ORDERINGS, CUTS AND FORMAL POWER SERIES

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To: Masoomeh, my wife, and
S. Fatemah and S. Zahra,
our little daughters
ABSTRACT

Each real closed field $R$ can be viewed as a subfield of the formal power series field $\kappa((G))$, where $G$ and $\kappa$ are respectively the value group and the residue field of the natural valuation $v$ on $R$. On the other hand, each ordering on $R(y)$ is in one-to-one correspondence with a cut $(A, B)$ in $R$ such that $A < y < B$. In this way one is led to the study of cuts in subfields of a given formal power series field. In the definition of a cut in a field, the operation of multiplication does not come into play; therefore, one is led to first study cuts and their realization in divisible ordered abelian groups. As every ordered abelian group can be embedded in a suitable Hahn product of archimedean ordered groups, the study of cuts in divisible ordered abelian groups reduces further to the study of cuts in a given Hahn product of archimedean groups.

One can prove that cuts in a given Hahn product $H$ might be realized by suitable elements of a certain bigger Hahn product. In this way, each cut in $R$, and hence each ordering on $R(y)$ corresponds to a canonically defined element $\psi$ in a certain bigger formal power series field $\kappa^a((G^a))$ which contains $\kappa((G))$. These elements $\psi$ do not belong to $R$. More generally, one can prove that orderings on the ring $R[y]$ are in one-to-one correspondence with canonically defined elements $\phi$, where $\phi$ is an element $\psi$ as above or an element of $R$. To find $\phi$, one first fixes an embedding $\iota$ of $R$ into $\kappa((G))$ which is also proper, i.e., the value group of $\iota(R)$ is $G$. Then it can be seen that there exists a certain extension $\kappa^a((G^a))$ of $\kappa((G))$ so that $\phi \in \kappa^a((G^a))$.

The correspondence between the orderings on the ring $R[y]$ and the elements $\phi$ can be generalized to the case of the orderings on the ring $R[y_1, \cdots, y_n]$. Actually, one can obtain an $n$-tuple $(\phi_1, \cdots, \phi_n)$ corresponding to an ordering on $R[y_1, \cdots, y_n]$, where all the $\phi_i$'s belong to a certain Hahn product.

If $F$ is an ordered field having $R$ as its real closure such that $\iota(F) \subseteq \kappa((V))$, where $V$ is the value group of $F$, then it can be proved that the value group $W$
of \( F(\phi_1, \ldots, \phi_n) \) is generated over \( V \) by all the exponents \( \gamma \) appearing in some \( \phi_i, 1 \leq i \leq n \). One can also find all the possible forms of \( W/V \). Furthermore, it is possible to show that if an ordered abelian group extension \( W \) of \( V \) is given so that \( W/V \) has one of those forms, then there exists \( (\phi_1, \ldots, \phi_n) \) such that the value group of \( F(\phi_1, \ldots, \phi_n) \) is \( W \).
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INTRODUCTION

Let $R$ be a real closed field. Our aim is to understand the orderings on $R(y_1, \ldots, y_n)$ or, more generally on $R[y_1, \ldots, y_n]$. Orderings are not only interesting in themselves but they come also into play in other areas, for example in the proof of Hilbert's 17-th problem: Let $Q$ be a rational function of $n$ variables with rational coefficients such that $Q(a_1, \ldots, a_n) \geq 0$ for all real numbers $a_1, \ldots, a_n$ for which $Q$ is defined. Then is $Q$ necessarily a sum of squares of rational functions with rational coefficients? ([Hi, Ja, BCR]) In 1927 Artin gave an affirmative answer to Hilbert's question [Ar, LT]. In his proof, he used the properties of ordered fields. By an ordered field we mean a field $F$ together with a total ordering $\geq$ on $F$ which is compatible with the field operations. Not every field has an ordering. Actually, an ordering can be defined on a field $F$ if and only if $-1$ is not a sum of squares of elements of $F$ [Ja]. An ordered field $R$ is said to be real closed if there is no ordered field $K \supseteq R$ which is algebraic over $R$. The most common example of a real closed field is the field of real numbers $\mathbb{R}$. More recently, orderings in $R[y_1, \cdots, y_n]$ have been shown to play an important role in real algebraic geometry [BCR].

The orderings on $\mathbb{R}(x)$ are well-known [BCR]. In fact, they can easily be characterized. Actually, we just need to know whether $x$ is larger than any real number, smaller than any real number, or there exists a real number $a$ such that $x$ is, roughly speaking, adjacent to $a$ and in this case we should also know if $x$ is to the right or to the left of $a$. In other words, each ordering on $\mathbb{R}(x)$ determines a cut in $\mathbb{R}$. In general, if $R$ is a real closed field then there is also such a nice relation between the orderings on $R(x)$ and the cuts in $R$ [Gi]. The orderings on $\mathbb{R}(x, y)$ are discussed in [AGR] (also see [Br]). In [AGR], some sort of "limit process" is used to determine how close $y$ is to $x$ (assuming $0 < y < x < \varepsilon$, for all real $\varepsilon > 0$). This method leads to an approximation of $y$ in terms of $x$. This approximation is 0; a finite sum $\sum_{i=1}^{n} a_i x^{r_i}$, where $a_i \in \mathbb{R}$ for $i \leq n$, $r_i \in \mathbb{Q}$ (= field of rational numbers) for $i < n$, $a_1 > 0$, $r_n \in \mathbb{R} \setminus \mathbb{Q}$, and $r_1 < \cdots < r_n$; or a power series of the form $\sum_{i=1}^{\infty} a_i x^{r_i}$, where
\(a_i \in \mathbb{R}, \ a_i > 0, \) and \(\{r_i\}_{i=1}^{\infty}\) is an increasing sequence of rational numbers (also see example 3.2.5).

Since any form of the sum that we saw in the previous paragraph, as an approximation of \(y\), belongs to the formal power series field \(\mathbb{R}((\mathbb{R}))\), we are naturally led to understand and use the necessary properties of such fields. The formal power series field \(k((G))\), where \(k\) is a subfield of \(\mathbb{R}\) and \((G, +, \leq)\) is an ordered abelian group, consists of all formal sums \(\sum_{r \in G} a_r x^r\), where \(a_r \in k\) for all \(r \in G\), and \(\{r \in G : a_r \neq 0\}\) is well-ordered. Such fields were first considered by H. Hahn [Ha]. Many properties of such fields and also equivalent definitions of them were developed later, see for example [Kr, Ka, Fu].

We find that valuation theory is a very nice and useful means to understand orderings. It is a well-known result [Mr, Ro] that there exists a natural valuation \(v : R \to G \cup \{\infty\}\), where \(G\) is a divisible ordered abelian group, with residue field \(\kappa\) which is (an isomorphic image of) a real closed subfield of \(\mathbb{R}\). Also it is known (and we will prove it in the first chapter) that there exists an ordered field embedding \(\iota : R \hookrightarrow \kappa((G))\) which is also proper (for the definition of a proper embedding, see Remark following Theorem 1.4.8). In general there are many such embeddings, but we choose one and fix it. As we saw above, an ordering on \(R(y)\) gives rise to a cut in \(R\). We will try "to fill" this cut in \(\kappa((G))\) or, if this is not possible, in a certain bigger power series field.

In the first chapter of the thesis, we review the preliminaries which we will need throughout the thesis such as the properties of real closed fields, valuations, maximally complete fields, etc. In chapter two we discuss the cuts in an ordered abelian group. This subject is not only interesting by itself but also paves the way for the discussion of the cuts in real closed fields. In fact, any cut \((A, B)\) in \(R\) gives rise to a cut \((S, T)\) in the ordered abelian group \(G\). Actually, the set \(S = \{v(b-a) : b \in B, a \in A\}\) is a lower cut in \(G\) and \(T = G \setminus S\) is its associated upper cut. Thus we are led to study the cuts in an arbitrary (divisible) ordered abelian group \(G\). We first embed \(G\) in its associated Hahn product [Fu]. This is the Hahn product \(H_G = \mathcal{H}_{i \in I} G_i\) of a family \(\{G_i\}_{i \in I}\) of archimedean ordered abelian groups
(also see section 2.1.2). The procedure is as follows: There exists a set valuation $w : G \to I \cup \{\infty\}$, where $I$ is a totally ordered set, such that for any $a, b \in G \setminus \{0\}$ we have $w(a) \leq w(b)$ if and only if there exists a positive integer $n$ such that $|b| \leq n|a|$. In this case, $G_i$ is the ordered group obtained by factoring the subgroup $D_i = \{ a \in G : w(a) \geq i \}$ of $G$ by the convex subgroup $C_i = \{ a \in G : w(a) > i \}$. As each $G_i$ is an archimedean (divisible) ordered group, $G$ can actually be embedded in $\mathbb{H}_G = \bigwedge_{i \in I} \mathbb{R}$ which is the Hahn product of $I$ copies of $\mathbb{R}$.

Now we use the properties of the valuation to define an element $\theta$ of $\mathbb{H}_G$ corresponding to the given cut $(S, T)$ in $G$. In short, the procedure is as follows: The cut $(S, T)$ gives rise to a lower cut $U$ in $I$, namely $U = \{ w(\beta - \alpha) : \beta \in T, \alpha \in S \}$. For each $i \in U$, there exists $\alpha \in S, \beta \in T$ such that $w(\beta - \alpha) = i$. Then $\theta_j$, i.e., $j$-th slot of $\theta$, is the same as $\alpha_j$ (or equivalently $\beta_j$) for $j \in I$ with $j < i$, where $\alpha_j$ (resp., $\beta_j$) is the $j$-th slot of $\alpha$ (resp., $\beta$). Of course, if $U$ has a last element $k$ then we should employ another method to define $\theta_k$. Moreover, we define $\theta_i$ to be 0 if $i \in I \setminus U$. This element $\theta$ together with two more pieces of data characterizes the given cut $(S, T)$. One of these two is $U$. The other one is knowing whether there exists $\gamma \in G$ such that $w(\theta - \gamma) \notin U$ and, if such $\gamma$ exists then whether $\gamma \in S$, or $\gamma \in T$. It will be proved that the set consisting of these three pieces of data, which we call the cut symbol associated to $(S, T)$, completely characterizes the cut $(S, T)$ (see Theorem 2.2.6). Then we will prove, depending on the form of the cut symbol, that either the obtained $\theta$ fills the cut $(S, T)$ in $\mathbb{H}_G$, or we can go to a certain bigger Hahn product $\mathbb{H}_G$ so that a certain element $\theta \pm 1_U \in \mathbb{H}_G$ fills the cut $(S, T)$.

The description of cuts in real closed fields follows immediately from that in ordered abelian groups. We show that, given a cut $(A, B)$ of $R$ with a cut symbol, say $a$, there exists a canonically defined extension of the proper embedding $\iota : R \hookrightarrow \kappa((G))$ to a proper embedding $\iota^a : R(y) \hookrightarrow \kappa^a((G^a))$, where $\kappa^a$ is a certain real closed archimedean extension of $\kappa$, $G^a$ is a certain divisible ordered abelian group extension of $G$. The ordering on $R(y)$ induced by the embedding $\iota^a$ is the ordering on $R(y)$ corresponding to the cut $(A, B)$, $\kappa^a$ is the residue field of the real closure of $R(y)$ in $\kappa^a((G^a))$ and $G^a$ is the value group of the real closure of the image of $R(y)$
in $\kappa^a((G^a))$. Clearly the embedding $e^a$ on $R(y)$ extends to the real closure of $R(y)$. We denote by $\phi = \phi^a$ the image of $y$ in $\kappa^a((G^a))$. Obviously, knowing $\phi$ determines the embedding $e^a$ and consequently the ordering on $R(y)$ and the cut $(A,B)$ of $R$, i.e., $\phi$ carries the same information as the cut symbol $a$ (see Proposition 3.2.3).

In chapter three after the discussion of the cuts in real closed fields, we have the opportunity to discuss the orderings on the ring $R[y_1,\ldots,y_n]$, where $R$ is a real closed field as before. Orderings on a general commutative ring $A$ with identity were first introduced by [CR] (see also [Be]). Suppose that there exist a subset $P \subseteq A$ and a prime ideal $\mathfrak{p} \subseteq A$ such that $P \cup -P = A$, $P \cap -P = \mathfrak{p}$, $P + P \subseteq P$, and $PP \subseteq P$ (This same definition is also used to define the orderings on a non-commutative ring $A$ with identity [LMZ]). Then we say that $P$ is an ordering on $A$ with support $\mathfrak{p}$. As $\mathfrak{p}$ is prime, the quotient ring $A/\mathfrak{p}$ is an integral domain and therefore, we can talk about the field of quotients of $A/\mathfrak{p} (= \text{f.q.}(A/\mathfrak{p}))$. It is not difficult to see that, corresponding to the given ordering on $A$, there is an induced ordering on the field $\text{f.q.}(A/\mathfrak{p})$. Conversely, if $\mathfrak{p}$ is a prime ideal in the ring $A$ then any ordering on $\text{f.q.}(A/\mathfrak{p})$ induces an ordering on $A$ with support $\mathfrak{p}$ [BCR]. If $P$ is an ordering of $A$ with support $\mathfrak{p}$ then the extensions of $P$ to $A[y]$ are in natural one-to-one correspondence with the orderings on $S[y]$, where $S$ denotes the real closure of $\text{f.q.}(A/\mathfrak{p})$ at the ordering induced by $P$. This is well-known [Be], but we give the proof. An ordering on $S[y]$ either has support $\{0\}$ (so corresponds to a cut in $S$) or has support generated by $y-a$ for some $a \in S$ (so corresponds to an element $a \in S$).

By induction on $n$, we transform the characterization of the orderings on the ring $R[y_1,\ldots,y_n]$ to the discussion of cuts in (or elements of) certain real closed fields which are defined inductively. We will find that the orderings on $R[y_1,\ldots,y_n]$ are in one-to-one correspondence with certain $n$-tuples $(\phi_1,\ldots,\phi_n)$ which we call order symbols, where for each $1 \leq i \leq n$, $\phi_i$ is either a power series corresponding to a cut symbol or an element of the inductively defined real closed field. The number of former elements tells us how many of the $\phi_1,\ldots,\phi_n$ are algebraically independent.

