The Dynamic Modelling of an Axial Piston Hydraulic Pump

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mechanical Engineering, University of Saskatchewan.

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Abstract

As part of the study involving the design and implementation of axial piston pump control systems, an accurate, dynamic model of a axial piston pump is required. In this thesis, the complete derivation of such a model is presented. The derivation is performed symbolically, using general geometrical coordinates and parameters of a typical axial piston pump, rather than specific values. Therefore, the developed model should be applicable to other types of axial piston pumps with similar configurations.

The developed model is expressed by a set of highly nonlinear mathematical equations, these equations being functions of pump parameters and operating conditions. When the specific values of these various parameters were substituted into the describing equations, it was found, for the pump used in this study, that the model could be simplified to a set of linear expressions. In addition, the experimental results indicate that the developed model accurately predicts the dynamic response of a particular pump - a Vickers PVB5 axial piston pump.
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Nomenclature

$a$ The axial distance from the piston-slipper yoke pivot. (m)

$a_r$ The perpendicular distance from the yoke pivot to the shoe plate. (m)

$a_{sp}$ The axial distance from the spring cap pivot point to the yoke pivot. (m)

$A$ General area (m$^2$)

$A_c$ Effective control piston area (m$^2$)

$A_c'$ Actual control piston area (m$^2$)

$A_n$ Notch area (m$^2$)

$A_o$ Overlap area (m$^2$)

$A_p$ Effective piston area (m$^2$)

$A_p'$ Actual piston area (m$^2$)

$b$ The distance between the center line of the control piston guide and the swashplate pivot. (m)

$b_{sp}$ The radial distance from the spring cap pivot point to the yoke pivot. (m)

$B$ Dimensionless pressure gradient

BDC Bottom dead center

$C$ Swashplate angle correction constant (rad)
$C_1, C_2$ Integration constants

$C_1$ Complete pump model coefficient (Nm)

$C_2$ Complete pump model coefficient (Nms)

$C_1$ Complete pump model coefficient (Nms$^2$)

$C_d$ General discharge coefficient

$C_{dn}$ Discharge coefficient for the relief notch

$C_{do}$ Discharge coefficient for the overlap area

$d$ Depth of the relief notch (m)

$D_h$ Hydraulic diameter (m)

$f_{\text{min}}$ Minimum high frequency fluctuation (Hz)

$F$ General force (N)

$F_{ap}$ Force applied to the swashplate by a slipper (N)

$F_c$ Force applied to the yoke by the control piston (N)

$F_{cs}$ Viscous shear force applied to the control piston (N)

$F_{ps}$ Viscous shear force applied to a piston (N)

$F_r$ Force applied by the shoe plate to a slipper (N)

$F_{sc}$ Force applied by the return spring to the spring cap (N)

$F_{sp}$ Force applied by the spring cap to the yoke (N)

$F_{sp_o}$ Force constant used in the calculation of $F_{sp}$ (N)

$h_c$ One half the radial clearance between the control piston guide and the control piston. (m)

$h_p$ One half the radial clearance between the pistons and the cylinders. (m)
H
Total angular momentum vector (kgm²/s)

H_z
Component of the total angular momentum vector in the z direction. (kgm²/s)

i
Unit vector in the z direction

I
Yoke moment of inertia (includes swashplate and pintles) (kgm²)

I_r
Shoe plate moment of inertia about the yoke pivot (kgm²)

I_r'
Centroidal shoe plate moment of inertia (kgm²)

I_e
Total effective moment of inertia (kgm²)

I̅_e
Average total effective moment of inertia (kgm²)

I_p
Total pintle moment of inertia (kgm²)

I_{sw}
Swashplate moment of inertia (kgm²)

I_y
Yoke moment of inertia (kgm²)

j
Unit vector in the axial direction

k
Unit vector in the z direction

K
General constant (units vary)

K_d, K_{d_1}
Yoke damping torque constant (Nms)

K_{d_2}
Yoke damping torque constant (Nm)

K_{d_3}
Yoke damping torque constant (s⁻¹)

K_l
Leakage constant for one piston (m³/s/Pa)

K_L
Total leakage flow constant (m³/s/Pa)

K_o
Overlap area constant (m²)

K_{pr_1}
Pressure torque constant (Nm)
\[ K_{pr_2} \] Pressure torque constant (Nm/Pa)

\[ K_{pr_3} \] Pressure torque constant (Nm/Pa)

\[ K_{pr_4} \] Pressure torque constant (Nms)

\[ K_{sp} \] Return spring spring rate (N/m)

\[ l \] General length (m)

\[ l_{cl} \] Length of the control piston chamber leakage path. (m)

\[ l_{cl_o} \] Length of the control piston chamber leakage path when \( y_c = 0 \). (m)

\[ l_{cs} \] Length of the control guide covered by the control piston. (m)

\[ l_{cs_o} \] Length of the control piston shear surface for \( y_c = 0 \). (m)

\[ l_p \] Length of a piston inserted inside the pump barrel. (m)

\[ l_{po} \] Length of piston insertion for \( y_p = 0 \). (m)

\[ l_{sc} \] Length of the spring cap (m)

\[ l_{sp_o} \] Length of spring compression for \( y_{sp} = 0 \). (m)

\[ m \] Mass of one piston (kg)

\[ M \] Mass of a control volume (kg)

\[ m_c \] Mass of the control piston (kg)

\[ m_r \] Mass of the shoe plate (kg)

\[ m_{sc} \] Mass of the spring cap (kg)

\[ M \] Moment vector (Nm)

\[ n \] General exponent (unitless)

\[ n_o \] Overlap area exponent
$N$  Number of pistons

$p$  General Pressure (Pa)

$\dot{p}$  Total hydrostatic pressure (Pa)

$P$  Cylinder pressure (Pa)

$P_c$  Control piston chamber pressure (Pa)

$P_D$  Discharge pressure (Pa)

$P_S$  Suction pressure (Pa)

$q$  General flow ($m^3/s$)

$Q$  General total flow ($m^3/s$)

$Q_{in}$  Total flow into a control volume ($m^3/s$)

$Q_{out}$  Total flow out of a control volume ($m^3/s$)

$q_{cl}$  Control piston leakage ($m^3/s$)

$q_l$  Individual piston leakage ($m^3/s$)

$Q_L$  Total piston cylinder leakage ($m^3/s$)

$q_n$  Flow through the relief notch ($m^3/s$)

$q_o$  Flow through the overlap area ($m^3/s$)

$r$  Displacement vector (m)

$r_c$  Control piston radius (m)

$r_p$  Piston radius (m)

$R$  Piston pitch radius (m)

$Re$  Reynolds Number

$S_1$  Simplified pump model constant (Nm)
$S_2$ Simplified pump model constant (Nm)

$S_3$ Simplified pump model constant (Nms)

t Time (s)

t' Thickness (m)

T General torque (Nm)

$T_{ap}$ Total torque applied to the swashplate by the slippers (Nm)

$\overline{T}_{ap}$ Average total torque applied to the swashplate by the slippers. (Nm)

$T_{ap_i}$ Torque applied to the swashplate by one slipper (Nm)

$\overline{T}_{ap_i}$ Average torque applied to the swashplate by one slipper. (Nm)

$T_c$ Torque applied to the yoke by the control piston (Nm)

$T_d$ Torque due to yoke damping (Nm)

$T_{pm_i}$ Torque effect of an individual piston (Nm)

$\overline{T}_{pm}$ Average effect of piston inertia on $\overline{T}_{ap}$. (Nm)

$\overline{T}_{ps}$ Average torque due to piston shear (Nm)

$\overline{T}_{pr}$ Average torque due to pressure (Nm)

$\overline{T}_r$ Average torque effect of the retaining ring (Nm)

$T_{sp}$ Torque applied to the yoke by the return spring (Nm)

TDC Top dead center

u Fluid velocity vector component in the axial direction (m/s)

$u^*$ Dimensionless velocity vector component

U General wall velocity (m/s)
$v$ General velocity (m/s)
$v$ Velocity vector (m/s)
$V$ General volume (m$^3$)
$V$ Fluid velocity vector (m/s)
$V_c$ Control cylinder volume (m$^3$)
$V_{c_0}$ Control cylinder volume for $y_c = 0$ (m$^3$)
$V_p$ Cylinder volume (m$^3$)
$V_{p_0}$ Cylinder volume for $y_p = 0$ (m$^3$)
$y$ General axial displacement (m)
$y_c$ Control piston displacement (m)
$\dot{y}_c$ Control piston velocity (m/s)
$\ddot{y}_c$ Control piston acceleration (m/s$^2$)
$y_p$ Piston displacement (m)
$\dot{y}_p$ Piston velocity (m/s)
$\ddot{y}_p$ Piston acceleration (m/s$^2$)
$y_{sp}$ Displacement of the end of the return spring (m)
$\dot{y}_{sp}$ Spring end velocity (m/s)
$\ddot{y}_{sp}$ Spring end acceleration (m/s$^2$)
$x$ General cartesian coordinate perpendicular to $y$ (m)
$x^*$ Dimensionless cartesian coordinate
$\Delta l_{spring}$ Amount of spring compression (m)
$\Delta P$ General pressure drop (Pa)
\( \Delta P_n \) Pressure drop across the relief notch (Pa)
\( \Delta P_o \) Pressure drop across the overlap area (Pa)
\( \Delta P_p \) Pump differential pressure (Pa)
\( \alpha \) Swashplate angle (rad)
\( \dot{\alpha} \) Swashplate angular velocity (rad/s)
\( \ddot{\alpha} \) Swashplate angular acceleration (rad/s^2)
\( \beta \) Bulk modulus of fluid (Pa)
\( \gamma \) Angle of relief notch (rad)
\( \theta \) General angular position of the barrel and pistons (rad)
\( \theta_i \) Angular position of the \( i^{th} \) piston (rad)
\( \theta_n \) Angular position of the start of the relief notches (rad)
\( \theta_o \) Angular position of the end of the pump ports (rad)
\( \theta_{1-6} \) Angular positions which delimit the pressure regions (rad)
\( \mu \) Fluid viscosity (Ns/m^2)
\( \rho \) Fluid density (kg/m^3)
\( \tau_{wall} \) Wall shear stress (N/m^2)
\( \omega \) Pump rotational speed, general rotational speed (rad/s)
\( \omega_n \) Approximate undamped natural frequency (rad/s)
\( \zeta \) Approximate Damping ratio
Chapter 1
Introduction

The fluid power industry faces competition not only from within but from competing technologies such as electric and electro-mechanical systems. To meet this competition, hydraulic firms must design and build hydraulic systems offering greater efficiency and performance than existing systems now provide. One method of achieving this is to improve the method of control of hydraulic systems.

One innovative method of control that is currently being investigated is the implementation of computer (or digital) control systems. The aim of using digital controllers is to utilize the inherent flexibility and power of computers to expand the limits now imposed on hydraulic controllers. These digital systems can offer greater performance than conventional control systems or they can perform functions, such as adaptive control, that conventional controllers cannot. Furthermore, computer control systems may be less expensive than existing controllers because they may not require the expensive mechanical elements often found in conventional control systems.

Hydraulic systems are traditionally analog in nature, while these new “smart” or adaptive controllers are digital. Consequently, one problem in using digital controllers in hydraulic applications is that the mechanical elements (pumps, motors, valves, etc.) must be interfaced with the computer. Generally, these interfacing elements fall into one of two categories: valve controllers or pump controllers. (Hydraulic motors are similar to pumps and are controlled by the same methods as pumps.) Both types of components
have been the subject of intense research [1-7]. Conclusions from many of these studies indicate that new innovative ideas must still be postulated and investigated before a true marriage between the computer and hydraulic components can be established. The research initiated in this study is a continuation towards this goal but is limited to the area of digital pump controllers as an overall research objective.

1.1. Control System Design Techniques

In order to implement computer control of hydraulic pumps, new interfacing devices must be designed and built. The approach used in this design and fabrication process is important. One method of determining the feasibility of various ideas and concepts is to design and build a digital control system with its interfacing device and then analyze their performance. The problem with this experimental approach is that if any modifications are made to the original design, the interfacing device will likely have to be rebuilt, or at least modified. Although this is a very common approach (dictated perhaps, by a lack of analytical skills of the designer), it could be an expensive and time consuming procedure.

An alternative method of designing these control systems is to use computer simulations. This approach utilizes mathematical models of the various components of a prospective control system to simulate, on a digital computer, its performance. This allows the control system to be analyzed and optimized using computer simulations instead of experimental testing. Once the simulation results indicate that the control system will perform satisfactorily, and that the optimal design parameters have been established, an actual system can be built and tested.

The advantages of this method are as follows:

†The numbers inside brackets refer to references at the end of the thesis.
Many types of control systems could be investigated because the designer is not restricted by available components (until an actual system is to be built).

The process of optimizing a given control system is simplified because the effects of a design modification can be readily determined.

This approach can be less expensive and less time consuming than the build, test, rebuild, retest, etc. experimental approach.

The disadvantage of this computer simulation based approach is that the results of this method depend upon the mathematical simulation of the object to be controlled. If the object is not simulated accurately and reliably, then the control system will be based upon an inaccurate model and will likely perform unsatisfactorily. Therefore, to utilize this method an accurate dynamic model of the object to be controlled is essential. In addition, to give confidence to any simulation results, the basic model of the object to be controlled should be verifiable.

As stated, this research is directed towards the development of a digital control system for a hydraulic pump. An approach to direct digital control of a hydraulic variable displacement axial piston pump has been postulated by the author and is shown in Appendix A. This system would use digital (open or closed) solenoid valves and pulse width modulated control signals to provide direct digital control of a hydraulic pump. However, in order to implement this system the control scheme and the required interfacing elements must be designed and developed. This could be done with either a trial and error or a simulation based approach. The experiences of other researchers have clearly indicated that the performance of hydraulic components varies widely and any control scheme must reflect these variations [7,8]. Therefore, a trial and error approach was not considered to be a satisfactory approach in establishing the feasibility of this new control scheme and interfacing device. Consequently, it was believed that the the best approach to the design of this control system was to simulate the
system on a computer before any actual building and testing takes place. However, before any control system could be simulated and studied, it was imperative that an accurate and verifiable dynamic model of a variable displacement axial piston pump was developed.

A literature search was undertaken to determine what models and what modelling techniques had been used to describe variable displacement axial piston pumps. There have been numerous papers which have discussed the components and parameters which make up a mathematical model of an axial piston pump [9-16]. However, to fully understand these models and their limitations, an understanding of the operation of an axial piston pump is required. An axial piston pump is explained in Chapter 2 and therefore, the specific details and limitations of the available pump models will be fully described in Chapter 2.

As will be shown in Chapter 2, the available models are inadequate and could not be used to simulate confidently and accurately the dynamic characteristics of an axial piston pump throughout the pump's operating range. Thus, an accurate dynamic model of a variable displacement pump had to be developed.

1.2. Research Goals

In order to examine the feasibility of the digital control device postulated by the author (Appendix A) it was necessary that an accurate model of a variable displacement pump be established. The objective of this particular research project was therefore to develop an accurate model of a variable displacement hydraulic pump which would include as many factors which affect dynamic response as possible.

Therefore, a dynamic model of a hydraulic pump was developed using the following philosophy:

- The model must be applicable to any axial piston pump with a
similar configuration. Therefore, the model was developed using general coordinates for an axial piston pump, rather than the coordinates for a specific pump.

- During the development of the model, every attempt was made to include all forces, torques, etc. that may affect the pump dynamics. Only once the model was completed were the various terms compared to each other to determine which terms could be neglected. This comparison was based upon a specific pump.

- During the development of the model, the effect of a particular component or element was kept separate. This provided insight into the operation of the pump and allowed the various terms to be compared once the model was completed.

- The model is intended for use in the design of any type of pump control system. Therefore, a specific type of control system was not modelled. The completed model only describes the dynamic characteristics of the pump itself.

- Many computer simulation packages can easily simulate nonlinear systems with no more difficulty than linear systems. Therefore, during its development, no attempt was made to linearize the model.

- To allow this model to be used for control system design, it had to be verified experimentally.

Subject to these guidelines, the development of an accurate dynamic model of a variable displacement pump is presented in this thesis.

1.3. Thesis Outline

This thesis is organized as follows. The operation of a variable displacement axial piston pump is described and the previously used pump models, and their limitations, are explained. An axial piston pump is then examined and the components or elements of the pump that have an effect on its dynamic response are determined. The equations which mathematically describe each of these effects are then fully derived. Once the derivation of these equations has been completed, the various equations are combined to produce an overall dynamic pump model. A simplified pump
model is also developed. The results of the verification testing are then presented. Finally, conclusions and recommendations for further research are presented.
Chapter 2
Components of an Axial Piston Pump

A variable displacement axial piston pump converts mechanical power into hydraulic power. This chapter will describe the components and operation of variable displacement axial piston pumps. The specific details and limitations of previous pump models will also be presented.

2.1. Fixed Displacement Pump

A simplified diagram of a fixed displacement axial piston pump is shown in Figure 2-1. In the diagram the drive shaft passes through a hole in the swashplate, and the end of the shaft is supported by a bearing in the valve plate. Both the swashplate and the valve plate are fixed and do not rotate. The rotation arrow on the lower view of the valve plate only indicates the direction of barrel rotation with respect to the valve plate. The barrel of the pump is attached to a drive shaft. As the barrel rotates, the pistons slide along the swashplate on their slippers. The angle of the swashplate causes the pistons to reciprocate back and forth inside the cylinders in the pump barrel. This reciprocating action provides the pumping motion.

The pump barrel is held against the valve plate, which contains the suction and discharge ports. The suction port is connected to the pump inlet and the discharge port is connected to the pump outlet. The pump barrel contains cylinder ports on the end that contacts the valve plate. The movement of the cylinder ports against the valve plate provides the necessary valving to control the flow in and out of the cylinders.
Figure 2-1: A Fixed Displacement Axial Piston Pump
Consider the motion of an individual piston starting at top dead center (TDC). As the barrel rotates, the piston moves away from the valve plate. The cylinder port for the piston is open to the suction port on the valve plate, and hydraulic fluid is drawn into the cylinder chamber. Once the cylinder has reached bottom dead center (BDC), the cylinder port is no longer open to the suction port and the cylinder is full of fluid. With continued rotation, the swashplate forces the piston to move towards the valve plate and the cylinder port opens to the discharge port. This causes the fluid to flow out of the pump, until the piston reaches TDC, and then this process is repeated. There are a number of pistons in the pump and so the output flow is continuous and nearly constant. This is the pumping process. The swashplate converts the rotary motion into reciprocating motion and the valve plate and the pistons convert the reciprocating motion into an output flowrate.

For clarity, Figure 2-1 does not show the shoe plate assembly, which is shown in Figure 2-2. The shoe plate resembles a large washer with holes in it. The pistons can pass through the holes, but the bases of the slippers cannot. The shoe plate rotates with the barrel and reciprocates with the pistons. It is held in place with a spherical washer that slides along the shaft and rotates with it. When the pistons reciprocate, the shoe plate pivots about the center of the spherical washer. The spring that acts on the spherical washer forces the shoe plate against the slippers, holding the slippers against the swashplate. The spring force also acts on the pump barrel, holding it against the valve plate. It is the shoe plate that forces the pistons to move away from the valve plate during the pumping process.

Axial piston pumps are a flow source, not a pressure source. The pressure that is produced depends on the resistance to the output flowrate. Thus to control an axial piston pump, the flowrate has to be controlled. The output flowrate depends only on the pump displacement, and the input rotational speed. Therefore, the only ways to control the flowrate are to change the input rotational speed, or to change the pump displacement.
Variable speed electric AC motors are expensive and inefficient and so controlling the input speed is not economical. Therefore, to control an axial piston pump, the displacement must be controlled.

2.2. Variable Displacement Pump

Figure 2-3 shows a variable displacement axial piston pump. It is identical to the piston pump previously shown and discussed except that the swashplate sits in a yoke that can rotate. The yoke rotates about the yoke pivot on two short shafts (one on each side) called pinnles. This allows the angle of the swashplate to be changed and thus the pump displacement can be changed.

Figure 2-4 shows the difference between a large swashplate angle and a small swashplate angle. It shows that the pump stroke increases with the
swashplate angle. Therefore, increasing the swashplate angle will increase the pump displacement, and therefore, the output flowrate. Thus, to control a variable displacement axial piston pump, the swashplate angle must be controlled.

Figure 2-3 shows the control piston and the return spring that are used
to position the swashplate yoke assembly in many axial pumps. The control piston slides on the control piston guide, which is a hollow cylinder fixed to the pump case. When high pressure fluid is allowed to flow into the control piston chamber, the pump destrokes (the angle of the swashplate decreases) and the output flowrate drops. If the control chamber is vented to a low pressure reservoir, then the return spring will stroke the pump (the swashplate angle increases) and the output flowrate will increase. Figure 2-3 does not show any valving to regulate the net flow into the control piston chamber. Some type of valving must be present.

Numerous mechanical and electro-mechanical valve assemblies have been developed to control these pumps. However, there are disadvantages to these types of control systems. The different assemblies provide different control functions but generally, for each different type of control, a different valve
assembly is required. Usually, as the level of control increases, the complexity of the required valving increases greatly. This causes problems because the valves that are used to control conventional pumps are expensive and may be sensitive to contamination.

Many of the problems with conventional mechanical and electromechanical control devices could possibly be solved by implementing computer control systems. However, to design such systems, a dynamic model of a variable displacement axial piston pump must be developed.

2.3. Previous Pump Models

In general, two types of modelling techniques have been used to describe the dynamics of variable displacement pumps: linear analysis and small signal analysis. Linear analysis is based upon the assumption that the pump's behaviour can be described by a linear transfer function throughout its operating range, while the small signal technique assumes that the small variations, or perturbations, of the model output about a desired operating point can be described as a linear function of an input disturbance or perturbation. Both methods are very powerful analytical tools but they have limitations.

Using these techniques, numerous authors have researched the control of axial piston pumps in the past. Lewis and Stern [9], Merritt [10], Green and Crossley [11], and Baz [12] have analyzed the dynamic response of various types of conventional control systems for these pumps. However, a universal problem with these studies is that the major proportion of their modelling is devoted to the specific control system and the pump is only represented by very simple linear models. To achieve these simple models major factors which will affect the pump dynamics, such as the effects of load pressure, have been neglected. Therefore, it was decided that these models were inadequate.
Helgestad [17] has analyzed the pressure transients in the cylinder chambers that occur when a pump is operating. However, his work is directed towards the effect that these transients have on the noise of the pump and not the dynamic response of the pump.

There have been a few papers published involving the determination of the more complex parameters of hydraulic pump model. Yamaguchi [13,14] presents a linear model of a hydraulic pump. He considers the effects that the load pressure will have on the pumps dynamics, but he bases his analysis on a pump with a simplified and impractical configuration. Zeiger and Akers [15] and Lin [16] present a method for determining the effects of load pressure for a practical pump. However, their work is based upon an essentially steady state approach, and an overall dynamic pump model is not presented.

There are problems with each of the pump models that these papers represent. One problem is that these models are either a linear model or a small signal model. The linear analysis is limited to conditions of small nonlinearities and relatively simple applications. The geometry and the motion of the various components of an axial piston pump are complex and nonlinear. Therefore, it was decided that it could not be assumed, without justification, that a variable displacement axial piston pump could be accurately described by a linear function throughout its operating range. The problem with the small signal analysis is that it is only valid for a limited variation about an operating point. Thus, it is not feasible to use a small signal model to simulate the dynamic characteristics of a hydraulic pump throughout its operating range.

During the literature search, it was discovered that different authors neglected different pump components (such as piston inertia effects, load pressure effects, etc.), usually without justification, indicating that there are no guidelines stating which components have a large effect and which components can be neglected. Also, it was found that none of the presented
models included or mentioned the effect of viscous damping forces on the pistons or the control piston. All of the available pump models were incomplete in some respect, and were based upon simplifying assumptions. Therefore, it was decided that the available models could not be used to simulate confidently and accurately the dynamic response of an axial piston pump throughout its operating range. As a result, the research undertaken and presented in this thesis is the development of a mathematical model of an axial piston pump.
Chapter 3
Torques Applied to the Swashplate

To develop a mathematical model of an axial piston pump, the dynamics of the swashplate yoke assembly must be considered. The dynamics of the swashplate yoke assembly are controlled by the torques acting on this assembly. Therefore, to determine the mathematical model of an axial piston pump the torques that act on the swashplate and the yoke must be determined.

Figure 3-1 illustrates the various components or elements that have been identified to have an effect on the net torque applied to the swashplate yoke assembly. The torques that act on the swashplate and yoke are a consequence of:

- Pressure forces acting on the pistons
- Inertia of the pistons
- The forces applied by the shoe plate
- Viscous friction forces acting on the pistons
- The return spring force acting on the yoke
- Friction forces acting on the yoke (both stiction and viscous damping forces)
- The control force applied by the control piston.

These components can be split into two groups: those that affect the net torque applied to the swashplate and those that apply a torque directly
Figure 3-1: Components of the Swashplate Yoke Torques

to the yoke. This chapter will consider only the components of the net
torque applied to the swashplate.
3.1. Swashplate Torques

There is a torque applied to the swashplate by the slippers on the pistons. The force that is applied by these slippers to the swashplate is affected by the pressure forces acting on the pistons, the inertia of the pistons, the inertia of the shoe plate, and the viscous friction forces acting on the pistons. Therefore, the net torque applied to the swashplate is influenced by:

- The inertia of the pistons
- The shear forces acting on the pistons
- The forces that the shoe plate applies to the slippers
- The pressure forces acting on the pistons.

