An Analysis of Spin Correlations in $^{12}C(d, ^2He)^{12}B$ to Test the Bell and Wigner Inequalities: A Tale of Two Protons

A Thesis Submitted to the College of Graduate Studies and Research in Partial Fulfillment of the Requirements for the Degree of Master of Science in the Department of Physics, University of Saskatchewan

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Abstract

Arising out of the EPR debate, the Bell and Wigner inequalities of quantum theory are experimentally testable relations which address the question of quantum completeness. This thesis provides feasibility results for a fermionic test of the Bell inequality, and the first known attempt to investigate the completeness question through the Wigner relation. Such a test is made possible by the production of an entangled p-p singlet at the KVI research facility in Groningen, the Netherlands, through the reactions $^{12}C(d,^2He)^{12}B$ and $p(d,^2He)n$. The p-p spin-correlations are analyzable via the KVI's $2\pi$ polarimeter acceptance, which eliminates loopholes common to previous experiments. The results distinguish between a hidden variables and quantum mechanical description of the universe. Also presented is a critique of the GHZ argument against the existence of local hidden variables.
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Chapter 1

Introduction

1.1 Motivation

In 1935 Albert Einstein, Boris Podolsky, and Nathan Rosen, published an historic paper [1] in which these three Princeton physicists challenged a basic assumption about the theory of quantum mechanics: the assumption of its completeness. This paper, which has come to be known as EPR, lays out an argument based on fundamental conservation principles, attempting to show that the quantum mechanical description of an entangled system is lacking in its ability to describe such a system completely. (A definition of the concept of completeness according to EPR follows in section 1.2). A debate over EPR’s relevance was carried on by physicists until an experimentally testable framework [2] for its claim was finally put forward by John Bell in 1964.

The quantum mechanical description of the physical universe has certain qualities which are of particular interest to a fast-developing field of computational science: quantum computing. Quantum computing and quantum information theory are founded upon an understanding of entanglement that assumes quantum mechanical completeness. Existing quantum search algorithms, for instance, employ the unique qualities of quantum systems in a superposition of states where the overall wavefunction describes an “entangling” of otherwise independent particles. [3, 4]
The concept of an entangled state is absent from the formulations of classical physics. Considered in the context of the quantum commutation relations, it requires non-local interactions between entangled entities: an exchange of information through a seemingly Newtonian “action-at-a-distance” which guarantees an adherence to the restrictions put on the entangled system by its overall wave function. That this is the case became clear through the work of Bell, and of the experiments performed in response to his work.

Its history began with Einstein and was continued through the work of David Bohm [5, 6], John Bell (and more recently a handful of refinements on Bell’s ideas, notably the work of Wigner [7] and Hardy [8]; of Greenberger, Horne and Zeilinger [9]; and of Kochen and Specker [10]). The EPR study is not solely of philosophical or metaphysical interest (valuable and interesting as that might be), but has critical application in emerging and very promising fields of applied physics. A test of EPR’s challenge to the completeness of quantum theory is crucial to a proper and rigorous understanding of current work in quantum computing. [11] In the quest for an understanding of the way the universe works—its inherent qualities, principles, and mechanisms—the EPR debate over the completeness of the quantum theory is also of fundamental importance. It cuts to the core of what seems to be the source of much strangeness in modern physics.

1.2 Reality and Completeness

The aim of the theoretical scientist, according to the philosopher Karl Popper, is to develop *explanatory theories* that depict “certain structural properties of the world, and which permit us to deduce, with the help of initial conditions, the effects to be explained” [12]. Such a theory is naturally testable: its predictions must relate to quantities, events, or other information which are experimentally observable. If a theory makes a prediction which is fundamentally (or ultimately) untestable, then at least that portion of the theory is practically useless to the scientific community, for it speaks of “observables” of which we cannot know.
Moreover, one might introduce a further criterion, desiring one’s scientific theory to be not only explanatory, but also complete. A complete theory is one which addresses every quantity or observable in its realm of influence, leaving none out. If an experimenter can actually gather information about the system he studies, yet the theorists can only provide him with an explanatory theory which has no “knowledge” of this observable information, we might still have reason to consider such a theory to be good—but fundamentally incomplete. It may be very useful in describing the “real-world system” under study, but it cannot be relied upon to predict everything about the system with which the experimenter is dealing. The experimenter is left guessing about some of the observables he may encounter, since he has more information about the system than the theorist believed available.

In the context of a physical theory, therefore, completeness is a property of those theories which are able to assign an observable to every physical attribute in the “real” system they are designed to describe [13]. All attributes of the system occur as definite-valued entities in the theory.

Consideration of a “real” system, however, begs the question of what is meant by reality. Addressing this question on some level is essential to determining whether a given theory is complete or not. While there is much philosophical and metaphysical richness in pursuing rigorous definitions, here the primary concern is the practicalities of the scientific endeavour. To that end, the definition of physical reality need only be sufficient rather than necessary. Some elements of the physical reality may escape that definition, but those that are bound up in it will indeed be accepted as “real”. As a framework for this sufficiency, Einstein, Podolsky and Rosen proposed that “the elements of the physical reality...must be found by an appeal to results of experiments and measurements”.

A satisfactory and reasonable definition for reality was given by EPR in a single criterion:

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a
physical quantity, then there exists an element of physical reality corresponding to this physical quantity. [13]

For our practical purposes, something that is physically observable is considered to be an element of physical reality. It is a use of Occam’s Razor, that the best explanation of why a successful measurement at some particular time is made, is that there exists at that time some element of reality having that predictable value. Predictability arises from regularities underlying our elements of reality, and if there were no elements of reality extant at the time of measurement, we should not expect to have made a successful prediction [14]. Thus, a theory which purports to be a complete theory will make an appropriate prediction for everything which has been empirically observed in the system it describes, at some particular time.

In many disciplines there are good theories which are quite obviously incomplete. In economics, a model which predicts market trends may do well in anticipating macroscopic phenomena, but can say little about an individual investor. In physics, thermodynamics is an incomplete theory, able to describe temperatures and pressures, yet ignorant of the individual behaviour of the particles which produce the observed effects. Conversely, there is one theoretical masterpiece which has long been regarded as a complete theory: quantum mechanics. Unlike thermodynamics or economic theory, quantum mechanics seeks to explain fundamental reality. Physics students are often taught that a wave function contains all the information one can possibly know about the system which it describes [15, 16, 17]. This is the very essence of completeness, a view advocated by the founding fathers of the Copenhagen interpretation, among them Dirac and Heisenberg [18]. The theory of quantum mechanics, then, so long as it has been founded on this notion of the wave function, is popularly held to be complete.
1.3 Quantum Commutation Rules and Angular Momentum States

At the heart of quantum theory lies the commutation relations. It is these relations, which Dirac termed the "fundamental quantum conditions" [19], that are ultimately responsible for most of the paradoxical pronouncements for which quantum mechanics is famous. Though based on and derived from classical principles (by analogy), these relations inform us of whether or not we are permitted (at least insofar as the formalism is concerned) to have a simultaneous knowledge of two quantum mechanical operators.

The derivations following are an attempt to demonstrate in a concise way how the singlet state of two spin-$\frac{1}{2}$ fermions arises out of the algebra of the commutation relations. It is this singlet state which is at the heart of the EPRB argument described in section 2.2, as well as the basic formalism employed to deal with spin angular momentum. This formalism is used extensively in the GHZ argument of chapter 5, applied to a specific state of four particles rather than the two of EPRB. More thorough discussions of the commutation relations and/or angular momentum algebra are scattered in the texts, among others Dirac [20], Sakurai [21], Mandl [22], Biedenharn and Louck [23], Goldstein [24] and Liboff [25].

The algebra of the quantum commutation relation is closely related to that of a similar classical relation: the Poisson Bracket, defined for dynamical variables $u(q,p)$ and $v(q,p)$ as,

$$[u(q,p), v(q,p)] = \sum_s \left( \frac{\partial u}{\partial q_s} \frac{\partial v}{\partial p_s} - \frac{\partial v}{\partial q_s} \frac{\partial u}{\partial p_s} \right)$$

(1.1)

where $q_s$ and $p_s$ are $2s$ canonical variables satisfying Hamilton’s equations,

$$\dot{q}_s = \frac{\partial H}{\partial p_s}$$

(1.2)

$$-\dot{p}_s = \frac{\partial H}{\partial q_s}.$$  

(1.3)

The properties of the Poisson Bracket, resulting from this definition, are:
\[ [u, u] = 0, \]  
\[ [u, v] = -[v, u], \]  
\[ [u, c] = 0, \]  
(1.4)  
(1.5)  
(1.6)  
(where c is a constant),  
\[ [u_1 + u_2, v] = [u_1, v] + [u_2, v] \]  
\[ [u, v_1 + v_2] = [u, v_1] + [u, v_2] \]  
\[ (1.7) \]  
\[ [u_1 u_2, v] = [u_1, v] u_2 + u_1 [u_2, v] \]  
\[ [u, v_1 v_2] = [u, v_1] v_2 + v_1 [u, v_2] \]  
(1.8)  
and the Jacobi Identity,  
\[ [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \]  
(1.9)  

Following Dirac [26], we seek a quantum mechanical equivalent to the classical Poisson Bracket which will satisfy the above conditions and also those particular to the quantum theory. One such quantum demand is that the order of the variables must be protected in the relation \([u_1 u_2, v_1 v_2]\). Using the given conditions, this can be evaluated two ways:  
\[ [u_1 u_2, v_1 v_2] = [u_1, v_1 v_2] u_2 + u_1 [u_2, v_1 v_2] \]  
\[ = [u_1, v_1] v_2 u_2 + v_1 [u_1, v_2] u_2 + u_1 [u_2, v_1] v_2 + u_1 v_1 [u_2, v_2] \]  
(1.10)  
and  
\[ [u_1 u_2, v_1 v_2] = [u_1 u_2, v_1] v_2 + v_1 [u_1 u_2, v_1] \]  
\[ = [u_1, v_1] u_2 v_2 + u_1 [u_2, v_1] v_2 + v_1 [u_1, v_2] u_2 + v_1 u_1 [u_2, v_2]. \]  
(1.11)  

Equating these gives  
\[ [u_1, v_1] (u_2 v_2 - v_2 u_2) = (u_1 v_1 - v_1 u_1) [u_2, v_2]. \]  
(1.12)
Since \( u_1 \) and \( v_1 \) can be independent of \( u_2 \) and \( v_2 \), the result is

\[
\begin{align*}
    u_1v_1 - v_1u_1 &= C[u_1, v_1] \\
    u_2v_2 - v_2u_2 &= C[u_2, v_2].
\end{align*}
\]  

(1.13)

\( C \) must be a factor independent of the four variables, and if we wish the quantum Poisson Bracket to have a real value as with the classical Bracket then it must be multiplied by a factor of \( \iota \), since \( \eta \xi - \xi \eta \) by itself is not real-valued where the operators \( \eta \) and \( \xi \) are Hermitian. The constant \( C \) is redefined with a factor \( \iota \) as \( C = \iota \hbar \), giving the usual form of the commutation relation,

\[
\iota \hbar [u, v] = uv - vu
\]

(1.14)

where \( \hbar \) is Planck's universal constant divided by \( 2\pi \), arising from experimental considerations rather than the basic tenets of the theory.

It is apparent from this derivation that the value of any quantum mechanical commutation relation can be arrived at from the corresponding classical Poisson Bracket via the relation,

\[
[u, v]_{\text{classical}} \rightarrow \frac{[u, v]_{\text{QM}}}{\iota \hbar},
\]

(1.15)

allowing that we can interpret the classical parameters as operators in the quantum theory.

Thus, for the canonical coordinates, \( q_i \), and momenta, \( p_i \), (where \( i, j, k \) denote the directions along the three space axes) one finds the commutation relations among the corresponding quantum operators to be

\[
[q_i, q_j]_{\text{QM}} = \iota \hbar [q_i, q_j]_{\text{classical}} = 0
\]

(1.16)

\[
[p_i, p_j]_{\text{QM}} = 0
\]

(1.17)

\[
[q_i, p_j] = \iota \hbar \delta_{ij}.
\]

(1.18)

Further, one can derive the relations for a quantum mechanical angular momentum operator by analogy to the classical case where the angular mo-
momentum, \( \vec{L} \), is found from the vector cross-product of the Cartesian position and its conjugate linear momentum:

\[
\vec{L} = \vec{x} \times \vec{p}.
\]

(1.19)

Thus, the classical angular momentum Poisson Bracket is \([L_i, L_j] = \epsilon_{ijk} L_k\) giving the quantum result,

\[
[L_i, L_j] = \hbar \epsilon_{ijk} L_k.
\]

(1.20)

In the quantum theory, the total angular momentum operator, \( \mathbf{J} \), is the sum of two components: the orbital angular momentum, \( \mathbf{L} \), which is equivalent to the classical case; and the spin angular momentum, \( \mathbf{S} \), which has no classical analogue. Thus, \( \mathbf{J} = \mathbf{L} + \mathbf{S} \). Unrelated to the spatial coordinates of a system, the algebra of spin must be asserted rather than derived from a classical analogue. The form of the commutation relation for \( \mathbf{S} \) is inferred from the operator’s formal similarity to \( \mathbf{L} \) and it is usually given as a postulate that

\[
[J_i, J_j] = \hbar \epsilon_{ijk} J_k
\]

(1.21)

to arrive at a commutator for the total quantum angular momentum observable [27]. The same type of relation holds for \( \mathbf{S} \) alone:

\[
[S_i, S_j] = \hbar \epsilon_{ijk} S_k.
\]

(1.22)

From (1.21) it follows that each component of \( \mathbf{J} \) commutes with itself (the value of the commutation relation is zero):

\[
[J_i, J_i] = 0
\]

(1.23)

and each commutes with the operator \( \mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 \):

\[
[J^2, J_i] = 0.
\]

(1.24)

(1.21) also shows that different components of the \( \mathbf{J} \) operator are incompatible: they do not commute, since the value of their commutator is non-zero.
It is a basic principle of quantum theory that one cannot simultaneously and precisely specify a system’s eigenvalues corresponding to two incompatible operators. Given the non-commutativity of the angular momentum components, then, the empirical consequence is the uncertainty relation,

\[(\Delta J_i)(\Delta J_j) \geq \frac{\hbar}{2} |\langle J_k \rangle| . \tag{1.25}\]

Except for the case where all components are 0, systems for which any one component, \(J_i\), is precisely determined \([\langle \Delta J_i \rangle^2 = 0]\) will have random distributions of the other two components.

Following Sakurai [28] and Liboff [29], the commutation relations allow the derivation of angular momentum eigenvalues. Two ladder operators are defined as,

\[J_\pm = J_x \pm iJ_y, \tag{1.26}\]

where \(J_x\) and \(J_z\) (as well as \(J_z\)) are the projection operators of \(J\) along the indicated spatial axes. Although non-Hermitian,

\[J_- = J_+^\dagger. \tag{1.27}\]

It follows from the definition (1.26) along with (1.21) and (1.24) that

\[[J_+, J_-] = 2\hbar J_z, \tag{1.28}\]

\[[J^2, J_\pm] = 0 \tag{1.29}\]

and

\[[J_z, J_\pm] = \pm \hbar J_\pm. \tag{1.30}\]

Consider the as-yet undetermined eigenvalues of \(J^2\) and \(J_z\), denoted by \(\mathcal{A}\) and \(b\) for the eigenstate \(|a, b\rangle\):

\[J^2 |a, b\rangle = \mathcal{A} |a, b\rangle \tag{1.31}\]
\[ J_z |a, b\rangle = b |a, b\rangle. \] (1.32)

\( J_z \) acting on \( J_\pm |a, b\rangle \) gives,

\[
J_z(J_\pm) |a, b\rangle &= (J_z J_\pm + J_\pm J_z - J_\pm J_z) |a, b\rangle \\
&= ([J_z, J_\pm] + J_\pm J_z) |a, b\rangle \\
&= (\pm \hbar J_\pm + b J_\pm) |a, b\rangle \\
&= (\pm \hbar + b) J_\pm |a, b\rangle. \] (1.33)

An eigenstate of the operator \( J_z \) to which a ladder operator is applied remains an eigenstate of \( J_z \)—with its eigenvalue increased or decreased by one unit of \( \hbar \). Because of (1.29), however, the ladder operators do not affect the eigenvalues of \( J^2 \). Thus, the commutation algebra gives

\[
J_\pm |a, b\rangle = c_\pm |a, b \pm \hbar\rangle \] (1.34)

where \( c_\pm \) is some constant, and it is clear that \( n_\pm \) successive operations of \( J_\pm \) on the state will result in a series of eigenstates \( |a, b + n_+ \hbar - n_- \hbar\rangle \) common to both operators \( J^2 \) and \( J_z \).

Applying the eigenbra, \( \langle a, b \mid \), to (1.31),

\[
\langle a, b \mid J^2 \mid a, b \rangle = \mathcal{A} \\
= \langle a, b \mid (J_x^2 + J_y^2 + J_z^2) \mid a, b \rangle \\
= \langle J_x^2 \rangle + \langle J_y^2 \rangle + b^2 \] (1.35)

where \( \langle J_x^2 \rangle \) and \( \langle J_y^2 \rangle \) are the expectation values (ensemble averages) for the operators \( J_x^2 \) and \( J_y^2 \). Since these values must be positive\(^1\) it follows that \( \mathcal{A} \geq b^2 \) and there exists some maximum value of \( b, b_{\text{max}} \), so that

\[
J_+ |a, b_{\text{max}}\rangle = 0 \] (1.37)

and some minimum value, \( b_{\text{min}} \) such that

---

\(^1\)The eigenvalues, \( \xi \) of any Hermitian operator, \( \Xi \), are by necessity real-valued, and by definition \( \Xi = \Xi^\dagger \), so for any eigenstate \( \langle \xi \mid \), of this operator,

\[
\langle \xi \mid \Xi^2 \mid \xi \rangle = \langle \xi \mid \Xi \Xi \mid \xi \rangle = \langle \xi \mid \Xi^\dagger \Xi \mid \xi \rangle = \xi^2 > 0. \] (1.36)
\[ J_- |a, b_{\text{min}}\rangle = 0. \]  \hfill (1.38)

With (1.26) and (1.21) it follows that

\[
J_- J_+ |a, b_{\text{max}}\rangle = 0 \\
= [J_x^2 + J_y^2 - \imath(J_y J_x - J_x J_y)] |a, b_{\text{max}}\rangle \\
= (J^2 - J_x^2 - \hbar J_x) |a, b_{\text{max}}\rangle \\
= (\mathcal{A} - b_{\text{max}}^2 - \hbar b_{\text{max}}) |a, b_{\text{max}}\rangle. \]  \hfill (1.39)

Since the eigenket is not the null ket,

\[
\mathcal{A} - b_{\text{max}}^2 - \hbar b_{\text{max}} = 0 \\
\Rightarrow \mathcal{A} = b_{\text{max}}(b_{\text{max}} + \hbar) \]  \hfill (1.40)

and by similar argument

\[
\mathcal{A} = b_{\text{min}}(b_{\text{min}} - \hbar) \]  \hfill (1.41)

so that

\[
b_{\text{max}} = -b_{\text{min}}. \]  \hfill (1.42)

There should be some \( n \) number of applications of \( J_+ \) to \( |a, b_{\text{min}}\rangle \) that will result in the state \( |a, b_{\text{max}}\rangle \), so that

\[
b_{\text{max}} = b_{\text{min}} + n\hbar. \]  \hfill (1.43)

Thus,

\[
b_{\text{max}} = \frac{n\hbar}{2} \]  \hfill (1.44)

and with (1.40),

\[
\mathcal{A} = \frac{n(n + 2)\hbar^2}{4}. \]  \hfill (1.45)

Defining \( j = n/2 \),

\[
\mathcal{A} = \hbar^2 j(j + 1) \]  \hfill (1.46)
the eigenvalue of $\mathbf{J}^2$ where $j$ can take integer or half-integer values. Also defining $m$ such that $b \equiv m\hbar$, we have

\[
\begin{align*}
b_{\text{min}} & \leq b \leq b_{\text{max}} \\
-\hbar j & \leq \hbar m \leq \hbar j \\
-j & \leq m \leq j
\end{align*}
\]

(1.47)

giving $(2j + 1)$ values for $m$.

Using only the commutation relations (1.21) we arrive at the angular momentum eigenvalue equations,

\[
\mathbf{J}^2 \left| j, m \right\rangle = \hbar^2 j(j+1) \left| j, m \right\rangle
\]

(1.48)

and

\[
J_z \left| j, m \right\rangle = \hbar m \left| j, m \right\rangle
\]

(1.49)

showing the quantization of angular momentum in integral or half-integral steps.²

As with the commutators from which the eigenvalue equations are determined, the results all apply equally well to the spin operator $\mathbf{S}$. Thus,

\[
\mathbf{S}^2 \left| s, m \right\rangle = \hbar^2 s(s+1) \left| s, m \right\rangle
\]

(1.50)

and

\[
S_z \left| s, m \right\rangle = \hbar m \left| s, m \right\rangle
\]

(1.51)

are true for integral and half-integral values of $s$. Systems in nature which possess an overall non-zero spin can be grouped according to that spin being integral or half-integral. Those with integral spin are termed bosons, while the half-integral spin systems are fermions.

