On the Value Group of Exponential and Differential Ordered Fields

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Abstract

The first chapter comprises a survey of valuations on totally ordered structures, developing notation and properties. A contraction map is induced by the exponential map on the value group G of an ordered exponential field K with respect to the natural valuation v_G. By studying the algebraic properties of Abelian groups with contractions, the theory of these groups is shown to be model complete, complete, decidable and to admit elimination of quantifiers. Hardy fields provide an example of non-archimedean exponential fields and of differential fields and therefore, they play a very important role in our research.

In accordance with Rosenlicht we define asymptotic couples and then give a short exposition of some basic facts about asymptotic couples. The theory T_P of closed asymptotic triples, as defined in Section 2.4, is shown to be complete, decidable and to have elimination of quantifiers. This theory, as well as the theory T of closed H-asymptotic couples do not have the independence property. The main result of the second chapter is that there is a formal connection between asymptotic couples of H-type and contraction groups.

A given valuation of a differential field of characteristic zero is a differential valuation if an analogue of l'Hospital's rule holds. We present in the third chapter, a survey of the most important properties of a differential valuation. The theorem of M. Rosenlicht regarding the construction of a differential field with given value group is given with a detailed proof. There exists a Hardy field, whose value group is a given asymptotic couple of Hardy type, of finite rank. We also investigate the problem of asymptotic integration.

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To my parents Hermina and Ioan Haiaş

In Memoriam Prof. univ. dr. Gheorghe Nadiu (1941-1998), University of Oradea, Romania

"The geniuses consume themselves faster."

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Introduction

An exponential field is an ordered field $(K, <)$ that has an isomorphism from the additive group of K to the multiplicative group of positive elements of K. On ordered fields, a very important subclass of valuations are the convex valuations, that is, valuations, which are compatible with the ordering of the corresponding field.

The main focus of my thesis is on contraction groups which were firstly introduced in *Abelian Groups with Contractions* by F.-V. Kuhlmann [9]. More precisely, the aim was to construct for a given contraction group G, an exponential field for which the value group of the natural valuation is G.

A further important concept for the value group G of a convex valuation of an ordered field K is the asymptotic couple (G, ψ), where ψ is a map from G^* to G, satisfying certain axioms. The notion of asymptotic couple is due to M. Rosenlicht, see *On The Value Group Of A Differential Valuation II* [19]. Assume that K is a Hardy field and let v be the natural valuation on K, with value group G. Let the map $\psi : G^* \to G$ be given by $\psi(v(f)) := v(f'/f)$ for every $f \in K^*$ with $v(f) \neq 0$. Then ψ is well-defined, and (G, ψ) is an example of an asymptotic couple.

In *Some Remarks About Asymptotic Couples* [1], M. Aschenbrenner revealed a formal connection between asymptotic couples and contraction groups. M. Rosenlicht proved in *On The Value Group Of A Differential Valuation I* [17], that for certain asymptotic couples (G, ψ) there exists a differential field for which the value group of a differential valuation is G. One question that arises is the following: From a given contraction group how do we get a corresponding asymptotic couple, or vice-versa? Following the same steps as in the construction of a differential field with a given asymptotic couple as its value group, we wanted to construct an exponential field with a given contraction group as its value group. One approach was to consider exponential Hardy fields, since they are closed under both, exponentiation and derivative. My thesis gathers all the useful information for future research for answering our open question.

Both the algebraic and the model-theoretical properties of contraction groups and asymptotic couples are very interesting. This requires to study the theories of closed asymptotic triples (G, ψ, P), where (G, ψ) is an asymptotic couple and P a cut of (G, ψ), and of non-trivial divisible centripetal precontraction groups. These theories are complete, they are decidable, and they even admit elimination of quantifiers.

This thesis consists of three chapters. In the first chapter we introduce the contraction map, and we study the theory of contraction groups. In the second chapter we introduce the asymptotic couples and discuss important aspects of their algebraic and model-theoretic properties together with their relation to contraction groups, as presented by M. Aschenbrenner. The last chapter deals with the construction of a differential field with given value group (in particular a Hardy field) and also with the problem of asymptotic integration.

The first three sections of Chapter 1 are introductory in nature and most of the information we gathered comes from the lecture notes of "Ordered Structures" and "Valuation Theory", given by my supervisors Salma Kuhlmann and Franz-Viktor Kuhlmann. Then I show that the exponential function induces canonically a map (called a contraction) on the value group G of an ordered exponential field K with respect to the natural valuation. Some of the results which are discussed in this chapter can be found in *Ordered Exponential*

Fields, by Salma Kuhlmann [12]. The elementary class of non-trivial divisible centripetal contraction groups has a very well-behaved model-theory, as presented in detail in [9]. We selected only those results that are similar for the theory of closed asymptotic couples.

In Chapter 2 we essentially follow the presentation in the paper [1]. We expose the main algebraic facts about asymptotic couples. Then we present selected model-theoretic results about closed asymptotic couples. To be precise, the theory T_P of closed asymptotic triples does not have the independence property. A theory T in the language $\mathcal{L}$ is said to have the independence property if all $\mathcal{L}$-formulas $\varphi(x, y)$ have the independence property with respect to all models of T, as defined in section 2.5. In addition T_P is a complete and decidable theory, which admits elimination of quantifiers. In the last section we show that asymptotic couples resemble contraction groups and this is the crucial link to our open question mentioned above.

In Chapter 3, we start working with fields of functions of a real or complex variable, which are closed under differentiation. Moreover, we stipulate that each function approaches a limit, possibly ∞, as the variable approaches some fixed limit, in such a way that a certain version of l'Hospital's rule holds. This is given by a differential valuation v of a differential field k of characteristic zero. Further, we analyze algebraic properties of differential valuations as in *Differential Valuation*, by M. Rosenlicht [18]. Then we apply these properties to show that any asymptotic couple (G, ψ) arises from a differential valuation of a differential field, at least if the ordered subset $\psi(G^*)$ of G, in the opposite ordering, is well-ordered. This is the first main result of [17]. As an application of this result, we state the main theorem from [19] which says that under certain assumptions, there exists a Hardy field whose asymptotic couple of Hardy type is (G, ψ). The other main result of [17] uses the asymptotic couple (G, ψ) to solve the problem of asymptotic integration: for $a \in k^*$, we want to find an antiderivative $\int a$ of a in k, or, if this is not possible, to find an element $b \in k$ whose derivative is near a.

We intend that the present thesis, in which we summarize the results of our research in the area of value groups of exponential and differential field, provides a consistent achievement for more in-depth studies in this remarkable branch of ordered algebraic structures.

Chapter 1

Ordered Structures

We start this chapter working on ordered Abelian groups and then we analyze some conditions that an ordered exponential field K has to satisfy. The contraction map is introduced as a map that is induced by the exponential on the value group G of K with respect to the natural valuation. We present the axioms for contractions on ordered Abelian groups and the model theory of non-trivial divisible contraction groups. A review of notions and terminology about Hardy fields is exposed following *Hardy Fields* [20], by M. Rosenlicht.

1.1 Totally Ordered Abelian Groups

An *ordered Abelian group* G written additively is an Abelian group together with a total ordering (on the underlying set) which is compatible with the addition, i.e.:
for any $x, y, z \in G : x < y \rightarrow x + z < y + z$. An Abelian group G is *divisible* if it satisfies:

$$\forall x \in G \; \exists y \in G : ny = \underbrace{y + y + ... + y}_{n-times} = x, \text{ whenever } 0 \neq n \in \mathbb{N} \qquad .$$

Note that if G is divisible, then G is a $\mathbb{Q}$-vector space.

We put $|x| = \max \{x, -x\}$. For non-zero $x, y \in G$, we will say that x is *Archimedean equivalent to* y and write $x \sim^{+} y$ if there exists $n \in \mathbb{N}$ such that

$$n|x| \geq |y| \text{ and } n|y| \geq |x|.$$

We say that x is *infinitely smaller* than y and write $x \ll^+ y$ if $n|x| < |y|$ for all $n \in \mathbb{N}$.

Remarks: $\sim^+$ is an equivalence relation and $\ll^+$ is compatible with this equivalence relation:

$$x \ll^+ y \text{ and } x \sim^+ z \Rightarrow z \ll^+ y$$

$$x \ll^+ y \text{ and } y \sim^+ z \Rightarrow x \ll^+ z.$$

Further we have:

$$x \ll^+ y \text{ and } y \ll^+ z \Rightarrow x \ll^+ z.$$

We denote by $[x]$ the equivalence class of $x \neq 0$, where $[x] := \{y |\ y \sim^+ x\}$.

An ordered Abelian group G is *Archimedean* if $\sim^+$ has at most two equivalence classes, that is, for every x, y non-zero elements of G, there exists $n \in \mathbb{N}$ such that:

$$n|x| \geq |y| \text{ and } n|y| \geq |x|.$$

Theorem 1.1.1 ([8], Hölder)**.** *If G is an Archimedean ordered group, then it is isomorphic to a subgroup of* $(\mathbb{R}, +, 0, <)$.

Note that $\mathbb{R}$ is the largest Archimedean ordered group.

Denote by Γ the set of equivalence classes of nonzero elements. Then

$$\Gamma := G \setminus \{0\}/ \sim^+ = \{[x]|\ x \in G, x \neq 0\}.$$

We define an order on Γ in the following way:

$$[y] < [x] \text{ if and only if } x \ll^+ y.$$

By the properties mentioned above, this order is well defined, that is, if $x \sim^+ x'$ and $y \sim^+ y'$, then $x \ll^+ y$ if and only if $x' \ll^+ y'$. We can say that Γ is a *chain*, (i.e., a totally ordered set). Observe that for every non-zero $r \in \mathbb{Z}$, we have that $rx \sim^+ x$.

Let Γ be an ordered set. Then $\Gamma \cup \{\infty\}$ is the set Γ together with a new element ∞, the ordering extended such that $\gamma < \infty$ for all $\gamma \in \Gamma$. If Γ is also an ordered Abelian group then + is extended to $\Gamma \cup \{\infty\}$ by $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$

Let G be an Abelian group. A *(group) valuation* of G is a mapping v from G onto $\Gamma \cup \{\infty\}$, where Γ is a totally ordered set, such that for any $x, y \in G$, we have

1. $v(x) = \infty \Leftrightarrow x = 0$
2. $v(x + y) \geq \min\{v(x), v(y)\}$ (Ultrametric triangle law)
3. $v(-x) = v(x)$

We call $vG := \Gamma$ *the value set of the valued group* (G, v). Note that by this definition, $vG = \{v(g) | 0 \neq g \in G\}$.

Immediate consequences are: For $g_1, ..., g_n \in G$

- $v(\sum_{1\leq i\leq n} g_i) \geq \min_{1\leq i\leq n} v(g_i)$
- $v(\sum_{1\leq i\leq n} g_i) = \min_{1\leq i\leq n} v(g_i)$ if all $v(g_i)$ are distinct.

Proposition 1.1.1. *Let G be an ordered group. The map*

$$v_G : G \to \Gamma \cup \{\infty\}$$

$$x \mapsto [x]$$

is a valuation on G.

The proof is straightforward.

We will call v_G *the natural valuation* on G. Observe that v_G is compatible with $<$ on G. In fact,

if for any $x, y \in G$, $x > 0$, $y > 0$ and $x \not\sim^+ y$, then $v_G(x) < v_G(y) \Leftrightarrow y < x$

and

if for any $x, y \in G$, $x < 0$, $y < 0$ and $x \not\sim^+ y$, then $v_G(x) < v_G(y) \Leftrightarrow x < y$

Let (G_1, v_1), (G_2, v_2) be valued Abelian groups with value sets Γ_1 and Γ_2, respectively. Let

$$h : G_1 \to G_2$$

be an isomorphism of Abelian groups. We say that h *preserves the valuation* or h is an *isomorphism of valued Abelian groups* if there exists an isomorphism of totally ordered sets

$$\varphi : \Gamma_1 \to \Gamma_2$$

such that for all $x \in G_1$,

$$\varphi(v_1(x)) = v_2(h(x))$$

We say that (G_1, v_1) and (G_2, v_2) are *isomorphic as valued Abelian groups* if such an isomorphism exists.

Lemma 1.1.1 ([12], p.2)**.** *An isomorphism $h : G_1 \to G_2$ preserves the valuation if and only if the map*

$$\tilde{h} : \Gamma_1 \to \Gamma_2,\ \tilde{h}(v_1(x)) = v_2(h(x))$$

is well-defined and an isomorphism of chains.

1.2 Convex Subgroups

Let $(S, <)$ be a totally ordered set. A subset $T \subseteq S$ is called *convex* if for all $a, b \in T$ and $c \in S$ such that $a \leq c \leq b$ implies that $c \in T$.

A subgroup of an ordered Abelian group is called a *convex subgroup* if it is convex as a subset.

Facts:

- $\{0\}$ is a convex subgroup of every ordered Abelian group.

- The set $\mathcal{C}(G)$ of all convex subgroups of G is totally ordered by inclusion.

- $\mathcal{C}(G)$ is closed under unions and intersections.

Let $D, C \in \mathcal{C}(G)$ and assume that $D \subseteq C$. We say $D \subseteq C$ is a *jump* if for every $D' \in \mathcal{C}(G)$ such that $D \subseteq D' \subseteq C$, either $D' = D$ or $D' = C$. Let $x \in G \setminus \{0\}$. Consider the following two convex subgroups associated to x:

$$C_x = \cap\{C \in \mathcal{C}(G); x \in C\} \text{ the smallest convex subgroup containing } x$$

$$D_x = \cup\{D \in \mathcal{C}(G); x \notin D\} \text{ the largest convex subgroup not containing } x$$

C_x is called *the principal convex subgroup generated by* x.

Immediate consequences are:

- $D_x \subseteq C_x$ is a jump

- $C_x = \{y | y \sim^+ x \text{ or } y \ll^+ x\} = \{y | v_G(y) \geq v_G(x)\}$

  $D_x = \{y \ll^+ x\} = \{y | v_G(y) > v_G(x)\}$

- So $x \sim^+ y \Leftrightarrow C_x = C_y \Leftrightarrow D_x = D_y$ and

  $x \ll^+ y \Leftrightarrow C_x \subsetneqq C_y \Leftrightarrow D_x \subsetneqq D_y$

- $v_G(x) > v_G(y) \Leftrightarrow C_x \subsetneqq C_y$

The *rank* of an ordered Abelian group Γ is (the order type) of the chain of non-trivial convex subgroups of Γ, ordered by inclusion. The *principal rank* of Γ is (the order type) of the chain of non-trivial principal convex subgroups of Γ, ordered by inclusion. The *order type of an ordered set* is the equivalence class under order preserving bijections.

Lemma 1.2.1. *Let G be an ordered Abelian group. Then G is Archimedean if and only if it has no proper (non-trivial) convex subgroups.*

Set $B_x = C_x/D_x$ with the induced order from C_x, so B_x is an ordered Abelian group. Note that there is a one-to-one correspondence between convex subgroups of C_x/D_x and convex subgroups C of G with the property $D_x \subseteq C \subseteq C_x$. Therefore, by the previous lemma, $B_x = C_x/D_x$ is an Archimedean group. $B_{x_1} \simeq B_{x_2}$ if and only if $x_1 \sim^+ x_2$ and therefore we can define $B_\gamma = B_x$ for any $\gamma \in \Gamma$, where $x \in G$ such that $v_G(x) = \gamma$. Recall that $v_G(G) = \Gamma$.

Define the skeleton$(G) := [\Gamma; \{B_\gamma | \gamma \in \Gamma\}]$.

Skeleton(G) is an invariant for isomorphisms of ordered groups in the following sense: assume that there is an isomorphism $\varphi : G_1 \to G_2$, then there is an isomorphism of chains $\hat{\varphi} : v_G(G_1) \to v_G(G_2)$ and for every $\gamma \in v_G(G)$ we have an isomorphism of ordered groups

$$\hat{\varphi}_\gamma : B^1_\gamma \to B^2_{\hat{\varphi}(\gamma)}$$

where B^1_γ is the Archimedean component associated to γ in G_1 and $B^2_{\hat{\varphi}(\gamma)}$ is the Archimedean component associated to $\hat{\varphi}(\gamma) \in v_G(G_2)$ in G_2.

Let I and A be chains. The *lexicographic product* $I \coprod A$ is the chain obtained as follows: by ordering the Cartesian product $I \times A = \{(i,a)|\ i \in I,\ a \in A\}$ lexicographically from the left:

$$(i_1, a_1) < (i_2, a_2) \text{ if either } i_1 < i_2 \text{ in } I \text{ or } i_1 = i_2 \text{ and } a_1 < a_2 \text{ in } A$$

Remark: $I \coprod A \simeq \sum_I A$ (the product of I copies of A).

Proof. For each $i \in I$ consider the chain $A_i = \{(i,a)|\ a \in A\}$. If $a_1 < a_2$ in A, then $(i, a_1) < (i, a_2)$. So, all these chains are isomorphic to A. $I \times A = \cup A_i$ and moreover $I \coprod A = \sum_{i \in I} A_i \simeq \sum_I A$. □

For any $i \in I$ fix $0_i \in A_i$. Let $s \in \prod_{i \in I} A_i$ define the support of $s := \{i \in I|\ s(i) \neq 0_i\}$.

Given a chain Γ and $\{B_\gamma|\ \gamma \in \Gamma\}$ a family of Archimedean groups indexed by Γ, we construct the following groups:

1. $\coprod_{\gamma\in\Gamma} B_\gamma$: *the Hahn sum* i.e. the direct sum of the B_γ's, ordered lexicographically.

   Fact: $\text{skeleton}(\coprod_{\gamma\in\Gamma} B_\gamma) = [\Gamma; \{B_\gamma |\ \gamma \in \Gamma\}]$.

2. $\mathbf{H}_{\gamma\in\Gamma} B_\gamma$: *the Hahn product* $= \{x|\ x : \Gamma \to \cup B_\gamma$ such that $x(\gamma) \in B_\gamma$ and $\{\gamma|\ x(\gamma) \neq 0\}$ is well-ordered in $\Gamma\}$.

   Fact: $\text{skeleton}(\mathbf{H}_{\gamma\in\Gamma} B_\gamma) = [\Gamma; \{B_\gamma |\ \gamma \in \Gamma\}]$.

Note that $\coprod_{\gamma\in\Gamma} B_\gamma$ is an ordered subgroup of $\mathbf{H}_{\gamma\in\Gamma} B_\gamma$

Theorem 1.2.1 ([13], Hahn's Embedding Theorem)**.** *Let G be a divisible ordered Abelian group.*

1. *Assume* $\text{skeleton}(G) = [\Gamma; \{B_\gamma |\ \gamma \in \Gamma\}]$. Then $\coprod_{\gamma\in\Gamma} B_\gamma$ embeds as an ordered group in G.

$$\coprod_{\gamma\in\Gamma} B_\gamma \hookrightarrow G$$

2. Assume $\text{skeleton}(G) = [\Gamma; \{B_\gamma |\ \gamma \in \Gamma\}]$. Then G embeds in $\mathbf{H}_{\gamma\in\Gamma} B_\gamma$.

$$G \hookrightarrow \mathbf{H}_{\gamma\in\Gamma} B_\gamma$$

1.3 Totally Ordered Fields

1.3.1 The Natural valuation of an Ordered Field

Let K be a field. A *valuation* of K is a mapping v from K onto $G\infty$, where G is an ordered abelian group such that for any $a, b \in K$, we have:

1. $v(a) = \infty \Leftrightarrow a = 0$

2. $v(a - b) \geq \min\{v(a), v(b)\}$ (Ultrametric triangle law)

3. $v(ab) = v(a) + v(b)$.

We call $vK := \Gamma$ *the value group of the valued field* (K, v).

The following are consequences of the above axioms:

- $v(1) = v(-1) = 0$
- $v(a) = v(-a)$
- $v(a^{-1}) = -v(a)$, $a \neq 0$
- $v(\sum_{1\leq i\leq n} a_i) \geq \min_{1\leq i\leq n} v(a_i)$, for all $a_i \in K$.

A subring $\mathfrak{O}$ of a field K is called a *valuation ring* (of K) if $x \in \mathfrak{O}$ or $\frac{1}{x} \in \mathfrak{O}$, for any $x \in K^*$.

Proposition 1.3.1. *Every valuation ring is a local ring, i.e. it has a unique maximal ideal* $\mathfrak{M}$*. This consists of all non-units, in other words* $\mathfrak{O}^* = \mathfrak{O} \backslash \mathfrak{M}$

Proof. Let $\mathfrak{O}$ be the valuation ring of a field K and de fine $\mathfrak{M}$ to be the set of all non-units in $\mathfrak{O}$.

First, we show that $\mathfrak{M}$ is an ideal of $\mathfrak{O}$. Take $a, b \in \mathfrak{M}$. Since $\mathfrak{O}$ is a valuation ring, we have $\frac{a}{b} \in \mathfrak{O}$ or $\frac{b}{a} \in \mathfrak{O}$. We may assume that $\frac{a}{b} \in \mathfrak{O}$. Then $\frac{a+b}{b} = \frac{a}{b} + 1 \in \mathfrak{O}$. If $a + b$ were a unit in $\mathfrak{O}$, it would follow that $\frac{1}{b} \in \mathfrak{O}$, contradiction to our assumption that b is not a unit. Further, for every $c \in \mathfrak{O}$, ca is a non-unit. If ca were invertible, the same would hold for a. So $\mathfrak{M}$ is an ideal of $\mathfrak{O}$.

If $\mathfrak{I}$ is a proper ideal of $\mathfrak{O}$, then it does not contain any units, so $\mathfrak{I} \subseteq \mathfrak{M}$. Thus $\mathfrak{M}$ is the unique maximal ideal of $\mathfrak{O}$. □

Proposition 1.3.2 ([10])**.** *Let* v *be a valuation of a field* K*. Then* $\mathfrak{O} := \{x \in K^* | \ v(x) \geq 0\}$ *is a valuation ring of* K *with maximal ideal* $\mathfrak{M} = \{x \in K | \ v(x) > 0\}$ *and units* $\mathfrak{O}^\times = \{x \in$

$K^*|\ v(x) = 0\}$.

Proof. Take $0 \neq x \in K$, then $v(x) \geq 0$ or $v(x^{-1}) = -v(x) \geq 0$. This implies that $x \in \mathfrak{O}$ or $x^{-1} \in \mathfrak{O}$, so $\mathfrak{O}$ is a valuation ring of K. Now $x \in \mathfrak{O}^* \Leftrightarrow (x \in \mathfrak{O}$ and $x^{-1} \in \mathfrak{O}) \Leftrightarrow (v(x) \geq 0$ and $v(x^{-1}) = -v(x) \geq 0)$. This implies $v(x) = 0$. □

Fact: $\mathfrak{O}/\mathfrak{M}$ is a field, where $\mathfrak{M}$ denotes the unique maximal ideal in the valuation ring $\mathfrak{O}$. We call $\overline{K} = \mathfrak{O}/\mathfrak{M}$ the *residue field* of K with respect to $\mathfrak{O}$. A *place* P of a field K is a surjective mapping $\mathrm{P} : K \to k \cup \{\infty\}$ such that:

- (P1) there is a subring $\mathfrak{O}_\mathrm{P}$ of K such that $\mathrm{P}|_{\mathfrak{O}_\mathrm{P}}$ is a homomorphism and $P : K \backslash \mathfrak{O}_\mathrm{P} \to \{\infty\}$
- (P2) for any $x \in K^*$ we have $x\mathrm{P} = 0 \Leftrightarrow \frac{1}{x}\mathrm{P} = \infty$

We write $K\mathrm{P} := k$ and call it the residue field of (K, P).

