \rightarrow \infty$$

in the trace norm topology, where $\Gamma_{N_1, N_2, t}^{(k)}$ is the k -particle marginal density associated with $\psi_{N_1, N_2, t}$, and the time evolutions $\varphi_{1, t}$ and $\varphi_{2, t}$ satisfy the nonlinear coupled Hartree equations

$$i \frac{\partial}{\partial t} \varphi_{1, t} = \left(-\Delta + (V_1 * |\varphi_{1, t}|^2) + \frac{1}{a+1} (Q * |\varphi_{2, t}|^2) \right) \varphi_{1, t} \quad (3.9)$$

and

$$i \frac{\partial}{\partial t} \varphi_{2, t} = \left(-\Delta + (V_2 * |\varphi_{2, t}|^2) + \frac{a}{a+1} (Q * |\varphi_{1, t}|^2) \right) \varphi_{2, t}. \quad (3.10)$$

We also have the following definition.

Definition 3.1.6. The *one-particle density associated with ψ* has kernel given by

$$\Gamma_{N_1, N_2, t}^{(1)}(x; y) = \frac{1}{\langle \psi, \mathcal{N}_1 \psi \rangle \langle \psi, \mathcal{N}_2 \psi \rangle} \langle \psi, a_{1, y}^* a_{1, x} \psi \rangle \langle \psi, a_{2, y}^* a_{2, x} \psi \rangle.$$

3.2 Previous Results for the Case of an Initial Coherent State (Hepp, Rodnianski & Schlein)

Let us start by discussing the work of Hepp. In [9], Hepp studies the evolution of coherent states in the semi-classical limit (which is mathematically considered equivalent to the mean-field limit) from which Hartree dynamics emerge. His approach consists of using coherent states as initial data and embedding a many body system into the second-quantized Fock space representation. Let us look at the evolution operator for one species.

Definition 3.2.1. The unitary *evolution operator* $\mathcal{U}_N(t; s)$ is given by

$$\mathcal{U}_N(t; s) = W^*(\sqrt{N} \varphi_t) e^{-i\mathcal{H}_N(t-s)} W(\sqrt{N} \varphi_s).$$

From [19], \mathcal{U}_N satisfies the equation

$$i \frac{\partial}{\partial t} \mathcal{U}_N(t; s) = \mathcal{L}_N(t) \mathcal{U}_N(t; s), \quad \text{and } \mathcal{U}_N(s; s) = 1,$$

where $\mathcal{L}_N(t)$ is the generator

$$\begin{aligned} \mathcal{L}_N(t) &= \int dx \Delta_x a_x^* \Delta_x a_x + \int dx (V * |\varphi_t|^2)(x) a_x^* a_x + \int dx dy V(x-y) \bar{\varphi}_t(x) \varphi_t(y) a_y^* a_x + \\ &+ \frac{1}{2} \int dx dy V(x-y) (\varphi_t(x) \varphi_t(y) a_x^* a_y^* + \bar{\varphi}_t(x) \bar{\varphi}_t(y) a_x a_y) + \\ &+ \frac{1}{\sqrt{N}} \int dx dy V(x-y) a_x^* (\varphi_t(y) a_y^* + \bar{\varphi}_t(y) a_y) a_x + \frac{1}{2N} \int dx dy V(x-y) a_x^* a_y^* a_y a_x. \end{aligned}$$

Hepp's result shows that in the limit $N \rightarrow \infty$, the fluctuation dynamics $\mathcal{U}_N(t; s)$ approaches a limiting evolution $\mathcal{U}(t; s)$, which brings about the Hartree dynamics in the evolution of coherent states in the mean-field limit.

To summarize, Hepp's result shows convergence for the case of initial coherent states in the mean-field limit to Hartree dynamics. However, it does not show the rate of convergence or prove convergence results for the evolution of factorized initial states. In their paper [19], Rodnianski and Schlein use an approach based on Hepp in [9] to show rates of convergence of marginal densities for both initial coherent states and initial factorized states in the case of one species. In the case of an initial coherent state, to which this chapter is devoted, they proved

$$\text{Tr} \left| \Gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq \frac{C}{N} e^{Kt} \quad (3.11)$$

where C and K are constants, and $\Gamma_{N,t}^{(1)}$ is the one-particle marginal density associated with the time evolution of a coherent state $\psi(\sqrt{N}\varphi)$. The use of coherent states as initial data allowed Rodnianski and Schlein to obtain the optimal rate of convergence $1/N$ for all fixed times.

3.3 Statement of Main Theorem & Overview of Proof

Introduction to Main Theorem

We extend the result of Rodnianski of Schlein by looking at an initial coherent state in a system of N_1 bosonic particles of one species and N_2 bosonic particles of another species. We prove the following theorem, where we have obtained a formula similar to (3.11), for the rate of convergence in the case of this initial coherent state.

Theorem 3.3.1. *Suppose we have the operator inequalities*

$$V_1^2(x) \leq D_1(1 - \Delta_x), \quad V_2^2(x) \leq D_2(1 - \Delta_x), \quad Q^2(x) \leq D_3(1 - \Delta_x)$$

for constants $D_1, D_2, D_3 > 0$. Let $\Gamma_{N_1, N_2, t}^{(1)}$ be the one-particle marginal associated with $\psi(N_1, N_2, t) = e^{-iH_{N_1, N_2} t} \psi(\sqrt{N_1}\varphi_1, \sqrt{N_2}\varphi_2)$, where $\psi(\sqrt{N_1}\varphi_1, \sqrt{N_2}\varphi_2)$ is the coherent state given by

$$\psi(\sqrt{N_1}\varphi_1, \sqrt{N_2}\varphi_2) = W_1(\sqrt{N_1}\varphi_1) W_2(\sqrt{N_2}\varphi_2) \Omega_{\mathcal{F} \otimes \mathcal{F}}.$$

Here $\varphi_1, \varphi_2 \in H^1(\mathbb{R}^3)$ satisfy the nonlinear Hartree equations (3.9) and (3.10), where a is the proportionality constant between N_1 and N_2 . Also, let J_1 and J_2 be Hilbert-Schmidt operators over $L^2(\mathbb{R}^3)$. Then there exists constants $C, K > 0$ such that

$$\text{Tr} \left| (J_1 \otimes J_2) \left(\Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right| \leq C \frac{N_1 + N_2}{N_1 N_2} e^{Kt}, \quad t \geq 0. \quad (3.12)$$

Remark 3.3.2. Note (3.12) implies

$$\text{Tr} \left| \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right| \leq C \frac{N_1 + N_2}{N_1 N_2} e^{Kt}. \quad (3.13)$$

To see this, let

$$J_1 \otimes J_2 = \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}|.$$

Then (3.12) can be expressed as

$$\left\| \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right\|_{\text{HS}} \leq C \frac{N_1 + N_2}{N_1 N_2} e^{Kt}.$$

In general, the Hilbert-Schmidt norm is bounded above by the trace-norm; however, in this case, they differ at most by a factor of 2. In fact, since $|\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}|$ is a rank one projection, it has one non-zero eigenvalue equal to one, by Proposition (A.1.26)(3). Meanwhile, the operator $\Gamma_{N_1, N_2, t}^{(1)}$ is a density operator and so it has only non-negative eigenvalues, by Proposition (A.1.31)(3). It follows that the operator $\Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}|$ has one negative eigenvalue λ_{neg} .

Also, we have that

$$\left\| \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right\|_{\text{HS}} \leq \epsilon(N_1, N_2, t)$$

for some $\epsilon(N_1, N_2, t) > 0$, where as $N_1, N_2 \rightarrow \infty$ we have that $\epsilon(N_1, N_2, t) \rightarrow 0$.

It follows that, when N_1 and N_2 are finite, λ_{neg} should be equal, in absolute value, to the sum of the positive eigenvalues of $\Gamma_{N_1, N_2, t}^{(1)}$, up to an order of ϵ . That is, if we let $\{\lambda_{\text{pos}, i}\}_{i \geq 1}$ be a set containing all the positive eigenvalues of $\Gamma_{N_1, N_2, t}^{(1)}$, we have

$$|\lambda_{\text{neg}}| + \sum_{i \geq 1} \lambda_{\text{pos}, i} = 2|\lambda_{\text{neg}}| + \mathcal{O}(\epsilon).$$

Thus,

$$\text{Tr} \left| \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right| = |\lambda_{\text{neg}}| + \sum_{i \geq 1} \lambda_{\text{pos}, i} = 2|\lambda_{\text{neg}}| + \mathcal{O}(\epsilon).$$

Now, since $\Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}|$ is a bounded operator, its operator norm is given by

$$\left\| \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right\|_{\text{op}} = |\lambda_{\text{neg}}| + \mathcal{O}(\epsilon).$$

Since the operator norm is bounded above by its Hilbert-Schmidt norm (see (A.2)), we have

$$\text{Tr} \left| \Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right| = 2 \left\| \left(\Gamma_{N_1, N_2, t}^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right\|_{\text{HS}} + \mathcal{O}(\epsilon),$$

which shows (3.13).

The proof of Theorem (3.3.1) will formally be given in the last section of this chapter; however, it relies on the following information as well as on each of the lemmas given in the subsequent sections. Let us now give a basic overview of how the proof will go.

Overview of Proof

In the remainder of this section, we will calculate the kernel of the one-particle marginal $\Gamma_{N_1, N_2, t}$ associated with $\psi(N_1, N_2, t) = e^{-iH_{N_1, N_2} t} \psi(\sqrt{N_1} \varphi_1, \sqrt{N_2} \varphi_2)$, which we write in the form $\Gamma_{N_1, N_2, t}(x, y) = \gamma_1(x, y) \gamma_2(x, y)$. Then we introduce a new evolution $\tilde{\mathcal{U}}_{N_1, N_2}$, and we get the important inequalities

$$\begin{aligned} \left| \text{Tr } J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \right| &\leq \frac{\|J_1\|_{\text{HS}}}{N_1} \langle \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \mathcal{N}_1 \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\ &+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_1 + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| + \\ &+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_1 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\| \end{aligned}$$

and

$$\begin{aligned} \left| \text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right| &\leq \frac{\|J_2\|_{\text{HS}}}{N_2} \langle \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \mathcal{N}_2 \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\ &+ \frac{2\|J_2\|_{\text{HS}}}{\sqrt{N_2}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_2 + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| + \\ &+ \frac{2\|J_2\|_{\text{HS}}}{\sqrt{N_2}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\|. \end{aligned}$$

In the subsequent sections, we aim to find the bounds

$$\begin{aligned} \text{Tr } J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) &\leq \frac{C}{N_1} \|J_1\|_{\text{HS}} e^{Kt} \\ \text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) &\leq \frac{C}{N_2} \|J_2\|_{\text{HS}} e^{Kt} \end{aligned}$$

which enables us to prove the theorem. This is broken up into several pieces. First, in Section (3.4), we prove Proposition (3.4.1), which is the longest part of the proof of the theorem and requires proving several lemmas. This proposition enables us to bound the term $\|(\mathcal{N}_j + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\|$. In Section (3.5), we prove Lemma (3.5.1), which allows us bound the term $\left\| (\mathcal{N}_j + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \right\|$. In Section (3.6), we prove Lemma (3.6.1), which gives us a bound on $\left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\|$. Finally, in the last section, we show how each of these separate bounds come together to prove the theorem.

Basics for the Proof

Let us now calculate the one-particle marginal density. From Definition (3.1.6), the one-particle marginal associated with $\psi(N_1, N_2, t) = e^{-iH_{N_1, N_2} t} \psi(\sqrt{N_1} \varphi_1, \sqrt{N_2} \varphi_2)$, where $\psi(\sqrt{N_1} \varphi_1, \sqrt{N_2} \varphi_2)$ is the coherent state given by $\psi(\sqrt{N_1} \varphi_1, \sqrt{N_2} \varphi_2) = W_1(\sqrt{N_1} \varphi_1) W_2(\sqrt{N_2} \varphi_2) \Omega$, has kernel given by

$$\Gamma_{N_1, N_2, t}^{(1)}(x, y) = \gamma_1(x, y) \gamma_2(x, y),$$

where

$$\begin{aligned}\gamma_1(x, y) &= \frac{1}{N_1} \langle \psi(N_1, N_2, t), a_{1,y}^* a_{1,x} \psi(N_1, N_2, t) \rangle \\ \gamma_2(x, y) &= \frac{1}{N_2} \langle \psi(N_1, N_2, t), a_{2,y}^* a_{2,x} \psi(N_1, N_2, t) \rangle.\end{aligned}$$

Now we introduce the unitary evolution operator for two species.

Definition 3.3.3. The unitary *evolution operator for two species* is given by

$$U_{N_1, N_2}(t; s) = W_1^*(\sqrt{N_1}\varphi_{1,t})W_2^*(\sqrt{N_2}\varphi_{2,t})e^{-i\mathcal{H}_{N_1, N_2}(t-s)}W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s}).$$

It follows that

$$\begin{aligned}W_1^*(\sqrt{N_1}\varphi_{1,s})W_2^*(\sqrt{N_2}\varphi_{2,s})e^{i\mathcal{H}_{N_1, N_2}(t-s)}\left(a_{j,x} - \sqrt{N_j}\varphi_{j,t}(x)\right)e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \\ \cdot W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s}) = U_{N_1, N_2}^*(t; s)a_{j,x}U_{N_1, N_2}(t; s), \quad \text{for } j = 1, 2.\end{aligned}\tag{3.14}$$

It can be shown that

$$i\frac{\partial}{\partial t}U_{N_1, N_2}(t; s) = \mathcal{L}_{N_1, N_2}(t)U_{N_1, N_2}(t; s)$$

where the generator \mathcal{L}_{N_1, N_2} is given by

$$\mathcal{L}_{N_1, N_2}(t) = L1(t) + L2(t) + \frac{1}{N_1 + N_2} \int dx dy A(t),\tag{3.15}$$

where

$$\begin{aligned}L1(t) &= \int dx \Delta_x a_{1,x}^* \Delta_x a_{1,x} + \int dx (V_1 * |\varphi_{1,t}|^2)(x) a_{1,x}^* a_{1,x} + \\ &+ \int dx dy V_1(x-y) \bar{\varphi}_{1,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{1,x} + \\ &+ \frac{1}{2} \int dx dy V_1(x-y) (\varphi_{1,t}(x) \varphi_{1,t}(y) a_{1,x}^* a_{1,y}^* + \bar{\varphi}_{1,t}(x) \bar{\varphi}_{1,t}(y) a_{1,x} a_{1,y}) + \\ &+ \frac{1}{\sqrt{N_1}} \int dx dy V_1(x-y) a_{1,x}^* (\varphi_{1,t} a_{1,y}^* + \bar{\varphi}_{1,t}(y) a_{1,y}) a_{1,x} + \\ &+ \frac{1}{2N_1} \int dx dy V_1(x-y) a_{1,x}^* a_{1,y}^* a_{1,y} a_{1,x},\end{aligned}$$

$$\begin{aligned}L2(t) &= \int dx \Delta_x a_{2,x}^* \Delta_x a_{2,x} + \int dx (V_2 * |\varphi_{2,t}|^2)(x) a_{2,x}^* a_{2,x} + \\ &+ \int dx dy V_2(x-y) \bar{\varphi}_{2,t}(x) \varphi_{2,t}(y) a_{2,y}^* a_{2,x} + \\ &+ \frac{1}{2} \int dx dy V_2(x-y) (\varphi_{2,t}(x) \varphi_{2,t}(y) a_{2,x}^* a_{2,y}^* + \bar{\varphi}_{2,t}(x) \bar{\varphi}_{2,t}(y) a_{2,x} a_{2,y}) + \\ &+ \frac{1}{\sqrt{N_2}} \int dx dy V_2(x-y) a_{2,x}^* (\varphi_{2,t} a_{2,y}^* + \bar{\varphi}_{2,t}(y) a_{2,y}) a_{2,x} + \\ &+ \frac{1}{2N_2} \int dx dy V_2(x-y) a_{2,x}^* a_{2,y}^* a_{2,y} a_{2,x},\end{aligned}$$

and

$$\begin{aligned}
A(t) = Q(x-y) & \left[a_{2,x}^* a_{1,y}^* a_{1,y} a_{2,x} + \sqrt{N_2} \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* a_{1,y} + \sqrt{N_1} \varphi_{1,t}(y) a_{2,x}^* a_{1,y}^* a_{2,x} + \right. \\
& + \sqrt{N_1 N_2} \varphi_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* + \sqrt{N_1} \bar{\varphi}_{1,t}(y) a_{2,x}^* a_{1,y} a_{2,x} + \sqrt{N_1 N_2} \bar{\varphi}_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y} + \\
& + N_1 \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) a_{2,x}^* a_{2,x} + \sqrt{N_2} \bar{\varphi}_{2,t}(x) a_{1,y}^* a_{1,y} a_{2,x} + N_2 \bar{\varphi}_{2,t}(x) \varphi_{2,t}(x) a_{1,y}^* a_{1,y} + \\
& + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{2,x} + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) a_{1,y} a_{2,x} + \\
& \left. + N_1 N_2 \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) \varphi_{2,t}(x) \right].
\end{aligned}$$

Using (3.14), we can write $\gamma_1(x, y)$ as

$$\begin{aligned}
\gamma_1(x, y) & = \varphi_{1,t}(x) \overline{\varphi_{1,t}(y)} + \frac{\overline{\varphi_{1,t}(y)}}{\sqrt{N_1}} \langle \Omega, U_{N_1, N_2}^*(t; 0) a_{1,x} U_{N_1, N_2}(t; 0) \Omega \rangle + \\
& + \frac{\varphi_{1,t}(x)}{\sqrt{N_1}} \langle \Omega, U_{N_1, N_2}^*(t; 0) a_{1,y}^* U_{N_1, N_2}(t; 0) \Omega \rangle + \\
& + \frac{1}{N_1} \langle \Omega, U_{N_1, N_2}^*(t; 0) a_{1,y}^* a_{1,x} U_{N_1, N_2}(t; 0) \Omega \rangle
\end{aligned}$$

and $\gamma_2(x, y)$ as

$$\begin{aligned}
\gamma_2(x, y) & = \varphi_{2,t}(x) \overline{\varphi_{2,t}(y)} + \frac{\overline{\varphi_{2,t}(y)}}{\sqrt{N_2}} \langle \Omega, U_{N_1, N_2}^*(t; 0) a_{2,x} U_{N_1, N_2}(t; 0) \Omega \rangle + \\
& + \frac{\varphi_{2,t}(x)}{\sqrt{N_2}} \langle \Omega, U_{N_1, N_2}^*(t; 0) a_{2,y}^* U_{N_1, N_2}(t; 0) \Omega \rangle + \\
& + \frac{1}{N_2} \langle \Omega, U_{N_1, N_2}^*(t; 0) a_{2,y}^* a_{2,x} U_{N_1, N_2}(t; 0) \Omega \rangle.
\end{aligned}$$

Now we introduce another evolution \tilde{U}_{N_1, N_2} defined through the equation

$$i \frac{\partial}{\partial t} \tilde{U}_{N_1, N_2}(t; s) = \tilde{\mathcal{L}}_{N_1, N_2}(t) \tilde{U}_{N_1, N_2}(t; s), \quad \text{with } \tilde{U}_{N_1, N_2}(s; s) = 1.$$

with the generator

$$\tilde{\mathcal{L}}_{N_1, N_2} = \tilde{L}1 + \tilde{L}2 + \frac{1}{N_1 + N_2} \int dx dy \tilde{A}, \quad (3.16)$$

where

$$\begin{aligned}
\tilde{L}1 & = \int dx \Delta_x a_{1,x}^* \Delta_x a_{1,x} + \int dx (V_1 * |\varphi_{1,t}|^2)(x) a_{1,x}^* a_{1,x} + \int dx dy V_1(x-y) \bar{\varphi}_{1,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{1,x} + \\
& + \frac{1}{2} \int dx dy V_1(x-y) (\varphi_{1,t}(x) \varphi_{1,t}(y) a_{1,x}^* a_{1,y}^* + \bar{\varphi}_{1,t}(x) \bar{\varphi}_{1,t}(y) a_{1,x} a_{1,y}) + \\
& + \frac{1}{2N_1} \int dx dy V_1(x-y) a_{1,x}^* a_{1,y}^* a_{1,y} a_{1,x}, \\
\tilde{L}2 & = \int dx \Delta_x a_{2,x}^* \Delta_x a_{2,x} + \int dx (V_2 * |\varphi_{2,t}|^2)(x) a_{2,x}^* a_{2,x} + \int dx dy V_2(x-y) \bar{\varphi}_{2,t}(x) \varphi_{2,t}(y) a_{2,y}^* a_{2,x} + \\
& + \frac{1}{2} \int dx dy V_2(x-y) (\varphi_{2,t}(x) \varphi_{2,t}(y) a_{2,x}^* a_{2,y}^* + \bar{\varphi}_{2,t}(x) \bar{\varphi}_{2,t}(y) a_{2,x} a_{2,y}) + \\
& + \frac{1}{2N_2} \int dx dy V_2(x-y) a_{2,x}^* a_{2,y}^* a_{2,y} a_{2,x},
\end{aligned}$$

and

$$\begin{aligned} \tilde{A} = Q(x-y) & \left[a_{2,x}^* a_{1,y}^* a_{1,y} a_{2,x} + \right. \\ & + \sqrt{N_1 N_2} \varphi_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* + \sqrt{N_1 N_2} \bar{\varphi}_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y} + \\ & + N_1 \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) a_{2,x}^* a_{2,x} + N_2 \bar{\varphi}_{2,t}(x) \varphi_{2,t}(x) a_{1,y}^* a_{1,y} + \\ & \left. + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{2,x} + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) a_{1,y} a_{2,x} \right]. \end{aligned}$$

This generator was formed by discarding the terms in \mathcal{L}_{N_1, N_2} that did not commute with $(-1)^{\mathcal{N}_1 + \mathcal{N}_2}$ so that

$$\left[(-1)^{\mathcal{N}_1 + \mathcal{N}_2}, \tilde{\mathcal{L}}_{N_1, N_2} \right] = 0.$$

From this it follows $\tilde{\mathcal{U}}_{N_1, N_2}(t; s)$ commutes with $(-1)^{\mathcal{N}_1 + \mathcal{N}_2}$ also. Hence, using Proposition (1.3.14), we have that

$$\begin{aligned} \langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle & = \langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2} (-1)^{\mathcal{N}_1 + \mathcal{N}_2} \Omega \rangle \\ & = \langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,y} (-1)^{\mathcal{N}_1 + \mathcal{N}_2} \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle \\ & = -\langle \Omega, (-1)^{\mathcal{N}_1 + \mathcal{N}_2} \tilde{\mathcal{U}}_{N_1, N_2} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle \\ & = -\langle (-1)^{\mathcal{N}_1 + \mathcal{N}_2} \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle \\ & = -\langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle. \end{aligned}$$

Similarly, $\langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,x}^* \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle = -\langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2} a_{1,x}^* \tilde{\mathcal{U}}_{N_1, N_2} \Omega \rangle$. Thus it follows that

$$\langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \rangle = \langle \Omega, \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) a_{1,x}^* \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \rangle = 0.$$

Using this fact, we can write $\gamma_1^{(1)}(x, y) - \varphi_{1,t}(x) \bar{\varphi}_{1,t}(y)$ as

$$\begin{aligned} \gamma_1^{(1)}(x, y) - \varphi_{1,t}(x) \bar{\varphi}_{1,t}(y) & = \frac{1}{N_1} \langle \Omega, \mathcal{U}_{N_1, N_2}(t; 0) a_{1,y}^* a_{1,x} \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\ & + \frac{\varphi_{1,t}(x)}{\sqrt{N_1}} \left[\langle \Omega, \mathcal{U}_{N_1, N_2}(t; 0)^* a_{1,y}^* \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \rangle + \right. \\ & + \langle \Omega, \left(\mathcal{U}_{N_1, N_2}^*(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) \right) a_{1,y}^* \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \rangle \left. \right] + \\ & + \frac{\bar{\varphi}_{1,t}(y)}{\sqrt{N_1}} \left[\langle \Omega, \mathcal{U}_{N_1, N_2}^*(t; 0) a_{1,x} \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \rangle + \right. \\ & + \langle \Omega, \left(\mathcal{U}_{N_1, N_2}^*(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) \right) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) \Omega \rangle \left. \right] \end{aligned}$$

and $\gamma_2^{(1)}(x, y) - \varphi_{2,t}(x)\bar{\varphi}_{2,t}(y)$ as

$$\begin{aligned} \gamma_2^{(1)}(x, y) - \varphi_{2,t}(x)\bar{\varphi}_{2,t}(y) &= \frac{1}{N_2} \langle \Omega, \mathcal{U}_{N_1, N_2}(t; 0) a_{2,y}^* a_{2,x} \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\ &+ \frac{\varphi_{2,t}(x)}{\sqrt{N_2}} \left[\langle \Omega, \mathcal{U}_{N_1, N_2}(t; 0)^* a_{2,y}^* \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \rangle + \right. \\ &+ \langle \Omega, \left(\mathcal{U}_{N_1, N_2}^*(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) \right) a_{2,y}^* \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \rangle \left. \right] + \\ &+ \frac{\bar{\varphi}_{2,t}(y)}{\sqrt{N_2}} \left[\langle \Omega, \mathcal{U}_{N_1, N_2}^*(t; 0) a_{2,x} \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \rangle + \right. \\ &+ \langle \Omega, \left(\mathcal{U}_{N_1, N_2}^*(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) \right) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}^*(t; 0) \Omega \rangle \left. \right]. \end{aligned}$$

Now we multiply $\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}|$ by a Hilbert-Schmidt operator J_1 over $L^2(\mathbb{R}^3)$ with kernel $J_1(x, y)$ and take the trace (see [19]) to obtain

$$\begin{aligned} &\text{Tr} \left(J_1(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \right) \\ &= \frac{1}{N_1} \int dx dy J_1(x, y) \langle a_{1,y} \mathcal{U}_{N_1, N_2}(t; 0) \Omega, a_{1,x} \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle \\ &+ \frac{1}{\sqrt{N_1}} \int dx dy J_1(x, y) \varphi_{1,t}(x) \langle a_{1,y} \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \rangle \\ &+ \frac{1}{\sqrt{N_1}} \int dx dy J_1(x, y) \varphi_{1,t}(x) \langle \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega, a_{1,y}^* \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \rangle \\ &+ \frac{1}{\sqrt{N_1}} \int dx dy J_1(x, y) \bar{\varphi}_{1,t}(y) \langle a_{1,x}^* \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \rangle \\ &+ \frac{1}{\sqrt{N_1}} \int dx dy J_1(x, y) \bar{\varphi}_{1,t}(y) \langle \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega, a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} &\left| \text{Tr} \left(J_1(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \right) \right| \\ &\leq \frac{1}{N_1} \left(\int dx dy |J_1(x, y)|^2 \right)^{1/2} \int dx \|a_{1,x} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\|^2 \\ &+ \frac{1}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \int dx |\varphi_{1,t}(x)| \|a_1(J(x, \cdot)) \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| \\ &+ \frac{1}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \int dx |\varphi_{1,t}(x)| \|a_1^*(J(x, \cdot)) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\| \\ &+ \frac{1}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \int dy |\varphi_{1,t}(y)| \|a_1^*(J(\cdot, y)) \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| \\ &+ \frac{1}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \int dy |\varphi_{1,t}(y)| \|a_1(J(\cdot, y)) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \text{Tr} J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \right| &\leq \frac{\|J_1\|_{\text{HS}}}{N_1} \langle \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \mathcal{N}_1 \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\ &+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_1 + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| + \\ &+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_1 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\|. \end{aligned} \quad (3.17)$$

Similarly, we can multiply $\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}|$ by a Hilbert-Schmidt operator J_2 over $L^2(\mathbb{R}^3)$ with kernel $J_2(x, y)$ and take the trace to obtain

$$\begin{aligned} \left| \text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right| &\leq \frac{\|J_2\|_{\text{HS}}}{N_2} \langle \mathcal{U}_{N_1, N_2}(t; 0)\Omega, \mathcal{N}_2 \mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle + \\ &+ \frac{2\|J_2\|_{\text{HS}}}{\sqrt{N_2}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_2 + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0)\Omega\| + \\ &+ \frac{2\|J_2\|_{\text{HS}}}{\sqrt{N_2}} \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \|(\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\|. \end{aligned} \quad (3.18)$$

Our first step in proving Theorem (3.3.1) is to find bounds on the terms in (3.17) and (3.18). To do so, we first aim to prove Proposition (3.4.1), which gives upper bounds on the growth of powers of the number operator in norm.

3.4 Control of Growth of Number Operator

This section will be devoted to proving the following proposition.

Proposition 3.4.1. *Let $\mathcal{U}_{N_1, N_2}(t; s)$ be the unitary evolution operator given in Definition (3.2.1). Then for every $j \in \mathbb{N}$ there exists constants $A(j)$ and $C(j)$ such that*

$$\langle \mathcal{U}_{N_1, N_2}(t; s)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s)\psi \rangle \leq A(j) \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 e^{C(j)|t-s|} \quad (3.19)$$

for all $\psi \in D(\mathcal{H}_{N_1, N_2})$ and for all $t, s \in \mathbb{R}$.

Note the right-hand side of (3.19) is independent of N_1 and N_2 . Thus, Proposition (3.4.1) says that the number of particles produced by the dynamics \mathcal{U}_{N_1, N_2} is independent of N_1 and N_2 and grows in time with at most an exponential rate. The proof of this proposition requires proving several lemmas. Also, we need to introduce a new unitary evolution $\mathcal{U}_{N_1, N_2}^{(M)}$ with generator $\mathcal{L}_{N_1, N_2}^{(M)}(t)$, which is just like $\mathcal{L}_{N_1, N_2}(t)$ except it has a cutoff in the number of particles in the cubic terms. To do so, we first need to introduce the cutoff operator $\chi(\mathcal{N} \leq M)$ on \mathcal{F} .

Definition 3.4.2. Let $\psi = \{\psi_n\}_{n \geq 0} \in \mathcal{F}$, and let M be a non-negative integer. The *cutoff operator* $\chi(\mathcal{N} \leq M)$ is defined by

$$\chi(\mathcal{N} \leq M)\psi = \{\psi_0, \psi_1, \psi_2, \psi_3, \dots, \psi_M, 0, 0, \dots\}.$$

Proposition 3.4.3. *Let $M \in \mathbb{N}$. The cutoff operator $\chi(\mathcal{N} \leq M)$ satisfies*

$$(i) \quad \|\chi(\mathcal{N} \leq M)\|_{op} \leq 1 \quad (3.20)$$

$$(ii) \quad \|\mathcal{N} \chi(\mathcal{N} \leq M)\|_{op} \leq M \quad (3.21)$$

$$(iii) \quad \chi(\mathcal{N} \leq M)a_x = a_x \chi(\mathcal{N} \leq M + 1) \quad (3.22)$$

$$(iv) \quad \chi(\mathcal{N} \leq M)a_x^* = a_x^* \chi(\mathcal{N} \leq M - 1). \quad (3.23)$$

Proof. Let $\psi = \{\psi_n\}_{n \geq 0} \in \mathcal{F}$. Note (i) is obvious, since

$$\|\chi(\mathcal{N} \leq M)\psi\|^2 = \sum_{n=0}^M \|\psi_n\|_{\mathcal{H}^{\otimes n}}^2 \leq \|\psi\|^2.$$

(ii) Note

$$\|\mathcal{N} \chi(\mathcal{N} \leq M)\psi\|^2 = \sum_{n=0}^M \|n\psi_n\|_{\mathcal{H}^{\otimes n}}^2 \leq M^2 \|\psi\|^2.$$

Finally (iii) and (iv) are obvious since a_x annihilates and a_x^* creates. \square

Now we can introduce the cutoff operators that we will use for two species.

Definition 3.4.4. Let $\psi = \psi_1 \otimes \psi_2 \in \mathcal{F} \otimes \mathcal{F}$, and let M be a non-negative integer. The operators $\chi(\mathcal{N}_1 \leq M)$ and $\chi(\mathcal{N}_2 \leq M)$ are defined by

$$\chi(\mathcal{N}_1 \leq M)\psi = (\chi(\mathcal{N} \leq M)\psi_1) \otimes \psi_2 \quad \text{and} \quad \chi(\mathcal{N}_2 \leq M)\psi = \psi_1 \otimes (\chi(\mathcal{N} \leq M)\psi_2).$$

The generator $\mathcal{L}_{N_1, N_2}^{(M)}(t)$, which is just like $\mathcal{L}_{N_1, N_2}(t)$ except it has a cutoff in the number of particles in the cubic terms, is then given by

$$\mathcal{L}_{N_1, N_2}^{(M)}(t) = L1^{(M)}(t) + L2^{(M)}(t) + \frac{1}{N_1 + N_2} \int dx dy A^{(M)}(t)$$

where

$$\begin{aligned} L1^{(M)}(t) &= \int dx \Delta_x a_{1,x}^* \Delta_x a_{1,x} + \int dx (V_1 * |\varphi_{1,t}|^2)(x) a_{1,x}^* a_{1,x} + \\ &+ \int dx dy V_1(x-y) \bar{\varphi}_{1,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{1,x} + \\ &+ \frac{1}{2} \int dx dy V_1(x-y) \left(\varphi_{1,t}(x) \varphi_{1,t}(y) a_{1,x}^* a_{1,y}^* + \bar{\varphi}_{1,t}(x) \bar{\varphi}_{1,t}(y) a_{1,x} a_{1,y} \right) + \\ &+ \frac{1}{\sqrt{N_1}} \int dx dy V_1(x-y) a_{1,x}^* \left(\bar{\varphi}_{1,t}(y) a_{1,y} \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) + \right. \\ &\quad \left. + \varphi_{1,t}(y) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y}^* \right) a_{1,x} + \\ &+ \frac{1}{2N_1} \int dx dy V_1(x-y) a_{1,x}^* a_{1,y}^* a_{1,y} a_{1,x}, \\ L2^{(M)}(t) &= \int dx \Delta_x a_{2,x}^* \Delta_x a_{2,x} + \int dx (V_2 * |\varphi_{2,t}|^2)(x) a_{2,x}^* a_{2,x} + \\ &+ \int dx dy V_2(x-y) \bar{\varphi}_{2,t}(x) \varphi_{2,t}(y) a_{2,y}^* a_{2,x} + \\ &+ \frac{1}{2} \int dx dy V_2(x-y) \left(\varphi_{2,t}(x) \varphi_{2,t}(y) a_{2,x}^* a_{2,y}^* + \bar{\varphi}_{2,t}(x) \bar{\varphi}_{2,t}(y) a_{2,x} a_{2,y} \right) + \\ &+ \frac{1}{\sqrt{N_2}} \int dx dy V_2(x-y) a_{2,x}^* \left(\bar{\varphi}_{2,t}(y) a_{2,y} \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) + \right. \\ &\quad \left. + \varphi_{2,t}(y) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{2,y}^* \right) a_{2,x} + \\ &+ \frac{1}{2N_2} \int dx dy V_2(x-y) a_{2,x}^* a_{2,y}^* a_{2,y} a_{2,x}, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
A^{(M)}(t) = Q(x-y) & \left[a_{2,x}^* a_{1,y}^* a_{1,y} a_{2,x} + \sqrt{N_2} \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} + \right. \\
& + \sqrt{N_1} \varphi_{1,t}(y) a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{2,x} + \sqrt{N_1 N_2} \varphi_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* + \\
& + \sqrt{N_1} \bar{\varphi}_{1,t}(y) a_{2,x}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} a_{2,x} + \sqrt{N_1 N_2} \bar{\varphi}_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y} + \\
& + N_1 \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) a_{2,x}^* a_{2,x} + \sqrt{N_2} \bar{\varphi}_{2,t}(x) a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} a_{2,x} + \\
& + N_2 \bar{\varphi}_{2,t}(x) \varphi_{2,t}(x) a_{1,y}^* a_{1,y} + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{2,x} + \\
& \left. + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) a_{1,y} a_{2,x} + N_1 N_2 \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) \varphi_{2,t}(x) \right]. \tag{3.25}
\end{aligned}$$

The corresponding evolution $\mathcal{U}_{N_1, N_2}^{(M)}$ is then given by

$$i \frac{\partial}{\partial t} \mathcal{U}_{N_1, N_2}^{(M)}(t; s) = \mathcal{L}_{N_1, N_2}^{(M)}(t) \mathcal{U}_{N_1, N_2}^{(M)}(t; s), \quad \text{with } \mathcal{U}_{N_1, N_2}^{(M)}(s; s) = 1.$$

3.4.1 Lemma (3.4.5)

Lemma 3.4.5. *There exists a constant C such that, for all $N_1, N_2, M \in \mathbb{N}$, $\psi \in D(\mathcal{H}_{N_1, N_2})$, and $t, s \in \mathbb{R}$,*

$$\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; s) \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; s) \psi \rangle \leq \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \psi \rangle e^{4^j C |t-s| \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}\right)}.$$

Proof. To prove this lemma, we look at the time derivative of the LHS and then use Gronwall's Lemma. Note

$$\begin{aligned}
& \frac{d}{dt} \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle \\
& = \langle -i \mathcal{L}_{N_1, N_2}^{(M)} \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle + \\
& \quad + \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \left(-i \mathcal{L}_{N_1, N_2}^{(M)} \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \right) \psi \rangle \\
& = \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, \left[i \mathcal{L}_{N_1, N_2}^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle.
\end{aligned}$$

Expanding out $\mathcal{L}_{N_1, N_2}^{(M)}$ gives

$$\begin{aligned}
& \frac{d}{dt} \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle \\
& = \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, i \left[L1^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle + \\
& \quad + \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, i \left[L2^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle + \\
& \quad + \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, \frac{i}{N_1 + N_2} \left[\int dx dy A^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle.
\end{aligned}$$

Now, for the first two terms, we can use Proposition (1.3.12) (ii) to get the result

$$\begin{aligned}
& |\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, i \left[L1^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle| \\
& \leq 4^j C \left(1 + \sqrt{M/N_1} \right) \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \rangle,
\end{aligned}$$

and

$$\begin{aligned} & |\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi, i \left[L2^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi \rangle | \\ & \leq 4^j C \left(1 + \sqrt{M/N_2} \right) \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi \rangle \end{aligned}$$

where C represents some constant and is not necessarily the same constant. (See [19] for details on a similar analysis in the case of one species.)