Among the fermions electrons, protons and neutrons have spin eigenvalue $s = \hbar/2$ and the $2s + 1$ multiplicity of the projection eigenvalue along the quantization axis (usually labeled $z$) gives $s_z = \pm \hbar/2$. The two possible spin

²While $j$ may be half-integral, the orbital angular momentum quantum number, $l$, is always an integer. Thus a half-integral $j$ results from half-integral spin, as described below.
states of such a particle are

\[ \alpha \equiv |\frac{1}{2}, +\frac{1}{2}\rangle \]  

and

\[ \beta \equiv |\frac{1}{2}, -\frac{1}{2}\rangle. \]  

(1.52)

These states can be re-written in the spinor matrix formalism of Pauli as

\[ \alpha \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] \[ \beta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

(1.54)

called the up state, often simply denoted by \(|\uparrow\rangle\), and the down state, or \(|\downarrow\rangle\),

(1.55)

The ladder operators for spin affect these two states giving

\[ S_+ \alpha = 0 \quad S_+ \beta = \hbar \alpha \]

\[ S_- \alpha = \hbar \beta \quad S_- \beta = 0. \]

(1.56)

From (1.26), we then have

\[ S_x \alpha = \frac{\hbar}{2} \beta \quad S_x \beta = \frac{\hbar}{2} \alpha \]

\[ S_y \alpha = \frac{\hbar}{2} \beta \quad S_y \beta = -\frac{\hbar}{2} \alpha \]

(1.57)

along with

\[ S_z \alpha = \frac{\hbar}{2} \alpha \quad S_z \beta = -\frac{\hbar}{2} \beta. \]

(1.58)

The Pauli spin operators \(\sigma_i = (2/\hbar)S_i \ (i = x, y, z)\) can thus be determined in a matrix form which can act on an arbitrary spinor:

\[ \sigma_x = \frac{2}{\hbar} \left( \begin{pmatrix} \langle \alpha | S_x | \alpha \rangle \\ \langle \beta | S_x | \alpha \rangle \end{pmatrix}, \begin{pmatrix} \langle \alpha | S_x | \beta \rangle \\ \langle \beta | S_x | \beta \rangle \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

(1.59)

Similarly,

\[ \sigma_y = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]  

(1.60)
\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  (1.61)

Along some arbitrary direction, \( \hat{n}_i \) with angles \( \theta_i \) and \( \phi_i \), the Pauli matrix is given by

\[ (\sigma_i \cdot \hat{n}_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i e^{-i\phi_i} \\ \sin \theta_i e^{i\phi_i} & \cos \theta_i \end{pmatrix}. \]  (1.62)

A system composed of two protons will have a total spin operator

\[ \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2, \]  (1.63)

with total spin projection eigenvalue denoted \( m \) and their individual projections \( m_1 \) and \( m_2 \) such that \( m = m_1 + m_2 \). Defining an exchange operator, \( \mathcal{X} \), that exchanges the spatial and spin coordinates of the two particles in the system such that

\[ \mathcal{X} \left| s_1, m_1 \right\rangle \otimes \left| s_2, m_2 \right\rangle = \left| s_2, m_2 \right\rangle \otimes \left| s_1, m_1 \right\rangle, \]  (1.64)

(where \( \otimes \) indicates the direct product) we find that

\[ \mathcal{X}^2 \left| s_1, m_1 \right\rangle \otimes \left| s_2, m_2 \right\rangle = \left| s_1, m_1 \right\rangle \otimes \left| s_1, m_2 \right\rangle \]  (1.65)

so the exchange operator has eigenvalues \( \pm 1 \). One can construct a “symmetric” eigenfunction of the operator as

\[ \phi_s = \left| s_1, m_1 \right\rangle \otimes \left| s_2, m_2 \right\rangle + \left| s_2, m_2 \right\rangle \otimes \left| s_1, m_1 \right\rangle \]  (1.66)

with eigenvalue \( +1 \), and an “anti-symmetric” eigenfunction

\[ \phi_a = \left| s_1, m_1 \right\rangle \otimes \left| s_2, m_2 \right\rangle - \left| s_2, m_2 \right\rangle \otimes \left| s_1, m_1 \right\rangle \]  (1.67)

with eigenvalue \( -1 \).

\( \mathcal{X} \) commutes with \( \mathbf{S}^2 \) and \( S_z \) so these three operators have common eigenstates. With each particle individually in either of states (1.52) or (1.53), there are four possibilities for the overall state of the system to be an eigenstate of all three operators:
\[ |S = 1, M = 1\rangle \equiv |\uparrow\uparrow\rangle \]  \hspace{1cm} (1.68)

\[ |S = 1, M = 0\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \]  \hspace{1cm} (1.69)

\[ |S = 1, M = -1\rangle \equiv |\downarrow\downarrow\rangle \]  \hspace{1cm} (1.70)

\[ |S = 0, M = 0\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \]  \hspace{1cm} (1.71)

(The pair of arrows on the RHS of the formulas indicates the direct product of the individual spin-orientation of particles one and two in that order.)

The triplicity of \( S = 1 \) leads to these three states being collectively labeled the \textit{triplet state}, while (1.71) is called the \textit{singlet state}.

The singlet state and the triplet state with \( M = 0 \) both have two notable qualities:

1. The two particles co-exist in a \textit{superposition} of states that is non-separable into a product of simpler superpositions. Such a system is said to be \textit{entangled}, where properties of one particle are associated with the specific properties of the other to which it is coupled.

2. Since the total spin of these states is zero, the individual spin projections of the two particles along the quantization axis are opposite \((m_1 = -m_2)\) and there is an equal probability that a measurement on the system will leave it in either state \(|\uparrow\downarrow\rangle\) or \(|\downarrow\uparrow\rangle\).

The singlet has a third significant property lacking in the triplet: \textit{rotational invariance}, which can be seen by applying the general Pauli matrix (1.62), with \( \phi = 0 \), to the singlet state:

\[
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}_1 \otimes \begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}_2 \left[ \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \otimes \begin{pmatrix}
  0 \\
  1
\end{pmatrix} - \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \otimes \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \right]
\]

\[
= \begin{pmatrix}
  \cos \theta & \sin \theta
\end{pmatrix} \begin{pmatrix}
  1 \\
  0
\end{pmatrix} + \sin \theta \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \otimes \begin{pmatrix}
  -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  1 \\
  0
\end{pmatrix} + \cos \theta \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \otimes \begin{pmatrix}
  -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  0 \\
  1
\end{pmatrix}
\]

\[
- \begin{pmatrix}
  -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  1 \\
  0
\end{pmatrix} + \cos \theta \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \otimes \begin{pmatrix}
  \cos \theta & \sin \theta
\end{pmatrix} \begin{pmatrix}
  1 \\
  0
\end{pmatrix} + \sin \theta \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \otimes \begin{pmatrix}
  \cos \theta & \sin \theta
\end{pmatrix} \begin{pmatrix}
  0 \\
  1
\end{pmatrix}
\]
\[
\begin{align*}
&= \left[(\sin^2 \theta + \cos^2 \theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (\sin^2 \theta + \cos^2 \theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \\
&= \left[\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right]. \\
\end{align*}
\]

recovering the form of the singlet. In contrast, the triplet state transforms under rotation as

\[
\begin{align*}
\Sigma^{(\theta)} &\rightarrow \sin 2\theta \left[\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right] \\
&+ \cos 2\theta \left[\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right].
\end{align*}
\]

The rotational invariance of the singlet state ensures that, unlike the triplet, the spin projections of the two entangled particles are opposed to one another along every axis in space, and not just the quantization axis.

### 1.4 Interpretation and Significance

Starting with the quantum commutation relations, it has been shown in the previous section that one can derive a state in which two particles are coupled together with spin projections opposite to one another along every axis in space—the singlet state. Though arising from the classical Poisson Bracket, the commutation relations have a non-classical effect: they restrict our access to precise and simultaneous knowledge of certain non-compatible observables. In the classical framework, any such restriction would imply a measurement problem—an instrumental limitation—but the reality of the observables would not be in question. How should the problem be interpreted in the context of the quantum theory?

Two viewpoints could be taken to “make sense” of the quantum mysteries: either the quantum formalism is a complete and accurate description of the nature of an individual system, or it is only a description of an en-
semble of identical systems. It would say little about the individual entities comprising the ensemble. These issues, grown out of non-compatibility and entanglement, lead ultimately to questions about reality and the fundamental metaphysics of the universe. Are quantum-scale processes fundamentally probabilistic and “fuzzy” as a face-value interpretation of the entangled wave function would suggest; or are they (in reality) deterministic, and our mathematical understanding of them is simply unable to see through a statistical haze of identical ensembles?

One would be forgiven for assuming, without further consideration, that this question is academic, better left to the realm of philosophy or personal preference. Surprisingly, it is a question which can be probed experimentally. The development of these experimental means formed the basis of the EPR argument (section 2.1) and subsequently EPRB (section 2.2). Whatever the answer, it is fundamentally important to the interpretation of quantum theory as we have it now. As has been mentioned, a test of the quantum relations would serve to answer one significant question about the theory: that of its completeness. If the commutation relations tell us only about the “ignorance” of the quantum formalism, then one might imagine that somewhere in the future physicists shall discover a theoretical framework which is “better” than the current theory, which has the power to give more information about the universe and do so in a different way. If, however, they tell us of a fundamentally probabilistic character to the universe, then quantum mechanics certainly seems to retain a right to the title of completeness and metaphysicists must be satisfied with the conceptual results.
Chapter 2

EPR and Developments

2.1 The Einstein-Podolsky-Rosen “Paradox”

The quality of completeness (section 1.3) which has been ascribed to quantum mechanics by some of its founders, and largely endures to this day, was challenged in 1935 by the three Princeton physicists, Albert Einstein, Boris Podolsky, and Nathan Rosen in their controversial paper [1]. Central to the argument of the EPR paper are their particular definitions of completeness and reality, along with the quantum mechanical concept of non-commuting operators, recognized by the authors as a likely place to scrutinize the relatively new quantum theory. The core of the problem is given as this:

In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. [13] [emphasis added]

If the description of reality given by quantum mechanics were complete, then one would expect that the value of any measurable quantity (being an element of reality) should be precisely predictable. If some quantity is measurable, but not precisely predictable, then the theory must not be a complete description of reality.
Judging that their definition of reality was sufficient, and “in agreement with classical as well as quantum-mechanical ideas,” EPR laid out an argument from the quantum formalism to show that the wave function of quantum mechanics does not produce a complete description of physical reality. They considered two systems, I and II, whose states are known initially and individually, and then are allowed to interact over a period of time, T, after which they are separated so that no further interaction can take place (they become localised systems). The interaction of these two systems would result in an entangled state for the overall system, $S = I + II$.

System I, described by variables $x_1$ is defined in terms of some physical quantity, $A$, by eigenfunctions $u_1(x_1), u_2(x_1), \ldots$ with corresponding eigenvalues $a_1, a_2, \ldots$. Then, one defines $\Psi$ as a function of $x_1$ as:

$$\Psi(x_1, x_2) = \sum_{n=1}^{\infty} \psi_n(x_2) u_n(x_1),$$  \hspace{1cm} (2.1)

where $x_2$ are the variables which describe system II. $\psi_n(x_2)$ are coefficients of the $\Psi$ expansion, carrying the information on system II. If a measurement is performed on the entangled system to determine the value of $A$, one may find it to be $a_k$, and I is left in state $u_k(x_1)$ while II is in a state described by $\psi_k(x_2)$. The entangled state, $\Psi$, is thus collapsed by measurement to a single term $\psi_k(x_2) u_k(x_1)$.

Instead, suppose one chose to measure the value of a different quantity, $B$, in terms of which I is described by eigenfunctions $v_1(x_1), v_2(x_1), \ldots$ with eigenvalues $b_1, b_2, \ldots$ so that,

$$\Psi(x_1, x_2) = \sum_{s=1}^{\infty} \phi_s(x_2) v_s(x_1).$$  \hspace{1cm} (2.2)

If $B$ now has value $b_r$, then I is left in state $v_r(x_1)$ and II in state $\phi_r(x_2)$. Both measurements on $\Psi$ would be made on system I, with system II left unmeasured in either case, and since the systems are arbitrarily far apart, no physical interaction can conceivably take place between them. Thus the reality condition that system II not be disturbed “in any way” is satisfied. One finds that system II may be left in states denoted by two different wave
functions. Being assured of the locality of the systems, any measurement on
I cannot affect the properties or state of II. Thus, the wave functions \( \psi_k \) and
\( \phi_r \) must describe the same reality for II.

This result is entirely plausible, until one considers the special case in
which the physical quantities \( A \) and \( B \) correspond to two non-commuting
operators, such as \( \hat{P} \) and \( \hat{Q} \), the momentum and position operators. The two
systems are taken to be two particles, and thus the wave function for \( S \) is
given as:

\[
\Psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{(2\pi i/h)(x_1 - x_2 + x_0)p} dp = \int_{-\infty}^{\infty} \psi_p(x_2)u_p(x_1) dp, \tag{2.3}
\]

with the integration due to the continuous spectrum of momentum eigen-
values. \( x_0 \) is some constant. Here, if momentum is the physical quantity being
measured in particle I, it has momentum eigenfunction \( u_p(x_1) = e^{(2\pi i/h)p x_1} \)
with eigenvalue \( p \). The remaining \( \psi_p \), then, is

\[
\psi_p(x_2) = e^{-(2\pi i/h)(x_2 - x_0)p}, \tag{2.4}
\]

with eigenvalue \(-p\); the second particle’s eigenfunction of the momentum
operator,

\[
\hat{P} = \frac{h}{2\pi i} \frac{\partial}{\partial x_2}. \tag{2.5}
\]

Hence the momentum of the second particle is a well-defined element of
reality.

Instead of measuring momentum, one might measure the position of the
first particle, which has corresponding eigenfunctions

\[
u_x(x_1) = \delta(x_1 - x), \tag{2.6} \]

with eigenvalue, \( x \). The entangled wavefunction for \( S \) then becomes:

\[
\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \phi_x(x_2)v_x(x_1) dx, \tag{2.7}
\]

with
\[ \phi_x(x_2) = \int_{-\infty}^{\infty} e^{(2\pi i/h)(x-x_2+x_0)} dp = h\delta(x-x_2+x_0). \]  

\( \phi_x \), as a Dirac delta function, is the eigenfunction of the position operator, \( \hat{Q} \), with eigenvalue \( (x + x_0) \). The position of the second particle is also a well-defined element of reality.

Since \( \hat{P} \) and \( \hat{Q} \) do not commute, it is shown that the wave functions simultaneously describing the state of system II, \( \psi_k \) and \( \phi_r \), are in fact eigenfunctions of two non-commuting operators. The quantum theory insists that physical quantities corresponding to non-commuting operators cannot have a simultaneous reality: if they did, their values would simultaneously "enter into the complete description according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values."

Despite this, one can, in principle, predict with certainty position or momentum of particle II without in any way disturbing that system. "In accordance with our criterion of reality," EPR states, "in the first case we must consider the quantity \( \hat{P} \) as being an element of reality, in the second case the quantity \( \hat{Q} \) as an element of reality." Both wave functions, \( \psi_k \) and \( \phi_r \), describe the same reality. The quantum mechanical description of physical reality, then, is not complete.

This "paradox" of EPR depends on the way quantum mechanics describes interacting systems. Mathematically, their individual wavefunctions do not merely interact in a product, but cease to exist as individual entities until some measurement is made to separate them. In the interim, however, a new and single state \( \Psi \) takes their place: a function of all the variables of both systems. In the words of Schrödinger,

...the combined function ceases to be a product and moreover does not again divide up, after they have again become separated, into factors that can be assigned individually to the systems. ... Best possible knowledge of a whole does not include best possible knowledge of its parts—and that is what keeps coming back to haunt us. [30]
It is important to understand the logic that underlies the EPR argument. This logic ultimately consists of four principles which, if simultaneously true, results in the internal contradiction in quantum theory which the paper seeks to demonstrate. Therefore, to counter the questions raised by EPR, one must refute or reject at least one of the following doctrines, which will be termed the EPR Principles:\(^1\)

i. the completeness of the quantum theory;

ii. the reality of physical observables whose values can be predicted with certainty for some system, without disturbing that system;

iii. the local nature of quantum (or classical) interactions and effects;

iv. the conservation of linear momentum.

Holding principles ii, iii and iv to be true, \textit{a priori}, EPR concluded that quantum mechanics is vulnerable on point i: it must be an incomplete theory of reality. However, though it might be comfortable to do so, it is not obvious that one must keep all of the other EPR principles at the expense of quantum mechanical completeness. Retaining the momentum conservation principles at all costs, and assuming the validity of i \textit{a priori}, advocates of completeness are left to question the concept of universal, fundamental elements of reality (ii), and the principle of locality (iii).

Having stated its case for the incompleteness of the quantum mechanics, EPR anticipates and replies to those who might call Principle ii into question, the most obvious objection to their argument: “...that our criterion of reality is not sufficiently restrictive.” A more restrictive definition, which would save quantum mechanics from the paradox of the paper, would place limitations on the \textit{simultaneous reality} of two physical quantities. “Only when they can be simultaneously measured or predicted” can two quantities be considered \textit{simultaneously real}. Such a definition, which would cause the reality of \( \hat{P} \) or

\(^1\)These principles, with minor variation, were identified by [9] and termed the “EPR premises”.

\( \hat{Q} \) in the second system to depend on the measurement process carried out on the first system, EPR considers unreasonable.²

In the final word of the paper, it is suggested that although the wave function does not provide a complete description of reality, such a complete description does exist, waiting to be found. While EPR makes no mention of the relation such a theory would have to the notions of determinism or locality, they anticipate the need for the “hidden variables” ideas of Bohm which would motivate John Bell to develop his famous inequalities.

2.2 Einstein-Podolsky-Rosen-Bohm

David Bohm, in 1951, published a text on quantum theory which gave a modification to the EPR argument [5] in terms of eigenstates of spin angular momentum.³ Bohm’s version of EPR has come to be known as EPRB.

In quantum mechanical theory, angular momentum operators, \( J_i \) \((i = x, y, z)\), obey the commutation relation,

\[ [J_x, J_y] = i \hbar \hat{J}_z, \quad [J_y, J_z] = i \hbar \hat{J}_x, \quad [J_z, J_x] = i \hbar \hat{J}_y. \]

²While unreasonable to the authors, it is indeed this definition of reality that Niels Bohr calls into question (by insisting on an intervention by the measuring instrument in defining the reality of a quantum system) in his first attempt [31] to bring resolution to the EPR paradox. According to Redhead [32], one may consider three perspectives of the "reality" of physical observables for which the state of some system is not an eigenstate (as in EPR):

i. The observable has a sharp but unknown value (the view of EPR that leads naturally to a hidden-variables theory);

ii. The observable has an unsharp or "fuzzy" value where reality is latent in potentiality;

iii. The value of the observable is undefined or "meaningless" (the Complementarity view of Bohr and the Copenhagen interpretation, hinging on the fact that microphysics is always described via macrophysics terminology, which is inadequate to the task, yet "all we have").

The usual recourse in denying EPR Principle ii has been to follow Bohr and propose perspective iii: that the non-compatible observable of which the system is not in an eigenstate, is (in reality) meaningless, and therefore so is the “contradiction” of EPR. (This is a naive interpretation of Bohr’s complicated philosophy which was certainly a mixture of both realist and positivist perspectives. For a short description of the difficulties, see [33].)

³Initially, the appearance of this modification served only as an easier way for Bohm to explain the EPR paradox in his text so that he could refute it more clearly. Later on, he came to see merit in the argument and put much work into fighting on its side against the orthodox interpretation of quantum mechanics.
\[ [J_i, J_j] = \imath \hbar \epsilon_{ijk} J_k. \tag{2.9} \]

As with the position and momentum operators, this non-commutativity of the mutually orthogonal components of \( J \) has an important interpretation: one cannot possess a knowledge of the simultaneous eigenvalues corresponding to the projections of a particle's spin vector on two mutually orthogonal axes. Bohm realized that this claim of the quantum mechanical formalism provided a more experimentally feasible framework from which the EPR problem could be considered. Whereas position and momentum are continuous variables and therefore too difficult (if not impossible) to measure within useful precision, spin angular momentum projections along orthogonal axes take on discrete values only, so that the precision of some measurement is no longer a concern—only the accuracy of a measurement becomes a factor in an experimental design. The use of orthogonal spin components falls within the framework of the original EPR argument which was derived for the case of any non-commuting operators, using position/momentum only as an example.

Bohm conceived of a molecule having zero total spin, \( S = 0 \), made of two atoms with spins \( \hbar/2 \) pointing in opposite directions. If this molecule can be disintegrated in such a way that no torque acts on the system (hence, orbital angular momentum, \( L \) is zero), then the total angular momentum, \( J \), of the two-atom system will remain zero even as the pieces are separated from one another by their relative motion and are no longer interacting. The wave function of this entangled system is that of the singlet state (1.71).

The total angular momentum is an interference property of the two terms in the wave function. These terms imply that a physical correlation exists between the spin projections of both atoms along any arbitrary axis in space. The system is in an eigenstate of any operator \( \sigma_1 \cdot \hat{n} \otimes \sigma_2 \cdot \hat{n} \) (where \( \hat{n} \) is any unit vector), with eigenvalue \(-1\). Accordingly, a measurement of the spin projection value of one particle along some axis will automatically yield that of the other particle, without it being probed in any way. By the EPR Reality Criterion, then, all three of the spin projections for each atom in the system
are elements of reality. Since, however, their quantum mechanical operators do not commute, the quantum theory cannot be a complete representation of reality. [34]

2.3 Hidden Variables

Without assuming a priori that quantum mechanical ideas of non-commutation are an objective natural law (such an assumption, in a sense, is at stake in the very argument of EPR), one could suppose that the usual, classical conservation principles of position and momentum are clearly at work behind the two particles of the EPR entangled state. Were this the case, then one would further suppose that if the two particles have precise initial positions and momenta (whether known or not by some scientist), then a measurement of either quantity for either particle at some later time would simply uncover the conservation principle previously at work. Quantum mechanics would obviously be incomplete, by any definition of reality, because it cannot predict these quantities. It cannot do so, because it is not classically deterministic in its handling of these individual systems once they form an entangled state.

Quantum theory can make fine predictions for one quantity or the other, but not both simultaneously. Presumably, some slightly improved theory might be discovered that incorporates the deterministic effects of the classical conservation principles. This “deeper” theory would have information to which quantum theory is not privy: “hidden variables” which do not find themselves in the usual formalism, causing the conservations to be worked out in a predictable, non-probabilistic (i.e. deterministic) way to which quantum mechanics is blind.