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. Denote by $K^{>0}$ the set of positive elements of K. Then $(K, +, 0, <)$ and $(K^{>0}, \cdot, 1, <)$ are ordered Abelian groups and $(K, +, 0, <)$ is divisible.

Let v denote the natural valuation on the ordered Abelian group $(K, +, 0, <)$ which we obtain by mapping $a \in K$, $a \neq 0$ to its *archimedean equivalence class* $[a]$. Denote by $v(K) = G =: \{[x]|\ x \neq 0\}$ its value set. Recall that G is totally ordered by

$$[a] < [b] \text{ if and only if } b \ll^+ a.$$

Define an addition on G: $[x]+[y] = [xy]$ or $v(x)+v(y) = v(xy)$ and note that $[x]+[1] = [x]$. Equipped with the above addition and order, G becomes an ordered Abelian group with neutral element $0 = [1] = v(1)$.

It is easy to prove that the natural valuation on $(K, +, 0, <)$

$$v : K \to G \cup \{\infty\}$$

$$a \mapsto [a]$$

is a valuation of the field K, called the *natural valuation on the ordered field* K. Here, $[0] := \infty$.

Then

$$\mathfrak{O} = \{x | \ [x] \geq [1]\} = \{x | \ x \ll^{+} 1 \text{ or } x \sim^{+} 1\} = C_1$$

is the *valuation ring*, or the *ring of finite elements* and

$$\mathfrak{M} = \{x | \ x \ll^{+} 1\}$$

is the *valuation ideal*, or the *ideal of infinitesimals.*

The field $\mathfrak{O}/\mathfrak{M}$, denoted by $\overline{K}$, is the *residue field.* The *group of units* of the valuation ring is the subgroup

$$\mathfrak{O}^{\times} = \{x | \ x \in K \text{ and } v(x) = 0\}$$

of the multiplicative group of $\mathfrak{O}$, and the *group of* 1*-units* is the subgroup

$$1 + \mathfrak{M} = \{x | \ v(x-1) > 0\}.$$

Assume that K is a totally ordered field. A valuation v on K is said to be *convex* if $\mathfrak{O}$ is convex. Fact: v is a convex valuation since $\mathfrak{O} = C_1$ which is the principal convex subgroup generated by 1. Moreover, v is the finest convex valuation on K, i.e., whenever v is a convex valuation, then $\mathfrak{O} \subseteq \mathfrak{O}_w$.

Recall that v is compatible with the order, that is,

$$\text{if } a > 0 \text{ and } b > 0, \text{ then } v(a) < v(b) \Rightarrow b < a.$$

We set $\text{sign}(0) = 0$ and for $a \in K$, we set $\text{sign}(a) = 1$ if $a > 0$ and $\text{sign}(a) = -1$ if $a < 0$. It follows that for all $a, b \in K$:

$$a < b < 0 \Rightarrow v(a) \leq v(b)$$

$$v(a) > v(b) \Rightarrow |a| < |b|.$$

We also get:

$$v(a-b) > v(a) \Rightarrow \text{sign}(a) = \text{sign}(b)$$

and

$$\text{sign}(a) = \text{sign}(b) \Rightarrow v(a+b) = \min\{v(a), v(b)\}.$$

1.3.2 The Decomposition Theorems

Fact: Let V be a divisible ordered Abelian group and therefore an ordered $\mathbb{Q}$-vector space. Assume that C is a convex divisible subgroup and let E be a *vector space complement* of C in V^*, that is, $V = E \oplus C$. Then $V = E \coprod C$, i.e. if for $x, y \in V$ we write $x = e_x + c_x$, $y = e_y + c_y$ then $x < y \Leftrightarrow (e_x, c_x) <_{lex} (e_y, c_y)$. We find the following theorem in [13]

Theorem 1.3.1 (Additive Decomposition)**.** *Let K be a totally ordered field and let $\mathbf{A}$ be any vector space complement to $\mathfrak{O}$. Then*

$$(K, +, 0, <) = \mathbf{A} \coprod \mathfrak{O}$$

Moreover $\mathbf{A}$ is unique up to isomorphism of ordered groups. The value set of the divisible ordered Abelian group $\mathbf{A}$ is $v(\mathbf{A}) = G^{<0}$, where $G := v(K)$.

Now we present $(K, +, 0, <)$ as a lexicographic sum of three summands, as in ([12], p.18)

Theorem 1.3.2 (Additive Lexicographic Decomposition)**.** *There exist a group complement $\mathbf{A}$ of $\mathfrak{O}$ in $(K, +, 0, <)$ and a group complement $\mathbf{A}'$ of $\mathfrak{M}$ in $\mathfrak{O}$ such that*

$$(K, +, 0, <) = \mathbf{A} \coprod \mathbf{A}' \coprod \mathfrak{M}.$$

Both $\mathbf{A}$ and $\mathbf{A}'$ are unique up to order preserving isomorphism, and $\mathbf{A}'$ is order isomorphic to the Archimedean group $(\overline{K}, +, 0, <)$. Furthermore, the value set of $\mathbf{A}$ is $G^{<0}$, the one of $\mathfrak{M}$ is $G^{>0}$, and the non-zero components of $\mathbf{A}$ and $\mathfrak{M}$ are all isomorphic to $(\overline{K}, +, 0, <)$.

For the multiplicative group $(K^{>0}, \cdot, 1, <)$ of positive elements, we will find similar decompositions as we have done for the additive group. Additive groups are always divisible, but the divisibility condition for multiplicative groups is not always given. In order to have this, we require the property that K is *root closed for positive elements*, that is, for every $a \in K$ and for every $n \in \mathbb{N}$, there is some $b \in K$ such that $b^n = a$ or $b = \sqrt[n]{a}$. Every real closed field has this property. Note that in this case vK is divisible. The next theorem can be found in ([13]).

Theorem 1.3.3 (Multiplicative Decomposition). *Assume K is root closed for positive elements. Then*

$$(K^{>0}, \cdot, 1, <) = \mathbf{B} \coprod (\mathfrak{O}^{\times})^{>0}$$

where $\mathbf{B}$ is a complement to the positive units. Moreover, $\mathbf{B} \simeq G$ by $-v$, as ordered Abelian groups.

As for the additive case, we can present $(K^{>0}, \cdot, 1, <)$ as a lexicographic sum of three summands, as in ([12], p.19).

Theorem 1.3.4 (Multiplicative Lexicographic Decomposition). *If the group $(K^{>0}, \cdot, 1, <)$ is divisible, then there exist a group complement $\mathbf{B}$ of $(\mathfrak{O}^{\times})^{>0}$ in $(K^{>0}, \cdot, 1, <)$ and a group complement $\mathbf{B}'$ of $1 + \mathfrak{M}$ in $((\mathfrak{O}^{\times})^{>0}, \cdot, 1, <)$ such that*

$$(K^{>0}, \cdot, 1, <) = \mathbf{B} \coprod \mathbf{B}' \coprod (1 + \mathfrak{M}, \cdot, 1, <).$$

Every group complement $\mathbf{B}$ of $(\mathfrak{O}^{\times})^{>0}$ in $(K^{>0}, \cdot, 1, <)$ is order isomorphic to G through the isomorphism $-v$. Every group complement $\mathbf{B}'$ of $1 + \mathfrak{M}$ in $((\mathfrak{O}^{\times})^{>0}, \cdot, 1, <)$ is isomorphic to $(\overline{K}^{>0}, \cdot, 1, <)$.

I give an example of contraction groups, introducing first exponential fields and then the natural contraction arising from an exponential.

Let K be an ordered field, root closed for positive elements. We say that K is *formally exponential field* if there exists an isomorphism of groups:

$$f : (K, +, 0, <) \to (K^{>0}, \cdot, 1, <)$$

We will call a map f with these properties, an *exponential* on K. The compositional inverse ℓ of an exponential f is called a *logarithm* on K. We say that (K, f) is an *exponential field* if K is an ordered field and f is an exponential on K. $(\mathbb{R}, \exp)$ is an example of an exponential field, where exp is the usual exponential function defined on the reals.

We say that v and f are *compatible* or that f is *v-compatible* if f also satisfies that

$$f(\mathfrak{O}) = (\mathfrak{O}^{\times})^{>0} \text{ and } f(\mathfrak{M}) = 1 + \mathfrak{M}.$$

Similarly, ℓ is *v-compatible* if

$$\ell((\mathfrak{O}^{\times})^{>0}) = \mathfrak{O} \text{ and } \ell(1 + \mathfrak{M}) = \mathfrak{M}.$$

An exponential field (K, f) such that f is v-compatible is called a *v-compatible exponential field*.

1.3.3 Lexicographic Decomposition of Exponentials

Let K be a formally exponential field. Then there exists an isomorphism $f : (K, +, 0, <) \to (K^{>0}, \cdot, 1, <)$. Since $(K, +, 0, <)$ is divisible, then also the multiplicative group of K is divisible and therefore, we can fix a decomposition as in Theorem 1.3.4. If f is a v-compatible exponential on K, then $f(\mathfrak{O}) = (\mathfrak{O}^{\times})^{>0}$ and since $\mathbf{B}$ is a group complement of $(\mathfrak{O}^{\times})^{>0}$ in $(K^{>0}, \cdot, 1, <)$, we obtain that $\mathbf{A} = f^{-1}(\mathbf{B})$ is a group complement of $\mathfrak{O}$ in $(K, +, 0, <)$. Similarly, we have that $f(\mathfrak{M}) = 1 + \mathfrak{M}$ and that $\mathbf{B}'$ is a group complement of $1 + \mathfrak{M}$ in $(U_v^{>0}, \cdot, 1, <)$, so $\mathbf{A}' = f^{-1}(\mathbf{B}')$ is a group complement of $\mathfrak{M}$ in $\mathfrak{O}$. Having the groups $\mathbf{A}$ and $\mathbf{A}'$, we obtain a decomposition as in Theorem 1.3.2. Denote the restriction of f to $\mathbf{A}$ by f_L, the restriction to $\mathbf{A}'$ by f_M and the restriction to $\mathfrak{M}$ by f_R. We get the following isomorphisms:

$$f_L : \mathbf{A} \to \mathbf{B}$$

$$f_M : \mathbf{A}' \to \mathbf{B}'$$

$$f_R : \mathfrak{M} \to 1 + \mathfrak{M}$$

The following definitions, as stated in ([12]) are motivated by the display of the lexicographic sums mentioned above. We will call a *v-left exponential* an isomorphism $f_L : \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}$ is a group complement of $\mathfrak{O}$ in $(K, +, 0, <)$ and $\mathbf{B}$ is a group complement of $(\mathfrak{O}^\times)^{>0}$ in $(K^{>0}, \cdot, 1, <)$. Similarly is defined a *v-left logarithm.* Since $\mathbf{B}$ is unique up to order preserving isomorphisms and isomorphic to G through the isomorphism $-v$, as stated in Theorem 1.3.4, a v-left exponential induces an isomorphism from $\mathbf{A}$ onto G. Conversely, every isomorphism between $\mathbf{A}$ and G induces a v-left exponential, or a v-left logarithm. More precisely, if we take the isomorphism $-v : \mathbf{B} \to G$, or equivalently $(-v)^{-1} : G \to \mathbf{B}$, and a v-left logarithm

$$\ell : \mathbf{B} \to \mathbf{A} \qquad ,$$

then $h_\ell : G \to \mathbf{A}$ such that

$$h_\ell := \ell \circ (-v)^{-1}$$

is an isomorphism of ordered groups. If $f = \ell^{-1}$, then we shall denote this isomorphism by h_f. Let f be an exponential and $f_L : \mathbf{A} \to \mathbf{B}$ the left v-left exponential corresponding to a given composition. By abuse of notation, we shall write h_f instead of h_{f_L}. Then replacing ℓ by f_L^{-1} in the last formula, we obtain:

$$h_f := f_L^{-1} \circ (-v)^{-1}.$$

An isomorphism f_M from a group complement $\mathbf{A}'$ of $\mathfrak{M}$ in $\mathfrak{O}$ onto a group complement $\mathbf{B}'$ of $1+\mathfrak{M}$ in $(\mathfrak{O}^\times)^{>0}$ will be called a *v-middle exponential.* By Theorems 1.3.2 and 1.3.4,

$\mathbf{A}'$ is order isomorphic to $(\overline{K}, +, 0, <)$ and $\mathbf{B}'$ to $(\overline{K}^{>0}, \cdot, 1, <)$. Since the group complements are unique up to order preserving isomorphisms, the isomorphism $f_M : \mathbf{A}' \to \mathbf{B}'$ induces an isomorphism between $(\overline{K}, +, 0, <)$ and $(\overline{K}^{>0}, \cdot, 1, <)$, that is, an exponential on $\overline{K}$.

An isomorphism f_R from $\mathfrak{M}$ onto $1 + \mathfrak{M}$ will be called a *v-right exponential.* We call $f_L \coprod f_M \coprod f_R$ the *lexicographic product of the exponentials* f_L, f_M and f_R, where

$$\forall a \in \mathbf{A} \forall a' \in \mathbf{A}' \forall \varepsilon \in \mathfrak{M} : (f_L \coprod f_M \coprod f_R)(a + a' + \varepsilon) := f_L(a) \cdot f_M(a') \cdot f_R(\varepsilon).$$

Recall from Lemma 1.1.1 that the isomorphism $h : G \to \mathbf{A}$ induces an isomorphism of ordered sets $\tilde{h}_f : v_G(G) \to G^{<0}$, which is defined by

$$\tilde{h}_f(v_G(g)) = v(h_f(g))$$

for any $g \in G$. We denote this isomorphism by $\tilde{h}_\ell$ if $f^{-1} = \ell$.

An isomorphism of ordered sets

$$\tilde{h} : v_G(G) \to G^{<0}$$

is called *group exponential* on G. We say that h is a *lifting* of $\tilde{h}$ if $h : G \to \mathbf{A}$ is an isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{h} & \mathbf{A} \\ \downarrow{\scriptstyle v_G} & & \downarrow{\scriptstyle v} \\ v_G(G) & \xrightarrow{\tilde{h}} & G^{<0} \end{array}$$

1.4 Natural Contraction

An important class of contraction groups is obtain by exponential fields.

Let G be an ordered Abelian group. If we compose $\tilde{h} : v_G(G) \to G^{<0}$ with the natural valuation $v_G : G \to v_G(G)$, then we obtain another map $\chi : G^{<0} \to G^{<0}$, which we want to analyze. Here G is considered to be an ordered Abelian group. A map χ from $G^{<0}$ into

$G^{<0}$ is called a *contraction* if it satisfies the following axioms:

(C1) χ is surjective,

(C2) χ preserves the ordering,

(C3) if g is archimedean equivalent to g', then $\chi(g) = \chi(g')$.

Note that for negative group elements: g is archimedean equivalent to g' if and only if there is $n \in \mathbb{N}$ such that $ng < g'$ and $ng' < g$.

We call χ a *natural contraction* if $\chi(x) = \chi(y)$ implies that x and y are archimedean equivalent.

Lemma 1.4.1 ([12], p.45)**.** *Let $\tilde{h} : v_G(G) \to G^{<0}$ be a group exponential on G. Define $\chi : G^{<0} \to G^{<0}$ by*

$$\chi = \tilde{h} \circ v_G$$

Then χ is a natural contraction on G.

We call $\chi = \tilde{h} \circ v_G$ the *natural contraction induced by $\tilde{h}$.*

Lemma 1.4.2 ([12], p.45)**.** *Let χ be a contraction on G. Define $\tilde{h} : v_G(G) \to G^{<0}$ by*

$$\tilde{h}(v_G(g)) = \chi(g),$$

for $g \in G^{<0}$. Then $\tilde{h}$ is well-defined, surjective and order preserving. Moreover, $\tilde{h}$ is an isomorphism (that is, a group exponential) if and only if χ is a natural contraction.

We will call $\tilde{h}$, constructed in this lemma, the *group exponential induced by χ.*

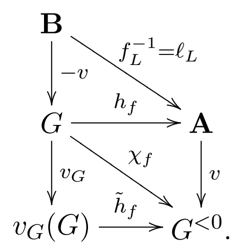
Let f be a v-compatible exponential on K and

$$\chi_f = \tilde{h}_f \circ v_G$$

the natural contraction induced by the group exponential $\tilde{h}_f$ as in Lemma 1.3.1. We will call χ_f the *natural contraction induced by f.*

If we set $\ell = f^{-1}$, we want to compute $\chi_f(g)$ in terms of ℓ, for $g \in G^{<0}$. Fix any decomposition of $(K, +, 0, <)$. Let $f_L : \mathbf{A} \to \mathbf{B}$ be the induced v-left exponential, so

$f_L^{-1} := \ell_L$. Let also $h_f : G \to \mathbf{A}$ be the induced isomorphism. Take $a > 0$ such that $v(a) = g < 0$ and let $b \in \mathbf{B}$ be the uniquely determined element for which $v(b) = g$. We get $\chi_f(g) = \tilde{h}_f(v_G(g))$ (by the definition) and $\tilde{h}_f(v_G(g)) = v(h_f(g))$, by the commutativity of the diagram bellow:



Further, $v(h_f(g)) = v((\ell_L \circ (-v)^{-1})(v(b)) = v(\ell_L(b^{-1})) = v(-\ell_L(v)) = v(\ell_L(b))$. Let $a = bu$, where u is a unit. Then $\ell_L(a) = \ell_L(bu) = \ell_L(b) + \ell_L(u)$ since ℓ_L is an isomorphism, and $v(\ell_L(a)) = v(\ell_L(b) + \ell_L(u))$. Now $\ell_L(b) \in \mathbf{A}$ and by Theorem 1.3.2, $v(\mathbf{A}) = G^{<0}$, so $v(\ell_L(b)) < 0$. Since ℓ_L is v-compatible, we have $\ell_L(u) \in \mathfrak{O}$ and therefore $v(\ell_L(u)) \geq 0$. By the ultrametric inequality it follows that $v(\ell_L(b) + \ell_L(u)) = \min\{v(\ell_L(b)), v(\ell_L(u))\} = v(\ell_L(b))$. Thus, $v(\ell_L(a)) = v(\ell_L(b))$.

In conclusion, we get:

$$\chi_f(v(a)) = v(\ell_L(a)) \text{ for all } a > 0 \text{ such that } v(a) < 0.$$

Observe that χ_f depends only on f and does not depend on the chosen decomposition. The only disadvantage is that we can obtain the defining formula for χ_f: $\chi_f(v(a)) = v(\ell_L(b))$ (for $a > 0$ such that $v(a) < 0$) only by using the uniquely determined element $b \in \mathbf{B}$ for which $v(b) = v(a)$.

Let $\tilde{h} : v_G \to G^{<0}$ be a group exponential on G and $h : G \to \mathbf{A}$ be a lifting of $\tilde{h}$. Then

$$\tilde{h}(v_G(g)) > g \Leftrightarrow v(h(g)) > g$$

for every $g \in G^{<0}$.

We will say that an exponential group $(G, \tilde{h})$ is a *strong exponential group* if for all

$g \in G^{<0}$, we have

$$\tilde{h}(v_G(g)) > g.$$

In this case, $\tilde{h}$ is called a *strong group exponential*. Let $\varphi = \tilde{h}^{-1}$ and observe that $\tilde{h}^{-1}(\tilde{h}(v_G(g)) = v_G(g) > \tilde{h}^{-1}(g) = \varphi(g)$, that is, $\varphi(g) < v_G(g)$ for all $g \in G^{<0}$. Note that if $\tilde{h}$ is a strong exponential on G, then for any $g \in G^{<0}$, $g < \tilde{h}(v_G(g)) = \chi(g)$, so χ maps towards the center of the ordered group, which is the element 0.

We say that a contraction χ is *centripetal* if it satisfies:

(CP) $$\forall g \in G^{<0}\colon\ g < \chi(g),$$

and *centrifugal*, if it satisfies

(CF) $$\forall g \in G^{<0}\colon\ g > \chi(g).$$

Lemma 1.4.3 ([12], p.46)**.** *Let $(G, \tilde{h})$ be an exponential group and χ the contraction induced by $\tilde{h}$. Then:*

1. *χ is centripetal if and only if $(G, \tilde{h})$ is a strong exponential group.*

2. *χ is centrifugal if and only if $\tilde{h}$ satisfies*

$$\forall g \in G^{<0}\colon\ \tilde{h}(v_G(g)) < g.$$

The proof is straightforward: χ is centripetal if and only if $\forall g \in G^{<0}\colon\ g < \chi(g) = \tilde{h}(v_G(g))$ (by Lemma 1.4.1), so the exponential group $(G, \tilde{h})$ is a strong exponential group. Similarly, we can prove 2.

A v-compatible exponential is called a *(GA)-exponential* if it satisfies the *growth axiom* scheme:

(GA) $$a \geq n^2 \Rightarrow f(a) > a^n \ (n \geq 1).$$

Corollary 1.4.1 ([13], p.46)**.** *Let f be a v-left exponential on K. Then f is a (GA)-exponential if and only if the induced contraction χ_f is centripetal.*

1.5 Hardy Fields

Given a property $P(x)$, with x ranging over $\mathbb{R}$, we say that $P(x)$ *holds ultimately* (or *ultimately* $P(x)$) if there exists $x_0 \in \mathbb{R}$ such that $P(x)$ holds for all $x > x_0$. We define an equivalence relation on the collection of real-valued functions defined on positive half-lines of $\mathbb{R}$ (i.e. their domain is either $\mathbb{R}$, or an interval of the form $[a, \infty]$ or (a, ∞) for some $a \in \mathbb{R}$), by saying that f and g are equivalent if ultimately $f(x) = g(x)$. We denote the equivalence class of such a function f by $\bar{f}$, and call it the *germ* of f (at $+\infty$). We can add and multiply functions in this collection because this respects the equivalence relation, so we can add and multiply germs by $\overline{f} + \overline{g} = \overline{f+g}$ and $\overline{f} \cdot \overline{g} = \overline{f \cdot g}$. Therefore, the set of germs become a commutative ring. From now on, we omit the bar and use the same letter for a function and its germ.

A *Hardy field* H is a set of germs at $+\infty$ of real valued functions on positive half-lines of $\mathbb{R}$ which is closed under differentiation and forms a field under ordinary addition and multiplication of germs.