Now we look at the term $\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi, \frac{i}{N_1 + N_2} \left[\int dx dy A^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi \rangle$. Looking at $A^{(M)}$ given in (3.25), we see that the integral of the first term, i.e. $\int dx dy Q(x - y) a_{2,x}^* a_{1,y}^* a_{1,y} a_{2,x}$, as well as the integral of the terms $N_2 Q(x - y) \bar{\varphi}_{2,t}(x) \varphi_{2,t}(x) a_{1,y}^* a_{1,y}$, $N_1 Q(x - y) \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) a_{2,x}^* a_{2,x}$, $\sqrt{N_1 N_2} \bar{\varphi}_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}$, and $\sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{2,x}$ commutes with $\mathcal{N}_1 + \mathcal{N}_2 + 1$, which follows from Proposition (1.2.17).

Therefore,

$$\begin{aligned} & \frac{i}{N_1 + N_2} \left[\int dx dy A^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] = \\ & = \frac{i}{N_1 + N_2} \left[\int dx dy Q(x - y) \left(\sqrt{N_2} \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} + \right. \right. \\ & + \sqrt{N_1} \varphi_{1,t}(y) a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{2,x} + \sqrt{N_1 N_2} \varphi_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* + \\ & + \sqrt{N_1} \bar{\varphi}_{1,t}(y) a_{2,x}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} a_{2,x} + \\ & + \sqrt{N_2} \bar{\varphi}_{2,t}(x) a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} a_{2,x} + \\ & \left. \left. + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) a_{1,y} a_{2,x} \right), (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right]. \end{aligned}$$

Note the integral contains 6 terms in total, which make up three pairs of terms and their complex conjugates. Let us now prove the following lemma.

Lemma 3.4.6. *Let A and B be operators on a Hilbert space where B be self-adjoint. Let $\psi \in D(A) \cap D(B)$. Then*

$$\langle \psi, i [A + A^*, B] \psi \rangle = -2 \operatorname{Im} \langle \psi, [A, B] \psi \rangle. \quad (3.26)$$

Proof. To show (3.26), note

$$\begin{aligned} -2 \operatorname{Im} \langle \psi, [A, B] \psi \rangle & = -2 \frac{1}{2i} \left(\langle \psi, [A, B] \psi \rangle - \overline{\langle \psi, [A, B] \psi \rangle} \right) \\ & = -\frac{1}{i} \left(\langle \psi, [A, B] \psi \rangle - \langle [A, B] \psi, \psi \rangle \right) \\ & = -\frac{1}{i} \left(\langle \psi, [A, B] \psi \rangle - \langle \psi, [B^*, A^*] \psi \rangle \right) \\ & = -\frac{1}{i} \left(\langle \psi, [A, B] \psi \rangle + \langle \psi, [A^*, B^*] \psi \rangle \right). \end{aligned}$$

Now since B is self adjoint, $[A^*, B^*] = [A^*, B]$. Thus,

$$\begin{aligned} -2 \operatorname{Im} \langle \psi, [A, B]\psi \rangle &= -\frac{1}{i} \langle \psi, [A + A^*, B]\psi \rangle \\ &= \langle \psi, i[A + A^*, B]\psi \rangle. \end{aligned}$$

□

Now, using Lemma (3.4.6), we can write

$$\begin{aligned} &\langle \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(t; 0)\psi, \frac{i}{\mathcal{N}_1 + \mathcal{N}_2} \left[\int dx dy A^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(t; 0)\psi \rangle = \\ &= \frac{2}{\mathcal{N}_1 + \mathcal{N}_2} \left(\sqrt{\mathcal{N}_2} \operatorname{Im} \left\langle \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi, \int dx dy \varphi_{2,t}(x)Q(x-y) \cdot \right. \right. \\ &\quad \cdot \left. \left[a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi \right\rangle + \\ &+ \sqrt{\mathcal{N}_1} \operatorname{Im} \left\langle \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi, \int dx dy \varphi_{1,t}(y)Q(x-y) \cdot \right. \\ &\quad \cdot \left. \left[a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{2,x}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi \right\rangle + \\ &\left. + \sqrt{\mathcal{N}_1 \mathcal{N}_2} \operatorname{Im} \left\langle \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi, \int dx dy \varphi_{1,t}(y) \varphi_{2,t}(x)Q(x-y) \left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi \right\rangle \right). \end{aligned} \tag{3.27}$$

We wish to find an upper bound on each term. We start by looking at the first term in (3.27). Using Proposition (1.3.12) (iv), note that

$$\begin{aligned} &\left[a_{2,x}^* a_{1,y}^* a_{1,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] = \sum_{k=0}^{j-1} \binom{j}{k} \left((\mathcal{N}_1 + \mathcal{N}_2 - 1)^{k/2} a_{2,x}^* a_{1,y}^* a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} + \right. \\ &+ (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{2,x}^* a_{1,y}^* a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} + \\ &\left. + (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{2,x}^* a_{1,y}^* a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \right). \end{aligned}$$

Hence,

$$\begin{aligned} &\int dx dy \varphi_{2,t}(x)Q(x-y) \left[a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi = \\ &= \sum_{k=0}^{j-1} \binom{j}{k} \int dx dy \varphi_{2,t}(x)Q(x-y) \cdot \\ &\quad \cdot \left((\mathcal{N}_1 + \mathcal{N}_2 - 1)^{k/2} a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} + \right. \\ &\quad + (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} + \\ &\quad \left. + (-1)^{j-k} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \right) \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}\psi. \end{aligned}$$

Therefore, the first term can be written as

$$\begin{aligned}
& \frac{2\sqrt{N_2}}{N_1 + N_2} \operatorname{Im} \left\langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, \int dx dy \varphi_{2,t}(x) \left[a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \\
&= \frac{2\sqrt{N_2}}{N_1 + N_2} \sum_{k=0}^{j-1} \binom{j}{k} \operatorname{Im} \left(\int dy \left\langle a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 - 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, \right. \right. \\
&\quad \left. \left. a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle + \right. \\
&\quad \left. + \int dy \left\langle (-1)^{j-k} a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, \right. \right. \\
&\quad \left. \left. a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle + \right. \\
&\quad \left. + \int dy \left\langle (-1)^{j-k} a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, \right. \right. \\
&\quad \left. \left. a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right).
\end{aligned}$$

Note

$$\begin{aligned}
& \left| \int dy \left\langle a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 - 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, \right. \right. \\
&\quad \left. \left. a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq \int dy \|a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 - 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \cdot \\
&\quad \cdot \|a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M)\| \|a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \\
&\leq \int dy \|a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2 \cdot \\
&\quad \cdot \|Q(\cdot - y) \varphi_{2,t}\|_{L^2} \|(\mathcal{N}_2 + 1)^{1/2} \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M)\| \\
&\leq (M+1)^{1/2} \sup_y \|Q(\cdot - y) \varphi_{2,t}\|_{L^2} \|\mathcal{N}_1^{1/2} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2 \\
&\leq (M+1)^{1/2} C \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \left| \int dy \left\langle (-1)^{j-k} a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, \right. \right. \\
&\quad \left. \left. a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq (M+1)^{1/2} C \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int dy \left\langle (-1)^{j-k} a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, \right. \right. \\
&\quad \left. \left. a_2^*(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{1,y} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq (M+1)^{1/2} C \|(\mathcal{N}_1 + \mathcal{N}_2 + 2)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2.
\end{aligned}$$

Therefore, we have the following bound for the absolute value of the first term in (3.27):

$$\begin{aligned}
& \left| \frac{2\sqrt{N_2}}{N_1 + N_2} \operatorname{Im} \left\langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, \int dx dy \varphi_{2,t}(x) \cdot \right. \right. \\
& \quad \left. \left. \cdot \left[\chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{2,x}^* a_{1,y}^* a_{1,y}, (\mathcal{N}_1 + \mathcal{N}_2 + 3)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
& \leq C \frac{\sqrt{N_2}}{N_1 + N_2} (M+1)^{1/2} \sum_{k=0}^{j-1} \binom{j}{k} \|(\mathcal{N}_1 + \mathcal{N}_2 + 3)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2 \\
& = C \frac{\sqrt{N_2}}{N_1 + N_2} (M+1)^{1/2} \sum_{k=0}^{j-1} \binom{j}{k} \langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k+1} \mathcal{U}_{N_1, N_2}^{(M)} \psi \rangle \\
& \leq C \frac{\sqrt{N_2}}{N_1 + N_2} (M+1)^{1/2} \sum_{k=0}^j \binom{j}{k} \langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 3)^k \mathcal{U}_{N_1, N_2}^{(M)} \psi \rangle \\
& \leq C \frac{\sqrt{N_2}}{N_1 + N_2} (M+1)^{1/2} 4^j \langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)} \psi \rangle.
\end{aligned}$$

Similarly, for the absolute value of the second term in (3.27), we get

$$\begin{aligned}
& \left| \frac{2\sqrt{N_1}}{N_1 + N_2} \operatorname{Im} \left\langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, \int dx dy \varphi_{1,t}(y) Q(x-y) \cdot \right. \right. \\
& \quad \left. \left. \cdot \left[a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 \leq M) \chi(\mathcal{N}_2 \leq M) a_{2,x}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
& \leq C \frac{\sqrt{N_1}}{N_1 + N_2} (M+1)^{1/2} \sum_{k=0}^{j-1} \binom{j}{k} \langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k+1} \mathcal{U}_{N_1, N_2}^{(M)} \psi \rangle \\
& \leq C \frac{\sqrt{N_1}}{N_1 + N_2} (M+1)^{1/2} 4^j \langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)} \psi \rangle.
\end{aligned}$$

Now let's look at the third term in (3.27), i.e. the term

$\frac{2\sqrt{N_1 N_2}}{N_1 + N_2} \operatorname{Im} \left\langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, \int dx dy \varphi_{1,t}(y) \varphi_{2,t}(x) Q(x-y) \left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle$. Note from Proposition (1.3.12) (iii),

$$\begin{aligned}
\left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] &= \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} \left((\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} + \right. \\
&\quad \left. + (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int dx dy \varphi_{1,t}(y) \varphi_{2,t}(x) Q(x-y) \left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \\
& = \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} \left((\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \int dx \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* (\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} + \right. \\
& \quad \left. + (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \int dx \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* (\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \right) \mathcal{U}_{N_1, N_2}^{(M)} \psi.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{2\sqrt{N_1 N_2}}{N_1 + N_2} \operatorname{Im} \left\langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, \int dx dy \varphi_{1,t}(y) \varphi_{2,t}(x) Q(x-y) \left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \\
&= \frac{2\sqrt{N_1 N_2}}{N_1 + N_2} \operatorname{Im} \sum_{k=0}^{j-1} \binom{j}{k} (-1)^{j-k} \cdot \\
&\quad \cdot \left(\int dx \left\langle \varphi_{2,t}(x) a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, a_1^*(\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle + \right. \\
&\quad \left. + \int dx \left\langle \varphi_{2,t}(x) a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, a_1^*(\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right).
\end{aligned}$$

Note

$$\begin{aligned}
& \left| \int dx \left\langle \varphi_{2,t}(x) a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, a_1^*(\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq \int dx |\varphi_{2,t}(x)| \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \|a_1^*(\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \\
&\leq \sup_x \|\varphi_{1,t} Q(x-\cdot)\|_{L^2} \|(\mathcal{N}_1 + \mathcal{N}_2)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \int dx |\varphi_{2,t}(x)| \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|.
\end{aligned}$$

Now, note that

$$\begin{aligned}
& \int dx |\varphi_{2,t}(x)| \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \\
&= \langle \varphi_{2,t}(x), \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\| \rangle_{L^2} \\
&\leq \left(\int dx \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2 \right)^{1/2} \\
&= \|\mathcal{N}_2^{1/2} (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left| \int dx \left\langle \varphi_{2,t}(x) a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, a_1^*(\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq C \|(\mathcal{N}_1 + \mathcal{N}_2 + 2)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \left| \int dx \left\langle \varphi_{2,t}(x) a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi, a_1^*(\varphi_{1,t} Q(x-\cdot)) (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq C \|(\mathcal{N}_1 + \mathcal{N}_2 + 3)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2.
\end{aligned}$$

So we get that the third term in (3.27) can be bounded as follows:

$$\begin{aligned}
& \left| \frac{2\sqrt{N_1 N_2}}{N_1 + N_2} \operatorname{Im} \left\langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, \int dx dy \varphi_{1,t}(y) \varphi_{2,t}(x) Q(x-y) \left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)} \psi \right\rangle \right| \\
&\leq C \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \sum_{k=0}^{j-1} \binom{j}{k} \|(\mathcal{N}_1 + \mathcal{N}_2 + 3)^{(k+1)/2} \mathcal{U}_{N_1, N_2}^{(M)} \psi\|^2 \\
&\leq C \frac{\sqrt{N_1 N_2}}{N_1 + N_2} 4^j \langle \mathcal{U}_{N_1, N_2}^{(M)} \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)} \psi \rangle.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
& \left| \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi, \frac{i}{N_1 + N_2} \left[\int dx dy A^{(M)}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \right] \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi \rangle \right| \\
& \leq C \frac{1}{N_1 + N_2} \left(\sqrt{N_2} + \sqrt{N_1} + \sqrt{N_1 N_2} \right) 4^j \langle \mathcal{U}_{N_1, N_2}^{(M)}\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}\psi \rangle \\
& \leq C \left(1 + \sqrt{\frac{M+1}{N_1 + N_2}} \right) 4^j \langle \mathcal{U}_{N_1, N_2}^{(M)}\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}\psi \rangle.
\end{aligned}$$

So we get

$$\begin{aligned}
& \left| \frac{d}{dt} \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)\psi \rangle \right| \\
& \leq C \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1 + N_2}} \right) 4^j \langle \mathcal{U}_{N_1, N_2}^{(M)}\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}\psi \rangle
\end{aligned}$$

and Gronwall's Lemma gives

$$\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi \rangle \leq \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \psi \rangle e^{4^j C |t-s| (1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1 + N_2}})},$$

as desired. □

3.4.2 Lemma (3.4.7)

Lemma 3.4.7. *For arbitrary $t, s \in \mathbb{R}$ and $\psi \in D(\mathcal{H}_{N_1, N_2})$, we have*

$$\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{N_1, N_2}^*(t; s)\psi \rangle \leq 6 \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2 + 1)\psi \rangle. \quad (3.28)$$

Also, for every $j \in \mathbb{N}$, there exist constants $C_1(j)$ and $C_2(j)$ such that

$$\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j} \mathcal{U}_{N_1, N_2}^*(t; s)\psi \rangle \leq C_1(j) \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j}\psi \rangle, \quad (3.29)$$

$$\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j+1} \mathcal{U}_{N_1, N_2}^*(t; s)\psi \rangle \leq C_2(j) \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j+1} (\mathcal{N}_1 + \mathcal{N}_2 + 1)\psi \rangle. \quad (3.30)$$

Proof. First we prove (3.28). Note, by the definition of $\mathcal{U}_{N_1, N_2}(t; s)$ and using the notation $\Phi_j(\varphi) = a_j^*(\varphi) +$

$a_j(\varphi)$, and from the properties of the Weyl operators and the fact that $[e^{i\mathcal{H}_{N_1, N_2}(t-s)}, \mathcal{N}_1 + \mathcal{N}_2] = 0$, we have

$$\begin{aligned}
& \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{N_1, N_2}(t; s) \\
&= \int dx \mathcal{U}_{N_1, N_2}^*(t; s) (a_{1,x}^* a_{1,x} + a_{2,x}^* a_{2,x}) \mathcal{U}_{N_1, N_2}(t; s) \\
&= \int dx W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left[(a_{1,x}^* - \sqrt{N_1} \bar{\varphi}_{1,t}) (a_{1,x} - \sqrt{N_1} \varphi_{1,t}(x)) + \right. \\
&\quad \left. + (a_{2,x}^* - \sqrt{N_2} \bar{\varphi}_{2,t}(x)) (a_{2,x} - \sqrt{N_2} \varphi_{2,t}(x)) \right] e^{-i\mathcal{H}_{N_1, N_2}(t-s)} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \\
&= W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\mathcal{N}_1 - \sqrt{N_1} a_1^*(\varphi_{1,t}) - \sqrt{N_1} a_1(\varphi_{1,t}) + N_1 + \right. \\
&\quad \left. + \mathcal{N}_2 - \sqrt{N_2} a_2^*(\varphi_{2,t}) - \sqrt{N_2} a_2(\varphi_{2,t}) + N_2 \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \\
&= W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) \left[\mathcal{N}_1 + \mathcal{N}_2 + \right. \\
&\quad \left. - e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(\varphi_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} + N_1 + N_2 \right] W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}).
\end{aligned} \tag{3.31}$$

Therefore,

$$\begin{aligned}
\langle \psi, \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{N_1, N_2}(t; s) \psi \rangle &= \langle \psi, W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) \left[\mathcal{N}_1 + \mathcal{N}_2 + \right. \\
&\quad \left. - e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(\varphi_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} + N_1 + N_2 \right] W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \rangle.
\end{aligned}$$

We look at the term with $\Phi_1(\varphi_{1,t})$. To simplify notation, define $\gamma := i\mathcal{H}_{N_1, N_2}(t-s)$. Note

$$\begin{aligned}
& \sqrt{N_1} \langle \psi, W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) e^\gamma \Phi_1(\varphi_{1,t}) e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \rangle \\
&= \sqrt{N_1} \langle e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi, \Phi_1(\varphi_{1,t}) e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \rangle \\
&\leq \sqrt{N_1} \|e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi\| \|\Phi_1(\varphi_{1,t}) e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi\|.
\end{aligned}$$

Now, note from (1.15) it follows that if $f \in L^2(\mathbb{R}^3)$, we have $\|\Phi_j(f)\psi\| \leq 2\|f\|_{L^2} \|(\mathcal{N}_j + 1)^{1/2} \psi\|$. Thus,

$$\begin{aligned}
& \sqrt{N_1} \langle \psi, W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) e^\gamma \Phi_1(\varphi_{1,t}) e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \rangle \\
&\leq 2\sqrt{N_1} \|e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi\| \|\varphi_{1,t}\|_{L^2} \|(\mathcal{N}_1 + 1)^{1/2} e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi\| \\
&= 2 \left\langle W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi, N_1 W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \right\rangle^{1/2} \\
&\quad \cdot \left\langle W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi, (\mathcal{N}_1 + 1) W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \right\rangle^{1/2}.
\end{aligned}$$

Note it follows from $(\sqrt{x} - \sqrt{y})^2 \geq 0$ that $2\sqrt{xy} \leq x + y$. Hence,

$$\begin{aligned}
& \sqrt{N_1} \langle \psi, W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) e^\gamma \Phi_1(\varphi_{1,t}) e^{-\gamma} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \rangle \\
&\leq \left\langle W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi, (N_1 + \mathcal{N}_1 + 1) W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \right\rangle.
\end{aligned}$$

The term with $\Phi_2(\varphi_{2,t})$ can be computed similarly; we get

$$\begin{aligned} & \sqrt{N_2} \langle \psi, W_1^*(\sqrt{N_1}\varphi_{1,s})W_2^*(\sqrt{N_2}\varphi_{2,s})e^\gamma\Phi_2(\varphi_{2,t})e^{-\gamma}W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s})\psi \rangle \\ & \leq \langle W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s})\psi, (N_2 + \mathcal{N}_2 + 1) W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s})\psi \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} & \langle \psi, \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{N_1, N_2}(t; s) \psi \rangle \\ & \leq \langle \psi, W_1^*(\sqrt{N_1}\varphi_{1,s})W_2^*(\sqrt{N_2}\varphi_{2,s}) \cdot \\ & \quad \cdot (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + \mathcal{N}_1 + 1 + N_2 + \mathcal{N}_2 + 1 + N_1 + N_2) W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s})\psi \rangle \\ & = 2 \langle \psi, W_1^*(\sqrt{N_1}\varphi_{1,s})W_2^*(\sqrt{N_2}\varphi_{2,s}) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2 + 1) W_1(\sqrt{N_1}\varphi_{1,s})W_2(\sqrt{N_2}\varphi_{2,s})\psi \rangle. \end{aligned}$$

Now, from $W_1^*(f_1)W_2^*(f_2)a_{j,x}W_1(f_1)W_2(f_2) = a_{j,x} + f_j$, for $j = 1, 2$, we get that

$$W_1^*(f_1)W_2^*(f_2)\mathcal{N}_jW_1(f_1)W_2(f_2) = \mathcal{N}_j + \Phi_j(f_j) + \|f_j\|_{L^2}^2.$$

So

$$\begin{aligned} & \langle \psi, \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{N_1, N_2}(t; s) \psi \rangle \\ & \leq 2 \langle \psi, \left(\mathcal{N}_1 + \mathcal{N}_2 + \Phi_1(\sqrt{N_1}\varphi_{1,s}) + \Phi_2(\sqrt{N_2}\varphi_{2,s}) + \|\sqrt{N_1}\varphi_{1,s}\|_{L^2}^2 + \|\sqrt{N_2}\varphi_{2,s}\|_{L^2}^2 + N_1 + N_2 + 1 \right) \psi \rangle \\ & = 2 \langle \psi, \left(\mathcal{N}_1 + \mathcal{N}_2 + \sqrt{N_1}\Phi_1(\varphi_{1,s}) + \sqrt{N_2}\Phi_2(\varphi_{2,s}) + 2N_1 + 2N_2 + 1 \right) \psi \rangle. \end{aligned}$$

Now note

$$\begin{aligned} \langle \psi, \sqrt{N_j}\Phi_j(\varphi_{j,s})\psi \rangle & \leq \sqrt{N_j}\|\psi\| \|\Phi_j(\varphi_{j,s})\psi\| \\ & \leq 2\sqrt{N_j}\|\psi\| \|\varphi_{j,s}\|_{L^2} \|(\mathcal{N}_j + 1)^{1/2}\psi\| \\ & \leq 2\langle \psi, N_j\psi \rangle^{1/2} \langle \psi, (\mathcal{N}_j + 1)\psi \rangle^{1/2} \\ & \leq \langle \psi, (N_j + \mathcal{N}_j + 1)\psi \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} & \langle \psi, \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{N_1, N_2}(t; s) \psi \rangle \\ & \leq 2 \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + \mathcal{N}_1 + 1 + N_2 + \mathcal{N}_2 + 1 + 2N_1 + 2N_2 + 1) \psi \rangle \\ & \leq 6 \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2 + 1) \psi \rangle, \end{aligned}$$

which shows (3.28). Now we prove (3.29). To do so, we define

$$X_{t,s} = \mathcal{N}_1 + \mathcal{N}_2 - e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1}\Phi_1(\varphi_{1,t}) + \sqrt{N_2}\Phi_2(\varphi_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} + N_1 + N_2.$$

Note

$$\begin{aligned}
& \langle \psi, X_{t,s} \psi \rangle \leq 2 \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2 + 1) \psi \rangle \\
\Rightarrow & X_{t,s} \leq 2 (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2 + 1) \\
& \leq 4 (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) \\
\Rightarrow & X_{t,s}^2 \leq C (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^2. \tag{3.32}
\end{aligned}$$

We also want to show $|\text{ad}_{X_{t,s}}^m(\mathcal{N}_1 + \mathcal{N}_2)| \leq C(\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)$, for all $m \in \mathbb{N}$. First, let us observe that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
& \left| \text{ad}_{X_{t,s}}^{2n}(\mathcal{N}_1 + \mathcal{N}_2) \right| = \left| e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(a_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \right|, \\
& \left| \text{ad}_{X_{t,s}}^{2n-1}(\mathcal{N}_1 + \mathcal{N}_2) \right| = \left| e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} (a_1^*(\varphi_{1,t}) - a_1(\varphi_{1,t})) + \right. \right. \\
& \quad \left. \left. + \sqrt{N_2} (a_2^*(\varphi_{2,t}) - a_2(\varphi_{2,t})) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \right|. \tag{3.33}
\end{aligned}$$

To see this, note that

$$\begin{aligned}
\text{ad}_{X_{t,s}}(\mathcal{N}_1 + \mathcal{N}_2) &= [\mathcal{N}_1 + \mathcal{N}_2, X_{t,s}] \\
&= \left[\mathcal{N}_1 + \mathcal{N}_2, \mathcal{N}_1 + \mathcal{N}_2 + \right. \\
& \quad \left. - e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(\varphi_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} + N_1 + N_2 \right] \\
&= -e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} [\mathcal{N}_1, \Phi_1(\varphi_{1,t})] + \sqrt{N_2} [\mathcal{N}_2, \Phi_2(\varphi_{2,t})] \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \\
&= -e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} (a_1^*(\varphi_{1,t}) - a_1(\varphi_{1,t})) + \right. \\
& \quad \left. + \sqrt{N_2} (a_2^*(\varphi_{2,t}) - a_2(\varphi_{2,t})) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)}, \\
\text{ad}_{X_{t,s}}^2(\mathcal{N}_1 + \mathcal{N}_2) &= \left[\mathcal{N}_1 + \mathcal{N}_2, [\mathcal{N}_1 + \mathcal{N}_2, X_{t,s}] \right] \\
&= - \left[\mathcal{N}_1 + \mathcal{N}_2, e^{i\mathcal{H}_{N_1, N_2}(t-s)} \cdot \right. \\
& \quad \left. \cdot \left(\sqrt{N_1} (a_1^*(\varphi_{1,t}) - a_1(\varphi_{1,t})) + \sqrt{N_2} (a_2^*(\varphi_{2,t}) - a_2(\varphi_{2,t})) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \right] \\
&= -e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} [\mathcal{N}_1, a_1^*(\varphi_{1,t}) - a_1(\varphi_{1,t})] + \right. \\
& \quad \left. + \sqrt{N_2} [\mathcal{N}_2, a_2^*(\varphi_{2,t}) - a_2(\varphi_{2,t})] \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \\
&= -e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(a_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)}
\end{aligned}$$

where we have used Proposition (1.2.16). Now (3.33) follows. Next, note for all $n \in \mathbb{N}$,

$$\begin{aligned}
& \left\langle \psi, |\text{ad}_{X_{t,s}}^{2n}(\mathcal{N}_1 + \mathcal{N}_2)| \psi \right\rangle = \left\langle \psi, e^{i\mathcal{H}_{N_1, N_2}(t-s)} \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(a_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi \right\rangle \\
&= \left\langle e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi, \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(a_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi \right\rangle \\
&\leq \|e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi\| \left\| \left(\sqrt{N_1} \Phi_1(\varphi_{1,t}) + \sqrt{N_2} \Phi_2(a_{2,t}) \right) e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi \right\| \\
&\leq \|e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi\| \left(\sqrt{N_1} \|\varphi_{1,t}\|_{L^2} \|(\mathcal{N}_1 + 1)^{1/2} e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi\| + \right.
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{N_2} \|\varphi_{2,t}\|_{L^2} \|(\mathcal{N}_2 + 1)^{1/2} e^{-i\mathcal{H}_{N_1, N_2}(t-s)} \psi\|) \\
& = \langle \psi, N_1 \psi \rangle^{1/2} \langle \psi, (\mathcal{N}_1 + 1) \psi \rangle^{1/2} + \langle \psi, N_2 \psi \rangle^{1/2} \langle \psi, (\mathcal{N}_2 + 1) \psi \rangle^{1/2} \\
& \leq \frac{1}{2} \langle \psi, (N_1 + \mathcal{N}_1 + 1) \psi \rangle + \frac{1}{2} \langle \psi, (N_2 + \mathcal{N}_2 + 1) \psi \rangle \\
& = \frac{1}{2} \langle \psi, (N_1 + N_2 + \mathcal{N}_1 + \mathcal{N}_2 + 2) \psi \rangle \\
& \leq \langle \psi, (N_1 + N_2 + \mathcal{N}_1 + \mathcal{N}_2 + 1) \psi \rangle.
\end{aligned}$$

$$\Rightarrow |\text{ad}_{X_{t,s}}^{2n}(\mathcal{N}_1 + \mathcal{N}_2)| \leq (N_1 + N_2 + \mathcal{N}_1 + \mathcal{N}_2 + 1) \leq C(N_1 + N_2 + \mathcal{N}_1 + \mathcal{N}_2).$$

Similarly, one can show that $|\text{ad}_{X_{t,s}}^{2n-1}(\mathcal{N}_1 + \mathcal{N}_2)| \leq C(N_1 + N_2 + \mathcal{N}_1 + \mathcal{N}_2)$ for all $n \in \mathbb{N}$. Hence,

$$|\text{ad}_{X_{t,s}}^m(\mathcal{N}_1 + \mathcal{N}_2)| \leq C(N_1 + N_2 + \mathcal{N}_1 + \mathcal{N}_2) \quad \text{for all } m \in \mathbb{N}. \quad (3.34)$$

From (3.32) and (3.34), it follows that for every $j \in \mathbb{N}$, there exists constants $A(j)$ and $B(j)$ such that

$$\begin{aligned}
X_{t,s}^{j-1} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^2 X_{t,s}^{j-1} & \leq A(j) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j}, \text{ and} \\
X_{t,s}^{2j} & \leq B(j) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j}.
\end{aligned} \quad (3.35)$$

Now, we prove (3.35) by induction. Note if $j = 1$ (3.35) reduces to (3.32). Now suppose (3.35) holds for $j \leq k-1$. Then for the case $j = k$ first we show the inequality

$$\begin{aligned}
X_{t,s}^{k-1} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^2 X_{t,s}^{k-1} \\
\leq 2(\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) X_{t,s}^{2(k-1)} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) + 2 \|[X_{t,s}^{k-1}, \mathcal{N}_1 + \mathcal{N}_2]\|^2.
\end{aligned} \quad (3.36)$$

Note that (3.36) follows from the fact for any two arbitrary operators a and x , we have

$$|ax|^2 \leq 2|xa|^2 + 2|[x, a]|^2 \quad (3.37)$$

and from the fact that the operators $X_{t,s}$ and $\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2$ are self-adjoint and that $X_{t,s}^k$ commutes with $N_1 + N_2$. To show (3.37), note that

$$\begin{aligned}
|ax|^2 & = ax(ax)^* \\
& = (xa + [a, x])(xa + [a, x])^* \\
& = |xa|^2 + xa[a, x]^* + [a, x](xa)^* + |[a, x]|^2 \\
& \leq 2|xa|^2 + 2|[a, x]|^2,
\end{aligned}$$

where the last step follows from Proposition (A.2.4)(i). Next, we will show the inequality

$$2 \|[X_{t,s}^{k-1}, \mathcal{N}_1 + \mathcal{N}_2]\|^2 \leq C(k) \sum_{m=0}^{k-2} \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) |X_{t,s}^m|^2 \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2). \quad (3.38)$$

First, we prove the following commutator expansion formula.

