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An exposition on
"Linear Associative Algebra and Quaternions", submitted to the Committee on Graduate Studies, in candidacy for the degree of Master of Arts

by

Jean Stewart

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Part I

Linear Associative Algebra

Introduction.

Addition and Subtraction.

Associative Law for Addition
Commutative Law for Addition

Multiplication and Division

Associative Law for Multiplication.
Commutative Law for Multiplication

Distributive Law for Addition and Multiplication

Modulus.

Basis.

Idemfactors and Nilfactors. Idemfaciend; idemfacient;
Nilfaciend; nilfacient; idempotent; nilpotent.

Theorem

Separation of elements into four groups.
Part I.

Linear Associative Algebra.

Introduction.

Let us assume that there exists a system $E$ of distinct elements $a_1, a_2, a_3, \ldots \ldots$ and that there also exist processes of combinations in a specified order of any two of these elements or of any one with itself, each of which yields a determinate result which is also an element of the system $E$. These processes we shall denote by

1. the 'first direct operation' or as it is commonly called in Algebra 'addition'.
2. the 'second direct operation' or 'multiplication'
3. the 'first inverse operation' or 'subtraction'
4. the 'second inverse operation' or 'division'.

Two combinations of elements of the system are said to be equal when either may be substituted for the other in all relations between the elements without destroying any such relation.

Addition and Subtraction.

The process of finding the sum will be denoted, as it is in ordinary Algebra, by addition. Addition may obey the two laws, known as Associative Law and Commutative Law.
(1) Associative Law: For all values of i, j, and k which are significant in this discussion

\[(a_i + a_j) + a_k = a_i + (a_j + a_k)\]

(2) Commutative Law: For all values of i and j which are significant in this discussion.

\[a_i + a_j = a_j + a_i\]

It can be shown that if the associative law applies to three elements it applies to any number of elements, and also that if the commutative law applies to two elements it applies to any number of elements.

Of every pair of elements of \(\mathbb{E}\) one is the greater and the other the less, the relation being such that if

\[a_i > a_j\quad \text{and} \quad a_j > a_k\]

then \[a_i > a_k\]

If \[a_i > a_j\quad \text{and} \quad a_i > a_k\]

then \[a_i + a_j + a_k > a_j + a_k\]

If \[a_i > a_j, a_j > a_k, \ldots, a_{2m-1} > a_{2m}\]

then \[a_i + a_j + a_k + \ldots + a_{2m-1} > a_j + a_k + \ldots + a_{2m}\]

For any elements \(a_i\) and \(a_j\) possibly the same, of the system \(\mathbb{E}\) there is either just one element or no element which substituted for \(x\) satisfies the relation.

\[a_i + x = a_j\]

When such an element exists we denote it by \(a_j - a_i\) and the operation of performing it by subtraction.
Multiplication and Division

The process of finding the product will be denoted as it is in ordinary algebra by multiplication. The applicability of the following laws can be determined.

(1) Associative Law.

If $a_i$, $a_j$ and $a_k$ are three elements of $\mathbb{E}$

\[(a_i a_j) a_k = a_i (a_j a_k)\]

(2) Commutative Law.

If $a_i$ and $a_j$ are two elements of $\mathbb{E}$

\[a_i a_j = a_j a_i\]

(3) Distributive Law for Multiplication and Addition

If $a_i$, $a_j$ and $a_k$ are three elements of $\mathbb{E}$

then \[(a_i + a_j)a_k = a_i a_k + a_j a_k\]

and \[a_k(a_i + a_j) = a_k a_i + a_k a_j\]

In Linear Associative Algebra the associative law is assumed. It can be shown that if the associative law applies to three elements it applies to any number of elements; if the communicative law applies to two elements it applies to any number of elements and that if the distributive law applies to three elements it applies to any number.

For any elements $a_i$ and $a_j$, possibly the same, of the system $\mathbb{E}$, there is either just one element or no element which substituted for $x$ satisfies the relation $a_i x = a_j$.

When such an element exists we shall denote it by $a_j^{-1} a_i$ and the operation of performing it by division.
Modulus.

Is there an element $x$ such that
\[ a_i + x = a_i \]
or
\[ x + a_i = a_i \]
Such an element when it exists is called a modulus or principle element and will be denoted by $a_o$.