Let K be a Hardy field. For $f \in K$, $f \neq 0$, there is $g \in K$ with $f \cdot g = 1$, so ultimately $f(x) \neq 0$ and therefore either ultimately $f(x) < 0$ or ultimately $f(x) > 0$ (by ultimate continuity of f). We define an order on K in the following way: $f > 0$ for $f \in K$ if ultimately $f(x) > 0$. Given $f \in K$ we also have $f' \in K$, so either $f' < 0$, or $f' = 0$, or $f' > 0$. Then f is either ultimately strictly decreasing, or ultimately constant, or ultimately strictly increasing, hence the limit $\lim_{x \to +\infty} f(x)$ always exists and it is either a real number, or $+\infty$, or$-\infty$. For this reason a Hardy field never contains an oscillating function such as $\sin(x)$.

Example: The fields $\mathbb{Q}$, $\mathbb{R}$ are Hardy fields consisting just of constant germs. Both fields are archimedean. Denote by x the germ of identity function. Then $x > \mathbb{R}$ (i.e. $x > r$ for every constant germ $r \in \mathbb{R}$. Examples of Hardy fields are any subfield of $\mathbb{R}$, the field of rational functions in one variable $\mathbb{R}(x)$ and any field of (germs of) functions obtained from $\mathbb{R}(x)$ by repeated adjunctions of real-valued algebraic functions, logarithms of positive functions and exponentials of functions such as the field $\mathbb{R}(x, \sqrt{\log x}, e^x, \exp(x\sqrt{logx}+e^x))$. On the other hand, if K is any subfield of $\mathbb{R}$, then $K(x^2)$, and $K(\log(x))$ are not Hardy fields because they are not closed under differentiation.

More generally, if H is a Hardy field and f a germ such that f is algebraic over H or $f' \in H$ or $f'/f \in H$, then $H(f)$ is a Hardy field.

This observation allows us to introduce a valuation on H. Let f, $g \in H$ such that f, $g \neq 0$. We set $f \simeq g$ if and only if $\lim_{x\to\infty} f(x)/g(x)$ is a non-zero real number. We can easily check that this is an equivalence relation. For $f \in H$, $f \neq 0$, we denote the equivalence class of f by $v(f)$; that is

$$v(f) = v(g) \Leftrightarrow \lim_{x\to\infty} f(x)/g(x) \in \mathbb{R} \setminus \{0\}.$$

Let $\Gamma =: \{v(f) : f \in H, f \neq 0\}$ be the set of all equivalence classes on $H \setminus \{0\}$. If a, b, c, $d \in H \setminus \{0\}$ and $a \simeq b$, $c \simeq d$, then clearly $ac \simeq bd$, so that multiplication on $H \setminus \{0\}$ induces a composition of elements of Γ, Γ becoming an Abelian group and the map $v : H \setminus \{0\} \to \Gamma$ a homomorphism. We follow the convention of writing the composition law on Γ additively.

Define an addition on the set of classes Γ: $v(f) + v(g) = v(fg)$.

If f, $g \in H \setminus \{0\}$ we write

$$v(f) > v(g) \Leftrightarrow \lim_{x\to\infty} f(x)/g(x) = 0.$$

This definition clearly depends only on the equivalence classes $v(f)$ and $v(g)$ of f and g

and it induces a total ordering of the set Γ.

For $f \in H \setminus \{0\}$, $v(f) > 0$ $(= v(1))$ means that $\lim_{x\to\infty} f(x) = 0$ and it follows that if f, $g \in H \setminus \{0\}$ and $v(f)$, $v(g) > 0$ then also $v(f) + v(g) = (v(fg)) > 0$. Therefore Γ is an ordered Abelian group.

If f, $g \in H \setminus \{0\}$ then $v(f) \geq v(g)$ means simply that $\lim_{x\to\infty} f(x)/g(x)$ is finite.

So we have associated with the field H an ordered Abelian group Γ and a surjective map $v : H \setminus \{0\} \to \Gamma$ such that:

(1) if f, $g \in H \setminus \{0\}$, then $v(fg) = v(f) + v(g)$

(2) if f, $g \in H \setminus \{0\}$, $f \neq -g$, then $v(f + g) \geq \min\{v(f), v(g)\}$.

Then the map $f \mapsto v(f)$ is a valuation on H called the canonical valuation on H with value group Γ. To extend the applicability of (1) and (2) to all f, $g \in H$, it is convenient to write $v(0) = \infty$.

Note that if f, $g \in H \setminus \{0\}$ and $v(f) \neq v(g)$, then $v(f + g) = \min\{v(f), v(g)\}$. If $f \in H \setminus \{0\} \cap \mathbb{R}$ then $v(f) = 0$.

We now describe the valuation ring $\mathfrak{O}$ of finite elements, the ideal of infinitesimals $\mathfrak{M}$ and the units $\mathcal{U}_v$ of the valuation ring in terms of limits.

$$\mathfrak{O} = \{f : v(f) \geq 0\} = \{f : \lim_{x\to\infty} f(x) \in \mathbb{R}\}$$

$$\mathfrak{M} = \{f : v(f) > 0\} = \{f : \lim_{x\to\infty} f(x) = 0\}$$

$$\mathfrak{O}^{\times} = \{f : v(f) = 0\} = \{f : \lim_{x\to\infty} f(x) \in \mathbb{R} \setminus \{0\}\}.$$

The set of positive infinite elements of H is denoted by

$$\mathbf{P}_H = \{f | \lim_{x\to\infty} f(x) = \infty\}$$

Let f, $g \in H \setminus \{0\}$ with $v(f)$, $v(g) \neq 0$. In particular, f, $g \notin \mathbb{R}$ and f', $g' \neq 0$. Then $v(f) \geq v(g)$ if and only if $v(f') \geq v(g')$ as follows from l'Hospital's rule for $x \to \infty$, in its indeterminate forms $0/0$, ∞/∞, $0/\infty$, $\infty/0$. Let us prove this. We know that $v(f) \geq v(g)$ if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} \in \mathbb{R}$. By the l'Hospital's rule we have that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \in \mathbb{R}$ and the latter relation is equivalent to $v(f') \geq v(g')$. Here $\lim_{x\to\infty} f(x) \notin \mathbb{R}$ gives us the condition from the hypothesis, that $v(f) \neq 0$.

We can assert that if $f,\, g \in H$ and $v(f) > v(g) \neq 0$, then $v(f') > v(g')$. To show this it remains to consider the case $v(f) = 0$, in which case there exists $c \in \mathbb{R}$ such that $v(f-c) > 0 > v(g)$ and therefore $v(f') = v((f-c)') = v(g')$. We can summarize as follows:

Theorem 1.5.1 ([20], p.304)**.** *Let H be a Hardy field. Then there exists a map v from the set of nonzero elements $H \setminus \{0\}$ of H onto an ordered Abelian group such that:*

1. *If $f,\, g \in H \setminus \{0\}$, then $v(fg) = v(f) + v(g)$;*

2. *If $f \in H \setminus \{0\}$, then $v(f) \geq 0$ if and only if $\lim_{x\to\infty} f(x) \in \mathbb{R}$;*

3. *Writing symbolically $v(0) = +\infty$, if $f,\, g \in H$, then $v(f+g) \geq \min\{v(f),\, v(g)\}$, with equality if $v(f) \neq v(g)$;*

4. *If $f,\, g \in H \setminus \{0\}$ and $v(f),\, v(g) \neq 0$, then $v(f) \geq v(g)$ if and only if $v(f') \geq v(g')$;*

5. *If $f,\, g \in H$ and $v(f) > v(g) \neq 0$, then $v(f') > v(g')$.*

Choose f and $g \neq 0$ to be real valued functions defined on positive half-lines of $\mathbb{R}$. We say that f is *asymptotic* to g and write $f \sim g$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Lemma 1.5.1 ([12], p.96)**.** *Let $f,\, g \neq 0$ be elements of the Hardy field H. Then f is asymptotic to g if and only if*

$$v(f-g) > v(g).$$

Proof. By definition f is asymptotic to g if and only if $\lim_{x\to\infty}(\frac{f(x)}{g(x)} - 1) = 0$, which implies $v(\frac{f}{g} - 1) > 0$. This is also equivalent to $v(\frac{f-g}{g}) = v(f-g) - v(g) > 0$, and to

$v(f - g) > v(g)$. $\square$

We call a Hardy field H an *exponential Hardy field* if it is real closed and satisfies the following conditions:

- (EH1) $\mathbb{R}(x) \subset H$,
- (EH2) if $f \in H$, then $\exp(f) \in H$,
- (EH3) if $f \in H$, $f > 0$ then $\log(f) \in H$.

The following lemma shows that exponential Hardy fields are incorporated in the class of ordered exponential fields.

Lemma 1.5.2 ([12], p.94)**.** *Let H be an exponential Hardy field. Then the map*

$$f \mapsto \exp(f)$$

is a (GA)-exponential on H.

1.6 The Model Theory of Contraction Groups

Our approach to the model theory of contraction groups is based on [9], with the help of the Model Theoretic Algebra class and the lecture notes from ([11]). In this section we will present the axiom system for the Abelian groups with contractions. Moreover, we will show that the theory of Abelian groups with contractions is complete, decidable and admits quantifier elimination.

1.6.1 Basic Definitions

A *language* is defined to be a triple $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$ where

- $\mathcal{R}$ is a set of relation symbols,
- $\mathcal{F}$ is a set of function symbols,
- $\mathcal{C}$ is a set of constant symbols.

A language $\mathcal{L}' = (\mathcal{R}', \mathcal{F}', \mathcal{C}')$ is an *expansion* of $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, if $\mathcal{R} \subseteq \mathcal{R}'$, $\mathcal{F} \subseteq \mathcal{F}'$, and $\mathcal{C} \subseteq \mathcal{C}'$. For example, $\mathcal{L}_R := \{+, -, \cdot, 0, 1\}$ is an expansion of $\mathcal{L}_G := \{+, -, 0\}$, but not of $\mathcal{L}_{OG} := \{+, -, 0, <\}$.

For a given language $\mathcal{L}$, an $\mathcal{L}$ *-structure* is a quadruple $\mathfrak{A} = (A, \mathcal{R}_\mathfrak{A}, \mathcal{F}_\mathfrak{A}, \mathcal{C}_\mathfrak{A})$ where

- A is a set, called the *universe of* $\mathfrak{A}$,
- $\mathcal{R}_\mathfrak{A} = \{R_\mathfrak{A} | R \in \mathcal{R}\}$ such that every $R_\mathfrak{A}$ is a relation on A of the same arity as the relation symbol R,
- $\mathcal{F}_\mathfrak{A} = \{f_\mathfrak{A} | f \in \mathcal{F}\}$ such that every $f_\mathfrak{A}$ is a function on A of the same arity as the function symbol f,
- $\mathcal{C}_\mathfrak{A} = \{c_\mathfrak{A} | c \in \mathcal{C}\}$ such that every $c_\mathfrak{A}$ is an element of A (called a *constant*).

We call $R_\mathfrak{A}$ the *interpretation of* R *on* A, and similarly for the functions $f_\mathfrak{A}$ and the constants $c_\mathfrak{A}$. If we have an $\mathcal{L}$-structure $\mathfrak{A}$ with nonempty universe A, we can extend the language $\mathcal{L}$ to a language $\mathcal{L}'$ by interpreting the new relation, function and constant symbols on A. For the new relation and function symbols, we choose any relations and functions on A of the same arity. Since we assume A to be nonempty, we can choose arbitrarily elements of A for the interpretation of new constant symbols. A structure $\mathfrak{A}'$, obtained in this way, is called an $\mathcal{L}'$*-expansion of* $\mathfrak{A}$, and $\mathfrak{A}$ is called the $\mathcal{L}$*-reduct* of $\mathfrak{A}'$.

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and S a subset of A. Then there is a very useful extension of the language $\mathcal{L}$, where we can give a name to every element of S. This extended language, denoted by $\mathcal{L}(S)$, is obtained by adjoining to $\mathcal{L}$ a constant symbol c_a for every $a \in S$. The interpretation of the symbol c_a in $\mathfrak{A}$ is meant to be the element a. Then the structure $\mathfrak{A}$ is expended in this way to an $\mathcal{L}(S)$-structure, which is denoted by $(\mathfrak{A}, S)$. In the case where $S = A$, we add to the language a name for every element of A.

Take $\mathfrak{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$. Then $\mathfrak{A}$ is said to be a *substructure of* $\mathfrak{B}$ and $\mathfrak{B}$ is called an *extension* of $\mathfrak{A}$ if $A \subseteq B$ and

- the restriction of $R_{\mathfrak{B}}$ to A is equal to $R_{\mathfrak{A}}$, for every relation symbol $R \in \mathcal{R}$,
- the restriction of $f_{\mathfrak{B}}$ to A is equal to $f_{\mathfrak{A}}$, for every function symbol $f \in \mathcal{F}$,
- $c_{\mathfrak{A}} = c_{\mathfrak{B}}$ for every constant symbol $c \in \mathcal{C}$.

A map $\sigma : A \to B$ is called a *homomorphism from* $\mathfrak{A}$ *into* $\mathfrak{B}$ if it satisfies:

- **(HOMR)** $R_{\mathfrak{A}}(a_1, ..., a_n) \Rightarrow R_{\mathfrak{B}}(\sigma(a_1), ..., \sigma(a_n))$ for all $n \in \mathbb{N}$, all n-ary relation symbols $R \in \mathcal{R}$ and all $(a_1, ..., a_n) \in A^n$,
- **(HOMF)** $\sigma(f_{\mathfrak{A}}(a_1, ..., a_n)) = f_{\mathfrak{B}}(\sigma(a_1), ..., \sigma(a_n))$ for all $n \in \mathbb{N}$, all n-ary function symbols $f \in \mathcal{F}$ and all $(a_1, ..., a_n) \in A^n$,
- **(HOMC)** $\sigma(c_{\mathfrak{A}}) = c_{\mathfrak{B}}$ for all constant symbols $c \in \mathcal{C}$.

A *strong homomorphism* is a homomorphism which also satisfies:

- **(HOMS)** $R_{\mathfrak{A}}(a_1, ..., a_n) \Leftrightarrow R_{\mathfrak{B}}(\sigma(a_1), ..., \sigma(a_n))$ for all $n \in \mathbb{N}$, all n-ary relation symbols $R \in \mathcal{R}$ and all $(a_1, ..., a_n) \in A^n$.

An *embedding* is an injective strong homomorphism.

Take a language $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$. An $\mathcal{L}$-*term* is a syntactically correct string built up from constant, function and variable symbols. A *constant* $\mathcal{L}$-*term* is an $\mathcal{L}$-term that does not contain variable symbols.

An *atomic $\mathcal{L}$-formula* is

- $R(t_1, ..., t_n)$, where $R \in \mathcal{R}$ (n-ary relation) and $t_1, ..., t_n$ are $\mathcal{L}$-terms.

- $t_1 = t_2$, where t_1 and t_2 are $\mathcal{L}$-terms.

A *free variable* is a variable which appears in an $\mathcal{L}$-formula, but is not bound by a quantifier.

An *(atomic) $\mathcal{L}$-sentence* is an (atomic) $\mathcal{L}$-formula without free variables.

Let ϕ be an $\mathcal{L}$-sentence and $\mathfrak{A}$ an $\mathcal{L}$-structure. We say that $\mathfrak{A}$ *is a model of* ϕ and write $\mathfrak{A} \models \phi$, if and only if ϕ is true in $\mathfrak{A}$.

If a property of $\mathcal{L}$-structures can be described by a set of elementary $\mathcal{L}$-sentences, then we call it an *elementary property.* An *elementary $\mathcal{L}$-theory* is a set of elementary $\mathcal{L}$-sentences. Take an $\mathcal{L}$-structure $\mathfrak{A}$. $Th(\mathfrak{A}) :=$ the set of all sentences that hold in $\mathfrak{A}$. Theories arise naturally as we try to axiomatize properties of mathematical structures. For example, the theory T_F of fields consists of all axioms for fields.

Two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* and write $\mathfrak{A} \equiv \mathfrak{B}$ if they satisfy the same $\mathcal{L}$-sentences. In other words, $Th(\mathfrak{A}) = Th(\mathfrak{B})$.

We say that an $\mathcal{L}$-theory is *complete* if for all $\mathcal{L}$-sentences ϕ, either $\mathbf{T} \models \phi$ or $\mathbf{T} \models \neg\phi$. Another way of saying this is that a theory is complete if any two models are elementarily equivalent.

We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent over* $\mathfrak{S}$, written as $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$, if $(\mathfrak{A}, S) \equiv (\mathfrak{B}, S)$. Here S is the universe of $\mathfrak{S}$.

Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$. Then we say that $\mathfrak{B}$ is *an elementary extension of* $\mathfrak{A}$, or $\mathfrak{A}$ is *an elementary substructure of* $\mathfrak{B}$, written $\mathfrak{A} \prec \mathfrak{B}$, if $\mathfrak{A} \equiv_{\mathfrak{A}} \mathfrak{B}$. An embedding of $\mathfrak{A}$ in $\mathfrak{B}$ is called *elementary embedding* if the image of $\mathfrak{A}$ in $\mathfrak{B}$ is an elementary substructure of $\mathfrak{B}$.

Lemma 1.6.1. *Assume that there is an isomorphism from a substructure* $\mathfrak{A}_0$ *of* $\mathfrak{A}$ *onto a* *substructure* $\mathfrak{B}_0$ *of* $\mathfrak{B}$. *Then there exists an* $\mathcal{L}$*-structure* $\mathfrak{A}' \cong \mathfrak{A}$ *which also has* $\mathfrak{B}_0$ *as a substructure.*

Let S be a set of $\mathcal{L}$-sentences and $\mathfrak{A}$, $\mathfrak{B}$ two $\mathcal{L}$-structures. We write $\mathfrak{A} \Rightarrow_S \mathfrak{B}$ if

$$\mathfrak{A} \models \varphi \Rightarrow \mathfrak{B} \models \varphi \text{ for all } \varphi \in S$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures such that $\mathfrak{A} \Rightarrow_S \mathfrak{B}$ with S the set of all existential (resp. all universal) $\mathcal{L}$-sentences, then we write $\mathfrak{A} \Rightarrow_\exists \mathfrak{B}$ (resp. $\mathfrak{A} \Rightarrow_\forall \mathfrak{B}$). We say that $\mathfrak{A}$ is *existentially closed* in $\mathfrak{B}$, and write $\mathfrak{A} \prec_\exists \mathfrak{B}$, if $(\mathfrak{B}, A) \Rightarrow_\exists (\mathfrak{A}, A)$, or equivalently $(\mathfrak{A}, A) \Rightarrow_\forall (\mathfrak{B}, A)$.

Let $\mathbf{T}$ be an $\mathcal{L}$-theory. Two $\mathcal{L}$-formulas ϕ, φ are $\mathbf{T}$-*equivalent* if $\mathbf{T} \models \forall(\phi \leftrightarrow \varphi)$. An $\mathcal{L}$-theory *admits quantifier elimination* if every $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$ is $\mathbf{T}$-equivalent to a quantifier free $\mathcal{L}$-formula $\phi(x_1, ..., x_n)$.

An $\mathcal{L}$-theory $\mathbf{T}$ is *substructure complete* if for all $\mathbf{T}$-models $\mathfrak{A}$ and $\mathfrak{B}$, and every common substructure $\mathfrak{S}$ of $\mathfrak{A}$ and $\mathfrak{B}$, we have $\mathfrak{A} \equiv_\mathfrak{S} \mathfrak{B}$.

Theorem 1.6.1. *A theory* $\mathbf{T}$ *admits (quantifier elimination) if and only if it is substructure complete.*

A theory $\mathbf{T}$ is called *model complete* if for all models $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{S}$ of $\mathbf{T}$ with $\mathfrak{S}$ a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$, we have that $\mathfrak{A} \equiv_\mathfrak{S} \mathfrak{B}$.

Corollary 1.6.1. *If a theory admits quantifier elimination, then it is model complete.*

If $\mathfrak{S}$ is a substructure of $\mathfrak{A}$ and both $\mathfrak{A}$ and $\mathfrak{S}$ are models of a theory $\mathbf{T}$, then $\mathfrak{S}$ is called a *submodel of* $\mathfrak{A}$.

Assume $\mathbf{T}$ is model complete, $\mathfrak{A} \models \mathbf{T}$ and $\mathfrak{S}$ a submodel of $\mathfrak{A}$. Take $\mathfrak{B} := \mathfrak{A}$ to obtain $\mathfrak{A} \equiv_\mathfrak{S} \mathfrak{B}$, or equivalently $\mathfrak{S} \prec \mathfrak{A}$.

This observation gives us an alternate definition for model complete. $\mathbf{T}$ is *model complete* if for every model $\mathfrak{A}$ of $\mathbf{T}$ and every submodel $\mathfrak{S}$ of $\mathfrak{A}$, we have $\mathfrak{S} \prec \mathfrak{A}$.

Theorem 1.6.2 (Robinson's Test)**.** *If for any two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathbf{T}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ we have that $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$, that is $\mathfrak{A} \prec_{\exists} \mathfrak{B}$, then $\mathbf{T}$ is model complete.*

A model $\mathfrak{A}_p$ of $\mathbf{T}$ is a *prime model* for $\mathbf{T}$ if it can be embedded in every other model of $\mathbf{T}$.

Proposition 1.6.1 (Prime Model Test)**.** *If $\mathbf{T}$ is model complete and admits a prime model, then $\mathbf{T}$ is complete.*

Take an $\mathcal{L}$-theory $\mathbf{T}$. An $\mathcal{L}$-structure $\mathfrak{S}$ is called a *prime structure* of $\mathbf{T}$ if it embeds in every model of $\mathbf{T}$.

Proposition 1.6.2 (Prime Structure Test)**.** *If the theory $\mathbf{T}$ admits quantifier elimination and a prime structure, then it is complete.*

Proof. If $\mathbf{T}$ admits quantifier elimination, then by Theorem 1.6.1 T is substructure complete.

Take any two models $\mathfrak{A}$, $\mathfrak{B}$ of $\mathbf{T}$ and let $\mathfrak{A}_p$ be the prime substructure of $\mathfrak{A}$ and $\mathfrak{B}$, so $\mathfrak{A}_p \hookrightarrow \mathfrak{A}$ and $\mathfrak{A}_p \hookrightarrow \mathfrak{B}$. By Lemma 1.6.1, we can assume that $\mathfrak{A}_p$ is a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$.

Since $\mathbf{T}$ is substructure complete, then $\mathfrak{A} \equiv_{\mathfrak{A}_p} \mathfrak{B}$, i.e. $(\mathfrak{A}, A_p) \equiv (\mathfrak{B}, A_p)$. Therefore $\mathfrak{A} \equiv \mathfrak{B}$, so $\mathbf{T}$ is complete. $\square$

If there is an algorithm to produce all sentences in an elementary scheme, then this scheme is called *recursive*. An $\mathcal{L}$-theory is called *decidable* if there is a recursive algorithm which tells us for every $\mathcal{L}$-sentence φ whether $\mathrm{T} \models \varphi$ or $\mathrm{T} \models \neg\varphi$.

Suppose that $\mathbf{T}$ and $\mathbf{T}'$ are $\mathcal{L}$-theories. We say that $\mathbf{T}'$ is the *model companion* of $\mathbf{T}$ if:

1. $\mathbf{T}'$ is model-complete,

2. every model of $\mathbf{T}$ has an extension that is a model of $\mathbf{T}'$, and

3. every model of $\mathbf{T}'$ has an extension that is a model of $\mathbf{T}$.