Lemma 3.4.8. *Let A and B be operators. For all $n \geq 1$*

$$[A^n, B] = \sum_{m=0}^{n-1} \binom{n}{m} \text{ad}_A^{n-m}(B) A^m. \quad (3.39)$$

Proof. We give a proof by induction. Clearly (3.39) holds when $n = 1$. Now suppose (3.39) holds for all $k \leq n$ for some $n \in \mathbb{N}$. To prove the $k = n + 1$ case, note

$$\begin{aligned}
[A^{n+1}, B] &= A[A^n, B] + [A, B]A^n \\
&= A \sum_{m=0}^{n-1} \binom{n}{m} \text{ad}_A^{n-m}(B)A^m + [A, B]A^n \\
&= \sum_{m=0}^{n-1} \binom{n}{m} (\text{ad}_A^{n-m+1}(B)A^m + \text{ad}_A^{n-m}(B)A^{m+1}) + [A, B]A^n \\
&= \sum_{m=0}^n \binom{n}{m} \text{ad}_A^{n-m+1}(B)A^m + \sum_{m=0}^{n-1} \binom{n}{m} \text{ad}_A^{n-m}(B)A^{m+1} \\
&= \sum_{m=1}^n \left(\binom{n}{m} + \binom{n}{m-1} \right) \text{ad}_A^{n-m+1}(B)A^m + \text{ad}_A^{n+1}(B) \\
&= \sum_{m=0}^n \binom{n+1}{m} \text{ad}_A^{n-m+1}(B)A^m
\end{aligned}$$

where in the last line we used the fact that $\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}$. □

Now we prove (3.38). Note

$$\begin{aligned}
2|[X_{t,s}^{k-1}, \mathcal{N}_1 + \mathcal{N}_2]|^2 &= \left| \sum_{m=0}^{k-2} \binom{k-1}{m} \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) X_{t,s}^m \right|^2 \\
&\leq C(k) \left| \sum_{m=0}^{k-2} \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) X_{t,s}^m \right|^2 \\
&\leq C(k)(k-1) \sum_{m=0}^{k-2} \left| \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) X_{t,s}^m \right|^2 \\
&= C(k) \sum_{m=0}^{k-2} \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) |X_{t,s}^m|^2 \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2)
\end{aligned}$$

where we have used the commutator expansion (3.39) in the first line, and Proposition (A.2.4)(iii) in the third line. Thus we have shown (3.38). Now, using the inequality (3.34) and the induction hypothesis, we get that

$$\begin{aligned}
2|[X_{t,s}^{k-1}, \mathcal{N}_1 + \mathcal{N}_2]|^2 &\leq C(k) \sum_{m=0}^{k-2} \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2m} \text{ad}_{X_{t,s}}^{k-1-m}(\mathcal{N}_1 + \mathcal{N}_2) \\
&\leq C(k) \sum_{m=0}^{k-2} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2(m+1)} \\
&\leq C(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2(k-1)}. \tag{3.40}
\end{aligned}$$

From (3.36) and (3.40), it follows that

$$\begin{aligned}
&X_{t,s}^{k-1} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^2 X_{t,s}^{k-1} \\
&\leq 2(\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) X_{t,s}^{2(k-1)} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) + C(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2(k-1)}.
\end{aligned}$$

Now, using the induction hypothesis once more, we get

$$\begin{aligned}
& X_{t,s}^{k-1} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^2 X_{t,s}^{k-1} \\
& \leq 2C(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2(k-1)} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2) + \\
& \quad + C(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2(k-1)} \\
& = 2C(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2k} + C(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2(k-1)} \\
& \leq A(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2k}
\end{aligned} \tag{3.41}$$

which proves the first inequality in (3.35). To show the second inequality, note by (3.32) and (3.41), we have

$$\begin{aligned}
X_{t,s}^{2k} &= X_{t,s}^{k-1} X_{t,s}^2 X_{t,s}^{k-1} \\
&\leq C X_{t,s}^{k-1} (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^2 X_{t,s}^{k-1} \\
&\leq B(k) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2k}.
\end{aligned}$$

Thus we have shown (3.35). Now, by (3.31) and (3.35) we have that

$$\begin{aligned}
& \left\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j} \mathcal{U}_{N_1, N_2}^*(t; s) \psi \right\rangle \\
&= \left\langle \psi, W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) X_{t,s}^{2j} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \right\rangle \\
&\leq C(j) \left\langle \psi, W_1^*(\sqrt{N_1} \varphi_{1,s}) W_2^*(\sqrt{N_2} \varphi_{2,s}) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j} W_1(\sqrt{N_1} \varphi_{1,s}) W_2(\sqrt{N_2} \varphi_{2,s}) \psi \right\rangle \\
&= C(j) \left\langle \psi, \left(\mathcal{N}_1 + \mathcal{N}_2 + \sqrt{N_1} \Phi_1(\varphi_{1,s}) + \sqrt{N_2} \Phi_2(\varphi_{2,s}) + 2N_1 + 2N_2 \right)^{2j} \psi \right\rangle.
\end{aligned} \tag{3.42}$$

Now, analogously to (3.35), one can prove that for every $j \in \mathbb{N}$, there exists a constant $C(j)$ such that

$$\left(\mathcal{N}_1 + \mathcal{N}_2 + \sqrt{N_1} \Phi_1(\varphi_{1,s}) + \sqrt{N_2} \Phi_2(\varphi_{2,s}) + 2N_1 + 2N_2 \right)^{2j} \leq C(j) (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j}. \tag{3.43}$$

Now, from (3.42) and (3.43), we get (3.29), as desired.

Finally, we prove (3.30). First, note

$$\begin{aligned}
0 &\leq |\mathcal{N}_1 + \mathcal{N}_2 - (N_1 + N_2)|^2 \\
&= (\mathcal{N}_1 + \mathcal{N}_2)^2 + (N_1 + N_2)^2 - 2(\mathcal{N}_1 + \mathcal{N}_2)(N_1 + N_2) \\
\Rightarrow (\mathcal{N}_1 + \mathcal{N}_2) &\leq \frac{1}{2(N_1 + N_2)} (\mathcal{N}_1 + \mathcal{N}_2)^2 + \frac{N_1 + N_2}{2}.
\end{aligned}$$

Multiplying both sides by $(\mathcal{N}_1 + \mathcal{N}_2)^{2j}$ gives

$$(\mathcal{N}_1 + \mathcal{N}_2)^{2j+1} \leq \frac{1}{2(N_1 + N_2)} (\mathcal{N}_1 + \mathcal{N}_2)^{2j+2} + \frac{N_1 + N_2}{2} (\mathcal{N}_1 + \mathcal{N}_2)^{2j}. \tag{3.44}$$

From (3.44) and (3.29), we have

$$\begin{aligned}
& \left\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j+1} \mathcal{U}_{N_1, N_2}^*(t; s) \psi \right\rangle \\
& \leq \frac{1}{2(N_1 + N_2)} \left\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j+2} \mathcal{U}_{N_1, N_2}^*(t; s) \psi \right\rangle +
\end{aligned}$$

$$\begin{aligned}
& + \frac{N_1 + N_2}{2} \left\langle \psi, \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j} \mathcal{U}_{N_1, N_2}^*(t; s) \psi \right\rangle \\
& \leq \frac{C(j+1)}{2(N_1 + N_2)} \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j+2} \psi \right\rangle + \frac{C(j)(N_1 + N_2)}{2} \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j} \psi \right\rangle \\
& \leq C_2(j) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j+1} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \psi \right\rangle,
\end{aligned}$$

which proves (3.30). \square

3.4.3 Lemma (3.4.9)

Lemma 3.4.9. *For every $j \in \mathbb{N}$, there exist constants $A(j)$, $B(j)$, and $C(j)$ such that*

$$\begin{aligned}
& \left| \langle \mathcal{U}_{N_1, N_2}(t; s) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j (\mathcal{U}_{N_1, N_2}(t; s) - \mathcal{U}_{N_1, N_2}^{(M)}(t; s)) \psi \rangle \right| \\
& \leq A(j) \left(\sqrt{\frac{N_1 + N_2}{N_1}} + \sqrt{\frac{N_1 + N_2}{N_2}} + 1 \right) \left(\frac{N_1 + N_2}{2M} \right)^j \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \cdot \\
& \quad \cdot \frac{1}{1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1 + N_2}}} e^{C(j)|t-s|(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1 + N_2}})}.
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
& \left| \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; s) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j (\mathcal{U}_{N_1, N_2}(t; s) - \mathcal{U}_{N_1, N_2}^{(M)}(t; s)) \psi \rangle \right| \\
& \leq B(j) \left(\frac{1}{M} \right)^j \left(\sqrt{\frac{N_1 + N_2}{N_1}} + \sqrt{\frac{N_1 + N_2}{N_2}} + 1 \right) \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \cdot \\
& \quad \cdot \frac{1}{1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1 + N_2}}} e^{C(j)|t-s|(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1 + N_2}})}
\end{aligned} \tag{3.46}$$

for all $\psi \in D(\mathcal{H}_{N_1, N_2})$ and for all $t, s \in \mathbb{R}$.

Proof. We consider the case $s = 0$ and $t > 0$ to simplify the proof, but all other cases can be treated similarly.

To start the proof of (3.45), we show

$$\begin{aligned}
& \langle \mathcal{U}_{N_1, N_2}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j (\mathcal{U}_{N_1, N_2}(t; 0) - \mathcal{U}_{N_1, N_2}^{(M)}(t; 0)) \psi \rangle = \\
& = -i \int_0^t ds \langle \mathcal{U}_{N_1, N_2}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) (\mathcal{L}_{N_1, N_2}(s) - \mathcal{L}_{N_1, N_2}^{(M)}(s)) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi \rangle.
\end{aligned} \tag{3.47}$$

To prove (3.47), note

$$\begin{aligned}
& \mathcal{U}_{N_1, N_2}(t; s) \left(\mathcal{L}_{N_1, N_2}(s) - \mathcal{L}_{N_1, N_2}^{(M)}(s) \right) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) = \\
& = \mathcal{U}_{N_1, N_2}^*(s; t) \mathcal{L}_{N_1, N_2}(s) \mathcal{U}_{N_1, N_2}(s; t) \mathcal{U}_{N_1, N_2}^*(s; t) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) - i \mathcal{U}_{N_1, N_2}(t; s) \frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \\
& = i \mathcal{U}_{N_1, N_2}^*(s; t) \left(\frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}(s; t) \right) \mathcal{U}_{N_1, N_2}^*(s; t) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) - i \mathcal{U}_{N_1, N_2}(t; s) \frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0),
\end{aligned}$$

where we have used the fact that $i \partial_t \mathcal{U}_{N_1, N_2}(t; s) = \mathcal{L}_{N_1, N_2}(t) \mathcal{U}_{N_1, N_2}(t; s)$ and

$i\partial_t \mathcal{U}_{N_1, N_2}^{(M)}(t; s) = \mathcal{L}_{N_1, N_2}^{(M)}(t) \mathcal{U}_{N_1, N_2}^{(M)}(t; s)$. Now, note

$$\begin{aligned} 0 &= \frac{\partial}{\partial s} (\mathcal{U}_{N_1, N_2}^*(s; t) \mathcal{U}_{N_1, N_2}(s; t)) = \left(\frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}^*(s; t) \right) \mathcal{U}_{N_1, N_2}(s; t) + \mathcal{U}_{N_1, N_2}^*(s; t) \left(\frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}(s; t) \right) \\ \Rightarrow \mathcal{U}_{N_1, N_2}^*(s; t) \left(\frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}(s; t) \right) &= - \left(\frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}^*(s; t) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{U}_{N_1, N_2}(t; s) \left(\mathcal{L}_{N_1, N_2}(s) - \mathcal{L}_{N_1, N_2}^{(M)}(s) \right) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) &= \\ &= -i \left(\frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}^*(s; t) \right) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) - i \mathcal{U}_{N_1, N_2}(t; s) \frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \\ &= -i \frac{\partial}{\partial s} \left(\mathcal{U}_{N_1, N_2}(t; s) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \right). \end{aligned}$$

Thus,

$$\begin{aligned} -i \int_0^t ds \frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}(t; s) \left(\mathcal{L}_{N_1, N_2}(s) - \mathcal{L}_{N_1, N_2}^{(M)}(s) \right) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \\ = - \int_0^t \frac{\partial}{\partial s} \left(\mathcal{U}_{N_1, N_2}(t; s) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \right) ds = \mathcal{U}_{N_1, N_2}(t; 0) - \mathcal{U}_{N_1, N_2}^{(M)}(t; 0), \end{aligned} \quad (3.48)$$

which proves (3.47). Now, note

$$\begin{aligned} \mathcal{L}_{N_1, N_2}(s) - \mathcal{L}_{N_1, N_2}^{(M)}(s) &= \\ &= \frac{1}{\sqrt{N_1}} \int dx dy V_1(x-y) a_{1,x}^* \left(\bar{\varphi}_{1,t}(y) a_{1,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \right. \\ &\quad \left. + \varphi_{1,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y}^* \right) a_{1,x} + \\ &+ \frac{1}{\sqrt{N_2}} \int dx dy V_2(x-y) a_{2,x}^* \left(\bar{\varphi}_{2,t}(y) a_{2,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \right. \\ &\quad \left. + \varphi_{2,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{2,y}^* \right) a_{2,x} + \\ &+ \frac{1}{N_1 + N_2} \int dx dy \left(A(t) - A^{(M)}(t) \right), \end{aligned}$$

where

$$\begin{aligned} A(t) - A^{(M)}(t) &= Q(x-y) \left(\sqrt{N_2} \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y} + \right. \\ &\quad + \sqrt{N_1} \varphi_{1,t}(y) a_{2,x}^* a_{1,y}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{2,x} + \\ &\quad + \sqrt{N_1} \bar{\varphi}_{1,t}(y) a_{2,x}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y} a_{2,x} + \\ &\quad \left. + \sqrt{N_2} \bar{\varphi}_{2,t}(x) a_{1,y}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y} a_{2,x} \right). \end{aligned}$$

Thus,

$$\begin{aligned}
& \langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \left(\mathcal{U}_{N_1, N_2}(t; 0) - \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \right) \psi \rangle = \\
& = -\frac{i}{\sqrt{N_1}} \int_0^t ds \int dx dy V_1(x-y) \langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{1,x}^* \cdot \\
& \quad \cdot \left(\bar{\varphi}_{1,t}(y) a_{1,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \varphi_{1,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y}^* \right) \cdot \\
& \quad \cdot a_{1,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle + \tag{3.49}
\end{aligned}$$

$$\begin{aligned}
& -\frac{i}{\sqrt{N_2}} \int_0^t ds \int dx dy V_2(x-y) \langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{2,x}^* \cdot \\
& \quad \cdot \left(\bar{\varphi}_{2,t}(y) a_{2,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \varphi_{2,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{2,y}^* \right) \cdot \\
& \quad \cdot a_{2,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle + \tag{3.50}
\end{aligned}$$

$$\begin{aligned}
& -i \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \int dx dy Q(x-y) \langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{1,y}^* \cdot \\
& \quad \cdot \left(\varphi_{2,t}(x) a_{2,x}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \bar{\varphi}_{2,t}(x) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{2,x} \right) \cdot \\
& \quad \cdot a_{1,y} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle + \tag{3.51}
\end{aligned}$$

$$\begin{aligned}
& -i \frac{\sqrt{N_1}}{N_1 + N_2} \int_0^t ds \int dx dy Q(x-y) \langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{2,x}^* \cdot \\
& \quad \cdot \left(\varphi_{1,t}(y) a_{1,y}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \bar{\varphi}_{1,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y} \right) \cdot \\
& \quad \cdot a_{2,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle. \tag{3.52}
\end{aligned}$$

Let us look at the term (3.49); (3.50) will follow almost identically. Note

$$\begin{aligned}
& \frac{-i}{\sqrt{N_1}} \int_0^t ds \int dx dy V_1(x-y) \langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{1,x}^* \cdot \\
& \quad \cdot \left(\bar{\varphi}_{k,t}(y) a_{1,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \varphi_{1,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y}^* \right) \cdot \\
& \quad \cdot a_{1,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle \\
& = \frac{-i}{\sqrt{N_1}} \int_0^t ds \int dx \langle a_{1,x} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi, \\
& \quad a_k(V_1(x - \cdot) \varphi_{1,t}) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle + \\
& \quad - \frac{i}{N_k} \int_0^t ds \int dx \langle a_{1,x} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi, \\
& \quad \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_k^*(V_1(x - \cdot) \varphi_{1,t}) a_{1,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \rangle.
\end{aligned}$$

Now, using the fact that $\chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,x} \psi = a_{1,x} \chi(\mathcal{N}_1 > M + 1) \chi(\mathcal{N}_2 > M)$ and

$\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)a_{1,x}^*\psi = a_{1,x}^*\chi(\mathcal{N}_1 > M - 1)\chi(\mathcal{N}_2 > M)$, we obtain

$$\begin{aligned}
& \left| \frac{-i}{\sqrt{N_1}} \int_0^t ds \int dx dy V_1(x-y) \left\langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s)a_{1,x}^* \right. \right. \\
& \quad \cdot \left. \left(\bar{\varphi}_{k,t}(y)a_{1,y}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M) + \varphi_{1,t}(y)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)a_{1,y}^* \right) \right. \\
& \quad \left. \cdot a_{1,x}\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \right\rangle \Big| \\
& \leq \frac{1}{\sqrt{N_1}} \int_0^t ds \int dx \|a_{1,x}\mathcal{U}_{N_1, N_2}^*(t; s)(\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi\| \\
& \quad \cdot \|a_1(V_1(x - \cdot)\varphi_{1,t})a_{1,x}\chi(\mathcal{N}_1 > M + 1)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\| + \\
& \quad + \frac{1}{\sqrt{N_1}} \int_0^t ds \int dx \|a_{1,x}\mathcal{U}_{N_1, N_2}^*(t; s)(\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi\| \\
& \quad \cdot \|a_1^*(V_1(x - \cdot)\varphi_{1,t})a_{1,x}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\|.
\end{aligned}$$

Next we use Proposition (1.2.20) to obtain

$$\begin{aligned}
& \left| \frac{-i}{\sqrt{N_1}} \int_0^t ds \int dx dy V_1(x-y) \left\langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s)a_{1,x}^* \right. \right. \\
& \quad \cdot \left. \left(\bar{\varphi}_{k,t}(y)a_{1,y}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M) + \varphi_{1,t}(y)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)a_{1,y}^* \right) \right. \\
& \quad \left. \cdot a_{1,x}\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \right\rangle \Big| \\
& \leq \frac{1}{\sqrt{N_1}} \sup_x \|V_1(x - \cdot)\varphi_{1,t}\|_{L^2} \int_0^t ds \int dx \|a_{1,x}\mathcal{U}_{N_1, N_2}^*(t; s)(\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi\| \\
& \quad \cdot \|\mathcal{N}_1^{1/2}a_{1,x}\chi(\mathcal{N}_1 > M + 1)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\| + \\
& \quad + \frac{1}{\sqrt{N_1}} \sup_x \|V_1(x - \cdot)\varphi_{1,t}\|_{L^2} \int_0^t ds \int dx \|a_{1,x}\mathcal{U}_{N_1, N_2}^*(t; s)(\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi\| \\
& \quad \cdot \|(\mathcal{N}_1 + 1)^{1/2}a_{1,x}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\|.
\end{aligned}$$

Using the fact that $\mathcal{N}_1^{1/2}a_{1,x} = a_{1,x}(\mathcal{N}_1 - 1)^{1/2} \leq a_{1,x}\mathcal{N}_1^{1/2}$ and $\sup_x \|V_1(x - \cdot)\varphi_{1,t}\|_{L^2} = C$, we get

$$\begin{aligned}
& \left| \frac{-i}{\sqrt{N_1}} \int_0^t ds \int dx dy V_1(x-y) \left\langle \mathcal{U}_{N_1, N_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s)a_{1,x}^* \right. \right. \\
& \quad \cdot \left. \left(\bar{\varphi}_{k,t}(y)a_{1,y}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M) + \varphi_{1,t}(y)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)a_{1,y}^* \right) \right. \\
& \quad \left. \cdot a_{1,x}\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \right\rangle \Big| \\
& \leq \frac{C}{\sqrt{N_1}} \int_0^t ds \int dx \|a_{1,x}\mathcal{U}_{N_1, N_2}^*(t; s)(\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi\| \\
& \quad \cdot \|a_{1,x}\mathcal{N}_1^{1/2}\chi(\mathcal{N}_1 > M + 1)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\| + \\
& \quad + \frac{C}{\sqrt{N_1}} \int_0^t ds \int dx \|a_{1,x}\mathcal{U}_{N_1, N_2}^*(t; s)(\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0)\psi\| \\
& \quad \cdot \|a_{1,x}\mathcal{N}_1^{1/2}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\|.
\end{aligned}$$

Finally, we use the fact that $\int dx \|a_x \psi_1\| \|a_x \psi_2\| \leq \int dx \|a_x \psi_1\| \int dx \|a_x \psi_2\|$ as well as $\int dx \|a_{1,x} \psi\| = \|\mathcal{N}_1^{1/2} \psi\|$ and $\chi(\mathcal{N}_1 > M+1) < \chi(\mathcal{N}_1 > M)$ to obtain

$$\begin{aligned}
& \left| \frac{-i}{\sqrt{N_1}} \int_0^t ds \int dx dy V_1(x-y) \left\langle \mathcal{U}_{N_1, N_2}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{1,x}^* \right\rangle \right. \\
& \quad \cdot \left(\bar{\varphi}_{1,t}(y) a_{1,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \varphi_{1,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{1,y}^* \right) \cdot \\
& \quad \left. \cdot a_{1,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi \right| \\
& \leq \frac{C}{\sqrt{N_1}} \int_0^t ds \|\mathcal{N}_1^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi\| \|\mathcal{N}_1 \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\| \\
& \leq \frac{C}{\sqrt{N_1}} \int_0^t ds \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi\| \cdot \\
& \quad \cdot \|(\mathcal{N}_1 + \mathcal{N}_2) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\|.
\end{aligned}$$

We proceed the same way for the term (3.50) and we get that

$$\begin{aligned}
& \left| \frac{-i}{\sqrt{N_2}} \int_0^t ds \int dx dy V_2(x-y) \left\langle \mathcal{U}_{N_1, N_2}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) a_{2,x}^* \right\rangle \right. \\
& \quad \cdot \left(\bar{\varphi}_{2,t}(y) a_{2,y} \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \varphi_{2,t}(y) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{2,y}^* \right) \cdot \\
& \quad \left. \cdot a_{2,x} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi \right| \\
& \leq \frac{C}{\sqrt{N_2}} \int_0^t ds \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi\| \cdot \\
& \quad \cdot \|(\mathcal{N}_1 + \mathcal{N}_2) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\|. \tag{3.53}
\end{aligned}$$

Now we look at the term (3.51). Note

$$\begin{aligned}
& \left| -i \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \int dx dy Q(x-y) \left\langle \mathcal{U}_{N_1, N_2}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s) \right\rangle \right. \\
& \quad \cdot a_{1,y}^* \left(\varphi_{2,t}(x) a_{2,x}^* \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + \bar{\varphi}_{2,t}(x) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) a_{2,x} \right) a_{1,y} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi \left. \right| \\
& = \left| -i \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \int dy \left\langle a_{1,y} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi, \left(a_2^*(Q(\cdot - y) \varphi_{2,t}) \right. \right. \right. \\
& \quad \left. \left. \cdot \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) + a_2(Q(\cdot - y) \varphi_{2,t}) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M+1) \right) a_{1,y} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi \right. \left. \right| \\
& \leq \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \int dy \|a_{1,y} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi\| \cdot \\
& \quad \cdot \|Q(\cdot - y) \varphi_{2,t}\|_{L^2} \|a_{1,y} (\mathcal{N}_2 + 1)^{1/2} \chi(\mathcal{N}_1 > M+1) \chi(\mathcal{N}_2 > M) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\| + \\
& \quad + \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \int dy \|a_{1,y} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi\| \cdot \\
& \quad \cdot \|Q(\cdot - y) \varphi_{2,t}\|_{L^2} \|a_{1,y} \mathcal{N}_2^{1/2} \chi(\mathcal{N}_1 > M+1) \chi(\mathcal{N}_2 > M+1) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\| \\
& \leq C \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \|\mathcal{N}_1^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; 0) \psi\|.
\end{aligned}$$

$$\begin{aligned}
& \cdot \|\mathcal{N}_1^{1/2}(\mathcal{N}_2 + 1)^{1/2}\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(s; 0)\psi\| \\
& \leq C \frac{\sqrt{\mathcal{N}_2}}{\mathcal{N}_1 + \mathcal{N}_2} \int_0^t ds \|\mathcal{N}_1 + \mathcal{N}_2\|^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\| \cdot \\
& \quad \cdot \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(s; 0)\psi\|.
\end{aligned}$$

It follows that for the term (3.52), we have

$$\begin{aligned}
& \left| -i \frac{\sqrt{\mathcal{N}_1}}{\mathcal{N}_1 + \mathcal{N}_2} \int_0^t ds \int dx dy Q(x - y) \langle \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; s) \cdot \right. \\
& \quad \cdot a_{2,x}^* \left(\varphi_{1,t}(y) a_{1,y}^* \chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M) + \bar{\varphi}_{1,t}(y) \chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M) a_{1,y} \right) a_{2,x} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(s; 0)\psi \rangle \left. \right| \\
& \leq C \frac{\sqrt{\mathcal{N}_1}}{\mathcal{N}_1 + \mathcal{N}_2} \int_0^t ds \|\mathcal{N}_1 + \mathcal{N}_2\|^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\| \cdot \\
& \quad \cdot \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(s; 0)\psi\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left| \langle \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \left(\mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0) - \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(t; 0) \right) \psi \rangle \right| \\
& \leq C \left(\frac{1}{\sqrt{\mathcal{N}_1}} + \frac{1}{\sqrt{\mathcal{N}_2}} + \frac{1}{\sqrt{\mathcal{N}_1 + \mathcal{N}_2}} \right) \int_0^t ds \|\mathcal{N}_1 + \mathcal{N}_2\|^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\| \cdot \quad (3.54) \\
& \quad \cdot \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^{(M)}(s; 0)\psi\|.
\end{aligned}$$

Now we look at the term $\|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\|$. Note

$$\begin{aligned}
& \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\|^2 = \\
& = \left\langle (\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi, \right. \\
& \quad \left. (\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi \right\rangle \\
& = \left\langle (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi, \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; s) (\mathcal{N}_1 + \mathcal{N}_2) \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi \right\rangle \\
& \leq 6 \left\langle (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_1 + \mathcal{N}_2 + 1) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi \right\rangle
\end{aligned}$$

where we have used equation (3.28) from Lemma (3.4.7) in the last line. Thus,

$$\begin{aligned}
& \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\|^2 \\
& \leq 6 \left\langle \psi, \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j (\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_1 + \mathcal{N}_2 + 1) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2} \psi \right\rangle \\
& = 6 \left\langle \psi, \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j+1} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2} \psi \right\rangle + 6(\mathcal{N}_1 + \mathcal{N}_2 + 1) \left\langle \psi, \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^{2j} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2} \psi \right\rangle.
\end{aligned}$$

Now we use equations (3.30) and (3.29) in Lemma (3.4.7) to get

$$\begin{aligned}
& \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{\mathcal{N}_1, \mathcal{N}_2}(t; 0)\psi\|^2 \\
& \leq 6C_2(j) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_1 + \mathcal{N}_2)^{2j+1} (\mathcal{N}_1 + \mathcal{N}_2 + 1)\psi \right\rangle + \\
& \quad + 6C_1(j)(\mathcal{N}_1 + \mathcal{N}_2 + 1) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_1 + \mathcal{N}_2)^{2j} \psi \right\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq 6C_2(j) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j+1} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \psi \right\rangle + 6C_1(j) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j+1} \psi \right\rangle + \\
&\quad + 6C_1(j) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j} \psi \right\rangle \\
&\leq C(j) \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + N_1 + N_2)^{2j+1} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \psi \right\rangle \\
&\leq C(j) (N_1 + N_2)^{2j+1} \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{2j+2} \psi \right\rangle \\
&= C(j) (N_1 + N_2)^{2j+1} \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \tag{3.55}
\end{aligned}$$

for an appropriate constant $C(j)$.

Now we look at the term $\|(\mathcal{N}_1 + \mathcal{N}_2 + 1)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\|$. First, we observe that

$$\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M) \leq \left(\frac{\mathcal{N}_1 + \mathcal{N}_2}{2M}\right)^{2j} \quad \text{for all } j \in \mathbb{N}. \tag{3.56}$$

To see this, note that for $\psi \in \mathcal{F}$, we have $\chi(\mathcal{N} > M) \leq \frac{\mathcal{N}}{M}$, since $\mathcal{N}\psi \geq \chi(\mathcal{N} > M)\mathcal{N}\psi \geq M\chi(\mathcal{N} > M)\psi$.

Thus, for $\psi = \psi_1 \otimes \psi_2 \in \mathcal{F} \otimes \mathcal{F}$, we have

$$\begin{aligned}
&2M\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\psi \\
&= (M + M)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\psi \\
&\leq M\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\psi + M\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\psi \\
&\leq \|(M\chi(\mathcal{N} > M)\psi_1) \otimes (\chi(\mathcal{N} > M)\psi_2) + (\chi(\mathcal{N} > M)\psi_1) \otimes (M\chi(\mathcal{N} > M)\psi_2)\| \\
&\leq (\mathcal{N}\psi_1) \otimes \psi_2 + \psi_1 \otimes (\mathcal{N}\psi_2) \\
&= (\mathcal{N}_1 + \mathcal{N}_2)\psi.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|(\mathcal{N}_1 + \mathcal{N}_2 + 1)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\| \\
&= \left\langle \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^2 \chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \right\rangle^{1/2} \\
&\leq \left\langle \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^2 \left(\frac{\mathcal{N}_1 + \mathcal{N}_2}{2M}\right)^{2j} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \right\rangle^{1/2} \\
&\leq \left(\frac{1}{2M}\right)^j \left\langle \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{2j+2} \mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi \right\rangle^{1/2}.
\end{aligned}$$

Recall from Lemma (3.4.5) that

$$\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi \rangle \leq \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^j \psi \rangle e^{4^j C|t-s|(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})}.$$

Thus, from Lemma (3.4.5) we obtain

$$\begin{aligned}
&\|(\mathcal{N}_1 + \mathcal{N}_2 + 1)\chi(\mathcal{N}_1 > M)\chi(\mathcal{N}_2 > M)\mathcal{U}_{N_1, N_2}^{(M)}(s; 0)\psi\| \\
&\leq \left(\frac{1}{2M}\right)^j \left\langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{2j+2} \psi \right\rangle^{1/2} e^{K(j)s(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})}
\end{aligned}$$

$$= \left(\frac{1}{2M} \right)^j \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\| e^{C(j)s(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})}. \quad (3.57)$$

From equations (3.54), (3.55) and (3.57) we get

$$\begin{aligned} & \left| \langle \mathcal{U}_{N_1, N_2}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \left(\mathcal{U}_{N_1, N_2}(t; 0) - \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \right) \psi \rangle \right| \\ & \leq C \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) \left(C(j) (N_1 + N_2)^{2j+1} \right)^{1/2} \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\| \cdot \\ & \quad \cdot \int_0^t ds \left(\frac{1}{2M} \right)^j \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\| e^{K(j)s(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})} \\ & = C(j) \left(\frac{1}{2M} \right)^j \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) \left((N_1 + N_2)^{2j+1} \right)^{1/2} \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \cdot \\ & \quad \cdot \int_0^t ds e^{K(j)s(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})} \\ & \leq C(j) \left(\frac{1}{2M} \right)^j \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) \left((N_1 + N_2)^{2j+1} \right)^{1/2} \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \cdot \\ & \quad \cdot \frac{1}{1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}} e^{K(j)t(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})}. \end{aligned}$$

Finally, note that

$$\begin{aligned} & \left(\frac{1}{2M} \right)^j \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) \left((N_1 + N_2)^{2j+1} \right)^{1/2} \\ & = \left(\frac{1}{2M} \right)^j \frac{\sqrt{N_2(N_1 + N_2)} + \sqrt{N_1(N_1 + N_2)} + \sqrt{N_1 N_2}}{\sqrt{N_1 N_2 (N_1 + N_2)}} \sqrt{(N_1 + N_2)^{2j+1}} \\ & = \left(\sqrt{\frac{N_1 + N_2}{N_1}} + \sqrt{\frac{N_1 + N_2}{N_2}} + 1 \right) \left(\frac{N_1 + N_2}{2M} \right)^j \end{aligned}$$

which proves (3.45). To prove (3.46), we proceed exactly as before and get

$$\begin{aligned} & \left| \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \left(\mathcal{U}_{N_1, N_2}(t; 0) - \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \right) \psi \rangle \right| \\ & \leq C \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) \int_0^t ds \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi\| \cdot \quad (3.58) \\ & \quad \cdot \|(\mathcal{N}_1 + \mathcal{N}_2 + 1) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\|. \end{aligned}$$

Now, also as before, we use (3.28) to obtain

$$\begin{aligned} & \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi\|^2 \\ & \leq 6 \left\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^{2j+1} \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \right\rangle + \\ & \quad + 6(N_1 + N_2 + 1) \left\langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^{2j} \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi \right\rangle. \end{aligned}$$

Now, however, we need to use Lemma (3.4.5) to continue. We get

$$\begin{aligned} & \|(\mathcal{N}_1 + \mathcal{N}_2)^{1/2} \mathcal{U}_{N_1, N_2}^*(t; s) (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi\|^2 \\ & \leq 6 \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{2j+1} \psi \rangle e^{K_1(j)t(1+\sqrt{\frac{M}{N_1}}+\sqrt{\frac{M}{N_2}}+\sqrt{\frac{M+1}{N_1+N_2}})} + \end{aligned}$$

$$\begin{aligned}
& + 6(N_1 + N_2 + 1) \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{2j} \psi \rangle e^{K_2(j)t \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}\right)} \\
& \leq C(N_1 + N_2 + 1) \langle \psi, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{2j+2} \psi \rangle e^{C(j)t \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}\right)}.
\end{aligned}$$

From before, we also have

$$\begin{aligned}
& \|(\mathcal{N}_1 + \mathcal{N}_2 + 1) \chi(\mathcal{N}_1 > M) \chi(\mathcal{N}_2 > M) \mathcal{U}_{N_1, N_2}^{(M)}(s; 0) \psi\| \\
& \leq \left(\frac{1}{2M}\right)^j \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\| e^{C(j)s \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left| \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \left(\mathcal{U}_{N_1, N_2}(t; 0) - \mathcal{U}_{N_1, N_2}^{(M)}(t; 0) \right) \psi \rangle \right| \\
& \leq C \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) \sqrt{N_1 + N_2 + 1} \left(\frac{1}{2M} \right)^j \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \\
& \quad \cdot \frac{1}{1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}} e^{C(j)t \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}\right)} \\
& \leq B(j) \left(\frac{1}{M} \right)^j \left(\sqrt{\frac{N_1 + N_2}{N_1}} + \sqrt{\frac{N_1 + N_2}{N_2}} + 1 \right) \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1} \psi\|^2 \\
& \quad \cdot \frac{1}{1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}} e^{C(j)t \left(1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}\right)}
\end{aligned}$$

which proves (3.46). □

3.4.4 Proof of Proposition (3.4.1)

Now we can finally prove our proposition. The proof is short now that we have proved the previous lemmas.