Is there an element $y$ such that
\[ a_i \cdot y = a_i \]
or
\[ y \cdot a_i = a_i \]
Such an element, $y$, when it exists is also called a modulus and will be denoted by $a_o$. Some systems have moduli, while others do not.

Basis.

Among the elements of $E$ there is a set consisting of a finite number of elements such that:

1. no linear function of these elements with coefficients ordinary complex numbers, not all zero, is equal to $a_o$.
2. every element of $E$ is expressible as a linear function of these elements with coefficients that are ordinary complex numbers.
3. every linear function of these elements with coefficients ordinary complex numbers is an element of the system $E$.

The set of elements minimum in number in terms of which every element of the system may be expressed linearly is called a basis of the system. The order of the system is
the number of the elements of the basis.

The axiom which states that if neither of the elements $a_i$ nor $a_j$ is equal to $a_*$, then $a_i \cdot a_j \neq a_*$ will be assumed in most cases.

**Idemfactors and Nilfactors.**

The terms 'facient', 'faciend' and 'factum' are often used instead of multiplier, multiplicand and product. When an expression used as a factor in certain combinations gives a product which vanishes, it is called a nilfactor. If it is the multiplier it is called nilfacient, if the multiplicand it is nilfaciend.

For example, if $\alpha \beta = \beta \alpha = 0$, $\alpha$ is a nilfactor. with respect to $\beta$, or vice versa.

In the first relation $\alpha \beta = 0$, $\alpha$ is nilfacient.
In the second relation $\beta \alpha = 0$, $\alpha$ is nilfaciend.

When an expression used as a factor in certain combinations yields a product which is the same as itself, it is called an idemfactor. When it is the multiplier, it is idemfacient, when the multiplicand, it is idemfaciend.

For example, if $\alpha \beta = \beta \alpha = \alpha$, $\alpha$ is an idemfactor.
If $\alpha \beta = \alpha$, $\alpha$ is idemfacient.
If $\beta \alpha = \alpha$, $\alpha$ is idemfaciend.

When an expression raised to some power vanishes, it is called nilpotent, but when raised to a square it gives itself as a result, it is called idempotent.
Theorem: In every linear associative algebra unless all the elements of a system are divisors of zero, there is at least one idempotent element.

Proof: If we have the elements \( \alpha, \alpha^2, \alpha^3, \ldots, \alpha^k \) in our system, some \( \alpha^r \) can be expressed linearly in terms of preceding elements, that is,

\[
\alpha^r + M_1 \alpha^{r-1} + M_2 \alpha^{r-2} + \cdots + M_r = 0
\]

Suppose \( \alpha \) is such a number that it is not a divisor of zero, that is, \( M_k \neq 0 \) then

\[
\alpha^r + M_1 \alpha^{r-1} + M_2 \alpha^{r-2} + \cdots + M_k = -M_k.
\]

or

\[
\frac{\alpha^r + M_1 \alpha^{r-1} + M_2 \alpha^{r-2} + \cdots + M_k}{-M_k} = 1.
\]

Thus there is a number \( \beta \) such that \( \alpha \beta = \alpha \).

Operating upon each of these by \( \beta \)

\[
\alpha \beta \cdot \beta = \alpha \beta.
\]

\[
\alpha \beta^2 = \alpha \beta.
\]

\[
\beta^2 = \beta,
\]

which proves that \( \beta \) is an idempotent element.

Thus systems may be of two sorts:

1. containing elements which are all divisors of zero.
2. containing at least one idempotent element.

The Four Groups

When there is an idempotent expression it may be
chosen as one of the elements of the basis. It shall now be shown that the remaining elements of the basis may be so selected as to be separable into four distinct groups. With reference to the idempotent element, which shall be chosen as an element of the basis, the elements of the first group are idemfactors; those of the second group are idemfacient and nilfacient; those of the third group are idemfacient and nilfaciend; and those of the fourth group are nilfactors.

Let \( a \) be the idempotent element.

Suppose that \( \alpha \) is a number such that

\[
\alpha = a, \beta = a, \beta = \alpha, a, \beta = a, \alpha.
\]

... \( \alpha \) is idemfaciend with respect to \( a \).

Assume that there are \( r \) linearly independent numbers which are idemfacient with respect to \( a \), \( a \), being one of them.