If $\mathbf{T}'$ is a model companion of $\mathbf{T}$ and $\mathbf{T}' \cup$ Diag($\mathfrak{A}$) is complete for any model $\mathfrak{A}$ of $\mathbf{T}$, then $\mathbf{T}'$ is a *model completion* of $\mathbf{T}$. Here, the *elementary diagram* of $\mathfrak{A}$, written Diag($\mathfrak{A}$), is the set of all atomic $\mathcal{L}$-sentences and negations of atomic $\mathcal{L}$-sentences that hold in $\mathfrak{A}$.

Examples: The theory of algebraically closed fields is the model companion of the theory of integral domains (subrings of fields) and the theory of real closed fields is the model companion of the theory of ordered domains.

1.6.2 Ordered Abelian Groups with Contractions

Recall that the natural valuation v_G on an ordered Abelian group satisfies the following axioms:

(V0) $v_G(x) = \infty \Leftrightarrow x = 0$,

(V1) $v_G(x - y) \geq \min\{v_G(x), v_G(y)\}$.

Axiom (V1) can be viewed as the *ultrametric triangle inequality* for valued Abelian groups. Setting $x = 0$ and replacing y by x in (V1), we obtain $v_G(-x) \geq \min\{v_G(0), v_G(x)\} = v_G(x)$, since by (V0), $v_G(0) = \infty$. Doing this once more, but replacing y by $-x$ we get $v_G(x) \geq \min\{v(_G 0), v_G(-x)\} = v_G(-x)$. Combining this two results, we obtain $v_G(-x) \geq v_G(x) \geq v_G(-x)$ and therefore we obtain the "symmetry":

(V2) $v_G(x) = v_G(-x)$, for all $x \in G$.

From these rules we may deduce

(V3) $v_G(\sum_{1\leq i\leq n} x_i) = \min_{1\leq i\leq n} v_G(x_i)$ if all non-zero x_i have different values,

(V4) $v_G(x-y) > \min\{v_G(x), v_G(y)\} \Rightarrow v_G(x) = v_G(y)$.

Observe that for every $a \in G$ and every $n \in \mathbb{Z} \setminus \{0\}$, the element $na \in G$ is archimedean equivalent to a, therefore the natural valuation satisfies the axiom scheme:

(NV1) $v_G(nx) = v_G(x)$, where $(0 \neq n \in \mathbb{Z})$.

Recall that v_G is compatible with the order:

(NV2) $v_G(x) < v_G(y) \Rightarrow |x| > |y|$.

Also note the following:

(NV3) $\mathrm{sign}(\sum_{1\leq i\leq n} x_i) = \mathrm{sign}(x_m)$ if $v_G(x_m) < v_G(x_i)$ for all $i \neq m$,

(NV4) $v_G(x-y) > v_G(x) \Rightarrow \mathrm{sign}(x) = \mathrm{sign}(y)$.

Further, we will present and discuss the axioms for contractions on ordered Abelian groups. We will use the language $\mathfrak{L}_{cg} = \{+, -, 0, <, \chi\}$, where $<$ is a binary relation symbol, $+$ is a binary function symbol, and $-$ and χ are unary function symbols. An $\mathfrak{L}_{cg}$-structure $(G, +, -, 0, <, \chi)$ will be called a *precontraction group* if it satisfies the following axioms:

(OAG) $(G, +, -, 0, <)$ is an ordered Abelian group,

(C0) $\chi(x) = 0 \Leftrightarrow x = 0$,

(C≤) χ preserves the ordering,

(C−) $\chi(-x) = -\chi(x)$,

(CA) If x is archimedean equivalent to y and sign(x) = sign(y), then $\chi(x) = \chi(y)$.

If these axioms hold, then we call χ a *precontraction.* If in addition,

(CS) χ is surjective,

then we call χ a *contraction* and the group G is a *contraction group.* Axioms (CA) and (CS) together show that every archimedean ordered contraction group must be trivial.

$(G, +, -, 0, <, \chi)$ will be called *centripetal* if it satisfies one more axiom:

(CP) $x \neq 0 \Rightarrow |x| > |\chi(x)|$,

and it will be called *centrifugal* if it satisfies

(CF) $x \neq 0 \Rightarrow |x| < |\chi(x)|$.

Using axiom (C−), axiom (CA) may be expressed by the following recursive elementary axiom scheme:

$$x \geq y > 0 \wedge ny \geq x \Rightarrow \chi(x) = \chi(y), \text{ for all } n \in \mathbb{N}.$$

We can observe that (C0) together with (C≤) imply

(CSN) $\mathrm{sign}(\chi(\mathrm{x})) = \mathrm{sign}(\mathrm{x})$

and that axioms (C0), together with (C−) and (C≤) imply

(CZ) $\chi(zx) = \mathrm{sign}(\mathrm{z}) \cdot \chi(\mathrm{x})$, where $z \in \mathbb{Z}$.

Moreover, by axiom (C≤), we have

(CC) $(x \leq y \leq z \wedge \chi(x) = \chi(z)) \Rightarrow \chi(y) = \chi(z)$

More generally, we can say that the preimage of every convex set under χ is convex. In the presence of axiom (C≤), the axiom scheme (CA) may be replaced by a single axiom

(CA') $\chi(2x) = \chi(x)$.

All these axioms are elementary in the language $\mathfrak{L}_{cg}$. Except for the surjectivity axiom (CS), all are universal and thus, all properties described by them will be inherited by substructures.

Lemma 1.6.2. *Every substructure S of a precontraction group (G, χ) is again a precontraction group. If (G, χ) is centripetal (resp. centrifugal), then so is S.*

The next axiom scheme says that $(G, +, 0)$ is divisible:

(D) $\forall x \exists y$: $ny = x$ $(0 \neq n \in \mathbb{N})$.

Note that (D) is not universal.

We will consider the theory of non-trivial divisible centripetal contraction groups. The axiom that guarantees that the groups are non-trivial is: $\exists x$: $x \neq 0$. Axiom (CA) can be expressed as follows:

(CV1) $(v_G(x) = v_G(y) \wedge$ sign(x)=sign$(y)) \Rightarrow \chi(x) = \chi(y)$

because x is Archimedean equivalent to y may be express using v_G in the following way $v_G(x) = v_G(y)$. Note that v_G is not definable in the theory of divisible centripetal or centrifugal group and it is not a symbol in our language. From now on, we may write $a = \pm b$ instead of $|a| = |b|$, if we will be only interested in equality up to the sign. Then (CV1) becomes:

(CV2) $v_G(x) = v_G(y) \Rightarrow \chi(x) = \pm\chi(y)$

(CV1) together with (V3) and (NV3) gives us:

(CV3) $\chi(\sum_{1 \leq i \leq n} x_i) = \chi(x_m)$ if $v_G(x_m) < v_G(x_i)$, for all $i \neq m$,

and from (CV1) together with (V4) and (NV4) one may deduce the following:

(CV4) $v_G(x - y) > v_G(x) \Rightarrow \chi(x) = \chi(y)$.

From (CV2), (NV2) and (C$\leq$) we obtain:

(CV5) $v_G(x) \leq v_G(y) \Rightarrow |\chi(x)| \geq |\chi(y)|$. Here is a general result about precontraction:

Lemma 1.6.3 ([9], p. 7)**.** *Let (G, χ) be a precontraction group. Then the following assertions hold:*

1. sign(a)=sign$(\chi(a))$ for every $a \in G$, hence $\chi(G^{<0}) \subset G^{<0}$ and $\chi(G^{>0}) \subset G^{>0}$. Moreover, $\chi(G^{<0}) = -\chi(G^{>0})$.

2. χ is centripetal if and only if $v_G(\chi(a)) > v_G(a)$ for all $a \in G \setminus \{0\}$. Similarly, χ is centrifugal if and only if $v_G(\chi(a)) < v_G(a)$ for all $a \in G \setminus \{0\}$.

3. Every non-trivial centripetal precontraction group is densely ordered. The same is

true for every non-trivial centrifugal contraction group.

Proof. 1) Take $a \in G$, $a < 0$. By (C$\leq$) we get that $\chi(a) < \chi(0)$, and by (C0) that $\chi(0) = 0$. Therefore, $a < 0 \Rightarrow \chi(a) < 0$. Similarly, we can prove $a > 0 \Rightarrow \chi(a) > 0$. Now let $b \in G^{>0}$ and by (C$-$), $\chi(-b) = -\chi(b)$, where $-b \in G^{<0}$. Similarly follows the other direction, so $\chi(G^{<0}) = -\chi(G^{>0})$.

2) Assume that χ is centripetal. Then by (CP), for every $a \in G$, $a \neq 0$, we have $|\chi(a)| < |a| \Rightarrow v_G(\chi(a)) \geq v_G(a)$, since v_G is compatible with the order on G. Now for every $b \in G$ such that $v_G(a) = v_G(b)$, that is, a is Archimedean equivalent to b, we get from (CA), that $\chi(a) = \pm\chi(b)$ and thus $|\chi(a)| = |\chi(b)|$. Moreover, since χ is centripetal, (CP) gives us $|\chi(a)| = |\chi(b)| < |b|$. Hence, $v_G(\chi(a)) = v_G(a)$ is impossible. For the converse, assume that $v_G(\chi(a)) > v_G(a)$, for all $a \in G \setminus \{0\}$. Then by (NV2) $|a| > |\chi(a)|$, and therefore χ is centripetal. Similarly, one can prove for centrifugal precontractions.

3) We want to show that a precontraction group cannot have a least positive element 1. Take $0 < g$, $g \in G$. Then by (C$\leq$) we have $\chi(0) < \chi(g)$ and by (C0) $\chi(0) = 0$. Combining them and using also (CP), we obtain $0 < \chi(g) < g$ for every $g \in G^{>0}$. For the second part of the proof, assume that 1 is the least positive element in a centrifugal contraction group (G, χ). Then there must be some $a \in G$ for which $\chi(a) = 1$. As we showed above, using (C$\leq$), (C0) and (CF), $0 < a < 1$, contradiction with our assumption. This completes the proof. □

Consider the Hahn sums $\coprod_{\mathbb{N}} \mathbb{Z}$ and $\coprod_{-\mathbb{N}} \mathbb{Z}$, where $\mathbb{Z}$ is the ordered group $(\mathbb{Z}, +, -, 0, <)$, $\mathbb{N}$ the positive integers and $-\mathbb{N}$ the negative integers with their usual ordering. Let us define a precontraction on $\coprod_{\mathbb{N}} \mathbb{Z}$ in the following way: if $(a_i)_{i\in\mathbb{N}} \in \coprod_{\mathbb{N}} \mathbb{Z}$ and if i_0 is the minimal index such that $a_{i_0} \neq 0$, then we set

$$\chi(a_i)_{i\in\mathbb{N}} = \mathrm{sign}(a_{i_0}) \cdot e_{i_0+1}$$

where $e_i \in \coprod_{\mathbb{N}} \mathbb{Z}$ has a 1 at the index i and zeros everywhere else. We will denote the resulting centripetal precontraction group by $\mathcal{P}_{cp}$. Similarly, if $(a_i)_{i \in -\mathbb{N}} \in \coprod_{-\mathbb{N}} \mathbb{Z}$ and if i_0 is the maximal index such that $a_{i_0} \neq 0$, then we set

$$\chi(a_i)_{i \in \mathbb{N}} = \text{sign}(a_{i_0}) \cdot e_{i_0 - 1}$$

where e_i is an element of $\coprod_{-\mathbb{N}} \mathbb{Z}$, defined as above. We will denote the resulting centrifugal precontraction group by $\mathcal{P}_{cf}$.

Lemma 1.6.4 ([9], p. 8)**.** *$\mathcal{P}_{cp}$ is the prime structure of the elementary class of non-trivial centripetal precontraction groups. Analogously, $\mathcal{P}_{cf}$ is the prime structure of the elementary class of non-trivial centrifugal precontraction groups.*

The next lemma shows that every precontraction group is embeddable in a divisible contraction group.

Lemma 1.6.5 ([9], p. 10)**.** *Every (centripetal resp. centrifugal) precontraction group (G, χ) is embeddable in a divisible (centripetal resp. centrifugal) contraction group (H, χ).*

Define a new map $\rho_\chi : v_G(G) \to G^{<0}$ as follows: if $\alpha = v_G(a) \in v_G(G)$, $a \in G^{<0}$, then $\rho_\chi(\alpha) = \chi(a)$. This is a well-defined map by (CV1). Note that ρ_χ is surjective if and only if χ is. If $b = \chi(a) \in \chi(G)$, then $b = \rho_\chi(v_G(a))$ and thus $\rho_\chi(v_G(G)) = \chi(G^{<0})$. Moreover, ρ_χ preserves $\leq$: by (C$\leq$), χ preserves $\leq$ and $v_G(a) < v_G(a')$ implies $a < a'$, for $a, a' \in G^{<0}$.

We will call (G', χ) a *divisible contraction hull* of (G, χ) if it is a divisible contraction group and has the following universal property:

(CHD) if $(G, \chi) \subset (H, \chi)$ is any extension of precontraction groups and (H, χ) is a divisible contraction group, then there is an embedding of (G', χ) in (H, χ) over (G, χ).

Lemma 1.6.6 ([9], p. 10)**.** *For every precontraction group (G, χ), there exists a contraction hull and a divisible contraction hull. Such a hull (G', χ) may be chosen such that ρ_χ induces* *an order preserving bijection $v_G(G') \setminus v_G(G) \to G'^{<0} \setminus \chi(G)$. Moreover, we can assume that for every $b \in G'$ there is some $n \in \mathbb{N}$ such that $\chi^n(b) \in G$.*

Now we apply the last lemma to the prime structures $\mathcal{P}_{cp}$ and $\mathcal{P}_{cf}$ and obtain:

Lemma 1.6.7 ([9], p. 11)**.** *The elementary classes of non-trivial centripetal (resp. centrifugal) contraction groups and of non-trivial divisible centripetal (resp. centrifugal) contraction groups have prime models.*

Lemma 1.6.8 ([9], p. 19)**.** *Let $(G, \chi) \subset (H, \chi)$ be an extension of centripetal precontraction groups and assume that (G, χ) is a non-trivial divisible contraction group. Then (G, χ) is existentially closed in (H, χ).*

Applying Robinson's Test and the previous lemma we obtain:

Theorem 1.6.3 ([9], p. 20)**.** *The elementary theory of non-trivial divisible centripetal contraction groups is model complete.*

Assume that (G, χ) is a common substructure of two divisible centripetal contraction groups (H, χ) and (H', χ). Since the properties described by universal axioms are inherited by their substructures (Lemma 1.6.2), (G, χ) is a centripetal precontraction group. By Lemma 1.6.6, there exists a divisible contraction hull (G', χ) of (G, χ) in (H, χ) which embeds in (H', χ) over (G, χ). Now let us identify (G', χ) with its image in (H', χ). By Lemma 1.6.4, $\mathcal{P}_{cp}$ is the prime structure of the theory of centripetal contraction groups. If G is the trivial group, then we may replace (G, χ) by the prime structure $\mathcal{P}_{cp}$, in order to get that (G', χ) is non-trivial. Thus (G', χ) is a non-trivial divisible centripetal contraction group. Since the theory of non-trivial divisible centripetal contraction group is model complete, we can say that (H, χ) and (H', χ) are elementary equivalent over (G', χ) and also over (G, χ). We assumed that (G, χ) is a common substructure of (H, χ) and (H', χ), so we have shown that the theory of divisible centripetal contraction groups is substructure complete. Then by Theorem 1.6.1 we have:

Theorem 1.6.4 ([9], p. 20)**.** *The elementary theory of non-trivial divisible centripetal contraction groups admits elimination of quantifiers in the language* $\mathcal{L}_{cg}$.

The preceding theorem and Lemma 1.6.5 give us:

Theorem 1.6.5 ([9], p. 20)**.** *The elementary theory of non-trivial divisible centripetal contraction groups is the model completion of the theory of centripetal precontraction groups.*

By Proposition 1.6.2 (The Prime Structure Test), we obtain:

Theorem 1.6.6 ([9], p. 20)**.** *The elementary theory of non-trivial divisible centripetal contraction groups is complete.*

The axiom system for the theory of non-trivial divisible centripetal contraction groups {(OAG), (C0), (CS), (C$\leq$), (C$-$), (CA$'$), (CP), (D), and $\exists x : x \neq 0$} is recursive. Combining this result with the foregoing theorem, we can conclude:

Theorem 1.6.7 ([9], p. 20)**.** *The elementary theory of non-trivial divisible centripetal contraction groups is decidable.*

Chapter 2

Closed Asymptotic Couples

In this chapter, we explore the structure induced by the derivation of a Hardy field K on the value group Γ of the natural valuation on K. The logarithmic derivative on K gives us a function $\psi : \Gamma \setminus \{0\} \to \Gamma$, such that $\psi(v(f)) = v(f'/f)$, where $f \in K \setminus \{0\}$ with $v(f) \neq 0$. By Rosenlicht's terminology, the pair (Γ, ψ) is called the *asymptotic couple* of K. We continue with the study of some algebraic and model-theoretic aspects of asymptotic couples, as in [1] and [2]. One of the most valuable results comes from [1] and it is shown in the last section of this chapter: that the asymptotic couples resemble the contraction groups and there is an important connection between them.

2.1 Definitions and Basic Properties

An important class of asymptotic couples is obtained from Hardy fields.

Let K be a Hardy field. The valuation

$$v : K^* = K \setminus \{0\} \to G = v(K^*)$$

associated to the place $f \mapsto \lim_{x\to\infty} f(x)$ (here we identify $+\infty$ and $-\infty$) has the important property that $v(f')$ depends only on $v(f)$, for $f \in K^*$ with $v(f) \neq 0$. Therefore we have a well-defined map $\psi : G^* = G \setminus \{0\} \to G$ given by $\psi(v(f)) := v(f'/f)$ for any $f \in K^*$

with $v(f) \neq 0$. The pair (G, ψ) called the asymptotic couple of K is very important to understand the interaction of the ordering and the derivation of K. It has the following properties: for all elements $f, g \in K^*$ with $a = v(f) \neq 0$, $b = v(g) \neq 0$,

- (A1) $\psi(ra) = \psi(a)$ for all $r \in \mathbb{Z}, r \neq 0$,
- (A2) $\psi(a+b) \geq \min\{\psi(a), \psi(b)\}$, where $\psi(0) := \infty > G$,
- (A3) $\psi(a) < \psi(b) + |b|$.

Moreover, ψ is decreasing on the set of positive elements of G, that is axiom (H): For all $a, b \in G$,

- (H) $0 < a \leq b \Rightarrow \psi(a) \geq \psi(b)$.

Therefore, by $(A1)$, the map ψ is increasing on the set of negative elements of G.

Axiom (A2) expresses the fact that ψ is a valuation on the ordered Abelian group G. For a, b as above, it follows that $\psi(a+b) = \min\{\psi(a), \psi(b)\}$ if $\psi(a) \neq \psi(b)$. Note that if the Hardy field K contains the germ x of the identity function on $\mathbb{R}$ and because $\psi(v(x^{-1})) = v((x^{-1})'/x^{-1}) = v(-x^{-2} \cdot x) = v(x^{-1})$, then we get $\psi(1) = 1$, where we put $1 =: v(x^{-1}) > 0$.

By an *asymptotic couple* we mean a pair (G, ψ) consisting of an ordered Abelian group G and a map $\psi : G^* \to G$ satisfying for all $a, b \in G^*$, the axioms $(A1)$-$(A3)$ above. Observe that because of $(A2)$ the map ψ is a valuation. We say that an asymptotic couple (G, ψ) is *of H-type* or an *H-asymptotic couple* if axiom (H) holds for all a, $b \in G$.

Notations: If G is an ordered Abelian group, then we put $S^{>0} = \{v \in S : v > 0\}$, $S^{<0} = \{v \in S : v < 0\}$ for any subset S of G.

By $[G]$ we denote the set of archimedean classes of G, and we set $[G^*] := [G] \setminus \{[0]\}$. We linearly order [G] by setting

$$[v] < [w] :\Leftrightarrow n|v| < |w| \text{ for all n} \Leftrightarrow [v] \neq [w] \text{ and } |v| < |w|.$$

Here are some facts about archimedean classes: for $v, w \in G$, $r \in \mathbb{Z} \setminus \{0\}$, we have:

1. $[v] = 0 \Leftrightarrow v = 0$,

2. $[v] = [rv]$,

3. $[v + w] \leq \max\{[v], [w]\}$, with $[v + w] = \max\{[v], [w]\}$, if $[v] \neq [w]$.

From now on, we use the version of M. Aschenbrenner for the third property of Archimedean classes. We want to be consistent with the source and to make it easier in order to continue future research based on this paper.

A *Hahn space* is an ordered vector space V over an ordered field k such that, for all vectors v, $w \in G^*$, we have that

$$[v] = [w] \Rightarrow \exists \lambda \in k \text{ such that } [v - \lambda w] < [w]$$

An *H-couple* (G, ψ) consists of a Hahn space G over an ordered field k, a distinguished positive element $1 \in G$, and a function $\psi : G^* \to G$, such that, for all v, $w \in G^*$,

1. $\psi(1) = 1$

2. $\psi(v) \leq \psi(w) \Leftrightarrow [v] \geq [w]$ (hence $\psi(v) = \psi(w) \Leftrightarrow [v] = [w]$

3. $\psi(v) < \psi(w) + |w|$

Note that H-asymptotic couples are H-couples, since axiom (A1) is included in property 2. of H-couples.

In the case of Hardy fields, and also in the case of differential fields of germs of real-valued functions at ∞, we have the important principle:

- (H$'$) $0 < \alpha \leq \beta \Rightarrow \psi(\alpha) \geq \psi(\beta)$. If $\alpha > n\beta$ for all positive integers n, then $\psi(\alpha) < \psi(\beta)$, and in the contrary case $\psi(\alpha) = \psi(\beta)$.

In other words, $0 < \alpha \leq \beta \Leftrightarrow \psi(\alpha) \geq \psi(\beta)$. Observe that this is principle is included in property 2. of H-couples. By this principle, the asymptotic couple corresponding to a Hardy field that contains $\mathbb{R}$ is of Hardy type. We now prove the principle in the case of Hardy fields.