Proof of Proposition (3.4.1). We choose $M = N_1 = aN_2$, for some constant a . Then

$$\begin{aligned}
& \frac{1}{1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}}} \\
& = \frac{1}{1 + 1 + \sqrt{a} + \sqrt{\frac{a}{a+1} + \frac{1}{(a+1)N_2}}} \\
& \leq \frac{1}{2 + \sqrt{a} + \sqrt{\frac{a}{a+1}}} = C
\end{aligned}$$

and

$$\begin{aligned}
& 1 + \sqrt{\frac{M}{N_1}} + \sqrt{\frac{M}{N_2}} + \sqrt{\frac{M+1}{N_1+N_2}} \\
& = 1 + 1 + \sqrt{a} + \sqrt{\frac{a}{a+1} + \frac{1}{(a+1)N_2}} \\
& \leq 2 + \sqrt{a} + \sqrt{\frac{a}{a+1} + \frac{1}{a+1}} = C.
\end{aligned}$$

Also,

$$\begin{aligned} \sqrt{\frac{N_1 + N_2}{N_1}} + \sqrt{\frac{N_1 + N_2}{N_2}} + 1 &= \sqrt{\frac{a+1}{a}} + \sqrt{a+1} + 1 = C, \\ \left(\frac{N_1 + N_2}{2M}\right)^j &= \left(\frac{a+1}{2a}\right)^j = C(j), \\ \left(\frac{1}{M}\right)^j &= \left(\frac{1}{N_1}\right)^j \leq 1. \end{aligned}$$

Now, from Lemma (3.4.9) and Lemma (3.4.5), we obtain

$$\begin{aligned} &\langle \mathcal{U}_{N_1, N_2}(t; s)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}(t; s)\psi \rangle \\ &= \langle \mathcal{U}_{N_1, N_2}(t; s)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j (\mathcal{U}_{N_1, N_2}(t; s) - \mathcal{U}_{N_1, N_2}^{(M)}(t; s))\psi \rangle + \\ &\quad + \langle (\mathcal{U}_{N_1, N_2}(t; s) - \mathcal{U}_{N_1, N_2}^{(M)}(t; s))\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi \rangle + \\ &\quad + \langle \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi, (\mathcal{N}_1 + \mathcal{N}_2)^j \mathcal{U}_{N_1, N_2}^{(M)}(t; s)\psi \rangle \\ &\leq A(j) \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{j+1}\|^2 e^{C(j)|t-s|}. \end{aligned}$$

□

3.5 Lemma (3.5.1)

Recall the evolution $\tilde{\mathcal{U}}_{N_1, N_2}$ which was defined through the equation

$$i \frac{\partial}{\partial t} \tilde{\mathcal{U}}_{N_1, N_2}(t; s) = \tilde{\mathcal{L}}_{N_1, N_2}(t) \tilde{\mathcal{U}}_{N_1, N_2}(t; s), \quad \text{with } \tilde{\mathcal{U}}_{N_1, N_2}(s; s) = 1$$

with the generator

$$\tilde{\mathcal{L}}_{N_1, N_2} = \tilde{L}1 + \tilde{L}2 + \frac{1}{N_1 + N_2} \int dx dy \tilde{A},$$

where

$$\begin{aligned} \tilde{L}1 &= \int dx \Delta_x a_{1,x}^* \Delta_x a_{1,x} + \int dx (V_1 * |\varphi_{1,t}|^2)(x) a_{1,x}^* a_{1,x} + \int dx dy V_1(x-y) \bar{\varphi}_{1,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{1,x} + \\ &\quad + \frac{1}{2} \int dx dy V_1(x-y) (\varphi_{1,t}(x) \varphi_{1,t}(y) a_{1,x}^* a_{1,y}^* + \bar{\varphi}_{1,t}(x) \bar{\varphi}_{1,t}(y) a_{1,x} a_{1,y}) + \\ &\quad + \frac{1}{2N_1} \int dx dy V_1(x-y) a_{1,x}^* a_{1,y}^* a_{1,y} a_{1,x}, \end{aligned}$$

$$\begin{aligned} \tilde{L}2 &= \int dx \Delta_x a_{2,x}^* \Delta_x a_{2,x} + \int dx (V_2 * |\varphi_{2,t}|^2)(x) a_{2,x}^* a_{2,x} + \int dx dy V_2(x-y) \bar{\varphi}_{2,t}(x) \varphi_{2,t}(y) a_{2,y}^* a_{2,x} + \\ &\quad + \frac{1}{2} \int dx dy V_2(x-y) (\varphi_{2,t}(x) \varphi_{2,t}(y) a_{2,x}^* a_{2,y}^* + \bar{\varphi}_{2,t}(x) \bar{\varphi}_{2,t}(y) a_{2,x} a_{2,y}) + \\ &\quad + \frac{1}{2N_2} \int dx dy V_2(x-y) a_{2,x}^* a_{2,y}^* a_{2,y} a_{2,x}, \end{aligned}$$

and

$$\begin{aligned} \tilde{A} = Q(x-y) & \left[a_{2,x}^* a_{1,y}^* a_{1,y} a_{2,x} + \right. \\ & + \sqrt{N_1 N_2} \varphi_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* + \sqrt{N_1 N_2} \bar{\varphi}_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y} + \\ & + N_1 \bar{\varphi}_{1,t}(y) \varphi_{1,t}(y) a_{2,x}^* a_{2,x} + N_2 \bar{\varphi}_{2,t}(x) \varphi_{2,t}(x) a_{1,y}^* a_{1,y} + \\ & \left. + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \varphi_{1,t}(y) a_{1,y}^* a_{2,x} + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) a_{1,y} a_{2,x} \right]. \end{aligned}$$

Now we have the following lemma.

Lemma 3.5.1. *There exists a constant $C > 0$ such that*

$$\left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \leq e^{Ct}. \quad (3.59)$$

Proof. We calculate the time derivative of $\left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle$ and then use Gronwall's Lemma. Note

$$\begin{aligned} & \frac{d}{dt} \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\ & = \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, \left[i\tilde{\mathcal{L}}_{N_1, N_2}(t), (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\ & = \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\tilde{L}1, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle + \\ & \quad + \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\tilde{L}2, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle + \\ & \quad + \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\frac{1}{N_1 + N_2} \int dx dy \tilde{A}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle. \end{aligned} \quad (3.60)$$

For the terms $\left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\tilde{L}1, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle$ and $\left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\tilde{L}2, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle$ in (3.60), we follow the paper [19] of Rodnianski and Schlein on pp. 51-52 to obtain

$$\begin{aligned} & \left| \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\tilde{L}1, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \right| \\ & \leq C \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \end{aligned}$$

and

$$\begin{aligned} & \left| \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\tilde{L}2, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \right| \\ & \leq C \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle. \end{aligned}$$

Now we look at the last term in (3.60). Note by removing the terms in \tilde{A} that commute with $\mathcal{N}_1 + \mathcal{N}_2$ we get

$$\begin{aligned} & \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\frac{1}{N_1 + N_2} \int dx dy \tilde{A}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\ & = \frac{1}{N_1 + N_2} \int dx dy Q(x-y) \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\sqrt{N_1 N_2} \varphi_{1,t}(y) \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* + \right. \right. \\ & \quad \left. \left. + \sqrt{N_1 N_2} \bar{\varphi}_{2,t}(x) \bar{\varphi}_{1,t}(y) a_{1,y} a_{2,x}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle. \end{aligned}$$

Using Proposition (3.4.6), we get

$$\begin{aligned} & \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\frac{1}{N_1 + N_2} \int dx dy \tilde{A}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\ &= -2 \operatorname{Im} \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \int dx dy Q(x - y) \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, \right. \\ & \quad \left. \varphi_{1,t}(y)\varphi_{2,t}(x) \left[a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle. \end{aligned}$$

From Proposition (1.3.12), we have that

$$\begin{aligned} [a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3] &= \sum_{k=0}^2 \binom{3}{k} (-1)^{3-k} \left((\mathcal{N}_1 + \mathcal{N}_2)^{k/2} a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{k/2} + \right. \\ & \quad \left. + (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{k/2} a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{k/2} \right) \\ &= - \left(a_{2,x}^* a_{1,y}^* + a_{2,x}^* a_{1,y}^* \right) + 3 \left((\mathcal{N}_1 + \mathcal{N}_2)^{1/2} a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2)^{1/2} + \right. \\ & \quad \left. + (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{1/2} a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 3)^{1/2} \right) + \\ & \quad - 3 \left((\mathcal{N}_1 + \mathcal{N}_2) a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2) + (\mathcal{N}_1 + \mathcal{N}_2 + 1) a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 3) \right). \end{aligned}$$

Using the push-through formula $(\mathcal{N}_1 + \mathcal{N}_2)a_{j,x}^* = a_{j,x}^*(\mathcal{N}_1 + \mathcal{N}_2 + 1)$ from Proposition (1.3.12)(i), we get that

$$\begin{aligned} [a_{2,x}^* a_{1,y}^*, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3] &= -2a_{2,x}^* a_{1,y}^* + 3a_{2,x}^* a_{1,y}^* (2(\mathcal{N}_1 + \mathcal{N}_2) + 5) + \\ & \quad + -3a_{2,x}^* a_{1,y}^* ((\mathcal{N}_1 + \mathcal{N}_2)^2 + 4(\mathcal{N}_1 + \mathcal{N}_2) + 4 + (\mathcal{N}_1 + \mathcal{N}_2)^2 + 6(\mathcal{N}_1 + \mathcal{N}_2) + 9) \\ &= -2a_{2,x}^* a_{1,y}^* - 3a_{2,x}^* a_{1,y}^* (2(\mathcal{N}_1 + \mathcal{N}_2)^2 + 8(\mathcal{N}_1 + \mathcal{N}_2) + 8) \\ &= -2(a_{2,x}^* a_{1,y}^* + 3a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 2)^2) \\ &= -2(a_{2,x}^* a_{1,y}^* + 3a_{2,x}^* a_{1,y}^* ((\mathcal{N}_1 + \mathcal{N}_2 + 1)^2 + 2(\mathcal{N}_1 + \mathcal{N}_2 + 1) + 1)) \\ &= -2(a_{2,x}^* a_{1,y}^* + 3(\mathcal{N}_1 + \mathcal{N}_2 - 1)a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1) + 6a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1) + 3a_{2,x}^* a_{1,y}^*) \\ &= -2(4a_{2,x}^* a_{1,y}^* + 3(\mathcal{N}_1 + \mathcal{N}_2 + 1)a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1)). \end{aligned}$$

Therefore,

$$\begin{aligned} & \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\frac{1}{N_1 + N_2} \int dx dy \tilde{A}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\ &= 4 \operatorname{Im} \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \int dx dy Q(x - y) \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, \right. \\ & \quad \left. \varphi_{1,t}(y)\varphi_{2,t}(x) (4a_{2,x}^* a_{1,y}^* + 3(\mathcal{N}_1 + \mathcal{N}_2 + 1)a_{2,x}^* a_{1,y}^* (\mathcal{N}_1 + \mathcal{N}_2 + 1)) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\ &= 16 \operatorname{Im} \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \int dx \varphi_{2,t}(x) \left\langle a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, a_1^*(Q(x - \cdot)\varphi_{1,t}) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle + \\ & \quad + 12 \operatorname{Im} \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \int dx \varphi_{2,t}(x) \left\langle a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, \right. \\ & \quad \left. a_1^*(Q(x - \cdot)\varphi_{1,t}) (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle. \end{aligned}$$

It follows that

$$\begin{aligned}
& \left| \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\frac{1}{N_1 + N_2} \int dx dy \tilde{A}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \right| \\
& \leq 16 \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \int dx |\varphi_{2,t}(x)| \|a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \|a_1^*(Q(x - \cdot)\varphi_{1,t}) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| + \\
& \quad + 12 \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \int dx |\varphi_{2,t}(x)| \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \cdot \\
& \quad \cdot \|a_1^*(Q(x - \cdot)\varphi_{1,t}) (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \\
& \leq 16 \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \sup_x \|Q(x - \cdot)\varphi_{1,t}\|_{L^2} \|(\mathcal{N}_1 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \int dx |\varphi_{2,t}(x)| \|a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| + \\
& \quad + 12 \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \sup_x \|Q(x - \cdot)\varphi_{1,t}\|_{L^2} \|(\mathcal{N}_1 + 1)^{1/2} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \cdot \\
& \quad \cdot \int dx |\varphi_{2,t}(x)| \|a_{2,x} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\|.
\end{aligned}$$

Using (1.12) we obtain

$$\begin{aligned}
& \left| \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, i \left[\frac{1}{N_1 + N_2} \int dx dy \tilde{A}, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \right] \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \right| \\
& \leq C \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \|(\mathcal{N}_1 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \|\mathcal{N}_2^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| + \\
& \quad + C \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \|(\mathcal{N}_1 + 1)^{1/2} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \|\mathcal{N}_2^{1/2} (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\| \\
& \leq C \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^{3/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega\|^2 \\
& \leq C \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left| \frac{d}{dt} \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \right| \\
& \leq \left(C_1 + C_2 \frac{\sqrt{N_1 N_2}}{N_1 + N_2} \right) \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\
& \leq \left(C_1 + C_2 \frac{\sqrt{(N_1 + N_2)(N_1 + N_2)}}{N_1 + N_2} \right) \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle \\
& = C \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)\Omega \right\rangle.
\end{aligned}$$

Using Gronwall's Lemma, we obtain (3.59). □

3.6 Lemma (3.6.1)

Lemma 3.6.1. *There exist constants $C, K > 0$ such that*

$$\|(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0))\Omega\| \leq C \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{Kt}.$$

Proof. Just as we showed (3.48) in the proof of Lemma (3.4.9), we can also show

$$-i \int_0^t ds \frac{\partial}{\partial s} \mathcal{U}_{N_1, N_2}(t; s) \left(\mathcal{L}_{N_1, N_2}(s) - \tilde{\mathcal{L}}_{N_1, N_2}(s) \right) \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) = \mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0).$$

Using this fact and the definitions of $\mathcal{L}_{N_1, N_2}(s)$ and $\tilde{\mathcal{L}}_{N_1, N_2}(s)$ given in (3.15) and (3.16), we obtain

$$\begin{aligned} & \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \\ &= -i \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \left(\mathcal{L}_{N_1, N_2}(s) - \tilde{\mathcal{L}}_{N_1, N_2}(s) \right) \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \\ &= -i \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \left(L1 + L2 + \frac{1}{N_1 + N_2} \int dx dy A - \tilde{L}1 - \tilde{L}2 + \right. \\ & \quad \left. - \frac{1}{N_1 + N_2} \int dx dy \tilde{A} \right) \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega. \end{aligned}$$

Observe that

$$\begin{aligned} L1 - \tilde{L}1 &= \frac{1}{\sqrt{N_1}} \int dx dy V_1(x-y) a_{1,x}^* (\varphi_{1,t} a_{1,y}^* + \bar{\varphi}_{1,t}(y) a_{1,y}) a_{1,x}, \\ L2 - \tilde{L}2 &= \frac{1}{\sqrt{N_2}} \int dx dy V_2(x-y) a_{2,x}^* (\varphi_{2,t} a_{2,y}^* + \bar{\varphi}_{2,t}(y) a_{2,y}) a_{2,x}, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{N_1 + N_2} \int dx dy (A - \tilde{A}) = \frac{1}{N_1 + N_2} \int dx dy Q(x-y) \left(\sqrt{N_2} \varphi_{2,t}(x) a_{2,x}^* a_{1,y}^* a_{1,y} + \right. \\ & \quad \left. + \sqrt{N_1} \varphi_{1,t}(y) a_{2,x}^* a_{1,y}^* a_{2,x} + \sqrt{N_1} \bar{\varphi}_{1,t}(y) a_{2,x}^* a_{1,y} a_{2,x} + \sqrt{N_2} \bar{\varphi}_{2,t}(x) a_{1,y}^* a_{1,y} a_{2,x} \right) \\ &= \frac{\sqrt{N_2}}{N_1 + N_2} \int dx dy Q(x-y) a_{1,y}^* \left(\varphi_{2,t}(x) a_{2,x}^* + \bar{\varphi}_{2,t}(x) a_{2,x} \right) a_{1,y} + \\ & \quad + \frac{\sqrt{N_1}}{N_1 + N_2} \int dx dy Q(x-y) a_{2,x}^* \left(\varphi_{1,t}(y) a_{1,y}^* + \bar{\varphi}_{1,t}(y) a_{1,y} \right) a_{2,x}. \end{aligned}$$

Thus,

$$\begin{aligned} & \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \\ &= -\frac{i}{\sqrt{N_1}} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx dy V_1(x-y) a_{1,x}^* (\varphi_{1,t} a_{1,y}^* + \bar{\varphi}_{1,t}(y) a_{1,y}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega + \\ & \quad - \frac{i}{\sqrt{N_2}} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx dy V_2(x-y) a_{2,x}^* (\varphi_{2,t} a_{2,y}^* + \bar{\varphi}_{2,t}(y) a_{2,y}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega + \\ & \quad - \frac{i\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx dy Q(x-y) a_{1,y}^* \left(\varphi_{2,t}(x) a_{2,x}^* + \bar{\varphi}_{2,t}(x) a_{2,x} \right) \cdot \\ & \quad \cdot a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega + \\ & \quad - \frac{i\sqrt{N_1}}{N_1 + N_2} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx dy Q(x-y) a_{2,x}^* \left(\varphi_{1,t}(y) a_{1,y}^* + \bar{\varphi}_{1,t}(y) a_{1,y} \right) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega. \end{aligned}$$

Using the notation $\Phi_j(f) = a_j(f) + a_j^*(f)$, we can write the above as

$$\begin{aligned}
& \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \\
&= -\frac{i}{\sqrt{N_1}} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx a_{1,x}^* \Phi_1(V_1(x - \cdot) \varphi_{1,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega + \\
&\quad -\frac{i}{\sqrt{N_2}} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx a_{2,x}^* \Phi_2(V_2(x - \cdot) \varphi_{2,t}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega + \\
&\quad -\frac{i\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dy a_{1,y}^* \Phi_2(Q(\cdot - y) \varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega + \\
&\quad -\frac{i\sqrt{N_1}}{N_1 + N_2} \int_0^t ds \mathcal{U}_{N_1, N_2}(t; s) \int dx a_{2,x}^* \Phi_1(Q(x - \cdot) \varphi_{1,t}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left\| \left(\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \right) \Omega \right\| \\
&\leq \frac{1}{\sqrt{N_1}} \int_0^t ds \left\| \int dx a_{1,x}^* \Phi_1(V_1(x - \cdot) \varphi_{1,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\| + \\
&\quad + \frac{1}{\sqrt{N_2}} \int_0^t ds \left\| \int dx a_{2,x}^* \Phi_2(V_2(x - \cdot) \varphi_{2,t}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\| + \tag{3.61} \\
&\quad + \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y) \varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\| + \\
&\quad + \frac{\sqrt{N_1}}{N_1 + N_2} \int_0^t ds \left\| \int dx a_{2,x}^* \Phi_1(Q(x - \cdot) \varphi_{1,t}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\|.
\end{aligned}$$

A bound for the first two terms of (3.61) follows from the analysis of Rodnianski and Schlein on pp. 52-54 of [19], and we obtain

$$\frac{1}{\sqrt{N_1}} \int_0^t ds \left\| \int dx a_{1,x}^* \Phi_1(V_1(x - \cdot) \varphi_{1,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\| \leq \frac{C}{\sqrt{N_1}} e^{Kt}, \tag{3.62}$$

$$\frac{1}{\sqrt{N_2}} \int_0^t ds \left\| \int dx a_{2,x}^* \Phi_2(V_2(x - \cdot) \varphi_{2,t}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\| \leq \frac{C}{\sqrt{N_2}} e^{Kt}. \tag{3.63}$$

Bounding the last two terms of (3.61) is slightly different and so we go through those steps now. We compute the bound on the third term and then the bound on the last term will follow. Note

$$\begin{aligned}
& \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y) \varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\|^2 \\
&= \left\langle \int dy a_{1,y}^* \Phi_2(Q(\cdot - y) \varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega, \int dx a_{1,x}^* \Phi_2(Q(\cdot - x) \varphi_{2,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\rangle \\
&= \int dy dx \left\langle a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega, \Phi_2(Q(\cdot - y) \varphi_{2,t}) a_{1,y} a_{1,x}^* \Phi_2(Q(\cdot - x) \varphi_{2,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\rangle.
\end{aligned}$$

Using the CCR $[a_{1,y}, a_{1,x}^*] = \delta(y - x)$ and then the fact that $a_{1,x}^*$ and $a_{1,y}$ commute with $\Phi_2(f)$, we obtain

$$\begin{aligned}
& \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y) \varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\|^2 \\
&= \int dy dx \left\langle a_{1,x} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega, \Phi_2(Q(\cdot - y) \varphi_{2,t}) \Phi_2(Q(\cdot - x) \varphi_{2,t}) a_{1,y} a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\rangle + \\
&\quad + \int dx \left\langle a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega, \Phi_2(Q(\cdot - x) \varphi_{2,t}) \Phi_2(Q(\cdot - x) \varphi_{2,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0) \Omega \right\rangle.
\end{aligned}$$

Since $\Phi_2(f)$ is a self-adjoint operator, we obtain

$$\begin{aligned}
& \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&= \int dy dx \left\langle \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,x} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega, \Phi_2(Q(\cdot - x)\varphi_{2,t}) a_{1,x} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\rangle + \\
&\quad + \int dx \left\langle \Phi_2(Q(\cdot - x)\varphi_{2,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega, \Phi_2(Q(\cdot - x)\varphi_{2,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\rangle \\
&\leq \int dy dx \left\| \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,x} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\| \left\| \Phi_2(Q(\cdot - x)\varphi_{2,t}) a_{1,x} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\| + \\
&\quad + \int dx \left\| \Phi_2(Q(\cdot - x)\varphi_{2,t}) a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2.
\end{aligned}$$

Using the inequality $\|\Phi_2(f)\psi\| \leq 2\|f\|_{L^2} \|(\mathcal{N}_2 + 1)^{1/2}\psi\|$, we have that

$$\begin{aligned}
& \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&\leq 4 \sup_x \|Q(x - \cdot)\varphi_t\|_{L^2}^2 \int dy dx \left\| (\mathcal{N}_2 + 1)^{1/2} a_{1,x} a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 + \\
&\quad + 4 \sup_x \|Q(x - \cdot)\varphi_t\|_{L^2}^2 \int dx \left\| (\mathcal{N}_2 + 1)^{1/2} a_{1,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&\leq C \int dy dx \left\| a_{1,x} a_{1,y} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 + \\
&\quad + C \int dx \left\| a_{1,x} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2.
\end{aligned}$$

From Proposition (1.11) we have that $\int dx \|a_{1,x}\psi\|^2 = \|\mathcal{N}_1^{1/2}\psi\|^2$. Thus,

$$\begin{aligned}
& \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&\leq C \int dy \left\| \mathcal{N}_1^{1/2} a_{1,y} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 + \\
&\quad + C \left\| \mathcal{N}_1^{1/2} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2.
\end{aligned}$$

Using $\mathcal{N}_1 a_{1,y} = a_{1,y}(\mathcal{N}_1 - 1)$ and then the equality $\int dx \|a_{1,x}\psi\|^2 = \|\mathcal{N}_1^{1/2}\psi\|^2$ again, we obtain

$$\begin{aligned}
& \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&= C \int dy \left\| a_{1,y} (\mathcal{N}_1 - 1)^{1/2} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 + C \left\| \mathcal{N}_1^{1/2} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&\leq C \left\| \mathcal{N}_1^{1/2} (\mathcal{N}_1 - 1)^{1/2} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 + C \left\| \mathcal{N}_1^{1/2} (\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&\leq C \left\| (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{3/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 + C \left\| (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \\
&\leq C \left\| (\mathcal{N}_1 + \mathcal{N}_2 + 1)^{3/2} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2.
\end{aligned}$$

Finally we use Lemma (3.5.1) to obtain

$$\left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\|^2 \leq C e^{Ks}. \quad (3.64)$$

Thus, for the third term of (3.61), we have

$$\begin{aligned}
& \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds \left\| \int dy a_{1,y}^* \Phi_2(Q(\cdot - y)\varphi_{2,t}) a_{1,y} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\| \\
& \leq \frac{\sqrt{N_2}}{N_1 + N_2} \int_0^t ds C e^{Ks} \\
& \leq C \frac{\sqrt{N_2}}{N_1 + N_2} e^{Kt} \\
& \leq C \frac{1}{\sqrt{N_1 + N_2}} e^{Kt}.
\end{aligned} \tag{3.65}$$

Similarly, for the fourth term of (3.61), we have

$$\frac{\sqrt{N_1}}{N_1 + N_2} \int_0^t ds \left\| \int dx a_{2,x}^* \Phi_1(Q(x - \cdot)\varphi_{1,t}) a_{2,x} \tilde{\mathcal{U}}_{N_1, N_2}(s; 0)\Omega \right\| \leq C \frac{1}{\sqrt{N_1 + N_2}} e^{Kt}. \tag{3.66}$$

From (3.61) and the bounds (3.62), (3.63), (3.65), and (3.66), we obtain

$$\left\| (\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0))\Omega \right\| \leq C \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{Kt},$$

as desired. \square

3.7 Proof of Theorem (3.3.1)

Proof. First we show that

$$\text{Tr } J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \leq \frac{C}{N_1} \|J_1\|_{\text{HS}} e^{Kt} \tag{3.67}$$

and

$$\text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \leq \frac{D}{N_2} \|J_2\|_{\text{HS}} e^{Kt}. \tag{3.68}$$

To do so, we will recall the inequalities (3.17) and (3.18) and use these together with Proposition (3.4.1) and Lemmas (3.5.1) and (3.6.1). We show this now in detail.

First note that, using Proposition (3.4.1), we have

$$\begin{aligned}
\langle \mathcal{U}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2)\mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle & \leq C_1 \|(\mathcal{N}_1 + \mathcal{N}_2 + 1)^2 \Omega\|^2 e^{K_1 t} \\
& = C_1 \|\Omega\|^2 e^{K_1 t} = C_1 e^{K_1 t}.
\end{aligned}$$

Thus, for $j = 1, 2$, we have

$$\begin{aligned}
\langle \mathcal{U}_{N_1, N_2}(t; 0)\Omega, \mathcal{N}_j \mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle & \leq \langle \mathcal{U}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2)\mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle \\
& \leq C_1 e^{K_1 t}
\end{aligned} \tag{3.69}$$

and

$$\begin{aligned}
\|(\mathcal{N}_j + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0)\Omega\|^2 & \leq \langle \mathcal{U}_{N_1, N_2}\Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1)\mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle \\
& = \langle \mathcal{U}_{N_1, N_2}(t; 0)\Omega, (\mathcal{N}_1 + \mathcal{N}_2)\mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle + \langle \mathcal{U}_{N_1, N_2}(t; 0)\Omega, \mathcal{U}_{N_1, N_2}(t; 0)\Omega \rangle \\
& \leq C_1 e^{K_1 t} + 1 \leq (C_1 + 1) e^{K_1 t} \\
\Rightarrow \|(\mathcal{N}_j + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0)\Omega\| & \leq ((C_1 + 1) e^{K_1 t})^{1/2} = C_2 e^{K_2 t}.
\end{aligned} \tag{3.70}$$

Now, using Lemma (3.5.1), we also obtain for $j = 1, 2$

$$\begin{aligned}
\left\| (\mathcal{N}_j + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \right\|^2 &\leq \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega, (\mathcal{N}_1 + \mathcal{N}_2 + 1) \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \right\rangle \\
&\leq \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega, (\mathcal{N}_1 + \mathcal{N}_2)^3 \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \right\rangle + \left\langle \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega, \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \right\rangle \\
&\leq e^{K_3 t} + 1 \leq C_3 e^{K_3 t} \\
\Rightarrow \left\| (\mathcal{N}_j + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega \right\| &\leq (C_3 e^{K_3 t})^{1/2} = C_4 e^{K_4 t}.
\end{aligned} \tag{3.71}$$

Also, we have from Lemma (3.6.1) that

$$\left\| (\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)) \Omega \right\| \leq C_5 \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{K_5 t}.$$

Thus, to show (3.67), first we recall (3.17)

$$\begin{aligned}
\text{Tr } J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle \langle \varphi_{1,t}| \right) &\leq \frac{\|J_1\|_{\text{HS}}}{N_1} \langle \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \mathcal{N}_1 \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\
&+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} \left\| (\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)) \Omega \right\| \|(\mathcal{N}_1 + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| + \\
&+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} \left\| (\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)) \Omega \right\| \|(\mathcal{N}_1 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\|.
\end{aligned}$$

Then we use (3.70), (3.71), and Lemma (3.6.1) to obtain

$$\begin{aligned}
\text{Tr } J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle \langle \varphi_{1,t}| \right) &\leq \frac{\|J_1\|_{\text{HS}}}{N_1} C_1 e^{K_1 t} + \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} C_5 \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{K_5 t} \cdot C_2 e^{K_2 t} \\
&+ \frac{2\|J_1\|_{\text{HS}}}{\sqrt{N_1}} C_5 \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{K_5 t} \cdot C_4 e^{K_4 t} \\
&\leq \frac{\|J_1\|_{\text{HS}}}{N_1} C_1 e^{K_1 t} + C_6 \|J_1\|_{\text{HS}} \left(\frac{1}{N_1} + \frac{1}{\sqrt{N_1 N_2}} + \frac{1}{\sqrt{N_1(N_1 + N_2)}} \right) e^{K_6 t} \\
&= C_7 \|J_1\|_{\text{HS}} e^{Kt} \left(\frac{1}{N_1} + \frac{1}{\sqrt{N_1 N_2}} + \frac{1}{\sqrt{N_1(N_1 + N_2)}} \right).
\end{aligned}$$

Now we have that $N_1 = aN_2 \Rightarrow N_2 = N_1/a = bN_1$, where $b = 1/a$. Thus,

$$\begin{aligned}
\text{Tr } J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle \langle \varphi_{1,t}| \right) &\leq C_7 \|J_1\|_{\text{HS}} e^{Kt} \left(\frac{1}{N_1} + \frac{1}{\sqrt{bN_1^2}} + \frac{1}{\sqrt{N_1^2(1+b)}} \right) \\
&= \frac{C_7}{N_1} \|J_1\|_{\text{HS}} e^{Kt} \left(1 + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{1+b}} \right) \\
&= \frac{C}{N_1} \|J_1\|_{\text{HS}} e^{Kt}
\end{aligned}$$

which shows (3.67). Similarly, to show (3.68), we first recall (3.18)

$$\begin{aligned}
\text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle \langle \varphi_{2,t}| \right) &\leq \frac{\|J_2\|_{\text{HS}}}{N_2} \langle \mathcal{U}_{N_1, N_2}(t; 0) \Omega, \mathcal{N}_2 \mathcal{U}_{N_1, N_2}(t; 0) \Omega \rangle + \\
&+ \frac{2\|J_2\|_{\text{HS}}}{\sqrt{N_2}} \left\| (\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)) \Omega \right\| \|(\mathcal{N}_2 + 1)^{1/2} \mathcal{U}_{N_1, N_2}(t; 0) \Omega\| + \\
&+ \frac{2\|J_2\|_{\text{HS}}}{\sqrt{N_2}} \left\| (\mathcal{U}_{N_1, N_2}(t; 0) - \tilde{\mathcal{U}}_{N_1, N_2}(t; 0)) \Omega \right\| \|(\mathcal{N}_2 + 1)^{1/2} \tilde{\mathcal{U}}_{N_1, N_2}(t; 0) \Omega\|.
\end{aligned}$$

Then we use (3.70), (3.71), and Lemma (3.6.1) to obtain

$$\begin{aligned}
\text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) &\leq \frac{\|J_2\|_{HS}}{N_2} C_1 e^{K_1 t} + \frac{2\|J_2\|_{HS}}{\sqrt{N_2}} C_5 \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{K_5 t} \cdot C_2 e^{K_2 t} \\
&+ \frac{2\|J_2\|_{HS}}{\sqrt{N_2}} C_5 \left(\frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}} + \frac{1}{\sqrt{N_1 + N_2}} \right) e^{K_5 t} \cdot C_4 e^{K_4 t} \\
&= \frac{\|J_2\|_{HS}}{N_2} C_1 e^{K_1 t} + C_6 \|J_2\|_{HS} \left(\frac{1}{N_2} + \frac{1}{\sqrt{N_1 N_2}} + \frac{1}{\sqrt{N_2(N_1 + N_2)}} \right) e^{K_6 t} \\
&= C_7 \|J_2\|_{HS} e^{K t} \left(\frac{1}{N_2} + \frac{1}{\sqrt{N_1 N_2}} + \frac{1}{\sqrt{N_2(N_1 + N_2)}} \right).
\end{aligned}$$

Using the fact that $N_1 = aN_2$, we obtain

$$\begin{aligned}
\text{Tr } J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) &\leq C_7 \|J_2\|_{HS} e^{K t} \left(\frac{1}{N_2} + \frac{1}{\sqrt{aN_2^2}} + \frac{1}{\sqrt{N_2^2(1+a)}} \right) \\
&= \frac{C_7}{N_2} \|J_2\|_{HS} e^{K t} \left(1 + \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{1+a}} \right) \\
&= \frac{D}{N_2} \|J_2\|_{HS} e^{K t}
\end{aligned}$$

which shows (3.68).

It follows that from (3.67) and (3.68) that

$$\text{Tr} \left[J_1 \otimes J_2 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \otimes \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right] \leq \frac{A}{N_1 N_2} \|J_1\|_{HS} \|J_2\|_{HS} e^{Bt}$$

for constants A and B . Indeed, note that

$$\begin{aligned}
&\text{Tr} \left[J_1 \otimes J_2 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \otimes \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right] \\
&= \text{Tr} \left[J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \right] \text{Tr} \left[J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right] \\
&\leq \frac{C}{N_1} \|J_1\|_{HS} e^{Kt} \frac{D}{N_2} \|J_2\|_{HS} e^{Kt} \\
&= \frac{A}{N_1 N_2} \|J_1\|_{HS} \|J_2\|_{HS} e^{Bt}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
& \text{Tr} \left(J_1 \otimes J_2 \left[\left(\gamma_1^{(1)} \otimes \gamma_2^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) - \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \otimes \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right] \right) \\
&= \text{Tr} (J_1 \gamma_1^{(1)}) \text{Tr} (J_2 \gamma_2^{(1)}) - \text{Tr} (J_1 |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \text{Tr} (J_2 |\varphi_{2,t}\rangle\langle\varphi_{2,t}|) + \\
&\quad - \text{Tr} J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) \text{Tr} J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \\
&= \text{Tr} (J_1 \gamma_1^{(1)}) \text{Tr} (J_2 \gamma_2^{(1)}) - \text{Tr} (J_1 |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \text{Tr} (J_2 |\varphi_{2,t}\rangle\langle\varphi_{2,t}|) + \\
&\quad - \left(\text{Tr} (J_1 \gamma_1^{(1)}) - \text{Tr} (J_1 |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \right) \left(\text{Tr} (J_2 \gamma_2^{(1)}) - \text{Tr} (J_2 |\varphi_{2,t}\rangle\langle\varphi_{2,t}|) \right) \\
&= \text{Tr} (J_1 \gamma_1^{(1)}) \text{Tr} (J_2 |\varphi_{2,t}\rangle\langle\varphi_{2,t}|) + \text{Tr} (J_1 |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \text{Tr} (J_2 \gamma_2^{(1)}) + \\
&\quad - 2 \text{Tr} (J_1 |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \text{Tr} (J_2 |\varphi_{2,t}\rangle\langle\varphi_{2,t}|) \\
&= \text{Tr} (J_2 |\varphi_{2,t}\rangle\langle\varphi_{2,t}|) \text{Tr} J_1 \left(\gamma_1^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \right) + \text{Tr} (J_1 |\varphi_{1,t}\rangle\langle\varphi_{1,t}|) \text{Tr} J_2 \left(\gamma_2^{(1)} - |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \\
&\leq \frac{C_1}{N_1} \|J_1\|_{\text{HS}} e^{Kt} + \frac{C_2}{N_2} \|J_2\|_{\text{HS}} e^{Kt}.
\end{aligned}$$

This implies

$$\begin{aligned}
& \text{Tr} \left[J_1 \otimes J_2 \left(\gamma_1^{(1)} \otimes \gamma_2^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \right] \\
&\leq \frac{C_1}{N_1} \|J_1\|_{\text{HS}} e^{Kt} + \frac{C_2}{N_2} \|J_2\|_{\text{HS}} e^{Kt} + \frac{A}{N_1 N_2} \|J_1\|_{\text{HS}} \|J_2\|_{\text{HS}} e^{Bt} \\
&\leq C \frac{N_1 + N_2}{N_1 N_2} e^{K_2 t},
\end{aligned}$$

which proves the theorem. By Remark (3.3.2), this means we also have

$$\text{Tr} \left(\gamma_1^{(1)} \otimes \gamma_2^{(1)} - |\varphi_{1,t}\rangle\langle\varphi_{1,t}| \otimes |\varphi_{2,t}\rangle\langle\varphi_{2,t}| \right) \leq C \frac{N_1 + N_2}{N_1 N_2} e^{K_2 t}.$$

□

CHAPTER 4

TIME EVOLUTION OF FACTORIZED STATES

4.1 Introduction

Now we consider our system of N_1 bosonic particles of one species and N_2 bosonic particles of another species. Let N_1 and N_2 scale linearly so that

$$N_1 = aN_2$$

for some (positive) rational number a .