Suppose that $F$ is an ordered field with $R$ as its real closure. As mentioned before, there exists a proper embedding $R \hookrightarrow \kappa((G))$. For brevity, we can identify
$R$ and $F$ with their images in $\kappa((G))$. Let $\phi$ be a cut symbol in $R$. In chapter four we discuss how the value group $V'$ of $F(\phi)$ is related to the value group $V$ of $F$. Actually, if $F \subseteq \kappa((V))$, then $V'$ is generated over $V$ by the exponents appearing in $\phi$. This problem has long been of interest from various points of view [MS, ZS]. We have generalized the problem so that one can find the value group of $F(\phi_1, \ldots, \phi_n)$, where $(\phi_1, \ldots, \phi_n)$ is an order symbol which corresponds to an ordering on $R[y_1, \ldots, y_n]$ (see Corollary 4.2.2 in the thesis and Theorem 4 and the final paragraph in [MS]).

If $F, R, \phi, V,$ and $V'$ are as in the previous paragraph then we can in general characterize the form of $V'$ relative to $V$. We can also characterize in some way the residue field $\kappa_0'$ of $F(\phi)$ relative to the residue field $\kappa_0$ of $F$. In fact, we will prove that one of these cases happens (see Theorem 4.4.1): (i) $V'/V$ is finite and $\kappa_0'$ is a finite extension of $\kappa_0$, (ii) $V'/V$ is finite and $\kappa_0'$ is finitely generated over $\kappa_0$ and $\text{trdeg}(\kappa_0 : \kappa_0) = 1$, (iii) $V' = W \oplus \mathbb{Z} \delta$ where $W \subseteq V$, $W/V$ is finite, $\mathbb{Z} \delta$ is an infinite cyclic group, and $\kappa_0'$ is a finite extension of $\kappa_0$, or (iv) $V'/V$ is torsion and $\kappa_0'$ is an algebraic extension of $\kappa_0$. These facts, except possibly the countability of the torsion case, might have been known already. We will then show that modulo these constraints, $V'$ can be arbitrarily prescribed. One thing should be pointed out here: The coefficients of $\phi$ might not all belong to $\kappa_0'$ even if we assume $F_0 \subseteq \kappa_0((G))$. We will show this by an example. This is somewhat a shortcoming to the theory, but if we assume that $R \subseteq F$ this will also be settled.

Suppose now that $R \subseteq \kappa((G))$. A question arises as whether $R$ is truncation closed. We say that $R$ is truncation closed if for each element $p = \sum_{r \in G} a_r x^r$ of $R$ and each element $s \in G$, the truncated series $(p)_{<s} = \sum_{r < s} a_r x^r$ also belongs to $R$. The field of Puiseux series with coefficients in $\mathbb{R}$, which is a subfield of $\mathbb{R}(\mathbb{Q})$, is truncation closed. The field $\kappa((G))$ is also truncation closed. On the other hand, there are simple examples of fields which are not truncation closed: There exists a Puiseux series $X = \sum_{i=1}^{\infty} a_i x^{r_i}$, $a_i \in \mathbb{R}$ for $i \geq 1$, which is transcendental over $\mathbb{R}(x)$. We can assume that $a_i \neq 0$ and $r_1$ is a non-zero integer. Now, if $\mathbb{R}(X)$ was truncation closed, then $x^{r_1}$ would belong to $\mathbb{R}(X)$. Thus $\text{trdeg}(\mathbb{R}(X) : \mathbb{R}) > 1$ which
is a contradiction. We got interested in the question of truncation-closedness when dealing with those elements that fill the cuts. We wanted to know when the proper truncations of these elements belonged to $R$. In [MR] it is shown that among the embeddings $R \hookrightarrow \kappa((G))$, there exists an embedding $\tau$ such that the image of $R$ under $\tau$ is truncation closed. Using our method, we include a proof of this fact here. The problem of truncation-closedness in the case of ordered abelian groups is also of interest, but much easier to handle. In fact, we use our method to highlight the truncation-closedness property in the proof of Hahn’s Embedding Theorem [Fu] (see section 5.6).

The results in chapter 1 are well-known. In chapter 2 the notion of cut symbols (Definition 2.2.1) and their correspondence with the cuts of a divisible ordered abelian group (Theorem 2.2.6) as well as the way the filling of cuts is introduced (Proposition 2.4.1) are new. In chapter 3 the notion of order symbols (Page 58) which can be used to codify the orderings on the ring $R[y_1, \ldots, y_n]$ (Proposition 3.3.3) is new. In chapter 4, Theorem 4.2.1 in its generality is new; moreover, Theorem 4.3.2 is also new. Furthermore, the results in section 4.4 (Theorem 4.4.1 and Theorem 4.4.2) seem, at least for some parts, to be new. The results in chapter 5 are already known.
Chapter 1

Introduction to Real Closed Fields and Valuations

In this chapter we state some properties of ordered fields and valuations which will be used later. The results of this chapter are well-known. However, we prove a few of these results. This is the case where we could not find proofs for such results or, at least, when the proofs given were interesting to us.

1.1 Real closed fields

By an ordered field $F$ we mean a field $F$ with a total ordering $\geq$ which is preserved under the operations of the field (i.e., if $a \geq b$ then $a + c \geq b + c$, and also $ac \geq bc$ if $c \geq 0$). We write $a \leq b$ (resp., $a < b$) if $b \geq a$ (resp., $b > a$ and $b \neq a$). It can easily be seen that the field $F$ is ordered if and only if there exists $P \subseteq F$ such that (1) $P \cup -P = F$, (2) $P \cap -P = \{0\}$, and (3) $P + P \subseteq P$, $PP \subseteq P$. In this case $P$ is called positive cone of the ordered field $F$. It is obvious that each ordered field $F$ contains an isomorphic copy of the $\mathbb{Q}$. Therefore, we may write $\mathbb{Q} \subseteq F$. An ordered field $F$ is said to be archimedean if for each $a \in F$ there exists a positive integer $n$ such that $n > a$. If $R$ is an ordered field and there is no algebraic ordered field extension of it, then $R$ is called a real closed field. It follows that any ordered field has an algebraic real closed field extension.

Proposition 1.1.1 Any real closed field has a unique ordering.
**Proof.** This is just a part of Theorem 2 in chapter VI in [Ja].  

**Theorem 1.1.2 (Artin-Schreier [AS])** A field $F$ is real closed if and only if $\sqrt{-1} \notin F$ and $F(\sqrt{-1})$ is algebraically closed.

**Proof.** This is just a part of Theorem 3.3 in [Pr], also see [Ja, chap. VI].  

**Example 1.1.1** The complex number field $\mathbb{C}$, as we know, is algebraically closed. Since $\mathbb{C} = \mathbb{R}(\sqrt{-1})$ and $\sqrt{-1} \notin \mathbb{R}$, it follows that $\mathbb{R}$ is real closed.

From the above theorem it follows that if $R$ is a real closed field, then any polynomial in $R[x]$ can be written as a product of linear and quadratic polynomials in $x$ with coefficients in $R$.

**Theorem 1.1.3 (Artin-Schreier [AS])** An ordered field $R$ is real closed if and only if the following conditions hold:

1. Every polynomial of odd degree with coefficients in $R$ has at least one root in $R$.
2. An element $a \in R$ has a square root in $R$ if and only if $a$ is non-negative.

**Proof.** Suppose that $R$ is real closed. Then every polynomial of odd degree with coefficients in $R$ has, by [Ja, Theorem 4, chap. VI], a root in $R$; therefore, condition 1 of the theorem is satisfied. Any element of a real closed field is, by [Ja, Theorem 1, chap. VI], either a square or negative of a square. Therefore, condition 2 of the theorem is obvious. Now suppose that conditions 1 and 2 of the theorem are satisfied. By condition 2, $\sqrt{-1} \notin R$. This together with condition 1 imply, by [Ja, Theorem 5, chap. VI], that $R(\sqrt{-1})$ is algebraically closed. Thus by Proposition 1.1.2, $R$ is real closed.  

**QED**

**Proposition 1.1.4** Suppose that $\phi : R \to F$ is a field isomorphism of the real closed field $R$ into an ordered field $F$. Then $\phi(R)$ with the ordering induced from $F$ is order isomorphic to $R$. In other words, $\phi$ is an order isomorphism.

**Proof.** Let $a \in R, a > 0$. Then, by Theorem 1.1.3, there exists $b \in R$ such that $b^2 = a$. Therefore, $\phi(a) = (\phi(b))^2$. But the positive cone of the ordering on an ordered field contains the squares of the elements of that field. Thus $\phi(a) > 0$ and we are done.  

**QED**
Proposition 1.1.5 Suppose that $R$ is a real closed field and $F$ is a subfield of $R$. If $F$ contains all $a \in R$ which are algebraic over $F$, then $F$ is real closed.

Proof. Let $f$ be a polynomial of odd degree with coefficients in $F$. $R$ is real closed; therefore by Theorem 1.1.3, $f$ has a root $a$ in $R$. Then $a \in F$ by the assumption. Now let $b \in F$. Using Theorem 1.1.3 again, the equation $x^2 = b$ has a root $c$ in $R$ if and only if $b$ is non-negative and, if this is the case, then $c \in F$ by the assumption. Thus $F$ is real closed by Theorem 1.1.3. QED

Suppose that $R$ is a real closed extension of the ordered field $F$. If $R$ is algebraic over $F$ then $R$ is called a real closure of $F$. For the proof of the following look at [Ja, Theorem 8, chap. VI].

Theorem 1.1.6 Every ordered field has a real closure. If $F_1$ and $F_2$ are ordered fields with real closures $R_1$ and $R_2$, respectively, then any order isomorphism of $F_1$ onto $F_2$ has a unique extension to an isomorphism of $R_1$ onto $R_2$. The extension is an order isomorphism.

Using the above theorem we immediately obtain the following.

Corollary 1.1.7 Suppose $F$ is an ordered field with $R$ as a real closure. Then the only automorphism of $R$ over $F$ is the identity.

The field of Puiseux series will be used a lot later, so the definition of such a field is given here:

Example 1.1.2 Suppose $\kappa$ is a field. Denote by $P_\kappa(x)$ the set of all formal power series $a = \sum_{i=l}^{\infty} a_i x^{i/n}$, where $l$ is an integer, $n$ is a positive integer, and $a_i \in \kappa$. Each $a_i x^{i/n}$, $l \leq i \leq \infty$, is called a term. Let $a, b \in P_\kappa(x)$. We can define $a + b$ termwise. Moreover, we can also define $a \cdot b$ in the usual way that the multiplication of two (convergent) series is defined (the explicit definition of multiplication is also given in a more general context in section 1.3). The set $P_\kappa(x)$ with the operations $+, \cdot$ forms a field [Wa]. This field is called the field of Puiseux series over $\kappa$ [PR]. If $\kappa$ is a real closed field, then by Proposition 1.1.2, $K = \kappa(\sqrt{-1})$ is algebraically closed.
and hence the field \( \mathcal{P}_K(x) \) is also algebraically closed [Wa, Theorem 3.1]. Note that \( \mathcal{P}_K(x)[\sqrt{-1}] \cong \mathcal{P}_K(x) \). Therefore, using Proposition 1.1.2 once again, we see that the field \( \mathcal{P}_K(x) \) is real closed.

### 1.2 Valuations

We will use valuations a lot. Therefore, we state properties that we will need later.

**Definition 1.2.1** Suppose that \( F \) is a field and \( B \) is a subring of \( F \). Then \( B \) is called a valuation ring of \( F \) if for any \( a \in F \) we have \( a \in B \) or \( 1/a \in B \).

**Definition 1.2.2** Suppose that \( A \) is a ring with identity. \( A \) is said to be a local ring if it has only one maximal ideal.

**Definition 1.2.3** Suppose that \( F \) is a field and \( B \) is a subring of \( F \). We say that \( B \) is integrally closed in \( F \) if whenever \( a \in F \) is a root of some monic polynomial \( f(x) \in B[x] \), then \( a \) is in \( B \).

**Proposition 1.2.1** Suppose that \( F \) is a field. Let \( B \) be a valuation ring of \( F \). Then

1. \( B \) is a local ring.
2. \( B \) is integrally closed in \( F \).

**Proof.** (1) This is standard [AM, La]. Let \( U \) denote the set of all units in \( B \) (i.e., those \( x \in B \) such that \( 1/x \in B \) as well). Let \( M = B \setminus U \). We show that \( M \) is a maximal ideal in \( B \). Suppose that \( a, b \in M \), \( a \neq 0 \neq b \). Then either \( b^{-1}a \in B \) or \( a^{-1}b \in B \). Without loss of generality, assume that \( b^{-1}a \in B \). So \( b^{-1}a + 1 \in B \). On the other hand, \( b \) is not a unit in \( B \). Therefore, \( a + b = b(b^{-1}a + 1) \in B \) is not a unit in \( B \) either. Thus \( a + b \in M \). Next assume that \( a \in M \) and \( t \in B \). Then \( at \in M \).

**Reason.** Assume, on the contrary, that \( at \notin M \). Then there exists \( s \in B \) such that \( ats = 1 \). Therefore, \( ts \in B \) is the inverse of \( a \), i.e., \( a \in U \). This contradicts our assumption that \( a \in M \). Thus \( at \in M \). So far, we have shown that \( M \) is an ideal in \( B \). To see that \( M \) is actually a maximal ideal, note that any proper ideal of \( B \)
does not contain any units of $B$. Therefore, for any proper ideal $I$ of $B$ we have that $I \subseteq M$. Thus $M$ is in fact the unique maximal of $B$ and (1) is proved.

(2) Suppose $a \in F$ is a root of some monic polynomial $f(x) = \sum_{i=0}^{n} b_{i}x^{i}$, where $n$ is a positive integer, $b_{i} \in B$ for all $0 \leq i \leq n$, and $b_{n} = 1$. If $a \in B$ then we are done. So assume that $1/a \in B$. Then $a = -\sum_{i=0}^{n-1} \frac{b_{i}}{a^{n-1-i}} \in B$, as required. QED

Definition 1.2.4 Suppose $F$ is a field, and $\Gamma = (\Gamma, +, \leq)$ is an ordered abelian group. Then a surjective map $v : F \setminus \{0\} \rightarrow \Gamma$ is called a valuation if the following conditions hold:

1. $v(ab) = v(a) + v(b)$.
2. $v(a + b) \geq \min\{v(a), v(b)\}$, if $a + b \neq 0$

The ordered group $\Gamma$ is called value group of the valuation $v$ (or value group of $F$ if the valuation is known). We also write $v(F) = \Gamma$. A valued field is a pair $(F, v)$ where $F$ is a field and $v$ is a valuation defined on $F^{*} = F \setminus \{0\}$. We can define $v$ at 0 by $v(0) = \infty$, where $\infty$ is larger than any element of $\Gamma$; moreover, we define $g + \infty = \infty + g = \infty$, for all $g \in G$, and $\infty + \infty = \infty$. Note. If $\Gamma$ is the trivial group $\{0\}$ then the corresponding valuation on $F$ is called the trivial valuation.