Proof. Take k to be a Hardy field of real-valued functions near $+\infty$. Choose f, $g \in k$ such that $\alpha = v(f)$, $\beta = v(g)$. Then $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \frac{f(x)}{g(x)} = 0$. We may take $f(x)$, $g(x) > 0$. Then for large x we have $f(x) < g(x)$. So, $\log f(x) < \log g(x)$, or $|\log f(x)| > |\log g(x)|$, since these are logarithms of very small positive numbers. We may assume that $\log f$, $\log g \in k$. Since for large x we have $|\log f(x)| > |\log g(x)|$, we get $v(\log f) \leq v(\log g)$. This implies that $v((\log f)') \leq v((\log g)')$, or $v(f'/f) \leq v(g'/g)$, which is equivalent to $\psi(\alpha) \leq \psi(\beta)$. This proves the first part of the principle. To prove the second statement, choose f, g as above and apply the same argument to f and g^n. Then for large x we obtain $|\log f(x)| > |n \log g(x)|$, or $|\log f(x)/\log g(x)| > n$. This is true for all integers, so $v(\log f/\log g) = v(\log f) - v(\log g) < 0$. Then $v(\log f) < v(\log g)$, so $v(f'/f) < v(g'/g)$, or $\psi(\alpha) < \psi(\beta)$. Finally, if there is a positive integer n, such that $\alpha < n\beta$, then $\psi(\alpha) \leq \psi(\beta) = \psi(n\beta) \leq \psi(\alpha)$. This proves the last statement $\psi(\alpha) = \psi(\beta)$. $\square$

The inclusion map $G \hookrightarrow G'$ induces an embedding $[G] \hookrightarrow [G']$, where $[G]$ and $[G']$ are linearly ordered sets. Now we identify G with its image under this embedding.

We consider G as a subgroup of the divisible Abelian group $\mathbb{Q}G = \mathbb{Q} \otimes_{\mathbb{Z}} G$ by taking the map $v \mapsto 1 \otimes v$, which is an embedding of G in the vector space $\mathbb{Q}G = \mathbb{Q} \otimes_{\mathbb{Z}} G$ over the ordered field $\mathbb{Q}$. Note that $[\mathbb{Q}G] = [G]$.

2.2 Properties of H-asymptotic Couples

In this section, (G, ψ) is an asymptotic couple. We set $\Psi := \psi(G^*)$ and we denote by id the identity function on G. Let $(id + \psi)(G^*) = \{x + \psi(x) : x \in G^*\}$ and similarly $(id+\psi)(G^{>0}) = \{x+\psi(x) : x \in G^{>0}\}$ and $(id+\psi)(G^{<0}) = \{x+\psi(x) : x \in G^{<0}\}$. Also, let $G_\infty := G \cup \{\infty\}$, with $\infty > v$ for all $v \in G$ and $v + \infty = \infty + v = -\infty = \infty$ for all $v \in G_\infty$. Then it is convenient to extend ψ to a map $G_\infty \to G_\infty$ by setting $\psi(0) := \psi(\infty) := \infty$.

Remark: Define $\psi + a : G^* \to G$ by $(\psi + a)(x) := \psi(x) + a$, for $a \in G$. Then $(G, \psi + a)$ is an asymptotic couple, with $(\psi+a)(G^*) = \psi(G^*)+a = \Psi+a$. We now check if $(G, \psi+a)$ satisfies (A1)-(A3), for $b, c \in G^*$:

(A1) $(\psi + a)(rb) = \psi(rb) + a = \psi(b) + a = (\psi + a)(b)$ for all $r \in \mathbb{Z}, r \neq 0$ and because $\psi(rb) = \psi(b)$ for all $r \in \mathbb{Z}, r \neq 0$, since (G, ψ) is an asymptotic couple.

(A2) $(\psi+a)(b+c) \geq \min\{(\psi+a)(b), (\psi+a)(c)\} \Leftrightarrow \psi(b+c)+a \geq \min\{\psi(b)+a, \psi(c)+a\} \Leftrightarrow \psi(b+c) \geq \min\{\psi(b), \psi(c)\}$, true since (G, ψ) is an asymptotic couple. We also have $(\psi + a)(0) = \psi(0) + a = \infty + a = \infty$.

(A3) $(\psi+a)(b) < (\psi+a)(c) + |c| \Leftrightarrow \psi(b) + a < \psi(c) + a + |c| \Leftrightarrow \psi(b) < \psi(c) + |c|$, true since the latter relation is exactly (A3) for the asymptotic couple (G, ψ).

For all positive elements $b, c \in G$, $(G, \psi + a)$ satisfies (H) if and only if (G, ψ) does:

(H) $(0 < b \leq c \Rightarrow (\psi + a)(b) \geq (\psi + a)(c)) \Leftrightarrow (0 < b \leq c \Rightarrow \psi(b) + a \geq \psi(c) + a) \Leftrightarrow (0 < b \leq c \Rightarrow \psi(b) \geq \psi(c))$.

Proposition 2.2.1 ([1], p. 3)**.** *Let v, $w \in G$.*

1. *If v, $w \neq 0$, then $n(\psi(w) - \psi(v)) < |v|$.*

2. *If v, w, $v-w \neq 0$, then* $[\psi(v)-\psi(w)] < [v-w]$.

3. *The map* $x \mapsto x+\psi(x) : G^* \to G$ *is strictly increasing.*

Proof. We want to prove 1. Let v, $w \in G^*$ and $n \in \mathbb{N}$. We may assume $\psi(w) > \psi(v)$, v, $w > 0$ and $n > 0$, because if $\psi(w) = \psi(v)$, or $n \leq 0$, our inequality is obvious. By passing from ψ to $\psi - \psi(v)$, if necessary, we can reduce to the case where $\psi(v) = 0 < \psi(w)$. Therefore we have to show $n\psi(w) < v$ and we proceed by induction on n. If $n = 1$, then $n\psi(w) = \psi(w) < v + \psi(v) = v$ holds by axiom (A3) for asymptotic couples. Induction step: Assume that $n\psi(w) < v$ and show that $(n+1)\psi(w) < v$. If $(n+1)\psi(w) \leq \psi(u)$ for some $u \in G^*$, we have $(n+1)\psi(w) \leq \psi(u) < v + \psi(v) = v$ again by axiom (A3). Now we can assume that $(n+1)\psi(w) > \psi(v)$ for all $v \in G^*$ i.e. $(n+1)\psi(w) > \Psi$. Observe that $\psi^2(w) > 0$ since $\psi(w) > 0$. Then $\psi(w) < \psi(w) + \psi(\psi(w))$ and hence

$$\psi(v-(n+1)\psi(w)) = \min\{\psi(v), \psi^2(w)\} = \psi(v) = 0$$

by (A2), so we get

$$\Psi < \psi(v-(n+1)\psi(w)) + |v-(n+1)\psi(w)| = |v-(n+1)\psi(w)|.$$

Suppose $v \leq (n+1)\psi(w)$. Then, $\Psi < (n+1)\psi(w) - v$ and in particular $\psi(w) < (n+1)\psi(w) - v = n\psi(w) + \psi(w) - v$. Hence $v < n\psi(w)$, contradiction with the induction hypothesis. Therefore $(n+1)\psi(w) < v$, completing the induction step.

To prove 2., let v, $w \neq 0$ with $d := v-w \neq 0$. Then we have to show $n|\psi(v)-\psi(w)| < |d|$ for all n. If $\psi(d) > \psi(w)$ and since ψ is a valuation on the ordered Abelian group G, we know that $\psi(d) = \psi(v-w) \geq \min\{\psi(v), \psi(w)\}$, and hence $\psi(v) = \psi(w)$. Suppose that $\psi(d) \leq \psi(w)$. If $\psi(v) < \psi(d)$, again because ψ is a valuation, we get $\psi(v) = \psi(w) < \psi(d)$ contradiction with our assumption. Therefore $\psi(d) \leq \psi(v)$. Multiplying by $n \in \mathbb{N}$ we have $n\psi(d) \leq n\psi(w)$ and $n\psi(d) \leq n\psi(v)$ and hence by (1):

$n\psi(d) \leq n\psi(w) < n\psi(d) + |d|$ and

$n\psi(d) \leq n\psi(v) < n\psi(d) + |d|$.

Thus $n|\psi(v) - \psi(w)| < |d|$ in all cases.

For Property 3. we have to show that $x < y \Rightarrow x + \psi(x) < y + \psi(y)$. By Property 2. we know that $n|\psi(x) - \psi(y)| < |x - y|$. Taking $n = 1$ we get $|\psi(x) - \psi(y)| < y - x$ i.e. $x - y < \psi(x) - \psi(y) < y - x$, and the last inequality completes the proof. □

Remark: By (A1) and Property 1. of the proposition above, ψ extends uniquely to a map $(\mathbb{Q}G)^* \to G$, which we denote also by ψ, such that $(\mathbb{Q}G, \psi)$ is an asymptotic couple, with $\psi((\mathbb{Q}G)^*) = \Psi$ since $\psi(\frac{p}{q}a) = \psi(q \cdot \frac{p}{q}a) = \psi(pa) = \psi(a)$ for all $a \in G^*$.

From now on, we will concentrate on H-asymptotic couples. Suppose that (G, ψ) is of H-type. Note that if $[v] \leq [w]$, then $|v| \leq n|w|$ for some $n > 0$ and for v, $w > 0$ we have by Axiom (H) that $0 < v \leq nw \Rightarrow \psi(v) \geq \psi(nw)$. Thus, by Axiom (A1) we have $\psi(nw) = \psi(w)$ for all $n \neq 0$. We can conclude that

$$[v] \leq [w] \Rightarrow \psi(v) \geq \psi(w) \text{ for all } v,\ w \in G^*. \tag{1.1}$$

In particular, ψ is constant on archimedean classes of G, i.e.

$$\text{for all } v,\ w \in G \text{ with } [v] = [w], \text{ we have } \psi(v) = \psi(w). \tag{1.2}$$

We know that $[v] = [w]$ if and only if $|v| \leq n|w|$ and $|w| \leq m|v|$ for some m, $n > 0$. By axiom (H) we have that $|v| \leq n|w| \Rightarrow \psi(v) \geq \psi(nw) = \psi(w)$ and $|w| \leq m|v| \Rightarrow \psi(w) \geq \psi(mv) = \psi(v)$. This gives us $\psi(v) = \psi(w)$.

We can use the same argument as we used for making $\mathbb{Q}G$ into an asymptotic couple and also (1.2), to show:

Corollary 2.2.1 ([1], p. 4)**.** *Let G' be an ordered Abelian group containing G as ordered subgroup such that $[G] = [G']$. Then there is a unique extension of ψ to a function $\psi' : (G')^* \to G'$ such that (G', ψ') is an H-asymptotic couple.*

Proof. Define ψ' such that $\psi'(x) = \psi(v)$ for $[x] = [v]$. This is well-defined since by (1.2) we have $[v] = [w] \Rightarrow \psi(v) = \psi(w)$. We have to verify that ψ' satisfies axioms (A1)-(A3) and also axiom (H).

First we show (A1) $\psi'(rx) = \psi'(x)$, for all $r \in \mathbb{Z}$, $r \neq 0$. We know that $[rx] = [x] = [v]$ by Property 2. of archimedean classes, and thus $\psi'(rx) = \psi(v) = \psi'(x)$.

Check (A2) $\psi'(x+y) \geq \min\{\psi'(x), \psi'(y)\}$, where $\min\{\psi'(x), \psi'(y)\} = \min\{\psi(v), \psi(w)\}$ for $[x] = [v]$ and $[y] = [w]$. Define $\psi'(x + y) = \psi(z)$ for $[x + y] = [z]$. We have that $[z] = [x+y] \leq \max\{[x], [y]\}$ by Property 3. of archimedean classes, and since $\max\{[x], [y]\} = \max\{[v], [w]\}$ we get $[z] \leq \max\{[v], [w]\}$. Then $\psi'(x + y) = \psi(z) \geq \min\{\psi(v), \psi(w)\} = \min\{\psi'(x), \psi'(y)\}$ and we are done.

Now we want to prove (A3) $\psi'(x) < \psi'(y) + |y|$. From Proposition 2.2.1 1. we know that if v, $w \in G$, v, $w \neq 0$, then $n(\psi(w) - \psi(v)) < |v|$. In this case $n(\psi(v) - \psi(w)) < |w|$ or $n(\psi'(x) - \psi'(y)) < |w|$ for $[x] = [v]$ and $[y] = [w]$. By the definition of equivalence relation, $[y] = [w]$ implies that $|w| \leq n|y|$ for some $n > 0$, so $|w|/n \leq |y|$. But $n(\psi'(x) - \psi'(y)) < |w|$, or $\psi'(x) - \psi'(y) < |w|/n \leq |y|$. Therefore we get $\psi'(x) < \psi'(y) + |y|$.

For axiom (H) we assume that $0 < x \leq y$ to show $\psi'(x) \geq \psi'(y)$, knowing that $0 < v \leq w \Rightarrow \psi(v) \geq \psi(w)$. We pick v, $w \in G$ such that $[x] = [v]$ and $[y] = [w]$, with v, $w > 0$. If $[x] \leq [y]$, then $[v] \leq [w]$ and by (1.1) we have $\psi(v) \geq \psi(w)$. Since $\psi(v) = \psi'(x)$ and $\psi(w) = \psi'(y)$, we get that $0 < x \leq y \Rightarrow \psi'(x) \geq \psi'(y)$ and therefore axiom (H) holds.

We have proved that (G', ψ') is an H-asymptotic couple. $\square$

Lemma 2.2.1 ([1], p. 4)**.** *Let* $w \in G^*$. *If* $[\psi(w)] \geq [w]$, *then* $[\psi(\psi(w))] = [\psi(w)]$.

Proof. By (1.2), we may suppose that $[\psi(w)] > [w]$. By Proposition 2.2.1 (2) we have $[\psi(w) - \psi(\psi(w))] < [w - \psi(w)]$ and by Property (3) of archimedean classes $[w - \psi(w)] = \max\{[w], [\psi(w)]\}$. We assumed that $[\psi(w)] > [w]$ so $\max\{[w], [\psi(w)]\} = [\psi(w)]$. Therefore

$[\psi(w) - \psi(\psi(w))] < [\psi(w)]$ and hence $[\psi(\psi(w))] = [\psi(w)]$. $\square$

Remark: Lemma 2.2.1 and (1.1) imply that if $w \in G^*$ satisfies $[w] \leq [\psi(w)]$, then $y + \psi(y) = 0$ for $y = -\psi(\psi(w))$.

Proof. Let $y = -\psi(\psi(w))$. By Lemma 2.2.1, if $[w] \leq [\psi(w)]$, then $[\psi(w)] = [\psi(\psi(w))]$ and since ψ is constant on archimedean classes of G, we have $\psi(\psi(\psi(w))) = \psi(\psi(w))$, which gives us $-\psi(\psi(w)) + \psi(\psi(\psi(w))) = -\psi(\psi(w)) + \psi(-\psi(\psi(w))) = y + \psi(y) = 0$. $\square$

We call a subset X of G *closed upward* (in G) if $a \in G$, $a > b \in X \Rightarrow a \in X$. A subset X of G is *closed downward* (in G), or a *cut in* G, if $a \in G$, $a < b \in X \Rightarrow a \in X$.

The following facts about $id + \psi$ are fundamental:

Corollary 2.2.2 ([1], p. 4)**.** *The set $(id + \psi)(G^{>0})$ is closed upward.*

The set $(id + \psi)(G^{<0})$ is closed downward, and

(1.3) $$(-id + \psi)(G^{>0}) = (id + \psi)(G^{<0}) = \{a \in G : a < \psi(x) \text{ for some } x \in G^*\}$$

There is at most one element $v \in G$ such that $\Psi < v < (id + \psi)(G^{>0})$. If Ψ has a largest element, then there is no $v \in G$ with $\Psi < v < (id + \psi)(G^{>0})$.

Proof. We want to show that the set $(id + \psi)(G^{>0}) = \{x + \psi(x) : x \in G^{>0}\}$ is closed upward, where $G^{>0} = \{x \in G : x > 0\}$. Let $a > x + \psi(x)$ for some $x > 0$; we want to show that $a \in (id + \psi)(G^{>0})$. Passing from (G, ψ) to $(G, \psi - a)$ if necessary, we reduce to case $a = 0$. Then $0 > x + \psi(x)$ for some $x > 0$, or $0 < x < -\psi(x)$. Since $0 < |x| < |\psi(x)|$, we have that $[x] \leq [\psi(x)]$. By the previous remark if $x \in G^*$ satisfies $[x] \leq [\psi(x)]$, then $a = y + \psi(y) = 0 \in (id + \psi)(G^{>0})$, for $y = -\psi(x)$. We also know from proposition 2.2.1 3. that $x \mapsto x + \psi(x) : G^* \to G$ is strictly increasing and thus $x < y \Rightarrow 0 = x + \psi(x) < y + \psi(y)$ which completes the proof that $(id + \psi)(G^{>0})$ is closed upward.

Now we want to show that the set $(id+\psi)(G^{<0}) = \{x+\psi(x) : x \in G^{<0}\}$ is closed downward, where $G^{<0} = \{x \in G : x < 0\}$. Let $a < x + \psi(x)$ for some $x < 0$; we want to show that $a \in (id+\psi)(G^{<0})$. Passing from (G, ψ) to $(G, \psi - a)$ if necessary, we reduce to case $a = 0$. Then $0 < x + \psi(x)$ for some $x < 0$, or $-\psi(x) < x < 0$. Since $|\psi(x)| > |x|$, we obtain that $[x] \leq [\psi(x)]$ and by the previous remark $y + \psi(y) = 0$ for $y = -\psi(x) \in G^{<0}$. Therefore $a = y+\psi(y) = 0 \in (id+\psi)(G^{<0})$ and because $x \mapsto x+\psi(x) : G^* \to G$ is strictly increasing, i.e, $y < x \Rightarrow y + \psi(y) < x + \psi(x) = 0$, we have that $(id + \psi)(G^{<0})$ is closed downward.

We want to prove the first equality in (1.3), i.e. $(-id+\psi)(G^{>0}) = (id+\psi)(G^{<0})$. Let $a \in (-id+\psi)(G^{>0})$, or $a = -x + \psi(-x)$ for some $x > 0$. Then $a \in (id + \psi)(G^{<0})$ and thus $(-id + \psi)(G^{>0}) \subseteq (id + \psi)(G^{<0})$. Let $b \in (id + \psi)(G^{<0})$, or $b = x + \psi(x)$ for some $x < 0$. We have that $b = x+\psi(x) = -(-x)+\psi(-x)$ with $-x > 0$, so $b \in (-id+\psi)(G^{<0})$. Therefore $(id + \psi)(G^{<0}) \subseteq (-id + \psi)(G^{>0})$ and we have proved the first equality.

Now prove $\{a \in G : a < \psi(x) \text{ for some } x \in G^*\} \subseteq (id + \psi)(G^{<0})$. Let a, $x \in G$, $x < 0$, with $a < \psi(x)$; we want to show that $a \in (id+\psi)(G^{<0})$. Replacing ψ by $\psi - a$ if necessary, we may assume that $a = 0$. If $[x] \leq [\psi(x)]$, then $a = 0 \in (id + \psi)(G^{<0})$ as we proved before. If $[\psi(x)] < [x]$, then $|\psi(x)| < |x|$, which gives us $\psi(x) > x$ and hence $0 < x+\psi(x)$, where $x < 0$ by our assumption. Therefore $0 \in (id+\psi)(G^{<0})$, since $(id+\psi)(G^{<0})$ is closed downward. Let us prove now $\{a \in G : a < \psi(x) \text{ for some } x \in G^*\} \supseteq (id + \psi)(G^{<0})$. Take $x + \psi(x) \in (id + \psi)(G^{<0})$, where $x < 0$. This gives us $x + \psi(x) < \psi(x)$, which proves the inclusion.

If u, $v \in G$ are two elements such that $\psi(w) \leq u < v < w + \psi(w)$ for all $w \in G^{>0}$, then $v < (v - u) + \psi(v - u)$. Choosing $w := v - u > 0$ in the last inequality and since $\psi(v-u) = \psi(w) \leq u$ yields $v < (v-u)+\psi(v-u) \leq (v-u)+u = v$, which is a contradiction.

So, there is at most one element $v \in G$ such that $\Psi \leq v < (id + \psi)(G^{>0})$. □

As a consequence of the last corollary, $G \backslash (id + \psi)(G^*)$ has at most one element, and $(id + \psi)(G^*) \neq G$ if and only if Ψ has a supremum in G, and in this case $G \backslash (id + \psi)(G^*) = \{\sup \Psi\}$.

2.3 Closed H-asymptotic Couples

A *cut* of an H-asymptotic couple (G, ψ) is a set $P \subseteq G$ which is closed downward, contains Ψ, and is disjoint from $(id + \psi)(G^{>0})$.

So $P < (id + \psi)(G^{>0})$. By Corollary 2.2.2, an H-asymptotic couple (G, ψ) has at most two cuts, and it has two cuts if and only if $\Psi < v < (id + \psi)(G^{>0})$ for some $v \in G$. If Ψ has a maximum, then (G, ψ) has exactly one cut $P = \{a \in G : a \leq \psi(x) \text{ for some } x \in G^*\}$.

An H-asymptotic couple (G, ψ) is *closed* if:

1. G is divisible (as an Abelian group)

2. $(id + \psi)(G^*) = G$, and

3. $\Psi = (id + \psi)(G^{<0})$.

In this case, $P = \Psi$ is the only cut of (G, ψ).

Example: Let K be a Hardy field containing $\mathbb{R}$ and closed under exponentiation (that is, $f \in K \Rightarrow \exp f \in K$) and integration (i.e. $f \in K \Rightarrow \exists g \in K : g' = f$). Then the asymptotic couple of K is a closed H-couple.

Proof. If $f \in K^{>0}$, then $\log f \in K$, since $(\log f)' = f'/f \in K$. The value group G of K is divisible since for all $v(x) \in G$, $n \in \mathbb{Z}^+$, there exists $v(y) \in G$ such that $n \cdot v(y) = v(x)$, because $n \cdot v(y) = v(y^n) = v(x)$, which implies that there exists $y \in \mathbb{R}$ such that $y = x^{1/n}$, true in a Hardy field containing $\mathbb{R}$. Let x, $\log x$, $1/n \log x$ be elements of a Hardy field containing $\mathbb{R}$. Then $\exp(1/\mathrm{n} \log \mathrm{x}) = \mathrm{x}^{1/\mathrm{n}}$ is also an element of this field. If $x < 0$, then $y = (-x)^{1/n}$.

We want to prove that, for $f \in K^*$

1. either $v(f) = v(g'/g)$ for some $g \in K^*$ with $g' = f$, $v(g) > 0$,

2. or $v(f) = v(g')$ for some $g \in K^*$, $v(g) > 0$.

Proof of 1) and 2):

Given $f \in K^*$, take $g \in K^*$ such that $f = g'$. If $v(g) \geq 0$, then take a constant c such that $v(g - c) > 0$ and we proved case 2). If $v(g) < 0$, then replace g by $-g$ if $g > 0$, so we can assume that $g < 0$ and it is infinite because $v(g) < 0$. Then $h := \exp g$ is positive infinitesimal, and thus $v(\exp g) > 0$. See that $\frac{h'}{h} = \frac{(\exp g)'}{\exp g} = g' = \pm f$, so $v(f) = v(\frac{h'}{h})$ and $v(h) > 0$, which proves case 1).