Let these numbers, represented by \( a, a_1, \ldots, a_\alpha \), form part of the basis. Let the rest of the basis consist of such numbers as \( a, a_{\alpha+1}, a_{\alpha+2}, \ldots, a_\lambda \).

Then the numbers of the set which are idemfaciend with respect to \( a \), can be written in the form

\[
x_1 a + x_1 a_1 + \ldots + x_\lambda a_\lambda
\]

Other numbers may be represented by

\[
x_1 a + x_1 a_1 + \ldots + x_\lambda a_\lambda + x_{\alpha+1} a_{\alpha+1} + \ldots + x_\lambda a_\lambda
\]
If \( s = r + k \) where \( k = 1, 2, 3, \ldots, n - r \),
\[
\begin{align*}
a_s & = (x, a_1 + x, a_2 + \ldots + x, a_r) + (x, a_{r+1} + \ldots + x, a_n) \\
    & = a_s (x, a_1 + x, a_2 + \ldots + x, a_r) + (x, a_{r+1} + \ldots + x, a_n)
\end{align*}
\]
from which we obtain
\[
a_s \left[ a_s - (x, a_1 + x, a_2 + \ldots + x, a_r) \right] = x, a_{r+1} + \ldots + x, a_n
\]
or \( a_s = x, a_{r+1} + \ldots + x, a_n \)

must be expressed linearly in terms of \( a_1, a_2, \ldots, a_r \), and the only way this can be done is to have all the coefficients, \( a_{r+1}, a_{r+2}, \ldots, a_n \), equal to zero.

Choose \( a_s' \) equal to \( a_s - (x, a_1 + x, a_2 + \ldots + x, a_r) \)

and change the rest of the basis after \( a_r \) to
\[
\begin{align*}
a_{r+1}' & \quad , a_{r+2}' & \quad \ldots \quad a_n'
\end{align*}
\]
Then \( a_s, a_{r+1}' \) \( = 0 \)

By a similar line of argument it can be shown that some of the \( a_s' \)s are idemfacient with respect to \( a_r \).
Suppose they are \( a_1, a_2, \ldots, a_k \)
where \( k < r \). The others from \( a_k \) to \( a_r \) are nilfacient.
Somewhere after \( a_k \) are idemfacient and the rest are nilfacient. Thus we have the four groups mentioned above.

Let the elements of the first group be denoted by
\[
a_1, a_2, a_3, \ldots, a_k
\]
those of the second group by \( e_1, e_2, e_3, \ldots, e_k \)
those of the third group by \( i_1, i_2, i_3, \ldots, i_m \)
Those of the fourth group by \( u_1, u_2, u_3, \ldots, u_m \)
To complete the multiplication table, it is necessary to find products such as \( a_r \cdot a_f \) - any 'a' by any other 'a', - , \( a_r \cdot e_f \) - any 'a' by any 'e', \( e_r \cdot a_f \) and others.

(1) Since \( a_r \), \( (a_r \cdot a_f ) = (a_r \cdot a_f ) a_r = a_r \cdot a_f \)
and \( (a_r \cdot a_f ) a_r = a_r (a_f \cdot a_r ) = a_r \cdot a_f \)
then \( a_r \cdot a_f \) is an idempotent with respect to \( a_r \)
and therefore can be expressed linearly in terms of the \( a_r \)s.
\[
d_r \cdot a_f = \sum_{s=1}^{s=K} x_s a_s.
\]

(2) \( a_r \cdot e_f \) may be expressed as
\[
= \sum_{s=1}^{s=K} x_s a_s + \sum_{s=1}^{s=h} y_s e_s + \sum_{s=1}^{s=m} z_s i_s + \sum_{s=1}^{s=n} w_s u_s.
\]
\( a_r (a_r \cdot e_f ) = a_r \sum_{s=1}^{s=K} x_s a_s + a_r \sum_{s=1}^{s=h} y_s e_s + a_r \sum_{s=1}^{s=m} z_s i_s + a_r \sum_{s=1}^{s=n} w_s u_s.\)

But \( a_r (a_r \cdot e_f ) = (a_r \cdot a_r ) e_f = a_r \cdot e_f \)
\( a_r \cdot e_f \) may be expressed as \( \sum_{s=1}^{s=K} x_s a_s + \sum_{s=1}^{s=h} y_s e_s.\)
\( (a_r \cdot e_f ) a_r = \sum_{s=1}^{s=K} x_s a_s a_r + \sum_{s=1}^{s=h} y_s e_s a_r = \sum_{s=1}^{s=K} x_s a_s.\)