This chapter will consider the development of the mathematical expressions that represent the contribution of each of these factors to the net torque applied to the swashplate. (The expression describing the effect of the pressure forces will be derived in detail in Chapter 4.)

3.2. Piston Geometry

To evaluate the various forces that act on the pistons, the geometrical relationships between the piston motion (displacement, velocity and acceleration) and the swashplate motion must be determined. The geometry and the notation used in this section are shown in Figure 3-2.

Considering Figure 3-2, the axial displacement of the piston is

\[ y_p = -\frac{a}{\cos \alpha} + R \sin \theta \tan \alpha \]  

(3.1)

or
Figure 3-2: Piston Geometry

\[
y_p = \frac{R \sin \theta \sin \alpha - a}{\cos \alpha}.
\]

(3.2)

Taking derivatives of Equation (3.1) gives

\[
\dot{y}_p = -\frac{a \dot{\alpha} \sin \alpha}{\cos^2 \alpha} + R \dot{\theta} \cos \theta \tan \alpha + \frac{R \dot{\alpha} \sin \theta}{\cos^2 \alpha},
\]

(3.3)

and
\[
\ddot{y}_p = -\frac{a\dot{\alpha}^2\cos \alpha - a\ddot{\alpha}\sin \alpha}{\cos^2 \alpha} - \frac{2a\dot{\alpha}^2\sin^2 \alpha}{\cos^3 \alpha}
+ R \left[ -\dot{\theta}^2 \sin \theta \tan \alpha + \frac{\dot{\alpha} \dot{\theta} \cos \theta}{\cos^2 \alpha} + \ddot{\theta} \cos \theta \tan \alpha \right] 
+ R \left[ \frac{\dot{\alpha} \dot{\theta} \cos \theta}{\cos^2 \alpha} + \frac{2\dot{\alpha}^2 \sin \alpha \sin \theta}{\cos^3 \alpha} + \frac{R \ddot{\alpha} \sin \theta}{\cos^2 \alpha} \right].
\]  

(3.4)

The time derivative of the angular position \( \theta \) is commonly denoted as \( \omega \). That is
\[
\dot{\theta} = \omega.
\]

(3.5)

Since the input shaft speed \( \omega \) is assumed constant
\[
\ddot{\theta} = \dot{\omega} = 0.
\]

(3.6)

Consequently, Equations (3.3), and (3.4) reduce to
\[
\dot{y}_p = \left( \frac{R \sin \theta - \alpha \sin \alpha}{\cos^2 \alpha} \right) \dot{\alpha} + R \omega \cos \theta \tan \alpha,
\]

(3.7)

and
\[
\ddot{y}_p = -\omega^2 R \sin \theta \tan \alpha + \frac{2\omega R \cos \theta \dot{\alpha}}{\cos^2 \alpha}
+ \left[ a \left( \cos^2 \alpha - 2 \sin^2 \alpha \right) + 2R \sin \alpha \sin \theta \right] \dot{\alpha}^2
+ \left[ \frac{R \sin \theta - \alpha \sin \alpha}{\cos^2 \alpha} \right] \ddot{\alpha}.
\]

(3.8)

Equations (3.2), (3.7), and (3.8) relate the position, velocity, and acceleration of the piston to various geometric angles and rotational speeds. These equations are used in the calculations of the net torque on the swashplate.
3.3. Piston Forces

Consider Figure 3-3, a free body diagram of a single piston.

![Free Body Diagram](image)

Figure 3-3: Piston Free Body Diagram

Applying Newton’s Second Law in the y direction gives

\[
\sum F_y = F_{ap} \cos \alpha - PA_p + F_{ps} - F_r \cos \alpha = m\ddot{y}_p, \tag{3.9}
\]

where \(F_{ap}\) is the force applied to the piston by the swashplate, \(P\) is the cylinder pressure, \(A_p\) is the effective piston area\(^\dagger\), \(F_{ps}\) is the shear force caused by viscous damping between the cylinder and the piston, and \(F_r\) is the force applied by the shoe plate. It has been assumed that stiction will not be a significant force because of the constant reciprocating motion of the piston, and the inherent vibration inside the pump. Rearranging this equation gives

\[
F_{ap} = \frac{m\ddot{y}_p + PA_p - F_{ps}}{\cos \alpha} + F_r. \tag{3.10}
\]

\(^\dagger\)The effective piston area is equal to the piston area plus one half the annular area between the piston and the cylinder. See page 116 in Appendix D for an explanation.
3.4. Torque Equations

From a control point of view what is important is not the force applied to each piston but the total torque that these piston forces apply to the swashplate. Figure 3-4 shows the applied force, $F_{ap}$, for one piston acting on the swashplate.

![Torque Arm Diagram](image)

**Figure 3-4:** Torque Relationships

Considering Figure 3-4 the torque applied by the $i^{th}$ piston to the swashplate is

$$T_{ap_i} = -F_{ap} \times \text{Torque Arm}_i$$  \hspace{1cm} (3.11)

where $T_{ap_i}$ is the torque produced by the $i^{th}$ piston. Note that a positive torque increases the swashplate angle. From geometry, the torque arm for the $i^{th}$ piston is described by

$$\text{Torque Arm}_i = \frac{R\sin\theta_i - asin\alpha}{\cos\alpha}$$  \hspace{1cm} (3.12)

where $\theta_i$ is the angle of the $i^{th}$ piston.
Substituting Equations (3.10) and (3.12) into Equation (3.11) gives the following expression:

$$T_{ap_i} = -\left(\frac{m\dot{y}_p + PA_p - F_p}{\cos \alpha} + F_r\right)\left(\frac{R\sin \theta_i - asin \alpha}{\cos \alpha}\right)$$  \hspace{1cm} (3.13)

The previous equation shows the torque applied to the swashplate by the $i^{th}$ piston. Since there are $N$ pistons the total torque $T_{ap}$ is simply the sum.

$$T_{ap}(\theta) = \sum_{i=1}^{N} T_{ap_i}(\theta_i)$$  \hspace{1cm} (3.14)

Equation (3.14) shows that the torque applied to the swashplate depends on the barrel angle $\theta$. This presents a problem because the barrel angle at any given time is not known and as a result, it is essentially a random variable. Therefore, an explicit mathematical expression to describe the swashplate torque at a given time cannot be determined. The effect of the barrel angle (or barrel rotation) is that it causes a high frequency fluctuation of the applied torque.

Equation (3.14) also shows that $T_{ap}(\theta)$ is the sum of $N$ functions. Each of these functions has a period of $2\pi$ radians and each function is separated from each other by a phase shift of $\frac{2\pi}{N}$ radians. This will cause $T_{ap}(\theta)$ to be a function that has a period of $\frac{2\pi}{N}$ radians.

Consider the case where the input shaft speed is 1750 rpm, and there are 9 pistons. The angular velocity $\omega$ is

$$\omega = 1750 \times \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{\text{rev}} \times \frac{\text{min}}{60 \text{ sec}} = 183.3 \frac{\text{rad}}{\text{sec}}$$ \hspace{1cm} (3.15)

Since $T_{ap}(\theta)$ is cyclic over every $\frac{2\pi}{N}$ radians, the period of $T_{ap}(\theta)$ is

$$\text{Period} = \frac{\frac{2\pi}{N} \text{ rad}}{1 \text{ cycle}} \times \frac{1 \text{ sec}}{183.3 \text{ rad}} = 3.8 \text{ msec/cycle}$$ \hspace{1cm} (3.16)
Therefore, the fundamental high frequency fluctuation (at 1750 rpm) of $T_{ap}(\theta)$ is

$$f_{\text{min}} = \frac{1}{\text{Period}} = \frac{1}{3.7 \text{ msec}/\text{cycle}} = 263 \text{ Hz} \quad \text{(3.17)}$$

In Appendix B, the approximate undamped natural frequency, for this pump, is calculated to be 17 Hz, and the damping ratio is shown to be 2.0. This high frequency fluctuation of 263 Hz is much larger than this undamped natural frequency, and since this pump is overdamped, it can be assumed that the effects of this high frequency fluctuation will be adequately filtered out. As a result, the effects of the random variable $\theta$ on the applied torque are not important and can be neglected.

The effects of the barrel angle are eliminated from subsequent equations by finding the average value of the applied torque for all possible barrel angles. This average value of applied torque for a single piston is

$$\bar{T}_{ap_i} = \frac{\int_0^{2\pi} T_{ap_i}(\theta) d\theta}{\frac{2\pi}{\int_0^{2\pi} d\theta}} = \frac{1}{2\pi} \int_0^{2\pi} T_{ap_i}(\theta) d\theta \quad \text{(3.18)}$$

Note that this average is taken with respect to the angle $\theta$ and not time. This is because it is the average value of all possible values of applied torque at a given instant in time that is required, not the average applied torque over an interval of time. This means that only those values that change directly with barrel rotation should be kept inside the integral. All other variables (those that vary with time but do not depend on the barrel angle, such as the swashplate angle) must be treated as constants, when evaluating the integral.
The average total applied torque is again the sum of the average torques for each piston.

\[
\overline{T_{ap}} = \sum_{i=1}^{N} \overline{T_{ap_i}} = N \overline{T_{ap_i}} = \frac{N}{2\pi} \int_{0}^{2\pi} T_{ap_i}(\theta) d\theta
\]  \hspace{1cm} (3.19)

Substituting Equation (3.13), which defines \( T_{ap_i}(\theta) \), into Equation (3.19) gives

\[
\overline{T_{ap}} = \frac{N}{2\pi} \int_{0}^{2\pi} \left( \frac{m j_p + PA_p - F_{ps}}{\cos \alpha} + F_r \right) \left( \frac{R \sin \theta - a \sin \alpha}{\cos \alpha} \right) d\theta
\]  \hspace{1cm} (3.20)

The first term inside the integral in Equation (3.20) is made up of four components. These components are: a piston inertia effect, a pressure force effect, a piston shear effect, and a shoe plate force effect. Defining \( \overline{T_{pm}} \) to be the effect of piston mass, \( \overline{T_{ps}} \) to be the average torque due to piston shear, \( \overline{T_r} \) to be the effect of the shoe plate, and \( \overline{T_{pr}} \) to be the average torque due to pressure gives

\[
\overline{T_{ap}} = \overline{T_{pm}} + \overline{T_{ps}} + \overline{T_r} + \overline{T_{pr}}
\]  \hspace{1cm} (3.21)

where,

\[
\overline{T_{pm}} = \frac{N}{2\pi} \int_{0}^{2\pi} \frac{-m j_p (R \sin \theta - a \sin \alpha)}{\cos^2 \alpha} d\theta
\]  \hspace{1cm} (3.22)

\[
\overline{T_{ps}} = \frac{N}{2\pi} \int_{0}^{2\pi} \frac{F_{ps} (R \sin \theta - a \sin \alpha)}{\cos^2 \alpha} d\theta
\]  \hspace{1cm} (3.23)
\[
\overline{T_r} = \frac{N}{2\pi} \int_0^{2\pi} -F_r(R\sin\theta - a\sin\alpha) \frac{d\theta}{\cos\alpha}, \tag{3.24}
\]

and

\[
\overline{T_{pr}} = \frac{N}{2\pi} \int_0^{2\pi} -PA_p(R\sin\theta - a\sin\alpha) \frac{d\theta}{\cos^2\alpha}. \tag{3.25}
\]

These four separate terms will be now considered individually. (The torque due to pressure will be completed in Chapter 4).

### 3.5. Piston Inertia Effects

Equation (3.22), rewritten here for clarity, shows the increase in the net torque that is applied to the swashplate when the effects of piston inertia are included.

\[
\overline{T_{pm}} = \frac{N}{2\pi} \int_0^{2\pi} -m\dot{\alpha}\sin\theta \frac{d\theta}{\cos^2\alpha}. \tag{3.26}
\]

In Appendix C, Equation (3.8), (representing \(\dot{\alpha}_p\)) is substituted into Equation (3.26) and the result is integrated. The average increase in net torque applied to the swashplate, as the result of piston inertia effects, is found to be

\[
\overline{T_{pm}} = \frac{mNR^2\omega^2\sin\alpha}{2\cos^3\alpha} - mN\left[\frac{R^2\sin\alpha + a^2\sin\alpha\cos^2\alpha - 2a^2\sin^3\alpha}{\cos^5\alpha}\right]\ddot{\alpha}^2 \\
- mN\left[\frac{R^2 + 2a^2\sin^2\alpha}{2\cos^4\alpha}\right]\dot{\alpha} \tag{3.27}
\]
Equation (3.27) above shows that the effect of piston mass depends on the swashplate angle, angular velocity, and angular acceleration as well as numerous physical constants. It also shows that the effect of piston inertia is not equal to zero even when the swashplate angle is constant.

3.6. Torque Due to Piston Shear

The torque due to piston shear is represented by Equation (3.23) and is rewritten here.

\[
\overline{T_{ps}} = \frac{N}{2\pi} \int_{0}^{2\pi} \frac{F_{ps}(R\sin \theta - \alpha \sin \alpha)}{\cos^2 \alpha} \, d\theta
\]  

\[\text{(3.28)}\]

In Appendix D, it is shown that for laminar flow, the viscous force on one piston is equal to

\[
F_{ps} = -\frac{\pi r_p \mu \dot{y}_p (l_{p_o} + y_p)}{h_p},
\]

\[\text{(3.29)}\]

where \(l_{p_o}\) is the length of the piston that is inserted in the cylinder when the axial displacement is zero, \(r_p\) is the piston radius, and \(h_p\) is one half the radial clearance between the piston and the cylinder. After substituting Equations (3.2) (representing \(y_p\)), (3.7) (representing \(\dot{y}_p\)), and (3.29) into Equation (3.28), the integral can be evaluated. This integration is performed in Appendix C and yields

\[
\overline{T_{ps}} = -\frac{\pi N r_p \mu \left[ l_{p_o} R^2 \cos \alpha + l_{p_o} a^2 \cos \alpha \sin^2 \alpha - 2a^2 R \sin^2 \alpha \right.}{h_p \cos^5 \alpha} \left. - \frac{1}{2} a R^2 \sin \alpha - \frac{1}{2} a^3 \sin^2 \alpha \right] \dot{\alpha}.
\]

\[\text{(3.30)}\]
The above equation shows that the torque due to piston shear depends on the swashplate angle and angular velocity. For a constant swashplate angle, Equation (3.30) shows that the average torque due to piston shear is zero. This indicates that the shear forces that act on the pistons are balanced about the yoke pivot when the swashplate angular velocity is zero.

3.7. Shoe Plate Effects

The shoe plate rotates with the barrel at an angle parallel to the swashplate. A spring inside the pump barrel acts on the shoe plate to hold the slippers against the swashplate. The effect of this spring is to increase the contact force for all the pistons, and as illustrated in Appendix C, this spring force does affect the torque on the swashplate. Only the inertia of the shoe plate affects the torque applied to the swashplate.

A theoretical analysis to determine the shoe plate effect on the net torque that is applied to the swashplate is performed in Appendix C. This effect is derived to be

\[
\overline{T_r} = -I_r \ddot{\alpha}.
\]  

(3.31)

where \( I_r \) is the mass moment of inertia about the yoke pivot for the ring. The effect of the shoe plate inertia varies with the swashplate angle, angular velocity, and angular acceleration. As with piston inertia, the effect of shoe plate inertia is not zero when the swashplate angle is constant.
3.8. Summary

From the analysis presented certain observations can be noted. They are summarized as follows.

1. The axial piston pump is a complex mechanical system. The equations that describe the various torques that act on the swashplate yoke assembly are highly nonlinear.

2. It was found that the inertia of the pistons affects the torque applied to the swashplate, even for a constant swashplate angle.

3. It was found that the torque due piston shear depends on the swashplate angular velocity. For zero angular velocity, the torque due to piston shear is zero.

4. It was found that the inertia of the shoe plate affects the torque applied to the swashplate, even for a constant swashplate angle.

5. To complete a dynamic analysis of the swashplate torque, the analysis of the torque due to the pressure forces on the pistons must be completed.
Chapter 4
Pressure Effects

The average torque that the pressure forces apply to the swashplate was shown by Equation (3.25), and is rewritten here for convenience.

\[
\bar{T}_{pr} = \frac{N}{2\pi} \int_{0}^{2\pi} \frac{-PA_p(R\sin \theta - \sin \alpha)}{\cos^2 \alpha} \, d\theta
\]  

(4.1)

To develop a simpler expression that describes this average torque, this integral must be evaluated. To evaluate this integral, the cylinder pressure distribution as a function of \( \theta \) must be known. This chapter will show how this pressure distribution was found, and how it was used to determine the average torque due to pressure forces.

As the barrel of the pump rotates, the pressure in the cylinder chamber changes. It was not feasible to experimentally measure the cylinder pressure during a complete revolution and so the pressure distribution was found by computer simulations. The following method was used.

1. A control volume analysis of the cylinder chamber was performed. This analysis produced differential equations that described the dynamics of the cylinder pressure for a complete revolution.

2. A set of operating conditions consisting of the swashplate angle \( \alpha \), the swashplate angular velocity \( \hat{\alpha} \), and the pump differential pressure \( \Delta P_p \) was chosen.

3. The differential equations describing the cylinder pressure were then numerically solved to produce a pressure distribution for a complete revolution, at the specified operating conditions.
4. This pressure distribution was then used to integrate numerically Equation (4.1). This gave the average torque due to pressure at the chosen operating conditions.

5. Steps 2 to 4 were then repeated for various operating conditions.

6. A simple empirical relationship was then found to relate the average torque due to pressure to the operating conditions.

To perform these numerical calculations, actual dimensions and operating conditions for an axial piston pump were required. The analysis was performed using data from a Vickers PVBS axial piston pump. Therefore, the results of this chapter are applicable only to that type of pump. However, the method outlined in this chapter could be repeated for any pump, possibly with some modifications to meet a particular pump's geometry.

4.1. Pressure Regions

There are four main regions of pressure along the path of a piston as the barrel makes one complete revolution. These regions are

- Load pressure along the discharge port of the valve plate,
- Suction pressure along the suction port of the valve plate,
- A transition region as the pressure falls from load pressure to suction pressure,
- A transition region as the pressure rises from the suction pressure to load pressure.

It is these transition regions that cause the resultant torque, due to pressure forces, on the swashplate to be nonzero. In addition, the dynamic behavior of the pressure in the transition regions makes the solution of the pressure effects a nontrivial problem.

These transition regions are caused by the configuration of the valve
plate. To reduce step changes in pressure, and subsequent shock loads and noise problems, the valve plate configuration of an axial piston pump is specially designed to produce these transition regions. Figure 4-1 shows the valve plate for a Vickers PVB5 pump with the relief notches that allow gradual pressure changes in the cylinders. These relief notches are V-shaped grooves, machined in the valve plate, which start at zero depth and reach a maximum depth at the start of the suction and discharge ports.

![Figure 4-1: Valve Plate](image)

Figure 4-1 shows the physical location of the ports and the relief notches on the valve plate. However, the point of application of the pressure induced force is the center of the piston. Therefore, the angular position of the center of the piston when the various regions start and end, is required.
The cylinder port, as shown in Figure 4-2, is a kidney shaped port that has an angle of 30°. This means that, for this pump, the center of the piston is 15° behind the leading edge of the cylinder port and 15° ahead of the trailing edge. When the barrel is rotating the effect of the relief notches and the effect of the ports starts when the leading edge of the cylinder port reaches these physical locations. Also, the end of the suction and discharge ports is realized when the trailing edge of the cylinder port reaches these physical locations. Note that the relief notches effectively end at the start of the valve plate ports, because once the leading edge of the cylinder port reaches the start of the valve plate port, the notches no longer have any effect. Figure 4-3 summarizes graphically the locations of the center of the pistons at the start and end of the various pressure regions.

Figure 4-3 shows that the start of the relief notches occurs before the
Figure 4-3: Valve Plate Geometry with Respect to Piston Centers

These figures graphically define the angular positions, $\theta_1$ to $\theta_6$, used to delimit the pressure regions. These angles represent the position of the piston centers at the start and end of the ports and relief notches.

end of the valve plate ports. This overlap is because the cylinder port is long enough to bridge the gap between the end of valve plate ports and the start of the relief notches. This produces six distinct pressure regions for
Figure 4-4: Pressure Regions as Seen by Piston Centers

This figure uses the angles $\theta_1$ to $\theta_6$, as defined in Figure 4-3, to define the boundaries of the pressure regions.

each revolution of the piston barrel, and they are defined in Figure 4-4. These regions are

Region 1. The cylinder is completely open to the discharge port and so the cylinder pressure $P$ is equal to the discharge pressure $P_D$.

Region 2. In this region the cylinder is open to both the discharge port and the relief notch before the suction port.

Region 3. In this region the cylinder is open only to the relief notch before the suction port.

Region 4. The cylinder is completely open to the suction port and so the cylinder pressure $P$ is equal to the suction pressure $P_S$. 
Region 5. In this region the cylinder is open to both the suction port and the relief notch before the discharge port.

Region 6. In this region the cylinder is open only to the relief notch before the discharge port.

4.2. Governing Equations

To simulate the pressure in the transient regions, a control volume consisting of the cylinder chamber is analyzed. The analysis is based on an instantaneous flow and volumetric rate of change balance for the control volume. Figure 4-5 shows the control volume in Region 2.

![Diagram of Cylinder Chamber Control Volume in Region 2]

**Figure 4-5:** Cylinder Chamber Control Volume in Region 2

The fundamental governing equation for any control volume is the continuity equation,
\[
\frac{dM}{dt} = \rho \frac{dV}{dt} + V \frac{d\rho}{dt}
\] (4.2)

where \(M\) is the fluid mass, \(V\) is the fluid volume, and \(\rho\) is the fluid density. The rate of change of mass is equal to the density times the net flow into the control volume or

\[
\frac{dM}{dt} = \rho (Q_{in} - Q_{out})
\] (4.3)

and from the definition of the fluid bulk modulus \(\beta\)

\[
\frac{d\rho}{dt} = \frac{\rho}{\beta} \frac{dP}{dt}
\] (4.4)

Substituting Equations (4.3) and (4.4) into Equation (4.2) gives

\[
\rho (Q_{in} - Q_{out}) = \rho \frac{dV}{dt} + V \frac{\rho}{\beta} \frac{dP}{dt}
\] (4.5)

where \(V_p\) is the volume of the cylinder chamber. Equation (4.5) is divided by the fluid density and rearranged to yield

\[
\frac{dP}{dt} = \frac{\beta}{V_p} \left( -\frac{dV_p}{dt} + Q_{in} - Q_{out} \right)
\] (4.6)

The volume \(V_p\) is dependent on the axial displacement of the piston \(y_p\). Defining \(V_{p_o}\) to be equal to the cylinder chamber volume when the axial displacement \(y_p\) equals zero, then

\[
V_p = V_{p_o} - A_p \cdot y_p
\] (4.7)

where \(A_p\) is the actual piston area and \(y_p\) is described by Equation (3.2). Taking the derivative of Equation (4.7) gives

\[
\frac{dV_p}{dt} = \dot{V}_p = -A_p \dot{y}_p
\] (4.8)

where \(\dot{y}_p\) is described by Equation (3.7).
From Equation (3.5),

\[
\frac{d\theta}{dt} = \omega \quad ,
\]  

(4.9)

or

\[
dt = \frac{d\theta}{\omega} \quad .
\]  

(4.10)

Substituting Equations (3.2), (3.7), (4.7), (4.8), and (4.10) into Equation (4.6) gives

\[
\frac{dP}{d\theta} = \frac{\beta}{\omega \left[ V_{p_o} - A_p \left( \frac{R \sin \theta \sin \alpha - a}{\cos \alpha} \right) \right]} \left[ A_p' \left\{ \left( \frac{R \sin \theta - a \sin \alpha}{\cos^2 \alpha} \right) \dot{\alpha} + \omega R \cos \theta \tan \alpha \right\} + Q_{in} - Q_{out} \right] .
\]  

(4.11)

To numerically integrate Equation (4.11), mathematical expressions describing $Q_{in}$ and $Q_{out}$ are required.

4.3. Flowrates

There are three flowrates to consider in the transition regions: flow through the relief notch, flow through an overlap area between the end of a valve port and the trailing edge of the cylinder port, and leakage flow out of the cylinder chamber.

The flows through the overlap area and the notch are defined by orifice flow equations. The general flow equation for an orifice is
\[ q = AC_d \sqrt{\frac{2}{\rho} \Delta P} \]  

(4.12)

where \( A \) and \( C_d \) represent the area and coefficient of discharge of the orifice, and \( \Delta P \) is the pressure drop across the orifice. Thus, the flow through the overlap area is

\[ q_o = A_o C_d o \sqrt{\frac{2}{\rho} \Delta P_o} \]  

(4.13)

and the flow through the notch is

\[ q_n = A_n C_d n \sqrt{\frac{2}{\rho} \Delta P_n} \]  

(4.14)

Note that both \( A_o \) and \( A_n \) represent variable areas that depend on the angular displacement \( \theta \). The discharge coefficient for the overlap area was found from McCloy [18], using an average Reynolds Number. The notch discharge coefficient was found from Zeiger [15].