It can easily be seen that $v(1) = 0$, $v(-a) = v(a)$ for all $a \in F$, and that $v(a + b) = \min\{v(a), v(b)\}$ for all $a, b \in F$ with $v(a) \neq v(b)$. Let $B_{v} := \{a \in F : v(a) \geq 0\}$ and $M_{v} := \{a \in F : v(a) > 0\}$. Then it is easily seen that $B_{v}$ is a valuation ring of $F$ with $M_{v}$ as its unique maximal ideal. The field $F_{v} := B_{v}/M_{v}$ is called the residue field of $(F, v)$. The set $U = \{x \in F : v(a) = 0\} \subseteq B$ is the set of units of $B$. The valuation $v$ is actually a homomorphism from $F^{*}$ onto $\Gamma$ with the kernel $U$. Thus we have $F^{*}/U \cong \Gamma$.

Suppose that $v_{1} : F^{*} \rightarrow \Gamma_{1}$, $v_{2} : F^{*} \rightarrow \Gamma_{2}$ are two valuations. We say that $v_{2}$ is equivalent to $v_{1}$ if there exists an order isomorphism of groups $\sigma : \Gamma_{1} \rightarrow \Gamma_{2}$ such that $v_{2} = \sigma \circ v_{1}$.

Proposition 1.2.2 Suppose that $F$ is a field and $B$ is a valuation ring of $F$. Then there exists a valuation $v$ defined on $F$ such that its corresponding valuation ring is
B. Moreover, if \( v' \) is another valuation defined on \( F \) with \( B \) as its valuation ring, then \( v' \) is equivalent to \( v \).

**Sketch of the proof.** Let \( U = \{ x \in B : 1/x \in B \} \). Then \( U \) is a subgroup of \( F^* \). \( F^*/U \) is an abelian group. The operation on \( F^*/U \) can be considered to be additive (i.e., define \( aU + bU = abU \)). Furthermore, we define \( aU > bU \) if \( a/b \in B \setminus U \). Then the natural map \( \sigma : F^* \rightarrow F^*/U \) is easily seen to define a valuation \( v \) on \( F^* \). Moreover, it is not difficult to see validity of the last assertion of the proposition.

QED

Suppose \( F \) is an ordered field with the positive cone \( P \). Let

\[
B = \{ a \in F : -n \leq a \leq n, \text{ for some natural number } n \}, \tag{1.2.1}
\]

\[
M = \{ a \in F : -1/n \leq a \leq 1/n, \text{ for all natural numbers } n \}. \tag{1.2.2}
\]

It is easily seen that \( B \) is a valuation ring of \( F \) having \( M \) as its maximal ideal. There exists a natural ordering on \( \bar{F} = B/M \) with the positive cone \( \bar{P} \) given by \( b + M \in \bar{P} \) if and only if \( b \in B \cap P \) or \( b \in M \). This ordering is easily seen to be archimedean. Therefore, there exists a natural embedding \( \bar{\tau} : \bar{F} \rightarrow \mathbb{R} \) such that \( \bar{\tau}^{-1}(\mathbb{R}^2) = \bar{P} \). Now if we compose \( \bar{\tau} \) with the natural homomorphism from \( B \) to \( \bar{F} \), then we get a map \( \tau : F \rightarrow \mathbb{R} \cup \{ \infty \} \) given by

\[
\tau(a) = \begin{cases} 
\bar{\tau}(a + M) & \text{if } a \in B \\
\infty & \text{if } a \notin B
\end{cases}
\]

This map \( \tau \) induces a ring homomorphism from \( B \) to \( \mathbb{R} \). Moreover, \( M \) is the kernel of this homomorphism.

**Definition 1.2.5** Suppose \( F \) is a field and \( B \) is a valuation ring of \( F \). Let \( \beta : B \rightarrow \mathbb{R} \) be a ring homomorphism having the maximal ideal of \( B \) as its kernel. Then the map \( \tau : F \rightarrow \mathbb{R} \cup \{ \infty \} \) defined by

\[
\tau(a) = \begin{cases} 
\beta(a) & \text{if } a \in B \\
\infty & \text{if } a \in F \setminus B
\end{cases}
\]

is said to be a real place.
If \( \tau \) is a real place defined on a field \( F \) then there is naturally a valuation on \( K \) associated to \( \tau \) so that its corresponding valuation ring is \( B \) as in Definition 1.2.5.

In order to state the next theorem we need the definition of preorderings. Suppose that \( F \) is field. A preorder on \( F \) is a subset \( T \) of \( F \) such that: \( T + T \subseteq T \), \( T \cdot T \subseteq T \), and \( F^2 \subseteq T \). For example, \( T = \sum F^2 \), i.e., the set of finite sums of the squares of the elements of \( F \), is a preorder on \( F \).

**Theorem 1.2.3** Let \( F \) be a field and \( \tau : F \to \mathbb{R} \cup \{\infty\} \) be a real place. Let \( U^+ = U_\tau^+ = \{a \in F : \tau(a) \neq \infty, \tau(a) > 0\} \). This is a subgroup of \( F^* = F \setminus \{0\} \) and \( -1 \notin F_\tau^2 U^+ \). \( F^2 U^+ = F^2 U^+ \cup \{0\} \) is a preorder on \( F \). Let \( P^* \) be any subgroup of \( F^* \) containing \( F_\tau^2 U^+ \), having index 2 in \( F^* \) and such that \( -1 \notin P^* \). Then \( P = P^* \cup \{0\} \) is an ordering. Every ordering \( P \) on \( F \) is obtained by this process, starting with some real place \( \tau \). The real place \( \tau = \lambda(P) \) is completely determined by the ordering \( P \) as follows: The valuation ring of \( \tau \) is \( B = \{a \in F : \tau(a) \neq \infty\} = \{a \in F : n + a, n - a \in P \text{ for some integer } n \geq 1\} \). The maximal ideal of \( \tau \) is \( M = \{a \in F : \tau(a) = 0\} = \{a \in F : \frac{1}{n} + a, \frac{1}{n} - a \in P \text{ for all integers } n \geq 1\} \). The induced embedding \( \bar{\tau} : B/M \to \mathbb{R} \) is the unique embedding such that \( \bar{\tau}^{-1}(\mathbb{R}^2) = \bar{P} \), where \( \bar{P} \) is the archimedean ordering on \( B/M \) induced by \( P \), i.e., \( \bar{P} = \{a + M : a \in B \cap P\} \).

**Proof.** See [Mr, Theorem 1.3.1]. QED

There are some more interesting relationships between a real place on a field \( F \) and the preorderings on \( F \) which correspond to that real place (see section 4.3).

If \( F \) is an ordered field, then there exists a unique valuation \( \nu \) defined on \( F \) satisfying \( \nu(b) \leq \nu(a) \) if and only if \( |a| \leq n|b| \) for some positive integer \( n \). In fact, by Proposition 1.2.2, there exist a unique (up to equivalence of the valuations) valuation \( \nu \) defined on \( F \) having \( B \), as defined in (1.2.1), as its valuation ring. Therefore, if \( a, b \in F^* \) then \( \nu(b) \leq \nu(a) \) if and only if \( a/b \in B \) or, if and only if \( |a/b| \leq n \) for some positive integer \( n \). Conversely, suppose that there exists a valuation \( \nu' \) such that for all \( a, b \in F \), \( \nu'(b) \leq \nu'(a) \) if and only if \( |a| \leq n|b| \) for some positive integer \( n \). Then for any \( a \in F \), \( \nu'(a) \geq 0 = \nu'(1) \) if and only if \( |a| \leq n \) for
some positive integer \( n \). That is, the set \( \{ a \in F : -n \leq a \leq n, \text{ for some positive integer } n \} \) is the valuation ring of \( v' \). Therefore, \( v' \) is equivalent to \( v \). The valuation \( v \) just defined is called the \textit{natural valuation} on \( F \). Therefore, as we saw above, there exists an archimedean ordering on the residue field \( \kappa = B/M \), where \( B \) is the valuation ring corresponding to the natural valuation \( v \) with \( M \) as its maximal ideal.

\textbf{Definition 1.2.6} The ordered abelian group \( G \) is said to be divisible if for any \( a \in G \) and any positive integer \( n \) there exists \( b \in G \) such that \( a = nb \).

If \( G \) is an ordered abelian group then there exists a unique smallest ordered abelian group \( \hat{G} \) (up to the isomorphism of ordered abelian groups) which is divisible and contains \( G \). \( \hat{G} \) is called the \textit{divisible hull} of \( G \).

\textbf{Theorem 1.2.4} Suppose that \( F \) is an ordered field having \( R \) as its real closure. Let \( V \) (resp., \( V_0 \)) be the value group, \( \kappa \) (resp., \( \kappa_0 \)) the residue field of \( R \) (resp., \( F \)) under the natural valuation \( v \). Then \( V \) is the divisible hull of \( V_0 \) and, \( \kappa \) is the real closure of \( \kappa_0 \).

\textbf{Proof.} We have \( V_0 \subseteq V \). Let \( \alpha \in V \) and \( n \) be a positive integer. Then \( \alpha = v(a) \) for some \( a \in R, a \geq 0 \). Since \( R \) is real closed, \( a^{1/n} \in R \). So \( \alpha/n \in V \). Therefore, \( V \) is divisible. Moreover, there exists a non-zero polynomial \( f \in F[x] \), say \( f = \sum_{i=0}^{k} b_ix^i \), such that \( f(a) = 0 \). From this it follows that there exist \( j \neq i, b_i \neq 0 \neq b_j \), such that \( v(b_ja^j) = v(b_ia^i) \). That is, \( v(a) = v(b_i/b_j)/(j - i) \). Therefore, \( (j - i)\alpha \in V_0 \). Thus \( V \) is the divisible hull of \( V_0 \).

To prove that \( \kappa \) is the real closure of \( \kappa_0 \), let \( B \) (resp., \( B_0 \)) denote the valuation ring corresponding to the natural valuation \( v \) on \( R \) (resp., \( v|_{F} \) on \( F \)). Moreover, let \( M \) (resp., \( M_0 \)) denote the maximal ideal of \( B \) (resp., \( B_0 \)). First note that there exists a natural embedding \( \kappa_0 \hookrightarrow \kappa \). Let \( \bar{f}(X) = \sum_{i=0}^{n} \bar{a}_iX^i \in \kappa[X] \), where \( n \) is a natural odd number, and \( \bar{a}_i \) is the image of some \( a_i \in B \) under the natural homomorphism \( B_0 \rightarrow \kappa_0 \). We can assume that \( a_n = 1 \). Now the polynomial \( \bar{f}(x) = \sum_{i=0}^{n} a_ix^i \in B[x] \) has a root \( x_0 \in R \). Furthermore by Proposition 1.2.1, \( B \) is integrally closed. Therefore, \( x_0 \in B \). Thus \( \bar{f} \) has a root \( \bar{x}_0 \). Similarly, the equation
$X^2 - \bar{a} = 0$, where $a \in B$, has a root in $\kappa$ if (and only if) $\bar{a} \geq 0$. Therefore by Theorem 1.1.3, $\kappa$ is real closed.

Now suppose that $\bar{a} \in \kappa$, where $a \in B \subseteq R$. Then there exists $0 \neq f = \sum_{i=0}^{n} a_i x^i \in F[x]$ having $a$ as a root. Dividing by $a_j$ where $v(a_j)$ is the smallest among $v(a_i)$, $0 \leq i \leq n$, if necessary, we can assume that the coefficients $a_i$ are all in $B_0$ and, not all $a_i$ belong to $M_0$. Thus $\bar{a}$ is a root of $0 \neq \bar{f}(X) = \sum_{i=0}^{n} \bar{a}_i X^i \in \kappa_0[X]$, and we are done. QED

Remark. The proof of Theorem 1.2.4 shows also that: If $\kappa$ (resp., $V$) is the residue field (resp., value group) of a real closed field $R$, then $\kappa$ is also real closed (resp., $V$ is divisible).

Finally in this section we state an important inequality and two more results which will be used in chapter four. Suppose that $F \subseteq K$ are fields with $v$ a valuation defined on $K$ (and hence on $F$). The value group of $K$ (resp., $F$) will be denoted by $v(K)$ (resp., $v(F)$) and the residue field of $K$ (resp., $F$) by $K_v$ (resp., $F_v$).

Remark. If we are dealing with the natural valuation $v$ on an ordered field $F$ then instead of $F_v$ we might also use notations such as $\kappa_F$, $\kappa_0$, etc.

**Theorem 1.2.5 (The fundamental inequality)** Suppose that $[K : F]$ is finite. Then

$$[K : F] \geq [v(K) : v(F)][K_v : F_v]$$

**Proof.** See Proposition 2 page 159 in [Ri]. QED

**Proposition 1.2.6** Suppose that $K = F(a)$ is a simple extension of $F$.

1. If $n v(a) \notin v(F)$ for all integers $n \geq 1$ then $v(K) = v(F) \oplus \mathbb{Z} \delta$, where $\delta = v(a)$. Moreover, we have $K_v = F_v$.

2. If $v(a) = 0$ and $\bar{a}$ (the image of $a$ in $K_v$) is transcendental over $F_v$ then $v(K) = v(F)$ and $K_v = F_v(\bar{a})$.

**Proof.** (1) Let $i \neq j$ be two non-negative integers and $b \neq 0, b' \neq 0$ be elements of $F^* = F \setminus \{0\}$. Then the assumption on $v(a)$ implies that $v(b' a^i) \neq v(ba^i)$. 