But \( (a_r \cdot e_f ) a_r = a_r (e_f \cdot a_r ) = 0 \)
\( a_r \cdot e_f = \sum_{s=1}^{s=h} y_s e_s.\)

(3) \( a_r \cdot i_f = (a_r \cdot a_i ) i_f = a_r (a_i , i_f ) = 0.\)

The remaining products may be found by similar processes, yielding the following table:
<table>
<thead>
<tr>
<th></th>
<th>$d_i$</th>
<th>$e_i$</th>
<th>$i_i$</th>
<th>$u_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_p$</td>
<td>$\sum_{s=1}^{s=m} x_s d_s$</td>
<td>$\sum_{s=1}^{s=k} y_s e_s$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$e_p$</td>
<td>0</td>
<td>0</td>
<td>$\sum_{s=1}^{s=k'} x_s' d_s'$</td>
<td>$\sum_{s=1}^{s=k} y_s' e_s$</td>
</tr>
<tr>
<td>$i_p$</td>
<td>$\sum_{s=1}^{s=m} z_s i_s$</td>
<td>$\sum_{s=1}^{s=m} w_s u_s$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$u_p$</td>
<td>0</td>
<td>0</td>
<td>$\sum_{s=1}^{s=m} z_s' i_s$</td>
<td>$\sum_{s=1}^{s=m} w_s' u_s$</td>
</tr>
</tbody>
</table>
Part II.

Quaternions

Introduction:

Basis

Addition and Subtraction

Associative Law for Addition
Commutative Law for Addition

Multiplication

Associative Law for Multiplication
Distributive Law for Multiplication and Addition
Commutative Law for Multiplication

Division

Idempotent element
Zero-product

Conclusion.
Introduction:

From a copy of a letter from Sir William R. Hamilton to John T. Graves we get some idea of Hamilton’s train of thought when he began to work upon an extension of his ‘Theory of Couplets’. It was his desire to possess a ‘Theory of Triplets’, and in his work in that respect he discovered a theory of quaternions which includes such a theory of triplets.

The following summarizes his line of reasoning. Since the $\sqrt{-1}$ may be regarded as a line perpendicular to a line represented by 1, it seemed natural that there should be some other imaginary to express a line perpendicular to both the former; and because the rotation from 1 to this being doubled conducts to -1, it also ought to be a square root of negative unity, though not to be confused with the former. Call the old root i and the new one j.

Now the question arises as to what laws ought to be assumed for multiplying together $a + ib + jc$ and $x + iy + jz$. Hamilton at first assumed all three laws - the Associative, Distributive and Commutative and thus obtained the product

$ax - by - cz + i(ay + bx) + j(az + cx) + ij(bz + cy)$

but for a time was confused as to what $ij$ should be. One would be tempted to think that its square ought to be 1, since $i^2 = j^2 = -1$, and if this were the case $ij$ would be 1 or -1.

Hamilton was tempted at one time to consider $ij = 0$, because he wished to have a certain condition respecting the modululi fulfilled, namely, that the sum of the squares of the coefficients of $l$, $i$ and $j$ in the product is equal to the product of the corresponding sums of squares in the factors. Let us consider the simple example in which we have a square:

$$(a + ib + jc)^2 = a^2 - b^2 - c^2 + 2i ab + 2j ac + 2ijbc.$$  

Then $(a^2 - b^2 - c^2)^2 + (2ab)^2 + (2ac)^2 = (a^2 + b^2 + c^2)$ which satisfies the condition respecting the modululi if the term in $ij$ is neglected. However he decided that such an assumption was too harsh, and instead let $ij = \text{some number } k$, and $ji = -k$, not specifying whether $k$ should or should not be zero.

Let us investigate the general product of two triplets:

$$(a + ib + jc) (x + iy + jz) = ax - by - cz + i (ay + bx) + j(az + cx) + k(bz - cy)$$

Does the law of the modululi apply if we suppress $k$? If so, then $(a^2 + b^2 + c^2) (x^2 + y^2 + z^2)$ would have to be equal to $(ax - by - cz)^2 + (ay + bx)^2 + (az + cx)^2$.