The variable areas \( A_o \) and \( A_n \) can be found from geometry. Consider Figure 4-6 which shows the variable overlap area \( A_o \). Note that for clarity the relief notch is not shown. The area \( A_o \) will be defined by the intersection of two circles. The geometrical relation that describes this area is complicated, and so, an empirical relationship to define this area was developed. This empirical relationship is

\[ A_o = K_o (\theta_o - \theta)^{n_o} \]  

(4.15)

where \( \theta_o \) denotes the angle corresponding to the end of the valve plate ports. The maximum error of this expression is less than 1.5% of the exact overlap area. From Figure 4-4, \( \theta_o \) will equal \( \theta_2 \) for the end of the discharge port, and it will equal \( \theta_5 \) for the end of the suction port.
The variable area $A_n$ is a triangular notch which increases with the angle $\theta$. This notch area is illustrated in detail in Figure 4-7. From geometry

$$d = R(\theta - \theta_n)\tan \gamma$$  \hspace{1cm} \text{(4.16)}

where $\theta_n$ denotes the angle corresponding to the start of the relief notches. From Figure 4-4, $\theta_n$ will equal $\theta_1$ for the notch before the suction port and it will equal $\theta_4$ for the notch before the discharge port.

With reference to Figure 4-7 the notch area is

$$A_n = d^2$$ \hspace{1cm} \text{(4.17)}

or
Figure 4-7: Notch Area

\[ A_n = R^2(\theta - \theta_n)^2 \tan^2 \gamma \quad . \] (4.18)

With the areas \( A_o \) and \( A_n \) defined, the generalized expressions for the overlap and notch flows become

\[ q_o = K_o C_{d_o} (\theta_o - \theta_o)^n_o \sqrt{\frac{2}{\rho} \Delta P_o} \quad , \] (4.19)

and

\[ q_n = R^2 C_{d_n} \tan^2 \gamma (\theta - \theta_n)^2 \sqrt{\frac{2}{\rho} \Delta P_n} \quad . \] (4.20)

The remaining flowrate that is required is the leakage flowrate \( q_l \). The leakage flowrate is the sum of the flow between the cylinders and the pistons, the flow out through the slippers, and the leakage between the cylinder barrel and the valve plate. All these flows leak into the pump case and are returned to the reservoir by the case drain. To determine this
leakage flow an experimental test was performed. Appendix E describes this test in detail plus the experimental results.

Appendix E also shows how the total leakage flow $Q_L$ can be approximated by

$$ Q_L = K_L P $$ \hspace{1cm} (4.21)

where $P$ is the cylinder pressure and $K_L$ is the total leakage constant. If it is assumed that leakage only occurs when the cylinders are exposed to the high pressure discharge port, then, on average, this total leakage will occur only from half of the cylinders. Thus the leakage flow per cylinder $q_l$ is

$$ q_l = \frac{Q_L}{N/2} $$ \hspace{1cm} (4.22)

or

$$ q_l = \frac{2K_L P}{N} $$ \hspace{1cm} (4.23)

Defining $K_l$ to be

$$ K_l = \frac{2K_L}{N} $$ \hspace{1cm} (4.24)

then Equation (4.23) becomes

$$ q_l = K_l P $$ \hspace{1cm} (4.25)

4.4. Pressure Equations

The previous section has developed expressions for the flowrates into the control volume of the piston chamber. The equations that describe the dynamic behavior of the cylinder chamber pressure can now be ascertained.
In Region 1, (refer to Figure 4-4) the cylinder is completely open to the discharge port. Thus for \( \theta_0 \leq \theta < \theta_1 \),

\[
\frac{dP}{d\theta} = 0
\]  

(4.26)

and

\[ P = P_D \]  

(4.27)

The control volume for Region 2 is shown in Figure 4-5. For this region, there is flow in through the overlap area.

\[ Q_{in} = q_o \]  

(4.28)

This overlap flow is described by Equation (4.19) where \( \theta_o \) is equal to \( \theta_2 \) and \( \Delta P_o \) is equal to the pressure difference between the discharge pressure \( P_D \) and the cylinder pressure \( P \). Substituting these into Equation (4.19) gives

\[ Q_{in} = K_o C_{d_o} (\theta_2 - \theta)^n_o \sqrt{\frac{2}{\rho}(P_D - P)} \]  

(4.29)

Also in Region 2, there is flow out through the notch and through leakage.

\[ Q_{out} = q_n + q_l \]  

(4.30)

The notch flow is described by Equation (4.20) with \( \theta_n \) equal to \( \theta_1 \) and the pressure drop \( \Delta P_n \) equal to the pressure difference between the cylinder pressure \( P \) and the suction pressure \( P_S \). Substituting these into Equation (4.20) yields

\[ q_n = R^2 C_{d_n} \tan^{2} \gamma (\theta - \theta_1)^2 \sqrt{\frac{2}{\rho}(P - P_S)} \]  

(4.31)

The leakage flow is described by Equation (4.25). Substituting Equations (4.31) and (4.25) into Equation (4.30) gives the flow out of the control volume for Region 2.
\[ Q_{out} = R^2 C_n \tan^2 \gamma (\theta - \theta_1)^2 \left( \frac{2}{\rho} (P - P_S) \right) + K_t P \]  

(4.32)

Therefore, by substituting Equations (4.30) and (4.32) into Equation (4.11) an expression that describes the dynamic behavior of the pressure in Region 2 results. Thus for \( \theta_1 \leq \theta < \theta_2 \),

\[
\frac{dP}{d\theta} = \frac{\beta}{\omega} \left[ A_p \left\{ \left( \frac{\sin \theta - \sin \alpha}{\cos \alpha} \right) \right\} \dot{\alpha} + \omega R \cos \theta \tan \alpha \right] \\
+ K_o C_d (\theta - \theta)^n \left( \sqrt{\frac{2}{\rho} (P_D - P)} - R^2 C_n \tan^2 \gamma (\theta - \theta_1)^2 \sqrt{\frac{2}{\rho} (P - P_S)} \right) \\
- K_l P \right] .
\]

(4.33)

In Region 3, the cylinder port is open only to the relief notch before the suction port. Therefore, there is no flow into the control volume. Hence,

\[ Q_{in} = 0 \]  

(4.34)

There is flow out of the cylinder chamber through the relief notch and through leakage. Therefore,

\[ Q_{out} = q_n + q_l \]  

(4.35)

The flow through the notch is again described by Equation (4.20) with \( \theta_n \) equal to \( \theta_1 \) and \( \Delta P_n \) equal to the difference between the cylinder pressure \( P \) and the suction pressure \( P_S \). This is the same as Region 2. Therefore, in Region 3
\[ Q_{out} = R^2 C_d \tan^2 \gamma (\theta - \theta_1)^2 \sqrt{\frac{2}{\rho} (P - P_s)} + K_l P \]  \hspace{1cm} (4.36)

Substituting Equations (4.34) and (4.36) into Equation (4.11) yields the pressure equation for Region 3. Therefore, for \( \theta_2 \leq \theta < \theta_3 \), Equation (4.11) becomes

\[
\frac{dP}{d\theta} = \frac{\beta}{\omega} \left[ V_{p_o} - A_p \left( \frac{R \sin \theta - R \sin \alpha}{\cos \alpha} \right) \right]
\left\{ A_p \left( \frac{R \sin \theta - R \sin \alpha}{\cos \alpha} \right) \right\} \dot{\alpha} + \omega R \cos \theta \tan \alpha

- R^2 C_d \tan^2 \gamma (\theta - \theta_1)^2 \sqrt{\frac{2}{\rho} (P - P_s)} - K_l P \right] . \hspace{1cm} (4.37)

In Region 4, the cylinder is completely open to the suction port. Thus for \( \theta_3 \leq \theta < \theta_4 \),

\[
\frac{dP}{d\theta} = 0 \hspace{1cm} (4.38)
\]

and

\[ P = P_s \] \hspace{1cm} (4.39)

In Region 5, the cylinder is open to both the relief notch before the discharge port and to the suction port. This will cause fluid to flow into the cylinder chamber from the discharge port and to flow out through the overlap area and through leakage. Thus

\[ Q_{in} = q_n \] \hspace{1cm} (4.40)

where \( q_n \) is defined by Equation (4.20) with \( \theta_n \) equal to \( \theta_4 \) and \( \Delta P_n \) is equal to the pressure difference between the discharge pressure and the cylinder pressure. With these substitutions, the notch flow becomes
\[ q_n = R^2 C_d \tan^2 \gamma (\theta - \theta_4)^2 \sqrt{\frac{2}{\rho} (P_D - P)} \] (4.41)

or

\[ Q_{in} = R^2 C_d \tan^2 \gamma (\theta - \theta_4)^2 \sqrt{\frac{2}{\rho} (P_D - P)} \] . (4.42)

The flow out is

\[ Q_{out} = q_o + q_l \] (4.43)

where \( q_o \) is defined by Equation (4.19) with \( \theta_o \) equal to \( \theta_5 \) and \( \Delta P_o \) equal to the pressure difference between the cylinder pressure and the suction pressure. Making these substitutions produces

\[ q_o = K_o C_d (\theta_5 - \theta)^n_0 \sqrt{\frac{2}{\rho} (P - P_5)} \] . (4.44)

The leakage is described by Equation (4.25). Substituting Equations (4.25) and (4.44) into (4.43) gives

\[ Q_{out} = K_o C_d (\theta_5 - \theta)^n_0 \sqrt{\frac{2}{\rho} (P - P_5)} + K_l P \] . (4.45)

Substituting Equations (4.42) and (4.45) into Equation (4.11) gives the pressure equation for Region 5. Thus for \( \theta_4 \leq \theta < \theta_5 \),
\[
\frac{dP}{d\theta} = \frac{\beta}{\omega \left[ V_{p_o} - A_p \left( \frac{R \sin \theta - a \sin \alpha}{\cos \alpha} \right) \right]} \left[ A_p \left\{ \left( \frac{R \sin \theta - a \sin \alpha}{\cos^2 \alpha} \right) \dot{\alpha} + \omega R \cos \theta \tan \alpha \right\} + R^2 C_d \tan^2 \gamma (\theta - \theta_4)^2 \sqrt{\frac{2}{\rho (P_D - P)}} - K_o C_d (\theta_5 - \theta)^2 \sqrt{\frac{2}{\rho (P - P_S)}} - K_t P \right].
\] (4.46)

In Region 6, the cylinder is only open to the notch before the discharge port. Therefore the flow into the control volume is equal to the notch flow and the flow out is equal to the leakage flow. That is

\[
Q_{in} = q_n
\] (4.47)

where \( q_n \) is described by Equation (4.20) with \( \theta_n \) equal to the start of the relief notch or \( \theta_4 \) and \( \Delta P_n \) equal to the pressure difference between the discharge pressure and the cylinder pressure. Equation (4.20) becomes

\[
q_n = R^2 C_d \tan^2 \gamma (\theta - \theta_4)^2 \sqrt{\frac{2}{\rho (P_D - P)}}
\] (4.48)

or

\[
Q_{in} = R^2 C_d \tan^2 \gamma (\theta - \theta_4)^2 \sqrt{\frac{2}{\rho (P_D - P)}} \quad .
\] (4.49)

As stated above the flow out will equal the leakage flow, as described by Equation (4.25). Thus

\[
Q_{out} = q_l
\] (4.50)

or
\[ Q_{out} = K_l P \quad . \] (4.51)

Substituting Equations (4.49) and (4.51) into Equation (4.11) gives the pressure equation for Region 6. Thus for \( \theta_5 \leq \theta < \theta_6 \),

\[
\frac{dP}{d\theta} = \frac{\beta}{\omega \left[ V_{p_o} - A_P \left( \frac{R \sin \theta - a \sin \alpha}{\cos \alpha} \right) \right]} \left[ A_P \left\{ \frac{R \sin \theta - a \sin \alpha}{\cos \alpha} \right\} \right. \\
+ \omega R \cos \theta \tan \alpha \right] \\
- R^2 C_d \tan^2 \gamma (\theta - \theta_4)^2 \sqrt{\frac{2}{\rho (P_D - P)}} - K_l P \quad . \] (4.52)

The differential equations that describe the dynamics of the cylinder pressure for one revolution of the barrel have now been determined. These equations are highly nonlinear and must be solved using numerical techniques.

4.5. Simulation Results

Equations (4.27), (4.33), (4.37), (4.39), (4.46), and (4.52) describe the dynamics of the cylinder pressure for a complete revolution. These expressions are differential equations that are nonlinear and cannot be solved explicitly. Therefore, to find the cylinder pressure distribution, these equations must be solved numerically. The numerical solution, or simulation, of these equations gives a pressure distribution that can then be used to find the average torque due to pressure, by substituting this distribution into the integral expression (4.1). This equation also cannot be solved explicitly and so again a numerical solution is required.

The aim of this research was to produce an mathematical model which accurately describes the dynamics of an axial piston pump. This model could then be used to simulate, on a computer, the behavior of a pump control
system. During a simulation it would be slow and cumbersome to have to solve Equations (4.27), (4.33), (4.37), (4.39), (4.46), (4.52), and (4.1) numerically at every instant to determine the effects of pressure.

To solve this problem, a simple expression, instead of seven nonlinear differential equations, was required that would relate the torque (due to pressure) to the operating conditions of the pump. The operating conditions consist of the pump differential pressure, the swashplate angle, and the swashplate angular velocity. A simple expression was found by numerically solving the equations for numerous operating conditions to find the average torque for these conditions. Using statistical analysis of the simulation results, a simple empirical relationship was determined.

The simulation results are shown in Figure 4-8 and Figure 4-9. In these figures, the solid lines represent the results of the computer simulation while the dashed lines represent the empirical expression that was found to fit the data. Figure 4-8 plots the average torque due to pressure as a function of the pump differential pressure $\Delta P_p$, and the swashplate angle $\alpha$. This graph is for zero swashplate angular velocity ($\dot{\alpha}=0$). Figure 4-9 plots the average torque due to pressure for various pump differential pressures and swashplate angular velocities. In this graph the swashplate angle is constant.

Figure 4-8 suggests that the pressure torque increases linearly with the pump differential pressure. It also suggests that the torque varies linearly with the swashplate angle, but that the slope of this linear relationship increases with the differential pressure. Figure 4-9 indicates that the torque due to pressure decreases almost linearly with the swashplate angular velocity. Therefore, Figures 4-8 and 4-9 indicate that the torque due to pressure can be adequately described by an empirical expression of the form

$$\bar{T}_{pr} = K_{pr_1} + K_{pr_2} \Delta P_p + K_{pr_3} \Delta P_p \alpha + K_{pr_4} \dot{\alpha} \quad . \quad (4.53)$$
Figure 4-8: Average Torque due to Pressure vs. Swashplate Angle

For $\alpha = 0$
Figure 4-9: Average Torque due to Pressure vs. Swashplate Angular Velocity

For $\alpha = 0.33$ rad.
The "best" constants for this empirical expression were found by analyzing the simulation results with multiple linear regression techniques. Using these "best" constants, Equation (4.53) is also plotted in Figures 4-8 and 4-9 (dashed lines).

For a complete description of the computer simulation, refer to Appendix F. Also, the numeric values of all of the constants used in this analysis can be found in Appendix G.

4.6. Summary

The torque applied to the swashplate due to pressure was found to be significant. This indicates that the pressure forces that act on the pistons are not symmetrical, or balanced, about the pintle axis. It was found that the pressure torque could be represented as a function of the pump differential pressure, the swashplate angle, and the swashplate angular velocity.

This torque was found using the valve plate geometry of a Vickers PVBS axial piston pump. The valve plate for this pump was symmetrical about the pintle axis and yet the pressure forces were not. This indicates that the torque due to pressure that is applied to the swashplate cannot be assumed to be zero for any axial piston pump, even if the valve plate is symmetric.
Chapter 5
Torques Applied to the Yoke

To complete a dynamic analysis of the swashplate yoke assembly, the torques that act directly on the yoke must be considered. These torques are caused by:

- The return spring force
- The damping on the yoke (both viscous damping and stiction)
- The control force applied by the control piston.

This chapter will develop mathematical expressions to describe each of these torques.

5.1. Torque Due to the Return Spring

The return spring is a spring mounted inside the pump that acts to stroke the pump. The object of this section is to develop an expression that describes the return spring torque as a function of the swashplate angle. One should ideally measure the torque versus angle relationship directly for a particular pump, and use this experimental relationship in the pump model. However, this is an inconvenient test because:

- It is difficult to apply a specific torque to the yoke that will be opposed only by the return spring.
- For most pumps, the angle of the swashplate can not be readily measured.

An alternative to using this experimental approach is to determine the
spring torque theoretically. This is done by relating the linear motion of the spring to the rotary motion of the yoke. The advantages of this approach are that:

- The only test that is required is a test to determine the spring constant of the return spring. This can be easily done by removing the spring from the pump.

- By using a theoretical development, dynamic effects (inertia) can be readily included.

- The resulting theoretical torque expression is applicable to any axial piston pump, once the spring rate and a few simple dimensions have been found.

There is a problem with the theoretical approach though. As the yoke rotates, the return spring will compress or expand, and it will also be bent longitudinally. The linear compression can be easily handled, but the bending of the spring greatly increases the complexity of the problem.

To allow the developed model to be applicable to any axial piston pump, it was decided that a theoretical approach should be used. However, to simplify the development of the return spring torque it was assumed that bending of the spring will have a negligible effect on the torque applied to the yoke. It was also decided that because of this assumption, the theoretical spring torque should be experimentally verified. Therefore, subject to this assumption, the return spring torque is theoretically derived in this section. The experimental verification is presented in Chapter 7.

The geometry of the spring and the notation used in this analysis are shown in Figure 5-1.

In general, springs are designed to be linear (a fact that was verified

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†This approach is not entirely theoretical because the resulting expression depends upon the measured spring constant.
Figure 5-1:  Return Spring Geometry

experimentally in this study).  Hence, the force that the spring applies to the spring cap is equal to

\[ F_{sc} = K_{sp} \Delta l_{spring} \]  \hspace{1cm} (5.1)

where \( F_{sc} \) is the force applied to the spring cap, \( K_{sp} \) is the measured spring rate, and \( \Delta l_{spring} \) is the amount of spring compression.  If the amount of spring compression when the spring displacement \( y_{sp} \) equals zero is set to \( l_{sp_o} \), then

\[ F_{sc} = K_{sp} (l_{sp_o} - y_{sp}) \]  \hspace{1cm} (5.2)

However, due to the inertia of the spring cap, the force applied to the
spring cap is not necessarily the force applied to the yoke. The actual force on the yoke is found by examining the motion of the spring cap. Consider the free body diagram of the spring cap shown in Figure 5-2.

![Free Body Diagram](image)

**Figure 5-2:** Spring Cap Free Body Diagram

The equation of motion for the spring cap is

\[
\sum F_y = F_{sc} - F_{sp} = m_{sc} \ddot{y}_{sp} ,
\]

where \( F_{sp} \) is the actual force applied to the yoke, and \( m_{sc} \) is the mass of the spring cap. Rearranging Equation (5.3),

\[
F_{sp} = F_{sc} - m_{sc} \ddot{y}_{sp} ,
\]

or

\[
F_{sp} = K_{sp}(l_{sp_0} - y_{sp}) - m_{sc} \ddot{y}_{sp} .
\]

Referring to Figure 5-1, the displacement of the spring \( y_{sp} \) is

\[
y_{sp} = b_{sp} \sin \alpha - a_{sp} \cos \alpha - l_{sc} ,
\]

where \( b_{sp}, a_{sp}, \) and \( l_{sc} \) are physical constants defined in Figure 5-1. Note that Figure 5-1 illustrates a negative \( y_{sp} \) value. Taking derivatives twice of Equation (5.6) gives the acceleration of the spring cap to be
\[ \ddot{y}_p = \left( -b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) \dot{\alpha}^2 + \left( b_{sp} \cos \alpha + a_{sp} \sin \alpha \right) \ddot{\alpha} \quad . \quad (5.7) \]

Substituting Equations (5.6) and (5.7) into Equation (5.5) yields

\[ F_{sp} = K_{sp} \left( l_{sp} - b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) \]
\[ - m_{sc} \left[ \left( -b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) \dot{\alpha}^2 + \left( b_{sp} \cos \alpha + a_{sp} \sin \alpha \right) \ddot{\alpha} \right] . \quad (5.8) \]

Defining \( F_{sp_o} \) to be

\[ F_{sp_o} = K_{sp} \left( l_{sp} + l_{sc} \right) \quad , \quad (5.9) \]

allows Equation (5.8) to be simplified to

\[ F_{sp} = F_{sp_o} + K_{sp} \left( -b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) - m_{sc} \left( -b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) \dot{\alpha}^2 \]
\[ - m_{sc} \left( b_{sp} \cos \alpha + a_{sp} \sin \alpha \right) \ddot{\alpha} \quad . \quad (5.10) \]

This spring force applies a torque to the yoke. Referring again to Figure 5-1, the torque arm is equal to

\[ \text{Torque Arm} = b_{sp} \cos \alpha + a_{sp} \sin \alpha \quad . \quad (5.11) \]

Therefore, the torque that the spring applies to the yoke is equal to the force times the torque arm, or
\[ T_{sp} = \left[ F_{sp_0} + K_{sp} \left( -b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) \right. \]
\[ \left. - m_{sc} \left( -b_{sp} \sin \alpha + a_{sp} \cos \alpha \right) \dot{\alpha} \right] \left[ b_{sp} \cos \alpha + a_{sp} \sin \alpha \right] \dot{\alpha}^2 \]

Evaluating Equation (5.12) and combining similar terms results in the following expression that describes the theoretical torque applied to the yoke by the return spring.

\[ T_{sp} = F_{sp_0} \left[ b_{sp} \cos \alpha + a_{sp} \sin \alpha \right] + K_{sp} \left[ a_{sp} b_{sp} \cos 2\alpha + \left( a_{sp}^2 - b_{sp}^2 \right) \sin \alpha \cos \alpha \right] \]
\[ - m_{sc} \left[ a_{sp} b_{sp} \cos 2\alpha + \frac{a_{sp}^2}{2} \sin \alpha \cos \alpha - b_{sp}^2 \sin \alpha \cos \alpha \right] \dot{\alpha}^2 \]
\[ - m_{sc} \left[ 2a_{sp} b_{sp} \sin \alpha \cos \alpha + \frac{a_{sp}^2}{2} \sin^2 \alpha + b_{sp}^2 \cos^2 \alpha \right] \ddot{\alpha} \]  \hspace{1cm} (5.13)

### 5.2. Torque Due to Yoke Damping

The yoke rotates inside the pump case which is filled with hydraulic fluid. This causes a viscous damping torque to act on the yoke which opposes motion. The yoke also rubs the inside of the pump case, causing a stiction torque to act on the yoke which also opposes motion.

An analytical derivation of this particular torque could not be readily developed. As such, the yoke damping had to be experimentally determined for a particular pump. This was done for a Vickers PVB5 pump and is presented in Appendix E. There were two limitations to this testing. It was not possible to measure the damping in both directions and so it was assumed that the damping torque acted equally in both directions. Also, the
pump could not be operating during the testing. The results of the experiments are shown in Figure 5-3.

![Diagram showing yoke damping torque vs. swashplate angular velocity](image)

**Figure 5-3:** Yoke Damping Torque

Appendix E also shows that the damping torque that acts on the yoke can be represented by an empirical relationship consisting of a linear term (which corresponds to viscous damping) and an exponential decay term (which corresponds to stiction). This expression is
\[ T_d = K_{d_1} \dot{\alpha} + K_{d_2} \frac{\dot{\alpha}}{|\dot{\alpha}|} e^{-\frac{|\dot{\alpha}|}{K_{d_3}}} \quad \text{for} \quad \dot{\alpha} \neq 0. \]  

(5.14)

where \( \dot{\alpha} \) is the angular velocity of the yoke. Note that the damping torque is defined to be in the negative direction. As Figure 5-3 shows, when the control piston velocity is zero, \( T_d \) is not a function of \( \dot{\alpha} \) and the value of \( T_d \) for zero velocity is determined by stiction.

As stated, the pump was not operating during the damping tests. When it is operating, vibration inside the pump will tend to eliminate the stiction effects. Therefore, it was assumed that stiction effects are negligible, causing the describing expression for the yoke damping torque (in the negative direction) to become

\[ T_d = K_{d_1} \dot{\alpha} \quad , \]  

(5.15)

or simply

\[ T_d = K_d \dot{\alpha} \quad . \]  

(5.16)

As will be shown, the results of the verification testing support this assumption and indicate that to include any stiction effects would be unnecessary.

5.3. Torque Due to the Control Piston

The control piston is the actuator that controls the displacement of the pump. It is shown in Figure 5-4. Note that the torque due to the control piston, \( T_c \), is defined to be in the negative direction.