Therefore, if $c = \sum_{i=0}^{n} b_i a^i \in F[a], c \neq 0$, where $b_i \in F^*, 0 \leq i \leq n$, then there exists $0 \leq j \leq n$ such that $v(c) = v(b_j) + jv(a)$. This implies that: (I) $v(K) = v(F) \oplus \mathbb{Z}\delta$, where $\delta = v(a)$. (II) $a$ is transcendental over $F$. In order to show that $K_v = F_v$, let $c \in F(a), v(c) = 0$. We can write $c = c_1/c_2$, where $c_j = \sum_{i=0}^{n} b_{ij} a^i, b_{ij} \in F, 0 \leq i \leq n, j = 1, 2$. Dividing the numerator and denominator of $c$, if necessary, by some $b_{ij}$ which has the least value among $\{v(b_{ij}) : 0 \leq i \leq n, j = 1, 2\}$, we may assume that $v(b_{ij}) \geq 0$, for $0 \leq i \leq n, j = 1, 2$. The assumption $v(c) = 0$ implies that there exists $0 \leq i \leq n$ such that $v(c_1) = v(b_{i1} a^i) = v(c_2) = v(b_{i2} a^i)$. Therefore, we can write $c = (b_{i1} a^i + \text{terms of higher value})/(b_{i2} a^i + \text{terms of higher value})$. Dividing the numerator and denominator by $b_{i2} a^i$, we obtain $c = ((b_{i1}/b_{i2}) + g_1)/(1 + g_2)$, where $g_{j}, j = 1, 2$, have positive values. Thus $\bar{c} = b_{i1}/b_{i2} \in F_v$ and we are done.

(2) We first show that:

**Claim.** If $b_i \in F^*, 0 \leq i \leq n$, then $v(\sum_{i=0}^{n} b_i a^i) = \text{Min}\{v(b_i a^i) : 0 \leq i \leq n\}$.

**Proof of the claim.** If not then we have $v(\sum_{i=0}^{n} b_i a^i) > \text{Min}\{v(b_i a^i), 0 \leq i \leq n\}$. By dividing $\sum_{i=0}^{n} b_i a^i$ by a suitable $b_j, 0 \leq j \leq n$, we can assume that for all $0 \leq i \leq n$ we have $v(b_i) \geq 0$ with at least one $b_i$ such that $v(b_i) = 0$. Therefore, $v(\sum_{i=0}^{n} b_i a^i) > 0$. Then $\sum_{i=0}^{n} b_i a^i = 0$, where at least one of $b_i$ is not 0. But this contradicts the assumption that $\bar{a}$ is transcendental over $F_v$. Thus the claim is proved.

It is obvious from the claim that $v(K) = v(F)$. To prove the assertion about the residue fields, let $c \in K^*, v(c) = 0$. Write $c = c_1/c_2$, where $c_j = \sum_{i=0}^{n} b_{ij} a^i, b_{ij} \in F^*, 0 \leq i \leq n, j = 1, 2$. So $v(c_2) = v(c_1)$. We may assume that for $0 \leq i \leq n, j = 1, 2$ we have $v(b_{ij}) \geq 0$ with at least one $b_{i1}$ and one $b_{i2}$ such that $v(b_{i1}) = v(b_{i2}) = 0$. Decompose $c_j, j = 1, 2$, as $\Sigma_{j1} + \Sigma_{j2}$, where $\Sigma_{j1}$ consists of those terms in $c_j$ each having value equal to 0 and $\Sigma_{j2}$ of those terms each of which has value strictly bigger than 0. Then $v(c_2) = v(\Sigma_{21}) = v(c_1) = v(\Sigma_{11}) = 0$. Moreover, $c = (\Sigma_{11} + \Sigma_{12})/(\Sigma_{21} + \Sigma_{22})$. Then $c(\Sigma_{21} + \Sigma_{22}) = \Sigma_{11} + \Sigma_{12}$. Therefore, $c\Sigma_{21} = \Sigma_{11}$. Thus $\bar{c} = \Sigma_{11}/\Sigma_{21} \in F_v(\bar{a})$ and we are done. QED

**Corollary 1.2.7** Suppose that $K = F(a)$ is a simple extension of $F$. 
(1) If there exists $b \in K$ such that $n v(b) \notin v(F)$ for all integers $n \geq 1$ then $v(K) \cong W \oplus \mathbb{Z}$ for some finite extension $W$ of $V$. Moreover, $K_v$ is a finite extension of $F_v$.

(2) If there exists $b \in K$ such that $v(b) = 0$ and $\bar{b}$ is transcendental over $F_v$ then $v(K)$ is a finite extension of $v(F)$ and $K_v$ is a finite extension of $F_v(\bar{b})$.

**Proof.** (1) Since $b$ is transcendental over $F$, $[K : F(b)]$ is finite. So by the fundamental inequality, $[v(K) : v(F(b))]$ is finite. Hence by using part (1) of Proposition 1.2.6 we see that $v(K)$ is a finite extension of $v(F) \oplus \mathbb{Z}\delta$, where $\delta = v(b)$. This means, by considering the divisible hull of $v(F) \oplus \mathbb{Z}\delta$, that there exists a positive integer $n$ such that $v(K) \subseteq 1/n(v(F) \oplus \mathbb{Z}\delta)$. Therefore, $v(K)/v(F)$ is finitely generated and $v(K) \cong W \oplus \mathbb{Z}$, where $W$ is a finite extension of $v(F)$. Now since $[K : F(b)]$ is finite, the fundamental inequality implies that $[K_v : (F(b))_v]$ is also finite. On the other hand, using part (1) of Proposition 1.2.6, we have $(F(b))_v = F_v$. Therefore, $K_v$ over $F_v$ is finite.

(2) As $\bar{b}$ is transcendental over $F_v$, $b$ is transcendental over $F$. Therefore, $[K : F(b)]$ is finite. Hence by the fundamental inequality, $[v(K) : v(F(b))]$ is finite. But by part (2) of Proposition 1.2.6 we have $v(F(b)) = v(F)$. Thus $[v(K) : v(F)]$ is finite. To prove the assertion about the residue fields, note that $[K_v : (F(b))_v]$ is finite. But $(F(b))_v = F_v(\bar{b})$. Thus $[K_v : F_v(\bar{b})]$ is finite and we are done. **QED**

### 1.3 Pseudo-convergent sequences

Suppose that $(F, v)$ is a valued field with the valuation ring $B$ and value group $\Gamma$. For the proofs of the results in this section see [Ka], also see [Shi, Ri].

**Definition 1.3.1** Let $A = \{a_\rho\}_{\rho \in \Lambda}$ be a subset of $F$ where the index set $\Lambda$ is a well-ordered set without a last element. Then we say $\{a_\rho\}$ is pseudo-convergent if $v(a_\sigma - a_\rho) < v(a_\tau - a_\rho)$ whenever $\rho < \sigma < \tau$.

**Lemma 1.3.1** Suppose that $\{a_\rho\}_{\rho \in \Lambda}$ is pseudo-convergent set. Then either

1. $v(a_\rho) < v(a_\sigma)$ for all $\rho < \sigma$, or
(2) $v(a_\rho)$ is ultimately constant, i.e., there exists some $\lambda \in \Lambda$ such that $v(a_\rho) = v(a_\sigma)$ whenever $\rho, \sigma > \lambda$.

**Lemma 1.3.2** If $\{a_\rho\}$ is pseudo-convergent, then $v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho)$ for all $\rho < \sigma$.

Fix $\rho$ and let $\sigma > \rho$. Then Lemma 1.3.2 tells us that $v(a_\sigma - a_\rho)$ does not depend on $\sigma$. Let $\gamma_\rho$ denote $v(a_\sigma - a_\rho)$. Thus we have an increasing set $\{\gamma_\rho\}$ of elements of $\Gamma$.

**Definition 1.3.2** An element $x$ of the field $F$ is called a pseudo-limit of the pseudo-convergent set $\{a_\rho\}$ if $v(x - a_\rho) = \gamma_\rho$ for all $\rho$.

Let $K$ be a field and $G$ an ordered abelian group. By $K((G))$ we mean the set of all formal power series with coefficients in $K$ and exponents in $G$, i.e., $K((G)) := \{f : G \to K : \text{Supp}(f) \text{ is well-ordered}\}$, where $\text{Supp}(f) := \{\alpha \in G : f(\alpha) \neq 0\}$ is the support of $f$. An element $c \in K((G))$ is simply denoted by $\sum_{\gamma \in G} c_\gamma x^\gamma$, where $c_\gamma = c(\gamma)$. The set $K((G))$ might also be denoted by $K((x^G))$. We can make $K((G))$ into a field. Let $c, d \in K((G))$. Then we define $c + d = \sum_{\gamma \in G} (c_\gamma + d_\gamma) x^\gamma$ and, $c \cdot d = \sum_{\gamma \in G} e_\gamma x^\gamma$ where $e_\gamma = \sum_{\delta_1 + \delta_2 = \gamma} c_{\delta_1} d_{\delta_2}$. The fact that $K((G))$ is a field was shown by H. Hahn [Ha] (also see [Fu]). $K((G))$ is called the field of formal power series with coefficients in $K$ and exponents in $G$. There is a canonical valuation $v$ on $K((G))$ defined by $v(c) = \text{the least } \gamma \in G \text{ such that } c_\gamma \neq 0$. The restriction of $v$ to the ground field $K$ is the trivial valuation on $K$. It is easy to see that $G$ and $K$ are respectively the value group and the residue field of the valued field $(K((G)), v)$. If $K$ is an ordered field, then there is a natural order on $K((G))$: Let $P$ be the set consisting of $0$ and those $\sum_{\gamma \in G} c_\gamma x^\gamma$ which have a positive first coefficient. Now it is easy to see that $P$ is the positive cone of an ordering on $K((G))$.

**Remarks.** If $K$ is an archimedean ordered field then the canonical valuation $v$ defined on $K((G))$ is the same as the natural valuation $v$ on the ordered field $K((G))$. In the thesis, as $K$ is always archimedean, we prefer to use the expression natural valuation on $K((G))$.  

Example 1.3.1 Suppose $F = R((Q))$. Then $\{\sum_{i=1}^{n} x^{i-\frac{1}{i}}\}_{n \geq 1}$ is a pseudo-convergent subset of $F$ having $\sum_{i=1}^{\infty} x^{i-\frac{1}{i}}$ as a pseudo-limit. Actually, any element of the form $(\sum_{i=1}^{\infty} x^{i-\frac{1}{i}}) + t$ where $t \in F$ with $v(t) \geq 1$ is a pseudo-limit of the given sequence.

Definition 1.3.3 The set of all $y \in F$ such that $v(y) > \gamma_{\rho}$ for all $\rho$ is called the breadth of the pseudo-convergent set $\{a_{\rho}\}$.

It is easily seen that the breadth of a pseudo-convergent set is a $B$-module in $F$.

Remark. When there is no ambiguity we sometimes simply use limit for pseudo-limit.

Lemma 1.3.3 Let $\{a_{\rho}\}$ be a pseudo-convergent set with the breadth $u$. Let $x$ be a limit of $\{a_{\rho}\}$. Then an element $y \in F$ is a limit of $\{a_{\rho}\}$ if and only if $y - x \in u$.

1.4 Maximal completeness

Suppose that $(F, v)$, $(K, v')$ are valued fields. We say that the valued field $(K, v')$ is an extension of the valued field $(F, v)$ if $F \subseteq K$ and $v'|_F = v$. In this case we denote the value map $v'$ on $K$ simply by $v$. If $(K, v')$ is an extension of $(F, v)$ and moreover, $K$ is algebraic over $F$ then we say that the valued field $(K, v)$ is an algebraic extension of $(F, v)$.

Let $(F, v)$ be a valued field. Moreover, suppose that $K$ be an extension field of $F$. Let $B$ denote the valuation ring corresponding to $(F, v)$. Then there exists at least one valuation ring $B'$ of $K$ such that $B = B' \cap F$ (see Theorem 13.2 in [En]). Therefore, by Proposition 1.2.2, there exists a valuation $v'$ defined on $K$ with the corresponding valuation ring $B'$. Since $B = B' \cap F$, $v'|_F$ has $B$ as its valuation ring; therefore, by using Proposition 1.2.2 once more, we see that $v'|_F$ is equivalent to $v$. So we have the following:

Proposition 1.4.1 Suppose that $(F, v)$ is a valued field and $K$ is an extension of $F$. Then there exists at least one valuation on $K$ extending the valuation $v$ on $F$. 
**Definition 1.4.1** Let the valued field $K$ be an extension of $F$. We say $K$ is an immediate extension of $F$ if $K$ and $F$ have the same value groups and the same residue fields.

**Proposition 1.4.2** Suppose that the valued field $K$ is an algebraic extension of the valued field $F$. Moreover, suppose that the residue field of $F$ is algebraically closed and the value group of $F$ is divisible. Then $K$ is an immediate extension of $F$.

**Proof.** An argument similar to that used in the proof of Theorem 1.2.4 proves the result. QED

**Theorem 1.4.3** (Krull) Let $F$ be a valued field. Then there is a cardinal number $\alpha$ such that $|K| \leq \alpha$ holds for all immediate extensions $K$ of $F$.

**Proof.** Actually, for all immediate extensions $K$ of $F$ we have $|K| \leq |\kappa((V))|$, where $\kappa$ is the residue field and $V$ the value group of the valued field $F$ (see [Ri, Lemma 1, page 80]). QED

Using Zorn’s Lemma and the above theorem we obtain:

**Corollary 1.4.4** (Krull) If $F$ is a valued field, then $F$ has at least one maximal immediate extension.

**Definition 1.4.2** A valued field $K$ is called maximally complete if it has no proper immediate extensions.

There is a very nice relationship between the notion of pseudo-convergence and maximal completeness:

**Theorem 1.4.5** (Kaplansky, [Ka]) Let $(F, v)$ be a valued field. Then $F$ is maximally complete if and only if every pseudo-convergent sequence in $F$ has at least one pseudo-limit in $F$.

**Proof.** See Theorem 4 in [Ka]. QED
Example 1.4.1 The set \( \{\sum_{i=1}^{n} x_i^{-i+1}\}_{n \geq 1} \) (of Example 1.3.1) is pseudo-convergent in the field of Puiseux series \( \mathcal{P}_a(x) \). But this sequence has no limit in \( \mathcal{P}_a(x) \). Therefore, the field \( \mathcal{P}_a(x) \) is not maximally complete (we can also see this result by noting that \( \mathbb{R}((\mathbb{Q})) \) is a proper immediate extension of \( \mathcal{P}_a(x) \)).

Theorem 1.4.6 (Krull) \( K((G)) \) with the natural valuation is maximally complete.

Proof. It not difficult to prove this as a corollary to Theorem 1.4.5, see [Ri, page 103, Corollaire]. QED

Corollary 1.4.7 Suppose that the ordered group \( G \) is divisible. Then:

1. If \( K \) is algebraically closed, then \( K((G)) \) is also algebraically closed.
2. If \( K \) is real closed, then \( K((G)) \) is also real closed.