As an example from the previous section, the formalism of quantum mechanics provides no value for a particle’s spin projections orthogonal to the z-direction (however it be defined). But this might not necessarily mean that $s_x$ and $s_y$ have no value. They may be well-defined by some parameters hidden to quantum mechanics which would allow for the precise definition of $s_x$ and $s_y$ at all times, or allow for them to become well-defined in some
deterministic way at some point in their history. The predictions of quantum theory would constitute an “averaging over” of these hidden parameters, resulting in probabilistic results.

The alleged incompleteness of quantum mechanics motivated the pursuit of just such an alternative framework by which completeness could be salvaged. In [6] (later re-adressed by John Bell in 1966 [35]) David Bohm put forward a pair of papers on the topic of hidden variables wherein he laid out a framework for reinterpreting quantum theory that would allow for these parameters. “In contrast to the usual interpretation [of quantum theory],” he wrote,

this alternative interpretation permits us to conceive of each individual system as being in a precisely definable state, whose changes with time are determined by definite laws, analogous to (but not identical with) the classical equations of motion. Quantum mechanical probabilities are regarded (like their counterparts in classical statistical mechanics) as only a practical necessity and not as a manifestation of an inherent lack of complete determination in the properties of matter at the quantum level. [36]

“The usual interpretation of quantum theory,” Bohm suggested, centers on the uncertainty principle and stands on two assumptions:

1. The wave function with its probability interpretation determines “the most complete possible specification” of the state of an individual system.

2. The process of transfer of a single quantum from observed system to measuring apparatus is inherently unpredictable, uncontrollable, and unanalyzable.

He was critical of this interpretation, since

it requires us to give up the possibility of even conceiving precisely what might determine the behavior of an individual system at the quantum level, without providing adequate proof that such
a renunciation is necessary. ... As a matter of fact, whenever we have previously had recourse to statistical theories, we have always ultimately found that the laws governing the individual members of a statistical ensemble could be expressed in terms of...hidden variables. [37]

Bohm’s theorem considered the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi$$

(2.10)

as a wave equation whose complex solution, $\psi$, can be expressed as

$$\psi = R \exp(iS/\hbar)$$

(2.11)

where $R$ and $S$ are real. Through substitution, one gets as equations for these parameters:

$$\frac{\partial R}{\partial t} = -\frac{1}{2m} \left( R \nabla^2 S + 2 \nabla R \cdot \nabla S \right)$$

(2.12)

$$\frac{\partial S}{\partial t} = - \left[ \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \right].$$

(2.13)

Taking the probability density of the particle position to be $R(x) = P^2(x)$, Bohm obtained from (2.13),

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{4m} \left[ \frac{\nabla^2 P}{P} - \frac{1}{2} \frac{(\nabla P)^2}{P^2} \right] = 0.$$ 

(2.14)

Of the form

$$H + \frac{\partial S}{\partial t} = 0,$$

(2.15)

(2.14) is the Hamilton-Jacobi equation with solution $S(x)$, if the total potential of the Hamiltonian is taken to be $V(x) + U(x)$. The first term represents the “classical potential” while the second acts on the system as a “quantum mechanical potential” of the form

$$U(x) = \frac{-\hbar^2}{4m} \left[ \frac{\nabla^2 P}{P} - \frac{1}{2} \frac{(\nabla P)^2}{P^2} \right] = \frac{-\hbar^2}{2m} \frac{\nabla^2 R}{R}.$$ 

(2.16)
This, Bohm said, is “the nucleus of an alternative interpretation for Schrödinger’s equation.” The particle moves under the influence of both a classical force and a force derivable from the “quantum mechanical potential,” and \( \psi \) is “regarded as a mathematical representation of an objectively real force field.” [38]

Bohm went on to show that the predictions of his new interpretation were consistent with those of quantum mechanics in experimental situations. The use of a statistical ensemble is “only a practical necessity” rather than “an inherent limitation” introduced to respect the uncertainty principle. The uncertainty principle becomes a *practical limitation* on the precision possible with physical measurements. [39] Whereas the usual interpretation of quantum mechanics regards the probability density as “inherent in the structure of matter,” in Bohm’s interpretation it arises “because from one measurement to the next, we cannot in practice predict or control the precise location of a particle, as a result of corresponding unpredictable and uncontrollable disturbances introduced by the measuring apparatus.” [40]

The key strength of the hidden variables interpretation was that it allowed for “an infinite number of ways of modifying the mathematical form of the theory,” such as the addition of short-range force terms, which would maintain consistency with the results of the usual quantum interpretation, yet “permit the particle position and momentum to be measured simultaneously.” [41]

Bohm’s theorem allowed for violations to the uncertainty principle while agreeing with the observed predictions of the quantum formalism. This opened the door, in principle, for hidden variables, and motivated Bell’s famous 1964 paper [2]. As it turned out, there is one experimental condition under which it can be shown, via Bell’s inequality, that any local hidden variables theory such as Bohm’s is “incompatible with the statistical predictions of quantum mechanics.” [42]
2.4 Bell’s Inequality

In 1964, John Bell, an Irish physicist, put forward a paper which suggested an experimental test of the questions surrounding quantum mechanics raised by EPR. Using the EPRB scenario of two entangled spin-$\frac{1}{2}$ particles, Bell considered the consequences of a hidden variable model on an experiment measuring the correlations between the spin polarizations of such particles projected along arbitrary axes. He developed an inequality which, for local hidden variables theories, put an upper limit on the value these correlations could take for any choice of angles between the axes. Quantum mechanics, which has its own predictions of these correlations in terms of expectation values violates the hidden variables limit. In this way, Bell was able to demonstrate the incompatibility between local hidden variables theories and quantum mechanics. This was the doorway to an experimental test of quantum completeness.

Bell’s argument considered a system of two entangled spin-half particles in the $^1S_0$ state, wherein the spins of the two particles, $\sigma_I$ and $\sigma_{II}$, are oppositely oriented. The particles move freely away from one another in opposite directions. Along some arbitrary space axes $\hat{a}$ and $\hat{b}$, the spin projections of the particles are measured respectively, giving results $A(\hat{a}) = \sigma_I \cdot \hat{a}$ for one particle and $B(\hat{b}) = \sigma_{II} \cdot \hat{b}$ for the other. It is assumed that the measurements are taken at some time when the particles are well-separated, such that no subluminal communication might occur between them. The values taken by $A$ and $B$ are ±1, in keeping with the dichotomic quality of spin-half projections.\(^4\)

A vital assumption must be made regarding the apparatus: that of locality. If the two measurements are performed at places remote from one another (which is reasonable if the particles are travelling in opposite directions) then the settings made for one measuring instrument (that is, what

\(^4\)Crucial to Bell’s derivation is the fact that $A$ and $B$ be dichotomic variables, taking only one of two opposite values. In fact, the Bell inequality here derived is applicable for any experiment testing the locality and deterministic evolution of a process where the final results of measurements are dichotomic values.
projection vector it is examining) do not influence the results obtained by the other, and vice-versa. The two devices cannot communicate with one another to influence the results of any experiment. Thus, the quantum mechanical violation of the Bell inequality is a demonstration of the incompatibility between the quantum predictions and the principle of locality.

From the logic of EPRB, it is clear that without taking a measurement we can know the value of \( B \), if only we take a measurement on particle I to find \( A \). The value of \( B \) seems predetermined, in contrast to the probabilistic viewpoint of quantum mechanics. Suppose, then, that a complete specification of the singlet state can be made by employing one or more hidden parameters, \( \lambda \), not accessible to the quantum mechanical formalism. This variable could represent a single hidden parameter, a set of parameters, or even a set of functions; its values could be either continuous or discrete, but it will be assumed for generality that they are continuous. The results \( A = \sigma_1 \cdot \hat{a} \) and \( B = \sigma_\Pi \cdot \hat{b} \), then, are dependent not only on the vectors \( \hat{a} \) and \( \hat{b} \) (the orientation of the measuring devices for each particle) but also on \( \lambda \): \( A = A(\hat{a}, \lambda) \) and \( B = B(\hat{b}, \lambda) \).

Bell showed his remarkable contradiction between the predictions of local hidden variables theories and those of quantum mechanics, by examining the expectation value for the product \( AB \).

By definition the expectation value of some operator, \( O \), acting on a state \( | \omega \rangle \) is \( \langle O \rangle = \langle \omega \mid O \mid \omega \rangle \), where \( \langle \omega \mid \omega \rangle \) is assumed to be normalized. Conceptually, the expectation value is the average result one expects from the same measurement performed on a very large number (\( N \)) of identical systems. One measures \( O \) and—assuming it takes discrete values—finds the results \( O_1, O_2, \ldots, O_i \), with probabilities \( P(O_1), P(O_2), \ldots, P(O_i) \). Then,

\[
\langle O \rangle = \sum_i O_i P(O_i). \tag{2.17}
\]

For continuous values of \( O \),

\[
\langle O \rangle = \int O P(O) dO. \tag{2.18}
\]

The expectation value of the product of measurements on \( \sigma_1 \cdot \hat{a} \) and \( \sigma_\Pi \cdot \hat{b} \)
is, then,

$$P(\hat{a}, \hat{b}) = \int \rho(\lambda)A(\hat{a}, \lambda)B(\hat{b}, \lambda)d\lambda,$$

(2.19)

when the results of measurements are dependent on the parameter $\lambda$ as in the hidden variables regime. $\rho(\lambda)$ is the probability distribution of the hidden parameters (normalized under integration over $\lambda$).

For quantum mechanics the product $AB$ corresponds to the operator

$$(\sigma_1 \cdot \hat{a})_1 \otimes (\sigma_2 \cdot \hat{b})_2$$

(2.20)

and the expectation value for the singlet state, $\chi_{\text{singlet}}$ (1.71), is given by,

$$P^{QM}(\hat{a}, \hat{b}) = \langle \chi_{\text{singlet}} | (\sigma_1 \cdot \hat{a})_1 \otimes (\sigma_2 \cdot \hat{b})_2 | \chi_{\text{singlet}} \rangle$$

(2.21)

Explicitly,

$$P^{QM} = \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_1 \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_2 - \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_1 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_2 \right] \left[ \left( \begin{array}{cc} \cos \theta_a & \sin \theta_a e^{-i\phi_a} \\ \sin \theta_a e^{i\phi_a} & -\cos \theta_a \end{array} \right)_1 \otimes \left( \begin{array}{cc} \cos \theta_b & \sin \theta_b e^{-i\phi_b} \\ \sin \theta_b e^{i\phi_b} & -\cos \theta_b \end{array} \right)_2 \right]$$

(2.22)

where $\theta_a$, $\phi_a$, $\theta_b$, and $\phi_b$ are the polar and azimuthal angles of the vectors $\hat{a}$ and $\hat{b}$.

Working through the matrix algebra, the quantum expectation value comes out as

$$P^{QM}(\hat{a}, \hat{b}) = -\cos(\phi_a - \phi_b) = -\cos(\Phi)$$

(2.23)

where $\Phi$ is the angle between the vectors $\hat{a}$ and $\hat{b}$, along which the spin-projections of the two singlet-state particles are measured.

Looking again at (2.19), one can easily derive the series of correlation functions,
\[ P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}') + P(\hat{a}', \hat{b}) + P(\hat{a}', \hat{b}') = \]
\[ \int \rho(\lambda) A(\hat{a}, \lambda)[B(\hat{b}, \lambda) - B(\hat{b}', \lambda)] A(\hat{a}', \lambda)[B(\hat{b}, \lambda) + B(\hat{b}', \lambda)] d\lambda \quad (2.24) \]

The measurement axes of both particles are varied. Since \( A(\hat{a}, \lambda) \) and \( A(\hat{a}', \lambda) \) are each maximally +1, it follows from (2.24) that

\[
| \Delta | \equiv | P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}') + P(\hat{a}', \hat{b}) + P(\hat{a}', \hat{b}') | \quad (2.25)
\[
\leq \int \rho(\lambda) | B(\hat{b}, \lambda) - B(\hat{b}', \lambda) + B(\hat{b}, \lambda) + B(\hat{b}', \lambda) | d\lambda
\]
\[
= \int \rho(\lambda) | 2B(\hat{b}, \lambda) | d\lambda
\]
\[
| \Delta | \leq 2 \int \rho(\lambda) d\lambda
\]

since the maximum value of \(| B(\hat{b}, \lambda) | \) is 1. With the precondition that \( \rho(\lambda) \) is normalized under the integration, the result is

\[-2 \leq \Delta \lambda = P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}') + P(\hat{a}', \hat{b}) + P(\hat{a}', \hat{b}') \leq 2 \quad (2.26)\]

This result for the expression \( \Delta \) follows from the assumed existence of some parameter(s), \( \lambda \), deterministically fixing the observable values for spin projections. It is known as the CHSH [43] version of the Bell inequality, first deduced in [2].

The correlation function (2.23) for the quantum theory can be shown to violate the condition of (2.26) for appropriate choice of co-planar unit vectors. The Bell inequality can be rewritten in terms of the angles between the four unit vectors:

\[
\cos \alpha = \hat{a} \cdot \hat{b} \quad (2.27)
\]
\[
\cos \beta = \hat{a} \cdot \hat{a}' \quad (2.28)
\]
\[
\cos \gamma = \hat{a} \cdot \hat{b}' \quad (2.29)
\]
Figure 2.1: Orientation of unit vectors along which spin projections are measured for Bell’s inequality, with arbitrary angles between the vectors.

illustrated in Figure 2.1. Then one has

\[
\Delta_{QM} = P_{QM}^{\alpha} - P_{QM}^{\beta} + P_{QM}^{\gamma + \beta} + P_{QM}^{\gamma - \beta} \\
= - \cos(\alpha) + \cos(\gamma) - \cos(\beta - \alpha) - \cos(\gamma - \beta). \tag{2.30}
\]

Choosing the angles to be \(\alpha = \pi/4\), \(\beta = \pi/2\), and \(\gamma = 3\pi/4\) (ie. equal inter-angular separations of \(\pi/4\)), one gets

\[
\Delta_{QM} = - \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) \\
= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\
= -2\sqrt{2} \\
\leq -2 \tag{2.31}
\]

demonstrating the violation of (2.26). [44]

Result (2.31) demonstrates quite simply that the quantum mechanical expectation value for the singlet system (2.23) cannot be reconciled with any hidden variables regime in which the parameters \(\lambda\) “pre-set” the results of the measurement, explaining entanglement as a strictly statistical phenomena of ensembles, and preserving locality. An experimental test of (2.26)
will show that either the quantum theory is not complete (in which case an experimental result will disagree with [2.30]) or that it is complete but the interactions governing the entangled singlet state are non-local, violating the third EPR Principle. The lesson of Bell is that quantum completeness and locality are mutually exclusive.

### 2.5 Wigner’s Inequality

E.P. Wigner, in 1969, proposed [7] an equivalent to the Bell inequality which considers singlet correlations arising from spin projections along three unit vectors rather than four.

Assume rotations along three arbitrary directions, $\hat{a}$, $\hat{b}$ and $\hat{c}$, which are taken as co-planar (in the $x$-$y$ plane) to simplify the problem. Consider the action of any two of these rotations (say, along $\hat{a}$ and $\hat{b}$) on the singlet state:

$$
\frac{1}{\sqrt{2}} \left( \begin{array}{c} \cos \frac{\alpha}{2} \\
\sin \frac{\alpha}{2} \end{array} \right) \otimes \left( \begin{array}{c} \cos \frac{\beta}{2} \\
-\sin \frac{\beta}{2} \end{array} \right)_1 \otimes \left[ \left( \begin{array}{c} 1 \\
0 \end{array} \right)_1 \otimes \left( \begin{array}{c} 0 \\
1 \end{array} \right)_2 - \left( \begin{array}{c} 0 \\
1 \end{array} \right)_1 \otimes \left( \begin{array}{c} 1 \\
0 \end{array} \right)_2 \right]
$$

$$
= \frac{1}{\sqrt{2}} \left[ \cos \frac{a}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin \frac{a}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_1 \otimes \left[ -\sin \frac{b}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos \frac{b}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_2
$$

$$
- \left[ -\sin \frac{a}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos \frac{a}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_1 \otimes \left[ \cos \frac{b}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin \frac{b}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_2
$$

(2.32)

giving the result,

$$
\Psi(a, b)_{\text{singlet}}
$$

$$
= \frac{1}{\sqrt{2}} \left( \sin \frac{a}{2} \cos \frac{b}{2} - \cos \frac{a}{2} \sin \frac{b}{2} \right) (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)
$$

$$
+ \frac{1}{\sqrt{2}} \left( \cos \frac{a}{2} \cos \frac{b}{2} + \sin \frac{a}{2} \sin \frac{b}{2} \right) (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
$$

$$
= \frac{1}{\sqrt{2}} \sin \left( \frac{a - b}{2} \right) (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) + \frac{1}{\sqrt{2}} \cos \left( \frac{a - b}{2} \right) (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
$$

(2.33)

The change of notation to arrows shows explicitly the weighting associated with each substate. The probabilities of finding the system in one of the
states where the particle spins are in the same relative direction along their measurement axes (|↑↑⟩ or |↓↓⟩) are equal, just the square of that substates’ coefficient:

\[ P(\uparrow\uparrow)_{a,b} = \frac{1}{2} \sin^2 \left( \frac{a - b}{2} \right) = P(\downarrow\downarrow)_{a,b} \]  

(2.34)

while the probabilities for the spin projections being in one of the substates (|↑↓⟩ or |↓↑⟩) are

\[ P(\uparrow\downarrow)_{a,b} = \frac{1}{2} \cos^2 \left( \frac{a - b}{2} \right) = P(\downarrow\uparrow)_{a,b}. \]  

(2.35)

Similar relations apply for the probabilities \( P(a, c) \) and \( P(b, c) \), for some third angle, \( c \).

Employing Wigner’s notation, the overall state of the system—assuming all the spin-projection values, \( m \), on all axes are pre-determined by hidden variables—can be written as \( \langle m_{1a}, m_{1b}, m_{1c}; m_{2a}, m_{2b}, m_{2c} \rangle \). The probability \( P(\uparrow\uparrow)_{a,c} \) applies to the state

\[ \langle \uparrow, m_{1b}, m_{1c}; m_{2a}, m_{2b}, \uparrow \rangle. \]  

(2.36)

Because the overall system is the singlet state, all the components in the two particles must be opposed, giving

\[ \langle \uparrow, m_{1b}, \downarrow; \downarrow, m_{2b}, \uparrow \rangle \]  

(2.37)

with two possibilities for the projections on \( b \). This results in a probability with two terms applying to the singlet state configuration

\[ \langle \uparrow, \uparrow; \downarrow, \downarrow, \uparrow \rangle + \langle \uparrow, \downarrow; \downarrow, \uparrow, \uparrow \rangle. \]  

(2.38)

The first term of this expression is simply the state

\[ \langle m_{1a}, \uparrow, \downarrow; m_{2a}, \downarrow, \uparrow \rangle - \langle \downarrow, \uparrow, \downarrow; \uparrow, \downarrow, \uparrow \rangle \]  

(2.39)

while the second term is

\[ \langle \uparrow, \downarrow, m_{1c}; \downarrow, \uparrow, m_{2c} \rangle - \langle \uparrow, \downarrow, \downarrow, \uparrow, \downarrow \rangle. \]  

(2.40)
The probability of the first term in (2.39) is

\[ P(\uparrow\uparrow)_{b,c} \]  

(2.41)

and that of the leading term in (2.40) is

\[ P(\uparrow\uparrow)_{a,b}. \]  

(2.42)

Thus, the probability of (2.39) is something less than the sum of these two expressions, giving

\[ P(\uparrow\uparrow)_{a,c} \leq P(\uparrow\uparrow)_{a,b} + P(\uparrow\uparrow)_{b,c}. \]  

(2.43)

The same statement could be made in terms of the probabilities \( P(\downarrow\downarrow)_{a,b} \), and thus the sum of these two inequalities would yield an expression identical in form to that given in terms of the overall probability of particles having spin aligned.

Looking again at the predictions of quantum mechanics, this inequality is

\[ \frac{1}{2} \sin^2 \left( \frac{c - a}{2} \right) \leq \frac{1}{2} \sin^2 \left( \frac{b - a}{2} \right) + \frac{1}{2} \sin^2 \left( \frac{b - c}{2} \right). \]  

(2.44)

Thus, if the predictions of the quantum theory are to be in agreement with the hidden variables assumptions on which (2.43) is based, then (2.44) must hold. It does not for many cases, such as all those where \( b \) intersects \( a + c \) (or \( b = \frac{a+c}{2} \)). This is made evident by setting \( (c - b) = (b - a) = (c - a)/2 \). Then, (2.44) becomes

\[
\sin^2 \left( \frac{b-a}{2} \right) \geq \frac{1}{2} \sin^2 \left( \frac{c-a}{2} \right) \\
= \frac{1}{2} \sin^2 \left( 2 \frac{b-a}{2} \right) \\
= \frac{1}{2} \left[ 2 \sin \left( \frac{b-a}{2} \right) \cos \left( \frac{b-a}{2} \right) \right]^2 \\
= 2 \sin^2 \left( \frac{b-a}{2} \right) \cos^2 \left( \frac{b-a}{2} \right)
\]

(2.45)

and

43
\[
\cos^2 \left( \frac{b - a}{2} \right) \leq \frac{1}{2}.
\]  
(2.46)

Since the angles \( a, b \) and \( c \) are taken only from 0 to \( \pi \), this limits the value of \((b - a)\) to the region

\[
\frac{\pi}{2} \leq (b - a) \leq \pi
\]  
(2.47)

which implies that

\[
(c - a) \leq \pi.
\]  
(2.48)

This result shows that for co-planar angles, with \( b \) bisecting \( a \) and \( c \), (2.44) is violated for all cases except \((b - a) = \pi/2, (c - a) = \pi\). The predictions of quantum mechanics cannot be reconciled with those of a hidden variables theory where the spin-projections of the two particles are uniquely fixed by the conservation principles so as to exclude entanglement.