We can see that 1) is equivalent to $v(f) = \psi(v(g)) \in \psi(G^{>0}) \in (id + \psi)(G^{<0})$ and 2) is equivalent to $v(f) = v(g') = v(\frac{g'}{g} \cdot g) = v(\frac{g'}{g}) + v(g) = \psi(v(g)) + v(g) \in (id + \psi)(G^{>0})$. Then by 1) and 2) we get that $G \in (id + \psi)(G^{<0})$

Now we want to prove that $\Psi = (id + \psi)(G^{<0})$. First, we want to show that $v(K^*)$ has no least element. For any $r \in \mathbb{R}^{>0}$, we have $x^{\frac{1}{r}} > \log x$. Take $x = \frac{1}{f}$ since $v(f) > 0$. Then $\frac{1}{f}^{\frac{1}{r}} > \log \frac{1}{f}$ and therefore $(-v(f^{\frac{1}{r}}) < v(\log \frac{1}{f}) = v(\log f)) \Leftrightarrow (\frac{1}{r} v(f) > -v(\log f))$, or $v(f) > -r \cdot v(\log f) = r \cdot v(\frac{1}{\log f}) > 0$. The ordered set $[v(K^*)^*]$ has no least element since, for any $f \in K^{>0}$ with $v(f) > 0$, we have $0 < r \cdot v(\frac{1}{\log f}) < v(f)$ for all $r \in \mathbb{R}^{>0}$. This fact, together with property 2. of H-couples give us the strict inequality $[\alpha] < [\beta] \Rightarrow \psi(\alpha) > \psi(\beta)$. The fact that $v(K^*)$ has no least element, gives us that $\psi(v(K^*))$ has no maximal element, since ψ acts injectively on Archimedean classes. If z is the maximal element of

Ψ, then by (1.3) we have that $\Psi \setminus \{z\} \subseteq (id+\psi)(G^{<0})$. Since Ψ has no maximal element, we get that $\Psi \subseteq (id+\psi)(G^{<0})$. To prove the other inclusion, we take $\alpha \in (id+\psi)(G^{<0})$. From 1. and 2. we have that $\alpha \in \psi(G^{>0})$ or $\alpha \in (id+\psi)(G^{>0})$. Since $(id+\psi)(G^{<0})$ and $(id+\psi)(G^{>0})$ are disjoint, we get that $\alpha \in \psi(G^{<0})$. We know that $\psi(G^{<0}) \subseteq \psi(G^*) = \Psi$, which completes the proof. □

An *asymptotic triple of H-type*, or *H-asymptotic triple*, is a triple (G, ψ, P) where (G, ψ) is an H-asymptotic couple and P a cut of (G, ψ), such that:

(1) G is divisible, and

(2) there exists a positive element 1 of G with $\psi(1) = 1$. Equivalently, $0 \in (id+\psi)(G^{<0})$.

2.4 The Model Theory of H-asymptotic Couples

We can consider asymptotic couples (G, ψ) as model-theoretic structures (G_∞, ψ) in the first-order language $\mathcal{L} = \{0, +, -, \psi, \infty\}$. Then the H-asymptotic couples are models of a universal theory in $\mathcal{L}$. In the case of H-asymptotic triples (G, ψ, P) considered as model-theoretic objects, we construct them as $\mathcal{L}_P$-structures $(G_\infty, \psi, 1, P, \delta_n)$, where $\mathcal{L}_P$ is an extension of $\mathcal{L}$ considering

1. a constant symbol 1 for the element $1 \in G^{>0}$ with $\psi(1) = 1$,
2. a unary predicate symbol for P, and
3. unary function symbols δ_n for each $n > 0$, to be interpreted on G as the scalar multiplication by $1/n$ (and $\delta_n(\infty) := \infty$).

The H-asymptotic triples are models of a *universal theory* (consists only of universal sentences) in $\mathcal{L}_P$. Let T be the theory of closed H-asymptotic couples in the language $\mathcal{L}$ and let T_P be the theory of closed H-asymptotic triples in the language $\mathcal{L}_P$.

Theorem 2.4.1 ([1], p. 6)**.** *The theory T_P is complete, decidable, and has elimination of quantifiers. It is the model completion of the theory of H-asymptotic triples.*

From this we get:

Theorem 2.4.2 ([1], p. 6)**.** *The theory T is the model companion of the theory of H-asymptotic couples.*

2.5 The Independence Property for Closed H-asymptotic Couples

Let $\mathcal{L}$ be a language (in the sense of first-order logic) and $\varphi(x, y)$ an $\mathcal{L}$-formula, where $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_n)$. We say that the formula $\varphi(x, y)$ *has the independence property* with respect to an $\mathcal{L}$-structure $\mathbf{A} = (A, ...)$ if for every $k \in \mathbb{N}$ there is a sequence $(a_1, ..., a_k)$ of elements of A^m such that for all subsets I of $\{1, ..., k\}$, there exists $b_I \in A^n$ with

$$\mathbf{A} \models \varphi(a_i, b_I) \Leftrightarrow i \in I,$$

for all $i = 1, ..., k$.

A theory T in the language $\mathcal{L}$ is said to have the independence property if all formulas $\varphi(x, y)$ as above have the independence property with respect to all $\mathbf{A} \models T$.

Suppose that $\mathcal{L}$ contains a binary relation symbol $<$ and that T is a complete theory with quantifier elimination such that all models $\mathbf{A} = (A, <, ...)$ of T are expansions of a dense linear ordering $(A,<)$ without endpoints. A *cut in* $(A,<)$ is a downward closed subset $C \subseteq A$. The following lemma is a special case of a criterion due to Poizat [16]:

Lemma 2.5.1 ([1], p. 9)**.** *The theory T does not have the independence property if for all models $\boldsymbol{A}$ and $\boldsymbol{B}$ of T with $\boldsymbol{A} \preceq \boldsymbol{B}$ and all cuts C of A, there exist at most $2^{|A|}$ simple extensions $\boldsymbol{A} \subseteq \boldsymbol{A}\langle c\rangle \subseteq \boldsymbol{B}$ with $C < c < A \setminus C$, up to isomorphism over $\boldsymbol{A}$.*

Let $(G,\psi,P) \subseteq (G',\psi',P')$ be closed asymptotic triples and let C be a cut in G. It is proved in ([2], section 6), that there exist at most two simple extensions of (G,ψ,P) inside (G',ψ',P') with generator $c \in G'$ such that $C < c < G \setminus C$, up to isomorphism over G.

This result and the previous lemma imply:

Corollary 2.5.1 ([1], p. 9)**.** *The theory T_P of closed asymptotic triples does not have the independence property. (Hence the theory T of closed H-asymptotic couples also does not have the independence property).*

2.6 The Relation of Asymptotic Couples to Contraction Groups

Let K be a Hardy field closed under taking logarithms (i.e. $f \in K^{>0} \Rightarrow \log f \in K$), with its valuation $v : K^* \to G = v(K^*)$. The logarithm map induces a so-called contraction map $\chi : G^{<0} \to G^{<0}$ by

$$\chi(v(f)) := v(\log f) \text{ for all } f \in K^{>0} \text{ with } v(f) < 0,$$

which we extend to a map $G \to G$ by requiring $\chi(-y) = -\chi(y)$. If K is closed under exponentiation (i.e. $f \in K \Rightarrow \exp f \in K$), then G is divisible, and χ is surjective ($\chi(G) = G$).

Let us show that G is divisible in this case. For $0 < f \in K$ and $0 < n \in \mathbb{N}$, we want to show that $\frac{1}{n}v(f) \in G$. We know that $g = \exp(\frac{1}{n}\log f) \in K$ since K is closed under exponentiation. So $g = f^{\frac{1}{n}}$ and thus $g^n = f$. Therefore, $nv(g) = v(g^n) = v(f)$. Since $v(g) \in G$, we get that $\frac{1}{n}v(f) \in G$.

This means that the pair (G, χ), where G is an ordered group and χ the contraction map, is a divisible centripetal contraction group, as axiomatized in Chapter 1.

In the example above, for $f \in K^{>0}$, with $y = v(f) < 0$, we have:

$$\psi(y) = \psi(v(f)) = v(f'/f) = v((\log f)') = v((\log f)'/\log f) + v(\log f) = \psi(\chi(y)) + \chi(y).$$

Let (G, ψ) be any closed H-asymptotic couple. For $y < 0$ in G, let $\chi(y) = z$ be the unique solution in G^* of the equation

$$z + \psi(z) = \psi(y) \tag{1.3}$$

For $y > 0$, set $\chi(y) := -\chi(-y)$, and $\chi(0) := 0$. Then (G, χ) is a non-trivial divisible centripetal contraction group. Therefore, χ is definable (without parameters) in (G, ψ).

Here is a negative result: we cannot definably reconstruct ψ in (G, χ), even allowing parameters.

Proposition 2.6.1 ([1], p. 10)**.** *There is no divisible centripetal contraction group (G, χ) such that one can define, a function $\psi : G^* \to G$ such that (G, ψ) is a closed H-asymptotic couple and $\chi + \psi \circ \chi = \psi$ on $G^{<0}$, even allowing parameters.*

We need some lemmas in order to prove this proposition. Let (G, ψ) be a closed asymptotic couple and assume that $0 \in \Psi$, so there exists $1 \in G^*$ such that $\psi(1) = 1 > 0$.

Iterates of ψ. Let $\psi^n : G_\infty \to G_\infty$, where $n > 0$, be the n-fold functional composition $\psi \circ \psi \circ \cdots \circ \psi$. Put

$$D_n := \{v \in G : \psi^n(v) \neq \infty\}.$$

For example $D_1 = G^*$, $D_2 = G^* \backslash \psi^{-1}(0)$, etc. We see that $\psi(D_1) = \psi(G^*) = \Psi$. By induction on n we see that $\psi^n(D_n) = \Psi$.

Lemma 2.6.1 ([1], p. 10). *Let $v \in G^*$ and $n > 0$ such that $\psi^n(v) < 0$. Then $\psi^i(v) < 0$ for all $i = 1, ..., n$, and*

$$[\psi^n(v)] < [\psi^{n-1}(v)] < \cdots < [\psi(v)] < [v].$$

Proof. We prove by induction. Induction basis: for $n = 1$, suppose that $[v] \leq [\psi(v)]$ and (1.1) imply $\psi(v) \geq \psi(\psi(v))$, hence $-\psi(v) + \psi(-\psi(v)) \leq 0 < (id + \psi)(G^{>0})$. Thus, $\psi(v) > 0$, contradiction.

Induction hypothesis: Assume that the lemma holds for a certain $n > 0$. Let $v \in D_{n+1}$ with $\psi^{n+1}(v) < 0$. Now we apply the case $n = 1$ to $\psi^n(v)$ instead of v. If $[\psi^n(v)] \leq [\psi^{n+1}(v)]$, then by (1.1) we get $\psi^{n+1}(v) \geq \psi(\psi^{n+1}(v))$, hence $-\psi^{n+1}(v) + \psi(-\psi^{n+1}(v)) \leq 0 < (id + \psi)(G^{>0})$. Thus $\psi^{n+1}(v) > 0$, a contradiction. Therefore, $[\psi^{n+1}(v)] < [\psi^n(v)]$. By the induction hypothesis, the remaining inequalities follow from $\psi^n < 0$. We suppose $\psi^n(v) \geq 0$, so $\psi^n(v) \in \Psi^{>0}$ and we want to show $[\psi^n(v)] \leq [1]$. We know that $\psi(a) < \psi(b) + |b|$, where $a = v(f) \neq 0$ and $b = v(g) \neq 0$ (axiom (A3)). For $n = 1$, $[\psi(v)] \leq [1]$, where $\psi(v) \geq 0$. By (A3) we have $\psi(v) < \psi(1) + 1 = 1 + 1 = 2$, so $\psi(v) < 2$. Moreover, $\psi(\psi(v)) < \psi(2)$ and $\psi(2) \geq \min \psi(1), \psi(1) = 1$. Then we get $\psi^n < 1$, and therefore $[\psi^n(v)] \leq [1]$.

Hence by (1.1) we get that $0 > \psi^{n+1}(v) \geq \psi(1) = 1$, a contradiction. □

Let $D_\infty := \cap_{n>0} D_n$ and

$$G_{\text{inf}} := \{v \in D_\infty : \psi^n(v) < 0 \text{ for all } n > 0,$$

$$G_{\text{fin}} := G \setminus G_{\text{inf}}.$$

Observe that $[v_0] < [v]$ for all $v_0 \in G_{\text{fin}}$, $v \in G_{\text{inf}}$ and that $G_{\text{inf}} \cap G^{>0}$ is closed upward and $G_{\text{inf}} \cap G^{<0}$ is closed downward.

Remark: The previous lemma and the fact that $\psi^n(D_n) = \Psi$, implies that for all $n > 0$, we can find an element $v \in D_n$ such that all iterates $\psi(v), \psi^2(v), ..., \psi^n(v)$ are negative.

Lemma 2.6.2 ([1], p. 10)**.** *G_{fin} is a convex subspace of G, and $(G_{\text{fin}}, \psi \backslash G^*_{\text{fin}})$ is a closed H-asymptotic couple. Moreover, $\psi(G_{\text{inf}}) = G_{\text{inf}} \cap G^{<0}$.*

Proof. We want to show that G_{fin} is convex. Observe that $G_{\text{fin}} = \{v \in G | \; \exists n > 0$ such that $\psi^n(v) \geq 0\} \cup \{0\}$. Let $v_1, v_2 \in G_{fin}$ such that $v_1 < v_2$ and $v_1 < v < v_2$, with $v \in G$. We want to show that $v \in G_{\text{fin}}$. Without loss of generality we assume $v_1 = 0$. Then $0 < v < v_2$. We know that $\psi^n(v_2) \geq 0$ for some $n > 0$ since $v_2 \in G_{\text{fin}}$. We want to show that $0 > \psi(v) \geq \psi(v_2)$. Assume that $\psi(v) < 0$. Then $0 > \psi^2(v) \geq \psi^2(v_2) ... > \psi^n(v) \geq \psi^n(v_2) \geq 0$, contradiction. This implies $\psi(v) \geq 0$, so $v \in G_{\text{fin}}$. □

Let χ be the contraction map defined by $\psi(v) = \chi(v) + \psi(\chi(v))$ for all $v < 0$.

Lemma 2.6.3 ([1], p. 10)**.** *Let $v \in G^{<0}$ and $\psi^3(v) < 0$. Then $\chi(v) = \psi(v) - \psi^2(v)$.*

Proof. From Lemma 2.6.1 we know that $[v] > [\psi(v)]$, so $\psi(v) < \psi^2(v)$ and hence $\psi(v) - \psi^2(v) < 0$. Similarly, $\psi^2(v) < \psi^3(v)$. Here $\psi(\psi(v) - \psi^2(v)) = \min\{\psi(\psi(v)), \psi(\psi^2(v))\} = \min\{\psi^2(v), \psi^3(v)\} = \psi^2(v)$. Then $(\psi(v) - \psi^2(v)) + \psi(\psi(v) - \psi^2(v)) = (\psi(v) - \psi^2(v)) + \psi^2(v) = \psi(v)$. By the defining equation (1.3) of χ we obtain that $\chi(v) = \psi(v) - \psi^2(v)$. □

Proof of Proposition 2.6.1. Suppose (G, ψ) is a closed H-asymptotic couple such that we can define ψ in (G, χ). For ease of notation we may assume that ψ is defined without parameters in (G, χ). If $0 \in (id + \psi)(G^{<0})$, then (G, ψ, Ψ) is a closed H-asymptotic triple. Otherwise, let $1 \in G^{>0}$ be the unique solution to the equation $x + \psi(x) = 0$, and pass from (G, ψ) to (G, ψ_0), where we define $\psi_0 := \psi + 1 - \psi(1)$, so that $\psi_0(1) = \psi(1) + 1 - \psi(1) = 1 > 0$. Thus, we may assume that (G, ψ, Ψ) is a closed H-asymptotic triple, with a distinguished element 1.

We modify ψ to a function $\widetilde{\psi} : G^* \to G$ by setting

$$\widetilde{\psi}(v) := \begin{cases} \psi(v), & \text{if } v \in G^*_{fin} \\ \psi(v) + 1, & \text{if } v \in G_{inf}. \end{cases}$$

Then $(G, \widetilde{\psi})$ is an H-asymptotic couple and $\widetilde{\psi}(G_{inf}) = \psi(G_{inf})$. Therefore $\Psi = \widetilde{\psi}(G^*)$ and $(G, \widetilde{\psi})$ is a closed H-asymptotic couple. Let $\widetilde{\chi}$ be the contraction map associated to $(V, \widetilde{\psi})$.

By the completeness of the theory of closed H-asymptotic triples, the same formula defines ψ in (G, χ) and $\widetilde{\psi}$ in $(G, \widetilde{\chi})$. By Lemma 2.6.3 $\chi = \widetilde{\chi}$ and hence $\psi = \widetilde{\psi}$, contradiction.

□

Chapter 3

On the Value Group of a Differential Valuation

We start this chapter by defining a differential valuation for differential fields of characteristic zero and then we show some fundamental properties of differential valuations, as presented in [18]. Let G be any ordered Abelian group and ψ a function satisfying the following conditions:

(i) If $\alpha \in G^*$ and $n \in \mathbb{Z}$, $n \neq 0$, then $\psi(n\alpha) = \psi(\alpha)$.

(ii) If α, $\beta \in G^*$, $\beta \neq -\alpha$, then $\psi(\alpha + \beta) \geq \min\{\psi(\alpha), \psi(\beta)\}$.

(iii) For any $\alpha, \beta \in G^*, \psi(\beta) < \psi(\alpha) + |\alpha|$.

We will show that there exists a differential field K and a differential valuation of K whose corresponding asymptotic couple is (G, ψ), at least if reasonable extra conditions hold ([17], Theorem 1). Further, we use the pair (G, ψ) associated with a differential valuation v of a differential field k to find the nonzero elements of k which are "asymptotically integrable".

3.1 Differential Valuation

A *differential field* is a field k, together with a derivation $(')$. Note that for any two elements x, y of the field, one has:

$(xy)' = xy' + x'y$

since multiplication on the field is commutative. The derivation on the field must also be distributive over addition in the field:

$(x+y)' = x' + y'$

The subfield of all elements with vanishing derivative $C = \{a \in k |\ a' = 0\}$ is called the *field of constants.*

Theorem 3.1.1 ([18], Theorem 1)**.** *Let k be a differential field of characteristic zero, C its subfield of constants, $(')$ its derivation, v a valuation of k that induces the trivial valuation on C, and let $\mathfrak{O}$ and $\mathfrak{M}$ be respectively the valuation ring of v and its maximal ideal. Then the following statements are equivalent:*

(1) *If $a \in \mathfrak{O}, b \in \mathfrak{M}, b \neq 0$, then $a'b/b' \in \mathfrak{M}$*

(2) *If $a, b \in k^*$ and $0 < v(a) \leq v(b)$, then $(b/a - b'/a') \in \mathfrak{M}$*

(3) *If $a, b \in k^*$ and $v(a) \leq v(b) < 0$, then $(b/a - b'/a') \in \mathfrak{M}$*

(4) *If $a \in \mathfrak{O}, b \in k^*, 1/b \in \mathfrak{M}$, then $a'b/b' \in \mathfrak{M}$.*

Note that $C \cap \mathfrak{M} = \{0\}$, so that if $b \in \mathfrak{M}$ and $b \neq 0$, then $b \notin C$ and therefore $b' \neq 0$.

Proof. Proof of (1) $\Rightarrow$ (2). Assume that $a, b \in k^*$ and $0 < v(a) \leq v(b)$. We can write $b = ac$, with $c \in \mathfrak{O}$. Since $a \in \mathfrak{M}$, we obtain

$\frac{b}{a} - \frac{b'}{a'} = c - \frac{a'c+ac'}{a'} = -\frac{c'a}{a'},$

By (1) we have that $\frac{c'a}{a'} \in \mathfrak{M}$, so $-\frac{c'a}{a'} \in \mathfrak{M}$, which gives (2).

Proof of (2) $\Rightarrow$ (3). Assume that $a, b \in k^*$ and $v(a) \leq v(b) < 0$. If we write $\alpha = \frac{1}{a}$ and $\beta = \frac{1}{b}$, we get $v(\alpha) = v(\frac{1}{a}) = -v(a)$ and $v(\beta) = v(\frac{1}{b}) = -v(b)$, and $v(a) \leq v(b) < 0$ equivalent to $0 < v(\beta) \leq v(\alpha)$, so that $0 < v(\alpha) \leq v(\frac{\alpha^2}{\beta})$. Hence $\frac{b}{a} - \frac{b'}{a'} = \frac{(1/\beta)}{(1/\alpha)} - \frac{(1/\beta)'}{(1/\alpha)'} = \frac{\alpha}{\beta} - \frac{\alpha^2\beta'}{\beta^2\alpha'} = \frac{(\alpha^2/\beta)'}{\alpha'} - \frac{\alpha^2/\beta}{\alpha}$. Since $\frac{(\alpha^2/\beta)'}{\alpha'} - \frac{\alpha^2/\beta}{\alpha}$ is in $\mathfrak{M}$ by (2), then $(b/a - b'/a') \in \mathfrak{M}$

Proof of (3) $\Rightarrow$ (4). Assume that $a \in \mathfrak{O}, b \in k^*, 1/b \in \mathfrak{M}$. Replacing a by a+1 if necessary, to reduce to the special case in which a is a unit in $\mathfrak{O}$, we have $v(b) =$

$v(a)+v(b)=v(ab)<0$, so by (3) we obtain that $(ab)/b-(ab)'/b' \in \mathfrak{M}$.

Moreover, $\frac{ab}{b}-\frac{(ab)'}{b'}=\frac{ab}{b}-\frac{a'b+b'a}{b'}=-\frac{a'b}{b'}$ and since $-a'b/b' \in \mathfrak{M}$, we have that $\frac{a'b}{b'} \in \mathfrak{M}$.

Proof of (4) $\Rightarrow$ (1). Assume that $a \in \mathfrak{O}, b \in \mathfrak{M}, b \neq 0$. Then $\frac{a'(1/b)}{(1/b)'} \in \mathfrak{M}$. Using the identity $b/b'=-(1/b)/(1/b)'$, we obtain that $a'b/b' \in \mathfrak{M}$. □

Corollary 3.1.1 ([18], Corollary 1)**.** *Under the same assumptions as in Theorem 3.1.1, each of the statements* (1)*,* (2)*,* (3)*, and* (4) *implies*

(5) *If $a,b \in k^*$ and $v(a),v(b) \neq 0$, then $v(a) \leq v(b)$ if and only if $v(a') \leq v(b')$.*

Conversely, under the same assumption, if $\mathfrak{O}=C+\mathfrak{M}$, then (5) *implies* (1)*,* (2)*,* (3)*, and* (4)*.*

Proof. For the proof, instead of (5) consider the equivalent statement

($5'$) If $a,b \in k^*$ and $v(a),v(b) \neq 0$, then $(v(a)=v(b)) \Rightarrow (v(a')=v(b'))$ and $(v(a)<v(b)) \Rightarrow (v(a')<v(b'))$.