Proof. (1) Let \( K^* \) be an algebraic extension of \( K((G)) \). By Proposition 1.4.1, there exists a valuation on \( K^* \) extending the natural valuation on \( K((G)) \). By Proposition 1.4.2, \( K^* \) is an immediate extension of \( K((G)) \). But by Theorem 1.4.6, \( K((G)) \) is maximally complete. Therefore, \( K((G)) \) has no proper algebraic extension, i.e., it is algebraically closed. Also see [Mc].

(2) It is not difficult to see that \( K((G))[\sqrt{-1}] \cong K[\sqrt{-1}]((G)) \). By Theorem 1.1.2, \( K[\sqrt{-1}] \) is algebraically closed. Therefore, by part (1), \( K[\sqrt{-1}]((G)) \), and hence \( K((G))[\sqrt{-1}] \) is algebraically closed. \( K((G)) \) is an ordered field, so \( \sqrt{-1} \not\in K((G)) \). Thus \( K((G)) \) is real closed by Theorem 1.1.2. QED

Theorem 1.4.8 Suppose that \( R \) is a real closed field. Let \( G \) and \( \kappa \) denote respectively the value group and the residue field of the natural valuation \( v \) on \( R \). Then there exists a field embedding \( R \hookrightarrow \kappa((G)) \) preserving the ordering.

Proof. Claim. Any maximal archimedean subfield of \( R \) is isomorphic to \( \kappa \).

Proof of the claim. There exists a natural ring homomorphism \( \Psi \) from \( B \) onto \( \kappa \), where \( B \) is the valuation ring of the natural valuation on \( R \). Use Zorn’s Lemma to pick a maximal archimedean subfield \( F \) of \( R \). \( F \) is real closed. Reason: Suppose that \( b \in R \) is algebraic over \( F \). By Proposition 1.2.1, \( B \) is integrally closed. Moreover, as
\( F \) is archimedean, \( v(F) = \{0\} \), hence \( F \subseteq B \). Therefore, \( b \in B \). So \( F[b] \subseteq B \). Since \( b \) is algebraic over \( F \), we see that: (1) \( F[b] \) is a field, and (2) \( F[b] \) is archimedean. Then by the maximality of \( F \) we have \( F[b] = F \). That is \( b \in F \). Therefore, by Proposition 1.1.5, \( F \) is real closed. Now as the only element \( b \in F \) with \( \Psi(b) = 0 \) is 0, we see that \( \Psi|_F \) is injective. So it is enough to show that \( \Psi(F) = \kappa \). If not, then let \( a \in \kappa \setminus \Psi(F) \) and choose \( b \in B \setminus F \) such that \( \Psi(b) = a \). Since \( F \) (resp., \( \Psi(F) \)) is real closed, \( b \) (resp., \( a \)) is transcendental over \( F \) (resp., over \( \Psi(F) \)). Therefore, \( \Psi \) induces an order isomorphism between \( F(b) \) and the archimedean subfield \( \Psi(F)(a) \) of \( \kappa \). But \( F(b) \supseteq F \) which contradicts the choice of \( F \). Thus \( \Psi(F) = \kappa \) and the claim is proved.

Without loss of generality we may assume that \( \kappa \subseteq R \). There exists a subgroup \( G' \) of the multiplicative group \( R^* = R \setminus \{0\} \) such that \( G' \cong G \). Reason. \( G \) is a vector space over \( \mathbb{Q} \). So we can choose a \( \mathbb{Q} \)-basis for \( G \), say \( \{v(a_i)\}_{i \in I}, a_i \in R \). Without loss of generality, we may assume that \( a_i > 0 \), for \( i \in I \). Let \( G' \) be the set of all finite products \( a_i^{r_i}, i \in I, r_i \in \mathbb{Q} \). \( G' \) is obviously a multiplicative subgroup of \( R^* = R \setminus \{0\} \); moreover, the map \( a_i \mapsto v(a_i) \) gives the isomorphism \( G' \cong G \). Now by our construction of \( G' \), we have the field \( \kappa(G') \subseteq R \) obtained by adjoining to \( \kappa \) all elements of \( G' \). Note that \( R \) is an immediate extension of \( \kappa(G') \). On the other hand, the map \( a_i^{r_i} \mapsto x^{v(a_i)} \) gives an embedding of \( \kappa(G') \) into \( \kappa((G)) \). Therefore, there exists a maximal immediate extension \( M \) of \( \kappa(G') \) isomorphic to \( \kappa((G)) \). Let \( N \) be a maximal immediate extension of \( R \). Then \( N \) is also a maximal immediate extension of \( \kappa(G') \). Since \( \kappa \) has characteristic 0, by Kaplansky [Ka] (also see Theorem 4.1.7), \( N \) and \( M \) are isomorphic. Thus we have the embedding \( \iota : R \hookrightarrow \kappa((G)) \). Now use Proposition 1.1.4 to see that \( \iota \) is order preserving. QED

Remark. Suppose that \( \iota : R \hookrightarrow \kappa((G)) \) is an order preserving embedding, where \( G \) and \( \kappa \) are respectively the value group and the residue field of the natural valuation on \( R \). We say that \( \iota \) is proper if for each \( \gamma \in G \) there exists \( x \in R \) such that \( v(\iota(x)) = \gamma \). The embedding \( R \hookrightarrow \kappa((G)) \) which was found in the proof of the previous theorem is actually a proper embedding.
1.5 Hensel's Lemma

Suppose that \((F, v)\) is a valued field, and \(K\) is an algebraic field extension of \(F\). According to Proposition 1.4.1, \(K\) has at least one valuation defined on it which extends \(v\). We say that \((F, v)\) is Henselian if every algebraic extension of \(F\) admits only one valuation extending \(v\).

Proposition and Definition 1.5.1 Any valued field \((F, v)\) admits a unique (up to an isomorphism preserving the valuation) smallest Henselian valued extension field \(F^h\). \(F^h\) is called the Henselization of \(F\).

Proof. See [PR, page 21], also see [En, Theorem 17.10]. QED

Theorem 1.5.2 Suppose \((F, v)\) is a valued field. Let \(B\) and \(\kappa\) denote respectively the valuation ring and the residue field of \((F, v)\). Then the following conditions are equivalent:

(1) \((F, v)\) is Henselian.

(2) Suppose that \(f(X) = \sum_{i=0}^{n} a_i X^i \in B[X]\) is monic. Let \(\tilde{f}(x) = \sum_{i=0}^{n} \tilde{a}_i x^i\), \(\tilde{a}_i \in \kappa\). Then corresponding to any simple root \(\tilde{t} \in \kappa, t \in B\), of \(\tilde{f}\) there exists a root \(s \in B\) of \(f\); moreover \(v(s - t) > 0\).

Proof. See [PR, page 20], also [Pr, page 85]. QED

Remarks. (1) The implication \(1 \implies 2\) in the above theorem is referred to as the Hensel’s Lemma.

(2) It can easily be seen that the element \(s \in B\) in part (2) of Theorem 1.5.2 is unique, i.e., there exists a unique \(s \in B\) such that \(\bar{s} = \bar{t}\) and \(f(s) = 0\).

Theorem 1.5.3 Assume the set-up in Theorem 1.5.2. Moreover, assume that characteristic of \(\kappa\) is 0. Then

(1) \((F, v)\) is Henselian if and only if there is no proper immediate algebraic extension of \((F, v)\).

(2) There is a unique (up to a value preserving isomorphism) immediate algebraic extension \((K, v)\) of \((F, v)\) which is Henselian.
Proof. See Proposition 8.1 in [Pr]. QED

The extension $(K, v)$ in part (2) of the above theorem is actually the Henselization of $(F, v)$. Note that Theorem 1.5.3 implies that the Henselization of $(F, v)$ is the maximal immediate algebraic extension of $(F, v)$.

Corollary 1.5.4 Suppose that $R$ is a real closed field and $v$ is the natural valuation on $R$. Then $R$ with the valuation $v$ is Henselian.

Proof. The only algebraic extension of $R$ is the field $L = R(\sqrt{-1})$. Let the valuation $v'$ be an extension of $v$ to $L$. Then an argument as in the proof of Theorem 1.2.4 shows that the residue field of $(L, v')$ is the algebraic closure $\kappa(\sqrt{-1})$ of $\kappa$, where $\kappa$ is the residue field of $(R, v)$. Therefore, $(L, v')$ is not an immediate extension of $(R, v)$. Thus by Proposition 1.5.3, $(R, v)$ is Henselian.

Finally we have the following theorem:

Theorem 1.5.5 Suppose that the field $K$ is a finite extension of the field $F$. Moreover, suppose that $v$ is a valuation defined on $K$ and $(F, v)$ is Henselian. Then

$$[K : F] = [v(K) : v(F)][K_v : F_v] \delta,$$

where $\delta$ is equal to 1 if the characteristic $p$ of $F_v$ is 0 and is a power of $p$ if $p \neq 0$.

Proof. This is just Theorem 2 page 236 in [Ri]. QED

Remark. $[v(K) : v(F)]$ is called the ramification index of $K$ over $F$. $[K_v : F_v]$ is called the residue degree of $K$ over $F$ (see [En]).
Chapter 2

Cuts in Ordered Abelian Groups

Suppose $I$ is an ordered set. Let $U, V \subseteq I$ be such that $U \cup V = I$ and for all $a \in U, b \in V$ we have $a < b$. Then $(U, V)$ is said to be a cut in $I$. Studying cuts in an ordered abelian group is a basis for our later study of cuts in formal power series. To study the cuts in a general ordered abelian group $G$, we use Hahn's Theorem to "decompose" $G$. This will be done in section 2.1. Such a decomposition leads to an ordered set $I$. Each cut $(S, T)$ in $G$ defines a cut $(U, V)$ in $I$. The lower cut $U$, as we will see in section 2.2, plays an important role in characterization of the cut $(S, T)$ in the ordered abelian group $G$. Then in section 2.3 we will discuss when the lower cut $S$ (resp., the upper cut $T$) of a cut $(S, T)$ has a maximal element (resp., a minimal element). Later we will need some kind of cut completion of an ordered abelian groups; therefore, filling the cuts of an ordered abelian group will be discussed in section 2.4.

2.1 Decomposition of an ordered abelian group

In this section we first study decomposition of a divisible ordered abelian group of finite rank, which is of special interest. Then we shall generalize these results to an arbitrary divisible ordered abelian group. Throughout our discussion we assume, unless otherwise stated, that the ordered abelian groups we consider are divisible (although in some parts, i.e., in defining rank of ordered abelian groups, divisibility of $G$ is not needed). Let $G$ be an abelian group and $C$ be a (divisible) subgroup of
G. We say that C is a convex subgroup of G if for all \( a \in G \), and \( b, c \in C \) with \( b < a < c \), we have \( a \in C \). It is easily seen that the set of all convex subgroups of G forms a chain under inclusion. By the order type of an ordered set \( I \) we mean the equivalence class of all the ordered sets which are order isomorphic to \( I \). The order type of the chain of all non-zero convex subgroups of G is called the rank of G. If the chain of all non-zero convex subgroups of G is order isomorphic to the set \( \{1, \cdots, n\} \), where \( n \) is a positive integer, then we say that G has finite rank \( n \).

### 2.1.1 Ordered abelian groups of finite rank

Let G be a divisible ordered abelian group of rank \( n \). Therefore, there exists a chain \( G = C_1 \not\subseteq C_2 \not\subseteq \cdots \not\subseteq C_{n+1} = \{0\} \) of convex subgroups of G. Each \( C_i \) is divisible as it is convex. Then \( C_i, \ 1 \leq i \leq n \), is a vector space over \( \mathbb{Q} \). Therefore, for each \( 1 \leq i \leq n \), there exists a (non-unique) divisible ordered subgroup \( G_i \) of \( C_i \) such that \( C_i = G_i \oplus C_{i+1} \). Thus we have \( G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \).

**Proposition 2.1.1** Let each \( G_i \) have the ordering inherited from G. Then the ordering on G is just the lexicographical ordering on \( G_1 \oplus \cdots \oplus G_n \). Moreover, each \( G_i \) is archimedean.

**Proof.** Suppose that \( a, b \in G, \ a \neq b \). We can write \( a = \sum_{i=1}^{n} a_i \) and \( b = \sum_{i=1}^{n} b_i \), where \( a_i, b_i \in G_i \) for \( 1 \leq i \leq n \). Let \( k, 1 \leq k \leq n \), be the smallest integer such that \( a_k \neq b_k \). Without loss of generality assume that \( a_k < b_k \). We must show that \( a < b \). Assume, on the contrary, that \( a \geq b \). We have \( a - b = \sum_{i=k}^{n} (a_i - b_i) \). Therefore, \( a - b \in G_k \oplus \cdots \oplus G_n \subseteq C_k \). On the other hand, \( (a - b) - (a_k - b_k) = \sum_{i=k+1}^{n} (a_i - b_i) \in G_{k+1} \oplus \cdots \oplus G_n \subseteq C_{k+1} \). Thus \( 0 < a - b < (a - b) - (a_k - b_k) \in C_{k+1} \). Since \( C_{k+1} \) is convex, \( a - b \in C_{k+1} \). This together with the fact that \( C_k = G_k \oplus C_{k+1} \) imply that \( a_k - b_k = 0 \). But this is a contradiction to our assumption that \( a_k < b_k \). So we are done.

To prove the second assertion assume, on the contrary, that there exists \( k, 1 \leq k \leq n \), such that \( G_k \) is not archimedean. Therefore, there exist \( a, b \in G_k, a, b > 0 \) such that for all positive integers \( n \) we have \( a > nb \). Let D be the subset of \( C_k \).
consisting of all elements $c \in C_k$ having the property $a > n|c|$, for all positive integers $n$. Clearly, $D$ contains $b$ but not $a$. It is easily seen that $D$ is a non-zero convex subgroup of $G$. Moreover, we have $C_k \supsetneq D$. On the other hand, $D \neq C_i$ for $i \geq k + 1$ (otherwise, $b \in D = C_i \subseteq C_{k+1}$ which contradicts $C_k = G_k \oplus C_{k+1}$). Thus the rank of $G$ is greater than $n$ which is a contradiction to the assumption. QED

2.1.2 Ordered abelian groups of arbitrary rank

Let $G$ be a divisible ordered abelian group. Associated to $G$, we are going to build a natural (surjective group) valuation $w : G \to I \cup \{\infty\}$, where $I$ is a suitable ordered set. By a group valuation $w$ we mean a surjective map $w : G \to I \cup \{\infty\}$ such that (1) $w(a) = \infty$ if and only if $a = 0$, (2) $w(a + b) \geq \text{Min}\{w(a), w(b)\}$, and (3) $w(ra) = w(a)$, for each $r \in \mathbb{Q} \setminus \{0\}$. Let $a, b \in G \setminus \{0\}$, then we say that $a$ is archimedean equivalent to $b$ if and only if there exist natural numbers $m, n$ such that $|a| \leq m|b|$ and $|b| \leq n|a|$. This defines an equivalence relation on $G$. Assume that $[a], [b]$ are two equivalent classes. If $[b] \neq [a]$, then either $|a| > n|b|$ for all natural numbers $n$, or else $|b| > n|a|$ for all natural numbers $n$. That is, either $|a| > |b|$ (i.e., $|a|$ is infinitely larger than $|b|$), or $|b| > |a|$. We define an order on the set of equivalence classes by letting $[a] > [b]$ if $|b| > |a|$. This is obviously well-defined.