We consider an initial factorized state given by

$$\psi_{N_1, N_2} = \varphi_1^{\otimes N_1} \otimes \varphi_2^{\otimes N_2} \in \mathcal{F} \otimes \mathcal{F}$$

for some $\varphi_1, \varphi_2 \in H^1(\mathbb{R}^3)$ where $\|\varphi_j\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 1, 2$. Here \mathcal{F} is the bosonic Fock space described in (1.1). Alternatively, we can write this initial state as a wavefunction (see Remark (1.3.2))

$$\psi_{N_1, N_2}(x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2}) = \prod_{j=1}^{N_1} \varphi_1(x_j) \prod_{k=1}^{N_2} \varphi_2(y_k) \in L^2(\mathbb{R}^{3(N_1+N_2)}). \quad (4.1)$$

Note $\prod_{j=1}^{N_1} \varphi_1(x_j) \in L_s^2(\mathbb{R}^{3N_1})$ and $\prod_{k=1}^{N_2} \varphi_2(y_k) \in L_s^2(\mathbb{R}^{3N_2})$.

The time evolution $\psi_{N_1, N_2, t}$ is then given by the Schrödinger equation

$$i\partial_t \psi_{N_1, N_2, t} = H_{N_1, N_2} \psi_{N_1, N_2, t}, \quad \psi_{N_1, N_2, 0} = \psi_{N_1, N_2} \quad (4.2)$$

where the mean-field Hamiltonian H_{N_1, N_2} is given by

$$\begin{aligned} H_{N_1, N_2} &= \sum_{j=1}^{N_1} -\Delta_{x_j} + \sum_{k=1}^{N_2} -\Delta_{y_k} + \frac{1}{N_1} \sum_{1 \leq r < s \leq N_1} V_1(x_s - x_r) + \frac{1}{N_2} \sum_{1 \leq p < q \leq N_2} V_2(y_q - y_p) \\ &+ \frac{1}{N_1 + N_2} \sum_{\ell=1}^{N_1} \sum_{m=1}^{N_2} W(x_\ell - y_m). \end{aligned} \quad (4.3)$$

Here V_1, V_2 , and W are symmetric interaction potentials.

The time evolutions $\varphi_{1,t}$ and $\varphi_{2,t}$ satisfy the nonlinear coupled Hartree equations

$$i\frac{\partial}{\partial t} \varphi_{1,t} = h_1^H \varphi_{1,t}, \quad i\frac{\partial}{\partial t} \varphi_{2,t} = h_2^H \varphi_{2,t} \quad (4.4)$$

where

$$\begin{aligned} h_1^H &= -\Delta + (V_1 * |\varphi_{1,t}|^2) + \frac{1}{a+1} (W * |\varphi_{2,t}|^2), \\ h_2^H &= -\Delta + (V_2 * |\varphi_{2,t}|^2) + \frac{a}{a+1} (W * |\varphi_{1,t}|^2). \end{aligned}$$

Let

$$\begin{aligned} h_{1,j}^H &= -\Delta_{x_j} + (V_1 * |\varphi_{1,t}|^2)(x_j) + \frac{1}{a+1} (W * |\varphi_{2,t}|^2)(x_j), \\ h_{2,j}^H &= -\Delta_{y_j} + (V_2 * |\varphi_{2,t}|^2)(y_j) + \frac{a}{a+1} (W * |\varphi_{1,t}|^2)(y_j) \end{aligned}$$

so that

$$H_1^H = \sum_{j=1}^{N_1} h_{1,j}^H, \quad H_2^H = \sum_{j=1}^{N_2} h_{2,j}^H$$

are each a sum of the Hartree Hamiltonians for each particle.

Notation 4.1.1. In this chapter, to simplify notation we use $\|\cdot\|_r$ to mean $\|\cdot\|_{L^r}$, for any $r \geq 1$. Also, again we use an inner product or a norm without a subscript to correspond to the norm or inner product in $\mathcal{F} \otimes \mathcal{F}$ where \mathcal{F} is the bosonic Fock space defined in (1.1). All other norms or inner products will be indicated with a subscript.

The main goal of this chapter is to use a method introduced by Pickl in [17] to show the following theorem. (See also [12].)

Theorem 4.1.2. *Let $V_1, V_2, W \in L^{2r}$ for some $r \geq 1$. Let $\psi_{N_1, N_2, t}$ be a solution of the Schrödinger equation (4.2) with initial wave function (4.1), and let $\varphi_{1,t}$ and $\varphi_{2,t}$ be solutions to the coupled Hartree equations (4.4) with $\|\varphi_{1,t}\|_{2s} < \infty$ and $\|\varphi_{2,t}\|_{2s} < \infty$ for $s = \frac{r}{r-1}$. Let $\Gamma_{N_1, N_2, t}^{(1)}$ be the one-particle density associated with $\psi_{N_1, N_2, t}$. Let $T < \infty$ be any fixed number. Then for any time $\tau < T$*

$$\lim_{N_1, N_2 \rightarrow \infty} \Gamma_{N_1, N_2, \tau}^{(1)} = |\varphi_{1,\tau}\rangle\langle\varphi_{1,\tau}| \otimes |\varphi_{2,\tau}\rangle\langle\varphi_{2,\tau}| \quad \text{in the trace norm topology.}$$

The method of Pickl involves a biased counting algorithm which counts the number of “bad” particles of the evolved state $\psi_{N_1, N_2, t}$ (i.e. the particles which become entangled and disrupt the product structure of the state). This method introduces a counting measure $\alpha_{N_1, N_2}(t)$ such that for $\alpha \approx 0$ most particles are good. The equation for $\alpha_{N_1, N_2}(t)$ will show that if $\alpha_{N_1, N_2}(0) \approx 0$ then $\alpha_{N_1, N_2}(t) \approx 0$. The counting measure α will also have the property that $\lim_{N_1, N_2 \rightarrow \infty} \alpha = 0$ implies convergence of the one particle marginal density.

4.2 Preliminary Definitions

Let us start by pointing out a convention that we will be using; then we will go on to give several important definitions.

Remark 4.2.1. Let $\psi = \psi_{N_1} \otimes \psi_{N_2} \in \mathcal{F} \otimes \mathcal{F}$ where \mathcal{F} is the bosonic Fock space described in (1.1). We have that

$$\psi_{N_1} = \varphi_1 \otimes_s \varphi_2 \otimes_s \cdots \otimes_s \varphi_{N_1}, \quad \psi_{N_2} = \rho_1 \otimes_s \rho_2 \otimes_s \cdots \otimes_s \rho_{N_2}$$

where $\varphi_k, \rho_\ell \in L^2(\mathbb{R}^3)$, for $k = 1, 2, \dots, N_1$ and $\ell = 1, 2, \dots, N_2$. Recall we can write ψ_{N_1} as a function of $L_s^2(\mathbb{R}^{3N_1})$ and we can write ψ_{N_2} as a function of $L_s^2(\mathbb{R}^{3N_2})$ so that $\psi_{N_1} \otimes \psi_{N_2} \in L_s^2(\mathbb{R}^{3N_1}) \otimes L_s^2(\mathbb{R}^{3N_2})$. When we do this, let us make the following convention. Let x variables be associated with the element ψ_{N_1} of the first Fock space, and let y variables be associated with the element ψ_{N_2} of the second Fock space, so that

$$\psi_{N_1} = \varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_{N_1}(x_{N_1}), \quad \psi_{N_2} = \rho_1(y_1)\rho_2(y_2)\cdots\rho_{N_2}(y_{N_2}).$$

Then we can further write $\psi_{N_1} \otimes \psi_{N_2}$ as a function in $L^2(\mathbb{R}^{3(N_1+N_2)})$ as follows:

$$\psi_{N_1, N_2} = \varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_{N_1}(x_{N_1})\rho_1(y_1)\rho_2(y_2)\cdots\rho_{N_2}(y_{N_2}).$$

With the convention described in Remark (4.2.1) we can now make use of the following definitions.

Definition 4.2.2. For any $\varphi \in L^2(\mathbb{R}^3)$ where $\|\varphi\| = 1$, the projectors $p_{1,j}^\varphi$ and $p_{2,j}^\varphi$ are given by

$$p_{1,j}^\varphi = |\varphi(x_j)\rangle\langle\varphi(x_j)| \quad \text{and} \quad p_{2,j}^\varphi = |\varphi(y_j)\rangle\langle\varphi(y_j)|$$

and the projectors $q_{1,j}^\varphi$ and $q_{2,j}^\varphi$ are given by

$$q_{1,j}^\varphi = 1 - p_{1,j}^\varphi \quad \text{and} \quad q_{2,j}^\varphi = 1 - p_{2,j}^\varphi.$$

The operators are defined on $L^2(\mathbb{R}^{3(N_1+N_2)})$.

Definition 4.2.3. Let

$$\begin{aligned} \mathcal{A}_{1,k} &= \{(a_1, a_2, \dots, a_{N_1}) : a_j \in \{0, 1\} \text{ and } \sum_{j=0}^{N_1} a_j = k\}, \\ \mathcal{A}_{2,k} &= \{(a_1, a_2, \dots, a_{N_2}) : a_j \in \{0, 1\} \text{ and } \sum_{j=0}^{N_2} a_j = k\}. \end{aligned}$$

Then, for any $0 \leq k \leq N_1$, the projector $P_{1,k}^\varphi : L^2(\mathbb{R}^{3(N_1+N_2)}) \rightarrow L^2(\mathbb{R}^{3(N_1+N_2)})$ is given by

$$P_{1,k}^\varphi = \sum_{a \in \mathcal{A}_{1,k}} \prod_{j=1}^{N_1} (p_{1,j}^\varphi)^{1-a_j} (q_{1,j}^\varphi)^{a_j}$$

and, for any $0 \leq k \leq N_2$, the projector $P_{2,k}^\varphi : L^2(\mathbb{R}^{3(N_1+N_2)}) \rightarrow L^2(\mathbb{R}^{3(N_1+N_2)})$ is given by

$$P_{2,k}^\varphi = \sum_{a \in \mathcal{A}_{2,k}} \prod_{j=1}^{N_2} (p_{2,j}^\varphi)^{1-a_j} (q_{2,j}^\varphi)^{a_j}.$$

Definition 4.2.4. Let \hat{n}_1^φ and \hat{n}_2^φ be defined so that

$$\hat{n}_1^\varphi = \sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^\varphi, \quad \hat{n}_2^\varphi = \sum_{\ell=0}^{N_2} \frac{\ell}{N_2} P_{2,\ell}^\varphi.$$

Now we can define our counting measure α_{N_1, N_2} , which counts (in a biased way) the number of bad particles.

Definition 4.2.5. For any $N_1, N_2 \in \mathbb{N}$, we define the counting measure $\alpha_{N_1, N_2} : L^2(\mathbb{R}^{3(N_1+N_2)}) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \rightarrow \mathbb{R}_0^+$ to be

$$\alpha_{N_1, N_2}(\psi_{N_1, N_2}, \varphi_1, \varphi_2) = \left\langle \psi_{N_1, N_2}, (\hat{n}_1^{\varphi_1} + \hat{n}_2^{\varphi_2}) \psi_{N_1, N_2} \right\rangle$$

where the inner product is over L^2 . For ease of notation, we also define

$$\alpha_{N_1, N_2}(t) = \alpha_{N_1, N_2}(\psi_{N_1, N_2, t}, \varphi_{1, t}, \varphi_{2, t})$$

as well as

$$\alpha_1(t) = \left\langle \psi_{N_1, N_2, t}, \hat{n}_1^{\varphi_{1, t}} \psi_{N_1, N_2, t} \right\rangle, \quad \alpha_2(t) = \left\langle \psi_{N_1, N_2, t}, \hat{n}_2^{\varphi_{2, t}} \psi_{N_1, N_2, t} \right\rangle.$$

Remark 4.2.6. In our definition of α_{N_1, N_2} we have chosen a particular weight (which makes our problem easy to handle). That is, the part of ψ_{N_1, N_2} where k of the N_1 particles behave badly and ℓ of the N_2 particles behave badly (i.e. $\langle \psi_{N_1, N_2}, (P_{1, k}^{\varphi_1} P_{2, \ell}^{\varphi_2}) \psi_{N_1, N_2} \rangle$) is given the weight $\frac{k}{N_1} + \frac{\ell}{N_2}$. Thus we see that α_{N_1, N_2} is counting, in a biased way, the number of particles that behave badly. While we will not use different weights, it sometimes can be advantageous to do so as described in [17].

4.3 Theorem (4.3.1)

Now we aim to show that $\lim_{N_1, N_2 \rightarrow \infty} \alpha_{N_1, N_2}(0) = 0$ implies $\lim_{N_1, N_2 \rightarrow \infty} \alpha_{N_1, N_2}(t) = 0$. To do so we prove the following theorem.

Theorem 4.3.1. *Let $V_1, V_2, W \in L^{2r}$ for some $r \geq 1$. For all $t > 0$ and any integers $N_1, N_2 > 1$, let $\psi_{N_1, N_2, t}$ be a solution of the Schrödinger equation (4.2) with initial wave function (4.1). Let $\varphi_{1, t}$ and $\varphi_{2, t}$ be solutions to the coupled Hartree equations (4.4) with $\|\varphi_{1, t}\|_{2s} < \infty$ and $\|\varphi_{2, t}\|_{2s} < \infty$ for $s = \frac{r}{r-1}$. Then*

$$\alpha_{N_1, N_2}(t) \leq e^{\int_0^t C(\tau) d\tau} \alpha_{N_1, N_2}(0) + \left(e^{\int_0^t C(\tau) d\tau} - 1 \right) \left(\frac{1}{N_1} + \frac{1}{N_2} \right)$$

where

$$C(t) = 24 \left(\|V_1\|_{2r} \|\varphi_{1, t}\|_{2s} + \|V_2\|_{2r} \|\varphi_{2, t}\|_{2s} + \|W\|_{2r} (\|\varphi_{1, t}\|_{2s} + \|\varphi_{2, t}\|_{2s}) \right).$$

Proof. First we prove $|\frac{\partial}{\partial t} \alpha_{N_1, N_2}(t)| \leq C \alpha_{N_1, N_2}(t) + \mathcal{O}\left(\frac{1}{N_1}\right) + \mathcal{O}\left(\frac{1}{N_2}\right)$; then the theorem will follow from Gronwall's Lemma.

Note

$$\begin{aligned}
\frac{\partial}{\partial t} \alpha_{N_1, N_2}(t) &= \left\langle \frac{\partial}{\partial t} \psi_{N_1, N_2, t}, \left(\sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^{\varphi_{1,t}} + \sum_{l=0}^{N_2} \frac{l}{N_2} P_{2,l}^{\varphi_{2,t}} \right) \psi_{N_1, N_2, t} \right\rangle \\
&+ \left\langle \psi_{N_1, N_2, t}, \left(\sum_{k=0}^{N_1} \frac{k}{N_1} \left(\frac{\partial}{\partial t} P_{1,k}^{\varphi_{1,t}} \right) + \sum_{l=0}^{N_2} \frac{l}{N_2} \left(\frac{\partial}{\partial t} P_{2,l}^{\varphi_{2,t}} \right) \right) \psi_{N_1, N_2, t} \right\rangle \\
&+ \left\langle \psi_{N_1, N_2, t}, \sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^{\varphi_{1,t}} \sum_{l=0}^{N_2} \frac{l}{N_2} P_{2,l}^{\varphi_{2,t}} \frac{\partial}{\partial t} \psi_{N_1, N_2, t} \right\rangle.
\end{aligned}$$

Let us calculate $\frac{\partial}{\partial t} P_{1,k}^{\varphi_{1,t}}$. Note

$$\begin{aligned}
\frac{\partial}{\partial t} (p_{1,j}^{\varphi_{1,t}}) &= \frac{\partial}{\partial t} (|\varphi_{1,t}(x_j)\rangle\langle\varphi_{1,t}(x_j)|) \\
&= -i h_{1,j}^H |\varphi_{1,t}(x_j)\rangle\langle\varphi_{1,t}(x_j)| + i |\varphi_{1,t}(x_j)\rangle\langle\varphi_{1,t}(x_j)| h_{1,j}^H \\
&= -i [h_{1,j}^H, p_{1,j}^{\varphi_{1,t}}]
\end{aligned}$$

and

$$\frac{\partial}{\partial t} (q_{1,j}^{\varphi_{1,t}}) = -\frac{\partial}{\partial t} (p_{1,j}^{\varphi_{1,t}}) = i [h_{1,j}^H, 1 - q_{1,j}^{\varphi_{1,t}}] = -i [h_{1,j}^H, q_{1,j}^{\varphi_{1,t}}].$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t} (P_{1,k}^{\varphi_{1,t}}) &= \frac{\partial}{\partial t} \left(\sum_{a \in \mathcal{A}_{1,k}} \prod_{j=1}^{N_1} (p_{1,j}^{\varphi_{1,t}})^{1-a_j} (q_{1,j}^{\varphi_{1,t}})^{a_j} \right) \\
&= -i [H_1^H, P_{1,k}^{\varphi_{1,t}}].
\end{aligned}$$

Similarly, we have that

$$\frac{\partial}{\partial t} (P_{2,k}^{\varphi_{2,t}}) = -i [H_2^H, P_{2,k}^{\varphi_{2,t}}].$$

It follows that

$$\begin{aligned}
\frac{\partial}{\partial t} \alpha_{N_1, N_2}(t) &= i \left\langle \psi_{N_1, N_2, t}, \left[H_{N_1, N_2}, \sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^{\varphi_{1,t}} + \sum_{l=0}^{N_2} \frac{l}{N_2} P_{2,l}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle \\
&- i \left\langle \psi_{N_1, N_2, t}, \left(\left[H_1^H, \sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^{\varphi_{1,t}} \right] + \left[H_2^H, \sum_{l=0}^{N_2} \frac{l}{N_2} P_{2,l}^{\varphi_{2,t}} \right] \right) \psi_{N_1, N_2, t} \right\rangle \\
&= i \left\langle \psi_{N_1, N_2, t}, \left[H_{N_1, N_2} - H_1^H - H_2^H, \sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^{\varphi_{1,t}} + \sum_{l=0}^{N_2} \frac{l}{N_2} P_{2,l}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle.
\end{aligned}$$

Now, note that

$$\begin{aligned}
& H_{N_1, N_2} - H_1^H - H_2^H \\
&= \frac{1}{N_1} \sum_{1 \leq r < s \leq N_1} V_1(x_s - x_r) + \frac{1}{N_2} \sum_{1 \leq p < q \leq N_2} V_2(y_q - y_p) + \frac{1}{N_1 + N_2} \sum_{\ell=1}^{N_1} \sum_{m=1}^{N_2} W(x_\ell - y_m) \\
&\quad - \sum_{j=1}^{N_1} (V_1 * |\varphi_{1,t}|^2)(x_j) - \sum_{j=1}^{N_2} (V_2 * |\varphi_{2,t}|^2)(y_j) \\
&\quad - \frac{1}{a+1} \sum_{j=1}^{N_1} (W * |\varphi_{2,t}|^2)(x_j) - \frac{a}{a+1} \sum_{j=1}^{N_2} (W * |\varphi_{1,t}|^2)(y_j).
\end{aligned}$$

Also, we have the following lemma.

Lemma 4.3.2. *For any $N_1, N_2 \in \mathbb{N}$, we have*

$$\hat{n}_1^\varphi = \frac{1}{N_1} \sum_{j=0}^{N_1} q_{1,j}^\varphi, \quad \hat{n}_2^\varphi = \frac{1}{N_2} \sum_{j=0}^{N_2} q_{2,j}^\varphi. \quad (4.5)$$

Proof. We show the first equality and the second follows analogously. Note $\bigcup_{k=0}^{N_1} \mathcal{A}_{1,k} = \{0, 1\}^{N_1}$ and thus

$\sum_{k=0}^{N_1} P_{1,k}^\varphi = 1$. Therefore,

$$\frac{1}{N_1} \sum_{j=1}^{N_1} q_{1,j}^\varphi = \sum_{k=0}^{N_1} \frac{1}{N_1} P_{1,k}^\varphi \sum_{j=1}^{N_1} q_{1,j}^\varphi = \sum_{k=0}^{N_1} \frac{k}{N_1} P_{1,k}^\varphi = \hat{n}_1^\varphi.$$

To get the second equality, we used the fact that for a particular value of k , the product $\prod_{i=1}^{N_1} (p_{1,i}^\varphi)^{1-a_i} (q_{1,i}^\varphi)^{a_i} q_{1,j}^\varphi$ is zero if $a_j \neq 1$. Since the product $\prod_{i=1}^{N_1} (p_{1,i}^\varphi)^{1-a_i} (q_{1,i}^\varphi)^{a_i}$ in $P_{1,k}^\varphi$ has $a_i = 1$ for k different values of i , we see that $P_{1,k}^\varphi \sum_{j=1}^{N_1} q_{1,j}^\varphi = k P_{1,k}^\varphi$. \square

Now, using (4.5) and symmetry of $\psi_{N_1, N_2, t}$, we get that

$$\begin{aligned}
& \left[H_{N_1, N_2} - H_1^H - H_2^H, \hat{n}_1^{\varphi_{1,t}} + \hat{n}_2^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \\
&= \left[\frac{1}{N_1} \sum_{1 \leq r < s \leq N_1} V_1(x_s - x_r) + \frac{1}{N_2} \sum_{1 \leq p < q \leq N_2} V_2(y_q - y_p) + \frac{1}{N_1 + N_2} \sum_{\ell=1}^{N_1} \sum_{m=1}^{N_2} W(x_\ell - y_m) \right. \\
&\quad - \sum_{j=1}^{N_1} (V_1 * |\varphi_{1,t}|^2)(x_j) - \sum_{j=1}^{N_2} (V_2 * |\varphi_{2,t}|^2)(y_j) \\
&\quad \left. - \frac{1}{a+1} \sum_{j=1}^{N_1} (W * |\varphi_{2,t}|^2)(x_j) - \frac{a}{a+1} \sum_{j=1}^{N_2} (W * |\varphi_{1,t}|^2)(y_j), q_{1,1}^{\varphi_{1,t}} + q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \\
&= \left[\frac{1}{N_1} \sum_{s=2}^{N_1} V_1(x_s - x_1) + \frac{1}{N_2} \sum_{q=2}^{N_2} V_2(y_q - y_1) + \frac{1}{N_1 + N_2} \sum_{\ell=1}^{N_1} \sum_{m=1}^{N_2} W(x_\ell - y_m) \right. \\
&\quad - (V_1 * |\varphi_{1,t}|^2)(x_1) - (V_2 * |\varphi_{2,t}|^2)(y_1) \\
&\quad \left. - \frac{1}{a+1} (W * |\varphi_{2,t}|^2)(x_1) - \frac{a}{a+1} (W * |\varphi_{1,t}|^2)(y_1), q_{1,1}^{\varphi_{1,t}} + q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t}
\end{aligned}$$

where to get the last equality we used the fact that any term containing neither x_1 nor y_1 commutes with $q_{1,1}^{\varphi_{1,t}} + q_{2,1}^{\varphi_{2,t}}$. Now we use symmetry to get

$$\begin{aligned} \left[H_{N_1, N_2} - H_1^H - H_2^H, \hat{n}_1^{\varphi_{1,t}} + \hat{n}_2^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} &= \left[A_1(x_1, x_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} + \left[A_2(y_1, y_2), q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \\ &+ \left[B_1(x_1, y_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} + \left[B_2(x_2, y_1), q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \\ &+ \frac{1}{N_1 + N_2} \left[W(x_1 - y_1), q_{1,1}^{\varphi_{1,t}} + q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \end{aligned}$$

where

$$\begin{aligned} A_1(x_1, x_2) &= \frac{N_1 - 1}{N_1} V_1(x_2 - x_1) - (V_1 * |\varphi_{1,t}|^2)(x_1), \\ A_2(y_1, y_2) &= \frac{N_2 - 1}{N_2} V_2(y_2 - y_1) - (V_2 * |\varphi_{2,t}|^2)(y_1), \\ B_1(x_1, y_2) &= \frac{N_2 - 1}{N_1 + N_2} W(x_1 - y_2) - \frac{1}{a + 1} (W * |\varphi_{2,t}|^2)(x_1), \\ B_2(x_2, y_1) &= \frac{N_1 - 1}{N_1 + N_2} W(x_2 - y_1) - \frac{a}{a + 1} (W * |\varphi_{1,t}|^2)(y_1). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial t} \alpha_{N_1, N_2}(t) &= i \left\langle \psi_{N_1, N_2, t}, \left[A_1(x_1, x_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle + i \left\langle \psi_{N_1, N_2, t}, \left[A_2(y_1, y_2), q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle \\ &+ i \left\langle \psi_{N_1, N_2, t}, \left[B_1(x_1, y_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle + i \left\langle \psi_{N_1, N_2, t}, \left[B_2(x_2, y_1), q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle \quad (4.6) \\ &+ \frac{i}{N_1 + N_2} \left\langle \psi_{N_1, N_2, t}, \left[W(x_1 - y_1), q_{1,1}^{\varphi_{1,t}} + q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle. \end{aligned}$$

Let us look at the first term of (4.6). Note

$$\begin{aligned} &i \left\langle \psi_{N_1, N_2, t}, \left[A_1(x_1, x_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle \\ &= i \left\langle \psi_{N_1, N_2, t}, A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle - i \left\langle \psi_{N_1, N_2, t}, q_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) \psi_{N_1, N_2, t} \right\rangle \\ &= i \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle + i \left\langle \psi_{N_1, N_2, t}, q_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\ &\quad - i \left\langle \psi_{N_1, N_2, t}, q_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle - i \left\langle \psi_{N_1, N_2, t}, q_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\ &= -2 \operatorname{Im} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\ &= -2 \operatorname{Im} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} (p_{1,2}^{\varphi_{1,t}} + q_{1,2}^{\varphi_{1,t}}) A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} (p_{1,2}^{\varphi_{1,t}} + q_{1,2}^{\varphi_{1,t}}) \psi_{N_1, N_2, t} \right\rangle \end{aligned}$$

where we have used the fact that $p_{1,1}^{\varphi_{1,t}} + q_{1,1}^{\varphi_{1,t}} = 1$ and $p_{1,2}^{\varphi_{1,t}} + q_{1,2}^{\varphi_{1,t}} = 1$. Expanding, we get

$$\begin{aligned} &i \left\langle \psi_{N_1, N_2, t}, \left[A_1(x_1, x_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle \\ &= -2 \operatorname{Im} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\ &\quad - 2 \operatorname{Im} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \quad (4.7) \\ &\quad - 2 \operatorname{Im} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\ &\quad - 2 \operatorname{Im} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle. \end{aligned}$$

Note the third term is zero, since $(p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}})^* = p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_2, x_1) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}}$ and since A_1 is symmetric.

Now let us make the following definition.

Definition 4.3.3. For any $j > 0$, let $(\hat{n}_1^\varphi)^{-j}$ be defined so that

$$(\hat{n}_1^\varphi)^{-j} = \sum_{k=1}^{N_1} \left(\frac{k}{N_1} \right)^{-j} P_{1,k}^\varphi.$$

Lemma 4.3.4. For any positive rational number j and any $i \in \{1, 2, \dots, N_1\}$

$$(\hat{n}_1^\varphi)^j (\hat{n}_1^\varphi)^{-j} q_{1,i}^\varphi = q_{1,i}^\varphi. \quad (4.8)$$

Proof. First, note that, for any $m \in \mathbb{N}$

$$\left(\sum_{k=1}^{N_1} \frac{k}{N_1} P_{1,k}^\varphi \right)^m = \sum_{k=1}^{N_1} \left(\frac{k}{N_1} \right)^m P_{1,k}^\varphi$$

where we used the fact that $P_{1,k}^\varphi P_{1,\ell}^\varphi = 0$ when $k \neq \ell$. Thus, it follows that for any $m, n \in \mathbb{N}$

$$\left(\sum_{k=1}^{N_1} \left(\frac{k}{N_1} \right)^{m/n} P_{1,k}^\varphi \right)^{n/m} = \left(\sum_{k=1}^{N_1} \left(\frac{k}{N_1} \right)^m P_{1,k}^\varphi \right)^{1/m} = \sum_{k=1}^{N_1} \frac{k}{N_1} P_{1,k}^\varphi$$

which implies

$$\left(\sum_{k=1}^{N_1} \frac{k}{N_1} P_{1,k}^\varphi \right)^{m/n} = \sum_{k=1}^{N_1} \left(\frac{k}{N_1} \right)^{m/n} P_{1,k}^\varphi.$$

It follows that for any positive rational number j

$$(\hat{n}_1^\varphi)^j (\hat{n}_1^\varphi)^{-j} q_{1,i}^\varphi = \sum_{k=1}^{N_1} \sum_{\ell=1}^{N_1} \left(\frac{k}{N_1} \right)^j \left(\frac{\ell}{N_1} \right)^{-j} P_{1,k}^\varphi P_{1,\ell}^\varphi q_{1,i}^\varphi = \sum_{k=1}^{N_1} P_{1,k}^\varphi q_{1,i}^\varphi = (\mathbb{1} - P_{1,0}^\varphi) q_{1,i}^\varphi = q_{1,i}^\varphi.$$

□

Using (4.8), we replace $q_{1,1}^\varphi$ with $(\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^\varphi$ in the second term of (4.7) and we get

$$\begin{aligned} & \left| i \left\langle \psi_{N_1, N_2, t}, \left[A_1(x_1, x_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle \right| \\ & \leq 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\ & \quad + 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) (\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^\varphi q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\ & \quad + 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right|. \end{aligned} \quad (4.9)$$

We start by looking at the first term of (4.9). Note

$$\begin{aligned} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) p_{1,2}^{\varphi_{1,t}} &= \frac{N_1 - 1}{N_1} p_{1,2}^{\varphi_{1,t}} V_1(x_2 - x_1) p_{1,2}^{\varphi_{1,t}} - p_{1,2}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)(x_1) p_{1,2}^{\varphi_{1,t}} \\ &= \frac{N_1 - 1}{N_1} |\varphi_{1,t}(x_2)\rangle \langle \varphi_{1,t}(x_2)| V_1(x_2 - x_1) |\varphi_{1,t}(x_2)\rangle \langle \varphi_{1,t}(x_2)| - p_{1,2}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)(x_1) \\ &= \left(1 - \frac{1}{N_1} \right) p_{1,2}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)(x_1) - p_{1,2}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)(x_1) \\ &= -\frac{1}{N_1} p_{1,2}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)(x_1). \end{aligned}$$

Hence,

$$\begin{aligned}
& 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
& \leq \frac{2}{N_1} \|(V_1 * |\varphi_{1,t}|^2)(x_1) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\| \|q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\| \\
& \leq \frac{2}{N_1} \|(V_1 * |\varphi_{1,t}|^2)(x_1) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\| \\
& \leq \frac{2}{N_1} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)^2(x_1) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle^{1/2}.
\end{aligned}$$

Note

$$\begin{aligned}
& \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)^2(x_1) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle^{1/2} \\
& = \left\langle \psi_{N_1, N_2, t}, |\varphi_{1,t}(x_1)\rangle \langle \varphi_{1,t}(x_1) | (V_1 * |\varphi_{1,t}|^2)^2(x_1) |\varphi_{1,t}(x_1)\rangle \langle \varphi_{1,t}(x_1) | \psi_{N_1, N_2, t} \right\rangle^{1/2} \\
& = \left\langle (V_1 * |\varphi_{1,t}|^2) \varphi_{1,t}, (V_1 * |\varphi_{1,t}|^2) \varphi_{1,t} \right\rangle^{1/2} \left\langle \psi_{N_1, N_2, t}, |\varphi_{1,t}(x_1)\rangle \langle \varphi_{1,t}(x_1) | \psi_{N_1, N_2, t} \right\rangle^{1/2} \\
& = \|(V_1 * |\varphi_{1,t}|^2) \varphi_{1,t}\| \|p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\| \\
& \leq \|(V_1 * |\varphi_{1,t}|^2)^2 \varphi_{1,t}\|_1^{1/2} \\
& \leq \|(V_1 * |\varphi_{1,t}|^2)^2\|_r^{1/2} \|\varphi_{1,t}^2\|_s^{1/2}
\end{aligned}$$

where in the last line we recalled $\frac{1}{s} + \frac{1}{r} = 1$ and used Hölder's inequality. Now, note, using Young's inequality $\|f * g\|_r \leq \|g\|_1 \|f\|_r$, we obtain

$$\|(V_1 * |\varphi_{1,t}|^2)^2\|_r = \|V_1 * |\varphi_{1,t}|^2\|_{2r}^2 \leq (\|\varphi_{1,t}^2\|_1 \|V_1\|_{2r})^2 = (\|\varphi_{1,t}\|_2^2 \|V_1\|_{2r})^2 = \|V_1\|_{2r}^2.$$

It follows that

$$\left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} (V_1 * |\varphi_{1,t}|^2)^2(x_1) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle^{1/2} \leq \|V_1\|_{2r} \|\varphi_{1,t}^2\|_s^{1/2} = \|V_1\|_{2r} \|\varphi_{1,t}\|_{2s}.$$

Therefore, we get the following bound for the first term of (4.9):

$$2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right| \leq \frac{2}{N_1} \|V_1\|_{2r} \|\varphi_{1,t}\|_{2s} = \frac{2}{N_1} C_1^{\varphi_{1,t}} \quad (4.10)$$

where

$$C_1^{\varphi_{1,t}} = \|V_1\|_{2r} \|\varphi_{1,t}\|_{2s}.$$

Now we estimate the second term of (4.9). Note

$$\begin{aligned}
& 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) (\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
& \leq 2 \left\| (\hat{n}_1^{\varphi_{1,t}})^{1/2} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \right\| \left\| (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\|.
\end{aligned}$$