But upon expanding we see that the first quantity is greater than the second by $(bz - cy)^2$ which is the square of the coefficient of $k$. At this time Hamilton decided that he
must admit a third distinct imaginary symbol \( k \), not to be confused with either \( i \) or \( j \), but equal to the product of the first as multiplier, and the second as multiplicand; and therefore was led to introduce quaternions, such as \( a + ib + jc + kd \), where \( a, b, c \) and \( d \) denote any real quantities, and are called the constituents of the quaternion, and \( i, j, \) and \( k \) are called vectors.

If the associative law is to apply when multiplying any two of, or the three of \( i, j, \) and \( k \), then, since \( i^2 = j^2 = -1 \), \( ij = k \) and \( ji = -k \).

\[
\begin{align*}
    ik &= i \cdot ij = ii \cdot j = -j. \\
    ki &= ij \cdot i = i \cdot ji = i(-k) = j. \\
    jk &= j \cdot ij = ji \cdot j = -ij \cdot j = -i \cdot jj = i \\
    kj &= ij \cdot j = i \cdot jj = -i. \\
    k &= ij \cdot ij = i \cdot ji \cdot j = -iijj = -1.
\end{align*}
\]

The assumptions concerning \( i, j \) and \( k \) now being complete may be summed up in the following table where we shall use the idempotent element 'e' instead of 1.
Let us now study the properties of quaternions of the form \( w + xi + yj + zk \) where \( e^2 = e \), and \( w, x, y \) and \( z \) are real quantities.

The two quaternions, \( w + xi + yj + zk \) and \( w'e + x'i + y'j + z'k \) are said to be equal

if \( w = w', x = x', y = y' \) and \( z = z' \)

**Addition and Subtraction**

The sums of the constituents of any two quaternions are the constituents of the sum of those two quaternions themselves.

If \( q = w + xi + yj + zk \)
and \( q' = w'e + x'i + y'j + z'k \)
then, \( q + q' = (w+w') e + (x+x')i + (y+y') j + (z+z') k \)

Does the associative law for addition hold, that is, does \((q+q') + q'' = q + (q'+q'')\)?

where \( q = we + xi + yj + zk \)

\[ q' = w'e + x'i + y'j + z'k \]

\[ q'' = w'' e + x'' i + y'' j + z'' k \]

\((q+q') + q'' = \left( [(w+w') + w''] e + [(x+x') + x''] i + [(y+y') + y''] j + [(z+z') + z''] k \right)\]

But \((w+w') + w'' = w + (w'+w'')\)

\((x+x') + x'' = x + (x'+x'')\) since the associative principle holds in ordinary algebra

\((y+y') + y'' = y + (y'+y'')\)

\((z+z') + z'' = z + (z'+z'')\)

\[ (q+q') + q'' = q + (q'+q'') \]

Does the commutative law for addition hold, that is does \( q+q' = q'+q \)?

\( q+q' = (w+w') e + (x+x')i + (y+y') j + (z+z') k \)

\( q'+q = (w'+w) e + (x'+x)i + (y'+y) j + (z'+z) k \)

But \( w+w' = w'+w \)

\( x+x' = x'+x \)

\( y+y' = y'+y \)

\( z+z' = z'+z \)

since the commutative principle applies in ordinary algebra.

\[ . . . . q+q' = q'+q \]
The differences of the constituents of any two quaternions are the constituents of the difference of those two quaternions themselves, that is,

\[ q - q' = (w-w')e + (x-x')i + (y-y')j + (z-z')k \]

**Multiplication:**

The product of \( q \cdot q' \) is defined by:

\[ q \cdot q' = (we + xi + yj + zk)(w'e + x'i + y'j + z'k) \]
\[ = w w'e^2 + w x'e'i + wy'e'j + wz'e'k + xw'ie + xx'i^2 \]
\[ + xy'i j + xz'ik + yw'je + yx'ji + yy'j^2 + yz'jk \]
\[ + zw'ke + zx'ki + zy'kj + zz'k^2 \]
\[ = (w'w - xx' - yy' - zz')e + (wx' + w'x + yz' -zy')i \]
\[ + (wy' + yw' - xz' + zx')j + (wz' + zw' + xy' -yx')k \]

Does the associative law for multiplication hold, that is, does \( (q \cdot q')q'' = q (q'q'') \)?