Let $I$ be the set of equivalent classes. So $I$ is an ordered set. For $a \in G \setminus \{0\}$ define $w(a) = [a]$. Moreover, let $w(0) = \infty$. We define $\infty > w(a)$ for all $a \in G \setminus \{0\}$. It is obvious that $w(a) = \infty$ if and only if $a = 0$. Furthermore, it is not difficult to see that for all $a, b \in G$ we have $w(a + b) \geq \text{Min}\{w(a), w(b)\}$. The validity of the condition (3) for group valuation is obvious by our construction of the map $w$. Note that we have $w(a + b) = \text{Min}\{w(a), w(b)\}$ if $w(b) \neq w(a)$. We call the order type of $(I, \leq)$ the principal rank of $G$. It is not difficult to see that $G$ has finite rank if and only if the principal rank of $G$ is finite, and in this case rank and principal rank of $G$ are equal. In general, the cardinal number of the rank of a given ordered abelian group $G$ is greater or equal to that of the principal rank of that group (see Example 2.3.2 and the remark preceding it).

Remark. The group valuation $w$ obtained above is called the natural (group) valu-
ation on the ordered abelian group $G$; moreover, $I$ is said to be the value set of the natural valuation $w$. If $F$ is an ordered field then the natural valuation $w$ on $F$, as an ordered abelian group, is the same as the natural valuation on $F$ as defined in section 1.2.

For each $i \in I$, let $G_i$ be obtained by factoring the subgroup $D_i = \{ \alpha \in G : w(\alpha) \geq i \}$ of $G$ by the convex subgroup $C_i = \{ \alpha \in G : w(\alpha) > i \}$. Then $G_i$ is a divisible abelian group. Since $C_i$ is a convex subgroup of $G$, there is an induced ordering on $G_i$: If $\beta \in D_i$, and $\tilde{\beta}$ is its image in $G_i$, then we define $\tilde{\beta} > 0$ if $\beta > \alpha$ for all $\alpha \in C_i$. This is a well-defined ordering as $C_i$ is convex. Now it is obvious that with this definition of order, $G_i$ is an ordered group. $G_i$ is called the residue group of $G$ at $i$. The pair $(I, (G_i)_{i \in I})$ is called the skeleton of $G$.

Let $\Pi_{i \in I} G_i$ denote the direct product of the family of ordered abelian groups $(G_i)_{i \in I}$. If $a \in \Pi_{i \in I} G_i$ then by the support of $a$ we mean the set $\text{Supp}(a) = \{ i \in I : a_i \neq 0 \}$. The Hahn product of the family of ordered abelian groups $(G_i)_{i \in I}$, denoted by $H_G = \mathcal{H}_{i \in I} G_i$, is defined to be the set of those elements of $\Pi_{i \in I} G_i$ which have well-ordered supports. It is obvious that $\mathcal{H}_{i \in I} G_i$ is a subgroup of $\Pi_{i \in I} G_i$. Actually, $\mathcal{H}_{i \in I} G_i$ is an ordered abelian group with the lexicographic ordering: If $\alpha \neq \beta$ are elements of $\mathcal{H}_{i \in I} G_i$, then there exists a smallest $i \in I$ such that $\alpha_i \neq \beta_i$. Then we define $\alpha < \beta$ if and only if $\alpha_i < \beta_i$. Note that the direct sum $\sum_{i \in I} G_i$, which also has the lexicographic ordering, is an ordered subgroup of $\mathcal{H}_{i \in I} G_i$.

One can define a group valuation on $\sum_{i \in I} G_i$ and $\mathcal{H}_{i \in I} G_i$ as follows: Let $w(0) = \infty$. Now suppose that $\alpha$ belongs to either of these groups, and $\alpha \neq 0$, then we let $w(\alpha)$ be the least $i$ for which $a_i \neq 0$. This definition of $w$ turns out to be compatible with that previously defined on $G$. By the Hahn Embedding Theorem [Ha, Fu] (also see Theorem 5.5.4), there exist (non-unique) order preserving embeddings

$$\sum_{i \in I} G_i \hookrightarrow G \hookrightarrow \mathcal{H}_{i \in I} G_i$$

with the composite being the natural embedding. Moreover, these embeddings preserve the valuation. Furthermore, $\sum_{i \in I} G_i$ and $\mathcal{H}_{i \in I} G_i$ have the same skeleton as $G$ [Fu].
Our construction shows that each $G_i$ is archimedean. Thus there exists a group embedding $G_i \hookrightarrow \mathbb{R}$ which preserves the order. Therefore, in particular, $G$ can be embedded in $H_{i \in I} \mathbb{R}$, the Hahn product of $I$ copies of $\mathbb{R}$. Throughout our discussion, unless otherwise stated, we fix such an embedding and identify $G$ with its image in $H_{i \in I} \mathbb{R}$. The Hahn product $H_{i \in I} \mathbb{R}$ corresponding to $G$ will be denoted by $H_G$ or just by $H$ if there is no ambiguity. Note that if $G$ is of finite rank $n$, then $H_G = \mathbb{R}^n$.

Remark. There is also another notion of rank, i.e., the rational rank, of a divisible ordered abelian group $G$. This is defined to be $\dim_{\mathbb{Q}}(G)$ (the $\mathbb{Q}$ dimension of $G$ as a vector space over $\mathbb{Q}$). If $G$ has finite rank, say $n$, then from Proposition 2.1.1 it follows that $\dim_{\mathbb{Q}}(G) = \sum_{i=1}^{n} \dim_{\mathbb{Q}}(G_i)$.

Example 2.1.1 (1) Consider the ordered abelian group $G = (\mathbb{Q} + \sqrt{2}\mathbb{Q}) \oplus (\mathbb{Q} + \sqrt{2}\mathbb{Q})$, ordered lexicographically. $G$ has rank 2 and rational rank 4.

(2) The rational rank of $\mathbb{R}$ is not finite, while its rank is 1.

(3) Let $G = \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ be the ordered abelian group of $n$ copies of $\mathbb{Q}$ ordered lexicographically. Then the rank and the rational rank of $G$ are the same, namely $n$.

Example 2.1.2 If $\kappa$ is an ordered field and $V$ is an abelian group then $\kappa((V))$, as an ordered abelian group, is the Hahn product $H_{i \in V} \kappa$.

2.2 Cuts in ordered abelian groups

Lemma 2.2.1 Suppose that $(S, T)$ is a cut in the ordered abelian group $G$ and $w : G \to I \cup \{\infty\}$ is the group valuation. Then $U = \{w(\alpha - \beta) : \alpha \in S, \beta \in T\}$ is a lower cut in $I$.

Proof. Suppose that $i \in U$, $j \in I$, and $j < i$. Then there exist $\alpha \in S$, $\beta \in T$, and $\gamma \in G$ such that $i = w(\alpha - \beta)$ and $j = w(\gamma)$. Let $\gamma' = \min\{\gamma, -\gamma\}$. Since $\gamma' \leq 0$, we have $\alpha + \gamma' \in S$. Thus $j = w(\gamma) = w(\gamma') = w((\alpha - \beta) + \gamma') = w((\alpha + \gamma') - \beta) \in U$.

QED
Lemma 2.2.2 Suppose that \( S, T, \) and \( w \) are as defined in Lemma 2.2.1. Moreover, suppose that \( \alpha, \alpha' \in S, \beta, \beta' \in T, \) \( i = w(\beta - \alpha), j = w(\beta' - \alpha') \). Then

\[
    w(\alpha' - \alpha) \geq \text{Min}\{i, j\}, \tag{2.2.4}
\]

\[
    w(\beta' - \beta) \geq \text{Min}\{i, j\}. \tag{2.2.5}
\]

Proof. To prove the inequality (2.2.4), assume, on the contrary, that \( w(\alpha' - \alpha) < \text{Min} \{i, j\} \). We can assume that the element \( \alpha - \alpha' \) is positive. Then \( \alpha - \beta' = (\alpha - \alpha') - (\beta - \alpha') \) has the same sign as \( \alpha - \alpha' \) (because \( w(\alpha - \alpha') < w(\beta' - \alpha') \)) so it is also positive, i.e., \( \alpha > \beta' \), which contradicts that \( (S, T) \) is a cut. The proof for inequality (2.2.5) is similar. QED

Proposition 2.2.3 Suppose that \( G \) is a divisible ordered abelian group, and \( S, T \), \( w, I, U \) are as in Lemma 2.2.1. Moreover, suppose that \( H \) is the Hahn product of \( I \) copies of \( \mathbb{R} \). Then there exists a unique element \( \theta = (\theta_i)_{i \in I} \) in \( H \) such that

1. If \( \alpha \in S, \beta \in T \), then \( w(\theta - \alpha) \geq w(\alpha - \beta) \) and \( w(\theta - \beta) \geq w(\alpha - \beta) \).
2. \( \theta_i = 0 \) if \( i \notin U \).
3. If \( U \) has a largest element \( k \), and \( \alpha \in S, \beta \in T \) with \( w(\alpha - \beta) = k \), then \( \alpha_k \leq \theta_k \leq \beta_k \).

Proof. Define \( \theta_i = 0 \) if \( i \notin U \). If \( i \in U \) and \( i \) is not the largest element of \( U \), then there exist \( \alpha \in S, \beta \in T \) with \( w(\alpha - \beta) > i \). In this case, we define \( \theta_i = \alpha_i (= \beta_i) \). Lemma 2.2.2 shows that this is well-defined. Now we define \( \theta_k \) in the case where \( U \) has a largest element \( k \). Let \( \alpha \in S, \beta \in T \) be such that \( w(\alpha - \beta) = k \). Since \( \alpha < \beta \), we see that \( \alpha_k < \beta_k \). Let \( M \) (respectively \( N \)) be the set of such \( \alpha_k \) (respectively \( \beta_k \)) obtained in this manner.

Claim. There exists a unique \( r \in \mathbb{R} \) such that \( M \leq r \leq N \).

Proof of claim. Let \( \alpha_k \in M, \beta_k \in N \). By the definition of \( M \), \( \alpha_k \in M \) implies that there exist \( \alpha \in S, \beta' \in T \) such that \( w(\alpha - \beta') = k \) and \( \alpha_k \) is the \( k \)-th component of \( \alpha \). Similarly, since \( \beta_k \in N \), so there exist \( \alpha' \in S, \beta \in T \) such that \( w(\alpha' - \beta) = k \) and \( \beta_k \) is the \( k \)-th component of \( \beta \). By Lemma 2.2.2, we have \( w(\alpha - \alpha') \geq k \). Since
\[ \alpha - \beta = (\alpha - \alpha') + (\alpha' - \beta), \] we have \( w(\alpha - \beta) = k. \) Therefore, \( \alpha_k < \beta_k. \) Therefore, \( M \) has a least upper bound \( r \in \mathbb{R}, \) and \( N \) has a greatest lower bound \( s \in \mathbb{R}. \) It is enough to show that \( r = s. \) If this is not the case, then \( r < s. \) Choose \( \alpha \in S, \beta \in T \) such that \( w(\beta - \alpha) = k. \) Also choose a rational number \( q, 0 < q < 1 \) such that

\[ r < q\alpha_k + (1 - q)\beta_k < s \]  
(2.2.6)

( i.e., \( \frac{\beta_k - s}{\alpha_k - \alpha_k} < q < \frac{\beta_k - r}{\alpha_k - \alpha_k} \)). Let \( \gamma = q\alpha + (1 - q)\beta. \) Then \( \gamma \) is either in \( S \) or in \( T. \) If \( \gamma \in T, \) then \( w(\gamma - \alpha) = w((1 - q)(\beta - \alpha)) = w(\beta - \alpha) = k. \) Similarly, if \( \gamma \in S \) then \( w(\beta - \gamma) = k. \) So \( \gamma_k = q\alpha_k + (1 - q)\beta_k \) is either in \( M \) which implies that \( \gamma_k \leq r, \) or is in \( N \) which implies that \( \gamma_k \geq s. \) But either case contradicts the choice of \( q \) as in the condition (2.2.5), and the claim is thus proved.

The unique element \( r \) found in the above claim is the \( \theta_k \) stated in the proposition.