Now,

$$\begin{aligned}
& \|(\hat{n}_1^{\varphi_{1,t}})^{1/2} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \\
&= \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A(x_1, x_2) \hat{n}_1^{\varphi_{1,t}} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\
&= \frac{N_1 - 2}{N_1} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A(x_1, x_2) q_{1,3}^{\varphi_{1,t}} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\
&\quad + \frac{2}{N_1} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \|(\hat{n}_1^{\varphi_{1,t}})^{1/2} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \\
&= \frac{N_1 - 2}{N_1} \left\langle q_{1,3}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}} q_{1,3}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\
&\quad + \frac{2}{N_1} \|q_{1,1}^{\varphi_{1,t}} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \\
&\leq \|p_{1,2}^{\varphi_{1,t}} q_{1,3}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \|p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}}\|_{\text{op}} + \frac{2}{N_1} \|p_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \|p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}}\|_{\text{op}}
\end{aligned}$$

where we used that for a bounded operator A

$$\frac{\langle x, Ax \rangle}{\|x\|} \leq \|Ax\| \quad \Rightarrow \quad \langle x, Ax \rangle \leq \|A\|_{\text{op}} \|x\|^2.$$

Now, note

$$\begin{aligned}
\|p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}}\|_{\text{op}} &= \| |\varphi_{1,t}(x_1)\rangle \langle \varphi_{1,t}(x_1)| A_1(x_1, x_2) |\varphi_{1,t}(x_1)\rangle \langle \varphi_{1,t}(x_1)| \|_{\text{op}} \\
&\leq \sup_{x_2} \|\varphi_{1,t}^2 A_1^2(\cdot, x_2)\|_1 \|p_{1,1}^{\varphi_{1,t}}\|_{\text{op}} \\
&\leq \sup_{x_2} \|\varphi_{1,t}^2\|_s \|A_1^2(\cdot, x_2)\|_r \\
&= \sup_{x_2} \|\varphi_{1,t}\|_{2s}^2 \|A_1(\cdot, x_2)\|_{2r}^2.
\end{aligned}$$

Now,

$$\begin{aligned}
\sup_{x_2} \|A_1(\cdot, x_2)\|_r &= \sup_{x_2} \left\| \frac{N_1 - 1}{N_1} V_1(x_2 - \cdot) - (V_1 * |\varphi_{1,t}|^2)(\cdot) \right\|_r \\
&\leq \frac{N_1 - 1}{N_1} \|V_1\|_r + \|V_1 * |\varphi_{1,t}|^2\|_r \\
&\leq \|V_1\|_r + \|\varphi_{1,t}^2\|_1 \|V_1\|_r = 2\|V_1\|_r.
\end{aligned}$$

It follows that

$$\|p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}}\|_{\text{op}} \leq \|\varphi_{1,t}\|_{2s}^2 (2\|V_1\|_{2r})^2 = 4(C_1^{\varphi_{1,t}})^2. \quad (4.11)$$

Therefore,

$$\begin{aligned}
\|(\hat{n}_1^{\varphi_{1,t}})^{1/2} A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 &\leq 4(C_1^{\varphi_{1,t}})^2 \|q_{1,3}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 + \frac{8}{N_1} (C_1^{\varphi_{1,t}})^2 \\
&= 4(C_1^{\varphi_{1,t}})^2 \alpha_1(t) + \frac{8}{N_1} (C_1^{\varphi_{1,t}})^2.
\end{aligned}$$

Next, note that

$$\begin{aligned}
N_1(N_1 - 1) \| (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \| &= N_1(N_1 - 1) \left\langle \psi_{N_1, N_2, t}, (\hat{n}_1^{\varphi_{1,t}})^{-1} q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\
&\leq \left\langle \psi_{N_1, N_2, t}, (\hat{n}_1^{\varphi_{1,t}})^{-1} \sum_{j,k=1}^{N_1} q_{1,j}^{\varphi_{1,t}} q_{1,k}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \\
&= N_1^2 \left\langle \psi_{N_1, N_2, t}, (\hat{n}_1^{\varphi_{1,t}})^{-1} (\hat{n}_1^{\varphi_{1,t}})^2 \psi_{N_1, N_2, t} \right\rangle = N_1^2 \alpha_1(t).
\end{aligned}$$

Also, note, for $N_1 > 1$,

$$\frac{N_1^2}{N_1(N_1 - 1)} = \frac{1}{1 - 1/N_1} = 1 + \frac{1}{N_1} + \frac{1}{N_1^2} + \dots = 1 + \frac{1}{N_1} \left(\frac{1}{1 - 1/N_1} \right) = 1 + \frac{1}{N_1 - 1} \leq 2.$$

Thus, we have the following bound for the second term of (4.9):

$$\begin{aligned}
&2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) (\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} q_{2,1}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
&\leq 2 \left(4 (C_1^{\varphi_{1,t}})^2 \alpha_1(t) + \frac{8}{N_1} (C_1^{\varphi_{1,t}})^2 \right)^{1/2} (2\alpha_1(t))^{1/2} \\
&\leq 2 C_1^{\varphi_{1,t}} \left(8\alpha_1(t) + \frac{16}{N_1} \right)^{1/2} (\alpha_1(t))^{1/2} \\
&\leq 8 (C_1^{\varphi_{1,t}}) \left(\alpha_1(t) + \frac{1}{N_1} \right). \tag{4.12}
\end{aligned}$$

Finally, we estimate the third term of (4.9). Note

$$\begin{aligned}
&2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
&\leq 2 \| A_1(x_1, x_2) p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \| \| q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \| \\
&\leq 2 \left\langle q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle^{1/2} \| q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \| \\
&\leq 2 (\| p_{1,1}^{\varphi_{1,t}} A_1^2(x_1, x_2) p_{1,1}^{\varphi_{1,t}} \|_{\text{op}})^{1/2} \| q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \|^2.
\end{aligned}$$

From (4.11), it follows that

$$2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} A_1(x_1, x_2) q_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} q_{2,1}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \leq 4 C_1^{\varphi_{1,t}} \alpha_1(t). \tag{4.13}$$

Now, from the bounds (4.10), (4.12), and (4.13), we can bound the first term of (4.6), i.e. the term (4.9), as follows:

$$\begin{aligned}
&\left| i \left\langle \psi_{N_1, N_2, t}, \left[A_1(x_1, x_2), q_{1,1}^{\varphi_{1,t}} \right] q_{2,1}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
&\leq \frac{2}{N_1} C_1^{\varphi_{1,t}} + 8 (C_1^{\varphi_{1,t}}) \left(\alpha_1(t) + \frac{1}{N_1} \right) + 4 C_1^{\varphi_{1,t}} \alpha_1(t) \\
&\leq 12 C_1^{\varphi_{1,t}} \left(\alpha_1(t) + \frac{1}{N_1} \right). \tag{4.14}
\end{aligned}$$

Similarly, the second term of (4.6)

$$\left| i \left\langle \psi_{N_1, N_2, t}, \left[A_2(y_1, y_2), q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle \right| \leq 12 C_2^{\varphi_{2,t}} \left(\alpha_2(t) + \frac{1}{N_2} \right) \tag{4.15}$$

where

$$C_2^{\varphi_{2,t}} = \|V_2\|_{2r} \|\varphi_{2,t}\|_{2s}.$$

Next, we look at the third term of (4.6). Proceeding similarly to how we obtained (4.9), we get

$$\begin{aligned} & \left| i \left\langle \psi_{N_1, N_2, t}, \left[B_1(x_1, y_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle \right| \\ & \leq 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\ & \quad + 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) (\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\ & \quad + 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right|. \end{aligned} \tag{4.16}$$

In the first term of (4.16), we have

$$\begin{aligned} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) p_{2,2}^{\varphi_{2,t}} &= p_{2,2}^{\varphi_{2,t}} \left(\frac{N_2 - 1}{N_1 + N_2} W(x_1 - y_2) - \frac{N_2}{N_1 + N_2} (W * |\varphi_{2,t}|^2)(x_1) \right) p_{2,2}^{\varphi_{2,t}} \\ &= \frac{N_2 - 1}{N_1 + N_2} p_{2,2}^{\varphi_{2,t}} (W * |\varphi_{2,t}|^2)(x_1) - \frac{1}{a + 1} p_{2,2}^{\varphi_{2,t}} (W * |\varphi_{2,t}|^2)(x_1) \\ &= -\frac{1}{N_1 + N_2} p_{2,2}^{\varphi_{2,t}} (W * |\varphi_{2,t}|^2)(x_1) \end{aligned}$$

where we used the fact that $N_1 = aN_2$. Thus,

$$\begin{aligned} & 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\ & \leq \frac{2}{N_1 + N_2} \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} (W * |\varphi_{2,t}|^2)(x_1) q_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t} \right\rangle \right|. \end{aligned}$$

Similarly to how we obtained (4.10), we get the following bound for the first term of (4.16):

$$2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \leq \frac{2}{N_1 + N_2} D^{\varphi_{1,t}} \tag{4.17}$$

where

$$D^{\varphi_{1,t}} = \|W\|_{2r} \|\varphi_{1,t}\|_{2s}.$$

Now, the second term of (4.16)

$$\begin{aligned} & 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) (\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\ & \leq 2 \|(\hat{n}_1^{\varphi_{1,t}})^{1/2} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\| \|(\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|. \end{aligned}$$

Note

$$\begin{aligned}
& \|(\hat{n}_1^{\varphi_{1,t}})^{1/2} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 \\
&= \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) \hat{n}_1^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \\
&= \frac{N_1 - 1}{N_1} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,2}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \\
&\quad + \frac{1}{N_1} \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,1}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \\
&= \frac{N_1 - 1}{N_1} \left\langle q_{1,2}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} B_1^2(x_1, y_2) p_{1,1}^{\varphi_{1,t}} q_{1,2}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \\
&\quad + \frac{1}{N_2} \|q_{1,1}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 \\
&\leq \|p_{2,2}^{\varphi_{2,t}} q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \|p_{1,1}^{\varphi_{1,t}} B_1^2(x_1, y_2) p_{1,1}^{\varphi_{1,t}}\|_{\text{op}} \\
&\quad + \frac{1}{N_1} \|p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 \|p_{1,1}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|_{\text{op}}.
\end{aligned}$$

Now, proceeding similarly to how we obtained (4.11), we get

$$\|p_{1,1}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|_{\text{op}} \leq 4(D^{\varphi_{1,t}})^2.$$

Hence,

$$\begin{aligned}
& \|(\hat{n}_1^{\varphi_{1,t}})^{1/2} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 \\
&\leq \|q_{1,2}^{\varphi_{1,t}} \psi_{N_1, N_2, t}\|^2 \|\varphi_{1,t}\|_{2s}^2 (2\|W\|_{2r})^2 + \frac{1}{N_1} \|p_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 \|\varphi_{1,t}\|_{2s}^2 (2\|W\|_{2r})^2 \\
&\leq 4\alpha_1(t) (D^{\varphi_{1,t}})^2 + \frac{4}{N_1} (D^{\varphi_{1,t}})^2.
\end{aligned}$$

Also,

$$\|(\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 = \alpha_2(t).$$

Thus, we get the following bound for the second term of (4.16):

$$\begin{aligned}
& 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} p_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) (\hat{n}_1^{\varphi_{1,t}})^{1/2} (\hat{n}_1^{\varphi_{1,t}})^{-1/2} q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
&\leq 2 \left(4\alpha_1(t) (D_1^{\varphi_{1,t}})^2 + \frac{4}{N_1} (D_1^{\varphi_{1,t}})^2 \right)^{1/2} (\alpha_2(t))^{1/2} \\
&\leq 4D^{\varphi_{1,t}} \left(\alpha_1(t) + \frac{1}{N_1} \right)^{1/2} (\alpha_2(t))^{1/2} \\
&\leq 4D^{\varphi_{1,t}} \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} \right). \tag{4.18}
\end{aligned}$$

The third term of (4.16)

$$\begin{aligned}
& 2 \left| \left\langle \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} B_1(x_1, y_2) q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle \right| \\
& \leq 2 \|B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\| \|q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\| \\
& = 2 \left\langle q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}, p_{1,1}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t} \right\rangle^{1/2} \|q_{1,1}^{\varphi_{1,t}} q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\| \\
& \leq 2 \|q_{2,2}^{\varphi_{2,t}} \psi_{N_1, N_2, t}\|^2 \|p_{1,1}^{\varphi_{1,t}} B_1(x_1, y_2) p_{1,1}^{\varphi_{1,t}}\|_{\text{op}}^{1/2} \\
& \leq 4 \alpha_2(t) D^{\varphi_{1,t}}.
\end{aligned} \tag{4.19}$$

Thus, from (4.17), (4.18), and (4.19), we get that the third term of (4.6)

$$\begin{aligned}
& \left| i \left\langle \psi_{N_1, N_2, t}, \left[B_1(x_1, y_2), q_{1,1}^{\varphi_{1,t}} \right] \psi_{N_1, N_2, t} \right\rangle \right| \\
& \leq \frac{2}{N_1 + N_2} D^{\varphi_{1,t}} + 4 D^{\varphi_{1,t}} \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} \right) + 4 \alpha_2(t) D^{\varphi_{1,t}}.
\end{aligned} \tag{4.20}$$

Similarly, the fourth term of (4.6)

$$\begin{aligned}
& \left| i \left\langle \psi_{N_1, N_2, t}, \left[B_2(x_1, y_2), q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle \right| \\
& \leq \frac{2}{N_1 + N_2} D^{\varphi_{2,t}} + 4 D^{\varphi_{2,t}} \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} \right) + 4 \alpha_1(t) D^{\varphi_{2,t}}.
\end{aligned} \tag{4.21}$$

Proceeding similarly, one can show that the last term of (4.6)

$$\begin{aligned}
& \left| \frac{i}{N_1 + N_2} \left\langle \psi_{N_1, N_2, t}, \left[W(x_1 - y_1), q_{1,1}^{\varphi_{1,t}} + q_{2,1}^{\varphi_{2,t}} \right] \psi_{N_1, N_2, t} \right\rangle \right| \\
& \leq \frac{2}{N_1 + N_2} \left(D^{\varphi_{1,t}} + D^{\varphi_{2,t}} + (D^{\varphi_{1,t}} + D^{\varphi_{2,t}}) \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} \right) \right. \\
& \quad \left. + (D^{\varphi_{1,t}} + D^{\varphi_{2,t}}) \alpha_{N_1, N_2}(t) \right).
\end{aligned} \tag{4.22}$$

Therefore, from the bounds (4.14), (4.15), (4.21), (4.20), (4.22), we get

$$\begin{aligned}
\left| \frac{\partial}{\partial t} \alpha_{N_1, N_2}(t) \right| & \leq 20 (C_1^{\varphi_{1,t}} + C_2^{\varphi_{2,t}} + D^{\varphi_{1,t}} + D^{\varphi_{2,t}}) \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} + \frac{1}{N_2} \right) \\
& \quad + \frac{4}{N_1 + N_2} (D^{\varphi_{1,t}} + D^{\varphi_{2,t}}) \left(1 + \alpha_{N_1, N_2}(t) + \frac{1}{N_1} \right) \\
& \leq 24 (C_1^{\varphi_{1,t}} + C_2^{\varphi_{2,t}} + D^{\varphi_{1,t}} + D^{\varphi_{2,t}}) \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} + \frac{1}{N_2} \right) \\
& = C(t) \left(\alpha_{N_1, N_2}(t) + \frac{1}{N_1} + \frac{1}{N_2} \right),
\end{aligned}$$

where $C(t) = 24 (C_1^{\varphi_{1,t}} + C_2^{\varphi_{2,t}} + D^{\varphi_{1,t}} + D^{\varphi_{2,t}})$. Now define $f(t) = \alpha_{N_1, N_2}(t) + \frac{1}{N_1} + \frac{1}{N_2}$. Then

$$\frac{\partial}{\partial t} f(t) = \frac{\partial}{\partial t} \alpha_{N_1, N_2}(t) \leq C(t) f(t).$$

Using Gronwall's Lemma, $f(t) \leq f(0) e^{\int_0^t C(\tau) d\tau}$. That is,

$$\alpha_{N_1, N_2}(t) + \frac{1}{N_1} + \frac{1}{N_2} \leq \left(\alpha_{N_1, N_2}(0) + \frac{1}{N_1} + \frac{1}{N_2} \right) e^{\int_0^t C(\tau) d\tau}$$

which implies the theorem. \square

4.4 Proposition (4.4.1)

Now we show that $\lim_{N_1, N_2 \rightarrow \infty} \alpha_{N_1, N_2} = 0$ implies convergence of the one particle marginal density and vice versa.

Proposition 4.4.1. *Let $\varphi_1, \varphi_2 \in L^2$ and let $\psi_{N_1, N_2} = \psi_{N_1} \otimes \psi_{N_2} \in L^2(\mathbb{R}^{3N_1}) \otimes L^2(\mathbb{R}^{3N_2})$ be such that $\psi_{N_1} \in L_s^2(\mathbb{R}^{3N_1})$ and $\psi_{N_2} \in L_s^2(\mathbb{R}^{3N_2})$. Let $\Gamma_{N_1, N_2}^{(1)}$ be the one particle marginal density associated with ψ_{N_1, N_2} . Then*

$$\lim_{N_1, N_2 \rightarrow \infty} \left\langle \psi_{N_1, N_2}, (\hat{n}_1^{\varphi_1} + \hat{n}_2^{\varphi_2}) \psi_{N_1, N_2} \right\rangle = 0 \Leftrightarrow \lim_{N_1, N_2 \rightarrow \infty} \Gamma_{N_1, N_2}^{(1)} = |\varphi_1\rangle\langle\varphi_1| \otimes |\varphi_2\rangle\langle\varphi_2| \text{ in operator norm.}$$

Proof. (\Rightarrow) Let $\lim_{N_1, N_2 \rightarrow \infty} \left\langle \psi_{N_1, N_2}, (\hat{n}_1^{\varphi_1} + \hat{n}_2^{\varphi_2}) \psi_{N_1, N_2} \right\rangle = 0$. Then by (4.5) and symmetry,

$$\lim_{N_1, N_2 \rightarrow \infty} \left(\langle \psi_{N_1, N_2}, q_{1,1} \psi_{N_1, N_2} \rangle + \langle \psi_{N_1, N_2}, q_{2,1} \psi_{N_1, N_2} \rangle \right) = 0.$$

This implies that $\lim_{N_1, N_2 \rightarrow \infty} \|q_{1,1} \psi_{N_1, N_2}\| = 0$ and $\lim_{N_1, N_2 \rightarrow \infty} \|q_{2,1} \psi_{N_1, N_2}\| = 0$, and also that $\lim_{N_1, N_2 \rightarrow \infty} \|p_{1,1} \psi_{N_1, N_2}\| = 1$ and $\lim_{N_1, N_2 \rightarrow \infty} \|p_{2,1} \psi_{N_1, N_2}\| = 1$. It follows that

$$\lim_{N_1, N_2 \rightarrow \infty} \|q_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 = 0. \quad (4.23)$$

Also, since

$$\|p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 + 1 = \|q_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 + \|p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 + \|p_{1,1}^{\varphi_1} \psi_{N_1, N_2}\|^2$$

it follows that

$$\lim_{N_1, N_2 \rightarrow \infty} \|p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\| = 1. \quad (4.24)$$

Also, $p_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} + q_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} = 1 - q_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} - p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2}$ and thus

$$\lim_{N_1, N_2 \rightarrow \infty} \left\| \left(p_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} + q_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \right) \psi_{N_1, N_2} \right\|^2 = \lim_{N_1, N_2 \rightarrow \infty} \left(1 - \|q_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2}\|^2 - \|p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2}\|^2 \right) = 0. \quad (4.25)$$

Now, the one particle marginal density $\Gamma_{N_1, N_2}^{(1)}$ has kernel given by

$$\begin{aligned} \gamma(x, y; x', y') &= \int \psi_{N_1, N_2}(x, x_2, \dots, x_{N_1}, y, y_2, \dots, y_{N_2}) \\ &\quad \cdot \bar{\psi}_{N_1, N_2}(x', x_2, \dots, x_{N_1}, y', y_2, \dots, y_{N_2}) dx_2 \cdots dx_{N_1} dy_2 \cdots dy_{N_2}. \end{aligned}$$

Using the fact that $(p_{1,1}^{\varphi_1} + q_{1,1}^{\varphi_1})(p_{2,1}^{\varphi_2} + q_{2,1}^{\varphi_2}) = 1$, we get

$$\gamma(x, y; x', y') = \gamma_1(x, y; x', y') + \gamma_2(x, y; x', y') + \gamma_3(x, y; x', y')$$

where

$$\begin{aligned}
\gamma_1(x, y; x', y') &= \int p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}(x, x_2, \dots, x_{N_1}, y, y_2, \dots, y_{N_2}) \\
&\quad \cdot \overline{p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}}(x', x_2, \dots, x_{N_1}, y', y_2, \dots, y_{N_2}) dx_2 \cdots dx_{N_1} dy_2 \cdots dy_{N_2}, \\
\gamma_2(x, y; x', y') &= \int (p_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} + q_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2}) \psi_{N_1, N_2}(x, x_2, \dots, x_{N_1}, y, y_2, \dots, y_{N_2}) \\
&\quad \cdot \overline{(p_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} + q_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2}) \psi_{N_1, N_2}}(x', x_2, \dots, x_{N_1}, y', y_2, \dots, y_{N_2}) dx_2 \cdots dx_{N_1} dy_2 \cdots dy_{N_2}, \\
\gamma_3(x, y; x', y') &= \int q_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} \psi_{N_1, N_2}(x, x_2, \dots, x_{N_1}, y, y_2, \dots, y_{N_2}) \\
&\quad \cdot \overline{q_{1,1}^{\varphi_1} q_{2,1}^{\varphi_2} \psi_{N_1, N_2}}(x', x_2, \dots, x_{N_1}, y', y_2, \dots, y_{N_2}) dx_2 \cdots dx_{N_1} dy_2 \cdots dy_{N_2}.
\end{aligned}$$

Therefore, we can write $\Gamma_{N_1, N_2}^{(1)}$ as

$$\Gamma_{N_1, N_2}^{(1)} = \Gamma_1 + \Gamma_2 + \Gamma_3 \quad (4.26)$$

where Γ_1 has kernel γ_1 , Γ_2 has kernel γ_2 , and Γ_3 has kernel γ_3 .

Since $\|\Gamma_{N_1, N_2}^{(1)}\|_{\text{op}} \leq \|\Gamma_{N_1, N_2}^{(1)}\|_{\text{HS}} = \|\gamma\|_{L^2}$, we see that

$$\lim_{N_1, N_2 \rightarrow \infty} \|\Gamma_2 + \Gamma_3\|_{\text{op}} \leq \lim_{N_1, N_2 \rightarrow \infty} (\|\Gamma_2\|_{\text{op}} + \|\Gamma_3\|_{\text{op}}) \leq \lim_{N_1, N_2 \rightarrow \infty} (\|\gamma_2\| + \|\gamma_3\|) = 0$$

where we used (4.25) and (4.23) to get the last equality. Therefore, $\lim_{N_1, N_2 \rightarrow \infty} \|\Gamma_2 + \Gamma_3\|_{\text{op}} = 0$. Now

$$\begin{aligned}
\lim_{N_1, N_2 \rightarrow \infty} (\Gamma_1 f)(x, y) &= \lim_{N_1, N_2 \rightarrow \infty} \int p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}(x, x_2, \dots, x_{N_1}, y, y_2, \dots, y_{N_2}) \\
&\quad \cdot \overline{p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}}(x', x_2, \dots, x_{N_1}, y', y_2, \dots, y_{N_2}) \\
&\quad \cdot f(x', y') dx' dx_2 \cdots dx_{N_1} dy' dy_2 \cdots dy_{N_2} \\
&= \lim_{N_1, N_2 \rightarrow \infty} \varphi_1(x) \varphi_2(y) \int \overline{\varphi_2}(x') \overline{\varphi_2}(y') f(x', y') dx' dy' \\
&\quad \cdot \int \left| \left\langle \varphi_1(x) \varphi_2(y), \psi_{N_1, N_2}(x, x_2, \dots, x_{N_1}, y, y_2, \dots, y_{N_2}) \right\rangle \right|^2 dx_2 \cdots dx_{N_1} dy_2 \cdots dy_{N_2} \\
&= \lim_{N_1, N_2 \rightarrow \infty} \|p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 |\varphi_1(x)\langle \varphi_1(x) | \otimes |\varphi_2(y)\langle \varphi_2(y) | f(x, y) \\
&= |\varphi_1(x)\langle \varphi_1(x) | \otimes |\varphi_2(y)\langle \varphi_2(y) | f(x, y).
\end{aligned}$$

It follows that

$$\lim_{N_1, N_2 \rightarrow \infty} \|\Gamma_{N_1, N_2}^{(1)} - |\varphi_1\rangle\langle \varphi_1 | \otimes |\varphi_2\rangle\langle \varphi_2 |\|_{\text{op}} \leq \lim_{N_1, N_2 \rightarrow \infty} (\|\Gamma_1 - |\varphi_1\rangle\langle \varphi_1 | \otimes |\varphi_2\rangle\langle \varphi_2 |\|_{\text{op}} + \|\Gamma_2 + \Gamma_3\|_{\text{op}}) = 0.$$

(\Leftarrow) Suppose $\lim_{N_1, N_2 \rightarrow \infty} \Gamma_{N_1, N_2}^{(1)} = |\varphi_1\rangle\langle \varphi_1 | \otimes |\varphi_2\rangle\langle \varphi_2 |$ in operator norm. It follows that

$$\lim_{N_1, N_2 \rightarrow \infty} \left\langle \varphi_1 \otimes \varphi_2, \Gamma_{N_1, N_2}^{(1)} \varphi_1 \otimes \varphi_2 \right\rangle = 1.$$

Writing $\Gamma_{N_1, N_2}^{(1)}$ as $\Gamma_{N_1, N_2}^{(1)} = \Gamma_1 + \Gamma_2 + \Gamma_3$ as in (4.26) and using the fact that $q_{1,1}^{\varphi_1} \varphi_1(x) = 0$ and $q_{2,1}^{\varphi_2} \varphi_2(y) = 0$, we see that $(\Gamma_2 + \Gamma_3) \varphi_1(x) \varphi_2(y) = 0$. Hence,

$$\Gamma_{N_1, N_2}^{(1)} \varphi_1(x) \varphi_2(y) = \Gamma_1 \varphi_1(x) \varphi_2(y) = \|p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 \varphi_1(x) \varphi_2(y).$$

It follows that

$$\lim_{N_1, N_2 \rightarrow \infty} \|p_{1,1}^{\varphi_1} p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 = 1.$$

Since $\psi_{N_1, N_2} = \psi_{N_1} \psi_{N_2}$, this implies $\lim_{N_1, N_2 \rightarrow \infty} \|p_{1,1}^{\varphi_1} \psi_{N_1, N_2}\|^2 = 1$ and $\lim_{N_1, N_2 \rightarrow \infty} \|p_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 = 1$. Hence

$$\lim_{N_1, N_2 \rightarrow \infty} \|q_{1,1}^{\varphi_1} \psi_{N_1, N_2}\|^2 = 0 \quad \text{and} \quad \lim_{N_1, N_2 \rightarrow \infty} \|q_{2,1}^{\varphi_2} \psi_{N_1, N_2}\|^2 = 0.$$

So $\lim_{N_1, N_2 \rightarrow \infty} \langle \psi_{N_1, N_2}, (q_{1,1} + q_{2,1}) \psi_{N_1, N_2} \rangle = 0$, and therefore

$$\lim_{N_1, N_2 \rightarrow \infty} \langle \psi_{N_1, N_2}, (\hat{n}_1^{\varphi_1} + \hat{n}_2^{\varphi_2}) \psi_{N_1, N_2} \rangle = 0.$$

□

4.5 Conclusion

Theorem (4.1.2) now follows. Clearly $\alpha_{N_1, N_2}(0) = 0$ for our initial wave function ψ_{N_1, N_2} given in (4.1). Let $T < \infty$ be fixed. Then, for all $\tau < T$, we have that $\alpha(\tau) \rightarrow 0$ as $N_1, N_2 \rightarrow \infty$, by Theorem (4.3.1). Now, by Proposition (4.4.1), we see

$$\lim_{N_1, N_2 \rightarrow \infty} \Gamma_{N_1, N_2, \tau}^{(1)} = |\varphi_{1, \tau}\rangle\langle\varphi_{1, \tau}| \otimes |\varphi_{2, \tau}\rangle\langle\varphi_{2, \tau}| \quad \text{in operator norm.}$$

Remark 4.5.1. Note convergence of $\Gamma_{N_1, N_2, \tau}^{(1)}$ in operator norm is equivalent to convergence in trace norm since the operator $|\varphi_{1, \tau}\rangle\langle\varphi_{1, \tau}| \otimes |\varphi_{2, \tau}\rangle\langle\varphi_{2, \tau}|$ is a rank one projection. (See Remark (3.3.2).)

REFERENCES

- [1] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, 2nd edition, Springer-Verlag, 1997.
- [2] H. Cohn and A. Kumar, *Algorithmic design of self-assembling structures*, PNAS **106** (2009), 9570-9575.
- [3] P.A.M. Dirac, *Principles of Quantum Mechanics*, Oxford University Press, Oxford, 1958.
- [4] A. Elgart, L. Erdős, and B. Schlein, *Gross-Pitaevskii equation as the mean field limit of weakly coupled bosons*, Arch. Rat. Mech. Anal., **179** (2006), no. 2, 265-283.
- [5] L. Erdős, B. Schlein, and H. Yao, *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate*, Annals of Mathematics, **172** (2010), no. 1, 291-370.
- [6] L. Erdős, B. Schlein, and H. Yao, *Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential*, J. Amer. Math. Soc., **22** (2009), 1099-1156.
- [7] W.N. Everitt and L. Markus, *Infinite Dimensional Symplectic Spaces*, American Mathematical Soc., 2004.
- [8] J. Fröhlich, A. Knowles, and A. Pizzo, *Atomism and quantization*, J. Phys. A: Math. Theor., **40** (2007), 3033-3045.
- [9] K. Hepp, *The classical limit for quantum mechanical correlation functions*, Commun. Math. Phys., **35** (1974), 265-277.
- [10] D.D. Holm and V. Putkaradze, *Aggregation of finite-size particles with variable mobility*, Phys. Rev. Lett. **95** (2005), 226106.
- [11] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.
- [12] A. Knowles and P. Pickl, *Mean Field Dynamics: Singular Potentials and Rate of Convergence*, Commun. Math. Phys., **298** (2010), no. 1, 101-138.
- [13] T. Liu, *Hydrophilic macroionic solutions: What happens when soluble ions reach the size of nanometer scale?*, Langmuir **26** (2009), 9202-9213.
- [14] T. Liu, M. Langston, D. Li, J.M. Pigga, C. Pichon, A.M. Todea, and A. Müller, *Self-recognition among different polyprotic macroions during assembly processes in dilute solutions*, Science **331** (2011), 1590-1592.
- [15] A. Mackey, T. Kolokolnikov, and A. Bertozzi, *Two-Species Particle Aggregation and Stability of Co-Dimension One Solutions*, preprint.
- [16] M. Merkli, *The Ideal Quantum Gas*, Lectures notes (2005).
- [17] P. Pickl, *A simple derivation of mean field limits for quantum systems*, Lett. Math. Phys., **97** (2010), no. 2, 151-164.
- [18] P. Pickl, *On the Time Dependent Gross Pitaevskii- and Hartree equation*, arXiv:0808.1178v1 (2008).
- [19] I. Rodnianski and B. Schlein, *Quantum Fluctuations and Rate of Convergence Towards Mean Field Dynamics*, Commun. Math. Phys., **291** (2009), 31-61.

- [20] B. Simon and M. Reed, *Methods of Modern Mathematical Physics I: Functional Analysis. Revised and Enlarged Edition*, Academic Press, 1980.
- [21] L.A. Takhtajan, *Quantum Mechanics for Mathematicians*, American Mathematical Society, 2008.

APPENDIX A

MATHEMATICAL & PHYSICAL PRELIMINARIES

A.1 Operators

In this section, we start by discussing basic notions and definitions that relate to operators on Hilbert spaces. Then we go on to discuss specific types of operators such as projections, integral operators, and density operators. Much of the information from this section can be found in books on operator theory or functional analysis; for example, see [11] and [20].

A.1.1 Operators on a Hilbert space

Definition A.1.1. A *Banach space* is a complete normed vector space.

Definition A.1.2. A *Hilbert space* is a complete vector space equipped with an inner product.

Remark A.1.3. Note that a Hilbert space is a Banach space equipped with an inner product, which induces the norm.

Example A.1.4.

1. The space \mathbb{R}^n is a Hilbert space with the inner product given by the dot product. That is, if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, then

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

2. The space $L^2[a, b]$ is a Hilbert space with $\langle x, y \rangle = \int_a^b x(t) \overline{y(t)} dt$.

Definition A.1.5. The Sobolev space on $\Omega \subset \mathbb{R}^n$ of order 1 is a Hilbert space given by

$$H^1(\Omega) = \left\{ f \in L^2(\Omega), \frac{\partial}{\partial x_i} f \in L^2(\Omega), i = 1, 2, \dots, n \right\}.$$

with inner product

$$\langle f, g \rangle = \int_{\Omega} \left(fg + \sum_{i=1}^n \frac{\partial}{\partial x_i} f \frac{\partial}{\partial x_i} g \right) dx.$$

Definition A.1.6. An *operator* is a mapping from one vector space to another. If V is a vector space and $T : V \rightarrow V$ is a mapping from V to V , we say that T is an *operator on V* .

Definition A.1.7. Let T be an operator on some Hilbert space \mathcal{H} . We use $D(T)$ to denote the domain of T which is given by

$$D(T) = \{x \in \mathcal{H} : \|Tx\| < \infty\}.$$

Definition A.1.8. Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces, and let $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be an operator. The *operator norm* of T is given by

$$\|T\|_{\text{op}} = \sup_{\substack{h \in \mathcal{H}_1 \\ \|h\|_{\mathcal{H}_1} \leq 1}} \|Th\|_{\mathcal{H}_2}.$$

Usually if T is an operator, $\|T\|$ denotes the operator norm of T . If $\|T\|_{\text{op}} < \infty$, T is said to be *bounded*.

Definition A.1.9. We let $\mathcal{B}(\mathcal{H})$ denote the family of bounded linear operators on a Hilbert space \mathcal{H} .

Definition A.1.10. Let T be a densely defined operator on some Hilbert space \mathcal{H} . Then we define

$$\varphi \in D(T^*) \iff \text{There exists a } \rho \in \mathcal{H} \text{ such that } \langle \varphi, T\psi \rangle = \langle \rho, \psi \rangle \text{ for all } \psi \in D(T). \quad (\text{A.1})$$

If (A.1) holds, then the *adjoint* T^* of T is defined by setting $T^*\varphi = \rho$.

Definition A.1.11. An operator T is *self-adjoint* if $T = T^*$. This means that $D(T) = D(T^*)$ and for any $\psi \in D(T)$ we have that $T\psi = T^*\psi$.

Definition A.1.12. A linear operator T on a Hilbert space \mathcal{H} is *Hermitian* if

$$\langle \psi, T\varphi \rangle = \langle T\psi, \varphi \rangle$$

for all $\psi, \varphi \in D(T)$. Note, in particular, this means $D(T) \subseteq D(T^*)$.

Remark A.1.13. Note that a self-adjoint operator is Hermitian, but the converse is not necessarily true. In the case of bounded operators, Hermitian and self-adjoint are interchangeable.