Consider Figure 5-5, a free body diagram of the control piston. Applying Newton’s Second Law in the axial direction yields

\[ \sum F_y = -P_c A_c' + F_{cs} + F_c = m_c \ddot{y}_c \quad , \]  

(5.17)
Figure 5-4: Control Piston Geometry

Figure 5-5: Control Piston Free Body Diagram
where $P_c$ is the chamber pressure, $A'_c$ is the actual piston area, $F_{cs}$ is the viscous shear force acting on the piston, $F_c$ is the force applied to the yoke, $m_c$ is the mass of the control piston, and $\ddot{y}_c$ is the axial acceleration of the piston. It has been assumed that since the surfaces of the piston and the guide are smooth and well lubricated, the inherent vibration inside the pump will eliminate any stiction. Rearranging Equation (5.17) gives

$$F_c = P_c A'_c - F_{cs} + m_c \ddot{y}_c .$$  \hspace{1cm} (5.18)

As shown in Appendix D, the viscous shear force on the control piston in the y direction was analytically found to be

$$F_{cs} = -\frac{\pi r_c \mu \dot{y}_c (l_{cs_0} + y_c)}{h_c} + 2\pi r_c h_c P_c ,$$  \hspace{1cm} (5.19)

where $r_c$ is the radius of the piston, $y_c$ is the piston displacement, $\dot{y}_c$ is the piston velocity, $l_{cs_0}$ is the length of the shear surface when the piston displacement $y_c$ is zero, and $h_c$ is one half of the radial clearance between the piston and the guide. Substituting Equation (5.19) into Equation (5.18),

$$F_c = P_c A'_c + \frac{\pi r_c \mu \dot{y}_c (l_{cs_0} + y_c)}{h_c} - 2\pi r_c h_c P_c + m_c \ddot{y}_c .$$  \hspace{1cm} (5.20)

If the effective piston area $A_c$ is defined to be

$$A_c = A'_c - 2\pi r_c h_c ,$$  \hspace{1cm} (5.21)

then Equation (5.20) reduces to

$$F_c = P_c A_c + \frac{\pi r_c \mu \dot{y}_c (l_{cs_0} + y_c)}{h_c} + m_c \ddot{y}_c .$$  \hspace{1cm} (5.22)
As Figure 5-4 shows, the torque arm for the control force is constant and equal to \( b \), the distance between the yoke pivot and the centerline of the control piston guide. Therefore, the torque that the control piston applies to the yoke in the negative direction is equal to

\[
T_c = P_c A_c b + \frac{\pi r_c \mu b \dot{y}_c (l_{c_o} + y_c)}{h_c} + m_c b \ddot{y}_c . \tag{5.23}
\]

Equation (5.23) is not compatible with the previously found torque equations because it is defined in terms of the axial displacement \( y_c \) instead of the swashplate angle \( \alpha \). Therefore, the axial displacement coordinates in Equation (5.23) must be converted to angular coordinates.

The actual relationship between the control piston displacement and the swashplate angle is complicated and it would be very cumbersome to use. However, a very accurate relationship is not required because the dominate factor of this torque is the pressure force term and it is independent of displacement. Therefore, the following approximate relationship can be used,

\[
y_c = b \tan(\alpha - C) , \tag{5.24}
\]

where \( C \) is an angle correction constant. For this pump, this expression is accurate to within \( \pm 0.4 \) mm over the operating range which corresponds to 2.5\% of the total possible length that the control piston can travel. Taking derivatives of Equation (5.24),

\[
\dot{y}_c = \frac{b \dot{\alpha}}{\cos^2(\alpha - C)} \tag{5.25}
\]

and

\[
\ddot{y}_c = b \left[ \frac{2 \dot{\alpha}^2 \sin(\alpha - C)}{\cos^3(\alpha - C)} + \frac{\ddot{\alpha}}{\cos^2(\alpha - C)} \right] . \tag{5.26}
\]
Substituting Equations (5.24), (5.25), and (5.26) into Equation (5.23) yields

\[
T_c = P_c A_c b + \frac{\pi r_c \mu b^2}{h_c} \left[ l_{c_o} \frac{b \tan (\alpha - C)}{\cos^2 (\alpha - C)} \right] \dot{\alpha}
\]

\[
+ m_c b^2 \left[ 2 \sin \frac{(\alpha - C)}{\cos^3 (\alpha - C)} \dot{\alpha}^2 + \frac{\ddot{\alpha}}{\cos^2 (\alpha - C)} \right].
\]  

Equation (5.27) represents the torque applied to the yoke due to the control piston (in the negative direction).

5.4. Summary

In this chapter the expressions to describe the torques applied directly to the yoke have been determined. Certain observations about these torques can now be made.

The theoretical torque applied by the return spring was found to be a nonlinear function of the angular displacement. This torque was also found to be influenced by the mass of the spring cap. However, to develop this theoretical torque, it was necessary to neglect bending effects.

The torque due to yoke damping was experimentally found to be the sum of a linear viscous term and a nonlinear stiction term. However, this experiment was performed on a pump that was not operating (vibration free). It has been assumed that the vibration that will occur when the pump is operating will eliminate any stiction effects.

The torque due to the control piston was found to be a linear function of the control piston chamber pressure. However, it also contained nonlinear control piston inertia and control piston damping effects.
Chapter 6
Pump Models

The previous chapters have developed mathematical expressions to describe the various torques that are applied to the swashplate yoke assembly. In this chapter, these expressions will be combined to produce an overall model to describe the dynamic response of an axial piston pump. This model will relate the dynamic behavior of the pump to a given control piston chamber pressure input.

6.1. Complete Pump Model

It has been determined that the torques that act on the swashplate yoke assembly consist of the swashplate torque $\overline{T_{ap}}$, the yoke damping torque $T_d$ (in the negative direction), the return spring torque $T_{sp}$ and the torque applied to the yoke by the control piston $T_c$ (in the negative direction). Therefore, taking a summation of the torques that act on the swashplate yoke assembly yields

$$\sum T_\alpha = \overline{T_{ap}} - T_d + T_{sp} - T_c = I \ddot{\alpha}, \quad (6.1)$$

where $I$ is the mass moment of inertia of the swashplate yoke assembly.

The swashplate torque $\overline{T_{ap}}$ is defined by (3.21),

$$\overline{T_{ap}} = \overline{T_{pm}} + \overline{T_{ps}} + \overline{T_r} + \overline{T_{pr}}. \quad (6.2)$$

The effect of piston inertia, $\overline{T_{pm}}$, is described by Equation (3.27), while the
effect of piston shear, $T_{ps}$, the shoe plate effect, $T_r$, and the load pressure
effect, $T_{pr}$, are described by Equations (3.30), (3.31), and (4.53) respectively.
The return spring torque, $T_{sp}$, is defined by Equation (5.13), the damping
torque, $T_d$, by Equation (5.16), and the control torque, $T_c$, by Equation
(5.27). Substituting all these equations into Equation (6.1) and grouping
similar terms yields,

$$-P_c A_c b + K_{pr_2} \Delta P_P + K_{pr_3} \Delta P_P \alpha$$

$$+ \left[ F_{sp_0} \left( b_{sp} \cos \alpha + a_{sp} \sin \alpha \right) + K_{sp} \left[ a_{sp} b_{sp} \cos 2\alpha + \left( a_{sp}^2 - b_{sp}^2 \right) \sin \alpha \cos \alpha \right] 
+ \frac{mNR^2\omega^2 \sin \alpha}{2 \cos^3 \alpha} + K_{pr_1} \right]$$

$$+ \left[ -\frac{\pi r_p \mu b^2}{h_c} \left[ \frac{l_{c\rho o} + b \tan (\alpha - C)}{\cos^2 (\alpha - C)} \right] - K_d + K_{pr_4} \right]$$

$$- \frac{\pi N_{r_p}^2 \mu}{h_p \cos^5 \alpha} \left[ l_{p_o} R^2 \cos \alpha + l_{p_o} a^2 \cos \alpha \sin^2 \alpha - 2a^2 R \sin^2 \alpha 
- \frac{1}{2} a R^2 \sin \alpha - \frac{1}{2} a^3 \sin^2 \alpha \right] \dot{\alpha}$$

(continues)
\[ -m_{sc} \left[ a_{sp} b_{sp} \cos 2\alpha + a_{sp}^2 \sin \alpha \cos \alpha - b_{sp}^2 \sin \alpha \cos \alpha \right] \\
+ m_e b^2 \left[ \frac{2 \sin (\alpha - C)}{\cos^3 (\alpha - C)} \right] + mN \left[ \frac{R^2 \sin \alpha + a^2 \sin \alpha \cos^2 \alpha - 2 a^2 \sin^3 \alpha}{\cos^5 \alpha} \right] \dot{\alpha}^2 \\
= \left[ m_{sc} \left[ 2 a_{sp} b_{sp} \sin \alpha \cos \alpha + a_{sp}^2 \sin^2 \alpha + b_{sp}^2 \cos^2 \alpha \right] \\
+ \frac{b^2}{\cos^2 (\alpha - C)} \right] + mN \left[ \frac{R^2 + 2 a^2 \sin^2 \alpha}{2 \cos^4 \alpha} \right] + I_r + I_e \right] \ddot{\alpha} \] (6.3)

Equation (6.3) represents the complete model of an axial piston pump. It can be more easily shown by

\[- P_e A_e b + K_{pr_2} \Delta P_p + K_{pr_3} \Delta P_p' + C_1 + C_2 \dot{\alpha} + C_3 \dot{\alpha}^2 = I_e \ddot{\alpha} \quad (6.4)\]

where \( C_1 \) includes all terms that depend only on the angular position, \( C_2 \) is the coefficient for the angular velocity term, \( C_3 \) is the coefficient for the square of the angular velocity term, and \( I_e \) is the total effective mass moment of inertia. Note that \( C_1, C_2, C_3, \) and \( I_e \) are not constants because they all vary with the angular position.

Equating Equations (6.3) and (6.4) defines the coefficients to be

\[ C_1 = F_{sp_o} \left( b_{sp} \cos \alpha + a_{sp} \sin \alpha \right) + K_{sp} \left[ a_{sp} b_{sp} \cos 2\alpha + \left( a_{sp}^2 - b_{sp}^2 \right) \sin \alpha \cos \alpha \right] \\
+ \frac{mN R^2 \omega^2 \sin \alpha}{2 \cos^3 \alpha} + K_{pr_1} \] (6.5)
\[ C_2 = - \frac{\pi r_c \mu b^2}{h_c} \left[ \frac{l_{cs_o} + b \tan (\alpha - C)}{\cos^2(\alpha - C)} \right] - K_d + K_{pr_4} \]

\[ - \frac{\pi N r_p \mu}{h_p \cos^5 \alpha} \left[ l_{p_o} R^2 \cos \alpha + l_{p_o} a^2 \cos \alpha \sin^2 \alpha - 2a^2 R \sin^2 \alpha \right. \]

\[ \left. - \frac{1}{2} a R^2 \sin \alpha - \frac{1}{2} a^3 \sin^2 \alpha \right] \]

(6.6)

\[ C_3 = - m_{sc} \left[ a_{sp} b_{sp} \cos 2\alpha + a_{sp}^2 \sin \alpha \cos \alpha - b_{sp}^2 \sin \alpha \cos \alpha \right] \]

\[ - m_e b^2 \frac{2 \sin (\alpha - C)}{\cos^3(\alpha - C)} - m_N \left[ \frac{R^2 \sin \alpha + a^2 \sin \alpha \cos^2 \alpha - 2a^2 \sin^3 \alpha}{\cos^5 \alpha} \right] \]

(6.7)

and

\[ I_c = m_{sc} \left[ 2a_{sp} b_{sp} \sin \alpha \cos \alpha + a_{sp}^2 \sin^2 \alpha + b_{sp}^2 \cos^2 \alpha \right] \]

\[ + m_e \frac{b^2}{\cos^2(\alpha - C)} + m_N \left[ \frac{R^2 + 2a^2 \sin^2 \alpha}{2 \cos^4 \alpha} \right] + I_r + I \]

(6.8)

Equations (6.4), (6.5), (6.6), (6.7), and (6.8) compose the complete model of an axial piston pump. However, the number of terms in the coefficients make it cumbersome to use. The model can be simplified by comparing the various terms in the coefficients and neglecting those terms that have an insignificant effect.
6.2. Simplification of the Pump Model

In this section, the numerical values for the various terms of the complete model coefficients will be evaluated, plotted, and compared. This will allow insignificant terms to be neglected and simplifications to be made. The terms are evaluated using the physical data for a Vickers PVB5 pump and so the conclusions made in this section are not necessarily applicable to other types of axial piston pumps. Note that in Appendix G, the four coefficients and the terms that make them up have been evaluated for three swashplate angles and the results are tabulated.

6.2.1. Simplification of $C_1$

The coefficient $C_1$ depends on the swashplate angle and is defined by Equation (6.5). A graphical representation of $C_1$ is shown in Figure 6-1. As indicated by Figure 6-1, $C_1$ can be well represented by a linear expression. Therefore, the defining equation for $C_1$ can be simplified to

$$C_1 \approx S_1 - S_2 \alpha \quad ,$$

where $S_1$ and $S_2$ are constants found using linear regression techniques. This linear expression is also shown in Figure 6-1. From Figure 6-1, the approximate maximum error of this linear expression is $\pm 0.3$ Nm, which corresponds to approximately 2% of the average $C_1$ value.

6.2.2. Simplification of $C_2$

The total damping coefficient $C_2$ is expressed by Equation (6.6). The variation of $C_2$ with swashplate angle is graphically illustrated is Figure 6-2. Figure 6-2 illustrates that $C_2$ is virtually constant over the operating range of the pump. Thus, to simplify $C_2$, it can be assumed constant and equal to

$$C_2 \approx S_3 \quad .$$

From Figure 6-2, the maximum error between this average value and the actual value is approximately $\pm 0.006$ Nms, which corresponds to 1% of the average value.
Figure 6-1: Plot of $C_1$ versus Swashplate Angle

In Appendix G, Table G-2 shows that yoke damping accounts for roughly 75% of the total damping, while the piston and control piston damping both contribute approximately 3%, and the effects of the load pressure account for about 18%.
6.2.3. Simplification of $C_3$

Coefficient $C_3$ is represented by Equation (6.7). The variation of $C_3$ with the swashplate angle is graphically shown in Figure 6-3.

Figure 6-3 indicates that $C_3$ could be well represented by a linear expression. However, it also shows that the magnitude of the coefficient $C_3$ is very small. From Figure 6-3, the maximum magnitude of $C_3$ is

$$|C_3|_{\text{max}} = 7.8 \times 10^{-5} \text{ Nm}^2$$

(6.11)
Consider the case where the pump is operating at its maximum pressure of 20 MPa, and that this pressure is applied to the control piston. This maximum pressure will cause the yoke to rotate at its maximum possible angular velocity. Using this pressure, Equation (6.4) gives the magnitude of this maximum velocity to be (also using $\alpha = 0$ and $\dot{\alpha} = 0$)

$$|\dot{\alpha}|_{\text{max}} = 26 \text{ rad/s}$$  \hspace{1cm} (6.12)

and therefore,
\[ \left| C_3 \dot{\alpha} \right|_{\text{max}} = 2.03 \times 10^{-3} \text{ Nms} \quad (6.13) \]

This maximum value is only 0.4\% of the virtually constant magnitude of \[ \left| C_2 \right| = 0.549 \text{ Nms} \]. Therefore, it can be concluded that

\[ \left| C_2 \right| \gg \left| C_3 \dot{\alpha} \right|_{\text{max}} \quad (6.14) \]

For other cases, this difference will only increase, indicating that

\[ \left| C_2 \right| \gg \left| C_3 \dot{\alpha} \right| \quad \text{for all cases} \quad (6.15) \]

or

\[ \left| C_2 \dot{\alpha} \right| \gg \left| C_3 \dot{\alpha}^2 \right| \quad \text{for all cases} \quad (6.16) \]

This implies that the \( C_3 \dot{\alpha}^2 \) term will have a negligible effect when compared to the damping torques. Thus it can be assumed that

\[ C_3 \approx 0 \quad (6.17) \]

### 6.2.4. Simplification of \( I_e \)

The effective inertia coefficient is defined by Equation (6.8) and plotted in Figure 6-4. Figure 6-4 shows that the effective inertia varies only slightly over the operating range of the pump. Therefore, to simplify the pump model, the variation of the coefficient \( I_e \) can be neglected. Thus, the effective inertia can be assumed constant and equal to the average effective inertia.

\[ I_e \approx \bar{I}_e \quad (6.18) \]

For this pump, the maximum error between this average value and the actual value is \( \pm 2 \times 10^{-5} \) kgm\(^2\) or 0.5\% of the average value.

In Appendix G, Table G-4 shows the relative magnitude of each component of the effective inertia for three swashplate angles. This table
Figure 6-4: Plot of $I_e$ versus Swashplate Angle

shows that the yoke inertia accounts for approximately 75% of the total inertia, while the control piston inertia accounts for between 11 and 12%, the spring cap has a contribution of roughly 9%, and the piston inertia contributes less than 4%. Table G-4 also points out that the shoe plate inertia has a negligible effect and it can be neglected.

6.3. Simplified Pump Model

Substituting these approximate coefficient expressions into the system Equation (6.4) results in the following simplified pump model.

$$- P_c A_e b + K_{p r_2} \Delta P_P + K_{p r_3} \Delta P_P \alpha + S_1 - S_2 \alpha + S_3 \dot{\alpha} = \overline{I_e} \ddot{\alpha} \quad (6.19)$$

Rearranging (6.19),
\[-P_c A_c b + K_{pr_2} \Delta P_P + S_1 = \left( S_2 - K_{pr_3} \Delta P_P \right) \alpha - S_3 \dot{\alpha} + \Gamma_e \ddot{\alpha} \quad (6.20)\]

Equation (6.20) represents the simplified pump model.

6.4. Summary

This chapter has presented the complete pump model and the simplified pump model. From the analysis presented certain conclusions can be made.

- The coefficient \( C_1 \) in Equation (6.4) can be approximated by a constant and a linear term (linear with respect to \( \alpha \)).

- The only significant damping effects were yoke damping and the damping effect of the pressure torque. The viscous shear forces acting on the pistons and the control piston have only a small effect on the pump dynamics.

- All terms involving the square of the angular velocity were found to be insignificant.

- The effective inertia was found to vary only slightly over the operating range and it can be assumed constant. The magnitude of the effective inertia is determined largely by the yoke inertia, but the mass of the pistons, the control piston mass, and the spring cap mass also affect the effective inertia. The shoe plate inertia has a negligible effect on the pump dynamics and it can be neglected.
Chapter 7
Experimental Verification of the Pump Models

The previous chapter developed the complete and simplified pump models. In this chapter the experimental verification of the steady state and dynamic response of these two models will be presented. The verification is based upon comparing the experimentally measured responses for a Vickers PVB5 axial piston pump to the theoretical responses predicted by the pump models.

7.1. Steady State Verification

The steady state response of the pump model can be evaluated from the relationship between a constant control piston chamber pressure and the constant swashplate angle (or pump flowrate) that this pressure will produce. This relationship is dominated by the return spring and the torque it applies to the yoke. Therefore, to verify the steady state characteristics of the pump models, the theoretical return spring torque developed in Chapter 5 had be experimentally verified.

The experimental test stand arrangement used to verify the theoretical return spring torque is illustrated in Figure 7-1. To perform this test, the pump was disconnected and mounted vertically. The compensator valve was removed to allow a rod to slide inside the control piston guide and rest against the control piston. A rotary potentiometer was attached to the yoke to measure the swashplate angle. By adding weights to the rod, and measuring the swashplate angle, the experimental return spring torque was determined.
Figure 7-1: Steady State Verification Test Stand

Since the pump was not operating, there were no pressure, inertia, or damping torques on the swashplate. However, because the pump was not operating, there was a stiction torque that had to be overcome before the yoke could rotate. This stiction effect was reduced by manually vibrating the pump before each measurement. Also, the swashplate angle was measured as weights were added to the rod (loading) until the pump was fully destroked. Then, the angles was measured again as the weights were removed (unloading). This allowed the effects of stiction to be “balanced out” by averaging the two swashplate angles that were measured for each applied torque.
A summary of the experimental results is shown in Figure 7-2, along with the theoretically determined spring torque developed in Chapter 5. The difference between the loading and unloading curves in Figure 7-2 is caused by stiction. Therefore, the average of the loading and unloading curves represents the experimental spring torque, because by taking the average, the effects of stiction should be eliminated. This average experimental curve is also shown in Figure 7-2.

Figure 7-2 shows that the general shape of the theoretical torque curve agrees with the average experimental results. However, the measured spring torque is larger than the theoretical spring torque for small swashplate angles. Figure 7-2 also shows that measured spring torque rapidly increases as the spring "bottoms out" at approximately -0.02 rad.

It should be recalled that during the development of the return spring torque, it was necessary to assume, without justification, that any bending of the spring would have a negligible effect on the applied torque. The difference between the average experimental and theoretical torques can be attributed to the error in this assumption. This is because, for the pump used in this study, the amount that the spring bends increases as the swashplate angle decreases, as does the error between the theoretical and experimental curves. Therefore, it can be concluded that the steady state response of the developed models will be in error.

To improve the steady state response of the pump models, the coefficient $C_1$ was redefined by replacing the spring torque terms in Equation (6.5) (the equation that defines $C_1$) with the average experimental curve in Figure 7-2. This experimental value for $C_1$ is shown in Figure 7-3, along with the previously developed theoretical value. Figure 7-3 also shows the redefined linear expression that was used to approximate the experimental values for the coefficient $C_1$. 
Figure 7-2: Experimental and Theoretical Return Spring Torque
Figure 7-3: Plot of Experimental $C_1$ vs. Swashplate Angle
7.2. Dynamic Response Verification

To verify the dynamic response of the model, the simulated dynamic responses of the pump were compared to those obtained experimentally. This was achieved with the test stand illustrated in Figure 7-4.

![Diagram of Dynamic Verification Test Stand Schematic](image)

**Figure 7-4:** Dynamic Verification Test Stand Schematic

For these tests the compensator valving was removed and the pump was modified so that the pump outlet was connected directly to the control piston chamber by a line that had only a small resistance. The pump outlet was also connected to a two position manual control valve. The two positions of the valve were:

1. The valve shown in Figure 7-4 is in the open position. In this position the output from the pump is allowed to flow freely to the reservoir tank, with the valve itself providing only a slight resistance.
2. In the bottom position, the output from the pump is blocked. This position provides, in effect, an infinite resistance to fluid flow.

The tests were conducted by manually shifting the valve from one position to another. This caused a sudden change in the flow resistance, and therefore, the line pressure would change quickly. The control piston chamber was connected directly to the pump outlet and so shifting of the valve also caused the control chamber pressure to change quickly.

The line pressure was measured with a pressure transducer connected to the pump outlet, and the swashplate angle was measured with a rotary potentiometer connected to a pintle. The outputs of both of these transducers were recorded with a digital storage oscilloscope. The oscilloscope was set up so that when the valve was shifted, the dynamic responses of both the line pressure and the swashplate angle were stored and displayed.

To determine the theoretical dynamic response of the pump models, the valve was quickly shifted and both the line pressure versus time history and the resulting experimental swashplate angle response were measured and stored. Subsequently, the developed models were used to predict the theoretical swashplate angle response for this measured line pressure versus time history. The resulting theoretical response was then compared to the measured response. For a complete discussion of the simulations, refer to Appendix F.

The dynamic verification results are shown in Figures 7-5 and 7-6. In these graphs, the measured swashplate angle is shown along with the simulated swashplate angle, using both the complete and simplified pump models. As well, the actual dynamic line pressure that caused experimental response, and that was used as the input to the simulation to produce the simulated responses, is shown.

Figures 7-5 and 7-6 show that the simulated dynamic responses of the
Figure 7-5: Experimental and Simulated Dynamic Response No. 1

The dynamic response occurring when the valve was shifted from the open to the closed position.
Figure 7-6: Experimental and Simulated Dynamic Response No. 2

The dynamic response occurring when the valve was shifted from the closed to the open position.
swashplate angle agree with the measured dynamic responses. The main error between the theoretical and experimental curves is a steady state error. However, this could be due to experimental error though, because a variation of the measured line pressure by less than 4% will correct this steady state error. Figures 7-5 and 7-6 also show that there is virtually no difference between the complete and simplified pump models.

7.3. Summary

From the results of the experimental verification tests, the following observations can be made:

1. During the development of the theoretical return spring torque, it was assumed that any bending of the spring, that may occur, would have a negligible effect. The results of the steady state verification indicate that this assumption is incorrect. Therefore, if the steady state aspects of the pump model are important, then the return spring torque should be experimentally determined.

2. The results of the dynamic verification indicate that the developed pump models accurately predict the dynamic response of the axial piston pump used in this study. The results also indicate that there is some steady state error in the models. However, this slight steady state error in the pump model is not considered to be a major problem. This is because this model is intended to be used in the design of pump control systems and these feedback control systems must be designed to be able to handle and correct steady state errors anyways.

3. There was virtually no difference between the response predicted by the complete and simplified pump models. Therefore, for this pump, the simplified model could be used in control system design and analysis.
Chapter 8
Conclusions and Recommendations

8.1. Conclusions

In this thesis, the derivation of a complete, dynamic, mathematical model of an axial piston pump has been presented. This model was developed symbolically using general coordinates and parameters of a typical axial piston pump. Therefore, it can be concluded that the developed model is applicable to any axial piston pump of similar configuration.

The model is expressed by a set of highly nonlinear equations. These equations were developed theoretically wherever possible. However, it was found that the experimental results indicated that longitudinal bending of the return spring can have a significant effect on the return spring torque. The theoretical expression for the return spring torque was developed by assuming that this bending would have a negligible effect. Therefore, if the steady state response of a pump model is important, or if there is significant bending of the return spring, the return spring torque should be experimentally determined. If this is not the case, then the theoretical expression for the return spring torque should be sufficient.

Overall, the developed pump model was found to have three major parts: a steady state effect, a damping effect, and an inertia effect. Each component of these major parts was developed separately. This allowed comparisons to be made between the various components, and these comparisons showed which components of an axial piston pump have a dominant effect, and which components have an insignificant effect. To
make these comparisons, it was necessary to evaluate the model using specific values of pump parameters. This was done using data from a Vickers PVB5 axial piston pump. Therefore, strictly speaking, the following conclusions/observations are applicable only to this specific type of axial piston pump.

1. The steady state effect was found to be dominated by the return spring in the pump, and influenced by load pressure effects.

2. The damping effect was found to be dominated by yoke damping. However, the viscous shear forces that act on the pistons and the control piston, and the load pressure effects, were found to contribute to the total damping of the system.

3. The total effective inertia of the developed model was found to be determined largely by the inertia of the yoke. However, the mass of the pistons, the control piston, and the spring cap did have a significant effect on the total effective inertia.