For these angular conditions \((a, b, c)\) co-planar with \( b \) bisecting \( a \) and \( c \) it should be possible to test the Wigner inequality experimentally using spin correlations of entangled particles. If the inequality (2.43) is obeyed, then the validity of the hidden variables regime is demonstrated. A violation of the angular conditions would be a vindication of the reality described by quantum mechanics. The apparatus described in section 4.1 is able to meet the requirement of measuring the necessary 3-angle correlations required in the Wigner problem.

### 2.6 The Classical Angular Momentum Correlation

In deriving his inequality, Bell ensured that his argument was fashioned in very general terms, avoiding as much interpretational baggage as possible. Thus, a completely flexible form is employed for the hidden parameter(s):

It is a matter of indifference...whether \( \lambda \) denotes a single variable
or a set, or even a set of functions, and whether the variables are discrete or continuous. [45]

Besides the assumption of locality, it is only taken for granted that the angular momentum conservation principle holds, and that the measuring apparatuses passively record dichotomic results.

It is obvious from the results of section 2.4 that no version of the hidden variables theories embraced by Bell’s work (i.e. those that assume locality) are equivalent to the quantum theory. Considering Bell’s generality, however, it would be a tempting mistake to suggest that the only other alternative is something *classical*. The very fact that Bell’s inequality was derived with spin angular momentum in mind should keep one away from this conclusion. The local hidden variables theories must be regarded as *hypothetical*, a new thing. The Bell Inequality simply (and literally) places some limitations on what we should expect from a local, deterministic theory.

This said, it is indeed true that classical physics obeying the locality of special relativity fits the category of “local determinism.” Ignoring for a moment the particular difficulty of *spin*, the classical perspective on angular momentum should fall in line with the limits of the Bell inequality (2.26).

The classical analog of the EPRB case is nicely illustrated by A. Peres in [46], where one imagines “a bomb, initially at rest, which explodes into two asymmetric parts, carrying angular momenta \( \mathbf{J}_1 \) and \( \mathbf{J}_2 = -\mathbf{J}_1 \).” An experiment is carried out \( N \) times in which one observer measures the \( j \)-th dynamical variable \( a_j = \text{sign}(\hat{a}_j \cdot \mathbf{J}_1) = \pm 1 \) for arbitrary unit vector \( \hat{a}_j \) and another measures \( b_j = \text{sign}(\hat{b}_j \cdot \mathbf{J}_2) = \pm 1 \) for some unit vector \( \hat{b}_j \). Assuming a random distribution of \( \mathbf{J}_1 \) and \( \mathbf{J}_2 \), the average correlation of the observers’ measurements will be

\[
P_{\text{classical}} = \langle ab \rangle = \frac{1}{N} \sum_j a_j b_j. \tag{2.49}
\]

The expected correlation can be determined by imagining the unit sphere, cut by one equatorial plane to which \( \hat{a} \) is normal and another to which \( \hat{b} \) is normal, as in Figure 2.2. These planes divide the sphere into four sectors
Figure 2.2: The unit sphere for obtaining the classical correlation (2.50). The correlation function is +1 in the shaded sectors and −1 in the unshaded sectors.

where the unit vector directions are (up-up), (down-down), (up-down), or (down-up). Between these sectors the sign of (2.49) alternates, being negative for the former two cases and positive for the latter two.

For a total surface area of $2\pi$, and an angle between the unit vectors of $\theta$, the total positive surface area of the sphere is equal to $2\theta/2\pi$, while the negative area is $2(\pi - \theta)/2\pi$. The correlation function will be the averaged difference of all positive and negative measurements, or the difference in surface areas for the two regions:

$$P_{\text{classical}} = \frac{2\theta}{2\pi} - \frac{2(\pi - \theta)}{2\pi} = \frac{2\theta}{\pi} - 1.$$  \hspace{1cm} (2.50)

A plot of this function for the angle is shown along with the quantum mechanical expectation value in Figure 2.3. Clearly, the classical and quantum correlation functions disagree over most of the angular phase space. Under the conditions of (2.31), the classical correlation function has a value of $\Delta = 1$, well within the Bell limit.
Figure 2.3: The classical (solid line) and quantum mechanical (dashed line) correlations for the entangled singlet state as a function of $\theta$. 
Chapter 3

The Lamehi-Rachti Mittig Experiment

3.1 Photon Experiments

Since John Bell’s paper, [2], two types of experiments have been performed to test inequality (2.26). Several studies [49-58] have been made using entangled photon pairs, while only one has employed the fermions of the original EPRB idea.

Like spin-$\frac{1}{2}$ particles, photon polarization is a dichotomic observable, only taking values of ±1. Bell-type inequalities can thus be formulated for photon pairs in an entangled state

$$|J^p = 0^-\rangle = \frac{1}{\sqrt{2}} [ |x_1\rangle |y_2\rangle - |y_1\rangle |x_2\rangle ]$$

(3.1)

where $|x\rangle$ and $|y\rangle$ are the linear polarization states of the individual photons along the x- and y-axes respectively. The inequalities are based on polarizer transmission probabilities. Nearly all of the photon experiments were found to agree with the predictions of quantum theory. The low efficiency of photon detectors (typically 10% to 20%) raises the difficulty that transmissions through the polarizers are not easily detected—especially joint transmissions for a photon pair. The inequality must be redefined in terms of joint transmission/detection probabilities which are too small to result in a violation
of Bell’s upper limit. [57] The alternative, in the words of Afriat and Selleri, “has traditionally been ... ad hoc assumptions concerning the nature of the transmission-detection process.” [58] These assumptions, though reasonable, open loopholes in the interpretation of results. Thus, the photon experiments are not strict tests of local realism as defined by EPRB. [59]

3.2 Lamehi-Rachti Mittig Experimental Design

The first test of the Bell inequality using spin polarization measurements on fermionic systems was performed by Lamehi-Rachti and Mittig (LRM) [60] at Saclay. They measured the spin correlations in low-energy proton scattering off a hydrogen target. At the time of their work, four notable experiments had been performed with photons, yielding contradicting results: Kasday [55] as well as Freedman and Clauser [47] violating Bell’s inequality; Faraci [56] as well as Holt and Pipkin [48] agreeing with the inequality. LRM was, therefore, a pioneering work which could shed light on the existing ambiguity via a new method.

The experimental device for LRM consisted of the Saclay tandem accelerator which produced a beam of protons at energies 13.2 and 13.7 MeV. The 1.5 μA beam was incident on a polyethylene (CH₂) foil of 9 mg/cm², as shown in Figure 3.2. Proton scattering off the hydrogen in the target resulted in proton pairs. These p-p pairs, in kinematical coincidence, entered the analyzers at a laboratory angle of 45°. The analyzers used were carbon foils of 18.6 and 29 mg/cm² for the proton energies of 13.2 and 13.7 MeV respectively. The analyzing carbon foil scattered the protons toward silicon detectors which noted their coincidences for chosen angles. Each analyzer was followed by a right and left detector. One set of detectors was positioned in the reaction plane and the other set was rotated by an angle, θ, around the momentum axis of its incoming proton.

The transmission of the analyzers was low, on the order of 10⁻⁵. This
resulted in an overall coincidence detection rate of 1 proton pair in $10^{10}$. To acquire a sufficient number of events for a reasonable analysis, the experimental design had to incorporate thick targets, high beam intensities, and large solid angle acceptance. In all, the LRM data consisted of about 10,000 true coincidence events.

Measurements were taken to test the Bell limit at five values of $\theta$: 0°, 30°, 45°, 60° and 90°. In order to compare their experimental findings with the values of Bell, LRM employed the measurement correlation function,

$$P_{\text{meas}}(\hat{a}, \hat{b}) = \frac{N_{LL} + N_{RR} - N_{RL} - N_{LR}}{N_0}$$  \hspace{1cm} (3.2)

where $N_{LL}$ is the number of coincidences between the left detectors for analyzers 1 and 2 (similar definitions apply to the other $N$'s), and $N_0$ is the total number of coincidences registered:

$$N_0 = N_{LL} + N_{RR} + N_{RL} + N_{LR}.$$  \hspace{1cm} (3.3)
For a sufficiently large number of events, one can define the probability of having a count in the left detector of analyzer 1 as,

\[ p^L_1 = \frac{N^L_1}{N_0} = \int P^L_1(\hat{a}, \lambda) \rho(\lambda) d\lambda \] (3.4)

which takes a value less than or equal to one. Defining the detection probabilities for the other detectors similarly, it is clear that \(-1 \leq p^L_1(\hat{a}, \lambda) - p^R_1(\hat{a}, \lambda) \leq 1\).

Forbidding communication between the protons once they have interacted in the target, the coincidences between the left counters in the two analyzers are

\[ \frac{N^L_{LL}}{N_0} = \int p^L_1(\hat{a}, \lambda) p^L_2(\hat{b}, \lambda) \rho(\lambda) d\lambda. \] (3.5)

With such a relation in place for all four types of coincidences between the detectors, (3.2) becomes,

\[ P_{meas}^L(\hat{a}, \hat{b}) = \int [p^L_1(\hat{a}, \lambda) - p^R_1(\hat{a}, \lambda)][p^L_2(\hat{b}, \lambda) - p^R_2(\hat{b}, \lambda)] \rho(\lambda) d\lambda. \] (3.6)

This has the form of Bell’s correlation function, (2.19), with

\[ A(\hat{a}, \lambda) = p^L_1(\hat{a}, \lambda) - p^R_1(\hat{a}, \lambda) \] (3.7)

and

\[ B(\hat{b}, \lambda) = p^L_2(\hat{b}, \lambda) - p^R_2(\hat{b}, \lambda). \] (3.8)

Therefore, the product of the asymmetries in proton scattering for each of the analyzers should obey Bell’s inequality, (2.26), and lead to a contradiction with the quantum theory if local hidden variables are at work in the system.

The correlation function, (3.2), cannot be directly compared with the limits imposed by the Bell inequality unless the apparatus used meets three conditions. For a real apparatus, the measurement result is dependent on the transmission of the device, \(T\), its analyzing power, \(P\), and possibly some other parameters, \(C\). The quantum expectation value for the di-proton sys-
tem, (2.23), applies for an ideal apparatus with \( T = P = C = 1 \). For values obtained from measurements made with a real apparatus, the quantum mechanical expectation for the correlation function becomes

\[
P_{Q,M}^\prime(\hat{a}, \hat{b}) = -P_1 P_2 T_1 T_2 C \cos(\theta).
\]  

(3.9)

The first condition for a true test of the Bell inequality is that the coefficient \( |P_1 P_2 T_1 T_2| \) must have a value exceeding \( \frac{1}{\sqrt{2}} \) (assuming \( C \) is unity). This is because the Bell limit, equation (2.26), is maximally 2 and an ideal apparatus obeying (2.23) has a maximum correlation value \( 2\sqrt{2} \). Thus, to have any contradiction with the Bell limit the condition

\[
|P_1 P_2 T_1 T_2| \geq \frac{1}{\sqrt{2}}
\]  

(3.10)

must hold.

The second and third conditions have to do with the geometry of the apparatus, which must ensure that no hypothetical information exchange occurs between the particles or analyzers which could bias the results toward the quantum mechanical predictions. The second condition is that

\[
t_w c < d
\]  

(3.11)

where \( t_w \) is the particle’s flight time between the analyzer entrance and the detector, \( c \) is the speed of light, and \( d \) is the distance between the pair of particles in the analyzers. This condition invokes the theory of relativity to ensure that any communication between particles “aimed at” collapsing their entangled state must be non-local.

The third condition is that the orientation of the analyzers (in this case the angle \( \theta \)) must be changed in a non-predictable way during the time the particles are in flight so that

\[
(t_{ch} - t_{det}) c < d.
\]  

(3.12)

Here, \( t_{ch} \) is the time at which the relative analyzer orientation is changed, and \( t_{det} \) is the time at which the particles are detected. This condition ensures
that actual analysis process of one particle cannot affect that of the other, so the analyzers remain unbiased.

LRM met condition (3.12), varying the analyzer directions \( \hat{a} \) and \( \hat{b} \), or the angle \( \theta \). The apparatus could not meet conditions (3.10) and (3.11):

The conditions for transmission and/or analyzing power and the spacelike separation of the particles...are not respected....

In the first case, the assumption was made that the analyzing power and transmission were intrinsic constants of the apparatus, and therefore were not determined in any way by possible hidden variables. In failing the second condition, it was simply assumed that this did not matter to the results. Consequently, the possibility of sub-luminal communication between particles was not overcome.

The value of the parameter, \( C \), in (3.9) deviated from the ideal \(-1\) due to a 2% contribution from triplet scattering in the target. An experimental value for this coefficient was interpolated from Catillon [61] to be \( C = -0.95 \pm 0.015 \) for the LRM proton energies, although the value .90 appears to have been used in assessing their final results. The triplet scattering also affects the value of the correlation function for the hidden variables case, equation (2.19). Assuming the correlation function can be decomposed into a sum of rotationally invariant and non-invariant parts, then the maximum contribution to \( P(\hat{a}, \hat{b}) \) from triplet scattering will be the non-invariant component. This LRM takes as the probability of triplet scattering, \( \beta \), found experimentally as \( \beta \leq 0.02 \pm 0.01 \) [61]. The upper limit of relation (2.26) was thus \( 2 + 4\beta = 2.08 \) for the LRM experiment.

The transmission coefficients, \( T \), for the analyzers were found to have some slight dependence on the angle, \( \tau \), of the target scattering. The measured correlation value, (3.2), was adjusted for this angular dependence by integrating over \( \tau \) and taking the first order of the correction term in the result. Accounting for the analyzing powers, the final derivation for \( P_{\text{meas}}(\theta) \) is,
29 mg/cm² & 18.6 mg/cm² & Bell’s Limit for the Absolute Value & QM  \\
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Analyzer</th>
<th>Analyzer</th>
<th>Mean</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>-0.99 ± 0.09</td>
<td>-0.85 ± 0.11</td>
<td>-0.93 ± 0.07</td>
<td>≤ 1.00</td>
</tr>
<tr>
<td>30°</td>
<td>-0.74 ± 0.08</td>
<td>-0.81 ± 0.10</td>
<td>-0.77 ± 0.06</td>
<td>≤ 0.69</td>
</tr>
<tr>
<td>45°</td>
<td>-0.69 ± 0.08</td>
<td>-0.63 ± 0.09</td>
<td>-0.66 ± 0.06</td>
<td>≤ 0.52</td>
</tr>
<tr>
<td>60°</td>
<td>-0.48 ± 0.07</td>
<td>-0.50 ± 0.10</td>
<td>-0.48 ± 0.06</td>
<td>≤ 0.38</td>
</tr>
<tr>
<td>90°</td>
<td>+0.07 ± 0.10</td>
<td>-0.01 ± 0.07</td>
<td>+0.02 ± 0.05</td>
<td>≤ 0.02</td>
</tr>
</tbody>
</table>

Table 3.1: LRM experimental correlation values, $P_{exp}(\theta)$ compared to bell’s limit and quantum mechanics for analyzer orientation $\theta$.

$$P_{meas}(\theta) = \mathcal{P}_1 \mathcal{P}_2 P_{exp}(\theta) + \mu [1 - |\mathcal{P}_1 \mathcal{P}_2 P_{exp}(\theta)|^2] \cos \theta$$ (3.13)

where

$$\mu = \frac{\int (T_{1L}T_{2L} + T_{1R}T_{2R} - T_{1R}T_{2L} - T_{1L}T_{2R}) d\tau}{\int (T_{1L}T_{2L} + T_{1R}T_{2R} + T_{1L}T_{2R} + T_{1R}T_{2L}) d\tau},$$ (3.14)

showing the left and right transmissions of the two analyzers. Using (3.13) the experimental correlation values were deduced from the measured correlations, the analyzing powers, and the value of $\mu$ which LRM determined via scattering off tantalum foils.

### 3.3 LRM Experimental Results

The results of the LRM study are shown in Table 3.1. Within experimental error, the values of the measured correlation function for both carbon targets are in agreement with the predictions of quantum mechanics, adjusted for the apparatus. The underlined values in the table are in agreement with the Bell inequality, within the stated experimental error. These values were determined from inequality (2.26) (plus the $\beta$ factor) by inserting special values for the angles between the four unit vectors $\hat{a}$, $\hat{d}$, $\hat{b}$ and $\hat{\theta}$. It is noteworthy that these values also follow directly from the classical correlation value (2.50). Thus, the Bell limit on individual correlation functions was found by LRM to be consistent with classical phenomena.
The disagreement with the inequality is slight, compared to the size of the error quantities which arise from one-standard-deviation errors together with the uncertainties in the analyzing powers. Moreover, the quantum mechanical values which were appropriately affected by the experimental conditions are seen to fall within the Bell limit for the endpoints, as seen in Figure 3.2. Thus, it is only from measurements at three angles that one can draw any conclusions on the EPRB problem. The figure shows the central angles to be a region in which the experimental data of LRM disagrees strongly with the Bell limit. The region is of particular interest to further study as it can accommodate larger error bars on experimental results.

The LRM paper concludes thus:

The measurement of the spin correlation of protons gave good agreement with QM. To compare with the limit of Bell, as in previous experiments with photons, some extra assumptions are necessary. With these assumptions which seem natural but cannot be tested in our device, a contradiction is obtained with the limit of Bell providing an argument against the validity of this
limit, and thus being in favor of nonlocal properties of microphysics.

3.4 Deficiencies of LRM

LRM was indeed a pioneering study demonstrating that EPRB experiments could be performed using the singlet state envisioned by Bohm. As mentioned in section 5.1, however, LRM were unable to meet two of the criteria necessary for a true test of the Bell Limit:

1. A sufficiently high product of analyzing powers and transmissions (condition 3.10);

2. sufficient spatial separations to eliminate hypothetical sub-luminal communication in the analyzing process (conditions 3.11 and 3.12).

There are, as well, a number of other improvements one might hope to make on the LRM experiment which may lead to results that better assess the pertinent questions. These improvements would be:

1. **Allow for Unbiased Phase Space**: The path taken by the two protons after scattering in the analyzer was realistically unlimited in terms of scattering angles. However, only those events wherein both protons scattered (in the analyzer) with azimuthal angle 90° to the vector $\hat{a}$ or $\hat{b}$ were actually counted in the correlations, introducing a potential for biased results. A more thorough investigation would allow for a detection phase space of at least 180°, counting coincidences at all angles to the left or right of the unit vectors.

2. **Eliminate Triplet Contributions**: An ideal test of the Bell Limit would allow for a sharp identification of the angular momentum state of each proton pair, whether singlet or triplet. There should be some confidence in the preparation of the singlet state that triplet background is nearly absent. This ensures the rotational invariance of the spin-states, and thus permits arbitrary orientation vectors to be chosen in the analyzers.
3. **More Data Points:** LRM limited to their acquisition to five angles for which they could easily deduce the maximum limit imposed on the correlation function by the Bell inequality. An improved test would allow one to test the inequality itself, rather than the deduced correlations, for an arbitrarily wide range of azimuthal angles in the analyzer planes.

4. **Analyze in Center-of-Momentum Frame of p-p System:** An important aspect of the EPRB scenario is the anti-parallel momenta of the entangled particles.

This criterion allows for an expectation value (2.23) of the p-p singlet state that is simple from the perspective of angular comparison. It is a benefit to determine the scattering angles in the center-of-momentum frame in order to accurately determine right/left correlations in keeping with the expectation value.
Chapter 4

Experiment and Results

4.1 KVI Apparatus

The experimental component of this thesis was performed at the Kernfysisch
Versneller Instituut (KVI) in Gröningen, Netherlands, in June of 2001. It
was designed primarily as a feasibility study of spin correlations in the $^1S_0$
state of $^2$He, composed of two quasi-bound protons. A study of these spin
correlations, as in [60], allows the testing of Bell’s inequality. Significantly,
the geometry of the KVI apparatus also permits a test of Wigner’s Inequality.

The run-time for the experiment was approximately three days, during
which 4,757,615 events were acquired for potential analysis. Only a fraction
of these were useful for the final analyses, after a set of significant cuts were
placed on the data to isolate physically interesting events. This is described
fully in Section 4.2.

The preparation of the proton pairs in a state with total spin 0 is crucial
to this experiment, given that only in such a state can one be assured (by
quantum mechanics and the classical conservation principles) that each spin
component of the two protons are oppositely oriented—a criterion on which
Bell developed his inequality. Assuming zero orbital angular momentum,
both the singlet and triplet states for the di-proton system can meet this
condition. However, the triplet state can take on total spin projection values
of $M_{total} = 0$ as well as ±1. The latter cases, where the spin projections of
the two protons are aligned along any axis, are obviously not useful in Bell’s analysis and must be screened out. The preparation of the singlet state, then, removes this challenge since all such pairs—properly prepared—are assured to meet the criterion of opposite orientation.

The singlet state was created from deuterons, produced by the KVI’s Polarized Ion Source (POLIS), described in [62, 63, 64]. The deuterons are fed into the AGOR (Accelerateur Groningen Orsay) cyclotron, built in a collaboration between the Institut de Physique Nucléaire d’Orsay and the KVI. AGOR is able to accelerate both light ions and protons, with proton energies up to about 200 MeV. Located in the cyclotron vault, it feeds out its beam to two experimental areas: one devoted to radiobiological experiments, and the second which is an experimental hall containing the main detectors for nuclear physics, including the Big-Byte Spectrometer (BBS) used for this experiment. The facility is shown in Figure 4.1.

![Figure 4.1: Overview of KVI facility.](image_url)

Deuterons produced by POLIS are accelerated by AGOR to 170 MeV kinetic energies, and a 1 nA beam current is ejected and guided to the BBS.
The proton-proton singlet state is created via two interactions between the deuterons and the carbon target.