Now we prove that the element \( \theta \) found above is actually in the Hahn product \( H. \) Let \( \Lambda \subseteq \text{Supp} (\theta), \) \( \Lambda \neq \emptyset. \) It is enough to show that \( \Lambda \) has a least element. Without loss of generality we can assume that \( \Lambda \) is infinite. Suppose that \( k \) is not the last element of \( U. \) The set \( \Lambda_0 = \{ i \in \Lambda : i \leq k \} \) is an initial segment of \( \Lambda, \) so we need only show that \( \Lambda_0 \) has a least element. By our construction of \( \theta, \) there exists \( \alpha \in S \) such that \( \theta_i = \alpha_i, \) for all \( i \leq k. \) Therefore, \( \Lambda_0 \) is a subset of the \( \text{Supp}(\alpha) \) which means that \( \Lambda_0 \) and hence \( \Lambda \) has a least element, and the proof of the proposition is thus complete. QED

**Lemma 2.2.4** Assume that \( (S, T) \) is a cut in \( G \) and \( U, \theta \) are associated to \( (S, T) \) as in Proposition 2.2.3. Then

1. If there are elements \( \gamma \in G \) with \( w(\gamma - \theta) \notin U, \) then either all such elements are in \( S \) or all such elements are in \( T. \)

2. Suppose \( \gamma \in G \) and \( w(\gamma - \theta) \in U. \) If \( \gamma > \theta \) then \( \gamma \in T \) and, if \( \gamma < \theta \) then \( \gamma \in S. \)

**Proof.** (1) Suppose that \( \gamma_1, \gamma_2 \) are in \( G \) with \( w(\gamma_i - \theta) \notin U, \) for \( i = 1, 2. \) Therefore, we have that \( w(\gamma_i - \theta) > k, \) for \( i = 1, 2, \) and for all \( k \in U. \) Since \( \gamma_2 - \gamma_1 = (\gamma_2 - \theta) - (\gamma_1 - \theta) , \) it follows that \( w(\gamma_2 - \gamma_1) \notin U, \) so \( \gamma_1 \) and \( \gamma_2 \) either both belong to \( S, \) or both belong to \( T. \)
(2) Assume first that \( w(\gamma - \theta) \in U \) is not the largest element of \( U \). Therefore, there exist \( \alpha \in S, \beta \in T \) such that \( w(\alpha - \beta) > w(\gamma - \theta) \). Using this and Lemma 2.2.3, we see that \( w(\theta - \beta) \) and \( w(\theta - \alpha) \) are both greater than \( w(\gamma - \theta) \). Thus, using the identity \( \gamma - \beta = (\theta - \beta) + (\gamma - \theta) \), we see that \( \gamma - \beta \) has the same sign as \( \gamma - \theta \). Therefore, if \( \gamma > \theta \) then \( \gamma > \beta \), that is \( \gamma \in T \). Similarly, using the identity \( \gamma - \alpha = (\theta - \alpha) + (\gamma - \theta) \), we see that \( \gamma - \alpha \) has the same sign as \( \gamma - \theta \). Therefore, if \( \gamma < \theta \) then \( \gamma < \alpha \), that is \( \gamma \in S \). Next assume that \( U \) has a largest element \( k \) and \( \gamma \in G \) is such that \( w(\gamma - \theta) = k \). So \( \gamma < \theta \) if and only if \( \gamma_k < \theta_k \), and \( \gamma > \theta \) if and only if \( \gamma_k > \theta_k \). Therefore, by the construction of \( \theta \), \( \gamma > \theta \) if and only if \( \gamma \in T \), and \( \gamma < \theta \) if and only if \( \gamma \in S \). Thus the proof is complete. QED

**Lemma 2.2.5** Suppose that \((S, T), (S', T')\) are cuts in \( G \) with the associated \( U \) and \( \theta \) as in Proposition 2.2.3,

1. If there exists \( \gamma \) with \( w(\gamma - \theta) \notin U \) and either \( \gamma \in S \cap S' \) or \( \gamma \in T \cap T' \) then \( S' = S \) and \( T' = T \). We also have: \( S = \{ \gamma \in G : \gamma < \theta \text{ or } w(\gamma - \theta) \notin U \} \) and \( T = \{ \gamma \in G : \gamma > \theta \text{ and } w(\gamma - \theta) \in U \} \), or \( S = \{ \gamma \in G : \gamma < \theta \text{ and } w(\gamma - \theta) \in U \} \) and \( T = \{ \gamma \in G : \gamma > \theta \text{ or } w(\gamma - \theta) \notin U \} \).

2. If for all \( \gamma \in G \) we have \( w(\gamma - \theta) \in U \) then \( S' = S \) and \( T' = T \). We also have: \( S = \{ \gamma \in G : \gamma < \theta \} \) and \( T = \{ \gamma \in G : \gamma > \theta \} \).

**Proof.** (1) Suppose that there exists \( \gamma \) with \( w(\gamma - \theta) \notin U \) and \( \gamma \in S \cap S' \). Let \( \delta \in S \). Then either \( w(\delta - \theta) \in U \) or \( w(\delta - \theta) \notin U \). In the former case we have, by part (2) of Lemma 2.2.4, that \( \delta < \theta \); therefore, \( \delta \in S' \). On the other hand, if \( w(\delta - \theta) \notin U \) then by Lemma 2.2.4 (1) we have \( \delta \in S' \). So \( S \subseteq S' \). Similarly, we obtain \( S' \subseteq S \). Thus \( S' = S \) and hence \( T' = T \). The other assertions in part (1) of the lemma also follow easily from Lemma 2.2.4.

(2) This is obvious from Lemma 2.2.4 (2). QED

In Proposition 2.2.3, corresponding to a cut \((S, T)\) in \( G \), we obtained a lower cut \( U \) in \( I \) and an element \( \theta \) in its corresponding Hahn product \( \mathbb{H}_G \) such that

\[
\theta_i = 0 \text{ if } i \notin U.
\]

(2.2.7)
Now a question arises here as to whether the converse is also true, namely if we are given a lower cut $U$ in $I$ and an element $\theta$ in the Hahn product $\mathbb{H}_G$ which satisfies the condition (2.2.7), then is it always true that we can find a cut $(S,T)$ in $G$ giving rise to $U$ and $\theta$ when we apply Proposition 2.2.3? Moreover, if the answer is positive, then is the cut found unique? The answer to these questions is "No" at the present situation! But it is possible to make some more restriction to make sure of the validity of the converse. First let us look at some examples:

Example 2.2.1 Let $G = G_1 \oplus G_2 \oplus G_3$, where $G_1 = G_2 = G_3 = \mathbb{Q}$. Then $G$ is an ordered abelian group with lexicographic ordering. Let $U = \{1, 2\}$, and $\theta = (\sqrt{2}, \sqrt{3}, 0)$. Then there is no cut $(S,T)$ in $G$ which gives rise to the data $U, \theta$. The reason is, according to the construction of $\theta$ in the proposition 2.2.3, that if an $i$-th coordinate of $\theta$ is not in $G_i$, then all the coordinates of $\theta$ following the $i$-th one should be 0.

Example 2.2.2 Let $G = \mathcal{P}_R(x)$ (= the field of Puiseux series with real coefficients) be considered as an additive ordered abelian group. Then the Hahn product $\mathbb{H}_G$ corresponding to $G$ is (isomorphic to) $\mathbb{R}((\mathbb{Q}))$. It is obvious that $I = \mathbb{Q}$. Let $U = \{r : r \leq 2\}$. Define $\theta$ in the following way: Let $\theta_j = 1$, if $j = \frac{2n-1}{n}$ for $n \in \mathbb{N}$ (N is the set of natural numbers), or $j = 2$, and $\theta_j = 0$ otherwise. In other words, $\theta = (\sum_{n=1}^{\infty} x^{2n-1}) + x^2$. Then there is no cut $(S,T)$ in $G$ which gives rise to the data $U, \theta$. The reason is that if we choose $i = 2$, then there is no element $\alpha \in G$ which agrees with $\theta$ in all coordinates $j$ with $j < 2$.

Example 2.2.3 Let $G = \mathbb{R}((\mathbb{Q}))$, $U = \mathbb{Q}$, and $\theta = 0$. Then the cuts $(S_1, T_1)$ and $(S_2, T_2)$ defined by $S_1 = \{t : t \leq 0\}$, $T_1 = G \setminus S_1$ and $S_2 = \{t : t < 0\}$, $T_2 = G \setminus S_2$ both give rise to the same $U = \mathbb{Q}$ and $\theta = 0$.

We can overcome the problems discussed in Examples 2.2.1 and 2.2.2 by imposing the following condition (which, according to the constructions given in Proposition 2.2.3, is a natural condition).

For every $i \in U$, there exists $\gamma \in G$ such that $w(\gamma - \theta) \geq i$. (2.2.8)
Now suppose that $U$ is a lower cut in $I$ and $\theta$ is an element of $H$ which satisfy conditions (2.2.7) and (2.2.8). In general, there might be more than one cut $(S,T)$ in $G$ which give rise to the same $U$ and the same $\theta$ if Lemma 2.2.1 and Proposition 2.2.3 are applied to $(S,T)$. However, if we add one more piece of information— as stated in the following definition— then we will be able to show the uniqueness of $(S,T)$ too.

**Definition 2.2.1** Suppose that $G$ is an ordered abelian group, $w : G \to I \cup \{\infty\}$ its corresponding natural valuation, and $H$ the Hahn product of $I$ copies of $\mathbb{R}$. Moreover, let $U$ and $\theta$ be respectively a lower cut in $I$ and an element of $H$ such that conditions (2.2.7) and (2.2.8) are satisfied. Then by a $G$-cut symbol we mean one of the following:

1. $(U, \theta)$, provided that for all $\gamma \in G$ we have $w(\gamma - \theta) \in U$.
2. $(U, \theta, +)$, or $(U, \theta, -)$, provided that there exists $\gamma \in G$ such that $w(\gamma - \theta) \notin U$.

Moreover, the $G$-cut symbol is:

1. $(U, \theta, +)$, if there exists $\gamma \in S$ such that $w(\gamma - \theta) \notin U$, where $S$ is the lower cut of a cut $(S,T)$ in $G$ with the corresponding $U$ and $\theta$ (as in Lemma 2.2.1 and Proposition 2.2.3).
2. $(U, \theta, -)$, if there exists $\gamma \in T$ such that $w(\gamma - \theta) \notin U$, where $T$ is the upper cut of a cut $(S,T)$ in $G$ with the corresponding $U$ and $\theta$ (as in Lemma 2.2.1 and Proposition 2.2.3).

Note that in view of part (1) of Lemma 2.2.4, the parts (21) and (22) in the above definition are meaningful. When there is no ambiguity, we refer to a $G$-cut symbol simply as a cut-symbol.

**Example 2.2.4** Suppose that $G$ is a divisible ordered subgroup of $(\mathbb{R}, +)$. Then $I$ is a singleton since $w(a) = 0$ for each $a \in G \setminus \{0\}$. Therefore, corresponding to each cut $(S,T)$ in $G$, the lower cut $U \subseteq I$ is either $I$ or the empty set. If $U = \emptyset$ then the cut symbol corresponding to $(S,T)$ is $(\emptyset, 0, +)$ (resp., $(\emptyset, 0, -)$) if $S = G$ (resp., $T = G$). If $U = I$, then $\theta$ is in fact the supremum of $S$ (or equivalently the
infinum of $T$). Depending on $\theta \notin G$, $\theta \in S$, or $\theta \in T$, we have the cut symbols $(I, \theta)$, $(I, \theta, +)$, or $(I, \theta, -)$ respectively.

**Theorem 2.2.6** Cuts in $G$ are naturally in one-to-one correspondence with the cut symbols introduced above.

**Proof.** Suppose that a cut $(S, T)$ in $G$ is given. Then using Lemma 2.2.1 and Proposition 2.2.3 we obtain a lower cut $U$ in $I$ and an element $\theta$ in $\mathbb{H}$ which satisfy (2.2.7) and (2.2.8). Now if for all $\gamma \in G$ we have $w(\gamma - \theta) \in U$ then we obtain the cut symbol $(U, \theta)$. So suppose that there exists $\gamma \in G$ such that $w(\gamma - \theta) \notin U$. Then by Lemma 2.2.5, we have the cut symbol $(U, \theta, +)$ (resp., $(U, \theta, -)$) if $\gamma \in S$ (resp., $\gamma \in T$).

Conversely, suppose that a $G$-cut symbol is given. Corresponding to this given cut symbol, we should find a (unique) cut $(S, T)$ so that if we apply Lemma 2.2.1 and Proposition 2.2.3 to $(S, T)$ then we obtain the given cut symbol. Note that Lemma 2.2.5 actually gives the possible form of the cut $(S, T)$ (as described in parts (1) through (3) of the following claim). Therefore, by proving this claim, we also obtain uniqueness of the corresponding cut $(S, T)$.

**Claim.** (I) Corresponding to the cut symbol $(U, \theta)$ there is a cut $(S, T)$ given by $S = \{\gamma \in G : \gamma < \theta\}$, $T = \{\gamma \in G : \gamma > \theta\}$.

(II) Corresponding to the symbol $(U, \theta, +)$ there is a cut $(S, T)$ given by $S = \{\gamma \in G : \gamma < \theta \text{ or } w(\gamma - \theta) \notin U\}$, $T = \{\gamma \in G : \gamma > \theta \text{ and } w(\gamma - \theta) \in U\}$.

(III) Corresponding to the symbol $(U, \theta, -)$ there is a cut $(S, T)$ given by $S = \{\gamma \in G : \gamma < \theta \text{ and } w(\gamma - \theta) \in U\}$, $T = \{\gamma \in G : \gamma > \theta \text{ or } w(\gamma - \theta) \notin U\}$.

**Proof of the claim.** Suppose that a cut symbol is given. If the cut symbol is of the form $(U, \theta)$, then define $S$ and $T$ as in (I) (i.e., $S = \{\gamma \in G : \gamma < \theta\}$, $T = \{\gamma \in G : \gamma > \theta\}$). Also, if the cut symbol is of the form $(U, \theta, +)$ (resp., $(U, \theta, -)$), then define $S$ and $T$ as in (II) (resp., (III)). First we show that $(S, T)$ forms a cut in $G$. For case (I), $\theta \notin G$ since otherwise by letting $\gamma = \theta$, we would get $w(\gamma - \theta) = \infty \notin U$ which is a contradiction to the meaning of cut symbol $(U, \theta)$. Therefore, $S \cup T = G$. Moreover, it is obvious that $S < T$. So $(S, T)$ is a cut in
$G$. For the case (II), it is easy to see that $S \cup T = G$. So we only have to show that $S < T$. Let $\gamma_1 \in S, \gamma_2 \in T$. Either $\gamma_1 < \theta$, or $w(\gamma_1 - \theta) \notin U$. If $\gamma_1 < \theta$, then we are done. So assume $w(\gamma_1 - \theta) \notin U$. As $\gamma_2 \in T$, we have $w(\gamma_2 - \theta) \in U$. Thus $w(\gamma_1 - \theta) > w(\gamma_2 - \theta)$. Therefore, using the identity $\gamma_2 - \gamma_1 = (\gamma_2 - \theta) - (\gamma_1 - \theta)$, we see that $\gamma_2 - \gamma_1$ has the same sign as $\gamma_2 - \theta$. On the other hand, as $\gamma_2 \in T$ we have $\gamma_2 > \theta$. So $\gamma_1 < \gamma_2$. Thus $S < T$. The proof that $(S, T)$ is a cut in case (III) is similar to that of case (II).