We want to show that the equivalent statements (1), (2), (3), (4) imply ($5'$). If $0<v(a) \leq v(b)$, then by (2) $v(b/a-b'/a')>0$. We know that $v(\frac{b}{a}-\frac{b'}{a'}) \geq \min\{v(b)-v(a), v(b')-v(a')\}$. This gives us $v(b)-v(a)=0 \Rightarrow v(b')-v(a')=0$ and $v(b)-v(a)>0 \Rightarrow v(b')-v(a')>0$. Similarly, if $v(a) \leq v(b)<0$, then statement ($5'$) is implied by (3). For the remaining case $v(a)<0<v(b)$, we use $v((1/a)')=v(-a'/a^2)=v(a')-2v(a)>v(a')$ since $v(a)<0$, and $v((1/b)')=v(b')-2v(b)<v(b')$. If $0<v(1/a) \leq v(b)$, we have that (2) implies $v(b') \geq v(1/a)')>v(a')$ so $v(a')<v(b')$. If $v(1/a)>v(b)$, then $-v(a)>v(b)>0$, and hence $v(a)<-v(b)<0$. Therefore we get $v(a)<v(1/b)<0$ and (3) implies $v(a')<v((1/b)')<v(b')$.

Conversely, assume (5) and suppose that $a,b \in \mathfrak{M}, a,b \neq 0$. Then $0<v(b)<v(a)+v(b)=v(ab)$, whence $v(b')<v((ab)')$. Also $v(b')<v(b')+v(a)=v(b'a)$, so that $v(b')<\min\{v((ab)'),v(b'a)\} \leq v((ab)'-b'a)=v(a'b)$. Hence $v(a'b/b')>0$, or $a'b/b' \in \mathfrak{M}$. This conclusion also holds if we suppose $a \in \mathfrak{O}$ instead of assuming $a \in \mathfrak{M}$, since $\mathfrak{O} = C + \mathfrak{M}$, and hence statement (1) holds. □

We know that $v(a'b/b') > 0 \Leftrightarrow v(a') + v(b) - v(b') > 0$ and hence $v(a') > v(b') - v(b) = v(b'/b)$. This implies that the subset $\{v(a') : a \in \mathfrak{O}, a \notin C\}$ of the value group $v(k^*)$ is bounded from below and the subset $\{v(b'/b) : b \in k^*, v(b) \neq 0\}$ is bounded from above. The next Corollary says that the set $\{v(a') : a \in k, a \notin C\}$ is bounded neither from above nor from below.

Corollary 3.1.2 ([18], Corollary 2)**.** *In the context of Theorem* 3.1.1*, if v is nontrivial then for any $a \in k^*$ there exist $x, y \in k^*$ such that $v(x') > v(a) > v(y')$.*

Proof. Fix some $b \in \mathfrak{O}, b \notin C$. Then from (1) and (4) of Theorem 3.1.1 we obtain that for any $y \in k^*$ such that $v(y) \neq 0$, we get $b'y/y' \in \mathfrak{M}$. Then $v(b'y/y') > 0$ and hence $v(y') < v(b'y)$. If we choose $y \in k$ such that $v(y) < v(a/b'), v(y) \neq 0$, then $v(y') < v(b') + v(y) < v(b') + v(a) - v(b') = v(a)$, and thus $v(y') < v(a)$, as desired. To obtain the other inequality, assume that $v(a) \neq 0$, and note that if $u \in k^*, v(u) > 0$, then $v(u'a/a') > 0$, or $v(u') > v(a'/a)$. Therefore $(v(u^2)') = v(2uu') = v(u) + v(u') > v(u) + v(a'/a)$. Take $x = u^2$, with $v(u) > \max\{0, v(a^2/a')\}$ to get $v(x') > v(a^2) - v(a') + v(a') - v(a) = 2v(a) - v(a) = v(a)$ which gives the first inequality. □

Let $(k, ')$ be a differential field of characteristic zero whose subfield of constants is $C = \{a \in k |\ a' = 0\}$. A valuation v of k is called a *differential valuation* if C is a field of representatives for the residue field of (k, v), (that is, v is trivial on C and for every $y \in k$ such that $v(y) = 0$, there is a unique $c \in C$ such that $v(y - c) > 0$), and if (1) holds.

Thus a valuation v on the differential field k with constant subfield C is a differential valuation of k if the equivalent conditions of Theorem 3.1.1 hold and the natural embedding of C into the residue field $\mathfrak{O}/\mathfrak{M}$ is surjective.

Another way of getting asymptotic couples is to consider a differential field.

Theorem 3.1.2 ([18], Theorem 4). *Let k be a differential field and let v be a differential valuation of k with value group G. Then there is a map $\psi : G^* \to G$ such that for all $a \in k^*$ with $v(a) \neq 0$ we have $\psi(v(a)) = v(a'/a)$, and ψ has the following properties:*

(i) If $\alpha \in G^*$ and $n \in \mathbb{Z}$, $n \neq 0$, then $\psi(n\alpha) = \psi(\alpha)$.

(ii) For any $\gamma \in G$, the set $\{\alpha \in G : \alpha = 0$ or $\psi(\alpha) \geq \gamma\}$ is a subgroup of G.

(iii) For any $\alpha, \beta \in G^*, \psi(\beta) < \psi(\alpha) + |\alpha|$.

Proof By Corollary 3.1.1 we have a well-defined map $\psi : G^* \to G$ given by $\psi(v(a)) = v(a'/a)$ for any $a \in k^*$ such that $v(a) \neq 0$. First, we want to prove (i). If $n \in \mathbb{Z}$, $n \neq 0$, then $\psi(nv(a)) = \psi(v(a^n)) = v((a^n)'/a^n) = v(na^{n-1}a'/a^n) = v(na'/a) = v(a'/a) = \psi(v(a))$. For $v(a) = \alpha$ we get $\psi(n\alpha) = \psi(\alpha)$. Now we want to prove (ii). If $a, b \in k^*$ are such that $\psi(v(a)), \psi(v(b)) \geq \gamma$, then $\psi(v(a) + v(b)) = \psi(v(ab)) = v((ab)'/ab) = v((a'b + ab')/ab) = v((a'b/ab) + (ab'/ab)) = v((a'/a) + (b'/b)) \geq \min\{v(a'/a), v(b'/b)\} = \min\{\psi(v(a)), \psi(v(b))\} \geq \gamma$, and if we take $n = -1$ in (i) we get the inverses $\psi(-v(a)) = \psi(v(a))$, showing that the set $\{\alpha \in G : \alpha = 0$ or $\psi(\alpha) \geq \gamma\}$ is a subgroup of G. To prove (iii), note that since $\psi(\alpha) = \psi(-\alpha)$ we may take $\alpha > 0$. Let $a, b \in k^*$ such that $v(a) = \alpha, v(b) = \beta$. So, $\psi(\alpha) + |\alpha| - \psi(\beta) = v(a'/a) + v(a) - v(b'/b) = v(a') - v(a) + v(a) - v(b') + v(b) = v(a'b/b') > 0$, by (1) and (4) of Theorem 3.1.1.

Theorem 3.1.3 ([18], Theorem 5). *Let G be an ordered Abelian group and $\psi : G^* \to G$ a map with properties (i), (ii) and (iii). Then for any $\alpha, \beta \in G^*$ and any positive integer n,*

$$n(\psi(\beta) - \psi(\alpha)) < |\alpha|.$$

3.2 Construction of Differential Fields with Given Asymptotic Couple as their Value Group

Let G be any torsion free Abelian group, and let $\mathbb{Q}G := G \bigotimes_{\mathbb{Z}} \mathbb{Q}$ called the divisible hull of G. If $\gamma \otimes q \in \mathbb{Q}G$, then $\frac{1}{n}(\gamma \otimes q) = \gamma \otimes \frac{q}{n} \in \mathbb{Q}G$, so $\mathbb{Q}G$ is a $\mathbb{Q}$-vector space. We can embed G into $\mathbb{Q}G$ via the map $\gamma \mapsto \gamma \bigotimes 1$. For every $\alpha \in G$ there is a unique element β in the divisible hull of G such that $n\beta = \alpha$. So we can write β as $\frac{\alpha}{n}$, viewing $\mathbb{Q}G$ as $\{\frac{\alpha}{n} | \ \alpha \in G, n \in \mathbb{N}\}$. Suppose that β is not unique, so there is an element $\beta^{'}$ in the divisible hull of G, $\beta^{'} \neq \beta$ such that $\alpha = n\beta^{'}$. Since $n\beta = \alpha$, then $n\beta - n\beta^{'} = n(\beta - \beta^{'})$. Therefore $\beta - \beta^{'}$ is a torsion element, so $\beta - \beta^{'} = 0$. Then $\beta = \beta^{'}$.

Lemma 3.2.1. *If G is an ordered Abelian group then the ordering has a unique extension to the divisible hull of G.*

Proof Let G be an ordered Abelian group and let $\mathbb{Q}G$ be its divisible hull. If $\mathbb{Q}G$ is an ordered group, then for all $a, b \in \mathbb{Q}G$ and $n \geq 1$, we have $na < nb \Leftrightarrow a < b$. Hence the only possibility to extend the ordering of G to an ordering of $\mathbb{Q}G$ is as follows: let $a, b \in \mathbb{Q}G$; define $a < b$ if and only if there exists a positive integer n such that $na, nb \in G$ and $na < nb$. This does not depend on the choice of n and is thus well defined. This defines an ordering, since antireflexivity, transitivity, comparability and compatibility with the group operation follow from the ordering on G as shown below. Let $a, b, c \in \mathbb{Q}G$, we may choose a positive integer n such that $na, nb, nc \in G$. Then we have:

- antireflexivity: $na \nless na$ implies $a \nless a$.

- transitivity: if $a < b \wedge b < c$, then $na < nb \wedge nb < nc$. So, $na < nc$ and therefore $a < c$.

- comparability: either $na < nb$, or $na = nb$, or $na > nb$, hence either $a < b$, or $a = b$, or $a > b$.

- compatibility: $na < nb \Rightarrow na + nc < nb + nc$, implies $a < b \Rightarrow a + c < b + c$.

Suppose now that $\psi : G^* \to G$ is a function that satisfies properties (i), (ii) and (iii) of Theorem 3.1.2. We claim that there is a unique function from $(\mathbb{Q}G)^*$ into $\mathbb{Q}G$, denoted also by ψ, that extends the given function ψ on G^* to $(\mathbb{Q}G)^*$, and that also satisfies properties (i), (ii) and (iii). Property (i) clearly gives a well-defined definition of ψ on $(\mathbb{Q}G)^*$. Property (ii) for $\mathbb{Q}G$ is an immediate consequence of (ii) for G. Property (iii) for $\mathbb{Q}G$ is an immediate consequence of Theorem 3.1.3, that is, prove that for any α, $\beta \in (\mathbb{Q}G)^*$, we have $\psi(\beta) < \psi(\alpha) + |\alpha|$. If α, $\beta \in (\mathbb{Q}G)^*$, then $n\alpha$, $n\beta \in G^*$, for some $n \in \mathbb{N}^*$. From Theorem 3.1.3 for $n\alpha$, $n\beta \in G^*$, we obtain that $n(\psi(n\beta) - \psi(n\alpha)) < |n\alpha|$. From Property (i) we get that $n(\psi(\beta) - \psi(\alpha)) < n|\alpha|$. Dividing by n, we obtain the required inequality.

Note that $\psi(G^*) = \psi((\mathbb{Q}G)^*)$ and that the cardinality of the latter set is at most the vector space dimension $\dim_{\mathbb{Q}} \mathbb{Q}G$, as proved in [22]. If (G, ψ) comes from a differential valuation v of a differential field k with subfield of constants C. The rational rank of G, $\dim_{\mathbb{Q}} \mathbb{Q}G$, is known to be at most trdeg k/C. Thus $\psi(G^*)$ is finite if the trdeg is finite, so k is generated as a differential extension field of C by a finite number of elements that are differentially algebraic over C. An element $x \in L$ is said to be *differentially algebraic over* K if, and only if, x satisfies an algebraic differential equation over K, i.e. $P(x, dx/dt, ..., d^n x/dt^n) = 0$, where P is a non-zero polynomial over K.

Theorem 3.2.1 ([17], Theorem 1)**.** *Let G be an ordered Abelian group and $\psi : G^* \to G$*

a function with properties (i), (ii), and (iii). Suppose that the ordered subset $\psi(G^*)$ of G, in the opposite ordering, is well-ordered. Let C be a field of characteristic zero such that $\dim_{\mathbb{Q}} C \geq \dim_{\mathbb{Q}} \mathbb{Q}G$. Then there exists a differential field K whose subfield of constants is C and a differential valuation v of K whose value group is G such that for each $a \in K^*$ with $v(a) \neq 0$ we have $v(a'/a) = \psi(v(a))$.

Proof. Choose a set $\Sigma_\gamma \subset G$ for each $\gamma \in (\mathbb{Q}G)^*$ such that the canonical image of Σ_γ in the $\mathbb{Q}$-vector space

$$\{\alpha \in \mathbb{Q}G : \alpha = 0 \text{ or } \psi(\alpha) \geq \gamma\}/\{\alpha \in \mathbb{Q}G : \alpha = 0 \text{ or } \psi(\alpha) > \gamma\}$$

is a $\mathbb{Q}$-basis of the latter vector space.

We shall prove the theorem for any field C of characteristic zero such that $\dim_{\mathbb{Q}} C \geq$ card Σ_γ for each $\gamma \in \psi(G^*)$, a slight weakening of the stated hypothesis. We have $\psi(\Sigma_\gamma) = \{\gamma\}$.

Let $\Sigma = \bigcup\{\Sigma_\gamma : \gamma \in \psi(G^*)\}$.

We want to prove that Σ is a $\mathbb{Q}$-basis of $\mathbb{Q}G$. First, we want to show that Σ is linearly independent. Suppose that Σ is linearly dependent, so we can find an integer $n \geq 1$ and elements $\alpha_1, ..., \alpha_n \in \Sigma$ and $c_1, ..., c_n \in \mathbb{Q}^*$ such that $c_1\alpha_1 + ... + c_n\alpha_n = 0$. Taking the canonical linear map of $\mathbb{Q}G$ with kernel $\{\alpha \in \mathbb{Q}G : \alpha = 0 \text{ or } \psi(\alpha) > \min\{\psi(\alpha_1), ..., \psi(\alpha_n)\}\}$ gives a relation of linear dependence among the images of those α_i's for which $\psi(\alpha_i) = \min\{\psi(\alpha_1), ..., \psi(\alpha_n)\}$, a contradiction. Therefore Σ is linearly independent. Secondly, if $\alpha_0 \in (\mathbb{Q}G)^*$ then we can find an element β_0 in the $\mathbb{Q}$-space spanned by $\Sigma_{\psi(\alpha_0)}$ such that either $\alpha_0 - \beta_0 = 0$ or $\psi(\alpha_0) < \psi(\alpha_0 - \beta_0)$. If $\alpha_0 - \beta_0 \neq 0$, we can find an element β_1 in the space spanned by $\Sigma_{\psi(\alpha_0 - \beta_0)}$ such that either $\alpha_0 - \beta_0 - \beta_1 = 0$ or $\psi(\alpha_0) < \psi(\alpha_0 - \beta_0 - \beta_1)$. We can continue this process, thus getting $\beta_2, \beta_3, ...,$ but because of our well-ordering hypothesis this process must come to an end. Therefore, we get $\alpha_0 = \beta_0 + \beta_1 + ... + \beta_n$ for some n, where each β_i is in the $\mathbb{Q}$-space spanned by Σ. So, we have proved that Σ is a

$\mathbb{Q}$-basis of $\mathbb{Q}G$.

Our desired field K will be an algebraic extension field of $C(X)$, where X is some fixed set of indeterminates such that card $X =$ card Σ.

Let $v : X \to \Sigma$ be a fixed bijection.

Fix a function $h : X \to C^*$ with the property that for each $\gamma \in \psi(G^*)$, h maps the set $\{x \in X : v(x) \in \Sigma_\gamma\}$ onto a set of elements of C that are $\mathbb{Q}$-linearly independent; that such an h exists follows from our cardinality assumption. The multiplicative group of an algebraically closed field is divisible if and only if, regarded as a $\mathbb{Z}$-module, it is injective, as in [[6], Theorem 2.5]. Thus for each $x \in X$ we can find a homomorphism $\varphi_x : \mathbb{Q} \to \overline{C(X)}^*$, where $\overline{C(X)}$ is some fixed algebraic closure of $C(X)$, such that $\varphi_x(1) = x$. For $x \in X$ and $a \in \mathbb{Q}$, we define the symbol x^a to be $\varphi_x(a)$. So, the set of all expressions $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, for n a positive integer, each $x_i \in X$ and each $a_i \in \mathbb{Q}$, forms a multiplicative subgroup U_1 of $\overline{C(X)}$ and $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ takes on its usual meaning if each $a_i \in \mathbb{Z}$. For any $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in U_1, we define $v(u) = \sum_{i=1}^{n} a_i v(x_i) \in \mathbb{Q}G$. The map $v : U_1 \to \mathbb{Q}G$ is a group isomorphism, since $v(u_1 \cdot u_2) = \sum_{i=1}^{n} a_i v(x_i) + \sum_{j=1}^{m} b_j v(y_j) = v(u_1) + v(u_2)$ for any $u_1 = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $u_2 = y_1^{b_1} y_2^{b_2} \cdots y_m^{b_m}$ in U_1 and it is also bijective because the elements of X are algebraically independent over C and $v(X) = \Sigma$ is a $\mathbb{Q}$-basis of $\mathbb{Q}G$, as we proved above. Observe that the ring $C[U_1]$ is a subring of $\overline{C(X)}$. Any element of $C[U_1]$ different from zero can be written in the form $c_1 u_1 + \cdots + c_r u_r$, where $u_1, ..., u_r$ are distinct elements of U_1 and each $c_i \in C^*$. By the algebraic independence of elements of X over C, this representation is unique except for the order of the terms.

We can extend the map $v : U_1 \to \mathbb{Q}G$ to a well-defined map of the nonzero elements of $C[U_1]^* = C[U_1] \setminus \{0\}$ into $\mathbb{Q}G$ by setting $v(c_1 u_1 + \cdots + c_r u_r) = \min\{v(u_1), ..., v(u_r)\}$. We can check that $v(y_1 y_2) = v(y_1) + v(y_2)$ and $v(y_1 + y_2) \geq \min\{v(y_1), v(y_2)\}$, for any non-zero elements $y_1, y_2 \in C[U_1]$. Therefore we can extend the map $v : (C[U_1])^* \to \mathbb{Q}G$ to a valuation $v : (C(U_1))^* \to \mathbb{Q}G$. Thus there is a unique valuation v on $C(U_1)$ that is trivial on C and agrees with the original v on X. Its value group is $\mathbb{Q}G$ and its residue field is C. Let $T_1 = \{u \in U_1 : v(u) \in G\}$. So, T_1 is a subgroup of U_1 and v induces an isomorphism between T_1 and G. If we set $K = C(T_1)$, then v induces a valuation of K that is trivial on C, with value group G and residue field C.

We now define a derivation on K. There is a well-defined function $\xi : X \to U_1$ such that for each $x \in X$ we have $v(\xi(x)) = \psi(v(x))$. Since $\psi : G^* \to G$ and T_1 is a subgroup of U_1 such that $T_1 = \{u \in U_1 : v(u) \in G\}$, we have $\xi(X) \subset T_1$. Since X is a transcendence base for K over C, there is a unique derivation of K that is trivial on C and such that for each $x \in X$ we have $x^{'} = h(x)\xi(x)x$. Then K becomes a differential field such that for each $x \in X$ we have $v(x^{'}/x) = v(h(x)\xi(x)) = v(h(x)) + v(\xi(x)) = v(\xi(x)) = \psi(v(x))$.

Let $T = C^*T_1$, a subgroup of the multiplicative group of K. Therefore $C^* \subset T$, $K = C(T)$, and the kernel of the homomorphism $v : T \to G$ is C^*. We want to show that any constant in T is in C. For this it suffices to prove that the only constant in T_1 is 1. Suppose the following: $x_1, x_2, ..., x_n$ are distinct elements of X, $a_1, a_2, ..., a_n$ are in $\mathbb{Q}^*$, $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in T_1$ and $(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n})^{'} = 0$. Then $(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n})^{'}/(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = ((x_1^{a_1})^{'} x_2^{a_2} \cdots x_n^{a_n})/(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) + (x_1^{a_1} (x_2^{a_2})^{'} \cdots x_n^{a_n})/(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) + ... + (x_1^{a_1} x_2^{a_2} \cdots (x_n^{a_n})^{'})/(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = (x_1^{a_1})^{'}/x_1^{a_1} + (x_2^{a_2})^{'}/x_2^{a_2} + ... + (x_n^{a_n})^{'}/x_n^{a_n} = (a_1 x_1^{a_1 - 1} x_1^{'})/x_1^{a_1} + ... + (a_n x_n^{a_n - 1} x_n^{'})/x_n^{a_n} = a_1 x_1^{'}/x_1 + ... + a_n x_n^{'}/x_n = a_1 h(x_1)\xi(x_1) + ... + a_n h(x_n)\xi(x_n) = 0$. Each $a_i h(x_i) \in C^*$ and each $\xi(x_i) \in U_1$, so there is a partition $\wp$ of the set $\{1, ..., n\}$ such that for $i, j \in \{1, ..., n\}$ we have $\xi(x_i) = \xi(x_j)$ if and only if i and j are in the same set in $\wp$. Thus for any set $I \in \wp$ we have $\sum_{i \in I} a_i h(x_i) = 0$. We get for $i \in I \in \wp$ that $\psi(v(x_i)) = v(\xi(x_i))$, where $v(x_i) \in \sum_{v(\xi(x_i))}$ depends only on I. But we constructed the function $h : X \to C^*$

such that the elements $\{h(x_i)\}$ are $\mathbb{Q}$-linearly independent. This implies that $a_i = 0$ for all $i \in I$, which gives us a contradiction. Therefore the only constant in T_1 is 1 and this proves that any constant in T is in C.

We want to check that for each $u \in T$ such that $v(u) \neq 0$, we obtain $v(u^{'}/u) = \psi(v(u))$. We know that $T = C^*T_1$ and since T_1 is a subgroup of U_1, so $T \subset C^*U_1$. Therefore we only need to prove $v(u^{'}/u) = \psi(v(u))$ for $u \in U_1$. Let $u = x_1^{a_1} \cdots x_n^{a_n}$, where $x_1, ..., x_n$ are distinct elements of X and $a_1, ..., a_n \in \mathbb{Q}^*$. As we showed already, $u^{'}/u = (x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n})^{'}/(x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}) = a_1x_1^{'}/x_1 + ... + a_nx_n^{'}/x_n$. Then $v(u^{'}/u) = v(\sum_{i=1}^n a_ix_i^{'}/x_i) = v(\sum_{i=1}^n a_ih(x_i)\xi(x_i))$. For $i = 1, ..., n$, $\xi(x_i)$ depends only on $\psi(v(x_i))$ and since $h : X \to C^*$ has the property that for each element γ of $\psi(G^*)$, h maps the set $\{x \in X : v(x) \in \Sigma_\gamma\}$ onto a set of $\mathbb{Q}$-linearly independent elements of C, we get $\sum_{i \in I} a_ih(x_i) \neq 0$ if I is any maximal subset of $\{1, ..., n\}$ for which $\xi(x_i)$ assumes a constant value. Thus $v(u^{'}/u) = \min_{i=1,...,n} v(\xi(x_i)) = \min_{i=1,...,n} \psi(v(x_i))$. By our construction of the set Σ and by the property that for any $\gamma \in G$, the set $\{\alpha \in G : \alpha = 0 \text{ or } \psi(\alpha) \geqslant \gamma\}$ is a subgroup of G, we get $\min_{i=1,...,n} \psi(v(x_i)) = \psi(\sum_{i=1}^n a_iv(x_i)) = \psi(v(u))$. Therefore we have proved that for any $u \in U_1$ such that $v(u) \neq 0$, we have $v(u^{'}/u) = \psi(v(u))$.