The first is the interaction $^{12}\text{C}(d,^{2}\text{He})^{12}B^*$ (where $^{2}\text{He}$ denotes two protons coupling in the $^1S_0$ state). The deuterons undergo charge exchange when incident on a carbon target of thickness 9 mg/cm$^2$, where the $\Delta S = 1$ Gamow-Teller transition causes the production of the proton pair and a (possibly excited) $^{12}\text{B}$. It has been shown [65] that the final $^2\text{He}$ states will be almost completely p-p $^1S_0$ states for relative internal energies less than 1 MeV. This internal energy, $\epsilon$, is defined as the difference between the invariant mass of the $^2\text{He}$ ($M_{^2\text{He}}$) system assumed to be in a “quasi-bound” state, and that expected for an unbound di-proton system ($M_{pp}$), such that

$$\epsilon = M_{pp} - M_{^2\text{He}}.$$  \hspace{1cm} (4.1)

In this 1 MeV range, contamination of higher order multipoles is at most only a few percent [65]. The fact that the incident deuteron has one excitation state (2.3 MeV), with the same quantum numbers ($J^\pi = 0^+, T = 1$) as the $^2\text{He} \, ^1S_0$ state adds weight to the conclusion that the quasi-bound protons are, in fact, in singlet configuration. It is a reasonably straightforward exercise in kinematics to find that the sum of the protons’ kinetic energies—in the quasi-bound state—arising from this interaction path is 155.6 MeV (see section A.1). A plot of proton kinetic energy sums produced experimentally is shown in Figure 4.2, where the sharp left-hand peak corresponds to this energy.

The second interaction producing proton pairs in the singlet configuration is $p(d,^{2}\text{He})n$. Here, the deuterons are incident on hydrogen contamination in the target material and a percentage of these interactions will yield a singlet proton pair, along with an undetected neutron. The existence of this singlet state production would be a sharp peak at the di-proton energy sum of 167.7 MeV, as found in section A.2. The presence of both peaks in Figure 4.2 confirms that quasi-bound di-proton states result from both interactions, as the kinematics suggest.

The normal distribution of events centered around the $p(d,^{2}\text{He})n$ peak
Figure 4.2: Sum of the kinetic energies of the two protons reconstructed by the analysis software. No cuts are in place on the data, hence the high proportion of background events. The left peak is caused by singlet events arising from the interaction channel $^{12}C(d,^{2}He)^{12}B$. Those in the right-hand peak arise from $p(d,^{2}He)n$.

is due to the uncorrelated p-p products of the deuteron break-up which may originate in the same or different beam bursts.

The small structure at about 153 MeV in Figure 4.2 corresponds to the small proportion of $^{12}C(d,^{2}He)^{12}B^*$ events which populate the 2$^-$ excited state of the boron atom, expected to fall at 152.2 MeV [66].

The detection system is shown in Figure 4.3. It has three main parts after the target: the Big-Bite Spectrometer (BBS), the Focal-Plane Detection System (FPDS) and the Focal-Plane Polarimeter (FPP).

The BBS consists of two focusing quadrupole magnets and one dipole with a radius of curvature of 220 cm and a maximum field of 1.4 T. It can be used in three modes which allow different positionings of the quadrupoles relative to the dipole, and a fixed distance between the target and the dipole entrance. A setting where the quadrupoles are closer to the dipole allows for a larger acceptance solid angle with decreased momentum acceptance. The opposite effect occurs by moving the quadrupoles away from the dipole.
Figure 4.3: KVI apparatus consisting of the carbon target, Big-Bite Spectrometer (BBS), Focal-Plane Detection System (FPDS) and Focal-Plane Polarimeter (FPP).

entrance. An intermediate setting (mode “B”) was used for this experiment, some parameters of which are given in Table 4.1.

The limited angular acceptance of the BBS is centered on zero degrees for the reaction kinematics. This constrains the internal energies of the $^2$He to the necessary $\epsilon \leq 1$ MeV range for dominance of singlet states. The dipole, having a fairly large momentum acceptance, allows for selection of the proton pairs with kinetic energies in the range 60 to 95 MeV. The proton trajectories are focused by the dipole onto a momentum focal plane in front of the FPDS.

The FPDS consists of two vertical drift chambers (VDCs) positioned at $39^\circ$ to the central beam trajectory, in parallel with the focal plane. The VDCs are separated by 23 cm and each has an x- and u-plane (the u-plane rotated at 32.86° to the x-plane) separated by 140 mm which are comprised of alternating sense and guard wires at high positive voltage. Cathode foils enveloping the wire planes are kept at a high negative voltage. The VDCs are filled with a 50/50 gas mixture of argon and isobutane. Charged protons
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Solid Angle</td>
<td>9.2 msr</td>
</tr>
<tr>
<td>Maximum Horizontal Opening Angle</td>
<td>66 mrad</td>
</tr>
<tr>
<td>Maximum Vertical Opening Angle</td>
<td>140 mrad</td>
</tr>
<tr>
<td>Momentum Bite ($\Delta p/p_0$)</td>
<td>19%</td>
</tr>
<tr>
<td>Horizontal Magnification</td>
<td>-0.45</td>
</tr>
<tr>
<td>Vertical Magnification</td>
<td>-10.1</td>
</tr>
<tr>
<td>Distance: Target to Quadrupole 1</td>
<td>114 cm</td>
</tr>
</tbody>
</table>

Table 4.1: Design parameters of the BBS in mode “B” [68]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Active Detection Area</td>
<td>$1030 \times 367$ mm$^2$</td>
</tr>
<tr>
<td>Angle Between VDCs and BBS Central Ray</td>
<td>$39^\circ$</td>
</tr>
<tr>
<td>Number of Sense Wires in X-Planes</td>
<td>240</td>
</tr>
<tr>
<td>Number of Sense Wires in U-Planes</td>
<td>240</td>
</tr>
<tr>
<td>Angle Between X- and U-Plane Wires</td>
<td>$32.86^\circ$</td>
</tr>
<tr>
<td>Sense Wire Spacing</td>
<td>4.2 mm</td>
</tr>
<tr>
<td>Guard Wire Spacing</td>
<td>4.2 mm</td>
</tr>
<tr>
<td>Sense Wire Thickness</td>
<td>20 $\mu$m</td>
</tr>
<tr>
<td>Guard Wire Thickness</td>
<td>50 $\mu$m</td>
</tr>
<tr>
<td>Distance Between Wire Plane and Cathode</td>
<td>15 mm</td>
</tr>
</tbody>
</table>

Table 4.2: Geometrical Paramters of VDCs. [69]

traversing the chambers ionize the argon atoms and the potential difference between cathode and wires directs the freed electrons towards the wires. The electrons gain energy, causing further ionizing collisions and creating an electron avalanche. The charge build-up is registered in the wires, producing an electrical signal. Photons produced by the de-excitation of the argon are absorbed by the quenching isobutane, preventing a signal spread out over the entire VDC wire plane. [67] The design parameters of the VDCs are given in Table 4.2.

Several adjacent wires will signal the passage of a single charged particle. On average 9 wires in the x-plane and 7.5 in the u-plane respond to a particle track. The arrival times of the electrical signals in each wire, measured relative to a common reference, determine the intersection of the particle tracks with the wire plane. Information from the sense wires is read out by
LeCroy 3377 TDCs, which allow for a window of 350 ns for the drift time within the plane. [70]

The primary uses of the VDCs are reconstructing the momenta of protons exiting the BBS and providing an initial coordinate for the particle trajectories. The VDC alignment with the focal plane ensures a direct correspondence between proton track location and proton momentum. The linearity of this correspondence is shown in Figure 4.4.

![Figure 4.4: Linear correspondence between proton momentum (y-axis) and coordinates of VDC 1 (x-axis), demonstrating the alignment of this detector along the focal plane of the BBS.](image)

The FPP consists of a set of four multi-wire proportional chambers (MWPCs), labeled D1 to D4, along with a carbon analyzer located between D1 and D2, and two scintillators (S1 and S2) which are used to trigger two-particle coincidences and determine time-of-flight. The carbon analyzer (40 mg/cm²) acts as a scatterer for the polarization analysis.

The scintillators, S1 and S2 are located before the analyzer and after the last MWPC, D4. They are segmented, consisting of five overlapping vertical paddles read out by Philips XP2262 photo-multipliers at the top and bottom. NE102A is used as the scintillating material. The paddles in S1 have dimensions of 210 mm × 414 mm, and thickness of 2 mm. The S2
<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Active Detection Area MWPC D1</td>
<td>840 × 400 mm²</td>
</tr>
<tr>
<td>Active Detection Area MWPC D2</td>
<td>1080 × 520 mm²</td>
</tr>
<tr>
<td>Active Detection Area MWPC D3 and D4</td>
<td>1240 × 960 mm²</td>
</tr>
<tr>
<td>Total Number of Channels</td>
<td>2976</td>
</tr>
<tr>
<td>Wire Spacing</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>Wire Thickness</td>
<td>20 μm</td>
</tr>
</tbody>
</table>

Table 4.3: Geometrical Parameters of MWPCs. [73]

paddles, which must have a larger acceptance for the outgoing tracks, have a thickness of 8 mm and dimensions of 275 mm × 1040 mm [71], less than half the 2500 mm attenuation length of the material [72].

Coincidences registered between S1 and S2 begin the read-out of the electronics, defining an event trigger. The trigger has a gate of 150 ns, permitting the creation of many background events which are reconstructed by the analysis software as p-p pairs, but arise from separate bursts of the cyclotron’s beam which occur every 23 ns. The coincidence of proton pairs detected in the scintillators gives time-of-flight information and thus could be used as a measure of the proton kinetic energy.

The MWPCs each contain X- and Y-planes with a wire spacing resolution of 2.5 mm. The geometrical parameters of MWPCs, given in Table 4.3, allow the determination of secondary polar scattering angles of the protons up to 20°. Like the VDCs, they are filled with a gas mixture of 50% argon and 50% isobutane. The wires are kept at zero potential and a negative high voltage is applied to the cathode foils which sandwich the wire planes. A lower voltage is applied to D1 and D2, which have a smaller wire-cathode separation distance than D3 and D4.

An average of one or two wires produce a signal when a proton passes through an MWPC wire plane. The signals are read out by CAMAC-based LeCroy PCOS III modules; they report which wires were fired in an event but give no timing information. These signals, along with those from the VDCs and scintillators, are transferred for real-time data processing and storage on tape. A schematic of the read-out system is shown in Figure 4.5.
Acquisition of the event data from the electronics is handled on the software side by an application called FPPDAQ, which performs the initial online analysis of all events. FPPDAQ extracts the relevant coordinates of particle tracks, and other observables on an event-by-event basis. The main steps are shown in Figure 4.6. During the experiment the results of the FPPDAQ analysis were written on an event-by-event basis to files in the NTUPLE format of the PAW [76] package from the CERN library. This format allows a convenient off-line analysis of the experimental observables.

During this off-line analysis, the information from the VDCs and MWPCs is used to determine the coordinates of the particle trajectories incoming \((v_{in1}, v_{in2})\) and outgoing \((v_{out1}, v_{out2})\) relative to the analyzer. As mentioned, the initial incoming coordinate, \(v_{in1}\), is provided by the first VDC in the FPDS. The second incoming coordinate, \(v_{in2}\), is given by D1. D2 and D4,
Figure 4.6: Overview of important steps performed by FPPDAQ software to analyze a single event accepted by the trigger logic. [75]

following the analyzer, yield $v_{out1}$ and $v_{out2}$, while D3 provides some redundancy for reconstruction of difficult outgoing tracks. The combined efficiency of the FPDS-FPP system is greater than 85% [77].

A typical problem is encountered with the polarimeter, in that the majority of p-C events are only Coulomb-scattered in the analyzer, producing no polarization information. Naturally, these events populate a narrow region of secondary polar scattering angles—typically taken as less than 5° [78]. The distribution of all events (no cuts applied) versus this scattering angle can be seen in Figure 4.7.

The design of the FPDS has a significant advantage over that of previous experiments, especially that of LRM. The planar geometry of the MWPCs eliminates the static nature of measurements that characterized LRM. As Redhead points out, in a static experiment “the choice of analyser setting is made well in advance of the emission of the particles from the source” [79]. There is thus the possibility that the static settings of the analyser are somehow communicated to the source (or target) such that the proton pairs will be produced in agreement with quantum predictions for the particular analyser settings. This potential loophole has been addressed in photon
Figure 4.7: Polar scattering angle, $\theta$, of p-C events outgoing from the carbon analyzer. Those events that are only Coulomb-scattered are assumed to lie in the region $\theta < 5^\circ$.

experiments by deciding the analyser settings after the time at which the photon pairs have left their source. [53] The MWPC design, however, is unbiased in terms of angular acceptance, being sensitive to protons scattering from the analyser through $360^\circ$ azimuthal angles.

The $360^\circ$ acceptance of the FPDS also eliminates the problem of counterfactuality from the measurement process—a lingering criticism of all EPR experiments. Peres [80] explains that in other EPR experiments only one angle can be measured for each particle in each event. This poses a possible difficulty when a quantity such as (2.25) is to be determined. Ideally, the value of (2.25) can be ascertained for each event, individually, which requires the measurement of up to four correlation values for each proton pair. In LRM and other EPR experiments, however, the limited acceptance of the detectors has demanded four separate measurements on four independent (though presumably similar) entangled systems. The counterfactuality problem involves the assumption (however reasonable it may seem) that these four systems are all “obeying the rules” to produce a result that is consistent with the correlations one would have observed on any one of the systems
by itself. If the analyser settings used for particle pair two had been used for particle pairs one or three or four, would they have given the same result? In this experiment, in principle, one event can give all the information needed for a value such as (2.25) to be determined\(^1\) and thus the problem of counterfactual reasoning is eliminated.

### 4.2 General Background Removal

Analysis of the proton tracks is done off-line. This assures that no obvious “conscious-observer-dependence” can influence the measurements taken, as might be a concern with such an experiment investigating entangled phenomena. 4,757,615 events were accepted by the trigger logic as p-p coincidences. Data from these were written to ntuples for further analysis with PAW. There were four main goals to this analysis:

1. Determining the count-rates of the p-p singlet production in both singlet peaks (adjusted for background);

2. Evaluating the analyzing power of the polarimeter system for the energy range of the singlet protons;

3. Measuring a value for expression (2.25) for comparison with (2.30).

4. Measuring a value for the Wigner Correlation, (2.43), under the conditions where it is violated by quantum mechanics.

The first step in the analysis involved cleaning up background in the data set. There are six cuts that must be applied to the data to remove events which are either not of singlet character, or whose proton scattering trajectories cannot be accurately reconstructed from the FPDS output. These are cuts on:

1. The relative time delay between the protons entering the analyzer.

\(^1\)Nonetheless, it is the statistical average (expectation value) of more than one event that is of interest in the comparison with quantum mechanics.
2. The internal energy of the proton pair.

3. Events which the analysis software cannot reconstruct in a valid way due to reasons of the apparatus.

4. Events which do not scatter in the analyzer at all.

5. Events where the spatial separation of the interpolated proton scattering vertices in the analyzer is too small, relative to the MWPC resolution.

6. Events where the spatial separation of the proton scattering vertices is too small to clearly distinguish one proton from the other.

Further cuts can be made to isolate those events whose individual proton kinetic energies are consistent with the values given in Appendix A.

The first cut is made on the many uncorrelated protons produced by deuteron break-up which are accepted by the trigger logic even though they may be created in different beam bursts of the cyclotron. Figure 4.8 shows the time delay between VDC measurements of apparently coinciding protons. Protons originating from the same pulse of the cyclotron are expected to have a relative time delay ($\Delta t$) of less than the 23 ns repetition rate of the AGOR facility (obvious from the regular peak locations on the horizontal axis). The middle or “prompt” peak, corresponding to $\Delta t \approx 0$, contains all genuinely correlated events, and so a cut is made on it. The prompt peak also contains a dose of background from deuteron break-up in the target, where two proton products are erroneously identified as a coupled pair from the desired interaction. Key to determining the effects of this background is the use of the adjacent, smaller peaks in the plot. They are composed solely of random deuteron break-up products from different beam bursts. The shape of the data from these shoulder peaks corresponds to the break-up background in the prompt peak, and can be used to estimate the background here. The effect of this prompt peak cut is shown in Figure 4.9.

A second cut can be made on the internal energy of the di-proton system, seen in Figure 4.10. While the acceptance of the BBS limits this quantity to
Figure 4.8: Time delay between protons identified by the analysis software as an “event”. The middle or “prompt” peak corresponds to events from the same beam burst, but contains both singlet events and uncorrelated protons from two deuterons breaking up in the target. The side peaks provide a means of “subtracting out” the influence of this background. No cuts applied.

small values, the tail of events exceeding $\epsilon = 1.0$ MeV can be removed. The result of this cut on the data in Figure 4.9 is shown in Figure 4.11.

A third cut is placed to remove events which the analysis software was unable to reconstruct in a valid way. There are several causes which might disqualify an event at this stage. The data acquired from the proton passage through the VDCs and MWPCs is used to interpolate the particle scattering vertices in the analyzer: the point at which their incoming and outgoing trajectories intersect. The reconstruction of each event yields a parameter, $D_{\text{min}}$, the minimum distance between the incoming and outgoing vectors for each proton. Events for which the VDC and MWPC data are ambiguous in terms of the linearity of their reconstructed tracks are assigned a $D_{\text{min}}$
value less than zero. This parameter must, then, take a value greater than or equal to zero. The location of the proton scattering vertices may also suffer from poor reconstruction, lying outside the physical dimensions of the analyzer. A small fraction of these events involve protons scattering off the first scintillator paddles (S1) or the second MWPC (D2), rather than off the carbon analyzer [81]. Each event whose scattering vertex falls outside the analyzer was also assigned a negative value of $D_{\text{min}}$ and was discarded in this cut. The effect of this cut on Figure 4.11 is shown in Figure 4.12.

The fourth cut eliminates events which did not scatter in the analyzer, or underwent a second scattering into forward angles. The incoming and outgoing trajectories of these events would be nearly identical and can be identified by the magnitude of the polar angle. Ideally, non-scattering events would have a value close to $\theta = 0^\circ$. However, as a conservative estimate for
the KVI apparatus the cut-off has usually been taken at 5°. [82] This limit derives in part from the unrelated effects of misalignment in the FPP wire chambers along the lab z-axis (the central axis of the apparatus). It has been shown that wire chamber offsets, not corrected for, can cause a shift in the secondary scattering angles of the protons which enter the analyzer at a small angle. This can result in asymmetries varying up to ±0.2 from their actual value, but only in the region 0° ≤ θ ≤ 5°. [83] Thus, a cut at this angular value eliminates the known issue of non-scattering events, as well as any potential inaccuracies from misalignment.

The drawback of such a conservative cut is a substantial loss in statistics. As figure 4.13 illustrates, most of the events at this stage of the analysis fall in the scattering region of 5°. A cut on this energy range would leave too few statistics in the present data set to be of value in any analysis. Therefore, a cut is chosen at 2°, providing a data set that is large enough to be useful,
Figure 4.11: Effect of the internal energy cut on the di-proton kinetic energy distribution on events with cut no. 1 already in place. The shaded region comprises events passing the first cut whose internal energies are also less than 1 MeV.

but sure to contain a proportion of events which did not scatter. It is to be noted that these non-scattering events have an average contribution to a correlation function similar to equation (3.2), which will be affected by the number of unscattered particles, $x$, so that

$$P'_{\text{meas}}(\hat{a}, \hat{b}) \rightarrow \frac{N_{LL} + \frac{x}{4} + N_{RR} + \frac{x}{4} - N_{RL} - \frac{x}{4} - N_{LR} - \frac{x}{4}}{N_0 + x}$$

$$= \frac{N_{LL} + N_{RR} - N_{RL} - N_{LR}}{N_0 + x}. \quad (4.2)$$

The systematic consequence of such a low cut on $\theta$ is to shrink the correlation values. This is a relevant consideration in assessing the results given in Section 4.4 where a correlation value is compared to a precisely defined quantum mechanical expectation value. The results of the Wigner test, in
Figure 4.12: Events are removed which the analysis software was unable to properly reconstruct in terms of analyzer scattering vertices. The shaded region comprises events passing the first, second and current cuts, as compared to the data passing only the first two cuts.

Section 4.5, however, are simply the comparison of three similar correlation values derived from the same data set and should remain largely unaffected by this contamination, except for magnitudes. The general correction factor of Section 4.3 (ideally) allows a means of adjusting for the non-scattering effects.

The result of cutting this parameter at 2° is shown in Figure 4.14 along with the cumulative effects of previous cuts. 51.4% of events are removed by this cut whereas one placed at 5° would have lost 96.7% of the total events at this stage in the analysis.

Figure 4.15 shows the proton trajectories in the vicinity of their analyzer scattering vertices, as reconstructed by the analysis software, using the MWPC and VDC data. The spatial proximity, A, of the two proton scat-
Figure 4.13: Distribution of events by polar scattering angle $\theta$ following interaction with the analyzer. The y-axis shows this angle for the lower energy particle in the pair, while the x-axis is that of the higher energy particle.

Scattering vertices must be larger in its components than the 2.5 mm resolution of the x- and y-planes in the spectrometer MWPCs (see Table 4.3). If this is not the case, there will be some ambiguity as to which particle trajectory belongs to which proton. This motivates the fifth cut, to ensure that a proper identification of protons is made with respect to resolution. The effects of this cut are shown in Figure 4.16.

Distinguishing between the two protons at the point of scattering also motivates the final major cut. It is applied to events for which the identification of one proton from the other is ambiguous due to the spatial proximity of their scattering vertices in the analyzer plane, as seen in Figure 4.15. Events for which
Figure 4.14: The unshaded region shows the energy sum of the correlated proton pair events remaining after the first three cuts are made on the data. Some of these events, however, may involve protons which did not scatter in the analyzer. A cut is made to eliminate events where one (or both) protons scattered to forward angles of less than 2°. What remains is the shaded region.

\[ D_{\text{min1}} + D_{\text{min2}} \leq A_{x,y} \]  

(4.3)

are removed. The effect of this final cut is to eliminate only 0.0015% of the remaining events leaving a spectrum nearly identical to that of Figure 4.16.

Beginning with 4,757,615 raw events, the six cuts described leave 36,951 events for analysis as proton pairs. The majority of these events do not, however, meet the kinematical criteria of the Appendix for truly correlated particle pairs. It is found that (discounting a layer of background) only those events composing the $^{12}C(d,^2He)^{12}B^*$ and $p(d,^2He)n$ peaks as described in Section 4.1 are true entangled p-p singlet pairs.