Example A.1.14. Consider the space $C^1([0, 1])$, the set of all continuously differentiable functions on $[0, 1]$, with inner product given by

$$\langle f, g \rangle = \int_0^1 \bar{f}(x)g(x)dx.$$

Now, let T be the operator on $C^1([0, 1])$ defined by

$$T(f)(x) = i \frac{d}{dx} f(x).$$

Let us look at $\langle f, Tg \rangle$. Using integration by parts, we get

$$\langle f, Tg \rangle = \int_0^1 \bar{f}(x) i \frac{d}{dx} g(x) dx = i (\bar{f}(1)g(1) - \bar{f}(0)g(0)) + \langle Tf, g \rangle.$$

If we try, for example, letting

$$D(T) = C^1([0, 1]) = D(T^*)$$

then we do not have $\langle f, Tg \rangle = \langle Tf, g \rangle$, since not all functions $g \in C^1([0, 1])$ have the property that $g(0) = g(1) = 0$. On the other hand, if we let

$$D(T) = \{g \in C^1([0, 1]) : g(0) = g(1) = 0\}, \quad D(T^*) = C^1([0, 1])$$

then we do have $\langle f, Tg \rangle = \langle Tf, g \rangle$, but $D(T) \neq D(T^*)$, and so T is Hermitian but not self-adjoint.

Definition A.1.15. Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces. An *isomorphism* from \mathcal{H}_1 onto \mathcal{H}_2 is a bijective mapping $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$T(\alpha x) = \alpha T(x), \quad T(x + y) = T(x) + T(y)$$

for all $x, y \in \mathcal{H}_1$ and any scalar α . Then \mathcal{H}_1 and \mathcal{H}_2 are said to be *isomorphic*.

Definition A.1.16. A linear operator T on a Hilbert space \mathcal{H} is called *positive* if for all $\psi \in D(T)$,

$$\langle T\psi, \psi \rangle \geq 0.$$

Often we will write $T \geq 0$ if T is positive.

Definition A.1.17. Let \mathcal{H} be a separable Hilbert space and let $\{\psi_n\}_{n \geq 0}$ be an orthonormal basis for \mathcal{H} . Then for any operator $T \in \mathcal{B}(\mathcal{H})$ we define the *trace* of T to be

$$Tr(T) = \sum_{n \geq 0} \langle \psi_n, T\psi_n \rangle$$

Definition A.1.18. An operator T is called *trace class* if and only if $Tr|T| < \infty$. The family of all trace class operators is denoted by \mathcal{T}_1 . If T is a trace class operator, then T has a norm given by $\|T\|_{\mathcal{T}_1} = Tr|T|$, called the trace-norm of T .

Remark A.1.19. We list some well-known properties of the trace. Let $A, B \in \mathcal{T}_1$. Then

1. $Tr(A)$ is independent of the choice $\{\psi_n\}$ of the orthonormal basis.
2. $Tr(A + B) = Tr(A) + Tr(B)$.
3. $Tr(\lambda A) = \lambda Tr(A)$ for all $\lambda \geq 0$.

Definition A.1.20. An operator T is called *Hilbert-Schmidt* if and only if $Tr(T^*T) < \infty$. The family of all Hilbert-Schmidt operators is denoted by \mathcal{T}_2 . If $T \in \mathcal{T}_2$, then T has a Hilbert-Schmidt norm given by $\|T\|_{HS} = \sqrt{Tr(T^*T)}$.

Proposition A.1.21. Let $T \in \mathcal{T}_1$ be a linear operator on a separable Hilbert space \mathcal{H} . Then its operator norm is bounded above by its Hilbert-Schmidt norm which is bounded above by its trace-class norm; that is

$$\|T\|_{op} \leq \|T\|_{HS} \leq \|T\|_{\mathcal{T}_1}. \quad (\text{A.2})$$

Proof. Let $T \in \mathcal{T}_1$ be a linear operator on a separable Hilbert space \mathcal{H} . Then there exist orthonormal bases $\{v_i\}$ and $\{v'_i\}$ such that for any $h \in \mathcal{H}$

$$T(h) = \sum_{i \geq 1} \alpha_i \langle h, v_i \rangle v'_i \quad (\text{A.3})$$

where $\alpha_i > 0$.

To show (A.3), note T^*T is a positive, Hermitian operator in \mathcal{T}_1 . Thus, we can find an orthonormal basis of eigenvectors v_i of T^*T with non-negative eigenvalues λ_i . Let $\{\alpha_i\}_{i \geq 1}$ be a set where $\alpha_i = \sqrt{\lambda_i}$ for each nonzero eigenvalue λ_i (these are the nonzero singular values of T). Now, note for any $h \in \mathcal{H}$, we can write h as $h = \sum_{i \geq 1} c_i v_i$ where c_i is a scalar. It follows that

$$T^*T(h) = \sum_{i \geq 1} c_i T^*T(v_i) = \sum_{i \geq 1} c_i \lambda_i v_i.$$

Observe that $\langle h, v_i \rangle = c_i$. Thus, we can write $T^*T(h)$ as

$$T^*T(h) = \sum_{i \geq 1} \alpha_i^2 \langle h, v_i \rangle v_i.$$

Now, let $v'_i = \alpha_i^{-1} T(v_i)$. Note $\{v'_i\}$ forms an orthonormal set. Observe $\alpha_i^2 v_i = T^*T(v_i) = \alpha_i T^*(v'_i)$. Hence, $T^*(v'_i) = \alpha_i v_i$. It follows that we can write T^* as

$$T^* = \sum_{i \geq 1} \alpha_i \langle \cdot, v'_i \rangle v_i.$$

Hence

$$\langle T v_\ell, v'_k \rangle = \langle v_\ell, T^* v'_k \rangle = \alpha_k \delta_{\ell k}.$$

Thus we can write T as

$$T = \sum_{i \geq 1} \alpha_i \langle \cdot, v_i \rangle v'_i$$

which shows (A.3). Now, note that

$$\|T\|_{op} = \sup_i \alpha_i, \quad \|T\|_{HS} = \left(\sum_{i \geq 1} \alpha_i^2 \right)^{1/2}, \quad \|T\|_{\mathcal{T}_1} = \sum_{i \geq 1} \alpha_i$$

from which the proposition follows. \square

A.1.2 Projections

Definition A.1.22. An *projection* \mathcal{P} on a Hilbert space \mathcal{H} is a linear map $\mathcal{P} : \mathcal{H} \rightarrow \mathcal{H}$ such that $\mathcal{P}^2 = \mathcal{P}$.

Definition A.1.23. An *orthogonal projection* \mathcal{P} on a Hilbert space \mathcal{H} is a linear map $\mathcal{P} : \mathcal{H} \rightarrow \mathcal{H}$ that satisfies

$$\mathcal{P}^2 = \mathcal{P} \quad \text{and} \quad \langle \mathcal{P}\varphi, \psi \rangle = \langle \varphi, \mathcal{P}\psi \rangle \quad \text{for all } \varphi, \psi \in D(\mathcal{P}).$$

(Note the second condition is the same as saying \mathcal{P} is Hermitian.)

Proposition A.1.24. Let \mathcal{P} be an orthogonal projection on a Hilbert space \mathcal{H} . Then

1. If \mathcal{P} is nonzero, then $\|\mathcal{P}\| = 1$.
2. \mathcal{P} is self-adjoint.
3. For any $f, g \in \mathcal{H}$, $\mathcal{P}f$ is orthogonal to $g - \mathcal{P}g$.
4. \mathcal{P} is a positive operator.

Proof. 1) Let $\varphi \in \mathcal{H}$ and let $\mathcal{P}\varphi \neq 0$. Using the Cauchy-Schwarz inequality, we get

$$\|\mathcal{P}\varphi\| = \frac{\langle \mathcal{P}\varphi, \mathcal{P}\varphi \rangle}{\|\mathcal{P}\varphi\|} = \frac{\langle \varphi, \mathcal{P}^2\varphi \rangle}{\|\mathcal{P}\varphi\|} = \frac{\langle \varphi, \mathcal{P}\varphi \rangle}{\|\mathcal{P}\varphi\|} \leq \|\varphi\|.$$

So $\|\mathcal{P}\| \leq 1$. On the other hand, note since $\mathcal{P} \neq 0$, there is an $f \in \mathcal{H}$ where $\mathcal{P}f \neq 0$. Let $g = \mathcal{P}f \neq 0$. Note $\|\mathcal{P}g\| = \|\mathcal{P}^2f\| = \|\mathcal{P}f\| = \|g\|$. So

$$\|\mathcal{P}\| = \sup_{\|g\| \leq 1} \|\mathcal{P}g\| \geq \|\mathcal{P}\left(\frac{g}{\|g\|}\right)\| = \frac{\|g\|}{\|g\|} = 1.$$

- 2) This follows from the fact that \mathcal{P} is bounded and Hermitian.
- 3) Note $\langle \mathcal{P}f, g - \mathcal{P}g \rangle = \langle \mathcal{P}^2f, g - \mathcal{P}g \rangle = \langle \mathcal{P}f, \mathcal{P}(g - \mathcal{P}g) \rangle = \langle \mathcal{P}f, \mathcal{P}g - \mathcal{P}g \rangle = 0$.
- 4) Let f be any element from \mathcal{H} . Note

$$\langle f, \mathcal{P}f \rangle = \langle f, \mathcal{P}^2f \rangle = \|\mathcal{P}f\|^2 \geq 0.$$

□

Definition A.1.25. Let $f, g, h \in \mathcal{H}$ for some Hilbert space \mathcal{H} . Then $|f\rangle\langle g| : \mathcal{H} \rightarrow \mathcal{H}$ is an operator defined by $|f\rangle\langle g|(h) = \langle g, h \rangle f$.

Proposition A.1.26. Let \mathcal{H} be a Hilbert space and let $\varphi \in \mathcal{H}$ such that $\|\varphi\| = 1$. The operator $\mathcal{P}_\varphi = |\varphi\rangle\langle\varphi| : \mathcal{H} \rightarrow \mathcal{H}$ has the following properties:

1. \mathcal{P}_φ is a rank one orthogonal projection on \mathcal{H} .
2. \mathcal{P}_φ is a positive trace-class operator with trace equal to 1.
3. \mathcal{P}_φ has one and only one nonzero eigenvalue equal to 1.

Proof. 1) Clearly $(\mathcal{P}_\varphi)^* = \mathcal{P}_\varphi$. Now, since $\|\varphi\| = 1$, we have

$$\mathcal{P}_\varphi^2 = (|\varphi\rangle\langle\varphi|)^2 = |\varphi\rangle\langle\varphi|\varphi\rangle\langle\varphi| = |\varphi\rangle\langle\varphi| = \mathcal{P}_\varphi,$$

so $|\varphi\rangle\langle\varphi|$ is an orthogonal projection. To show it is rank one, note for any $f \in \mathcal{H}$ we have

$$\mathcal{P}_\varphi(f) = |\varphi\rangle\langle\varphi|(f) = \langle\varphi, f\rangle\varphi$$

and so φ spans the image of \mathcal{P}_φ (so the dimension of the image is 1).

2) By Proposition (A.1.24)(4) \mathcal{P}_φ is positive since it is an orthogonal projection. Now, choose an orthonormal basis in \mathcal{H} given by $\{e_i\}$ such that $\varphi = e_1$. Then

$$\begin{aligned}\mathrm{Tr}(|\varphi\rangle\langle\varphi|) &= \sum_i \langle e_i, |\varphi\rangle\langle\varphi|e_i \rangle \\ &= \sum_i \langle e_i, \langle\varphi, e_i\rangle\varphi \rangle \\ &= \sum_i \langle e_1, e_i \rangle \langle e_i, e_1 \rangle = 1.\end{aligned}$$

3) Suppose \mathcal{P}_φ has an eigenvalue λ with an associated eigenvector v . Then $\mathcal{P}_\varphi v = \lambda v$. Since \mathcal{P}_φ is a projection,

$$\lambda^2 v = \mathcal{P}_\varphi^2 v = \mathcal{P}_\varphi v = \lambda v.$$

Hence, $\lambda = 0$ or $\lambda = 1$. Now suppose there exists (linearly independent) eigenvectors v_1 and v_2 with the same eigenvalue $\lambda = 1$. Then $\mathcal{P}_\varphi v_1 = v_1$ and $\mathcal{P}_\varphi v_2 = v_2$. However, note

$$\begin{aligned}\mathcal{P}_\varphi v_1 = v_1 &\Rightarrow v_1 = \langle\varphi, v_1\rangle\varphi \\ \mathcal{P}_\varphi v_2 = v_2 &\Rightarrow v_2 = \langle\varphi, v_2\rangle\varphi\end{aligned}$$

and so v_1 and v_2 are linearly dependent, which is a contradiction. \square

A.1.3 Integral Operators

Definition A.1.27. An *integral operator* is a Hilbert-Schmidt operator T given by

$$(Tf)(x) = \int k(x, y)f(y)dy,$$

where k is known as the *kernel* of the operator T .

Remark A.1.28. Let T be an integral operator with kernel $k \in L^1 \cap L^2$. Then

1. T has Hilbert-Schmidt norm given by $\|T\|_{\mathrm{HS}} = \|k\|_{L^2}$.
2. $\mathrm{Tr}(T) = \int k(x, x)dx$.

Example A.1.29. Consider the Hilbert space $L^2(\mathbb{R})$. Let $\varphi \in L^2(\mathbb{R})$ such that $\|\varphi\| = 1$. The projection operator $P_\varphi = |\varphi\rangle\langle\varphi|$ is also an integral operator with kernel k given by

$$k(x; y) = \varphi(x)\overline{\varphi}(y).$$

To see this, let $f \in L^2(\mathbb{R})$. Then

$$\begin{aligned}(P_\varphi f)(x) &= (|\varphi\rangle\langle\varphi|f)(x) \\ &= \langle\varphi, f\rangle\varphi(x) \\ &= \int \varphi(x)\overline{\varphi}(y)f(y)dy.\end{aligned}$$

A.1.4 Density Operators

Definition A.1.30. Suppose a physical system described by a Hamiltonian in a Hilbert space \mathcal{H} is in a normalized state ψ_n with probability p_n . The operator $\rho : \mathcal{H} \rightarrow \mathcal{H}$ given by $\rho = \sum_{n \geq 1} p_n |\psi_n\rangle\langle\psi_n|$ is called the *density operator*, or *density matrix*.

Now we list some properties of the density operator.

Proposition A.1.31. *The density operator $\rho : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the following properties:*

1. ρ is self-adjoint.
2. ρ is a trace-class, positive operator with trace 1.
3. The eigenvalues of ρ are all non-negative.

Proof. 1) This is obvious by the fact that p_n are all real values and $|\psi_n\rangle\langle\psi_n|$ is self-adjoint. 2) Note ρ is positive since $p_n \geq 0$ and $|\psi_n\rangle\langle\psi_n| \geq 0$ for each n . To show ρ is trace-class and that its trace is 1, note by linearity of the trace and Proposition (A.1.26)(2)

$$\begin{aligned} \text{Tr}(\rho) &= \text{Tr} \left(\sum_{n \geq 1} p_n |\psi_n\rangle\langle\psi_n| \right) \\ &= \sum_{n \geq 1} p_n \text{Tr} (|\psi_n\rangle\langle\psi_n|) \\ &= \sum_{n \geq 1} p_n = 1. \end{aligned}$$

3) Since ρ is a positive operator, we have that $\langle \rho f, f \rangle \geq 0$ for all $f \in \mathcal{H}$. Let λ be an eigenvalue of ρ . Since ρ is self-adjoint, λ must be real. Now $\rho v = \lambda v$ for some $v \in \mathcal{H}$. So

$$0 \leq \langle \rho v, v \rangle = \langle \lambda v, v \rangle = \lambda \|v\|^2 \Rightarrow \lambda \geq 0.$$

□

A.2 A Few Useful Lemmas & Propositions

The following lemma is a well-known result (and is a particular case of the BCH formula).

Lemma A.2.1. *If A and B are two operators on a Hilbert space \mathcal{H} and $[A, B]$ commutes with A and B , then for all $\psi \in D(e^{A+B})$, we have*

$$e^{A+B}\psi = e^{-\frac{1}{2}[A,B]}e^A e^B \psi.$$

It follows from Lemma (A.2.1) the following.

Proposition A.2.2. *If A , B and C are operators on a Hilbert space \mathcal{H} and $[A, B+C]$ commutes with A and $B+C$, then for all $\psi \in D(e^{A+B+C})$, we have*

$$e^A e^{B+C} \psi = e^{[A, B+C]} e^{B+C} e^A \psi.$$

Proof. Let $\psi \in D(e^{A+B+C})$. Note by Lemma (A.2.1), since $[A, B+C]$ commutes with A and $B+C$, we have

$$e^A e^{B+C} \psi = e^{\frac{1}{2}[A, B+C]} e^{A+B+C} \psi.$$

Reordering the operators in the exponential and using Lemma (A.2.1) again, we get that

$$\begin{aligned} e^A e^{B+C} \psi &= e^{\frac{1}{2}[A, B+C]} e^{C+A+B} \psi \\ &= e^{\frac{1}{2}[A, B+C]} e^{-\frac{1}{2}[B+C, A]} e^{B+C} e^A \psi \\ &= e^{[A, B+C]} e^{B+C} e^A \psi. \end{aligned}$$

□

Let us also prove the following useful lemma.

Lemma A.2.3. Let A and B be operators, and let ad_A be the operator defined by $ad_A B = [A, B]$. Then for all $\psi \in D(e^A B e^{-A})$

$$e^A B e^{-A} \psi = e^{ad_A} B \psi = \left(B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots \right) \psi \quad (\text{A.4})$$

Proof. Let $f(t) = e^{tA} B e^{-tA}$. We expand $f(t)$ as a Taylor series about $t = 0$:

$$f(t) = f(0) + f'(0)t + \frac{1}{2!} f''(0)t^2 + \dots$$

Note that the n th derivative of f can be written as

$$f^{(n)}(t) = e^{tA} A e^{-tA} f^{(n-1)}(t) + f^{(n-1)}(t) e^{tA} (-A) e^{-tA}, \quad n \geq 1.$$

Then

$$\begin{aligned} f^{(n)}(0) &= A f^{(n-1)}(0) + f^{(n-1)}(0) (-A) \\ &= [A, f^{(n-1)}(0)] \\ &= ad_A f^{(n-1)}(0) \\ &= (ad_A)^n f(0) \\ &= (ad_A)^n B. \end{aligned}$$

□

Lemma A.2.4. (i) For operators x, y , we have that

$$xy^* + yx^* \leq |x|^2 + |y|^2. \quad (\text{A.5})$$

(ii) For operators x, x_1, x_2, \dots, x_n , we have the inequality

$$(x_1 + x_2 + \dots + x_n) x^* + x (x_1 + x_2 + \dots + x_n)^* \leq |x_1|^2 + \dots + |x_n|^2 + n|x|^2, \quad \text{for all } n \in \mathbb{N}. \quad (\text{A.6})$$

(iii) Also, for operators x_1, x_2, \dots, x_n , we have the inequality

$$|x_1 + \dots + x_n|^2 \leq n(|x_1|^2 + \dots + |x_n|^2), \quad \text{for all } n \in \mathbb{N}. \quad (\text{A.7})$$

Proof. (i) Note

$$\begin{aligned} 0 &\leq |x - y|^2 \\ &= (x - y)(x - y)^* \\ &= |x|^2 - xy^* - yx^* + |y|^2 \\ \Rightarrow xy^* + yx^* &\leq |x|^2 + |y|^2. \end{aligned}$$

(ii) We prove (A.6) by induction. Note the case $n = 1$ holds by (A.5). Now suppose (A.6) holds for $n < k$. To show (A.6) holds for the case $n = k$, note

$$\begin{aligned} &(x_1 + x_2 + \dots + x_k) x^* + x (x_1 + x_2 + \dots + x_k)^* \\ &= (x_1 + x_2 + \dots + x_{k-1}) x^* + x (x_1 + x_2 + \dots + x_{k-1})^* + x_k x^* + x x_k^* \\ &\leq |x_1|^2 + \dots + |x_{k-1}|^2 + (k-1)|x|^2 + x_k x^* + x x_k^* \\ &\leq |x_1|^2 + \dots + |x_{k-1}|^2 + (k-1)|x|^2 + |x_k|^2 + |x|^2 \\ &= |x_1|^2 + \dots + |x_{k-1}|^2 + |x_k|^2 + k|x|^2, \end{aligned} \quad (\text{A.8})$$

$$(\text{A.9})$$

$$= |x_1|^2 + \dots + |x_{k-1}|^2 + |x_k|^2 + k|x|^2,$$

where we used the induction hypothesis in (A.8) and (i) in (A.9).

(iii) Again we use induction. The $k = 1$ case clearly holds. Now suppose (A.7) holds for $n < k$. To prove the $n = k$ case note

$$\begin{aligned}
|x_1 + \cdots + x_{k-1} + x_k|^2 &= (x_1 + \cdots + x_{k-1} + x_k)(x_1 + \cdots + x_{k-1} + x_k)^* \\
&= |x_1 + \cdots + x_{k-1}|^2 + (x_1 + \cdots + x_{k-1})x_k^* + x_k(x_1 + \cdots + x_{k-1})^* + |x_k|^2 \\
&\leq (k-1)(|x_1|^2 + \cdots + |x_{k-1}|^2) + |x_1|^2 + \cdots + |x_{k-1}|^2 + (k-1)|x_k|^2 + |x_k|^2 \quad (\text{A.10}) \\
&= k(|x_1|^2 + \cdots + |x_{k-1}|^2 + |x_k|^2)
\end{aligned}$$

where we used the induction hypothesis and (ii) in (A.10). \square

A.3 *-Algebras and C^* -Algebras

Definition A.3.1. Let \mathcal{A} be an algebra. An *involution* is a mapping $*$: $\mathcal{A} \rightarrow \mathcal{A}$ such that the properties

$$\begin{aligned}
(A^*)^* &= A \\
(aA^* + bB^*)^* &= \bar{a}A + \bar{b}B \\
(AB)^* &= B^*A^*
\end{aligned}$$

are satisfied for all $A, B \in \mathcal{A}$ and $a, b \in \mathbb{C}$.

Definition A.3.2. A **-algebra*, or *involution algebra*, is an algebra with an involution.

Definition A.3.3. A C^* -algebra is a complex Banach space \mathcal{A} which is also a *-algebra and has the properties

$$\begin{aligned}
\|AB\| &\leq \|A\| \|B\| \\
\|A^*A\| &= \|A\|^2
\end{aligned}$$

for all $A, B \in \mathcal{A}$.

Remark A.3.4. If \mathcal{A} is a C^* -algebra, then $\|A^*\| = \|A\|$ for all $A \in \mathcal{A}$.

Example A.3.5. Let \mathcal{H} be a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators on \mathcal{H} . If we define multiplication and addition of elements in $\mathcal{B}(\mathcal{H})$ in the standard way and equip this set with the operator norm, then the adjoint operation in \mathcal{H} defines an involution on $\mathcal{B}(\mathcal{H})$. Therefore, $\mathcal{B}(\mathcal{H})$ together with this involution and norm is an example of a C^* -algebra.

Remark A.3.6. *-algebras, and in particular C^* -algebras, play an important role in both classical and quantum mechanics. A C^* -algebra \mathcal{A} can be used to describe a physical system, whose observables are given by self-adjoint elements in \mathcal{A} . A state of the system can be described by a positive linear functional on \mathcal{A} .

A.4 Tensor Products

Definition A.4.1. The *tensor product* over two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 is another Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ equipped with a bilinear map

$$\gamma : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$$

defined by

$$\gamma(u, v) = u \otimes v$$

such that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is spanned by elements of the form $u \otimes v$. The inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle u_1, v_1 \rangle_{\mathcal{H}_1} \langle u_2, v_2 \rangle_{\mathcal{H}_2}.$$

Remark A.4.2.

1. The bilinearity of γ implies that for any $u_1, u_2 \in \mathcal{H}_1$, $v_1, v_2 \in \mathcal{H}_2$, and any scalar α

$$\begin{aligned}(u_1 + u_2) \otimes v_1 &= u_1 \otimes v_1 + u_2 \otimes v_1, \\ u_1 \otimes (v_1 + v_2) &= u_1 \otimes v_1 + u_1 \otimes v_2, \\ \alpha(u_1 \otimes v_1) &= (\alpha u_1) \otimes v_1 = u_1 \otimes (\alpha v_1).\end{aligned}$$

2. Let I and J be countable index sets. A basis $\{u_i\}_{i \in I}$ of \mathcal{H}_1 and $\{v_j\}_{j \in J}$ of \mathcal{H}_2 gives a basis $\{u_i \otimes v_j : (i, j) \in I \times J\}$ for $\mathcal{H}_1 \otimes \mathcal{H}_2$. For finite dimensional Hilbert spaces, if \mathcal{H}_1 has dimension m and \mathcal{H}_2 has dimension n , then $\mathcal{H}_1 \otimes \mathcal{H}_2$ has dimension mn .

3. Any element of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written uniquely as

$$\sum_{i \in I} \sum_{j \in J} \alpha_{i,j} u_i \otimes v_j$$

where $\alpha_{i,j}$ are scalars.

Often a tensor product of two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 is isomorphic to another Hilbert space \mathcal{H}_3 , and so an element $u \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can also be written as an element in \mathcal{H}_3 and vice versa. We show some examples to demonstrate this idea.

Example A.4.3. Consider the Hilbert space \mathbb{C}^2 with the standard basis $\{e_1, e_2\}$ where $e_1 = (1, 0)$, $e_2 = (0, 1)$. Note we can write the basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as $\{f_1, f_2, f_3, f_4\}$ where

$$f_1 = e_1 \otimes e_1, \quad f_2 = e_1 \otimes e_2, \quad f_3 = e_2 \otimes e_1, \quad f_4 = e_2 \otimes e_2.$$

Note there is an isomorphism between $\mathbb{C}^2 \otimes \mathbb{C}^2$ and \mathbb{C}^4 , i.e. $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. Hence we can represent any element of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as an element of \mathbb{C}^4 and vice-versa. That is, for any two elements $(x_1, x_2), (y_1, y_2) \in \mathbb{C}^2$, we can consider the tensor product $(x_1, x_2) \otimes (y_1, y_2)$ to be the same as $(x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2)$.

Example A.4.4. Consider the Hilbert space $L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n)$, for $m, n \in \mathbb{N}$. We have that $L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n) \cong L^2(\mathbb{R}^{m+n})$. Therefore, if $\psi_1 \in L^2(\mathbb{R}^m)$ and $\psi_2 \in L^2(\mathbb{R}^n)$, we can write $\psi_1 \otimes \psi_2 \in L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n)$ as a function in $L^2(\mathbb{R}^{m+n})$. That is, we can write

$$\psi_1 \otimes \psi_2 \equiv f \in L^2(\mathbb{R}^{m+n})$$

where $f(x, y) = \psi_1(x)\psi_2(y)$ and $x \in \mathbb{R}^m, y \in \mathbb{R}^n$.

Definition A.4.5. Let $\psi_1 \in L^2(\mathbb{R}^m)$ and $\psi_2 \in L^2(\mathbb{R}^n)$. We define $(\psi_1 \otimes \psi_2)(x, y)$, where $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, to be

$$(\psi_1 \otimes \psi_2)(x, y) = \psi_1(x)\psi_2(y).$$

Often we will drop the first (x, y) and just write $\psi_1 \otimes \psi_2 = \psi_1(x)\psi_2(y)$.

Definition A.4.6. Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces. Let S be a linear operator on \mathcal{H}_1 and T be a linear operator on \mathcal{H}_2 . Then the *tensor product of operators* S and T is a linear map

$$S \otimes T : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$$

defined by

$$S \otimes T(x \otimes y) = S(x) \otimes T(y).$$

We can also represent the tensor product of operators in a different way.

Example A.4.7. Consider the tensor product of the operators $|\varphi_1\rangle\langle\varphi_1|$ and $|\varphi_2\rangle\langle\varphi_2|$ acting on $L^2(\mathbb{R}^3)$. Note for any $f, g \in L^2(\mathbb{R}^3)$

$$\begin{aligned} \left(|\varphi_1\rangle\langle\varphi_1| \otimes |\varphi_2\rangle\langle\varphi_2| (f \otimes g)\right)(x, y) &= \langle\varphi_1, f\rangle\varphi_1(x)\langle\varphi_2, g\rangle\varphi_2(y) \\ &= |\varphi_1(x)\rangle\langle\varphi_1(x)| \cdot |\varphi_2(y)\rangle\langle\varphi_2(y)|f(x)g(y). \end{aligned}$$

It follows that for any $\psi \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$

$$\left(\left(|\varphi_1\rangle\langle\varphi_1| \otimes |\varphi_2\rangle\langle\varphi_2|\right)\psi\right)(x, y) = |\varphi_1(x)\rangle\langle\varphi_1(x)| \cdot |\varphi_2(y)\rangle\langle\varphi_2(y)|\psi(x, y)$$

Note on the left-hand side ψ is considered to be in $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ and on the right-hand side ψ is considered to be in $L^2(\mathbb{R}^6)$. It should be clear from the context which one is meant.

We can also talk about more than one tensor product. This can be easily seen and so we will just give a couple of important definitions below.

Definition A.4.8. Let \mathcal{H} be a Hilbert space. The n -fold tensor product $\mathcal{H}^{\otimes n}$ is given by

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}.$$

We can also talk about $\varphi^{\otimes n} \in \mathcal{H}^{\otimes n}$ which is the n -fold tensor product of φ with itself. That is, we define

$$\varphi^{\otimes n} = \underbrace{\varphi \otimes \varphi \otimes \cdots \otimes \varphi}_{n \text{ times}} \in \mathcal{H}^{\otimes n}.$$

Remark A.4.9. As in the case of a single tensor product, we can write an element $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \in L^2(\mathbb{R}^{\alpha_1}) \otimes L^2(\mathbb{R}^{\alpha_2}) \otimes \cdots \otimes L^2(\mathbb{R}^{\alpha_n})$ in the form of a function in $L^2(\mathbb{R}^{\alpha_1+\alpha_2+\cdots+\alpha_n})$.

Definition A.4.10. Let $\psi_1 \in L^2(\mathbb{R}^{\alpha_1})$, $\psi_2 \in L^2(\mathbb{R}^{\alpha_2})$, \dots , $\psi_n \in L^2(\mathbb{R}^{\alpha_n})$. We define $(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n)(x_1, x_2, \dots, x_n)$, where $x_1 \in \mathbb{R}^{\alpha_1}$, $x_2 \in \mathbb{R}^{\alpha_2}$, \dots , $x_n \in \mathbb{R}^{\alpha_n}$ to be

$$\left(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n\right)(x_1, x_2, \dots, x_n) = \psi_1(x_1)\psi_2(x_2) \cdots \psi_n(x_n).$$

Often we will just write $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n = \psi_1(x_1)\psi_2(x_2) \cdots \psi_n(x_n)$.

A.5 A Brief Introduction to Classical and Quantum Mechanics

In this section, we begin by discussing basic concepts in classical mechanics, such as the Hamiltonian formulation of classical mechanics and Hamilton's equations, the Poisson bracket, and symplectic manifolds as an abstraction of phase space. Then we discuss basic concepts in quantum mechanics, including the pictures of quantum mechanics and the quantum analogue to the Poisson bracket. Finally we give basic definitions for the classical limit and quantization, and we briefly introduce the phase space formulation of quantum mechanics. For further details, see, for example, [3], [7], and [21].

A.5.1 Hamiltonian mechanics

The Hamiltonian formulation of classical mechanics has been an invaluable tool in the development and understanding of quantum mechanics. In this formulation of classical mechanics, a system is described in terms of generalized coordinates and momenta. These generalized coordinates and momenta are thus referred to as the *canonical coordinates* of the system. The Hamiltonian, which in most cases can be thought of as the total energy of the system, can be used to predict how the system will evolve with time. The time evolution is given by Hamilton's equations

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \end{aligned} \tag{A.11}$$

where $H = H(\mathbf{q}, \mathbf{p}, t)$ is the Hamiltonian and $\mathbf{q} = (q_1, q_2, \dots, q_N)$, $\mathbf{p} = (p_1, p_2, \dots, p_N)$ are the generalized coordinates and momenta, respectively, for a system with N degrees of freedom.

Example A.5.1. Consider the problem of the simple harmonic oscillator in which a particle is moving in a quadratic potential field. We can think of a ball, experiencing no energy loss, sliding back and forth inside a parabolic shaped bowl on a vertical plane. We wish to find the ball's position $x(t)$ and momentum $p(t)$, given initial values $x(0)$ and $p(0)$.

In this case, the Hamiltonian H is given by the sum of the potential and kinetic energies. The potential energy is given by $U = \frac{1}{2}kx^2$, where k is a positive constant. The kinetic energy is expressed in terms of the momentum as $T = \frac{p^2}{2m}$. Then the Hamiltonian is given by $H(x, p) = U + T = \frac{1}{2}kx^2 + \frac{p^2}{2m}$. Substituting this into the first of Hamilton's equations, we get $\frac{dp}{dt} = -\frac{\partial}{\partial x} \left(\frac{1}{2}kx^2 + \frac{p^2}{2m} \right) = -kx$. The second of Hamilton's equations tells us that $\frac{dq}{dt} = \frac{\partial}{\partial p} \left(\frac{1}{2}kx^2 + \frac{p^2}{2m} \right) = \frac{p}{m}$. Now we have two coupled first-order ordinary differential equations which may be solved simultaneously to find $x(t)$ and $p(t)$.

Any one of the coordinates q_1, q_2, \dots, q_N or momenta p_1, p_2, \dots, p_N , is known as a dynamical variable, or an *observable*. It is a quantity which can be measured. Any other measurable quantity (observable) a of the system is a function $a(p, q)$ of p and q (and possibly also of time t).

A.5.2 Lagrangian Mechanics & Noether's Theorem

Suppose we have a classical system with N degrees of freedom and coordinates $\mathbf{q} = (q_1, q_2, \dots, q_N)$. We can define the Lagrangian \mathcal{L} as the kinetic energy T of the system minus its potential energy U ; that is,

$$\mathcal{L} = T - U$$

where

$$\mathcal{L} = \mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}}).$$

If the Lagrangian of the system is known, then the equations of motion of a system may be obtained by using the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) = 0, \quad j = 1, 2, \dots, N$$

which arise from what is known as *the principle of least action*, the fact that the true "path" of a system minimizes the action integral

$$\int_0^T \mathcal{L} dt$$

for some time T .

Example A.5.2. Consider the one-dimensional harmonic oscillator where $T = \frac{1}{2}m\dot{x}^2$ and $U = \frac{1}{2}kx^2$. In this case the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

and the Euler-Lagrange equation gives

$$0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = -kx + m\ddot{x}$$

which is the same equation of motion which results from using Newton's equation $F = ma$.

In a physical system, it is often very useful and important to determine which quantities are conserved.

Example A.5.3. In general, we can express the momentum p_j associated with the coordinate q_j as

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}.$$

Thus we can express the Euler-Lagrange equations as

$$\frac{\partial \mathcal{L}}{\partial q_j} = \frac{d}{dt} p_j, \quad j = 1, 2, \dots, N.$$

We see that if the Lagrangian is independent of q_j , then

$$\frac{d}{dt} p_j = 0$$

and thus p_j is a conserved quantity.