4. It was found that for the pump used in this study, the equations that mathematically describe each of these effects could be approximated by linear expressions, with very little error.

To verify the developed model, experimentally measured dynamic responses of a specific pump were compared to theoretically predicted dynamic responses. From the results of the verification testing it can be further concluded that the developed pump model accurately predicts the dynamic response of a Vickers PVB5 axial piston pump. It was found that there is a small steady state error in the model. However, as discussed in Chapter 7, it is felt that this is not a major problem.

In summary, the research presented in this thesis forms the basis for mathematical modelling of axial piston pumps. Using the equations that have been derived, and a few simple experiments, a mathematical model could be developed for any similar axial piston pump.
8.2. Recommendations for Further Research

The research initiated in this study is directed towards the development of new and innovative digital control systems for axial piston pumps. Therefore, the next step in this process would be to use this model to help design these pump control systems. Also, it would be interesting to determine the validity of this model when applied to other types of axial piston pumps.
References

[1] El-Ibiary, Y., Ukrainetz, P.R., and Nikiforuk P.N.  
Design and Performance of Some Digital Electrohydraulic Valves.  

Design and Performance of a Microprocessor-Based Digital Flow Control Valve.  

Micro Computer Control of a Variable Displacement Pump.  

[4] Ukrainetz, P.R., Ramachandran, S., and Nikiforuk, P.N.  
Design and Assessment of a Digital Proportional Flow Control Valve.  

[5] Rosa, S., Burton, R.T., and Nikiforuk, P.N.  
Feasibility Study of a Piezoelectric Jet Pipe Valve.  

The Xpert® Hydraulic Servo Actuator.  

[7] Chan, J., Ukrainetz, P.R., and Burton R.T.  
Stepper Motor Operated Digital Proportional Valve.  
[8] Ukrainetz, P.R., Bitner, D.V., and Nikiforuk, P.N.
Load Interaction in a Multi-Load, Load Sensing Pump Driven
Hydraulic System.

[9] Lewis, E. and Stern, H.
*Design of Hydraulic Control Systems.*

[10] Merritt, H.
*Hydraulic Control Systems.*

An Analysis of the Control Mechanism Used in Variable-Delivery
Hydraulic Pumps.

[12] Zaki, H. and Baz, A.
On the Dynamics of Pressure-Compensated Axial Piston Pumps.

[13] Yamaguchi, A.
Studies on the Characteristics of Axial Plunger Pumps and Motors.

Characteristics of Displacement Control Mechanisms in Axial Piston
Pumps.

Torque on the Swashplate of an Axial Piston Pump.

[16] Lin, S.J. et al.
Oil Entrapment in an Axial Piston Pump and its Effect Upon
Pressures and Swashplate Torques.

[17] Helgestad, B., Foster, K. and Bannister, F.
Pressure Transients in an Axial Piston Hydraulic Pump.
[18] McCloy, D. and Martin, H.
*Control of Fluid Power: Analysis and Design.*
John Wiley and Sons, 1980.
Appendix A

Solenoid Valve Control System

A possible digital control system for an axial piston pump is shown schematically in Figure A-1.

Figure A-1: Solenoid Valve Control System Schematic

For this type of control system, the displacement of the pump is controlled by a pair of two-way solenoid valves. These valves are digital in nature because they are either opened or closed. The two valves illustrated in Figure A-1 would shift to the left and open when an electrical signal is applied to them. If there is no electrical signal applied, the spring would force the valve to the right, and the valve would be closed. If the top valve
is energized, high pressure fluid would be allowed to flow into the control piston chamber and the pump would destroke. If the bottom valve is energized, the fluid would be allowed to flow out of the control piston chamber, and the return spring would increase the stroke of the pump. Thus the pump displacement is controlled by the two digital (on or off) control signals. As a result, this pump could be controlled directly from a digital computer, through some type of relay or electronic switch.

There are solenoid valves available that respond very quickly to electrical signals [1,2]. The fast response of these valves would allow them to be controlled by a stream of pulses. By modulating the width of the pulses, the net flow into the control piston chamber could be regulated. A digital computer would then be required to provide the two pulse width modulated (PWM) control signals. This computer could complete a feedback control system by sampling some type of measured parameter such as pressure or flowrate. The control problem becomes one of writing the software that would interpret the feedback signals and then produce the two PWM signals that would provide the desired type of pump control.

The advantages of this type of control system are:

- The mechanical elements are simple and inexpensive.
- No digital to analog converter is required.
- The type of control that is performed is determined solely by software, not the type of valve system. This system is flexible.
- This system would be easier than a conventional pump to integrate into a large, computer controlled process.
- Solenoid valves are more rugged and less sensitive to contamination than conventional valving schemes.

To design this type of swashplate actuation and control system, a mathematical model that represents the dynamic motion of the swashplate yoke assembly is required. An accurate mathematical model would simplify
the design of the control system and the computer software. Also, it would allow this control system to be simulated on a digital computer in order to analyze its performance.
A.1. References

[1] Post, K.H.
Electrohydraulic Valves with Fluidic Ball Elements.
In *Proceedings of the Fluid Power Systems and Controls Conference*.

[2] El-Ibiary, Y., Ukrainetz, P.R., and Nikiforuk, P.N.
Design and Assessment of a New Solenoid-Operated Ball Valve for
Digital Applications.
In *Proceedings of the 34th National Conference on Fluid Power*, pages
Appendix B
Calculation of Approximate Linear Pump Model Parameters

The approximate pump natural frequency and damping ratio for a Vickers PVB5 axial piston pump can be found from Equation (6.20). In this equation, the effective rotational spring constant, $K_\alpha$, is equal to

$$K_\alpha = S_2 - K_{pr \theta} \Delta P_P . \quad (B.1)$$

Therefore, the approximate undamped natural frequency of the pump is equal to the square root of this rotational spring constant divided by the rotational inertia, or

$$\omega_n = \sqrt{\frac{S_2 - K_{pr \theta} \Delta P_P}{I_e}} = 104 \text{ rad/s} = 17 \text{ Hz} . \quad (B.2)$$

Note that the pressure value used in the above calculation was 10 MPa, one half of the maximum allowable pump pressure. In addition, using the average magnitude of rotational damping, $S_3$, the approximate damping ratio, $\zeta$, can be calculated to be

$$\zeta = \frac{|S_3|}{2 \sqrt{(S_2 - K_{pr \theta} \Delta P_P)I_e}} = 2.0 . \quad (B.3)$$
Appendix C
Mathematical Relationships

C.1. Integration Examples

C.1.1. Piston Inertia Effects

This section will evaluate the integral that describes the effect of piston inertia on the net torque that is applied to the swashplate. This integral is represented by Equation (3.26) and is rewritten here.

\[
\overline{T}_{pm} = \frac{N}{2\pi} \int_0^{2\pi} \frac{-m \ddot{y}_p (R \sin \theta - a \sin \alpha)}{\cos^2 \alpha} d\theta,
\]  \hspace{1cm} (C.1)

or

\[
\overline{T}_{pm} = -\frac{mNR}{2\pi \cos^2 \alpha} \int_0^{2\pi} \ddot{y}_p \sin \theta d\theta + \frac{mN a \sin \alpha}{2\pi \cos^2 \alpha} \int_0^{2\pi} \ddot{y}_p d\theta.
\]  \hspace{1cm} (C.2)

Note that because this is an integral with respect to \( \theta \), not time, the swashplate angle \( \alpha \) is constant and can be taken outside the integral.

The expression for \( \ddot{y}_p \) is described by Equation (3.8) which states
\[ \ddot{y}_p = - R\omega^2 \sin \theta \tan \alpha + \frac{2R \cos \theta}{\cos^2 \alpha} \omega \dot{\alpha} + \left( -a(\cos^2 \alpha - 2\sin^2 \alpha) + 2R \sin \alpha \sin \theta \right) \dot{\alpha}^2 + \left( \frac{R \sin \theta - \alpha \sin \alpha}{\cos^2 \alpha} \right) \ddot{\alpha} \]  

(C.3)

Substituting this into Equation (C.2) yields

\[
\overline{T}_{pm} = -\frac{mNR}{2\pi \cos^2 \alpha} \left[ -\int_0^{2\pi} R \omega^2 \tan \alpha \sin^2 \theta d\theta + \int_0^{2\pi} \frac{2R \omega}{\cos^2 \alpha} \dot{\alpha} \cos \theta \sin \theta d\theta \right. \\
\left. - \int_0^{2\pi} \frac{a}{\cos \alpha} \dot{\alpha}^2 \sin \theta d\theta + \int_0^{2\pi} \frac{a \sin^2 \alpha}{\cos^3 \alpha} \dot{\alpha}^2 \sin \theta d\theta \right. \\
\left. + \int_0^{2\pi} \frac{2R \sin \alpha}{\cos^3 \alpha} \dot{\alpha}^2 \sin \theta d\theta + \int_0^{2\pi} \frac{R}{\cos^2 \alpha} \ddot{\alpha} \sin^2 \theta d\theta \right. \\
\left. - \int_0^{2\pi} \frac{a \sin \alpha}{\cos^2 \alpha} \ddot{\alpha} \sin \theta d\theta \right] \\
+ \frac{m \alpha \sin \alpha}{2\pi \cos^2 \alpha} \left[ -\int_0^{2\pi} R \omega^2 \tan \alpha \sin \theta d\theta + \int_0^{2\pi} \frac{2R \omega}{\cos^2 \alpha} \dot{\alpha} \cos \theta d\theta \right. \\
\left. - \int_0^{2\pi} \frac{a}{\cos \alpha} \dot{\alpha}^2 d\theta + \int_0^{2\pi} \frac{a \sin^2 \alpha}{\cos^3 \alpha} \dot{\alpha}^2 d\theta \right. \\
\left. - \int_0^{2\pi} \frac{R \sin \theta - \alpha \sin \alpha}{\cos^2 \alpha} \dot{\alpha} d\theta \right]

(cont.)
\[ + \int_{0}^{2\pi} 2R \sin \alpha \frac{\ddot{a} \sin \theta d\theta}{\cos^3 \alpha} + \int_{0}^{2\pi} \frac{R}{\cos^2 \alpha} \ddot{a} \sin \theta d\theta \]

\[ - \int_{0}^{2\pi} \frac{a \sin \alpha \ddot{a} d\theta}{\cos^2 \alpha} \]  

(C.4)

Using the following identities,

\[ \int_{0}^{2\pi} d\theta = 2\pi \quad , \quad \text{(C.5)} \]

\[ \int_{0}^{2\pi} \cos \theta d\theta = 0 \quad , \quad \text{(C.6)} \]

\[ \int_{0}^{2\pi} \sin \theta d\theta = 0 \quad , \quad \text{(C.7)} \]

\[ \int_{0}^{2\pi} \cos \theta \sin \theta d\theta = 0 \quad , \quad \text{(C.8)} \]

\[ \int_{0}^{2\pi} \sin^2 \theta d\theta = \pi \quad , \quad \text{(C.9)} \]

Equation (C.4) reduces to,
\[ T_{pm} = - \frac{mNR}{2\pi\cos^2 \alpha} \left[ - R \omega^2 \pi \tan \alpha + \frac{2R \sin \alpha}{\cos^3 \alpha} \dot{\alpha}^2 \pi + \frac{R}{\cos^2 \alpha} \ddot{\alpha} \right] \]

\[ + \frac{mN \sin \alpha}{2\pi\cos^2 \alpha} \left[ - \frac{a}{\cos^2 \alpha} \dot{\alpha}^2 \pi + \frac{2a \sin^2 \alpha}{\cos^3 \alpha} \dot{\alpha} \dot{\alpha} \pi - \frac{a \sin \alpha}{\cos^2 \alpha} \ddot{\alpha} \right] \] . \quad (C.10)

Rearranging Equation (C.10) results in the following expression.

\[ T_{pm} = \frac{mNR^2 \omega^2 \sin \alpha}{2\cos^3 \alpha} - mN \left[ \frac{R^2 \sin \alpha + a \sin \alpha \cos^2 \alpha - 2a \sin^3 \alpha}{\cos^5 \alpha} \right] \dot{\alpha}^2 \]

\[ - mN \left[ \frac{R^2 + 2a \sin^2 \alpha}{2\cos^4 \alpha} \right] \ddot{\alpha} \] . \quad (C.11)

This expression shows the average effect of piston mass on the net torque applied to the swashplate.

**C.1.2. Torque Due to Piston Shear**

This section will integrate the expression that describes the torque, applied to the swashplate, due to viscous shear forces acting on the pistons.

The torque due to shear is described by Equation (3.28) and is rewritten here.

\[ T_{ps} = \frac{N}{2\pi} \int_0^{2\pi} \frac{F_{ps}(R \sin \theta - \alpha \sin \alpha)}{\cos^2 \alpha} \, d\theta \] . \quad (C.12)

In Appendix D, on page 116, the shear force \( F_{ps} \) was found to be
\[ F_{ps} = - \frac{\pi r_p \mu (l_p + y_p) \dot{y}_p}{h_p} \]  \hspace{1cm} (C.13)

From Equation (3.2), the axial piston displacement \( y_p \) is defined to be

\[ y_p = \frac{R \sin \theta \sin \alpha - a}{\cos \alpha} \]  \hspace{1cm} (C.14)

Substituting this into Equation (C.13) gives

\[ F_{ps} = - \frac{\pi r_p \mu \left[ l_p + \left( \frac{R \sin \theta \sin \alpha - a}{\cos \alpha} \right) \right] \dot{y}_p}{h_p} \]  \hspace{1cm} (C.15)

The piston velocity \( \dot{y}_p \) is described by Equation (3.7). It states that

\[ \dot{y}_p = \left( \frac{R \sin \theta - a \sin \alpha}{\cos^2 \alpha} \right) \dot{\alpha} + R \omega \cos \theta \tan \alpha \]  \hspace{1cm} (C.16)

Substituting this into Equation (C.15) yields

\[ F_{ps} = - \frac{\pi r_p \mu}{h_p} \left[ l_p + \left( \frac{R \sin \theta \sin \alpha - a}{\cos \alpha} \right) \right] \left[ \left( \frac{R \sin \theta - a \sin \alpha}{\cos^2 \alpha} \right) \dot{\alpha} + R \omega \cos \theta \tan \alpha \right] \]  \hspace{1cm} (C.17)

Substituting Equation (C.17) for \( F_{ps} \) into Equation (C.12) gives
$$
\overline{T_{ps}} = -\frac{N_r p \mu}{2h_p} \int_0^{2\pi} \left\{ \left[ \frac{R \sin \theta - \sin \alpha}{\cos^2 \alpha} \right] \left[ l_p \left( \frac{R \sin \theta \sin \alpha - a}{\cos \alpha} \right) \right] \right. \\
\left. \left[ \left( \frac{R \sin \theta - \sin \alpha}{\cos^2 \alpha} \right) \dot{\alpha} + R \omega \cos \theta \tan \alpha \right] \right\} d\theta \quad . \quad (C.18)
$$

Expanding the expression yields

$$
\overline{T_{ps}} = -\frac{N_r p \mu}{2h_p} \left[ \int_0^{2\pi} \frac{R^2 l_p}{\cos^4 \alpha} \sin^2 \theta d\theta - \int_0^{2\pi} \frac{a R l_p}{\cos^4 \alpha} \sin \theta d\theta \right. \\
+ \int_0^{2\pi} \frac{R^2 l_p}{\cos^3 \alpha} \omega \sin \alpha \cos \theta d\theta - \int_0^{2\pi} \frac{a R l_p}{\cos^4 \alpha} \sin \theta d\theta \\
+ \int_0^{2\pi} \frac{a^2 l_p}{\cos^4 \alpha} \dot{\alpha} \sin^2 \alpha d\theta - \int_0^{2\pi} \frac{a R l_p}{\cos^3 \alpha} \omega \sin^2 \alpha d\theta \\
+ \int_0^{2\pi} \frac{R \sin \alpha}{\cos^5 \alpha} \sin^3 \theta d\theta - \int_0^{2\pi} \frac{a R^2 \sin^2 \alpha}{\cos^5 \alpha} \sin^2 \theta d\theta \\
+ \int_0^{2\pi} \frac{R^3 \omega \sin^2 \alpha}{\cos^4 \alpha} \sin \theta \cos \theta d\theta - \int_0^{2\pi} \frac{a R^2 \dot{\alpha}}{\cos^5 \alpha} \sin^2 \theta d\theta \\
+ \int_0^{2\pi} \frac{a^2 R \sin \alpha}{\cos^5 \alpha} \sin \theta d\theta - \int_0^{2\pi} \frac{a R^2 \omega \sin \alpha}{\cos^4 \alpha} \sin \theta \cos \theta d\theta
$$

(cont.)
\[ - \int_0^{2\pi} \frac{aR^2 \dot{\alpha} \sin^2 \alpha}{\cos^5 \alpha} \sin^2 \theta d\theta + \int_0^{2\pi} \frac{a^2 R \dot{\alpha} \sin \alpha}{\cos^5 \alpha} \sin \theta d\theta \]

\[ - \int_0^{2\pi} \frac{aR^2 \dot{\alpha} \sin^3 \alpha}{\cos^4 \alpha} \sin \theta \cos \theta d\theta + \int_0^{2\pi} \frac{a^2 R \dot{\alpha} \sin \alpha}{\cos^5 \alpha} \sin \theta d\theta \]

\[ - \int_0^{2\pi} \frac{a^3 \dot{\alpha} \sin^2 \alpha}{\cos^5 \alpha} d\theta + \int_0^{2\pi} \frac{a^2 R \dot{\alpha} \sin \alpha}{\cos^4 \alpha} \cos \theta d\theta \]  \quad (C.19)

Using the identities as shown by Equations (C.5) to (C.9) as well as

\[ \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta = 0 \]  \quad (C.20)

and

\[ \int_0^{2\pi} \sin^3 \theta d\theta = 0 \]  \quad (C.21)

Equation (C.19) reduces to

\[
\overline{T_{ps}} = - \frac{\pi N r_p \mu}{h_p \cos^5 \alpha} \left[ l_{p_o} R^2 \cos \alpha + l_{p_o} a^2 \cos \alpha \sin^2 \alpha - 2a^2 R \sin^2 \alpha \
- \frac{1}{2} a R^2 \sin \alpha - \frac{1}{2} a^3 \sin^2 \alpha \right] \dot{\alpha} \]  \quad (C.22)

This expression describes the average torque due to piston shear that is applied to the swashplate.
C.2. Shoe Plate Effects

As illustrated in Figure 2-2, the shoe plate rotates with the pump barrel and reciprocates with the pistons by pivoting about the spherical washer. The spring that acts on the washer applies a force to the shoe plate. However, the pump is constructed so that this spring force does not apply a torque to the swashplate. This is because the center of the spherical washer coincides with the yoke pivot. Thus the torque arm for this spring force is zero and as a result, produces no torque. Therefore, only the inertia of the shoe plate affects the average torque applied to the swashplate. This effect was theoretically determined by analyzing the motion of the shoe plate.

The shoe plate resembles a large flat washer with holes in it for each piston. The thickness of the plate is 4 mm or 6% of the diameter. Thus, it was assumed that the shoe plate could be analyzed as a thin plate. To further simplify this analysis, the shoe plate was treated as a flat washer with no holes in it, but with the same rectangular mass moment of inertia as the actual shoe plate. The result of using this idealized shoe plate was that, as desired, the average torque effect of the shoe plate was found.

Figure C-1 shows the idealized shoe plate with a differential element, \( dm \), located at an arbitrary radius \( r \) and an arbitrary angle \( \theta \). The mass of this element, \( dm \), is equal to

\[
 dm = \rho t^* r dr d\theta ,
\]

where \( \rho \) is the material density and \( t^* \) is the plate thickness. From Figure C-1, the cartesian coordinates of the differential element are:

\[
x_r = a_r \sin \alpha + r \cos \theta \cos \alpha ,
\]

\[
y_r = -a_r \sin \alpha + r \cos \theta \sin \alpha ,
\]

\[
z_r = -r \sin \theta ,
\]

where \( a_r \) is the perpendicular distance from the yoke pivot to the shoe plate,
as illustrated in Figure C-1. From Equations (C.24), (C.25), and (C.26), the displacement vector of the differential element, $r$, is

$$
    r = (a_r \sin \alpha + r \cos \theta \cos \alpha) \mathbf{i} + (-a_r \cos \alpha + r \cos \theta \sin \alpha) \mathbf{j} - r \sin \theta \mathbf{k}.
$$

(C.27)

The plate is rotating about an axis normal to its surface at the pump rotational velocity $\omega$, as illustrated in Figure C-1. Also, the plate is rotating about the yoke pivot at the swashplate angular velocity $\dot{\alpha}$. Therefore, the total angular velocity vector, $\Omega$, is equal to
\[ \Omega = -\omega \sin \alpha \mathbf{i} + \omega \cos \alpha \mathbf{j} + \dot{\alpha} \mathbf{k} \]  \hspace{1cm} (C.28)

The velocity vector of the differential element, \( \mathbf{v} \), is equal to the cross product

\[ \mathbf{v} = \Omega \times \mathbf{r} \]  \hspace{1cm} (C.29)

Substituting Equations (C.27) and (C.28) into Equation (C.29) and evaluating the cross product yields

\[
\mathbf{v} = \left(-r \omega \sin \theta \cos \alpha + a_r \dot{\alpha} \cos \alpha - r \dot{\alpha} \cos \theta \sin \alpha\right) \mathbf{i} \\
+ \left(a_r \dot{\alpha} \sin \alpha + r \dot{\alpha} \cos \theta \cos \alpha - r \omega \sin \theta \sin \alpha\right) \mathbf{j} \\
+ \left(-r \omega \cos \theta\right) \mathbf{k} \]  \hspace{1cm} (C.30)

The moment vector applied to the plate, \( \mathbf{M} \), is equal to the time rate of change of the total angular momentum vector of the plate, \( \mathbf{H} \), or,

\[ \mathbf{M} = \dot{\mathbf{H}} \]  \hspace{1cm} (C.31)

where the total angular momentum is equal to

\[ \mathbf{H} = \int (\mathbf{r} \times \mathbf{v}) \, dm \]  \hspace{1cm} (C.32)

The only torque that is of concern is the torque applied to the shoe plate in the \( z \) direction (the axis of yoke rotation). This torque is equal to the time rate of change of the \( z \) component of the angular momentum vector, \( H_z \). Therefore, only the \( z \) component of the total angular momentum vector has to be calculated.

From Equation (C.32), The \( z \) component of the total angular momentum vector is equal to
\[ H_z = \int (r \times v) \cdot k \, dm \quad . \] (C.33)

Substituting Equation (C.27), representing \( r \), and Equation (C.30), representing \( v \), into Equation (C.33) and evaluating the \( z \) components of the cross product gives

\[
H_z = \int \left[ a_r^2 \alpha \sin^2 \alpha + a_r \dot{r} \alpha \cos \theta \sin \alpha \cos \alpha \right. \\
\left. + a_r \dot{r} \cos \theta \sin \alpha \cos \alpha + r^2 \dot{\alpha} \cos^2 \theta \cos^2 \alpha - r^2 \omega \sin \theta \cos \theta \sin \alpha \cos \alpha \right. \\
\left. - a_r \dot{r} \omega \sin \theta \cos^2 \alpha + a_r^2 \dot{\alpha} \cos^2 \alpha - a_r \dot{r} \cos \theta \sin \alpha \cos \alpha \right. \\
\left. + r^2 \omega \sin \theta \cos \theta \sin \alpha \cos \alpha - a_r \dot{r} \cos \theta \sin \alpha \cos \alpha + r^2 \dot{\alpha} \cos^2 \theta \sin^2 \alpha \right] \, dm \quad .
\] (C.34)

Substituting Equation (C.23), representing \( dm \), into Equation (C.34) and integrating yields

\[
H_z = m_r a_r^2 \dot{\alpha} (\sin^2 \alpha + \cos^2 \alpha) + I_r \dot{\alpha} (\sin^2 \alpha + \cos^2 \alpha) \\
= \dot{\alpha} (m_r a_r^2 + I_r^*) \quad ,
\] (C.35)

where \( I_r^* \) is the rectangular mass moment of inertia of the shoe plate about its center, and \( m_r \) is the mass of the shoe plate. Therefore, from Equation (C.31), the average torque applied to the plate in the \( z \) direction, \( \overline{T_r} \), is equal to the time rate of change of the angular momentum in the \( z \) direction,

\[
\overline{T_r} = \dot{H}_z = (m_r a_r^2 + I_r^*) \ddot{\alpha} = I_r \ddot{\alpha} \quad ,
\] (C.36)

where \( I_r^* \) is the rectangular mass moment of inertia of the shoe plate about the yoke pivot.
Equation (C.36) describes the net torque in the $z$ direction that must be applied to the shoe plate, to overcome its inertia, while the pump is operating. This torque is applied to the plate by the swashplate, through the piston slippers. Therefore, the effect of shoe plate inertia on the net torque applied to the swashplate will be equal to and opposite to this torque. Therefore, the average effect of the shoe plate inertia, $\bar{T}_r$, on the swashplate torque (in the positive direction) is

$$\bar{T}_r = -I_r \ddot{\alpha} \quad .$$

(C.37)
Appendix D
Flow Theory

This appendix will derive the theoretical flow equations required to
determine the viscous shear forces on the pistons, the viscous shear forces on
the control piston, and the leakage from the control piston. The theory is
based on finding exact solutions for the fundamental flow equations of
incompressible newtonian flow.