In the interaction $^{12}C(d,^2He)^{12}B$, the ground state of the boron product produces the majority of the singlet events, and from the kinematics of
Figure 4.15: Proton trajectories incoming and outgoing with respect to the carbon analyzer, as reconstructed by the analysis software for a typical event. \( D_{\min 1} \) and \( D_{\min 2} \) are lines indicating the distance of closest approach for the proton incoming and outgoing trajectories. They should be larger than the spectrometer resolution. The midway point of these lines is taken as the scattering vertex for each proton, and the vector between these vertices, \( \mathbf{A} \), has x- and y- components in the lab frame. These components should be greater than the size of \( D_{\min 1} \) and \( D_{\min 2} \) (the z-component is irrelevant to the analysis).

Appendix A.1 the sum of the protons’ kinetic energies is fixed at 156.7 MeV for this transition. The lower energy proton is allowed an energy range of

\[
69.0 \text{ MeV} \leq T_2 \leq 77.8 \text{ MeV},
\]

while the higher energy particle has a possible energy range of

\[
77.8 \text{ MeV} \leq T_1 \leq 86.5 \text{ MeV}.
\]

Figure 4.17 reveals that the width of the corresponding peak is about 0.75 MeV, due to issues of resolution. The limits of the acceptable proton kinetic energy regions must be increased on both sides by this amount to accommodate all singlet events.
Figure 4.16: The shaded region indicates the effects of the fifth cut, made on those events for which the resolution of the spectrometer is too small for unambiguous particle identification.

Similarly, for the singlet pairs arising from the \( p(d,^2He)n \) interaction, Appendix A.2 gives the lower energy proton a kinetic energy range of

\[
74.8 \text{ MeV} \leq T_2 \leq 83.9 \text{ MeV}
\]  

(4.6)

and the higher energy particle

\[
83.9 \text{ MeV} \leq T_1 \leq 93.0 \text{ MeV},
\]

(4.7)

with an additional 0.75 MeV for the resolution of this peak as well.

Within the limits defined for the two singlet peaks lie all the events which are useful for the experimental test of both the Bell inequality (2.26) and the Wigner inequality (2.43), for they are largely singlet events (with underlying background of uncorrelated p-p pairs). 25,404 events underlie the \( p(d,^2He)n \) peak and 7918 events comprise the \(^{12}C(d,^2He)^{12}B\) peak, for a
Figure 4.17: The region surrounding the $^{12}C(d, ^2He)^{12}B$ singlet peak of the di-proton kinetic energy sum spectrum.

total of 33,322 useful events, or 0.70% of total raw events gathered by the apparatus.

No straightforward cuts were available to remove the remaining background events which were uncorrelated proton pairs meeting the kinematical requirements of the singlet states, but arising from deuteron-breakup in the target. Nonetheless, the effects of these background events on the spin correlations could be compensated for by determining an overall correction factor as follows in the next section.

4.3 The General Correction Factor

One challenge in working with the KVI apparatus at the proton energies employed was the lack of available data for the efficiency of the detector system and the carbon analyzing power. An estimate of the analyzing power was given in [77] as 0.2 in the polar range $5^\circ - 20^\circ$, based on data from [84]. The FPDS and FPP were estimated to have instrumental efficiencies greater than 92%. Nonetheless, more precise values were desirable for the analysis,
and indeed acquiring such values was a goal of the project.

Polarization measurements are based on the asymmetries that arise from scattering. It is crucial to account for any inherent asymmetries in the instrument which might bias the polarization results. Instrumental asymmetries may arise from the following systematic origins:

1. Limited geometrical acceptance of the detectors D2, D3 and D4. If the distribution of protons incident on the analyzer is not centered, some events may be lost to one side of the detector.

2. Misalignment of the detectors may bias the trajectory reconstructions.

3. Non-uniform detection efficiencies of either the wire chambers or the scintillators, which trigger the acceptance logic. [85]

While the asymmetries arising from each of these factors can be addressed by different methods (see [86]), the problem of the efficiencies remains. Fortunately, the physics of the proton singlet states being produced by the interactions $^{12}\text{C}(d,^2\text{He})^{12}\text{B}^*$ and $p(d,^2\text{He})n$ suggests a simple method of accounting for not just the analyzing power, but also the efficiencies and any other hypothetical biases which might influence the results of the experiment in a systematic way.

Any determination of the spin-correlations between the protons in the singlet pairs requires an examination of their “independent” scattering off the analyzer material. This can be accomplished by employing the known kinetic energies of the protons incoming to the analyzer, as well as their trajectories incoming and outgoing. One can calculate in a straightforward way the outgoing kinetic energies along with the scattering angles, both azimuthal ($\phi$) and polar ($\theta$), relative to the incoming trajectory. As shown in Figure 4.18, the outgoing trajectory can be projected onto the plane perpendicular to the incoming trajectory to determine (relative to some fixed lab axis\(^2\)) at what azimuthal angle the individual proton scattered.

\(^2\)The actual axis chosen in the lab is unimportant, since only the difference between the azimuthal angles is relevant to the analysis. Nonetheless, the same reference axis must be used for both particles.
Figure 4.18: Determination of polar and azimuthal scattering angles from incoming and outgoing particle trajectories. The azimuthal angle is measured from a fixed lab axis.

Consistent with the zero angular momentum of the proton singlet state, one would expect that for some suitably small ranges in energy and $\theta$, the scattering angles for the two protons, $\phi_1$ and $\phi_2$, would lie nearly back-to-back. Any exception to this must result from an instrumental asymmetry.

The analyzing power, $A$, for the scattering of an unpolarized particle beam off target particles is a straightforward spin-dependent observable which is calculable from the expression,

$$ A(\theta, T) = \frac{1}{P} \frac{N_R - N_L}{N_R + N_L}. \quad (4.8) $$

Here, $P$ is the polarization of the target. $N_R$ ($N_L$) is the number of beam particles with kinetic energy $T$ deflected to the right (left) at the angle $\theta$.

As given in (3.9), it is necessary for every measurement of spin polarizations to be weighted by the analyzing power and efficiencies, $T$, for the appropriate scattering angle and kinetic energy. Thus, the expected correlation between pairs of spin-$\frac{1}{2}$ particles scattering at angles $\theta_1$ and $\theta_2$ will have the form

$$ P(\theta_1, \theta_2) = \frac{1}{A(\theta_1)A(\theta_2)T(\theta_1)T(\theta_2)} \left( \frac{N_{RR} + N_{LL} - N_{RL} - N_{LR}}{N_{RR} + N_{LL} + N_{RL} + N_{LR}} \right). \quad (4.9) $$

82
Here, $N_{RL}$ (for instance) is the number of events where the first particle scatters to the right at an angle $\theta_1$ and the second to the left at an angle $\theta_2$.

As stated above, it seems reasonable that for an unbiased apparatus a spin correlation measurement for proton pairs in the singlet state (for some $T$ and $\theta$) should result in every scattered pair of particles deflecting to opposite sides of the detector. Any deviation from this provides a multiplicative factor, $\mathcal{A}'(\theta, T)$, a correction which quantifies the effects of asymmetry, efficiency and any other non-random phenomena [C in Equation (3.9)] as a function of the angle and energy parameters. With such a parameter in hand, the expected (theoretical) correlation value, $P_{th}(\theta_1, \theta_2, T_1, T_2)$, for particle 1 scattered at $\theta_1$ and particle 2 at $\theta_2$ will be related to the measured value, $M(\theta_1, \theta_2, T_2)$, by:

$$P_{th}(\theta_1, \theta_2, T_1, T_2) = \frac{M(\theta_1, \theta_2, T_1, T_2)}{\mathcal{A}'(\theta_1, T_1)\mathcal{A}'(\theta_2, T_2)}.$$  \hspace{1cm} (4.10)

Being certain, then, of the singlet character of events under the 155.6 MeV and 167.7 MeV peaks in the di-proton energy spectrum, these allow the determination of the values $\mathcal{A}'(\theta, T)$ from

$$\mathcal{A}'(\Delta\theta, T \pm \Delta T)\mathcal{A}'(\Delta\theta, T \pm \Delta T) = \frac{M(\Delta\theta, \Delta\theta, T_1, T_2)}{-1}.$$ \hspace{1cm} (4.11)

The measured correlation is for singlet events with both particles scattering within an angular bin of $\Delta\theta$, and having individual energies $T_1$ and $T_2$ in the range of $T \pm \Delta T$. The negative sign is introduced because the expected value (for unbiased apparatus) is $-1$. The actual value of $M(\Delta\theta, \Delta\theta, T_1, T_2)$ is found from

$$M(\Delta\theta, \Delta\theta, T_1, T_2) = \frac{N_s - N_d}{N_s + N_d}.$$ \hspace{1cm} (4.12)

$N_s$ is the number of particle pairs whose scattering angles, $\phi_1$ and $\phi_2$, have an absolute difference less than 90° (scattering to the same side of the detection apparatus—correlated). $N_d$ are events where this absolute difference is greater than 90° (scattering to different sides—anti-correlated).
An obvious concern is that the general correlation values appear to be imaginary if the apparatus is so biased as to result in a *positive* correlation measurement. Physically, this type of measurement would show one of two odd systematic phenomena: that particles passing through the analyzer and instruments are deflected to or detected in the same place, regardless of particle characteristics; or that one particle of each pair is systematically undergoing a deflection that is opposite to expectation, while the other is behaving normally. On average, the results of the polarization measurement on singlet pairs should yield a negative quantity, except perhaps in cases of large statistical error. If the result is irreconcilably positive, then the quality of the apparatus, itself, must be considered. It can thus be said, cautiously, that

\[
\mathcal{A}'(\Delta \theta, T \pm \Delta T) = \sqrt{-M(\Delta \theta, \Delta \theta, T_1, T_2)}
\]  \hspace{1cm} (4.13)

The general correlation expression, (4.9), becomes

\[
P(\theta_1, \theta_2, T_1, T_2) = \frac{(N_{RR} + N_{LL} - N_{RL} - N_{LR})/(N_{RR} + N_{LL} + N_{RL} + N_{LR})}{\sqrt{M(\Delta \theta_1, \Delta \theta_1, T_1 \pm \Delta T_1, T_1 \pm \Delta T_1)M(\Delta \theta_2, \Delta \theta_2, T_2 \pm \Delta T_2, T_2 \pm \Delta T_2)}}
\]  \hspace{1cm} (4.14)

Here, the angles \( \theta_i \) are contained in the range of \( \Delta \theta_i \), and the kinetic energies \( T_j \) in the range \( T_j \pm \Delta T_j \).

From the kinematics, the value of \( T \pm \Delta T \) for the 155.6 MeV peak will be \((77.75 \pm 8.75) \text{ MeV} \) and for the 167.7 MeV peak, \((83.9 \pm 9.1) \text{ MeV} \). Therefore, averaged over the angles, two values for \( \mathcal{A}' \) could be extracted from the data set, in an overlapping energy range. The width of the angular bins, \( \Delta \theta \), must depend on the statistics of the data set. A large number of events in each region of the angle parameter is critical to the overall usefulness of the technique—a criterion not suitably met with the current set of results. Nonetheless, the technique applied to this data is illustrated.

Performing the series of necessary cuts described in Section 4.2, the overall spectrum shown in Figure 4.19 is acquired. The events under the two singlet peaks are shouldered by background which, all things considered, should
Figure 4.19: Kinetic energy sum of the two protons showing the two peaks corresponding to the quasi-bound singlet states. The peak at 156 MeV arises from the interaction $^{12}C(d,^2He)^{12}B$ while that at 169 MeV is from $p(d,^2He)n$.

have a uniform polarization across the energy sum spectrum. Therefore, the background under the peaks can be removed by considering the average behavior of the background events to either side of each peak, assuming the binning and energy range of the shoulder segments are chosen to be identical to those of the peak, itself.$^3$

Figure 4.20 shows the events from Figure 4.19 which have passed the kinematical criteria for the process $^{12}C(d,^2He)^{12}B$ concerning the range of energy values (4.4) and (4.5). Similarly, Figure 4.21 are those events corresponding to the process $p(d,^2He)n$ with individual proton energies in the range (4.6) and (4.7).

The general correction factor (4.11) is determined for each peak in an identical way. As indicated, the total number of scattering events that can

$^3$ An alternative to this would be considering the events that arise from protons which the software reconstructs as pairs, but actually originate from different beam bursts of the cyclotron, illustrated in Figure 4.8. These uncorrelated pairs can be analyzed at the two energies corresponding to the peaks and appropriately scaled and binned to the background intensity of Figure 4.19.
Figure 4.20: Kinetic energy sum of the two protons which meet the criteria for the $^{12}C(d,^2He)^{12}B$ singlet state.

occur in the analyzer is the sum of the correlated pairs ($N_s$) and the anti-correlated pairs ($N_d$). The values, for the singlet peaks, must be adjusted by the average of the correlated and anti-correlated events in both left and right shoulder peaks. This way the background under the singlet peaks can be subtracted out leaving the general correlation value for the events of interest. If $N_{sl}$ ($N_{sr}$) is the number of correlated events in the left (right) shoulder of the peak, and $N_{dl}$ ($N_{dr}$) is the number of anti-correlated events in the left (right) shoulder, then the correction factor, following from (4.12), becomes the adjusted value,

$$
\mathcal{A}'(\Delta \theta, T \pm \Delta T) = \sqrt{-M(\Delta \theta, \Delta \theta, T_1, T_2)}
$$

$$
= \sqrt{-\frac{(N_s - \frac{N_{sl} + N_{sr}}{2}) - (N_d - \frac{N_{dl} + N_{dr}}{2})}{N_s + N_d - (\frac{N_{sl} + N_{sr} + N_{dl} + N_{dr}}{2})}}
$$

(4.15)

To find values for (4.15) the current data set was divided into three subsets of events based on the angles $\theta_1$ and $\theta_2$ of the scattered particle trajectories. The first set was events scattering in the range $2^\circ \leq (\theta_1, \theta_2) < 3^\circ$.\footnote{It must be remembered that many of the events in this sub-set (and even the next) will}

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the second in the range $3^\circ \leq (\theta_1, \theta_2) < 5^\circ$, and the third set is events scattering within $5^\circ \leq (\theta_1, \theta_2) < 10^\circ$. Above $10^\circ$, the number of events where both particles scatter at a similar angle is negligible. An ideally large number of statistics would permit the sub-sets to be of smaller angular range and thus the correction factor as a function of angle could be determined in a more meaningful way.

An analysis of the six regions of Figure 4.19 gives the results shown in Tables 4.4 and 4.5. The quoted errors are purely statistical. The results are plotted in Figure 4.22 showing the magnitude of the errors and the overlap of the data from the two singlet peaks. Substantial overlap is expected given that the proton energies between the peaks are similar. The spread of the points at $2.5^\circ \pm 0.5^\circ$ is probably due to the large number of non-scattering events in this range. Clearly, the size of the statistical errors is larger than desired, due again to the small amount of statistics, and will propagate through the results of the following sections as equation (4.10) is

\[\text{actually be } \text{unscattered} \text{ in the analyzer—a larger sample of data would allow this region to be cut out entirely.}\]
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<th>Left Background</th>
<th>Singlet Peak</th>
<th>Right Background</th>
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<td>408</td>
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<td></td>
<td>Adjusted Mean</td>
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Table 4.4: General Correlation Results for $^{12}C(d,^{2}He)^{12}B$ ($77.75 \pm 8.75$ MeV).

<table>
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</tr>
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<td>$4.0^\circ \pm 1.0^\circ$</td>
<td>$N_d$</td>
<td>1790</td>
<td>4540</td>
</tr>
<tr>
<td></td>
<td>Adjusted Mean</td>
<td>$A'$</td>
<td>$-0.089 \pm 0.018$</td>
</tr>
<tr>
<td></td>
<td>$N_s$</td>
<td>194</td>
<td>500</td>
</tr>
<tr>
<td>$7.5^\circ \pm 2.5^\circ$</td>
<td>$N_d$</td>
<td>220</td>
<td>560</td>
</tr>
<tr>
<td></td>
<td>Adjusted Mean</td>
<td>$A'$</td>
<td>$-0.071 \pm 0.048$</td>
</tr>
</tbody>
</table>

Table 4.5: General Correlation Results for $p(d,^{2}He)n$ ($83.9 \pm 9.1$ MeV).
Figure 4.22: The general correlation results with statistical errors, as a function of the polar scattering angle of protons outgoing from the analyzer. The lower (circle) points are from the $^{12}C(d,^2He)^{12}B$ peak, while the upper (square) points are from the $p(d,^2He)n$ events.

employed in deducing correlation functions. The inclusion of instrumental error would make it still larger. Nonetheless, the assumed figure of 0.2 quoted in [77] for the analyzing power of the system is seen to be reasonably valid, on average.

It is important to note that in Figure 4.22 the minimum correction factor value is 0.02, which cannot meet the condition of (3.10). The average of the six values, 0.24, cannot meet this stringent condition, either. Thus, the assumption of LRM must be taken, that the general correction factor is an intrinsic constant of the apparatus, not determined by any hidden parameters. Then the Bell limit may be multiplied to allow for a direct comparison with quantum mechanics.
Ideally, the general correction factors determined from this technique would be of suitable number, with small enough error, to construct an overall factor as a function of scattering angle. This would allow for experimental results to be adjusted by a reasonable interpolation when studying spin-correlation measurements.

4.4 Results for Bell’s Inequality

As explained in Section 2.4, relation (2.26) represents a key test of the EPRB argument of Section 2.2. The completeness of the quantum theory can be checked by data acquired in this experiment, adjusted by the general correction factor of the previous section. If (2.26) is violated and the results agree with the quantum predictions of relation (2.23) [more specifically, of (3.9)], then the quantum theory is found to be complete and no local hidden variables theory can account for the physical characteristics of the entangled p-p systems. Agreement with the Bell inequality, however, would require disagreement with the quantum theory and vindicate the view that quantum mechanics is an “averaging over” of the predictions of some more accurate hidden variables model.

A point of improvement on the LRM experiment mentioned in Section 3.4 is the analysis of the spin correlations in the center-of-momentum (c.m.) frame of the p-p pair. In this frame both the proton spin vectors and momenta are expected to be anti-parallel. Given the incoming proton trajectories in the lab frame, \( \hat{V}_1^i \) and \( \hat{V}_2^i \), the c.m. trajectory, \( \hat{V}_{cm} \) is defined as

\[
\hat{V}_{cm} = \frac{\hat{V}_1^i + \hat{V}_2^i}{|\hat{V}_1^i + \hat{V}_2^i|}
\]  

(4.16)

The trajectories incoming to the analyzer moved into the c.m. frame will be anti-parallel, and the scattering angle of the outgoing lab trajectories, \( \hat{V}_1^o \) and \( \hat{V}_2^o \), relative to the incoming is of interest. In the c.m. frame these four vectors become,
\[ \hat{V}_{j}^{ri, o} = \frac{\hat{V}_{j}^{ri} - \hat{V}_{cm}}{|\hat{V}_{j}^{ri} - \hat{V}_{cm}|} \]  

(4.17)

where \(^{t}\) (prime) indicates a vector in the c.m. frame and \(j\) identifies the particle as 1 (higher energy proton of the pair) or 2 (lower energy).

The vector cross-product of \(\hat{V}_{1}^{i}\) and \(\hat{V}_{2}^{i}\) defines a reference axis in the c.m. frame for each event that lies in the plane normal to these vectors. The azimuthal scattering angle, \(\phi_{j}\) of the particles can be defined with respect to this axis. This scattering angle itself is determined by the cross-product

\[ \hat{V}_{j}^{ri} \times \hat{V}_{j}^{ro} = \sin(\phi_{j} + \frac{\pi}{2}), \]  

(4.18)

as the projection of the outgoing c.m. vector onto the normal plane. The cross-product always lies at \(-\frac{\pi}{2}\) from the projection vector, requiring a systematic adjustment. Figure 4.23 illustrates the method described for finding \(\phi_{j}\).

Figure 4.23: The geometry for finding the proton scattering angle in the c.m. frame, from the incoming and outgoing vectors in the lab frame. The angle \(\phi_{j}\) is desired result from the vector operations.

Having acquired the angles \(\phi_{1}\) and \(\phi_{2}\) from the MWPC and VDC output, they can be compared to the given vectors \(\hat{a}, \hat{a}', \hat{b}\) and \(\hat{b}'\) of the Bell inequality (2.26) to find the correlation values \([P(\hat{a}, \hat{b})\) for instance]. The point of this comparison is to test whether the outgoing c.m. vector, \(\hat{V}_{j}^{ro}\), projects onto
the appropriate comparison vector (say \( \hat{a} \)) in the same direction or opposite; that is, whether the angle between the projection of \( \hat{V}_j^{\nu} \) onto the normal plane of \( \hat{V}_j^{\mu} \) and \( \hat{a} \) (found by the scalar product) is less than or greater than \( \frac{\pi}{2} \).

This analysis is performed for both protons, with respect to the given angles of the correlation function under consideration. The form of the correlation function for this apparatus is similar to that of LRM, given by (3.2) where the total number of analyzed events, \( N_0 \), is the sum of events scattered left/left or right/right minus events scattered left/right or right/left. While LRM based their results on this left-right scattering at strictly defined angles, the current results allow for a full \( 2\pi \) acceptance in the azimuthal scattering angle. Thus the correlation function is somewhat adjusted to depend on two variables: \( N_s \), the number of events in which the both particles scatter in the same way (along or against their respective comparison vectors), while \( N_d \) is the number of events where one proton scatters along its comparison vector and the other against its comparison vector, according to following conditions:

- \( \cos^{-1}(\hat{V}_1^{\nu} \cdot \hat{a}) \leq \frac{\pi}{2} \) and \( \cos^{-1}(\hat{V}_2^{\nu} \cdot \hat{b}) \leq \frac{\pi}{2} \implies N_s \)
- \( \cos^{-1}(\hat{V}_1^{\nu} \cdot \hat{a}) \leq \frac{\pi}{2} \) and \( \cos^{-1}(\hat{V}_2^{\nu} \cdot \hat{b}) \geq \frac{\pi}{2} \implies N_s \)
- \( \cos^{-1}(\hat{V}_1^{\nu} \cdot \hat{a}) \leq \frac{\pi}{2} \) and \( \cos^{-1}(\hat{V}_2^{\nu} \cdot \hat{b}) \leq \frac{\pi}{2} \implies N_d \)
- \( \cos^{-1}(\hat{V}_1^{\nu} \cdot \hat{a}) \geq \frac{\pi}{2} \) and \( \cos^{-1}(\hat{V}_2^{\nu} \cdot \hat{b}) \geq \frac{\pi}{2} \implies N_d \).