Now we have to verify that if $a, b \in T$ and $v(a)$, $v(b) > 0$, then $v(a^{'}b/b^{'}) > 0$. We have that $v(a^{'}b/b^{'}) = v(a^{'}) + v(b) - v(b^{'}) = v(a^{'}) - v(a) + v(a) - (v(b^{'}) - v(b)) = v(a^{'}/a) + v(a) - v(b^{'}/b) = \psi(v(a)) + v(a) - \psi(v(b))$. Since for any $\alpha, \beta \in G^*, \psi(\alpha) + |\alpha| - \psi(\beta) > 0$, and because $v(a) > 0$, we obtain $\psi(v(a)) + v(a) - \psi(v(b)) > 0$ and therefore $v(a^{'}b/b^{'}) > 0$. Then the following theorem can be applied to our situation, using the same v, C, T and K, and with k and S both taken to be C.

Theorem 3.2.2 ([18], Theorem 2)**.** *Let K be a differential field of characteristic zero, k a differential subfield of K, $\mathcal{C}$ a subfield of the field of constants of K, and v a valuation of*

K that is trivial on $\mathcal{C}$. Suppose that $C = \mathcal{C} \cap k$ maps surjectively to the image of k in the residue field of v. Let T be a subgroup of the multiplicative group K^* such that: $k^* \subset T$, $K = \mathcal{C}(T)$, any constant in T is in C, $\{a \in T | v(a) = 0\} \subseteq k^*$, and such that if $a, b \in T$ and $v(a), v(b) > 0$ then $v(a^{'}b/b^{'}) > 0$. Then v is a differential valuation of K, and $\mathcal{C}$ is the field of constants of K.

By this theorem, we can conclude that v is a differential valuation of K, and C is its subfield of constants.

Knowing that v has value group G, it remains to prove that if $a \in K^*$ and $v(a) \neq 0$, then $v(a^{'}/a) = \psi(v(a))$. We already proved this for $a \in T$. This statement is also true for any $a \in K^*$ such that $v(a) \neq 0$ because we can choose some $a_1 \in T$ such that $v(a) = v(a_1)$ and since v is a differential valuation of K we have $v(a^{'}) = v(a_1^{'})$ and we are done. □

Here is an example, so we can understand the procedure of the above proof. Let G be a subgroup of the lexicographically ordered group $\mathbb{R}^2$, given by

$$G = \{(m + n\pi, p) : m, n, p \in \mathbb{Z}\},$$

with ψ given by

$$\psi(m + n\pi, p) = \begin{cases} (-1, 0) & \text{if} \quad m \neq 0 \\ (-1, 1) & \text{if} \quad m = 0, (n, p) \neq (0, 0). \end{cases}$$

First, we have to check properties (i), (ii) and (iii) of Theorem 3.1.2. We want to verify property (i), i.e. if $\alpha \in G^*$ and $n \in \mathbb{Z}, n \neq 0$, then $\psi(n\alpha) = \psi(\alpha)$. Pick an element $\alpha = (a + b\pi, c)$, where $a, b, c \in \mathbb{Z}$.

Then $\psi(n\alpha) = \psi(na + nb\pi, nc) = \begin{cases} (-1, 0) & \text{if} \quad na \neq 0 \\ (-1, 1) & \text{if} \quad na = 0, (nb, nc) \neq (0, 0) \end{cases}$

$$= \begin{cases} (-1, 0) & \text{if} \quad a \neq 0 \\ (-1, 1) & \text{if} \quad a = 0, (b, c) \neq (0, 0) \end{cases} = \psi(a + b\pi, c) = \psi(\alpha).$$

Now we want to prove property (ii), that is, for any $\gamma \in G$, the set

$G' = \{\alpha \in G |\ \alpha = 0 \text{ or } \psi(\alpha) \geq \gamma\}$ is a subgroup of G.

Observe that $G = \mathbb{Z}\langle 1, \pi\rangle \coprod \mathbb{Z}$.

If $\gamma \leq (-1, 0)$, then $\psi(\alpha) \geq \gamma$ for all $\alpha \in G$ and G' is the group G.

If $(-1, 0) < \gamma \leq (-1, 1)$, then $\psi(\alpha) \geq \gamma$ if and only if $\psi(\alpha) = (-1, 1)$. In this case $G' = \{(m + n\pi, p) |\ \alpha = 0 \text{ or } \psi(\alpha) \geq (-1, 1)\} = \{(m + n\pi, p) |\ m = n = p = 0 \text{ or } (m = 0$ and $(n, p) \neq 0)\} = \{(n\pi, p) |\ n, p \in \mathbb{Z}\} = \mathbb{Z}\pi \coprod \mathbb{Z}$, which is again a subgroup of G.

If $\gamma > (-1, 1)$, then $G' = \{(0, 0\}$, and also in this case G' is a subgroup of G.

For property (iii) we need to check that for any $\alpha, \beta \in G^*$, $\psi(\beta) < \psi(\alpha) + |\alpha|$. Let $\alpha = (a + b\pi, c) \neq (0, 0)$ and $\beta = (m + n\pi, p) \neq (0, 0)$, with $a, b, c, m, n, p \in \mathbb{Z}$. Observe that our relation is true for the four possible cases:

1. If $m \neq 0$ and $a \neq 0$, then $(-1, 0) < (-1, 0) + |(a + b\pi, c)|$
2. If $m \neq 0$ and $a = 0, (b, c) \neq (0, 0)$, then $(-1, 0) < (-1, 1) + |(b\pi, c)|$
3. If $m = 0, (n, p) \neq (0, 0)$ and $a \neq 0$, then $(-1, 1) < (-1, 0) + |(a + b\pi, c)|$
4. If $m = 0, (n, p) \neq (0, 0)$ and $a = 0, (b, c) \neq (0, 0)$, then $(-1, 0) < (-1, 1) + |(b\pi, c)|$

Following the same procedure as in the proof, we can choose $\sum_{(-1,0)} = \{(1, 0)\}$, $\sum_{(-1,1)} = \{(\pi, 0), (0, 1)\}$. Let $C = \mathbb{C}$ and let x_1, x_2, x_3 be indeterminates over $\mathbb{C}$ with $v(x_1) = (1, 0)$, $v(x_2) = (\pi, 0)$ and $v(x_3) = (0, 1)$. The equation $v(\xi(x)) = \psi(v(x))$ gives us $v(\xi(x_1)) = \psi(v(x_1)) = \psi(1, 0) = (-1, 0) = -(1, 0) = -v(x_1) = v(x_1^{-1})$, $v(\xi(x_2)) = \psi(v(x_2)) = \psi(\pi, 0) = (-1, 1) = (0, 1) - (1, 0) = v(x_3) - v(x_1) = v(x_3 x_1^{-1})$ and $v(\xi(x_3)) = \psi(v(x_3)) = \psi(0, 1) = (-1, 1) = v(x_3 x_1^{-1})$. Therefore these equations give us $\xi(x_1) = x_1^{-1}$ and $\xi(x_2) = \xi_(x_3) = x_1^{-1} x_3$. Take $h(x_1) = h(x_3) = 1$ and $h(x_2) = a$, where a is an arbitrary element of $\mathbb{C} \setminus \mathbb{Q}$. Using the equation $x' = h(x)\xi(x)x$, we obtain $K =$

$\mathbb{C}(x_1, x_2, x_3)$, with $x_1' = h(x_1)\xi(x_1)x_1 = 1 \cdot x_1^{-1}x_1 = 1$, $x_2' = h(x_2)\xi(x_2)x_2 = ax_1^{-1}x_3x_2$ and $x_3' = h(x_3)\xi(x_3)x_3 = 1 \cdot x_1^{-1}x_3x_3 = x_1^{-1}x_3^2$. Solving the first equation $x_1' = 1$ we get $x_1 = z$, where z is a complex variable. The third one $x_3' = x_1^{-1}x_3^2$ is equivalent to $x_3'/x_3^2 = 1/z$ and $(-1/x_3)' = 1/z$, which gives us $-1/x_3 = \log z$, or $x_3 = -1/\log z$. Solving the second one $x_2' = ax_1^{-1}x_3x_2$, we get $(\log x_2)' = x_2'/x_2 = -a/z\log z$. Thus $\log x_2 = -a\log(\log z) + \mathcal{C} = \log(\log z)^{-a} + \mathcal{C}$, with the solution $x_2 = (1/\log z)^a$. Therefore, the given (G, ψ) is associated with a differential valuation of the field $\mathbb{C}(z, \log z, (1/\log z)^a)$.

3.3 The Case of Hardy Fields

An ordered Abelian group G of finite rank n can be embedded in the lexicographically ordered group $\mathbb{R}^n$. Let us show this. We can assume that $G = \mathbb{Q}G$. Let $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{0\}$ be the maximal chain of convex subgroups of G. So, we have an order-preserving homomorphism τ_i from G_i into $\mathbb{R}$ with kernel G_{i+1}. Let S_i, $i = 1, \cdots, n$ be a $\mathbb{Q}$-subvector space of G_{i-1} that is complementary to the subspace G_i, such that $G = S_1 \oplus \cdots \oplus S_n$. Therefore, any $s \in G$ can be written uniquely as $s = s_1 + \cdots + s_n$, where each $s_i \in S_i$. We obtain a well-defined map $G \to \mathbb{R}^n$ given by $s \mapsto (\tau_1(s_1), \cdots, \tau_n(s_n))$ which is an order-preserving embedding of G in $\mathbb{R}^n$. So, we can see that our ordered Abelian group G, of finite rank n, embeds in the lexicographically ordered group $\mathbb{R}^n$.

The following theorem is an application of Theorem 3.2.1 and it proves that there exists a Hardy field whose value group is a given asymptotic couple of Hardy type.

Theorem 3.3.1 ([19], Theorem 3)**.** *Let (G, ψ) be an asymptotic couple of Hardy type, with G of finite rank, and let G_0 be a non-zero subgroup of G such that $\psi(G_0) \subset G_0$. Let k be a Hardy field containing $\mathbb{R}$ whose corresponding asymptotic couple is $(G_0, \psi|_{G_0})$. Then* *there exists a Hardy field K containing k whose corresponding asymptotic couple is (G, ψ). Furthermore, K can be chosen to be an extension of k by repeated adjunctions of real powers of positive elements of non-zero value and logarithms and exponentials of positive elements of negative value, with the number of the last two types of extensions being* rank $G_1 -$ rank G_0.

3.4 The Problem of Asymptotic Integration

Let k be a differential field, C its subfield of constants and v a differential valuation of k. Given an element $a \neq 0$ in k, we want to find an integral of a in k, or, if this is not possible, an element $b \in k$ whose derivative is near a in an appropriate sense, i.e. $v(a - b^{'}) > v(a)$. If we can do this, we will be able to find an element $c \in k$ such that $v(a - b^{'}) < v((a - b^{'}) - c^{'}) = v(a - (b + c)^{'})$ and we will be able to find an infinite sequence $0 = b_0, b_1, b_2, ...,$ in k such that the sequence $v(a - b_i^{'})$ increases in $G = v(k^*)$. Whether or not the sequence $v(a - b_i^{'})$ is bounded, the various approximations b_i to $\int a$ may be useful. So, we want to see that, if we have $a \in k^*$, then is there an element $b \in k$ such that $v(a - b^{'}) > v(a)$? If we have such an element b, then $v(a) = v(b^{'})$. Conversely, if $b \in k$ is such that $v(a) = v(b^{'})$, then $v(a) - v(b^{'}) = v(a/b^{'}) = 0$, so there exists some $c \in C$ such that $v((a/b^{'}) - c) > 0$, or $v(a - (cb)^{'}) > v(b^{'}) = v(a)$. If $v(b) = 0$, then we have $v(b - c_1) > 0$ for some $c_1 \in C$, while $(b - c_1)^{'} = b^{'}$. Thus we have to find $b \in k$ such that $v(b) \neq 0$ and $v(b^{'}) = a$. Since $\psi(v(b)) = v(b^{'}/b) = v(b^{'}) - v(b)$, we have that $v(b^{'}) = \psi(v(b)) + v(b)$, so the question is to find $\beta = v(b) \in G^*$ such that $v(b^{'}) = \psi(\beta) + \beta = v(a)$. Therefore the nonzero elements of k which are "asymptotically integrable" are the elements whose values are of the form $\psi(\beta) + \beta$, for some $\beta \in G$.

Theorem 3.4.1 ([17], Theorem 2)**.** *Let G be an ordered Abelian group and $\psi : G^* \to G$ a function with properties (i), (ii), (iii). If $\psi(G^*)$ has a maximal element α, then there exists no $\beta \in G^*$ such that $\psi(\beta) + \beta = \alpha$. If the set $\psi(G^*)$ is well-ordered, then there is at most one element $\alpha \in G$ such that $\alpha \neq \psi(\beta) + \beta$ for any $\beta \in G^*$.*

Proof. Let α_0 be an element in G^* such that $\alpha = \psi(\alpha_0)$ is maximal in $\psi(G^*)$. If $\beta \in G^*$ is chosen such that $\psi(\beta) + \beta = \alpha = \psi(\alpha_0)$, then we must have $\psi(\beta) \leq \psi(\alpha_0)$, so $\beta = \psi(\alpha_0) - \psi(\beta) \geq 0$ contradicting property (iii): $\psi(\alpha_0) < \psi(\beta) + |\beta| = \psi(\alpha_0)$. This proves the first part of our theorem.

Now suppose that the set $\psi(G^*)$ is well-ordered and that $\alpha \in G$ is such that $\alpha \neq \psi(\beta)+\beta$ for any $\beta \in G^*$. If for some $\gamma \in \psi(G^*)$ we had $\alpha - \gamma \in \psi^{-1}(\gamma)$, then we would have $\psi(\alpha - \gamma) + (\alpha - \gamma) = \gamma + (\alpha - \gamma) = \alpha$, impossible since we supposed that α is not of the form $\psi(\beta) + \beta$ for any $\beta \in G^*$. Therefore $\alpha - \gamma \notin \psi^{-1}(\gamma)$ for each $\gamma \in \psi(G^*)$. We now claim that if $\gamma_1, \gamma_2 \in \psi(G^*)$ and $\gamma_1 \leq \gamma_2$, then $\alpha - \gamma_2 \notin \psi^{-1}(\gamma_1)$. For otherwise we could find $\gamma_1, \gamma_2 \in \psi(G^*)$ with $\gamma_1 \leq \gamma_2$ and $\alpha - \gamma_2 \in \psi^{-1}(\gamma_1)$ with γ_2 minimal for all pairs (γ_1, γ_2) satisfying these properties. We must have $\gamma_1 < \gamma_2$ since $\alpha - \gamma_2 \notin \psi^{-1}(\gamma_2)$. By the minimality property of γ_2, for each $\gamma \in \psi(G^*)$ such that $\gamma \leq \gamma_1$ we have $\alpha - \gamma_1 \notin \psi^{-1}(\gamma)$. But either $\alpha - \gamma_1 = 0$ or $\alpha - \gamma_1 \in \psi^{-1}(\psi(\alpha - \gamma_1)) \neq \psi^{-1}(\gamma)$. Then we get

$$\alpha - \gamma_1 \in \cup\{\psi^{-1}(\gamma) : \gamma \in \psi(G^*), \gamma > \gamma_1\} \cup \{0\}.$$

Since $\alpha - \gamma_2 \in \psi^{-1}(\gamma_1)$, by property (ii) we get that $\gamma_2 - \gamma_1 = (\alpha - \gamma_1) - (\alpha - \gamma_2) \in \psi^{-1}(\gamma_1)$, hence $\psi(\gamma_2 - \gamma_1) = \gamma_1$. Now property (iii) implies that $\gamma_2 < \psi(\gamma_2 - \gamma_1) + |\gamma_2 - \gamma_1| = \gamma_1 + (\gamma_2 - \gamma_1) = \gamma_2$, which is impossible. Therefore for each $\gamma_1 \leq \gamma_2$ in $\psi(G^*)$ we have $\alpha - \gamma_2 \notin \psi^{-1}(\gamma_1)$, so that

$$\alpha - \gamma_2 \in \cup\{\psi^{-1}(\gamma) : \gamma \in \psi(G^*), \gamma > \gamma_2\} \cup \{0\}.$$

Thus if $\alpha_1, \alpha_2 \in G$, $\alpha_1 \neq \alpha_2$ are neither of the form $\psi(\beta)+\beta$ for any $\beta \in G^*$, the statement

$\alpha_1 - \alpha_2 = (\alpha_1 - \gamma_2) - (\alpha_2 - \gamma_2)$ and property (ii) imply

$$\alpha_1 - \alpha_2 \in \cup\{\psi^{-1}(\gamma) : \gamma \in \psi(G^*), \gamma > \gamma_2\} \cup \{0\}$$

for all $\gamma_2 \in \psi(G^*)$. We obtain a contradiction if we take $\alpha_1 \neq \alpha_2$ by taking $\gamma_2 = \psi(\alpha_1 - \alpha_2)$. This completes our proof. □

A *differential field extension* L/K is given by two differential fields K and L such that $K \subseteq L$ and the restriction to K of the derivation of L is the derivation of K.

Suppose F is a field, K a differential field, $F \subseteq K$, and $f \in K$. To say that f is *differentially algebraic over* F means that $\{f, f', f'', \cdots\}$ is algebraically dependent over K, i.e., f is a root of a non-zero differential polynomial with coefficients in F.

Corollary 3.4.1 ([17], Corollary)**.** *Let k be a differential field, C its subfield of constants, and let v be a differential valuation of k. Suppose that $k \neq C$ and that each element of k is differentially algebraic over C. Then for any $a \in k^*$ there exists $b \in k$ such that $v(a - b^{'}) > v(a)$ except in the case where* $\max\{v(u^{'}/u) : u \in k^*, v(u) \neq 0\}$ *exists and $v(a)$ is this maximum.*

For example, consider the sequence of Hardy fields

$$\mathbb{R}(x) \subset \mathbb{R}(x, \log x) \subset \mathbb{R}(x, \log x, \log\log x) \subset ...$$

of germs of differentiable real-valued functions on neighborhoods of $+\infty$ in $\mathbb{R}$, differentiation being the usual differentiation with respect to x. According to axiom (H) in the case of Hardy fields, from a minimal positive element of the value group we obtain a maximal ψ.

Elements of the above sequence with minimal positive values are

$$1/x, 1/\log x, 1/\log\log x, ...$$

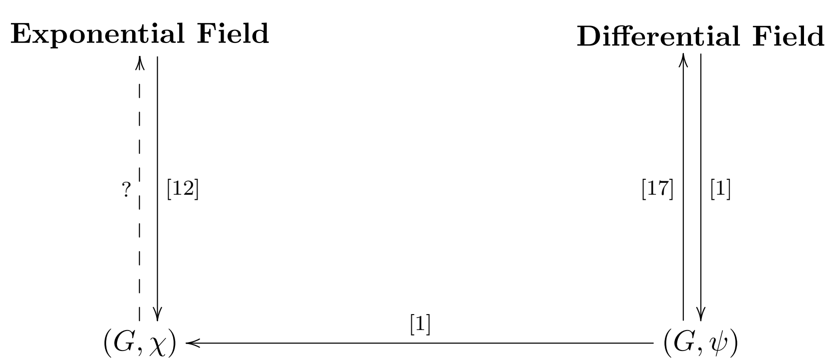
respectively, which have logarithmic derivatives

$$-1/x, -1/x \log x, -1/x \log x \log\log x, ...$$

respectively, whose values are precisely the values of those elements of the respective fields which cannot be asymptotically integrated in the same field, each element having an integral in the next field. In the union Hardy field $\mathbb{R}(x, \log x, \log\log x, ...)$ all elements can be asymptotically integrated.

Conclusion

The purpose of my thesis research, concerned with ordered algebraic structures, was how to construct an exponential field such that the contraction group described in [12] to be its value group. M. Aschenbrenner revealed a formal connection between asymptotic couples and contraction groups in his paper [1]. M. Rosenlicht proved in [17] that there exists a differential field whose valued group is a given asymptotic couple. One question that arises is this: If it is given a contraction group how do we get a corresponding asymptotic couple, or vice-versa? Following the same steps as in the construction of a differential field with a given asymptotic couple as its value group, we wanted to construct an exponential field with a given contraction group as its value group, as shown in the diagram bellow. One approach was to consider exponential Hardy fields, since they are closed under both, exponentiation and derivative.



This thesis gathers important notions and properties regarding asymptotic couples and contraction groups, in a manner I believe to be organized and efficient. I compared the algebraic and model-theoretical aspects of these structures, pointing the similarities between them. I proved in detail some of the proofs that were left to the reader, and some that only presented the ideas of the proof. I gave also some important examples, in order to sustain my theory. Through my work, resumed in this thesis, I set up the ground for further research in the area of ordered structures, as well as in analysis and real analytic geometry.

The significance of the notion of differential valuation is that various versions of L'Hospital's Rule hold with it, and one can do considerable work in asymptotic analysis. Hardy fields are very convenient for doing asymptotic analysis: if a Hardy field contains a germ of a function f, then this yields a lot of information about the growth of f.

Recently, Hardy fields have played an important role in model theory and its applications to real analytic geometry, via o-minimal structures on the real field.

In Section 2.5 we showed that the theory of closed H-asymptotic triples and hence the theory of H-asymptotic couples do not have the independence property. A theory T not having the independence property signifies on a model-theoretical level, a certain well-behavedness of T. There is a connection between the independence property and the notion of a Vapnik-Chernovenkis (VC) class from probability theory. A VC class is a collection $\mathcal{C}$ of subsets of a set X, if $f_{\mathcal{C}}(n) < 2^n$ for some n, where

$$f_{\mathcal{C}}(n) := \max\{|\mathcal{C} \cap F|, \text{ where } F \text{ is an } n\text{-element subset of } X\}.$$

Laskowski [14] proved that a formula $\varphi(x, y)$ does not have the independence property with respect to an $\mathfrak{L}$-structure $\mathbf{A}$ if and only if the collection $\mathcal{C}_\varphi = \{\varphi^{\mathbf{A}}(a, y)|\ a \in A^m\}$, where $\varphi^{\mathbf{A}}(a, y) := \{b \in A^n|\ \mathbf{A} \models \varphi(a, b)\}$, is a VC class.

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