\[ (q \cdot q')q'' = [(w'w' - xx' - yy' - zz')e + (wx' + xw' + yz' -zy')i \]
\[ + (wy' + yw' - xz' + zx')j + (wz' + zw' + xy' -yx')k \]
\[ (w''e + x''i + y''j + z''k)](w'e + x'i + y'j + z'k) \]
\[ = (w''w'' - xx'' - yy'' - zz'')e + (wx'' + xw'' + yz'' -zy'')i \]
\[ + (wy'' + yw'' - xz'' + zx'')j + (wz'' + zw'' + xy'' -yx'')k \]
\[ + (ww'w'' - xx'x'' - yy'y'' - zz'z'')e + (wx'x'' + xw'x'' + yz'x'' -zy'x'')i \]
\[ + (wy'x'' + yw'x'' - xz'x'' + zx'x'')j + (wz'x'' + zw'x'' + xy'x'' -yx'x'')k \]
\[ + (ww'z^n - xx'z^n - yy'z^n - zz'z^n + wz'w^n + zw'w^n + xy'w^n
- yx'w^n + wx'w^n + zw'w^n + yz'y^n - zy'y^n - wy'x^n - zw'x^n
+ xz'x^n - zx'x^n) k \]

\[ q(q' \cdot q'') = (we + xi + yj + zk) \left[ \left( w'w^n - x'x^n - y'y^n - z'z^n \right) e + \left( w'x^n + x'w^n
+ y'z^n - z'y^n \right) j + \left( w'y^n + y'w^n - x'z^n + z'x^n \right) i + \left( w'z^n + z'w^n + x'y^n - y'x^n \right) k \right] \]

\[ = (ww'w^n - wx'x^n - wy'y^n - wz'z^n - xx'w^n - x'y'z^n + xz'y^n - yw'y^n
- yy'w^n + yx'z^n - yz'x^n - zw'z^n - zz'w^n - zx'y^n - yx'z^n - wy'y^n - zz'z^n) e + \left( w'w^n + x'w^n + y'w^n + zw'w^n - xx'x^n - xy'y^n - xz'z^n
+ yw'z^n + yz'w^n + yx'y^n - yy'x^n - zw'y^n - zy'w^n + zx'z^n - zz'x^n \right) i + \left( w'y^n + yw'w^n - wx'z^n + wz'x^n + yw'w^n - yx'y^n - yz'z^n
- xw'z^n - xz'x^n - yx'y^n - zy'x^n - zw'x^n + zx'w^n + zy'z^n - zz'y^n \right) j + \left( w'z^n + wz'w^n + wx'y^n - wy'x^n + zw'w^n - zx'y^n - zy'y^n - zz'z^n
+ xw'y^n + xy'w^n - xz'x^n + xz'x^n - yw'x^n - yx'y^n - yz'y^n \right) k \]

\[ q(q' + q'') = (we + xi + yj + zk) \left[ \left( w'w^n + x'w^n - xx'x^n - yy'y^n - zz'z^n \right) e + \left( wx'w^n + xw'y^n + yz'z^n - zy'y^n \right) i
+ \left( wy'w^n + yw'w^n - xz'x^n + zx'x^n \right) j + \left( wz'w^n + zw'y^n + xy'y^n - yx'y^n \right) k \right] \]

\[ qq' + qq'' = \left[ \left( ww'w^n - xx'w^n - yy'y^n - zz'z^n \right) e + \left( wx'x^n + yz'z^n - zy'y^n \right) i
+ \left( wy'w^n - xz'x^n + zx'x^n \right) j + \left( wz'w^n + zw'y^n - yx'y^n - yx'x^n \right) k \right] + \left[ \left( ww''w^n - xx''w^n - yy'y^n - zz'z^n \right) e + \left( wx''w^n + yz''z^n - zy'y^n \right) i \right] \]