The experimental correlation function is simply

\[
P(\hat{a}, \hat{b}) = \frac{N_s - N_d}{N_s + N_d} \quad (4.19)
\]

as in (4.12). It is equivalent to the average of the individual correlation value (\( \pm 1 \)) for each event. As with (3.9), (4.19) must be adjusted for biases in the instrument and transmissions. This is accomplished through the general correlation factor of section 4.3) which requires that the protons' polar scattering angles in the lab frame, \( \theta_1 \) and \( \theta_2 \), be taken into account. Particularly, (4.14) becomes

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\[ P(\theta_1, \theta_2, T_1, T_2) = \frac{(N_s - N_d)/(N_s + N_d)}{A'((\Delta \theta_1, T_1 \pm \Delta T_1)A'(\Delta \theta_2, T_2 \pm \Delta T_2)).} \tag{4.20} \]

Each proton in an event is found in one of the three angular regions of the general correlation factor \((2.5 \pm 0.5^\circ, 4.0 \pm 1.0^\circ \text{ and } 7.5 \pm 2.5^\circ)\), and each singlet pair arises from one of the two singlet peaks of Figure 4.19, corresponding to different kinetic energy regions. There would be, then, nine groupings of events based on their two azimuthal angles, times two possible kinetic energy regions giving eighteen sets of events for which a particular correlation function (4.20) must be calculated. However, since expression (4.20) is insensitive to which proton scatters in which angular bin, it is assumed that \(P(\sigma, \tau, T_1, T_2)\) is equivalent to \(P(\tau, \sigma, T_1, T_2)\) and the statistics for such cases may be considered together. This supposes that the scattering angles of the protons are insensitive (within the errors considered) to the small relative kinetic energy difference between the protons. The final value of the function should be the average, then, of twelve preliminary values that arise because of the general correction factors taken under these groupings.

In order to test the quantum mechanical prediction (2.23) for the correlation function, the value of (4.20) was calculated from the data for nineteen angle combinations. The results for the \(^{12}C(d,^2He)^{12}B\) singlet events are given in Tables 4.6, 4.7, 4.8, 4.9 and 4.10. Those for the \(p(d,^2He)n\) events are given in Tables 4.11, 4.12, 4.13, 4.14 and 4.15. The quantum and classical predictions, from (2.23) and (2.50) respectively, are shown in each table for comparison.

A graphical comparison of the average values for both sets of interaction data, along with the theoretical values, is shown in Figure 4.24. The classical correlation line, it should be noted, is also the upper limit for the magnitude of the correlation values as determined by LRM from (2.26). Thus a data point in agreement with this Bell limit will fall between it and the zero line.

The large size of the error bars renders meaningless any real comparison of the experimental data with theoretical predictions. It is interesting that there
<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 0^\circ)$</td>
<td>$P(0^\circ, 15^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>1.07±0.99</td>
<td>0.08±0.97</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.49±0.57</td>
<td>-0.50±0.57</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>-1.36±1.12</td>
<td>-1.27±1.10</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-0.92±0.47</td>
<td>-0.27±0.45</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-0.44±0.78</td>
<td>-1.04±0.89</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.32±0.44</td>
<td>0.03±0.41</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.41±0.73</td>
<td>-0.49±0.73</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>-1.00</td>
<td>-0.97</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>-1.00</td>
<td>-0.83</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.6: Experimental and theoretical correlation values for $^{12}C(d, ^2He)^{12}B$ interaction.

<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 60^\circ)$</td>
<td>$P(0^\circ, 75^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>0.42±0.98</td>
<td>0.05±0.97</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.77±0.62</td>
<td>-0.69±0.61</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>-1.46±1.15</td>
<td>-1.27±1.10</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>0.47±0.46</td>
<td>0.52±0.46</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-1.19±0.93</td>
<td>-0.87±0.85</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.07±0.41</td>
<td>-0.12±0.42</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.43±0.76</td>
<td>-0.40±0.73</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>-0.50</td>
<td>-0.26</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>-0.33</td>
<td>-0.17</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.7: Experimental and theoretical correlation values for $^{12}C(d, ^2He)^{12}B$ , cont.
<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 120^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>0.19±0.97</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.60±0.59</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>-1.61±1.20</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>0.50±0.46</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>0.27±0.77</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.25±0.43</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.25±0.73</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>0.50</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>0.33</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.8: Experimental and theoretical correlation values for $^{12}C(d, ^2He)^{12}B$, cont.

<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 180^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>1.04±0.99</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.36±0.56</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>-0.14±0.92</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>0.59±0.46</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-0.15±0.76</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.15±0.42</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>0.14±0.68</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>1.00</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>1.00</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.9: Experimental and theoretical correlation values for $^{12}C(d, ^2He)^{12}B$, cont.

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<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(60^\circ, 120^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>0.78±0.98</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>0.77±0.63</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>0.47±0.94</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>0.82±0.47</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>1.65±1.06</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>0.74±0.53</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>0.87±0.77</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>-0.50</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>-0.33</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.10: Experimental and theoretical correlation values for $^{12}C(d, ^2He)^{12}B$, cont.

<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 0^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>0.02±0.49</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>0.04±0.39</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>-1.29±1.09</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>0.12±0.27</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-0.34±0.63</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>0.31±0.42</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.19±0.54</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>-1.00</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>-1.00</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.11: Experimental and theoretical correlation values for $p(d, ^2He)n$ interaction.
<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 60^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>-0.23±0.49</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.77±0.51</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>0.22±0.91</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>-0.28±0.27</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-1.09±0.78</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.14±0.40</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.38±0.56</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>-0.50</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>-0.33</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.12: Experimental and theoretical correlation values for $p(d, ^2 He)n$, cont.

<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 120^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>-0.18±0.49</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.73±0.50</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>0.11±0.90</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>-0.21±0.27</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-1.61±0.95</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.44±0.44</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.51±0.59</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>0.50</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>0.33</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.13: Experimental and theoretical correlation values for $p(d, ^2 He)n$, cont.
<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(0^\circ, 180^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>-0.53±0.50</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>-0.08±0.39</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>0.12±0.90</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>-0.30±0.27</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-0.72±0.69</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>-0.06±0.39</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>-0.26±0.52</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>1.00</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>1.00</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.14: Experimental and theoretical correlation values for $p(d, ^2He)n$, cont.

<table>
<thead>
<tr>
<th>Azimuthal Angle Bins*</th>
<th>Correlation Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(60^\circ, 120^\circ)$</td>
</tr>
<tr>
<td>2.5 and 2.5</td>
<td>0.76±0.51</td>
</tr>
<tr>
<td>4.0 and 4.0</td>
<td>1.39±0.70</td>
</tr>
<tr>
<td>7.5 and 7.5</td>
<td>0.63±0.95</td>
</tr>
<tr>
<td>2.5 and 4.0</td>
<td>1.16±0.33</td>
</tr>
<tr>
<td>2.5 and 7.5</td>
<td>-1.15±0.80</td>
</tr>
<tr>
<td>4.0 and 7.5</td>
<td>0.81±0.54</td>
</tr>
<tr>
<td>Average Experimental</td>
<td>0.60±0.64</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>-0.50</td>
</tr>
<tr>
<td>Classical/Bell Limit</td>
<td>-0.33</td>
</tr>
</tbody>
</table>

*Angular bin widths are: 2.5 ± 0.5, 4.0 ± 1.0, 7.5 ± 2.5

Table 4.15: Experimental and theoretical correlation values for $p(d, ^2He)n$, cont.
is overlap between the error bars and the region between the classical Bell limit and the zero axis, and no clear agreement with the quantum predictions. It is also reassuring that the order of magnitude of the experimental results agrees well with the theoretical. This might not be the case if the general correction factor, (4.14), were improperly determined. It is thus an indication that the methods of section 4.3 are largely correct, while the circumstances of their application might be less than ideal.

For interest, the actual Bell limit of (2.26) can be tested for the violating case of (2.31). The values of $P(0^\circ, 45^\circ)$ and $P(0^\circ, 135^\circ)$ are used. For the $^{12}C(d, ^2He)^{12}B$ case,

$$
\Delta[^{12}C(d, ^2He)^{12}B] = P(0^\circ, 45^\circ) - P(0^\circ, 135^\circ) + P(0^\circ, 45^\circ) + P(0^\circ, 45^\circ) \\
= -1.34 \pm 1.51
$$

and for $p(d, ^2He)n$ :

$$
\Delta[p(d, ^2He)n] = P(0^\circ, 45^\circ) - P(0^\circ, 135^\circ) + P(0^\circ, 45^\circ) + P(0^\circ, 45^\circ) \\
= -0.60 \pm 1.12
$$

The error on the first result allows for agreement with the quantum prediction ($\Delta_{QM} = -2.83$) and falls within the Bell limit $-2 \leq \Delta \leq +2$. The second result disagrees with the quantum value, though it falls nicely under the Bell limit.

The values resulting from the two singlet interactions have some overlap with one another as should be expected. A discrepancy is notable, though, between the six correlation values $P(\theta, \vartheta)$ where $\theta \neq 0^\circ$ (in Tables 4.9, 4.10, 4.14 and 4.15) and those where it is. Figures 4.24(a) and 4.24(b) show that somewhat different correlation values arise, even though the same relative angles are being studied.

The main factor contributing to the poor quality of the average results is the small number of statistics. This gives rise to the large statistical error and further results in angular bins for the correction factor of Section 4.3.
which are very rough and include events which might undergo double scattering in the analyser, or (as is more probable with these small angles) no scattering at all. The consequence of this could be dramatic, given that an inappropriate general correction factor, (4.14), might be applied to a high proportion of events. This would contribute physically meaningless values to the weighting of the average correlations. As mentioned previously, a functional correction factor with small associated error, spanning a conservative spectrum of azimuthal scattering angles (say $5^\circ - 10^\circ$), is desirable for improved results.

In a practical sense, the usefulness of any data set in testing the completeness of quantum theory by the present methods can be judged by comparing the error in the data to the magnitude of the maximum difference between the quantum and classical curves (0.21 at 39.54° and 129.54°). A definitive test would necessitate an error (both in statistics and apparatus) \textit{at least} smaller than $0.21/2 = 0.1$. Judging from the current results, and ignoring errors in the apparatus, this could be accomplished through an apparatus with general correction factor of about 0.5. Any increase in this factor has a significant effect on the size of the statistical errors.

Employing the current three-fold division of scattering angle bins, the 0.1 error could also be achieved with total statistics for each bin of 40,000 or 50,000 events. This would necessitate a ten-fold increase in run time for a similar experiment. With the current rate of 4.8 million raw events acquired in about three days running, the improved statistics would require a month of run-time at the current 1 nA current to gather nearly 50 million events for subsequent cuts.

### 4.5 Results for Wigner’s Correlation

The substantial ambiguity of the Bell results in the previous section suggests a similar difficulty will be met in the Wigner analysis as outlined in section 2.5. The Wigner inequality (2.43) has an advantage over that of Bell, however, in that it offers a strict either/or comparison between the hidden
<table>
<thead>
<tr>
<th></th>
<th>Wigner Comparison</th>
<th>$^{12}C(d,^2\text{He})^{12}B$</th>
<th>$p(d,^2\text{He})n$</th>
<th>Weighted Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P(0^\circ,30^\circ) - P(0^\circ,15^\circ)$</td>
<td>-0.04±1.25</td>
<td>0.35±1.00</td>
<td>0.20±0.78</td>
</tr>
<tr>
<td>2</td>
<td>$P(0^\circ,60^\circ) - P(0^\circ,30^\circ)$</td>
<td>-0.30±1.28</td>
<td>-0.42±0.96</td>
<td>-0.38±0.77</td>
</tr>
<tr>
<td>3</td>
<td>$P(0^\circ,90^\circ) - P(0^\circ,45^\circ)$</td>
<td>-0.35±1.32</td>
<td>-0.64±0.98</td>
<td>-0.54±0.79</td>
</tr>
<tr>
<td>4</td>
<td>$P(0^\circ,120^\circ) - P(0^\circ,60^\circ)$</td>
<td>-0.69±1.31</td>
<td>-0.73±1.04</td>
<td>-0.71±0.81</td>
</tr>
<tr>
<td>5</td>
<td>$P(0^\circ,150^\circ) - P(0^\circ,75^\circ)$</td>
<td>-0.69±1.27</td>
<td>-0.57±1.03</td>
<td>-0.62±0.80</td>
</tr>
<tr>
<td>6</td>
<td>$P(0^\circ,180^\circ) - P(0^\circ,90^\circ)$</td>
<td>-0.21±1.24</td>
<td>0.34±0.96</td>
<td>0.13±0.76</td>
</tr>
</tbody>
</table>

Table 4.16: Results for Wigner Inequality test.

variables predictions and those of quantum mechanics. The Bell inequality, on the other hand, only allowed for a small region of disagreement between the two frameworks.

Testing the Wigner inequality employs the average correlation values found in Tables 4.6-4.15. The functions $P(\uparrow\uparrow)_{a,c}$ of (2.43) are equivalent to the experimental values $P(a,c)$ which are simply the probabilities of both protons scattering in the same direction (i.e. having aligned spin vectors). Six cases can be tested from the available values for each interaction. The twelve results are tabulated in Table 4.16. They test the validity of the expression

$$P(0^\circ,c) - P(0^\circ,\frac{c}{2}) - P(\frac{c}{2},c) \leq 0$$  \hspace{1cm} (4.23)

A violation of the inequality is in keeping with the predictions of quantum mechanics, and would suggest its completeness. Conversely, the validity of the hidden variables perspective requires that the inequality be obeyed. Again, given the large experimental error of the results, one might consider them technically inconclusive.

Figure 4.25 shows the results of Table 4.16, compared with the lower limit
of agreement with quantum mechanics (the zero line). The pattern appears
to disagree with quantum mechanics, having a tendency to the negative side
of the figure. However, the error bars all enclose the zero line so that an
agreement with quantum theory is feasible.

As in the Bell case, a more conclusive result would follow from a larger
number of statistics. Nonetheless, the data sets for the two interactions are
consistent with one another. This offers promise that a similar experimental
geometry can, in fact, have meaningful input into the completeness debate
via the Wigner inequality.
Figure 4.24: A comparison of the experimental correlation values from (a) the $^{12}C(d,^2He)^{12}B$ and (b) the $p(d,^2He)n$ interaction, along with the predictions of the quantum and classical theories.
Figure 4.25: Results of Wigner’s inequality for the weighted mean of Table 4.16.
Chapter 5

GHZ

5.1 GHZ

In 1990 a significant argument was put forward by Greenberger, Horne, and Zeilinger (GHZ) \[9\] to show that the premises of EPR (section 2.1) are fundamentally inconsistent in the context of a system or three or more particles. The method of GHZ does not employ the statistical comparison of Bell’s two-particle case; rather, it claims to demonstrate a strict mathematical contradiction arising from the application of all four EPR premises to a chosen physical system.

Consider a single particle of spin 1 which decays into two spin-1 particles, each of which then decays into two more spin-\(\frac{1}{2}\) particles. We are left with a system of four entangled fermions of spin \(\frac{1}{2}\), and for simplicity we choose to study a system in which the momenta of the particles are directed along the positive or negative z-axis. The wave-function for the initial spin-1 particle is \(| J, M\rangle_{\text{initial}} = | 1, 0\rangle_{\text{initial}}\), and the first decay proceeds as

\[
| 1, 0\rangle_{\text{initial}} \rightarrow \frac{1}{\sqrt{2}}[| 1, +1\rangle_I | 1, -1\rangle_{II} - | 1, -1\rangle_I | 1, +1\rangle_{II}]. \tag{5.1}
\]

These intermediate spin-1 particles decay into the fermions as follows:

\[
| 1, +1\rangle_I \rightarrow | \frac{1}{2}, +\frac{1}{2}\rangle_1 | \frac{1}{2}, +\frac{1}{2}\rangle_2 \tag{5.2}
\]
\[ |1, -1\rangle_{II} \longrightarrow |\frac{1}{2}, -\frac{1}{2}\rangle_3 | \frac{1}{2}, -\frac{1}{2}\rangle_4 \]  \hspace{1cm} (5.3)

\[ |1, -1\rangle_I \longrightarrow |\frac{1}{2}, -\frac{1}{2}\rangle_1 | \frac{1}{2}, -\frac{1}{2}\rangle_2 \]  \hspace{1cm} (5.4)

\[ |1, +1\rangle_{II} \longrightarrow |\frac{1}{2}, \frac{1}{2}\rangle_3 | \frac{1}{2}, \frac{1}{2}\rangle_4 \]  \hspace{1cm} (5.5)

Thus, employing the states (1.52) and (1.53), the wavefunction of the final state is:

\[ |1, 0\rangle_{\text{initial}} \longrightarrow |\Psi\rangle = \frac{1}{\sqrt{2}} \left[ |\alpha\rangle_1 |\alpha\rangle_2 |\beta\rangle_3 |\beta\rangle_4 \right. \]

\[ \left. - |\beta\rangle_1 |\beta\rangle_2 |\alpha\rangle_3 |\alpha\rangle_4 \right]. \]  \hspace{1cm} (5.6)

A measurement of the spin projection for particle \(i\) along some arbitrary axis, \(\hat{n}_i\), utilizes the operator (1.62). The product of the four general spin projection operators,

\[(\sigma_1 \cdot \hat{n}_1) \otimes (\sigma_2 \cdot \hat{n}_2) \otimes (\sigma_3 \cdot \hat{n}_3) \otimes (\sigma_4 \cdot \hat{n}_4) \]  \hspace{1cm} (5.7)

is at the heart of the GHZ argument, as the product \((\sigma_a \cdot \hat{a}) \otimes (\sigma_b \cdot \hat{b})\) was the key to Bell’s.

The quantum expectation value for this product acting on the state \(|\Psi\rangle\) is found from:

\[ E_{QM}^{\Psi} = \langle \Psi | (\sigma_1 \cdot \hat{n}_1) \otimes (\sigma_2 \cdot \hat{n}_2) \otimes (\sigma_3 \cdot \hat{n}_3) \otimes (\sigma_4 \cdot \hat{n}_4) | \Psi \rangle \]  \hspace{1cm} (5.8)

where

\[(\sigma_i \cdot \hat{n}_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i e^{-i\phi_i} \\ \sin \theta_i e^{i\phi_i} & \cos \theta_i \end{pmatrix}. \]  \hspace{1cm} (5.9)

Explicitly, one has the state

\[ \langle \Psi | = \langle (1\ 0)_1 \otimes (1\ 0)_2 \otimes (0\ 1)_3 \otimes (0\ 1)_4 \]
\[
- \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right)_1 \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_3 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_4 \right|, \quad (5.10)
\]

with operator

\[
(s_1 \cdot \hat{n}_1)(s_2 \cdot \hat{n}_2)(s_3 \cdot \hat{n}_3)(s_4 \cdot \hat{n}_4) = \\
\left( \begin{array}{cc} \cos \theta_1 & -\sin \theta_1 e^{-i\phi_1} \\ \sin \theta_1 e^{i\phi_1} & \cos \theta_1 \end{array} \right)_1 \otimes \left( \begin{array}{cc} \cos \theta_2 & -\sin \theta_2 e^{-i\phi_2} \\ \sin \theta_2 e^{i\phi_2} & \cos \theta_2 \end{array} \right)_2 \otimes \\
\left( \begin{array}{cc} \cos \theta_3 & -\sin \theta_3 e^{-i\phi_3} \\ \sin \theta_3 e^{i\phi_3} & \cos \theta_3 \end{array} \right)_3 \otimes \left( \begin{array}{cc} \cos \theta_4 & -\sin \theta_4 e^{-i\phi_4} \\ \sin \theta_4 e^{i\phi_4} & \cos \theta_4 \end{array} \right)_4 \right) \quad (5.11)
\]

acting on the ket

\[
| \Psi \rangle = | \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_1 \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_2 \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right)_3 \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right)_4 \right) - \left( \begin{array}{c} 0 \\ 1 \end{array} \right)_1 \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right)_2 \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_3 \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_4 \right). \quad (5.12)
\]

The expectation value (5.8) works out to be,

\[
E_{QM}^\Psi (\theta_1, \theta_2, \theta_3, \theta_4, \phi_1, \phi_2, \phi_3, \phi_4) = \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4 - \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \cos(\phi_1 + \phi_2 - \phi_3 - \phi_4). \quad (5.13)
\]

Respectively, \( \theta_i \) and \( \phi_i \) give the polar (x-z plane) and azimuthal (x-y plane) orientations of the vector \( \hat{n}_i \). Thus, constraining the spin projection measurements to the x-y plane (\( \theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{2} \)) leaves (5.13) as:

\[
E_{QM}^\Psi (\phi_1, \phi_2, \phi_3, \phi_4) = -\cos(\phi_1 + \phi_2 - \phi_3 - \phi_4). \quad (5.14)
\]

For \( \phi_1 + \phi_2 - \phi_3 - \phi_4 = 0 \),

\[
E_{QM}^\Psi (\phi_1, \phi_2, \phi_3, \phi_4) = -1. \quad (5.15)
\]

For \( \phi_1 + \phi_2 - \phi_3 - \phi_4 = \pi \),
\[ E_{Q,M}^\Psi(\phi_1, \phi_2, \phi_3, \phi_4) = +1. \] (5.16)

The angular conditions imposed on the state vector to yield (5.15) and (5.16) are the conditions for eigenstates of \(\Psi\), so that for these instances the expectation value is identical with the eigenvalue.