D.1. Fundamental Equations

The basic equations of viscous flow are the continuity and momentum
equations. For the case of incompressible newtonian flow with constant
transport properties, these equations, as shown by White [1], are:

Continuity : $\nabla \cdot \mathbf{V} = 0$ \hspace{1cm} (D.1)

Momentum : $\rho \frac{D\mathbf{V}}{Dt} = -\nabla \mathbf{p} + \mu \nabla^2 \mathbf{V}$ \hspace{1cm} (D.2)

where $\mathbf{V}$ is the velocity vector, and $\mathbf{p}$ is the total hydrostatic pressure (which
includes the effect of gravity). These equations can be explicitly solved for
only a limited number of laminar flow problems.

Consider the flow between the piston and the cylinder. The radius of
the piston is 5.147 mm and the radius of the cylinder is 5.163 mm. This
gives a hydraulic diameter of

$$D_h = \frac{4 \cdot \text{area}}{\text{perimeter}} = \frac{4\pi \left( (5.163 \times 10^{-3} \text{ m})^2 - (5.147 \times 10^{-3} \text{ m})^2 \right)}{2\pi (5.163 \times 10^{-3} \text{ m} + 5.147 \times 10^{-3} \text{ m})} = 2 \mu \text{m} \hspace{1cm} (D.3)$$
Assuming laminar flow, the pressure drop is represented by

$$\Delta P = \frac{32\mu lv}{D_h^2},$$  \hspace{1cm} (D.4)

where \( l \) is the length of the passage and \( v \) is the fluid velocity. For the pistons the average length is 22.9 mm, and the viscosity at 40°C is \( 61 \times 10^{-3} \) Pa·s. For a pressure drop of 20 MPa, the maximum for this pump, Equation (D.4) can be solved to give a fluid velocity of

$$v = 1.79 \text{ mm/s} \hspace{1cm} (D.5)$$

This results in a Reynolds number of

$$Re = \frac{\rho v D_h}{\mu} = 54 \times 10^{-6} \hspace{1cm} (D.6)$$

Clearly, the flow is laminar as assumed.

When the piston is inserted into the cylinder the average length of this annular flow region is 22.9 mm while the thickness of the region is 16 \( \mu \)m, or just .07% of the length. Therefore, the flow pattern can be assumed one-dimensional, with no end effects. Also, this allows the annular region between the piston and the cylinder to be “unwrapped” to change the problem to that of flow between parallel plates.

**Figure D-1:** Flow Between Parallel Plates
Figure D-1 shows the coordinates used to describe the flow between parallel plates. Note that one of the plates is allowed to move at a velocity $U$, while the other is fixed. The fluid only flows in the axial direction and therefore, the velocity vector becomes

$$ \mathbf{V} = u \mathbf{j} $$  \hspace{1cm} (D.7)

where $u$ is the velocity in the $y$ direction. It is assumed that the piston is concentric inside the cylinder and so the axial velocity $u$ does not vary with the angular position. Therefore,

$$ u = u(x,y) $$  \hspace{1cm} (D.8)

Substituting Equation (D.7) into the Continuity equation produces

$$ \frac{\partial u}{\partial y} = 0 $$  \hspace{1cm} (D.9)

This implies that the velocity $u$ does not vary with $y$ and hence, the axial velocity $u$ varies only with the variable $x$, or

$$ u = u(x) $$  \hspace{1cm} (D.10)

Since the annular region between the pistons and the cylinders is so thin, the mass of the fluid in this region is small. This means that the forces required to accelerate the fluid will be insignificant. The surface area of the fluid is not small though, and so the shear forces that act on the fluid will be significant. This implies that the flow between the parallel plates will be dominated by shear forces, not inertia effects. This is also shown by the magnitude of the Reynolds number calculated in Equation (D.6), because the Reynolds number is the ratio of inertial forces to viscous forces. Therefore, it can be assumed that the inertia effects are negligible when compared to the shear forces. If the inertia of the fluid is neglected, then the forces required to accelerate the fluid become zero and the problem can be treated as a steady state problem. For steady state,
\[
\frac{DV}{\rho Dt} = 0 \quad .
\] (D.11)

Again, assuming that the piston and cylinder are concentric, the pressure becomes a function of the axial displacement alone. Therefore,

\[
- \nabla \hat{p} = - \frac{d\hat{p}}{dy} \quad .
\] (D.12)

Also, the total hydrostatic pressure is not affected by gravity for two reasons. Usually, axial piston pumps are mounted on their sides and so the change in vertical height is zero in the axial directions. In addition, the effect of gravity is negligible when compared to the pressure gradients involved when the pump is operating. Therefore

\[
\frac{d\hat{p}}{dy} = \frac{dp}{dy} \quad .
\] (D.13)

Since the velocity is a function of \( z \) alone, the shear force term in the momentum equation reduces to

\[
\mu \nabla^2 V = \mu \frac{d^2 u}{dz^2} \quad .
\] (D.14)

Substituting Equations (D.11), (D.12), (D.13), and (D.14) into the momentum equation (D.2) yields the simplified momentum equation,

\[
0 = -\frac{dp}{dy} + \mu \frac{d^2 u}{dz^2} \quad .
\] (D.15)

To solve an equation such as (D.15), it is desirable to introduce dimensionless variables. The dimensionless velocity can be defined as

\[
u^* = \frac{u}{U} \quad ,
\] (D.16)

where \( U \) is the physical velocity of one of the plates. The dimensionless displacement can be defined as
\[ z^* = \frac{x}{h}, \quad (D.17) \]

where \( h \) is one half the radial clearance between the plates. Consequently, \( \frac{du}{dz} \) becomes

\[ \frac{du}{dz} = \frac{du}{du^*} \frac{dx^*}{dx} = \frac{U}{h} \frac{du^*}{dx^*}, \quad (D.18) \]

and

\[ \frac{d^2u}{dx^2} = \frac{d}{dx} \left( \frac{du}{dx} \right) = \frac{d}{dx} \left( \frac{U}{h} \frac{du^*}{dx^*} \right) = \frac{d}{dx^*} \left( \frac{U}{h} \frac{du^*}{dx^*} \right) \frac{dx^*}{dx} = \frac{U}{h^2} \frac{d^2u^*}{dx^2}. \quad (D.19) \]

Substituting Equation (D.19) into Equation (D.15) yields

\[ 0 = \frac{dp}{dy} + \frac{\mu U d^2u^*}{h^2 dx^2}. \quad (D.20) \]

The pressure drop will vary linearly with axial displacement. Thus, rearranging Equation (D.20),

\[ \frac{d^2u^*}{dx^2} = \frac{h^2 dp}{\mu U dy} = -B = \text{const}. \quad (D.21) \]

Note that \( B \) is a constant with respect to the displacement coordinates \( x \) and \( y \), but it can vary with time. Equation (D.21), can be easily solved using double integration.

\[ \frac{du^*}{dx^*} = -Bx^* + C_1 \quad (D.22) \]

and

\[ u^* = -\frac{Bx^{*2}}{2} + C_1 x^* + C_2. \quad (D.23) \]

Equation (D.23) gives the velocity profile subject to the boundary conditions. This profile can be used to find the shear forces that act on the surfaces.
D.2. Piston Shear Force

Figure D-2: Piston Flow Boundary Conditions

Referring to Figure D-2, the boundary conditions for the case of the piston moving inside the cylinder are that the piston surface moves at the velocity $U$, while the cylinder surface is fixed. The cylinder surface is located at $(x = h)$ or $(x^* = 1)$. Similarly, the piston surface is located at $(x^* = -1)$. Therefore

$$u^*(-1) = 1$$  \hspace{1cm} (D.24)

and

$$u^*(+1) = 0$$  \hspace{1cm} (D.25)

Subject to these boundary conditions the velocity profile becomes

$$u^* = \frac{1}{2} (1 - x^*) + \frac{B}{2} (1 - x^{*2})$$  \hspace{1cm} (D.26)

The shear force acting on the piston surface in the $y$ direction is described by
\[ \tau_{\text{wall}} = \mu \frac{du}{dx} \bigg|_{x=-h} = \frac{\mu U du^*}{h \, dz^*} \bigg|_{z^*=-1} = \frac{\mu U}{h} \left[ -\frac{1}{2} - B z^* \right]_{z^*=-1} = \frac{\mu U}{h} \left[ -\frac{1}{2} + B \right] \]. \tag{D.27}

The pressure gradient \( \frac{dp}{dy} \) is equal to the cylinder pressure \( P \) divided by the length that the piston is inserted inside the cylinder or

\[ \frac{dp}{dy} = \frac{P}{l_p} \]. \tag{D.28}

Note that the pump case pressure is assumed to be zero. Substituting Equation (D.28) into Equation (D.21) gives

\[ B = -\frac{h^2 P}{\mu l_p U} \tag{D.29} \]

and so the shear stress becomes

\[ \tau_{\text{wall}} = \frac{\mu U}{h} \left[ -\frac{1}{2} - \frac{h^2 P}{\mu l_p U} \right] \tag{D.30} \]

The shear stress is constant over the entire surface area of the piston and so the viscous shear force on the piston is

\[ F_{\text{shear}} = \frac{\mu U}{h} \left[ -\frac{1}{2} - \frac{h^2 P}{\mu l_p U} \right] 2\pi r_p l_p \tag{D.31} \]

where \( r_p \) is the piston radius. Setting the general clearance \( h \) equal to the piston-cylinder clearance \( h_p \) and the general wall velocity \( U \) equal to the piston velocity \( \dot{y}_p \) causes Equation (D.31) to become

\[ F_{\text{shear}} = -\frac{\pi r_p l_p \mu \dot{y}_p}{h_p} - 2\pi r_p h_p P \]. \tag{D.32} \]
The second term of Equation (D.32) is equal to the cylinder pressure times one half the annular area. It is desirable to combine this term with the pressure force acting upon the pistons. The easiest way to do this is to define the effective piston area \( A_p \) to be equal to

\[
A_p = \pi r_p^2 + 2\pi r_p h_p .
\]

(D.33)

This causes the second term in Equation (D.32) to be included in the pressure force term \( PA_p \) in Equation (3.9). Therefore, the viscous shear force on the piston is

\[
F_{ps} = -\frac{\pi r_p l_p \mu \dot{y}_p}{h_p} .
\]

(D.34)

If the inserted length \( l_p \) is set to \( l_{po} \) when the axial piston displacement \( y_p \) is zero, then the inserted length can be defined by

\[
l_p = l_{po} + y_p .
\]

(D.35)

Therefore, the shear force on the piston is

\[
F_{ps} = \frac{\pi r_p \mu \dot{y}_p (l_{po} + y_p)}{h_p}
\]

(D.36)

### D.3. Control Piston Shear Force

Figure D-3 shows the control piston and the control piston guide. It shows the hole that is in the piston guide that prevents the pump from being destroked to an angle less than zero. Leakage will flow through the hole and out through the gap between the piston and the guide. Therefore, the length of the leakage path \( l_{cl} \) is only from the hole to the end of the piston, as illustrated. The length of the viscous shear surface \( l_{cs} \) is the total
inserted length, as illustrated. The area from the hole to the end of the piston \((l_{cl})\) is denoted as Area 1, and the area from the end of the guide to the hole \((l_{cs} - l_{cl})\) is denoted as Area 2.

Considering Figure D-3, the boundary conditions for the control piston are that the control piston moves at velocity \(U\) while the control piston guide is fixed. The control piston is located at \((x=h)\) or \((x^*=1)\). Similarly, the guide is located at \((x^*=-1)\). Therefore,

\[
u^*(-1) = 0 \quad (D.37)
\]

and

\[
u^*(+1) = 1 \quad (D.38)
\]

Subject to these boundary conditions, Equation (D.23) becomes, (for both areas),

\[
u^* = \frac{1}{2}(1 + x^*) + \frac{1}{2}B(1 - x^*^2) \quad (D.39)
\]
The shear stress, in the $y$ direction, on the outer wall is

\[
\tau_{\text{wall}} = -\mu \frac{dx}{dx} \bigg|_{x=h} = -\frac{\mu U du^*}{h} \bigg|_{x^*=1} = -\frac{\mu U}{h} \left[ \frac{1}{2} - B x^* \right] \bigg|_{x^*=1} = -\frac{\mu U}{h} \left[ \frac{1}{2} - B \right] . \tag{D.40}
\]

For the control piston, the wall velocity is $\dot{y}_c$, and the clearance $h$ is $h_c$. Therefore, the shear stress is equal to

\[
\tau_{\text{wall}} = -\frac{\mu \dot{y}_c}{h_c} \left[ \frac{1}{2} - B \right] . \tag{D.41}
\]

Referring again to Figure D-3, the pressure gradients for Areas 1 and 2 are

\[
\left( \frac{dp}{dy} \right)_1 = -\frac{P_c}{l_{cl}} \tag{D.42}
\]

\[
\left( \frac{dp}{dy} \right)_2 = 0 \tag{D.43}
\]

and therefore the constants $B_1$ and $B_2$ equal

\[
B_1 = -\frac{h_c^2}{\mu \dot{y}_c} \left( \frac{dp}{dy} \right)_1 = \frac{h_c^2 P_c}{\mu \dot{y}_c l_{cl}} , \tag{D.44}
\]

\[
B_2 = -\frac{h_c^2}{\mu \dot{y}_c} \left( \frac{dp}{dy} \right)_2 = 0 \tag{D.45}
\]

Substituting these into Equation (D.40) causes the shear stresses to become
\[
\tau_{wall_1} = -\frac{\mu \dot{y}_c}{h_c} \left[ \frac{1}{2} - \frac{h_c^2 P_c}{\mu \dot{y}_c l_{cl}} \right],
\]
(D.46)

and

\[
\tau_{wall_2} = -\frac{\mu \dot{y}_c}{h_c} \left[ \frac{1}{2} \right].
\]
(D.47)

The total shear force on the control piston is

\[
F_{cs} = \tau_{wall_1} A_1 + \tau_{wall_2} A_2,
\]
(D.48)
or

\[
F_{cs} = -\frac{\mu \dot{y}_c}{h_c} \left[ \frac{1}{2} - \frac{h_c^2 P_c}{\mu \dot{y}_c l_{cl}} \right] 2\pi r_c l_{cl} - \frac{\mu \dot{y}_c}{h_c} \left[ \frac{1}{2} \right] 2\pi r_c (l_{cs} - l_{cl}),
\]
(D.49)

where \( r_c \) is the control piston radius. Evaluating Equation (D.49) gives the viscous shear force on the control piston.

\[
F_{cs} = -\frac{\pi r_c l_{cs} \mu \dot{y}_c}{h_c} + 2\pi r_c h_c P_c
\]
(D.50)

If the length \( l_{cs} \) is set to \( l_{cs_0} \) when the axial displacement of the control piston \( y_c \) is zero, then

\[
l_{cs} = l_{cs_0} + y_c
\]
(D.51)

Therefore, the viscous shear force on the control piston is

\[
F_{cs} = -\frac{\pi r_c \mu \dot{y}_c (l_{cs_0} + y_c)}{h_c} + 2\pi r_c h_c P_c.
\]
(D.52)
D.4. Control Piston Leakage

The leakage from the control piston could not be experimentally measured because it was mixed with the leakage from the pistons in the pump case. Therefore the control piston leakage was theoretically determined using the theoretical velocity profile developed in the previous section.

The leakage occurs in Area 1 where the theoretical velocity profile is described by Equation (D.39).

\[ u^* = \frac{1}{2}(1 + x^*) + \frac{1}{2}B_1(1 - x^2^2) \]  \hspace{1cm} (D.53)

From equation (D.16),

\[ u = u^*U = u^i \dot{y}_c \]  \hspace{1cm} (D.54)

Substituting the constant \( B_1 \), as defined by Equation (D.44), and Equation (D.53) into (D.54) gives the following velocity profile.

\[ u = \frac{\dot{y}_c}{2} \left(1 + x^* \right) + \frac{h_c^2 P_c}{2\mu l_{cl}} \left(1 - x^2 \right) \]  \hspace{1cm} (D.55)

The total leakage flow is simply

\[ q_{cl} = \int_A u dA \]  \hspace{1cm} (D.56)

The incremental area is

\[ dA = 2\pi r_c dx = 2\pi r_c h_c dx^* \]  \hspace{1cm} (D.57)

and the area is from \((x^* = -1)\) to \((x^* = 1)\). Therefore, Equation (D.56) becomes
\[ q_{cl} = \int_{-1}^{+1} \left[ \frac{\dot{y}_c}{2} \left( 1 + x^* \right) + \frac{H_c^2 P_c}{2 \mu_l cl} \left( 1 - x^{*2} \right) \right] 2\pi r_c h_c dx^* . \quad (D.58) \]

Evaluating the integral yields

\[ q_{cl} = 2\pi r_c h_c \dot{y}_c + \frac{4\pi r_c h_c^3 P_c}{3 \mu_l cl} \quad (D.59) \]

This leakage is for a concentric annular area. However, the area will tend to become eccentric because an eccentric annular area offers less resistance to flow than a concentric area. If a 50% relative eccentricity is assumed, then from White [1], the flow for an eccentric annular passage is approximately 50% greater than that for a concentric passage of equal area. Using this assumed eccentricity, the control piston leakage is equal to

\[ q_{cl} = 1.5 \left[ 2\pi r_c h_c \dot{y}_c + \frac{4\pi r_c h_c^3 P_c}{3 \mu_l cl} \right] = 3\pi r_c h_c \dot{y}_c + \frac{2\pi r_c h_c^3 P_c}{\mu_l cl} . \quad (D.60) \]

Note that for the shear force calculations the annular area had to be assumed concentric to allow a solution, and so the effects of eccentricity were not taken into account. However, any eccentricity will tend to increase the shear force and so when the control piston is moving, it will tend to become concentric to minimize the shear force. As a result, the annular area will become eccentric only when it is not moving, which is when the shear force is zero. Therefore, eccentricity should have little effect on the shear force calculations.

For the leakage calculations, it was not possible to measure the eccentricity and so an assumed value had to be used. This assumed value may not be correct, but it is better than assuming the annular area to be concentric.
If the leakage length $l_{cl}$ is set to $l_{cl_o}$ when the control piston displacement $y_c$ is zero, then

$$l_{cl} = l_{cl_o} + y_c \quad , \tag{D.61}$$

and then control piston leakage is

$$q_{cl} = 3\pi r_c h_c \dot{y}_c + \frac{2\pi r_c h_c^3 P_c}{\mu (y_c + l_{cl_o})} \quad . \tag{D.62}$$
D.5. References

[1] White F.
*Viscous Fluid Flow.*
Appendix E
Experimental Tests

E.1. Pump Leakage

The pump leakage was experimentally determined using a graduated cylinder and stopwatch arrangement. The results are shown in Figure E-1.

Figure E-1: Plot of Total Pump Leakage vs. Pump Pressure
As indicated by Figure E-1 the total leakage can be represented by a linear expression of the form

\[ Q_L = K_L P \quad \text{(E.1)} \]

Using linear regression techniques the value of \( K_L \) was found to be

\[ K_L = 0.431 \times 10^{-12} \text{m}^3/\text{s/Pa} \]

Equation (E.1), using this value, is also plotted in Figure E-1.

**E.2. Yoke Damping**

The damping torque that acts on the yoke was experimentally determined using the test stand illustrated in Figure E-2. For this test, the pump was disconnected, filled with hydraulic fluid, and mounted vertically. Also, the return spring, the spring cap, the control piston, the barrel (including pistons), and the compensator valving were removed. This allowed the yoke to rotate freely, being influenced only by yoke damping.

As shown in Figure E-2, a gravitational load was used to apply a torque to the yoke. A rod was inserted into the control piston guide and allowed to rest against the yoke. The rod was also attached to a displacement transducer, that was connected to a strip chart recorder. The strip chart recorder produced displacement versus time plots and the rod velocity could be found by measuring the slope of these plots.

To find the damping torque, a weight was attached to the rod and the rod was allowed to fall. The rod velocity would start at zero and then increase to a constant value. Once the rod velocity reached a constant value, the only resistance to the rod motion was yoke damping. Therefore, the damping torque for this value of rod velocity was equal to the torque applied to the yoke by the weight\( ^\dagger \).

\( ^\dagger \)The torque due to the mass of the yoke was negligible compared to the torque applied by the weight.
Figure E-2: Yoke Damping Experimental Test Stand

There were two limitations to this testing procedure.

1. The test could only be performed in one direction and as a result, the yoke damping could not be measured in both directions. Therefore, it had to be assumed that the damping would be equal in both directions.

2. The pump could not be running during the tests. When the pump is running there is vibration inside the pump that could cause the yoke damping in a running pump to be different than the damping measured with this test stand.

The velocity of the rod was measured instead of the angular velocity of the yoke. To find the damping torque as a function of the yoke angular
velocity, as required, the angular velocity had to be calculated from the measured rod velocity. From Equation (5.25),

\[ \dot{\alpha} = \frac{b \dot{\alpha}}{\cos^2(\alpha - C)} \]  \hspace{1cm} (E.2)

Therefore, the relationship between the angular velocity and the rod velocity is

\[ \dot{\alpha} = \frac{\cos^2(\alpha - C)}{b} \dot{y}_c \]  \hspace{1cm} (E.3)

Equation (E.3) was used to calculate the angular velocity of the yoke. The value of \( \alpha \) that was used was 0.16 rad, the median value of \( \alpha \).

The results of the testing are shown graphically in Figure E-3. The experimental results indicate that the yoke damping, \( T_d \), can be represented by a linear viscous term and an exponentially decaying stiction term, as follows.

\[ T_d = K_{d1} \dot{\alpha} + K_{d2} \frac{\dot{\alpha}}{|\dot{\alpha}|} e^{\frac{-|\dot{\alpha}|}{K_{d3}}} \], \hspace{0.5cm} \text{for} \ \dot{\alpha} \neq 0. \]  \hspace{1cm} (E.4)

Using the statistical computer package BMDP \[1\], the constants were found to be:

\[ K_{d1} = 0.413 \ \text{Nms} \]
\[ K_{d2} = 2.25 \ \text{Nm} \]
\[ K_{d3} = 5.11 \times 10^{-3} \ \text{s}^{-1} \]

Using these values, Equation (E.4) is also plotted in Figure E-3.
**Figure E-3:** Experimentally Determined Yoke Damping

### E.3. Yoke Inertia

Due to the complex geometry of the yoke, its mass moment of inertia was experimentally determined. This was achieved by rotating the yoke in the test stand illustrated in Figure E-4. This test stand consisted of a horizontal shaft mounted in frictionless air bearings. A gear was mounted on the shaft and a proximity sensor was mounted next to the gear. This allowed the rotational speed of the shaft to be measured by recording the gear tooth passing frequency with a storage oscilloscope.
Figure E-4: Yoke Inertia Experimental Test Stand

A cord was wrapped around the shaft and attached to a weight. To measure the inertia, the weight was released and allowed to fall a specific distance. When the weight had fallen this specific distance, the oscilloscope was "triggered" and the gear tooth frequency at this instant was stored and displayed on the scope. This allowed the angular velocity of the shaft at this instant to be determined.

The initial energy of the system was equal to the potential energy of the weight. After the weight had fallen, the final energy of the system was equal to the kinetic energy of the weight plus the rotational kinetic energy of the shaft. (The length that the weight fell was set so that there was no change in the potential energy of the yoke between the initial and final conditions.) Equating these two,
\[ m_w gl = \frac{1}{2} I_t \omega^2 + \frac{1}{2} m_w v^2 \]  

(E.5)

where \( m_w \) is the mass of the weight, \( l \) is the height that the weight falls, \( I_t \) is the inertia of the shaft plus the inertia of the yoke, \( \omega \) is the final angular velocity of the shaft, and \( v \) is the linear velocity of the weight after it has fallen. This linear velocity is equal to

\[ v = r \omega \]  

(E.6)

where \( r \) is the shaft radius. Combining Equations (E.5) and (E.6) yields,

\[ I_t = \frac{2m_w gl}{\omega^2} - m_w r^2 \]  

(E.7)

This test was performed for different weights and using Equation (E.7), the total mass moment inertia for each test was calculated. Averaging these results and subtracting the shaft moment of inertia gave the average measured mass moment of inertia of the yoke, about the yoke pivot, to be

\[ I_y = 0.94 \times 10^{-3} \text{ kgm}^2 \]

The swashplate and the pintles also rotate with the yoke. Both the swashplate and the pintles have simple geometry, and standard formulas [2] can be used to determine the mass moment of inertia for these components. The swashplate moment of inertia about the yoke pivot was calculated to be

\[ I_{sw} = 62.9 \times 10^{-6} \text{ kgm}^2 \]

and the total pintle inertia was calculated to be

\[ I_p = 3.9 \times 10^{-6} \text{ kgm}^2 \].

Adding these two values to the yoke mass moment of inertia gives the following yoke assembly mass moment of inertia about the yoke pivot.

\[ I = 1.0 \times 10^{-3} \text{ kgm}^2 \]
E.4. References

*BMDP Statistical Software Manual*

*CRC Standard Math Tables.*
Appendix F

Computer Simulations

F.1. Introduction to the DARE-UNIX Simulation System

All simulations in this study were done using the DARE-UNIX simulation system on a Dual 83/20 minicomputer. This system is a package of FORTRAN programs that can solve a set of simultaneous first-order ordinary differential equations, such as the state equations of a control system.

To use this system, the user must provide an input to DARE. This input is in the form of a single file that may contain up to 5 parts:

1. A derivative block where the first-order differential equations that mathematically describe the system are expressed using valid FORTRAN statements. There is no requirement that the equations be linear.

2. A logic block written in FORTRAN. This block “controls” the simulation by performing such functions as the initialisation of variables, setting up multi-run simulations, etc.