Does the distributive law for multiplication apply, 

\[ q(q' + q'') = q(q' + q'') \]

\[ q(q' + q'') = (we + xi + yj + zk) \left[ \left( w'w^n + x'w^n - xx'x^n - yy'y^n - zz'z^n \right) e + \left( wx'w^n + xw'y^n + yz'z^n - zy'y^n \right) i
+ \left( wy'w^n + yw'w^n - xz'x^n + zx'x^n \right) j + \left( wz'w^n + zw'y^n + xy'y^n - yx'y^n \right) k \right] \]

\[ qq' + qq'' = \left[ \left( ww'w^n - xx'w^n - yy'y^n - zz'z^n \right) e + \left( wx'x^n + yz'z^n - zy'y^n \right) i
+ \left( wy'w^n - xz'x^n + zx'x^n \right) j + \left( wz'w^n + zw'y^n - yx'y^n - yx'x^n \right) k \right] + \left[ \left( ww''w^n - xx''w^n - yy'y^n - zz'z^n \right) e + \left( wx''w^n + yz''z^n - zy'y^n \right) i \right] \]
\[+(wy''+yw''-xz''+zx'')j + (wz''+zw''+xy''-yx'')k\]
\[=(ww''+ww''-xx''-xx''-yy''-yy''-zz''-zz'')e + (wx''+wx''+xw''+yw''+yz'')j + (wy''+wy''+yw''+yw''-xz''-xz''+zx''-zx'')j + (wz''+wz''+zw''+zw''+xy''+xy''-yx'')k\]
\[\therefore q(q'+q'') = q q' + q q''\]

The commutative principle did not hold when multiplying any two of the elements of the basis, so we can say that it does not apply in multiplying quaternions in general, since each element of the basis is a special case of a quaternion.

Saying that \(q \cdot q' = q' \cdot q\) would imply the following relations:
\[
\begin{align*}
ww' &-xx' -yy' -zz' = w'w -x'x -y'y -z'z \\
wx' +xw' -yz' +zy' & = w'x'x +w'y'z -z'y \\
wy' +yw' +xz' -zx' & = w'y'w -z'x +x'y \\
wz' +zw' +xy' -yx' & = w'z'z -x'w +y'x
\end{align*}
\]
whence \(\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}\).

Dealing with the three quaternions \(q, q'\) and \(q''\), note the special cases in which the commutative principle does apply:

when \(w = y = z = 0\), and \(x = e\); \(q = i\).
\[
w' = x' = z' = 0, \text{ and } y' = e \text{; } q' = j.
\]
\[
w'' = x'' = y'' = 0, \text{ and } z'' = e \text{; } q'' = k.
\]
\[
ijk = k \cdot k = k \cdot i \\
ik \cdot j = -j \cdot j = e = j \cdot (-j) = j \cdot k \\
jk \cdot i = i \cdot i = i \cdot k \\
j \cdot k = -k \cdot k = e = k \cdot (-k) = k \cdot j
\]
\[ ki \cdot j = j \cdot j = j \cdot ki \]
\[ kj \cdot i = -i \cdot i = e = i \cdot (-i) = i \cdot kj \]

**Division**

Division for quaternions is defined by

\[ \frac{q'}{q} = q'' \]

whence \( q = q' \cdot q'' \)

We have \( wi + xi + yj + zk = (w'e + x'i + y'j + z'k)(w''e + x''i + y''j + z''k) \)

\[ = (w'w'' - x'x'' - y'y'' - z'z'')e + (w'x'' + x'w'' + y'z'' + z'y'')i \]
\[ + (w'y'' + y'w'' - x'z'' + z'x'')j + (w'z'' + z'w'' + x'y'' - y'x'')k \]

\[ = w''w' - x'x'' - y'y'' - z'z'' = w \]
\[ x''w' + x'w'' - y'z'' + z'y'' = x \]
\[ y''w' + z'x'' + w'y'' - x'z'' = y \]
\[ z''w' - y'x'' + x'y'' + w'z'' = z \]

which can be solved for \( w'' \), \( x'' \), \( y'' \), and \( z'' \)

as long as

\[
\begin{vmatrix}
  w' & -x' & -y' & -z' \\
  x' & w' & -z' & y' \\
  y' & z' & w' & -x' \\
  z' & -y' & x' & w' \\
\end{vmatrix} \neq 0
\]

Expanding this determinate we get

\[ w'^{z} + x'^{z}z' + y'^{z} + x'^{z}z' + z'^{z} + w'^{z}y' + x'^{z} + w'^{z}y' \]

\[ = (w'^{z} + 2w'^{z}y' + y'^{z}) + (x'^{z} + 2x'^{z}z' + z'^{z}) \]

\[ = (w'^{z} + y'^{z}) + (x'^{z} + z'^{z})z' \]

which could not be zero unless \( w' = x' = y' = z' = 0 \), since \( w', x', y' \) and \( z' \) are all real quantities.