These specific cases are used to demonstrate a contradiction when the four EPR criteria are applied to (5.7). One infers the existence of four functions which give the \(\pm 1\) outcomes of each spin projection measurement along the azimuthal angles:

\[ A_\lambda(\phi_1), B_\lambda(\phi_2), C_\lambda(\phi_3), D_\lambda(\phi_4). \] (5.17)

These functions, in accordance with the locality and reality criteria, are independent of one another (hence the singular angular dependence) and take their values in accordance with the deterministic action of the parameter \(\lambda\). Clearly,

\[ A_\lambda^2(\phi_1) = B_\lambda^2(\phi_2) = C_\lambda^2(\phi_3) = D_\lambda^2(\phi_4) = +1 \] (5.18)

The contradiction is arrived at by equating the product of these independent functions (5.17) with the derived expectation value, thus “restating” (5.15) and (5.16) in terms of them. This correspondence is reasonable, since the system is in an eigenstate of the measurement operator. In the words of Dirac, “provided the state \(|x\rangle\) is normalized”, the quantum expectation value \(\langle x | \xi | x \rangle\) is “the average of all the results obtained...if the measurement of the observable \(\xi\) for the system in the state corresponding to \(|x\rangle\) is made a large number of times...” [87]. Under the current special conditions one can equate the results of a single measurement on an individual system [i.e. (5.17)] with the average result of a large number of measurements on an ensemble of identical systems (the expectation value).

Given that the functional arguments, \(\phi_i\), are non-separable in the cosine in equation (5.14) it is already obvious that these independent azimuthal functions are incompatible with the quantum expectation value. Nonetheless, the GHZ argument proceeds:
From the conditions of (5.15) it follows that,

\[
A_\lambda(0)B_\lambda(0)C_\lambda(0)D_\lambda(0) = -1 \\
A_\lambda(\phi)B_\lambda(0)C_\lambda(\phi)D_\lambda(0) = -1 \\
A_\lambda(\phi)B_\lambda(0)C_\lambda(0)D_\lambda(\phi) = -1 \\
A_\lambda(2\phi)B_\lambda(0)C_\lambda(\phi)D_\lambda(\phi) = -1
\]

Multiplying the first three of these equalities [and remembering (5.18)] gives:

\[
A_\lambda(0)A_\lambda(\phi)A_\lambda(\phi)B_\lambda(0)B_\lambda(0)B_\lambda(0)C_\lambda(0)C_\lambda(0)C_\lambda(\phi)D_\lambda(0)D_\lambda(\phi)D_\lambda(\phi) = \\
A_\lambda(0)B_\lambda(0)C_\lambda(\phi)D_\lambda(\phi) = -1.
\]

(5.19)

Comparing this with the fourth equality yields:

\[
A_\lambda(2\phi) = A_\lambda(0),
\]

(5.20)

an interesting preliminary result in itself, suggesting isotropy in the spin-projection value of each particle. From the conditions of (5.16), we see that

\[
A_\lambda(\Phi + \pi)B_\lambda(0)C_\lambda(\Phi)D_\lambda(0) = +1.
\]

(5.21)

Before, we found

\[
A_\lambda(\Phi)B_\lambda(0)C_\lambda(\Phi)D_\lambda(0) = -1.
\]

(5.22)

Thus we have

\[
A_\lambda(\Phi + \pi) = -A_\lambda(\Phi),
\]

(5.23)

as would be expected from the nature of the state vector (5.6). However, we see an apparent contradiction between this result and (5.20) for the case where \( \phi = \frac{\pi}{2} \) and \( \Phi = 0 \):

\[
A_\lambda(\pi) = +A_\lambda(0), \quad A_\lambda(\pi) = -A_\lambda(0).
\]

(5.24)
Showing this contradiction, GHZ concludes that they have “brought to
the surface an inconsistency hidden in” the four EPR parameters, not only for
the case of four particles, but also for three since the azimuthal angle of the
second particle (whose function is $B_\Lambda(\phi_2)$) was irrelevant to the operations
performed. GHZ further states,

But we know from Bell’s model...that the corresponding [EPR]
premises are consistent for a pair of spin-$\frac{1}{2}$ particles. Correlations
of three or more spin-$\frac{1}{2}$ particles involve at least one more degree
of freedom than one finds in correlations of two spin-$\frac{1}{2}$ particles,
and it is clear that the manipulation of an additional degree of
freedom is essential to the exhibition of a contradiction. [88]

5.2 Mermin’s Three-Particle Case

Immediately following the publication of GHZ, David Mermin presented the
outline of a simplified version [89] of the argument in the pages of Physics
Today. His version, developed more fully in [90] is a specialized case of
GHZ for a special choice of angles. There is a difference between the two
arguments, in that Mermin’s proceeds from the perspective of three particles
rather than four, taking advantage of the fact that one of the four particles
in the GHZ formulation remains unused throughout the proof.

Mermin relies on the careful choice of a particular entangled state for
three particles (A,B,C),

$$\Psi = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$

(5.25)

and three operators for which $\Psi$ is an eigenstate:

$$\hat{M} = \sigma_{Ax} \otimes \sigma_{By} \otimes \sigma_{Cy}$$

(5.26)

$$\hat{N} = \sigma_{Ay} \otimes \sigma_{Bz} \otimes \sigma_{Cy}$$

(5.27)

$$\hat{O} = \sigma_{Ay} \otimes \sigma_{By} \otimes \sigma_{Cx}$$

(5.28)
\[ \hat{P} = \sigma_{Ax} \otimes \sigma_{Bx} \otimes \sigma_{Cx}. \]  

(5.29)

The eigenvalues follow in a straightforward way,

\[ \hat{M} |\Psi\rangle = +1 |\Psi\rangle \]  

(5.30)

\[ \hat{N} |\Psi\rangle = +1 |\Psi\rangle \]  

(5.31)

\[ \hat{O} |\Psi\rangle = +1 |\Psi\rangle \]  

(5.32)

\[ \hat{P} |\Psi\rangle = -1 |\Psi\rangle \]  

(5.33)

One can now define six functions, of the same type as those in (5.17), corresponding to the results of spin projection measurements on the three particles along their x- and y- axes. These functions,

\[ A(x), A(y), B(x), B(y), C(x), C(y) \]  

(5.34)

must take the dichotomic values of ±1, which are pre-determined prior to any measurement at the creation of the entangled state. Their values are all either +1 or −1, but are independent of one another once assigned, obeying the condition of locality (thus the measurement of the spin projection of particle A cannot affect the value of B’s projection). The value of \( A(x) \) would arise from the measurement

\[ \sigma_{Ax} \otimes \mathbb{I}_B \otimes \mathbb{I}_C |\Psi\rangle, \]  

(5.35)

where \( \mathbb{I} \) is the identity operator.

From (5.30), the simultaneous measurement of \( A(x), B(y) \) and \( C(y) \) yields the relation

\[ A(x)B(y)C(y) = +1. \]  

(5.36)
Similary, from (5.31) and (5.32),

\[ A(y)B(x)C(y) = +1 \]  \hspace{1em} (5.37)
\[ A(y)B(y)C(x) = +1. \]  \hspace{1em} (5.38)

One expects, from the quantum operations of (5.33) that

\[ A(x)B(x)C(x) = -1. \]  \hspace{1em} (5.39)

However, remembering the dichotomic and independent character of the measurement functions (5.34), the product of (5.36), (5.37) and (5.38) gives

\[
\begin{align*}
A(x)B(y)C(y)A(y)B(x)C(y)A(y)B(y)C(x) \\
= +1 \\
= A(x)B(x)C(x)A(y)^2B(y)^2C(y)^2 \\
= A(x)B(x)C(x)(\pm 1)^2(\pm 1)^2 \\
= A(x)B(x)C(x)
\end{align*}
\]  \hspace{1em} (5.40)

which is in contradiction to (5.39).

The conclusion of Mermin’s argument is that the very formalism of quantum mechanics is incompatible with local realism as embodied in the independent functions (5.34). The EPR Principles cannot simultaneously hold.

### 5.3 Critique of GHZ/Mermin Argument

The work of GHZ and Mermin of the preceding sections attempt in each case to test the simultaneous validity of the EPR Principles. Common to each is the comparison of predictions made under the assertion of local determinism with those of quantum mechanics as expressed through the action of spin operators on an entangled eigenstate. Critical to these comparisons is the assumption that for some system of \( n \) entangled particles, the result of physically (i.e. experimentally) measuring an arbitrary spin component of one of the particles (with no other measurements being performed on other
parts of the system) will correspond formally to the operation of an appropriate quantum spin operator on the eigenstate of the system—specifically on that part of the wave function representing the particle that is being probed. The question must be raised as to what, if any, actual correspondence exists linking such a physical measurement with the mathematical operation.

The argument of Mermin will suffice to illustrate the issue, for here the supposed correspondence is most transparent. The wave function under consideration is

$$\Psi = \frac{1}{\sqrt{2}} (|↑↑↑⟩ - |↓↓↓⟩)$$

which is an eigenstate of the four composite operators (5.26)-(5.29) which are carefully chosen for this outcome. It is easily seen that for an operator representing a measurement on only one particle, such as $\mathbb{I}_A \otimes \sigma_{Bz} \otimes \mathbb{I}_C$, the wave function is left in some other state:

$$\begin{align*}
\mathbb{I}_A \otimes \sigma_{Bz} \otimes \mathbb{I}_C |\Psi\rangle &= \left(\frac{1}{\sqrt{2}}\right) \mathbb{I}_A \otimes \sigma_{Bz} \otimes \mathbb{I}_C (|↑↑↑⟩ - |↓↓↓⟩) \\
&= \left(\frac{1}{\sqrt{2}}\right) (|↑↓↑⟩ - |↓↑↓⟩) \\
&= |\Psi\rangle \\
&\neq |\Psi\rangle.
\end{align*}$$

(5.42)

In light of this, the composite operators cannot be thought of as representing a series of measurements in time, for in that case one must ask what condition the system is in between individual measurements as it undergoes a series of transformations. What becomes of the state if the experimenter forgets to perform all three measurements and only observes the first two values? It is only with respect to the combined action of all three operators that the system is an eigenstate.

The proof of Mermin (and GHZ) rests on the assertion that the product of three measurements, $A_iB_jC_k$ (where $i,j,k$ are one of $x,y$), should have the same value as the eigenvalue arising from the quantum operator $\sigma_{Ai} \otimes \sigma_{Bj} \otimes \sigma_{Ck}$ acting on its eigenstate. Can it be said that the value of $B_z$
alone is formally equivalent to the eigenvalue resulting from the operation $1_A \otimes \sigma_{Bx} \otimes 1_C |\Psi\rangle$? It cannot, because this is not an eigenvalue problem; only the expectation value of such an operation can be determined so that unless an ensemble of measurements on a single particle is considered, only the product of three measurements has any equivalence to the quantum prediction. Quantum theory is not able to deal with the result of an individual measurement in this case—only certain products of three measurements. Except in the case of ensemble probabilities, it is blind to what a measuring instrument would discover if applied to one of the three bodies in the system. The measurement products—(5.36)-(5.39)—if considered as equivalents to the quantum operations, should be seen as a single product of three measurements, such that

$$
A_x B_y C_y = m_{xyy} \\
A_y B_x C_y = n_{yxy} \\
A_y B_y C_x = o_{yyx} \\
A_x B_x C_x = p_{xxx}.
$$

In this way, the actual value of the individual particle spin projections, whether +1 or −1, is unknown. Only the product is useful in the comparison.

Recalling the eigenvalue operations of (5.30)-(5.33), the equivalence of measurement and quantum formalism becomes:

$$
\hat{M}|\Psi\rangle = \sigma_{Ax} \otimes \sigma_{By} \otimes \sigma_{Cy} |\Psi\rangle \\
= m_{xyy} |\Psi\rangle \\
\hat{N}|\Psi\rangle = \sigma_{Ay} \otimes \sigma_{Bx} \otimes \sigma_{Cy} |\Psi\rangle \\
= n_{yxy} |\Psi\rangle \\
\hat{O}|\Psi\rangle = \sigma_{Ay} \otimes \sigma_{By} \otimes \sigma_{Cx} |\Psi\rangle \\
= o_{yyx} |\Psi\rangle \\
\hat{P}|\Psi\rangle = \sigma_{Ax} \otimes \sigma_{Bx} \otimes \sigma_{Cx} |\Psi\rangle
$$
\[ = p_{xxx}|\Psi\rangle. \]  

(5.50)

Rather than the result of (5.40), this equivalence will give:

\[
m_{xgy}n_{gyx}o_{gyz}|\Psi\rangle = \hat{M}\hat{N}\hat{O}|\Psi\rangle
\]

\[
= \hat{M}\hat{N}(+1)|\Psi\rangle
\]

\[
= \hat{M}(+1)^2|\Psi\rangle
\]

\[
= (+1)^3|\Psi\rangle
\]

\[
= +1|\Psi\rangle
\]

(5.51)

\[
p_{xxx}|\Psi\rangle = \hat{P}|\Psi\rangle
\]

\[
= -1|\Psi\rangle.
\]

(5.52)

The interchange of measurement values on individual particles [which gives rise to the result of (5.40)] cannot be brought into the argument, because the individual values have no correspondence to any eigenvalue problem involving this state. The contradiction between (5.40) and (5.39) does not arise without assuming more than is allowed by the context of the formalism.

The argument of GHZ can also be analyzed in this light, and found to be wanting, as Mermin and GHZ employ the same underlying principles using different systems. The argument of Bell (Section 2.4), however, is not subject to the same criticism, since his comparison involved the expectation value of ensembles. There is an equivalence between the measurement of some spin component of many identical particles in an ensemble, and the quantum expectation value arising from the joint operation of the appropriate spinor along with the identity operator for the other particle.
Chapter 6

Conclusion

While the statistics of the experiment performed at KVI were small, giving technically inconclusive results from the data, several significant points can be drawn from this thesis.

First, it is recognised that the geometry of the KVI detectors has allowed an experiment to address two significant loop-holes in studying the EPR problem. The $2\pi$ acceptance of the apparatus allows for the passive recording of data from entangled singlet systems. This leaves no room for hypothetical observer interference in the experimental results. Such interference may arise (in other experimental geometries) when detectors must be positioned, post-analyser, at preferred angles. The $2\pi$ coverage of the detectors also allows for the spin correlations to be determined relative to any axis in the normal plane, long after the measurements have been taken. This eliminates the counterfactuality loophole, since any single event can supply sufficient information to test the full Bell or Wigner correlations. The apparatus represents a great improvement on those used in past EPR experiments, which had to rely on a limited, observer-selected angular acceptance.

In section 4.3, a technique was developed for determining the general correction factor. In a simple way, this factor accounts for any systematic biases in the instrument, and is based only on the fundamental angular momentum conservation principles for the spin singlet state. When a singlet state is employed in a spin correlation study, this factor could be used to account for
the analyzing power and other hypothetical biases of the apparatus.

To the best of our knowledge, this thesis presents the first experimental test of the Wigner inequality. By the trend of the data, the results indicate disagreement with quantum mechanics, suggesting the incompleteness of the theory. However, the statistical error of the data is too large to draw firm conclusions.

The power of Wigner’s strict test (compared to Bell’s) is seen in Figure 4.25: despite the present ambiguity, a modest improvement in statistics would allow for a conclusive application of the inequality to this problem. The quality of data necessary to distinguish between quantum and classical predictions in the context of Bell’s formulation exceed that necessary for one to make adequate use of Wigner’s relation. For that reason the latter test promises to be a good vehicle to settle the question of quantum completeness. It must be noted that the $2\pi$ acceptance of the KVI detectors make the Wigner analysis possible. Since Wigner devised this relation some thirty years ago it has been ignored in the experimental arena, in favour of testing Bell’s inequality which requires only a limited angular acceptance. Further effort should be made along the lines of Wigner.

Finally, the GHZ problem was considered. Questions were raised as to its fundamental applicability to experimental situations. In recent years, the argument of GHZ has been given great theoretical significance in considerations of the EPR problem. While mathematically it presents a beautiful and consistent argument that ends in a clear contradiction, it is suggested that this result has little relational to any situation which an experimenter might encounter in the “real” universe. This conclusion arises in consideration of the experimental operators employed in modeling very special multi-particle entangled states. If it is correct, then David Mermin may have too-soon recanted his conviction that “no set of experiments, real or gedanken, [is] known that could produce...an all-or-nothing demolition of the [hidden variables] elements of reality.” [91]
Bibliography


[13] In [1], 777.


[15] In [22], 3.

[16] In [25], 78.

[17] In [21], 11.


[19] In [20], 87.


[28] In [21], 187-191.

[29] In [25], 371-375.


[32] In [14], 44-51.


[34] In [5], 361-364.


[36] In [6], 369.


[38] Ibid, 372-373.


[40] Ibid, 374.

[41] Ibid, 382.

[42] In [2], 195.


[45] In [2], 195.


[55] L. Kasday, cited in [60].


[58] In [44], 24-25.

[59] Ibid, 23.


[67] In [62], 52.


[69] In [62], 53.

[70] Ibid, 52-56.

[71] Ibid, 59-60.


[73] In [62], 57.

[74] Ibid, 52.

[75] Ibid, 75.


[78] In [62], 56-57.

[79] In [14], 110.

[80] In [46], 167.

[81] In [62], 89-90.


[83] In [62], 97-98.


[85] In [62], 95.

[86] Ibid, 95-98.

[87] In [20], 46.
[88] In [9], 1135.


[91] In [89], 11.
Appendix A

A.1 $^{12}C(d,^2He)^{12}B^*$ Kinematics

Deuteron Momentum, $p_d$
Deuteron Kinetic Energy, $T_d = 170$ MeV
Deuteron Rest Mass, $m_d = 1875.58$ MeV/c²
$^2$He Momentum: $p_{He}$
$^2$He Kinetic Energy
$^2$He Rest Mass, $m_{He} = M_d + 1.44$ MeV/c² = 1875.58 MeV/c²
$^{12}B^*$ Momentum: $p_B$
$^{12}B^*$ Kinetic Energy: $T_B$
$^{12}B^*$ Rest Mass: $m_B = 11,191.42$ MeV/c²
$^{12}B^*$ Excitation Energy: $\epsilon = 0$ MeV
$^{12}C$ Rest Mass: $m_C = 11,178.0$ MeV/c²

Conservation of Momentum:

$$p_d = \sqrt{T_d^2 + 2T_d m_d} = \sqrt{T_{He}^2 + 2T_{He} m_{He}} + \sqrt{T_B^2 + 2T_B m_B} \quad (A.1)$$
$$816.45 \text{ MeV/c} = \sqrt{T_{He}^2 + 3,754.04 \text{ MeV/c}^2 T_{He}} + \sqrt{T_B^2 + 22,382.84 \text{ MeV/c}^2 T_B} \quad (A.2)$$

Conservation of Energy:

$$T_{He} = T_d + m_d + m_C - T_B - m_B - \epsilon - m_{He} \quad (A.3)$$
$$= 155.14 \text{ MeV} - T_B \quad (A.4)$$

Therefore:
\[ T_B = 0.06 \text{ MeV} \quad (A.5) \]
\[ T_{He} = 155.08 \text{ MeV} \quad (A.6) \]

Determining the \(^2\text{He}\) Proton Energies:

Proton 1 Momentum: \( p_1 \)
Proton 1 Kinetic Energy: \( T_1 \)
Proton 2 Momentum: \( p_2 \)
Proton 2 Kinetic Energy: \( T_2 \)
Proton Rest Mass: \( m = 938.27 \text{ MeV/}c^2 \)
Angle of Separation Between Proton Trajectories: \( \alpha \)

Conservation of Energy:
\[ T_{He} + m_{He} = T_1 + T_2 + 2m \quad (A.7) \]

Conservation of Momentum:
\[ p_{He}^2 = \sqrt{T_{He}^2 + 2T_{He} m_{He}} = p_1^2 + p_2^2 + p_1 p_2 \cos \alpha \quad (A.8) \]

For \( \cos \alpha = 1, \{T_1, T_2\} = \{86.54, 69.01\} \text{ MeV} \)
Minimum value of \( \cos \alpha \):
\[ \frac{\partial (\cos \alpha)}{\partial T_1} = 0 \implies \alpha = 6.22^\circ \quad (A.9) \]

and
\[ T_1 = T_2 = 77.78 \text{ MeV} \quad (A.10) \]

Therefore the kinetic energy conditions are:
\[ T_1 + T_2 = 155.55 \text{ MeV} \quad (A.11) \]
\[ 77.78 \text{ MeV} \leq T_1 \leq 86.54 \text{ MeV} \quad (A.12) \]
\[ 69.01 \text{ MeV} \leq T_2 \leq 77.78 \text{ MeV} \quad (A.13) \]
A.2 $p(d,^2He)n$ Kinematics

Deuteron Momentum, $p_d$
Deuteron Kinetic Energy, $T_d = 170$ MeV
Deuteron Rest Mass, $m_d = 1875.58$ MeV/c$^2$
$^2$He Momentum: $p_{^2He}$
$^2$He Kinetic Energy
$^2$He Rest Mass, $m_{^2He} = m_d + 1.44$ MeV/c$^2 = 1876.02$ MeV/c$^2$
Neutron Momentum: $p_n$
Neutron Kinetic Energy: $T_n$
Neutron Rest Mass: $m_n = 939.6$ MeV/c$^2$
Target Proton Rest Mass: $m_p = 938.3$ MeV/c$^2$

Following the same steps as Appendix A.1,

\[
T_n = 0.002 \text{ MeV} \quad (A.14)
\]
\[
T_{^2He} = 167.26 \text{ MeV} \quad (A.15)
\]

Kinetic energy conditions:

\[
T_1 + T_2 = 167.73 \text{ MeV} \quad (A.16)
\]
\[
83.87 \text{ MeV} \leq T_1 \leq 92.99 \text{ MeV} \quad (A.17)
\]
\[
74.75 \text{ MeV} \leq T_2 \leq 83.87 \text{ MeV} \quad (A.18)
\]