Noether's Theorem is a generalization of the above example. Noether's Theorem states that whenever we have a continuous symmetry of the Lagrangian, there is an associated conservation law. By "continuous symmetry" we mean a symmetry with a continuous constant parameter ϵ which is typically very small. In the above example, the symmetry was an infinitesimal shift $q_j \rightarrow q_j + \epsilon$.

Now we give the following proposition, which is a less general case of Noether's Theorem and does not take into account time translations.

Proposition A.5.4. Consider a Lagrangian system with N degrees of freedom and coordinates q_1, q_2, \dots, q_N . If for certain functions $\gamma_j(t)$ and for constant infinitesimal ϵ the transformation

$$q_j(t) \rightarrow q_j(t) + \epsilon \gamma_j(t), \quad \dot{q}_j(t) \rightarrow \dot{q}_j(t) + \epsilon \dot{\gamma}_j(t), \quad t \rightarrow t \tag{A.12}$$

is a symmetry (i.e. if it leaves the Lagrangian unaffected), then the quantity

$$\sum_{j=1}^N \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \gamma_j$$

is a constant of motion (i.e. it is conserved).

Proof. The Lagrangian must be invariant under the transformation (A.12). Thus, we must have

$$\mathcal{L}(t, q_j + \epsilon \gamma_j, \dot{q}_j + \epsilon \dot{\gamma}_j) - \mathcal{L}(t, q_j, \dot{q}_j) = 0.$$

Expanding $\mathcal{L}(t, q_j + \epsilon \gamma_j, \dot{q}_j + \epsilon \dot{\gamma}_j)$ to first order in ϵ , we obtain

$$\sum_{j=1}^N \left(\frac{\partial \mathcal{L}}{\partial q_j} \epsilon \gamma_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \epsilon \dot{\gamma}_j \right) = 0.$$

Rewriting the first term using the Euler-Lagrange equations of motion, we obtain

$$\sum_{j=1}^N \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \epsilon \gamma_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \epsilon \dot{\gamma}_j \right) = 0.$$

Note the left-hand side can be written as the total time derivative

$$\epsilon \frac{d}{dt} \left(\sum_{j=1}^N \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \gamma_j \right) = 0$$

which gives the desired result. □

Remark A.5.5. A time translation

$$q_j(t) \rightarrow q_j(t), \quad \dot{q}_j(t) \rightarrow \dot{q}_j(t), \quad t \rightarrow t + \epsilon,$$

for infinitesimal ϵ is a symmetry if and only if the partial time derivative of the Lagrangian vanishes. We can see this by using the same technique as above.

The Hamiltonian H of a system with N degrees of freedom can in general be defined through the Lagrangian as

$$H = \sum_{j=1}^N \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j - \mathcal{L}.$$

Let us look at under what conditions H is conserved. Note the total time derivative of the Lagrangian is given by

$$\frac{d}{dt} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \sum_{j=1}^N \left(\frac{\partial \mathcal{L}}{\partial q_j} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j \right).$$

From the Euler-Lagrange equations, we obtain

$$\frac{d}{dt} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \sum_{j=1}^N \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j \right).$$

The last two terms inside the sum can be written as a total time derivative, and we obtain

$$\frac{d}{dt} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt} \left(\sum_{j=1}^N \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j \right)$$

which implies

$$\frac{d}{dt} H = -\frac{\partial}{\partial t} \mathcal{L}.$$

We see that if the Lagrangian does not depend explicitly on time (i.e. it is invariant under time translations), then the Hamiltonian of the system is conserved.

A.5.3 Poisson bracket

In this section we consider a classical system with generalized coordinates q_1, q_2, \dots, q_N and momenta p_1, p_2, \dots, p_N .

Definition A.5.6. Let u and v be two observables. Then u and v have a *Poisson bracket*, denoted $\{u, v\}$, and given by

$$\{u, v\} = \sum_{r=1}^N \left(\frac{\partial u}{\partial q_r} \frac{\partial v}{\partial p_r} - \frac{\partial u}{\partial p_r} \frac{\partial v}{\partial q_r} \right).$$

The Poisson bracket has the following five main properties:

$$\{u, v\} = -\{v, u\} \tag{A.13a}$$

$$\{u, c\} = 0, \quad \text{where } c \text{ is some number} \tag{A.13b}$$

$$\{u_1 + u_2, v\} = \{u_1, v\} + \{u_2, v\} \tag{A.13c}$$

$$\{u_1 u_2, v\} = \{u_1, v\} u_2 + u_1 \{u_2, v\} \tag{A.13d}$$

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0 \tag{A.13e}$$

The last property (A.13e) is known as the Jacobi identity for Poisson brackets.

Let us look at the Poisson bracket between q_j and p_k . These are independent variables, and so we get the equations

$$\begin{aligned} \{q_j, q_k\} &= \{p_j, p_k\} = 0 \\ \{q_j, p_k\} &= \delta_{jk} \end{aligned} \tag{A.14}$$

The relations (A.14) are known as the *canonical Poisson bracket relations*.

We can rewrite Hamilton's equations in an equivalent form using the Poisson bracket. To do this, suppose $q = q(t)$ and $p = p(t)$ are solutions to Hamilton's equations. Note $\{q, H\} = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p}$. Similarly, $\{p, H\} = -\frac{\partial H}{\partial q}$. So Hamilton's equations become

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p} = \{q, H\} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} = \{p, H\} \end{aligned}$$

More generally, let $f = f(p, q, t)$ be any function of p, q and t . Using the chain rule, we can calculate the time derivative of f as follows:

$$\frac{d}{dt}f(q, p, t) = \frac{\partial f}{\partial q} \frac{dq}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial t}.$$

Using Hamilton's equations,

$$\frac{d}{dt}f(q, p, t) = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}.$$

Finally expressing $\frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}$ as a Poisson bracket, we get

$$\frac{d}{dt}f(q, p, t) = \{f, H\} + \frac{\partial f}{\partial t}. \tag{A.15}$$

Note if f does not depend explicitly on time, $\frac{\partial f}{\partial t} = 0$, and the above equation reduces to

$$\frac{d}{dt}f(q, p, t) = \{f, H\} \tag{A.16}$$

A.5.4 Phase Space & Symplectic Manifolds

A Hamiltonian system is associated with what is known as a *phase space* Γ , a manifold which represents all possible states of a system. For example, consider a system of N particles in three-dimensional space. Each particle is specified by six coordinates (three position coordinates and three momentum coordinates), and thus the phase space is given by \mathbb{R}^{6N} . However, in a more general setting (for example, if particles are constrained to lie on curved surfaces) there is no reason why Euclidean space should be the choice for the manifold, and thus one needs to describe phase space in a more abstract sense. It has been shown that the correct abstraction for phase space is through the notion of a symplectic manifold. We briefly discuss symplectic manifolds; however, we do not go into much detail as this would require quite some time.

First we introduce a complex symplectic vector space.

Definition A.5.7. A *complex symplectic vector space* V is a complex vector space together with a *symplectic form* which is a sesquilinear map

$$\omega : V \times V \rightarrow \mathbb{C}$$

which is skew-Hermitian and non-degenerate.

Remark A.5.8.

1. The sesquilinearity of ω implies that

$$\omega(\alpha u + \beta v, w) = \bar{\alpha}\omega(u, w) + \bar{\beta}\omega(v, w) \quad (\text{A.17})$$

$$\omega(u, \alpha v + \beta w) = \alpha\omega(u, v) + \beta\omega(u, w) \quad (\text{A.18})$$

for all $u, v, w \in V$ and any $\alpha, \beta \in \mathbb{C}$.

2. The fact that ω is skew-Hermitian means that for all $u, v \in V$.

$$\omega(u, v) = -\overline{\omega(v, u)}. \quad (\text{A.19})$$

Note (A.19) and (A.17) actually imply (A.18).

3. The non-degeneracy of ω means that for any $u \in V$

$$\omega(u, v) = 0 \text{ for all } v \in V \quad \Rightarrow \quad u = 0. \quad (\text{A.20})$$

Definition A.5.9. Let V_1 and V_2 be two symplectic vector spaces with symplectic forms ω_1 and ω_2 , respectively. We say that V_1 and V_2 are *symplectically isomorphic* if there exists a bijective map $T : V_1 \rightarrow V_2$ such that

$$\omega_1(u, v) = \omega_2(Tu, Tv)$$

for all vectors $u, v \in V_1$. This map T is called a *symplectic map* or a *symplectomorphism*. In the case when $V_1 = V_2$, we call T a *symplectic automorphism*.

Remark A.5.10. If T is a symplectic automorphism on a symplectic vector space V with symplectic form ω , we have that $\omega(Tu, Tv) = \omega(u, v)$, and so we see that the linear transformation T preserves the symplectic form.

Let us now give a basic definition of a manifold.

Definition A.5.11. A *manifold* M of dimension n is a Hausdorff topological space in which every point has a neighborhood homeomorphic to an open subset of \mathbb{R}^n .

One way of extending the definition of manifolds to infinite dimensions is to model them on Banach spaces.

Definition A.5.12. A *Banach manifold* is a topological space in which every point has a neighborhood homeomorphic to an open set in a Banach space.

Definition A.5.13. A *symplectic manifold* is a smooth manifold, M , equipped with a symplectic form ω .

Example A.5.14. The most basic example of a symplectic manifold is the manifold $M = \mathbb{R}^{2n}$ with coordinates

$x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ equipped with the symplectic form

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i,$$

known as the standard symplectic form.

Remark A.5.15. In Example (A.5.14), \wedge is what is known as a wedge or exterior product, and $\sum_{i=1}^n dx_i \wedge dy_i$ is an example of a differential 2-form. A symplectic form ω on a smooth manifold is a closed 2-form that is non-degenerate. We will not go into these details as they are out of the scope of this thesis.

Remark A.5.16. The phase space of any classical system is a symplectic manifold.

Example A.5.17. Consider a system consisting of one particle constrained to move along a straight line. The phase space for this system is given by the two-dimensional plane \mathbb{R}^2 with the bilinear form $A : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$A((x_1, p_1), (x_2, p_2)) = x_1 p_2 - x_2 p_1.$$

This function A is known as the *area form* since $|A(\mathbf{u}, \mathbf{v})|$ gives the area of the parallelogram spanned by the two vectors \mathbf{u} and \mathbf{v} . It is easy to show that the area form is a symplectic form. One can also show that the area form is the same as the wedge product of dx and dp ; that is, one can show that $A = dx \wedge dp$. This two-dimensional plane with area form is a symplectic manifold.

One might ask why we need the area form. It is because the position x and the momentum p are not unrelated; given the position as a function of time, one can calculate the momentum. This relationship is expressed mathematically through the area form.

A.5.5 The Mathematics of Quantum Mechanics

The mathematical formalism of classical mechanics and quantum mechanics are quite different. In classical mechanics, a state is a function in phase space and an observable a is a measurable quantity, a function $a(q, p)$ of the position and momentum; in quantum mechanics, a state is a vector in some Hilbert space, and Hermitian operators defined on some Hilbert space \mathcal{H} act as the observables.

In classical mechanics, we often want to find the position of some particle of mass m as a function of time. To do so, we can often apply Newton's second law $F = ma$. Once we have the position as a function of time, we can calculate the velocity, momentum, and so forth. In quantum mechanics, often the state of a system at a given time is described by a complex wave function ψ , which lives in some Hilbert space, whose solution can be found using the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(t) = \hat{H} \psi(t) \tag{A.21}$$

where \hat{H} is the Hamiltonian operator and describes the total energy of the system.

Remark A.5.18. In order for $\hat{H}\psi$ to make sense for some ψ in a Hilbert space \mathcal{H} , we need to ψ to be in the domain of \hat{H} . Let us look at a simple example. Suppose $\mathcal{H} = L^2(\mathbb{R}^3)$, and suppose we can write the Hamiltonian in the form $\hat{H} = \hat{T} + \hat{V}$, where $\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \Delta$ is the kinetic energy operator and $\hat{V} = V(\vec{x}, t)$ is the potential energy operator. Recall, in the \mathbb{R}^3 Cartesian coordinate system $\vec{x} = (x, y, z)$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Then

$$D(\hat{H}) = \{\psi \in \mathcal{H} : \|\hat{H}\psi\|_{\mathcal{H}} < \infty\} = \{\psi \in \mathcal{H} : \|\Delta\psi + V\psi\|_{L^2} < \infty\}.$$

For this to make sense we also need ψ to be twice continuously differentiable, and we need V to be a potential such that $V\psi$ belongs to $L^2(\mathbb{R}^3)$.

The wave function ψ allows us to calculate the probability of finding the particle near a certain point. Whereas in classical mechanics both position and momentum can be known exactly, in quantum mechanics, as the uncertainty principle says, the more precisely the position is determined, the less precisely is its momentum, and vice versa.

A.5.6 Pictures of Quantum Mechanics

In classical mechanics, the dynamics of some object in motion can be described differently depending on the frame of reference. One can either fix the coordinate vectors and look at the time development of the object, or one can make the object stationary and look at the time development of the coordinate vectors. These two different methods yield different mathematical formulations, but they give the same results. Similarly, in quantum mechanics, there are three commonly known different mathematical formulations, called the three pictures of quantum mechanics, to describe the dynamics of a system. All pictures must give the same results but differ in how they treat the time evolution of the system. These pictures are known as the Schrödinger picture, the Heisenberg picture, and the Interaction picture.

Schrödinger picture

In the Schrödinger picture, the state vectors evolve in time but the operators are stationary.

First let us look at the time evolution operator $\hat{U}(t)$. Given some initial state $\psi(0)$, the state $\psi(t)$ at some later time t is related to $\psi(0)$ by

$$\psi(t) = \hat{U}(t)\psi(0). \quad (\text{A.22})$$

In the Schrödinger picture, the state vector $\psi(t)$ satisfies the Schrödinger equation (A.21). Thus, to find $\hat{U}(t)$, we substitute (A.22) into (A.21) to get the following:

$$\frac{\partial}{\partial t} \left(\hat{U}(t)\psi(0) \right) = -\frac{i}{\hbar} \hat{H} \left(\hat{U}(t)\psi(0) \right) \quad (\text{A.23})$$

Suppose \hat{H} does not depend explicitly on time; then we have that $\psi(t) = e^{-\frac{i}{\hbar}\hat{H}t}\psi(0)$ is a solution to the Schrödinger equation (A.21), and we get that the solution to (A.23) is $\hat{U}(t)\psi(0) = e^{-\frac{i}{\hbar}\hat{H}t}\psi(0)$. Thus, the time evolution operator is given by

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}. \quad (\text{A.24})$$

Note that the evolution operator is unitary; that is, $\hat{U}^*\hat{U} = \mathbb{1}$, where $\mathbb{1}$ is the identity operator.

Now we consider the expectation value of an operator \hat{A} :

$$\begin{aligned} \langle \hat{A} \rangle_t &= \langle \psi(t) | \hat{A} | \psi(t) \rangle \\ &= \langle \hat{U}(t)\psi(0) | \hat{A} | \hat{U}(t)\psi(0) \rangle \\ &= \langle e^{-\frac{i}{\hbar}\hat{H}t}\psi(0) | \hat{A} | e^{-\frac{i}{\hbar}\hat{H}t}\psi(0) \rangle \end{aligned}$$

Since the state vector ψ evolves in time, the expectation values can also evolve in time, which is why we use the subscript t .

Heisenberg Picture

Let $\psi_H(t)$ be a state vector in the Heisenberg picture, and let $\psi(t)$ be a state vector in the Schrödinger picture so that $\psi(t) = \hat{U}(t)\psi(0)$. We define $\psi_H(t)$ so that $\psi_H(t) = \hat{U}^*(t)\psi(0)$. Now, note

$$\psi_H(t) = \hat{U}^*(t)\psi(t) = \hat{U}^*(t)\hat{U}(t)\psi(0) = \psi(0) = \psi_H(0).$$

So the state vector ψ_H is constant in time.

Now, an operator in the Heisenberg picture \hat{A}_H is related to the operator \hat{A} in the Schrödinger picture by

$$\hat{A}_H(t) = \hat{U}^*(t)\hat{A}\hat{U}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{A}e^{-\frac{i}{\hbar}\hat{H}t}. \quad (\text{A.25})$$

So operators in the Heisenberg picture evolve in time. Now, assuming \hat{H} and \hat{A} do not vary with time, we wish to find the Heisenberg equation of motion by calculating $i\hbar\frac{d}{dt}\hat{A}_H(t)$, but first we need to show that the time evolution operator commutes with \hat{H} . Using the fact that $[\hat{H}, t] = 0$, we see

$$\begin{aligned} \left[\hat{U}(t), \hat{H} \right] &= \left[e^{-\frac{i}{\hbar}\hat{H}t}, \hat{H} \right] \\ &= \left[\sum_{n=0}^{\infty} \left(\frac{-i\hat{H}t}{\hbar} \right)^n \frac{1}{n!}, \hat{H} \right] \\ &= \left[\sum_{n=0}^{\infty} \left(-\frac{it}{\hbar} \right)^n \frac{\hat{H}^n}{n!}, \hat{H} \right] \\ &= 0. \end{aligned} \quad (\text{A.26})$$

Now we are ready to find the Heisenberg equation of motion. We look at $i\hbar\frac{d}{dt}\hat{A}_H(t)$:

$$\begin{aligned} i\hbar\frac{d}{dt}\hat{A}_H(t) &= i\hbar\frac{d}{dt} \left(e^{\frac{i}{\hbar}\hat{H}t}\hat{A}e^{-\frac{i}{\hbar}\hat{H}t} \right) \\ &= i\hbar \left(\frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}\hat{H}t}\hat{A}e^{-\frac{i}{\hbar}\hat{H}t} + e^{\frac{i}{\hbar}\hat{H}t}\hat{A} \left(-\frac{i\hat{H}}{\hbar} \right) e^{-\frac{i}{\hbar}\hat{H}t} \right) \end{aligned}$$

where in the last step we used the product rule. Now, we multiply through by $i\hbar$ and use the fact that \hat{H} commutes with the time evolution operator, along with our definition of $\hat{A}_H(t)$ in (A.25):

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = -\hat{H} \hat{A}_H(t) + \hat{A}_H(t) \hat{H}$$

Now the last two terms can be combined into a commutator and we get the Heisenberg equation of motion:

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = [\hat{A}_H(t), \hat{H}] \quad (\text{A.27})$$

Note that this is exactly like (A.16) with the Poisson bracket replaced with the commutator divided by $i\hbar$. We see later that obtaining quantum equations from classical ones involves replacing the Poisson bracket with the commutator divided by $i\hbar$ (this is the quantum analogue to the Poisson bracket).

Finally, observe

$$\begin{aligned} \langle \hat{A} \rangle_t &= \langle \psi(t) | \hat{A} | \psi(t) \rangle \\ &= \langle \hat{U}(t)\psi(0) | \hat{U}(t)\hat{A}_H(t)\hat{U}^*(t) | \hat{U}(t)\psi(0) \rangle \\ &= \langle \psi(0) | \hat{U}^*(t)\hat{U}(t)\hat{A}_H(t)\hat{U}^*(t)\hat{U}(t) | \psi(0) \rangle \\ &= \langle \psi_H(0) | \hat{A}_H(t) | \psi_H(0) \rangle \end{aligned}$$

Thus, in the Heisenberg picture, the operators evolve in time while the state vectors remain stationary. However, the expectation value of an operator in the Heisenberg picture is the same as that of the expectation value of the operator in the Schrödinger picture.

Interaction Picture

Often we can divide the Hamiltonian into two parts so that $\hat{H} = \hat{H}_0 + \hat{V}$, where \hat{H}_0 is the unperturbed Hamiltonian while \hat{V} is some perturbation of the system. Usually if \hat{H} depends explicitly on time, the parts are also divided so that \hat{H}_0 is independent of time, while \hat{V} depends on time. Suppose this is the case, and also, suppose, for sake of simplicity, that \hat{V} is bounded.

Let $\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}}$ as before, and let $\hat{U}_0(t) = e^{-\frac{i\hat{H}_0 t}{\hbar}}$. Note $\hat{U}_0^*(t) = e^{\frac{i\hat{H}_0 t}{\hbar}}$. Now define $\hat{\Gamma}(t) := \hat{U}_0^*(t)\hat{U}(t)$, known as the interaction picture propagator.

In the interaction picture, both the state vectors and the operators depend on time. We will call the state vector in the interaction picture $\psi_I(t)$. Again, let $\psi(t)$ be the state vector in the Schrödinger picture. We define $\psi_I(t)$ as follows:

$$\psi_I(t) = \hat{U}_0(t)^* \psi(t) = \hat{U}_0(t)^* \hat{U}(t) \psi(0) = \hat{\Gamma}(t) \psi(0)$$

Note $\psi_I(t)$ is time dependent and if we know $\hat{\Gamma}(t)$, we know $\psi_I(t)$.

Now we aim to find a Schrödinger equation for the interaction picture. In other words, we want to find the interaction picture Hamiltonian \hat{H}_I so that

$$i\hbar \frac{\partial}{\partial t} \psi_I(t) = \hat{H}_I \psi_I(t).$$

We know that a state vector $\psi(t)$ in the Schrödinger picture satisfies the Schrödinger equation so that

$$i\hbar \frac{\partial}{\partial t} \psi(t) = \hat{H} \psi(t) = (\hat{H}_0 + \hat{V}) \psi(t).$$

Therefore, we see that

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \psi_I(t) &= -i\hbar \frac{\partial}{\partial t} \left(e^{\frac{i\hat{H}_0 t}{\hbar}} \psi(t) \right) \\
&= i\hbar e^{\frac{i\hat{H}_0 t}{\hbar}} \cdot \frac{i\hat{H}_0}{\hbar} \psi(t) + e^{\frac{i\hat{H}_0 t}{\hbar}} i\hbar \frac{\partial}{\partial t} \psi(t) \\
&= -e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}_0 \psi(t) + e^{\frac{i\hat{H}_0 t}{\hbar}} \left(\hat{H}_0 + \hat{V} \right) \psi(t) \\
&= e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{V} \psi(t) \\
&= e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{V} e^{-\frac{i\hat{H}_0 t}{\hbar}} \psi_I(t).
\end{aligned}$$

Thus we conclude that the interaction picture Hamiltonian \hat{H}_I is given by

$$\hat{H}_I = e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{V} e^{-\frac{i\hat{H}_0 t}{\hbar}} = \hat{U}_0^*(t) \hat{V} \hat{U}_0(t).$$

Next we observe that $\hat{\Gamma}(t)$ satisfies the interaction picture Schrödinger equation:

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \hat{\Gamma}(t) &= i\hbar \frac{\partial}{\partial t} \hat{U}_0^*(t) \hat{U}(t) \\
&= i\hbar \frac{\partial}{\partial t} \left(e^{\frac{i\hat{H}_0 t}{\hbar}} \right) \hat{U}(t) + i\hbar \hat{U}_0^*(t) \frac{\partial}{\partial t} \left(e^{-\frac{i\hat{H} t}{\hbar}} \right) \\
&= i\hbar \left(\frac{i\hat{H}_0}{\hbar} \right) e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{U}(t) + i\hbar \hat{U}_0^* e^{-\frac{i\hat{H} t}{\hbar}} \left(-\frac{i\hat{H}}{\hbar} \right) \\
&= -\hat{H}_0 \hat{U}_0^*(t) \hat{U}(t) + \hat{U}_0^*(t) \hat{U}(t) \hat{H} \\
&= \hat{U}_0^*(t) \left(\hat{H} - \hat{H}_0 \right) \hat{U}(t) \\
&= \hat{U}_0^*(t) \hat{V} \hat{U}(t) \\
&= \hat{U}_0^*(t) \hat{V} \hat{U}_0(t) \hat{U}_0^*(t) \hat{U}(t) \\
&= \hat{U}_0^*(t) \hat{V} \hat{U}_0(t) \hat{\Gamma}(t) \\
&= \hat{H}_I \hat{\Gamma}(t)
\end{aligned} \tag{A.28}$$

Next, we show that $\hat{\Gamma}(t)$ can be written as a time-ordered exponential. Note, by the Fundamental Theorem of Calculus, we have

$$\hat{\Gamma}(t) = \hat{\Gamma}(0) + \int_0^t \frac{\partial}{\partial s} \hat{\Gamma}(s) ds$$

Now, using (A.28), and proceeding iteratively, we get

$$\begin{aligned}
\hat{\Gamma}(t) &= \hat{\Gamma}(0) - \frac{i}{\hbar} \int_0^t \hat{H}_I(s_1) \hat{\Gamma}(s_1) ds_1 \\
&= \mathbb{1} - \frac{i}{\hbar} \int_0^t \hat{H}_I(s_1) \left(\mathbb{1} - \frac{i}{\hbar} \int_0^{s_1} \hat{H}_I(s_2) \hat{\Gamma}(s_2) ds_2 \right) ds_1 \\
&= \mathbb{1} - \frac{i}{\hbar} \int_0^t \hat{H}_I(s_1) ds_1 + \left(-\frac{i}{\hbar} \right)^2 \int_0^t \int_0^{s_1} \hat{H}_I(s_1) \hat{H}_I(s_2) \hat{\Gamma}(s_2) ds_2 ds_1 \\
&= \mathbb{1} - \frac{i}{\hbar} \int_0^t \hat{H}_I(s_1) ds_1 + \left(-\frac{i}{\hbar} \right)^2 \int_0^t \int_0^{s_1} \hat{H}_I(s_1) \hat{H}_I(s_2) ds_2 ds_1 + \dots \\
&+ \left(-\frac{i}{\hbar} \right)^n \int_0^t \int_0^{s_1} \int_0^{s_2} \dots \int_0^{s_{n-1}} \hat{H}_I(s_1) \hat{H}_I(s_2) \dots \hat{H}_I(s_n) ds_n ds_{n-1} \dots ds_2 ds_1 + \dots \\
&= \mathbb{1} + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar} \right)^n \int_0^t \int_0^{s_1} \int_0^{s_2} \dots \int_0^{s_{n-1}} \prod_{i=1}^n \hat{H}_I(s_i) ds_n ds_{n-1} \dots ds_1
\end{aligned}$$

Note that we have the string of inequalities $t > s_1 > s_2 > \dots > s_n > \dots > 0$.

The series above is called the Dyson series. Note that $\hat{H}(s_1)\hat{H}(s_2)\cdots\hat{H}(s_n)$ is time ordered, i.e. $\hat{H}(s_n)$ acts before $\hat{H}(s_{n-1})$, which acts before $\hat{H}(s_{n-2})$, and so on until $\hat{H}(s_1)$ acts last. We can rewrite the Dyson series in a more compact way using the time-ordering operator \mathcal{T} , which reorders operators A_1, \dots, A_n so that

$$\mathcal{T}(A_1(s_1)A_2(s_2)\cdots A_n(s_n)) = A_{i_1}(s_{i_1})A_{i_2}(s_{i_2})\cdots A_{i_n}(s_{i_n}), \quad s_{i_1} < s_{i_2} < \dots < s_{i_n}. \quad (\text{A.29})$$

First we try to find an expression for $\mathcal{T}\left(\int_0^t \hat{H}_I(r)dr\right)^n$. We begin by looking at the case $n = 2$:

$$\begin{aligned} \mathcal{T}\left(\int_0^t \hat{H}_I(r)dr\right)^2 &= \mathcal{T}\int_0^t \int_0^t ds_2 ds_1 \hat{H}_I(s_1)\hat{H}_I(s_2) \\ &= \int_0^t \int_0^t ds_2 ds_1 \mathcal{T}\hat{H}_I(s_1)\hat{H}_I(s_2) \\ &= \iint_{t>s_1>s_2>0} ds_2 ds_1 \hat{H}_I(s_1)\hat{H}_I(s_2) + \iint_{t>s_2>s_1>0} ds_2 ds_1 \hat{H}_I(s_2)\hat{H}_I(s_1) \end{aligned}$$

Due to symmetry, the last two integrals are equivalent, and so

$$\mathcal{T}\left(\int_0^t \hat{H}_I(r)dr\right)^2 = 2 \iint_{t>s_1>s_2>0} ds_2 ds_1 \hat{H}_I(s_1)\hat{H}_I(s_2)$$

or

$$\iint_{t>s_1>s_2>0} ds_2 ds_1 \hat{H}_I(s_1)\hat{H}_I(s_2) = \frac{1}{2}\mathcal{T}\left(\int_0^t \hat{H}_I(r)dr\right)^2$$

We can generalize this to get

$$\int \cdots \int_{t>s_1>s_2>\dots>s_n>0} ds_n \cdots ds_2 ds_1 \hat{H}_I(s_1)\hat{H}_I(s_2)\cdots\hat{H}_I(s_n) = \frac{1}{n!}\mathcal{T}\left(\int_0^t \hat{H}_I(r)dr\right)^n$$

Therefore, the Dyson series becomes a time-ordered exponential and we get the following expression for $\hat{\Gamma}$:

$$\begin{aligned} \hat{\Gamma}(t) &= 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\mathcal{T}\left(\int_0^t \hat{H}_I(r)dr\right)^n \\ &= \mathcal{T}e^{-\frac{i}{\hbar}\int_0^t \hat{H}(r)dr} \end{aligned} \quad (\text{A.30})$$

A.5.7 Quantum Analogue to the Poisson Bracket

Classical mechanics is a valid description for mechanical systems only when such systems are large enough. For quantum mechanics to be a valid description for any mechanical system, then, classical mechanics must be a limiting case of quantum mechanics. Thus there exists a quantum analogue to the Poisson bracket that also must obey properties (A.13).

To find this quantum analogue, we evaluate $\{u_1 u_2, v_1 v_2\}$ in two different ways. Note, using property (A.13d), we get

$$\begin{aligned} \{u_1 u_2, v_1 v_2\} &= \{u_1, v_1 v_2\}u_2 + u_1\{u_2, v_1 v_2\} \\ &= (\{u_1, v_1\}v_2 + v_1\{u_1, v_2\})u_2 + u_1(\{u_2, v_1\}v_2 + v_1\{u_2, v_2\}) \\ &= \{u_1, v_1\}v_2 u_2 + v_1\{u_1, v_2\}u_2 + u_1\{u_2, v_1\}v_2 + u_1 v_1\{u_2, v_2\} \end{aligned}$$

and using (A.13a) and (A.13a) we see

$$\begin{aligned} \{u_1 u_2, v_1 v_2\} &= \{u_1 u_2, v_1\}v_2 + v_1\{u_1 u_2, v_2\} \\ &= \{u_1, v_1\}u_2 v_2 + u_1\{u_2, v_1\}v_2 + v_1\{u_1, v_2\}u_2 + v_1 u_1\{u_2, v_2\}. \end{aligned}$$

Thus it follows that

$$\{u_1, v_1\} (u_2 v_2 - v_2 u_2) = (u_1 v_1 - v_1 u_1) \{u_2, v_2\}.$$

Now, in classical mechanics, any two dynamical variables commute. So in classical mechanics, the above equation simply says that $0 = 0$. However, in quantum mechanics, dynamical variables are represented by operators, and they do not necessarily commute.

Assuming u_1 and v_1 are independent of u_2 and v_2 , we must have

$$[u_1, v_1] = u_1 v_1 - v_1 u_1 = k \{u_1, v_1\} \quad \text{and} \quad [u_2, v_2] = u_2 v_2 - v_2 u_2 = k \{u_2, v_2\},$$

where k is some constant.

It has been determined that $k = i\hbar$. Formally, we get that for any two variables u and v and their corresponding pair of quantum operators \hat{u} and \hat{v}

$$\{u, v\} \rightarrow \frac{[\hat{u}, \hat{v}]}{i\hbar} \tag{A.31}$$

Hence, in quantum mechanics, the Poisson bracket $\{u, v\}$ is replaced by $\frac{[\hat{u}, \hat{v}]}{i\hbar}$. (Note that $\frac{[\hat{u}, \hat{v}]}{i\hbar}$ satisfies all properties (a)–(e).) In other words, the Poisson bracket of any two dynamical variables is given by taking the commutator of their corresponding pair of quantum operators and dividing it by $i\hbar$, so that $\{u, v\} \rightarrow \frac{[\hat{u}, \hat{v}]}{i\hbar}$. Note, however, that we can not necessarily obtain the classical Poisson bracket by calculating $\frac{[\hat{u}, \hat{v}]}{i\hbar}$.

Example A.5.19. Let us evaluate the Poisson bracket $\{x, p\}$ between the position and momentum variables, and compare this with the commutator $[\hat{x}, \hat{p}]$ of the corresponding position and momentum operators. Note

$$\{x, p\} = \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial p}{\partial x} = \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} = 1$$

In quantum mechanics, $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ and $\hat{x} = x$, so applying the commutator $[\hat{x}, \hat{p}]$ to some state φ we get

$$\begin{aligned} [\hat{x}, \hat{p}]\varphi &= x \left(-i\hbar \frac{\partial}{\partial x} \right) \varphi - \left(-i\hbar \frac{\partial}{\partial x} \right) (x\varphi) \\ &= -i\hbar x \frac{\partial \varphi}{\partial x} + i\hbar \varphi + i\hbar x \frac{\partial \varphi}{\partial x} \\ &= i\hbar \varphi \end{aligned}$$

Thus, $[\hat{x}, \hat{p}] = i\hbar$, and we see that $\{x, p\} \rightarrow \frac{1}{i\hbar} [\hat{x}, \hat{p}]$, which agrees with (A.31).

A.5.8 The Classical Limit, Quantization, and Phase-Space Formulation

Rewriting (A.14) using (A.31), we get the following set of quantum equations:

$$\begin{aligned} q_j q_k - q_k q_j &= p_j p_k - p_k p_j = 0 \\ q_j p_k - p_k q_j &= i\hbar \delta_{jk} \end{aligned}$$

Classically, q_j and p_k commute, so we see that by taking $\hbar \rightarrow 0$ we can recover classical equations. Recovering classical equations from quantum ones in such a way is known as a *classical limit*. Going the other way (from classical equations to quantum ones) is known as *quantization* and is usually understood in terms of deformation of algebras, which we discuss in Chapter (2). As we have seen, quantization involves replacing Poisson brackets with commutators multiplied by $\frac{1}{i\hbar}$.

Definition A.5.20. The *classical limit* refers to the procedure one takes to “recover” classical mechanics from quantum mechanics.

Definition A.5.21. *Quantization* refers a deformation procedure one takes to obtain quantum theory from classical theory.

Remark A.5.22. Quantization is often understood as a procedure in which one finds a mapping that associates to any classical observable (a real-valued function $a(q, p)$ of the momenta and position) a self-adjoint operator $\hat{A}(Q, P)$ on a Hilbert space. There is also such a thing known as *second quantization*, which is different than quantization. We discuss second quantization in Chapter (2).

Comparing quantum mechanics and classical mechanics is often difficult when the mathematical formulations of each theory are so different. Therefore, it becomes necessary to either describe quantum mechanics in a way that more closely resembles classical mechanics, or the other way around. Although both are possible, here we are only interested in the mathematical formulation that allows quantum mechanics to more closely resemble classical mechanics. This formulation of quantum mechanics is known as the *phase space formulation* of quantum mechanics.

Definition A.5.23. The *phase space formulation* of quantum mechanics is an alternate formulation of quantum mechanics that relies on the notion of a phase space (symplectic manifold) used in Hamiltonian mechanics in order to make logical comparisons between quantum mechanics and classical mechanics.