3. A subroutine block that contains any user defined FORTRAN subroutines.

4. A data block consisting of data inputs for the DARE system. This section is not written in FORTRAN and it must conform to a specific format.

5. An output block (or blocks) that allow the user to use DARE subroutines to produce listings and/or plots of the simulation results. Like the data block, this block is not written in FORTRAN.

All of these blocks are put together in a single input file and separated
by specific characters. When DARE is executed, the various blocks are split up and merged into the appropriate places in the existing DARE programs.

In return for providing many common routines, such as numerical integration and easy-to-use input/output procedures, DARE requires a particular naming convention for the variables used in the simulation. These conventions are:

- \( y(1) \) to \( y(100) \) - State Variables
- \( g(1) \) to \( g(100) \) - Derivatives of the State Variables.
  
  \[ g(1) \approx \frac{d}{dt} y(1) \]
- \( y(101) \) to \( y(200) \) - Defined Variables
- \( p(1) \) to \( p(100) \) - Defined Parameters

For a complete description of DARE refer to Reference [1].

F.2. Simulation of Pressure Effects

The torque due to pressure was found by simulating the cylinder pressure as it changed during the rotation of the barrel. This simulated pressure distribution was then used to numerically solve Equation (4.1), the integral that defines the torque due to pressure.

The equations that were used in this simulation are fully described and developed in Chapter 4. Some of the variables used in this simulation program are as follows:
State Variables:

\( y(1) \) - Cylinder Pressure \( P \)

\( y(2) \) - \[ \int \frac{-P_A (R \sin \theta - a \sin \theta)}{\cos^2 \alpha} d\theta \]

Defined Variables:

\( y(101) \) - Angular Position \( \theta \)

\( y(102) \) - \( \theta_1 \)

\( y(103) \) - \( \theta_2 \)

\( y(104) \) - \( \theta_3 \)

\( y(105) \) - \( \theta_4 \)

\( y(106) \) - \( \theta_5 \)

\( y(107) \) - \( \theta_6 \)

\( y(108) \) - \( P_S \)

\( y(109) \) - \( P_D \)

\( y(110) \) - Ang. velocity \( \dot{\alpha} \)

\( y(111) \) - \( V_p \)

\( y(112) \) - Actual Piston Area

\( y(113) \) - \( R \)

\( y(114) \) - Relief Notch Angle \( \gamma \)

\( y(115) \) - \( \omega \)

\( y(116) \) - \( K_l \)

\( y(117) \) - Swashplate Angle \( \alpha \)

\( y(118) \) - \( a \)

\( y(119) \) - Fluid Density

\( y(123) \) - \( K_o \)

\( y(124) \) - \( n_o \)

The following is a listing of the DARE input simulation file used to simulate the effects of pressure on the torque applied to the swashplate. Note that the "time" variable is the angular position \( \theta \), and that the derivatives are taken with respect to \( \theta \).

```
$dl
* Start of derivative block
* * Calculate angular position theta
*  y(101)=t
```
Geometry Calculations to speed up simulation

\[ \text{rsth} = y(113) \cdot \sin(y(101)) \]
\[ \text{sal} = \sin(y(117)) \]
\[ \text{cal} = \cos(y(117)) \]

Check cavitation

\[ \text{if}(y(1) \cdot 1.0 \cdot 1.0 \cdot 1.0) \cdot \text{y}(1) = -1.0 \cdot 1.0 \]

calculate derivatives depending upon the pressure region.

If in Region 1:

\[ \text{if}(y(101) \cdot 1.0 \cdot y(102)) \text{then} \]
\[ g(1) = 0 \]
\[ y(1) = y(109) \]

If in Region 2:

\[ \text{elseif}(y(101) > y(102) \cdot \text{and} \cdot y(101) > y(103)) \text{then} \]

calculate overlap flow:

\[ oarea = y(123) \cdot (y(103) - y(101)) \cdot y(124) \]
\[ dp = y(109) - y(1) \]
\[ \text{flow} = 0 \]
\[ \text{if}(dp \cdot \text{ne.} \cdot 0) \text{flow} = dp \cdot (\text{abs}(dp)) \cdot 0.5 / \text{abs}(dp) \]
\[ qin = oarea \cdot y(122) \cdot \text{flow} \]

calculate notch flow and leakage

\[ dp = y(1) - y(108) \]
\[ \text{flow} = 0 \]
\[ \text{if}(dp \cdot \text{ne.} \cdot 0) \text{flow} = dp \cdot (\text{abs}(dp)) \cdot 0.5 / \text{abs}(dp) \]
\[ qout = y(121) \cdot (y(101) - y(102)) \cdot 2 + \text{flow} \]
\[ \text{qleak} = 0 \]
\[ \text{if}(y(1) > y(106)) \text{qleak} = y(116) \cdot y(1) \]

Calculate dP/dw using Equation (4.33)

\[ g(1) = p(1) / (y(115) \cdot (y(111) - y(112)) \cdot (\text{rsth} \cdot \text{sal} - y(118) / \text{cal})) \]
\[ 1 \cdot (y(112) \cdot (\text{rsth} \cdot y(118) \cdot \text{sal} \cdot y(110) / \text{cal}) \cdot 2 + y(113) \cdot \cos(y(101)) \]
\[ 2 \cdot \tan(y(117)) \cdot y(115) + qin - qout - qleak \]

If in Region 3:
elseif(y(101).gt.y(103).and.y(101).le.y(104)) then

* * Calculate flowrates
*

qin=0
dp=y(1) - y(108)
flow=0
if(dp.ne.0)flow=dp*(abs(dp))*0.5/abs(dp)
qout=y(121)*(y(101)-y(102))**2*flow
qleak=0
if(y(1).gt.0)qleak=y(116)*y(1)

* * Calculate dP/dw using Equation (4.37)
*

g(1)=p(1)/(y(115)*y(111)-y(112)*(rsth*sal-y(118)/cal))
1 *(y(112)*(rsth-y(118)*sal)*y(110)/cal**2+y(113)*cos(y(101))
2 *(tan(y(117))*y(115))+qin-qout-qleak)

* * If in Region 4:
*
elseif(y(101).gt.y(104).and.y(101).le.y(105)) then

g(1)=0
y(1)=y(108)

* * If in Region 5:
*
elseif(y(101).gt.y(105).and.y(101).le.y(106)) then

* * Calculate flowrates
*
oarea=y(123)*(y(106)-y(101))**y(124)
dp=y(1) - y(108)
flow=0
if(dp.ne.0)flow=dp*(abs(dp))*0.5/abs(dp)
qout=oarea*y(122)*flow
dp=y(109)-y(1)
flow=0
if(dp.ne.0)flow=dp*(abs(dp))*0.5/abs(dp)
qin=y(121)*(y(101)-y(105))**2*flow
qleak=0
if(y(1).gt.0)qleak=y(116)*y(1)

* * Calculate dP/dw using Equation (4.46)
*
g(1)=p(1)/(y(115)*(y(111)-y(112)*(rsth*sal-y(118)/cal)))
1 *(y(112)*((rsth-y(118)*sal)*y(110)/cal**2+y(113)*cos(y(101))
2 *(tan(y(117))*y(115))+qin-qout-qleak)
* If in Region 6:

elseif(y(101).gt.y(106).and.y(101).le.y(107))then

* Calculate flowrates

qout=0
dp=y(109)-y(1)
flow=0
if(dp.ne.0)flow=dp*(abs(dp))**0.5/abs(dp)
qin=y(121)*(y(101)-y(105))**2*flow
qleak=0
if(y(1).gt.0)qleak=y(116)*y(1)

* Calculate dP/dw using (4.52)

g(1)=p(1)/(y(115)*(y(111)-y(112))*((rsth*sal-y(118)/cal)))
1*(y(112)*((rsth-y(118)*sal)*y(110)/cal**2+y(113)*cos(y(101)))
2*tan(y(117))*y(115)+qin-qout-qleak)

* If back in Region 1:

elseif(y(101).gt.y(107))then

g(1)=0
y(1)=y(109)
endif

* Calculate the integrand in Equation (4.1)
and set the derivative g(2) equal to it.

* g(2)=-y(1)*y(127)*(rsth-y(118)*sal)/cal**2

* End of derivative block

$1

* Logic block

* Initialise Variables

y(102)=1.344
y(103)=1.414
y(104)=1.728
y(105)=4.503
y(106)=4.555
y(107)=4.869
y(111)=1.05e-6
y(112)=82.23e-6
y(113)=.0224
\[ y(114) = 0.384 \]
\[ y(115) = 183.3 \]
\[ y(116) = 0.431 \times 10^{-12} / 4.5 \]
\[ y(118) = 1.45 \times 10^{-3} \]
\[ y(119) = 912. \]
\[ y(123) = 2.38 \times 10^{-4} \]
\[ y(124) = 1.4892 \]
\[ y(127) = 83.48 \times 10^{-3} \]

* Calculate some more constants

\[ y(120) = y(112) \times y(113) \]
\[ y(121) = (y(113) \times \tan(y(114))) \times 2 \times p(2) \times (2.0 / y(119)) \times 0.5 \]
\[ y(122) = 0.75 \times (2.0 / y(119)) \times 0.5 \]

* Open a data file for regression data

\[ \text{idata} = 9 \]
\[ \text{open(unit=9, file='press.rdat', status='new', buffered='unbuffered')} \]

* Open a data file for graphics data.

\[ \text{itell} = 8 \]
\[ \text{open(unit=itell, file='press.tag', status='new', buffered='unbuffered')} \]
\[ \text{write(itell,20)} \]
\[ \text{format(' INPUT DATA. ')} \]

* Do a multi-run simulation and vary the angular velocity from -25 rad/s to 25 rad/s.

\[ \text{call strof} \]
\[ \text{call nullog} \]
\[ \text{do 150,ivel=1,7} \]
\[ \text{if(ivel.eq.1)y(110)=-25.0} \]
\[ \text{if(ivel.eq.2)y(110)=-15.0} \]
\[ \text{if(ivel.eq.3)y(110)=-5.0} \]
\[ \text{if(ivel.eq.4)y(110)=0} \]
\[ \text{if(ivel.eq.5)y(110)=5.0} \]
\[ \text{if(ivel.eq.6)y(110)=15.0} \]
\[ \text{if(ivel.eq.7)y(110)=25.0} \]
\[ \text{write(itell,*)'y(110)=' ,y(110)} \]

* For each angular velocity, do a multi-run simulation and vary the pump differential pressure from 4 MPa to 20 MPa.

\[ \text{do 100,nsim=1,5} \]
Pump inlet and outlet pressures:

\[
y(108)=3000 \\
y(109)=3000+\text{nsim}\times4.0\times10^6 \\
diffp=(y(109)-y(108))/1.0\times10^6 \\
ndiffp=\text{int}(\text{diffp}) \\
\text{write(ite},125\text{)}\text{ndiffp} \\
\text{format(’”,12,” MPa”)}
\]

For each velocity and pressure combination, vary the swashplate angle, \(y(117)\), from 0 rad to 0.33 rad.

\[
do 100,\text{nalpha}=1,12 \\
y(117)=0.03*(\text{nalpha}-1)
\]

Reset simulation

\[
call \text{reset}
\]

Set initial cylinder pressure to discharge pressure

\[
y(1)=y(109)
\]

reset the integral to zero

\[
y(2)=0
\]

Run simulation

\[
call \text{run} \\
\text{write(*,*)’ Done simulation ’,isimul,’ ,ivel=’,ivel}
\]

After the simulation the state variable \(y(2)\) equals the integrand \(g(2)\) integrated over the simulation interval. Therefore, the torque is equal to:

\[
torque=9.0\times y(2)/(2.0\times3.14159)
\]

store output in special data files: One is for the linear regression analysis and one is for the plots

\[
diffpa=\text{diffp}\times y(117) \\
\text{write(0,70)}\text{torque,diffp,y(117),y(110),diffpa} \\
\text{write(ite},170\text{)}\text{y(117),torque} \\
\text{format(5e15.6)}
\]

100 continue
150 continue
write(itell,101)
101 format(' EOD. ')
*
*   End of logic block
*
$e
*   Subroutine block
*
subroutine nullog
  close(3)
  open(unit=3, file='/dev/null')
  return
end
*
*   End of subroutine block. The following is the data block
*
end
system
2,10,2
tmax
6.28319
dt
3.14159e-3
user
G P KAVANAGH
ident
PRESSURE SIMULATION-ALL
param
1.45e+9, 0.75
prange
1.5e+7, -1.0e+5, 1, 0, 1, 0
end

F.3. Dynamic Response Simulation

The simulation of the dynamic response was found by measuring a
dynamic line pressure and the resulting swashplate motion. Then, this
dynamic line pressure was used to simulate a theoretical swashplate motion
and this theoretical response was compared to the experimental response. To
facilitate this simulation, the control chamber pressure had to be determined
from the line pressure. This was done by performing a control volume
analysis of the control chamber.
From Equation (4.6), the equation that describes the dynamic behavior of the control piston chamber pressure is

$$\frac{dP_c}{dt} = \frac{\beta}{V_c} \left( -\frac{dV_c}{dt} + Q_{in} - Q_{out} \right)$$ \hspace{1cm} (F.1)

where $P_c$ is the chamber pressure, and $V_c$ is the chamber volume. The chamber volume is equal to

$$V_c = V_{c_0} - A_c y_c$$ \hspace{1cm} (F.2)

where $V_{c_0}$ is the chamber volume when the control piston displacement $y_c$ is zero. From Equation (F.2),

$$V_c = -A_c y_c$$ \hspace{1cm} (F.3)

The flow into the chamber was found by experimentally measuring the flow/pressure drop relationship for the fixed orifice that connects the chamber to the pump outlet line (refer to Figure 7-4). This relationship was found to be

$$Q_{in} = K_c \Delta P_c^n$$ \hspace{1cm} (F.4)

where:

$$\Delta P = P_{line} - P_c$$ \hspace{1cm} (F.5)

and the constants were empirically found to be:

$$K_c = 489.5 \times 10^{-6} \text{ m}^3/\text{s}/\text{Pa}^{0.536}$$
$$n_c = 0.536$$

To account for cases when the line pressure was less than the chamber pressure, Equation (F.4) was modified and the following expression was used to define $Q_{in'}$. 
\[
Q_{in} = \frac{\Delta P}{|\Delta P|} K_c |\Delta P|^n_c
\]  

(F.6)

The flow out of the chamber is leakage flow. This leakage flows out through the holes in the side of the control piston guide and out between the control piston and the guide. Depending on the location of the control piston, there are three cases of leakage.

**Case 1: Holes completely covered.**

When the holes are completely covered, the leakage from the chamber is a laminar flow between the control piston and the chamber. In Appendix D, on page 122, this leakage flow was analytically found to be,

\[
Q_{out} = 3\pi r_c h_c y_c + \frac{2\pi r_c h_c^3 P_c}{\mu(y_c + l_{cl_o})}
\]

For \( y_c \geq -l_{cl_o} + 1 \times 10^{-3} \text{m.} \)  

(F.7)

This is the expression that describes the leakage when the control piston covers the two holes in the sides of the control piston guide (Refer to Figure D-3 on page 117). Note that if the leakage path length \((y_c + l_{cl_o})\) becomes zero, then Equation (F.7) becomes infinite and invalid. Therefore, to prevent this, an arbitrary minimum leakage path length of 1mm was used, as indicated above by the restriction on \( y_c \).

**Case 2: Holes completely open.**

When the control piston does not cover the holes, the leakage increases greatly and it can be described by an orifice flow equation,

\[
Q_{out} = 2A_{hole} C_d \sqrt{\frac{2}{\rho}} P_c
\]

For \( y_c \leq -\left(l_{cl_o} + d_{hole}\right) \).  

(F.8)
Case 3: Holes partially open.

Between the cases of the holes completely covered or completely open, there is a transition region as the holes open. As the holes open, the leakage increases greatly. The actual relationship that describes this increase in leakage as a function of displacement is unimportant, because the length of this transition region is small. Therefore, in this transition region, the leakage was found by assuming that the leakage will increase linearly as the holes become progressively more open.

This region starts when the leakage path length equals $1\text{mm}$.  

$$y_e + l_{cl_o} = 1 \times 10^{-3} \text{ m} \quad , \quad \text{(F.9)}$$

or

$$y_e = -l_{cl_o} + 1 \times 10^{-3} \text{ m} = y_1 \quad . \quad \text{(F.10)}$$

At this point the leakage is described by Equation (F.7).

$$q_1 = 3 \pi r_c h_c y_e + \frac{2 \pi r_c h_c^3 P_c}{\mu (1 \times 10^{-3} \text{ m})} \quad \text{(F.11)}$$

This region ends when the holes are completely open, or

$$y_e + l_{cl_o} = -d_{\text{hole}} \quad , \quad \text{(F.12)}$$

or

$$y_e = -(l_{cl_o} + d_{\text{hole}}) = y_2 \quad . \quad \text{(F.13)}$$

The leakage at this point is defined by Equation (F.8) to be

$$q_2 = 2 A_{\text{hole}} C_d \sqrt{\frac{2}{\rho} P_c} \quad . \quad \text{(F.14)}$$
Between these two points a linear interpolation was used:

\[
Q_{out} = q_1 + \left( \frac{y_c - y_1}{y_2 - y_1} \right) \left( q_2 - q_1 \right)
\]

For \(-\left( l_{cl_o} + d_{hole} \right) < y_c < -l_{cl_o} + 1 \times 10^{-3} \text{ m.}\) \hspace{1cm} (F.15)

Substituting Equations (F.2), (F.3), and (F.6) into Equation (F.1) gives the following expression,

\[
\frac{dP_c}{dt} = \frac{\beta}{V_{c_o} - A_c y_c} \left( A_c \dot{y}_c + \frac{\Delta P}{|\Delta P|} K_c |\Delta P|^n_c - Q_{out} \right)
\]

where \(Q_{out}\) is described by either Equation (F.7), Equation (F.8), or Equation (F.15), depending upon the value of \(y_c\). Equation (F.16) was used to determine the chamber pressure from the measured line pressure.

To relate the motion of the swashplate to the chamber pressure, the models developed in Chapter 6 were used.

The following is a brief list of the variables used in the dynamic response simulation program.

<table>
<thead>
<tr>
<th>State Variables:</th>
<th>Defined Variables:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y(1)) - Swashplate Angle (\alpha)</td>
<td>(y(102)) - Fluid Viscosity (\mu)</td>
</tr>
<tr>
<td>(y(2)) - Ang. Velocity (\dot{\alpha})</td>
<td>(y(103)) - (\omega)</td>
</tr>
<tr>
<td>(y(3)) - Chamber Pressure (P_c)</td>
<td>(y(104)) - (\Delta P_P)</td>
</tr>
<tr>
<td></td>
<td>(y(105)) - (Q_P)</td>
</tr>
<tr>
<td></td>
<td>(y(106)) - (y_c)</td>
</tr>
</tbody>
</table>
**Derivatives:**

<table>
<thead>
<tr>
<th>$y(n)$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(107)$</td>
<td>$\dot{\gamma}_c$</td>
</tr>
<tr>
<td>$y(109)$</td>
<td>$P_{line}$</td>
</tr>
<tr>
<td>$y(110)$</td>
<td>$R$</td>
</tr>
<tr>
<td>$y(111)$</td>
<td>$a$</td>
</tr>
<tr>
<td>$y(112)$</td>
<td>$b$</td>
</tr>
<tr>
<td>$y(113)$</td>
<td>$C$</td>
</tr>
<tr>
<td>$y(114)$</td>
<td>$A'_p$</td>
</tr>
<tr>
<td>$y(115)$</td>
<td>$h_c$</td>
</tr>
<tr>
<td>$y(116)$</td>
<td>$l_{cl_0}$</td>
</tr>
<tr>
<td>$y(117)$</td>
<td>$l_{cs_0}$</td>
</tr>
<tr>
<td>$y(118)$</td>
<td>$r_p$</td>
</tr>
<tr>
<td>$y(119)$</td>
<td>$h_p$</td>
</tr>
<tr>
<td>$y(120)$</td>
<td>$l_{p_0}$</td>
</tr>
<tr>
<td>$y(121)$</td>
<td>$\alpha_{max}$</td>
</tr>
<tr>
<td>$y(122)$</td>
<td>$\alpha_{min}$</td>
</tr>
<tr>
<td>$y(124)$</td>
<td>$A'_c$</td>
</tr>
<tr>
<td>$y(127)$</td>
<td>$A'_p$</td>
</tr>
<tr>
<td>$y(128)$</td>
<td>$C_{d_{hole}}$</td>
</tr>
<tr>
<td>$y(129)$</td>
<td>$A_{hole}$</td>
</tr>
</tbody>
</table>

**Calculated Constants:**

<table>
<thead>
<tr>
<th>$y(n)$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(155)$</td>
<td>$bA_c$</td>
</tr>
<tr>
<td>$y(156)$</td>
<td>$mNR^2\alpha^2/2$</td>
</tr>
<tr>
<td>$y(157)$</td>
<td>$\pi r_c b^2/h_c$</td>
</tr>
<tr>
<td>$y(158)$</td>
<td>$-\pi Nr_p \mu/h_p$</td>
</tr>
<tr>
<td>$y(160)$</td>
<td>$3\pi r_c h_c^3$</td>
</tr>
<tr>
<td>$y(161)$</td>
<td>$\pi r_c h_c^3/\mu$</td>
</tr>
<tr>
<td>$y(162)$</td>
<td>$NA'<em>p R</em>\alpha/\pi$</td>
</tr>
<tr>
<td>$y(130)$</td>
<td>$K_s$</td>
</tr>
<tr>
<td>$y(131)$-$y(134)$</td>
<td>$K_{pr_1}$-$K_{pr_4}$</td>
</tr>
<tr>
<td>$y(135)$-$y(137)$</td>
<td>$K_{d_1}$-$K_{d_3}$</td>
</tr>
<tr>
<td>$y(138)$</td>
<td>$K_c$</td>
</tr>
<tr>
<td>$y(139)$</td>
<td>$n_c$</td>
</tr>
<tr>
<td>$y(140)$</td>
<td>$m_N$</td>
</tr>
<tr>
<td>$y(141)$</td>
<td>$I$</td>
</tr>
<tr>
<td>$y(142)$</td>
<td>$I_r$</td>
</tr>
<tr>
<td>$y(143)$</td>
<td>$m_{sc}$</td>
</tr>
<tr>
<td>$y(144)$</td>
<td>$m_c$</td>
</tr>
<tr>
<td>$y(145)$</td>
<td>$V_{e_0}$</td>
</tr>
<tr>
<td>$y(146)$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$y(147)$</td>
<td>$F_{sp_0}$</td>
</tr>
<tr>
<td>$y(148)$</td>
<td>$a_{sp}$</td>
</tr>
</tbody>
</table>
$d1$

check limits

\[
\text{if}(y(1).gt.y(122))y(1)=y(122) \\
\text{if}(y(1).lt.y(121))y(1)=y(121) \\
\text{if}(y(1).ge.y(122).\text{and}.y(2).gt.0)y(2)=0 \\
\text{if}(y(1).le.y(121).\text{and}.y(2).lt.0)y(2)=0 \\
\text{if}(y(3).le.0)y(3)=0.0
\]

geometry calculations to speed up simulation

\[
cosa=\cos(y(1)) \\
sina=\sin(y(1)) \\
tana=\tan(y(1)) \\
a2=2*y(1) \\
cos2a=\cos(a2) \\
cosa2=\text{cosa}^{\text{**}2} \\
cosa3=\text{cosa}^{\text{**}3} \\
cosa4=\text{cosa}^{\text{**}4} \\
cosa5=\text{cosa}^{\text{**}5} \\
sina2=\text{sina}^{\text{**}2}
\]

cora=y(1)-y(113) \\
cosca=\cos(cora)

\[
\text{calculate yc displacement } y(106) \text{ and velocity } y(107) \\
y(106)=y(112)\times\tan(cora) \\
y(107)=y(112)\times y(2)/\text{cosca}^{\text{**}2}
\]

\[
\text{find line pressure and set pump pressure equal to it} \\
y(109)=\text{pline}(t) \\
y(104)=y(109)
\]
calculate pump flowrate

\[ y(105) = y(162) \times \tan a \]

Calculate complete model coefficients

\[
\alpha = y(1) \\
c_1 = \tan(\alpha) + y(156) \times \sin a / \cos a + y(131)
\]

\[
c_2 = -y(157) \times ((y(117) + y(112) \times \tan(cora)) / \cos a) + y(135) + y(134) + \frac{y(158)}{\cos a}\times y(120) \times y(154) \times \cos a + y(120) \times y(153) \times \cos a \times \sin a - \frac{y(153)}{\cos a} \times y(110) \times \sin a - 0.5 \times y(111) \times y(154) \times \sin a - 0.5 \times y(111) \times \sin a \times 3 \times \sin a2
\]

\[
c_3 = -y(143) \times (y(148) \times y(149) \times \cos a + y(148) \times \sin a \times y(149) \times 2 \times \sin a \times \cos a - y(149) \times 2 \times \sin a) - \frac{y(140)}{\cos a} \times (y(154) \times \sin a \times y(153) \times \sin a \times \cos a - 2 \times y(153) \times \sin a \times 3) / \cos a5
\]

\[
\eta = y(143) \times (2 \times y(148) \times y(149) \times \sin a \times \cos a + y(148) \times \sin a \times y(149) \times 2 \times \sin a2 + \frac{y(149)}{\cos a} \times 2 \times \cos a + y(144) \times y(152) / \cos a \times 2 + y(140) \times (y(154) + 2 \times y(153) \times \sin a2 / (2 \times \cos a) + y(141) + y(142)
\]

calculate angular acceleration \( g(2) \)

\[
g(2) = (-y(3) \times y(155) + y(132) \times y(104) + y(133) \times y(104) + y(1) + c1 + c2 \times y(2) + c3 \times y(2) \times 2) / \eta
\]

\[
g(1) = y(2)
\]

check limits

if \((y(2) \leq 0)\) then
if \((y(1) \geq y(122) \text{ and } g(2) \geq 0)\) then
\(g(2) = 0\)
elseif \((y(1) \leq y(121) \text{ and } g(2) \leq 0)\) then
\(g(2) = 0\)
endif
endif

Calculate control piston chamber pressure

Calculate flow into chamber

\[
\text{flow} = 0 \\
\text{dp} = (y(109) - y(3)) \\
\text{if} (\text{dp} \neq 0) \text{flow} = (\text{abs} (\text{dp})) \times y(139) \times \text{dp} / \text{abs} (\text{dp}) \\
\text{qin} = y(138) \times \text{flow}
\]
Calculate leakage

\[
y_{\text{test}} = -y(116) + 1.0e-3
\]

if (y(106) .ge. y_{\text{test}}) then
  q_{\text{leak}} = y(160) \cdot y(107) + y(161) \cdot y(3) / (y(106) + y(116))
elseif (y(106) .ge. -14.9e-3) then
  q_1 = y(160) \cdot y(107) + y(161) \cdot y(3) / 1.0e-3
  q_2 = 2 \cdot y(128) \cdot y(129) \cdot y(3)^{0.5}
  q_{\text{leak}} = q_1 + (y(106) - y_{\text{test}}) \cdot (q_2 - q_1) / -3.1e-3
else
  q_{\text{leak}} = 2 \cdot y(128) \cdot y(129) \cdot y(3)^{0.5}
endif

Calculate the net flow into the chamber - dq_c

q_{\text{out}} = 0.0
q_{\text{move}} = -y(124) \cdot y(107)
dq_c = q_{\text{in}} - q_{\text{out}} - q_{\text{leak}} - q_{\text{move}}

Calculate chamber capacitance - cc

cc = (y(145) - y(124) \cdot y(106)) / y(146)

Calculate rate of change of chamber pressure - g(3)

g(3) = dq_c / cc

end of derivative block.