Thus, division is possible unless all the constituents are zero.
of the division are zero.

**Idempotent Element**

Can there be an idempotent element other than 1?

\[(we + xi + yj + zk)^2 = (w^2 - x^2 - y^2 - z^2)e + 2wx + 2wy + 2wz\]

If \(we + xi + yj + zk\) is to be idempotent

\[w^2 - x^2 - y^2 - z^2 = w\]

\[2wx = x\]

\[2wy = y\]

\[2wz = z\]

Then, either \(w = \frac{1}{2}\)

or \(x = y = z = 0\)

If \(w = \frac{1}{2}\), \(x^2 + y^2 + z^2 = -\frac{1}{4}\), which is impossible since \(x, y\) and \(z\) are real quantities.

If \(x = y = z = 0\), \(w^2 = w\), whence \(w = 0\) or 1.

Thus 1 is the only idempotent element.

**Zero-Product**

Could the product of two quaternions yield a zero-product, that is, could \(q \cdot q' = 0\)?

If \(q \cdot q' = 0\), \(ww' - xx' - yy' - zz' = 0\)

\[wx + xw' + yz - zy' = 0\]

\[wy + yw' - xz + zx' = 0\]

\[wz + zw' + xy - yx' = 0\]

\[ww' - xx' - yy' - zz' = 0\]

or \(xw' + wx' - zy' + yz' = 0\)

\[yw' + zx' + wy' - xz' = 0\]

\[zw' - yx' + xy' + wz' = 0\]
This set of equations has a solution provided,

\[
\begin{vmatrix}
  w & -x & -y & -z \\
  x & w & -z & y \\
  y & z & w & -x \\
  z & -y & x & w \\
\end{vmatrix} = 0
\]

or \((w^2 + y^2) + (x^2 + z^2) = 0\)

which can be true only if \(w = x = y = z = 0\).

Thus, the product of two quaternions, both different from zero, is not zero.

For this reason, there cannot be a nilpotent element in this system.

Conclusion.

The number of independent vectors in an algebra cannot be two, since the vector of \(ij\) is independent of \(i\) and \(j\). There may be no vector and in that case we have the ordinary algebra of reals; there may be only one vector, and in that case we have the ordinary algebra of imaginaries; or there might be three independent vectors, in which case we have the algebra of real quaternions.

It is now to be proven that we cannot have a fourth independent vector. Let us assume the two theorems, which can be proven:

1. American Journal of Mathematics - Volume IV.
   Extract on "Algebras in which Division is unambiguous" by C. S. Peirce
(1) The vector part of the product of two vectors is linearly independent of these vectors and of unity.

(2) If \( ij = s + u \), where \( s \) is a scalar and \( u \) a vector.

Then \( ji = s - u \).

Let \( i \) and \( j \) be two independent vectors such that \( ij = s + u \).

Let us substitute for \( j \)

\[ j' = s_i + j \]

Then \( ij' = u \)

\[ j'u = j,ij' = j^2i = i; \quad uj' = ij^2 = -i. \]

Suppose we have a fourth unit vector, \( k \), linearly independent of all the others, and let us write

\[ j_k = s' + u' \]

\[ k_i = s'' + u'' \]

Let us substitute for \( k \)

\[ k' = s''i + s'j' + k \]

and we get

\[ j_k' = -s''u + u'; \quad k_j' = s''u - u' \]

\[ k_i = s'u + u''; \quad ik' = s'u - u''. \]

Let us further suppose

\[ (ij,)k = s''' + u''' \]

Then because \( ij \) is a vector
$$k,(ij,) = a'' - u''$$

But

$$k,j, = -j,k, \quad k,i = -ik,$$

Hence

$$i\cdot j, k, = -i\cdot k,j, = -ik,j, = k,i j, = k,ij.$$ 

Therefore

$$a'' + u'' = s''' - u''.'$$

or $u''' = 0$, and the product of the two unit vectors is a scalar. These vectors cannot then be independent, or $k$ cannot be independent of $ij = u$. Thus it is proved that a fourth independent vector is impossible, and that ordinary real algebra, ordinary algebra with imaginaries, and real quaternions are the only associative algebras in which division by finites always yields an unambiguous quotient.