$1$

Logic block

First, read numerous constants out of the file 'constant.data'

open(unit=9, file='constant.data', status='old')
read(9, *, end=10) i, y(i)
goto 8
continue

Calculate further constants

\[
\begin{align*}
\pi &= 3.141592654 \\
y(150) &= \cos(y(113)) \\
y(151) &= \sin(y(113)) \\
y(152) &= y(112)^{2} \\
y(153) &= y(111)^{2} \\
y(154) &= y(110)^{2}
\end{align*}
\]
y(155)=y(112)*(y(124)-2*pi*y(114)*y(115))
y(156)=y(140)+y(154)*y(103)**2/2
y(157)=pi*y(114)*y(102)*y(112)**2/y(115)
y(158)=-pi+9*y(118)*y(102)/y(119)
y(159)=tan(y(113))
y(160)=3*pi*y(114)*y(115)
y(161)=2*pi*y(114)*y(115)**3/y(102)
y(162)=9*y(127)*y(110)*y(103)/pi

* call reset

* Get initial swashplate angle

write(*,*)'Initial Swashplate Angle?'
input y(1)

* Get initial line pressure y(109) and set
* pump pressure y(104) and chamber pressure y(3)
* equal to it.

y(109)=pline(0)
y(104)=y(109)
y(3)=y(109)
call strof

* Store initial variables in the Cross file.

* call store
write(*,*)'doing run number 1'

* do simulation for t=0 to t=1 msec and store results

* call run
call store
open(iolog,file='/dev/null',status='old')

* Repeat this until t=100 msec.

* do 100,i=2,100
write(*,*)'doing run number',i
call run
call store
100 continue

* End of logic block

$
Subroutine block
real function pline(point1)

Function that reads and interpolates the line pressure from
a look-up table

dimension table(500,2)
character*20 lfile,pfile
data npline/0/
data istart/1/

If this is the first call ask user for the file in which
the look-up table is stored

if(npline.eq.0)then
write(*,*)' Line Pressure file ?'
read(10,*)pfile
10 format(a12)
open(9,file=pfile,status='old')
do 15,i=1,500
read(9,*,err=16)table(i,1),table(i,2)
15 continue
16 npline=1
ir=i-1
endif

Perform interpolation.

if(point1.le.table(1,1)) goto 1
if(point1.ge.table(ir,1)) goto 2
if(point1.gt.table(istart,1))goto 6
istart=istart-1
goto 5
6 do 3 i=istart,ir
if((point1.ge.table(i,1)).and.(point1.lt.table(i+1,1))) goto 4
3 continue
4 pline=(table(i,2)+((table(i+1,2)-table(i,2))/(table(i+1,1)-table(i,1)))*(point1-table(i,1)))*1.0e+6
if(istart.ne.i)then
istart=i
endif
istart=i
return
1 pline=table(1,2)*1.0e+6
return
2 pline=table(ir,2)*1.0e+6
return
1000 write(iolog,10000)
write(ioerr,10000)
10000 format(' / ',10x,'<rpwl> ERROR -- two dimensional table.')</call stats
end
*
*
real function tsp(alpha)
*
* Look-up table for experimental spring torque
*
real spring(21,2)
data irow/21/
data icol/2/
data icall/0/
*
if(icall.eq.0)then
open(unit=10,file='spring.table',status='old')
do 10,ir=1,irow
read(10,*) spring(ir,1),spring(ir,2)
10 continue
icall=1
endif
*
* Call DARE interpolation routine
*
tsp=pwl(spring,21,2,1,alpha,0.0)
*
return
end
*
* End of subroutine block
*
end
system
3,4,0
tmax
100.0e-5
dt
10.0e-6
ident
Alpha vs. Line Pressure.
width
101
npoint
101
end
plotxt(cross,0.0,0.0,0)y(1)
Alpha vs Time.
end
plotxt(cross,0.0,0.0,0)y(3)
Comp. Press. vs. Time.
end
F.4. References

[1] Bolton, R.J.
_The DARE-VMS Simulation System_
Dept. of Electrical Engineering, University of Saskatchewan, 1985.
Appendix G
Physical Data

G.1. Coefficient Tables

The coefficients $C_1$ (theoretical), $C_2$, $C_3$, and $I_e$ are defined by Equations (6.5), (6.6), (6.7), and (6.8) respectively. In this section, the coefficients, the various terms in the describing equations, and the percentage effect of each term on the coefficient are calculated for three swashplate angles. These tables clearly show those terms that have a negligible effect on the dynamic and steady state response of a Vickers PVB5 axial piston pump.
<table>
<thead>
<tr>
<th>Term</th>
<th>$\alpha = 0$</th>
<th></th>
<th>$\alpha = 0.16 \text{ rad}$</th>
<th></th>
<th>$\alpha = 0.32 \text{ rad}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value (Nm)</td>
<td>% of $C_1$</td>
<td>Value (Nm)</td>
<td>% of $C_1$</td>
<td>Value (Nm)</td>
<td>% of $C_1$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>16.58</td>
<td>100</td>
<td>14.41</td>
<td>100</td>
<td>11.53</td>
<td>100</td>
</tr>
<tr>
<td>$F_{sp_0} b_{sp} \cos \alpha$</td>
<td>8.36</td>
<td>50.42</td>
<td>8.25</td>
<td>57.25</td>
<td>7.94</td>
<td>68.86</td>
</tr>
<tr>
<td>$F_{sp_0} a_{sp} \sin \alpha$</td>
<td>0</td>
<td>0</td>
<td>0.57</td>
<td>3.96</td>
<td>1.12</td>
<td>9.71</td>
</tr>
<tr>
<td>$K_{sp} a_{sp} b_{sp} \cos 2\alpha$</td>
<td>8.09</td>
<td>48.80</td>
<td>7.68</td>
<td>53.30</td>
<td>6.49</td>
<td>56.29</td>
</tr>
<tr>
<td>$K_{sp} a_{sp}^2 \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>0.54</td>
<td>3.75</td>
<td>1.03</td>
<td>8.93</td>
</tr>
<tr>
<td>$-K_{sp} b_{sp}^2 \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>-3.00</td>
<td>-20.82</td>
<td>-5.69</td>
<td>-48.35</td>
</tr>
<tr>
<td>$\frac{mNR^2 \omega^2 \sin \alpha}{2\cos^3 \alpha}$</td>
<td>0</td>
<td>0</td>
<td>0.23</td>
<td>1.60</td>
<td>0.52</td>
<td>4.51</td>
</tr>
<tr>
<td>$K_{p_{r1}}$</td>
<td>0.13</td>
<td>0.77</td>
<td>0.13</td>
<td>0.90</td>
<td>0.13</td>
<td>1.13</td>
</tr>
</tbody>
</table>

**Table G-1:** Comparison of the Terms of Coefficient $C_1$
<table>
<thead>
<tr>
<th>Term</th>
<th>$\alpha = 0$</th>
<th></th>
<th></th>
<th></th>
<th>$\alpha = 0.16 \text{ rad}$</th>
<th></th>
<th></th>
<th></th>
<th>$\alpha = 0.32 \text{ rad}$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value (Nms)</td>
<td>% of $C_2$</td>
<td>Value (Nms)</td>
<td>% of $C_2$</td>
<td>Value (Nms)</td>
<td>% of $C_2$</td>
<td>Value (Nms)</td>
<td>% of $C_2$</td>
<td>Value (Nms)</td>
<td>% of $C_2$</td>
<td>Value (Nms)</td>
</tr>
<tr>
<td>$C_2$</td>
<td>-0.544</td>
<td>100</td>
<td>-0.548</td>
<td>100</td>
<td>-0.555</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi r_c \mu b^2 l_{cs_0}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{\pi r_c \mu b^2 l_{cs_0}}{h_c \cos^2(\alpha - C)}$</td>
<td>-0.019</td>
<td>3.44</td>
<td>-0.018</td>
<td>3.27</td>
<td>-0.018</td>
<td>3.27</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{\pi r_c \mu b^2 \tan (\alpha - C)}{h_c \cos^2(\alpha - C)}$</td>
<td>0.006</td>
<td>-1.02</td>
<td>0.001</td>
<td>-0.21</td>
<td>-0.003</td>
<td>0.53</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- K_d$</td>
<td>-0.422</td>
<td>77.56</td>
<td>-0.422</td>
<td>76.96</td>
<td>-0.422</td>
<td>76.04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi N r_p \mu l_p R^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{\pi N r_p \mu l_p R^2}{h_p \cos^4 \alpha}$</td>
<td>-0.013</td>
<td>2.34</td>
<td>-0.013</td>
<td>2.45</td>
<td>-0.016</td>
<td>2.86</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi N r_p \mu l_p a^2 \sin^2 \alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{\pi N r_p \mu l_p a^2 \sin^2 \alpha}{h_p \cos^4 \alpha}$</td>
<td>0</td>
<td>0</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 \pi N r_p \mu a^2 R \sin^2 \alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{2 \pi N r_p \mu a^2 R \sin^2 \alpha}{h_p \cos^5 \alpha}$</td>
<td>0</td>
<td>0</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi N r_p \mu a R^2 \sin \alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{\pi N r_p \mu a R^2 \sin \alpha}{2 h_p \cos^5 \alpha}$</td>
<td>0</td>
<td>0</td>
<td>&lt; 0.001</td>
<td>-0.01</td>
<td>&lt; 0.001</td>
<td>-0.03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi N r_p \mu a^3 \sin^2 \alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$- \frac{\pi N r_p \mu a^3 \sin^2 \alpha}{2 h_p \cos^5 \alpha}$</td>
<td>0</td>
<td>0</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_{pr_4}$</td>
<td>-0.096</td>
<td>17.68</td>
<td>-0.096</td>
<td>17.54</td>
<td>-0.096</td>
<td>17.33</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table G-2:** Comparison of the Terms of Coefficient $C_2$
<table>
<thead>
<tr>
<th>Term</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.16$ rad</th>
<th>$\alpha = 0.32$ rad</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value (Nm$^2$)</td>
<td>% of $C_3$</td>
<td>Value (Nm$^2$)</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$2.2 \times 10^{-5}$</td>
<td>100</td>
<td>$-2.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>$-m_s c a_s p b_s p \cos 2\alpha$</td>
<td>$-4.5 \times 10^{-5}$</td>
<td>-204.07</td>
<td>$-4.2 \times 10^{-5}$</td>
</tr>
<tr>
<td>$-m_s c a_s^2 p \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>$-3.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>$m_s b_s^2 p \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>$1.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>$2m_r b_r^2 p \sin (\alpha-C)$</td>
<td>$6.7 \times 10^{-5}$</td>
<td>304.07</td>
<td>$1.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>$-m_s c a_s^2 p \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>$-1.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>$-m_s c a_s^2 \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>$-5.9 \times 10^{-8}$</td>
</tr>
<tr>
<td>$2m_s c a_s^2 \sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>$3.1 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Table G-3: Comparison of the Terms of Coefficient $C_3$
<table>
<thead>
<tr>
<th>Term</th>
<th>$\alpha = 0$</th>
<th>%</th>
<th>$\alpha = 0.16 \text{ rad}$</th>
<th>%</th>
<th>$\alpha = 0.32 \text{ rad}$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_e$</td>
<td>1.31\times10^{-3}</td>
<td>100</td>
<td>1.32\times10^{-3}</td>
<td>100</td>
<td>1.34\times10^{-3}</td>
<td>100</td>
</tr>
<tr>
<td>$2m_{sc}c_{sp}b_{sp}\sin \alpha \cos \alpha$</td>
<td>0</td>
<td>0</td>
<td>1.4\times10^{-5}</td>
<td>1.07</td>
<td>2.7\times10^{-5}</td>
<td>2.00</td>
</tr>
<tr>
<td>$m_{sc}c_{sp}^{2}\sin^{2}\alpha$</td>
<td>0</td>
<td>0</td>
<td>4.8\times10^{-7}</td>
<td>0.04</td>
<td>1.9\times10^{-6}</td>
<td>0.14</td>
</tr>
<tr>
<td>$m_{sc}b_{sp}^{2}\cos^{2}\alpha$</td>
<td>1.1\times10^{-4}</td>
<td>8.02</td>
<td>1.0\times10^{-4}</td>
<td>7.77</td>
<td>9.5\times10^{-5}</td>
<td>7.10</td>
</tr>
<tr>
<td>$m_{c}b^{2}$</td>
<td>1.5\times10^{-4}</td>
<td>11.79</td>
<td>1.5\times10^{-4}</td>
<td>11.24</td>
<td>1.5\times10^{-4}</td>
<td>11.21</td>
</tr>
<tr>
<td>$\cos^{2}(\alpha-C)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{m_{NR}^{2}}{2\cos^{4}\alpha}$</td>
<td>4.2\times10^{-5}</td>
<td>3.22</td>
<td>4.4\times10^{-5}</td>
<td>3.37</td>
<td>5.2\times10^{-5}</td>
<td>3.90</td>
</tr>
<tr>
<td>$\frac{m_{NA}^{2}\sin^{2}\alpha}{\cos^{4}\alpha}$</td>
<td>0</td>
<td>0</td>
<td>9.5\times10^{-9}</td>
<td>&lt;0.01</td>
<td>4.3\times10^{-8}</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>$I_r$</td>
<td>9.0\times10^{-6}</td>
<td>0.69</td>
<td>9.0\times10^{-6}</td>
<td>0.68</td>
<td>9.0\times10^{-6}</td>
<td>0.67</td>
</tr>
<tr>
<td>$I$</td>
<td>1.0\times10^{-3}</td>
<td>76.28</td>
<td>1.0\times10^{-3}</td>
<td>75.83</td>
<td>1.0\times10^{-3}</td>
<td>74.97</td>
</tr>
</tbody>
</table>

**Table G-4:** Comparison of the Terms of Coefficient $I_e$
G.2. Values of Physical Dimensions, Parameters, and Constants

The following table is a listing of all the physical dimensions, determined parameters, and constants for the pump used in this study, a Vickers PVB5 pressure compensated axial piston pump.

<table>
<thead>
<tr>
<th>Dimension or Constant</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>The axial distance from the piston-slipper joint axis point to the yoke pivot.</td>
<td>1.45x10^{-3} m</td>
</tr>
<tr>
<td>$a_{sp}$</td>
<td>The axial distance from the spring cap pivot point to the yoke pivot.</td>
<td>20.6x10^{-3} m</td>
</tr>
<tr>
<td>$A_c$</td>
<td>Effective control piston area</td>
<td>239.4x10^{-6} m^2</td>
</tr>
<tr>
<td>$A'_c$</td>
<td>Actual control piston area</td>
<td>239.9x10^{-6} m^2</td>
</tr>
<tr>
<td>$A_p$</td>
<td>Effective piston area</td>
<td>83.48x10^{-6} m^2</td>
</tr>
<tr>
<td>$A'_p$</td>
<td>Actual piston area</td>
<td>82.23x10^{-6} m^2</td>
</tr>
<tr>
<td>$b$</td>
<td>The distance between the center line of the control piston guide and the swashplate pivot.</td>
<td>50.0x10^{-3} m</td>
</tr>
<tr>
<td>$b_{sp}$</td>
<td>The radial distance from the spring cap pivot point to the yoke pivot.</td>
<td>48.5x10^{-3} m</td>
</tr>
<tr>
<td>$C$</td>
<td>Swashplate angle correction constant</td>
<td>0.212 rad</td>
</tr>
<tr>
<td>$C_{d_n}$</td>
<td>Discharge coefficient for the relief notch†</td>
<td>0.75</td>
</tr>
<tr>
<td>$C_{d_o}$</td>
<td>Discharge coefficient for the overlap area‡</td>
<td>0.75</td>
</tr>
<tr>
<td>$f_{\text{min}}$</td>
<td>Minimum high frequency fluctuation</td>
<td>263 Hz</td>
</tr>
<tr>
<td>$F_{sp_o}$</td>
<td>Force constant used in the calculation of $F_{sp}$</td>
<td>172.4 N</td>
</tr>
</tbody>
</table>

†From Zeiger [1]
‡From McCloy [2]
<table>
<thead>
<tr>
<th>Dimension or Constant</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_c$</td>
<td>One half the radial clearance between the control piston guide and the control piston</td>
<td>$8.5 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>$h_p$</td>
<td>One half the radial clearance between the pistons and the cylinders</td>
<td>$8 \times 10^{-6}$ m</td>
</tr>
<tr>
<td>$I$</td>
<td>Yoke assembly moment of inertia (includes swashplate and pintles)</td>
<td>$1.0 \times 10^{-3}$ kgm$^2$</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Pintles moment of inertia</td>
<td>$3.9 \times 10^{-6}$ kgm$^2$</td>
</tr>
<tr>
<td>$I_r$</td>
<td>Shoe plate moment of inertia</td>
<td>$9.0 \times 10^{-6}$ kgm$^2$</td>
</tr>
<tr>
<td>$I_{sw}$</td>
<td>Swashplate moment of inertia</td>
<td>$62.9 \times 10^{-6}$ kgm$^2$</td>
</tr>
<tr>
<td>$I_y$</td>
<td>Yoke moment of inertia</td>
<td>$0.94 \times 10^{-3}$ kgm$^2$</td>
</tr>
<tr>
<td>$\bar{I}_e$</td>
<td>Average total effective moment of inertia</td>
<td>$1.32 \times 10^{-3}$ kgm$^2$</td>
</tr>
<tr>
<td>$K_{d_1}, K_{d_1}$</td>
<td>Yoke damping torque constant</td>
<td>$0.422$ Nms</td>
</tr>
<tr>
<td>$K_{d_2}$</td>
<td>Yoke damping torque constant</td>
<td>$2.25$ Nm</td>
</tr>
<tr>
<td>$K_{d_3}$</td>
<td>Yoke damping torque constant</td>
<td>$5.11 \times 10^{-3}$ s$^{-1}$</td>
</tr>
<tr>
<td>$K_l$</td>
<td>Leakage constant for a piston</td>
<td>$95.8 \times 10^{-15}$ m$^3$s$^{-1}$Pa</td>
</tr>
<tr>
<td>$K_L$</td>
<td>Total leakage flow constant</td>
<td>$0.431 \times 10^{-12}$ m$^3$s$^{-1}$Pa</td>
</tr>
<tr>
<td>$K_o$</td>
<td>Overlap area constant</td>
<td>$0.238 \times 10^{-3}$ m$^2$</td>
</tr>
<tr>
<td>$K_{pr_1}$</td>
<td>Pressure torque constant</td>
<td>$0.128$ Nm</td>
</tr>
<tr>
<td>$K_{pr_2}$</td>
<td>Pressure torque constant</td>
<td>$0.725 \times 10^{-6}$ Nm/ Pa</td>
</tr>
<tr>
<td>$K_{pr_3}$</td>
<td>Pressure torque constant</td>
<td>$0.625 \times 10^{-6}$ Nm/ Pa</td>
</tr>
<tr>
<td>$K_{pr_4}$</td>
<td>Pressure torque constant</td>
<td>$-96.2 \times 10^{-3}$ Nms</td>
</tr>
<tr>
<td>$K_{sp}$</td>
<td>Return spring spring rate</td>
<td>$8100$ N/m</td>
</tr>
<tr>
<td>$l_{c1o}$</td>
<td>Length of the control piston chamber leakage path when $y_c = 0$</td>
<td>$11.8 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>Dimension or Constant</td>
<td>Description</td>
<td>Value</td>
</tr>
<tr>
<td>-----------------------</td>
<td>-------------</td>
<td>---------------</td>
</tr>
<tr>
<td>$l_{cs_o}$</td>
<td>Length of the control piston shear surface for $y_c = 0$</td>
<td>$36.3 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$l_{po}$</td>
<td>Length of piston insertion for $y_p = 0$</td>
<td>$22.9 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$l_{sc}$</td>
<td>Length of the spring cap</td>
<td>$7.58 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$l_{sp_o}$</td>
<td>Length of spring compression for $y_{sp} = 0$</td>
<td>$13.7 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass of one piston (average)</td>
<td>$0.01873$ kg</td>
</tr>
<tr>
<td>$m_c$</td>
<td>Mass of the control piston</td>
<td>$0.05911$ kg</td>
</tr>
<tr>
<td>$m_s$</td>
<td>Mass of the spring cap</td>
<td>$0.04471$ kg</td>
</tr>
<tr>
<td>$n_o$</td>
<td>Overlap area exponent</td>
<td>$1.489$</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of pistons</td>
<td>$9$</td>
</tr>
<tr>
<td>$r_c$</td>
<td>Control piston radius</td>
<td>$8.738 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$r_p$</td>
<td>Piston radius</td>
<td>$5.147 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$R$</td>
<td>Piston pitch radius</td>
<td>$22.38 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$S_1$</td>
<td>Simplified pump model constant$^\dagger$</td>
<td>$18.20$ Nm</td>
</tr>
<tr>
<td>$S_2$</td>
<td>Simplified pump model constant$^\dagger$</td>
<td>$20.29$ Nm</td>
</tr>
<tr>
<td>$S_3$</td>
<td>Simplified pump model constant</td>
<td>$-0.549$ Nms</td>
</tr>
<tr>
<td>$V_{c_o}$</td>
<td>Control cylinder volume for $y_c = 0$</td>
<td>$4.74 \times 10^{-6}$ m$^3$</td>
</tr>
<tr>
<td>$V_{p_o}$</td>
<td>Cylinder volume for $y_p = 0$</td>
<td>$1.05 \times 10^{-6}$ m$^3$</td>
</tr>
<tr>
<td>$\alpha_{\text{max}}$</td>
<td>Maximum swashplate angle</td>
<td>$0.313$ rad</td>
</tr>
<tr>
<td>$</td>
<td>\dot{\alpha}</td>
<td>_{\text{max}}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Angle of relief notch</td>
<td>$22^\circ$</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>Angular position which delimits the pressure regions</td>
<td>$77^\circ$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>Angular position which delimits the pressure regions</td>
<td>$81^\circ$</td>
</tr>
</tbody>
</table>

$^\dagger$ Based upon the experimental spring torque.
### Physical Dimensions and Constants

<table>
<thead>
<tr>
<th>Dimension or Constant</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_3$</td>
<td>Angular position which delimits the pressure regions</td>
<td>99°</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>Angular position which delimits the pressure regions</td>
<td>258°</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>Angular position which delimits the pressure regions</td>
<td>261°</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>Angular position which delimits the pressure regions</td>
<td>279°</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Pump rotational speed</td>
<td>183.3 rad/s</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>Approximate undamped natural frequency</td>
<td>104 rad/s</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Approximate damping ratio</td>
<td>2.0</td>
</tr>
</tbody>
</table>

**Table G-5: Values of Physical Dimensions and Constants**

#### G.3. Fluid Properties

The hydraulic fluid used in this study was Nuto H 68 hydraulic oil. The fluid properties that were used are listed in the following table. Note that all tests were performed at 40 °C, and therefore, the viscosity listed is the fluid viscosity at 40 °C.

<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Bulk Modulus</td>
<td>$1.45\times10^9$ Pa</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Viscosity</td>
<td>$61.0\times10^{-3}$ Ns/m²</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
<td>912 kg/m³</td>
</tr>
</tbody>
</table>

**Table G-6: Fluid Properties**
G.4. References

[1] Zeiger, G. and Akers, A.
Torque on the Swashplate of an Axial Piston Pump.

*Control of Fluid Power: Analysis and Design.*
John Wiley and Sons, 1980.