Now, we must show that $(S, T)$ gives rise to the given cut symbol. Let $U'$ be the lower cut in $I$, and $\theta'$ be the element of the Hahn product $H$ which correspond to the cut $(S, T)$ as in Proposition 2.2.3. First we show that $U' = U$. Let $i \in U$. By condition (2.2.8), there exists $\gamma \in G$ such that $w(\gamma - \theta) \geq i$. Choose $g_1, g_2 \in G_i$ such that $g_1 < \theta_i < g_2$. Let $\alpha \in S, \beta \in T$ be such that $\alpha_j = \beta_j = \gamma_j$ for all $j < i$, and $\alpha_i = g_1$, $\beta_i = g_2$. Then it is obvious that $\alpha \in S$ and $\beta \in T$ in all cases (I) through (III). Since $w(\alpha - \beta) = i$, we have $i \in U'$ by the definition of $U'$. Thus $U \subseteq U'$. Now assume $U' \notin U$, or equivalently, there exists $i \in U'$ such that for all $j \in U$ we have $j < i$. Since $i \in U'$, there exist $\alpha \in S, \beta \in T$ such that $i = w(\alpha - \beta)$. Therefore, $\alpha_j = \beta_j$, for all $j \in U$, which has two consequences: First that we are not in the situation of the case (I). Reason. Assume that we are in case (I). Then we have $\alpha < \theta < \beta$. Thus $\alpha$ and $\beta$ and $\theta$ agree on all coordinates $i \in U$. Therefore, $w(\alpha - \theta) \notin U$ which contradicts definition of the cut symbol $(U, \theta)$. Second that $w(\alpha - \theta) \notin U$ if and only if $w(\beta - \theta) \notin U$. But the conditions $w(\alpha - \theta) \notin U$ and $w(\beta - \theta) \notin U$ can not hold simultaneously if we are in the situation of case (II) or in that of case (III). Therefore, $U' \subseteq U$. Thus $U' = U$.

Now we show that $\theta' = \theta$. If not, let $i \in U' = U$ be the smallest element of $I$ such that $\theta_i' \neq \theta_i$. There are two cases to consider:

Case 1. $i$ is not the largest element of $U' = U$. Then there exist $\alpha \in S, \beta \in T$ such that $w(\beta - \alpha) > i$. Therefore by the definition of $\theta'$, we have that $\theta_i' = \alpha_i = \beta_i$. So we have $\alpha_i \neq \theta_i \neq \beta_i$. This implies that both $w(\alpha - \theta)$ and $w(\beta - \theta)$ belong to $U$. Hence in all cases of (I) through (III) we have that $\alpha < \theta < \beta$. Then $\alpha < \beta_i$ which is a contradiction. Therefore, this case is not possible.
Case 2. Assume that \( k = i \) is the largest element of \( U \). If \( \theta'_k \neq \theta_k \), then either \( \theta'_k < \theta_k \) or \( \theta'_k > \theta_k \). Assume first that \( \theta'_k < \theta_k \). Choose \( g \in G_k \) such that \( \theta'_k < g < \theta_k \). Condition (2.2.8) implies that there exists \( \gamma \in G \) such that \( w(\gamma - \theta) \geq k \). Therefore, there exists \( \alpha \in G \) such that \( \alpha_j = \gamma_j \) for all \( j \in I, j \neq k \), and \( \alpha_k = g \). Then \( w(\alpha - \theta) = k \in U \). So \( \alpha \in S \) in all cases of (I) through (III). But according to the construction in Proposition 2.2.3 (applied now to \( \theta' \)), we would have \( g = \alpha_k \leq \theta'_k \) which contradicts \( \theta'_k < g \). Therefore, we should consider \( \theta'_k > \theta_k \). But the assumption \( \theta'_k > \theta_k \) would similarly end up with a contradiction. So we must have that \( \theta'_k = \theta_k \). Since, by our conventions, we have that \( \theta_j = 0 \) as well as \( \theta'_j = 0 \), for all \( j \notin U = U' \); therefore, \( \theta'_j = \theta_j \), for all \( j \in I \). Thus \( \theta' = \theta \), and the proof of the claim and hence that of the proposition is complete. QED

Remarks and applications. (I) The condition (2.2.7) is a natural one; however, we could have chosen \( \theta_j \) to be any element in \( G_i \), for \( j \notin U \), provided: (1) \( \theta \in \mathbb{H}_C \), and (2) we made the corresponding change in condition (2) of Proposition 2.2.3.

(II) Theorem 2.2.6 tells us that corresponding to a given lower cut \( U \subseteq I \) and an element \( \theta \in H \) which satisfy (2.2.7) and (2.2.8) there is at least one and at most two cuts \( (S, T) \) in \( G \): One cut if for all \( \gamma \in G \) we have \( w(\gamma - \theta) \in U \), and two cuts otherwise.

(III) Suppose that \( (S, T) \) is a cut in \( G \) such that its associated lower cut \( U \subseteq I \) has a last element \( k \). Then the cut symbol corresponding to \( (S, T) \) is of the form \( (U, \theta) \) if and only if \( \theta_k \notin G_k \). Reason. Suppose that \( \theta_k \notin G_k \) but the cut symbol corresponding to \( (S, T) \) is, on the contrary, not of the form \( (U, \theta) \). Then there exists \( \gamma \in G \) such that \( w(\gamma - \theta) \notin U \). Therefore, as \( U \) is a lower cut in \( I \), \( \gamma_i = \theta_i \) for all \( i \in U \). So \( \theta_k = \gamma_k \in G_k \) which is a contradiction. Thus the cut symbol is of the form \( (U, \theta) \). Conversely, suppose that \( \theta_k \in G_k \). There exists \( \gamma \in G \) such that \( w(\gamma - \theta) \geq k \). We can modify the \( k \)-th slot of \( \gamma \), if necessary, so that \( \gamma_k = \theta_k \). Therefore, \( w(\gamma - \theta) > k \). Now if \( \gamma = \theta \) then \( w(\gamma - \theta) = \infty \notin U \), and the cut symbol is not of the form \( (U, \theta) \). On the other hand, if \( \gamma \neq \theta \) then there is a smallest \( i \in I \setminus U \) such that \( \gamma_i \neq \theta_i \). Then \( w(\gamma - \theta) = i \notin U \). Thus the corresponding cut symbol is again not of the form \( (U, \theta) \).
When $\theta_k \notin G_k$ then $\theta$ obviously does not belong to $G$. This leads us to the question of the possibility of generalization of the fact proved in the above paragraph. Namely, suppose that $(S, T)$ is a cut in $G$ then can we assert that $\theta \notin G$ if and only if the cut symbol corresponding to the cut $(S, T)$ is of the form $(U, \theta)$? One direction of this statement is always true: If the cut symbol is of the form $(U, \theta)$, then $\theta \notin G$. For the other direction, assume that the cut symbol is not of the form $(U, \theta)$. Therefore, there exists $\gamma \in G$ such that $w(\gamma - \theta) \notin U$. Thus $\theta_i = \gamma_i$, for all $i \in U$. This means that $\theta$ is a truncation of $\gamma$. Hence we could deduce $\theta \in G$ if we knew that the image of $G$ in its corresponding Hahn product $\mathbb{K}_G$ is "truncation closed". This is the reason why we are interested in the notion of truncation-closedness. We discuss more about the notion of truncation-closedness in chapter 5.

**Example 2.2.5** Let $G$ be a divisible ordered abelian group of finite rank $n$. We are going to study the cuts in $G$. $G$ can be written as the lexicographic ordered group $G_1 \oplus \ldots \oplus G_n$, where $G_i, 1 \leq i \leq n$ are archimedean groups. Denote the value set $I$ of the natural valuation on $G$ by $\{1, 2, \ldots, n\}$. So the lower cuts $U$ in $I$ are $\emptyset, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}$. Associated to $\emptyset$, we have the cut symbols $(\emptyset, 0, +)$ and $(\emptyset, 0, -)$ which correspond respectively to the cuts $(G, \emptyset)$ and $(\emptyset, G)$. On the other hand, if $(S, T)$ is a cut in $G$, $S \neq \emptyset \neq T$, then $U = \{1, \ldots, l\}$, for some $1 \leq l \leq n$. Moreover, the construction given in Proposition 2.2.3 shows that, the element $\theta$ must be an $n$-tuple of the form $\theta = (r_1, \ldots, r_l, 0, \ldots, 0)$, where $r_i \in G_i, 1 \leq i < l - 1, r_l \in \mathbb{R}$. Conversely, suppose that $U = \{1, \ldots, l\}$ and $\theta = (r_1, \ldots, r_l, 0, \ldots, 0)$, where $r_i \in G_i, 1 \leq i < l - 1, r_l \in \mathbb{R}$ are given. Now if $r_l \notin G_l$ then by Theorem 2.2.6, there is a cut $(S, T)$ corresponding to the cut symbol $(U, \theta)$. On the other hand, if $r_l \in G_l$ then there are two cuts $(S, T)$: One such that $\theta \in S$ which has $(U, \theta, +)$ as its cut symbol, and the other one such that $\theta \in T$ which will have $(U, \theta, -)$ as its cut symbol.

### 2.3 Maximal elements and translation invariance in cuts

Suppose that $U$ and $\theta$ are defined as in Proposition 2.2.3.
Proposition 2.3.1 Suppose that $G$ is an ordered abelian group and $(S,T)$ is a cut in $G$. Then $S$ has a largest element if and only if $U = I$ and $\theta \in S$ (in which case $\theta$ is the largest element of $S$). Similarly, $T$ has a smallest element if and only if $U = I$ and $\theta \in T$ (in which case $\theta$ is the smallest element of $T$).

Proof. Suppose that $U = I$, and $\theta \in S$. It is obvious that $S$ can not be as described in part (I) or (III) of the proof of Theorem 2.2.6. So $S$ is as in part (II).

But $w(\gamma - \theta) \notin U$ if and only if $\gamma = \theta$. Therefore, $S$ has a largest element which is equal to $\theta$. Conversely, assume that $S$ has a largest element. In cases (I) and (III) of Theorem 2.2.6, for any $\gamma \in S$ there exists $\gamma' \in S, \gamma' > \gamma$. Therefore, only in the case (II) might $S$ have a largest element. Moreover, if we are in case (II) but $U \neq I$, then again for any $\gamma \in S$, there exists $\gamma' \in S$ such that $\gamma' > \gamma$; therefore, $S$ can not have a last element. Hence we must have $U = I$. But then the only element of $H_G$ that satisfies $w(\theta - \gamma) \notin U$ is $\theta$ and we are done. Second assertion can be proved similarly. QED

Remark. From Proposition 2.3.1 it follows that a cut $(S,T)$ is realized by an element $\theta$ of $G$ if and only if its corresponding cut symbol is $(I, \theta, \pm)$ with $\theta \in G$.

Example 2.3.1 Let $G$ be an ordered abelian group such that its corresponding value set $I$ has more than one element. Then there exists a lower cut $U$ in $I$ such that $\emptyset \neq U \not\subseteq I$. Therefore, there is a cut $(S,T), S \neq \emptyset, T \neq \emptyset$, in $G$ such that neither does $S$ have a largest element nor $T$ a smallest element.

Let $(S,T)$ be a cut in $G$, and $0 \neq \delta \in G$. Then $(S,T)$ is called invariant under translation by $\delta$ if $S + \delta = S$ or, equivalently, $T + \delta = T$. Now a question arises as to under what conditions a cut is invariant under a translation. To answer this question, we first note that the map $w : G \rightarrow I$ sends each non-zero convex subgroup of $G$ into an upper cut in $I$, and vice versa.

Remark. The rank of an ordered abelian group $G$ (see section 2.1) is the order type of the set of all non-empty upper cuts of the value set $I$ under inclusion. Therefore, the cardinality of the rank of $G$ is greater or equal to $|I|$ which is the cardinal number of the principal rank of $G$. 

Example 2.3.2 The cardinal number of the rank of $\mathbb{R}((Q))$ is equal to the cardinal number of $\mathbb{R}$, while the cardinal number of the principal rank of $\mathbb{R}((Q))$ is the cardinal number of $\mathbb{Q}$.

Now, if $(S, T)$ is a cut in $G$ and $U$ is its corresponding lower cut in $I$, then $V = I \setminus U$ is an upper cut in $I$. Assume that $U \neq I$ (i.e., $V \neq \emptyset$), and let $G_0$ be the convex subgroup of $G$ corresponding to $V$. Then it is not difficult to see that for any $\delta \in G_0$, the cut $(S, T)$ is invariant under translation by $\delta$. More generally, we have the following:

**Proposition 2.3.2** Let $G_0 = \{ \delta \in G : w(\delta) \notin U \}$ and $G_1 = \{ \delta \in G : S + \delta = S \}$. Then $G_0$ and $G_1$ are convex subgroups of $G$ and $G_1 = G_0$.

**Proof.** It is not difficult to see that $G_0$ is a convex subgroup of $G$ and that $G_0 \subseteq G_1$ (see also [Ku, Theorem 3.1]). Therefore, it is enough to show that $G_1 \subseteq G_0$. Let $\delta \in G_1$. Assume, on the contrary, that $w(\delta) \in U$. It is obvious that $\delta \neq 0$. Replacing $\delta$ by $-\delta$ if necessary, we can assume that $\delta > 0$. There are two cases to consider:

1. $w(\delta)$ is not the largest element of $U$. Then there exist $\alpha \in S, \beta \in T$ such that $w(\alpha - \beta) > w(\delta)$. Therefore, $\beta - \alpha < \delta$. That is, $\beta < \delta + \alpha \in \delta + S = S$. But $S$ is a lower cut in $G$; therefore, $\beta \in S$ which is a contradiction.

2. $k = w(\delta)$ is the largest element of $U$. Recalling the proof of Proposition 2.2.3, we can choose $\alpha \in S, \beta \in T$ such that $w(\alpha - \beta) = k$ and $\beta_k - \alpha_k < \delta_k$, where $\beta_k, \alpha_k, \delta_k$ are respectively the $k$-th slot of $\beta, \alpha, \delta$. Then $\beta < \alpha + \delta$. On the other hand, $\alpha + \delta \in S + \delta = S$. Therefore, $\beta \in S$ which contradicts $\beta \in T$, and the proposition is thus proved. QED

The above proposition shows us that the smaller the set $U$ is, the more elements $\delta \in G, \delta \neq 0$ we can find so that the cut $(S, T)$ is invariant under translation by $\delta$. If $U$ and $\theta$ are as in the Proposition 2.2.3 then we call $U$ (resp., $\theta$) the circle (resp., center) of the cut $(S, T)$.

**Corollary 2.3.3** A cut $(S, T)$ in an ordered abelian group $G$ is translation invariant under some $\delta \in G, \delta \neq 0$ if and only if its corresponding circle is not the whole $I$. 
Combining the above corollary and Proposition 2.3.1, we obtain the following:

**Corollary 2.3.4** Suppose \((S, T)\) is a cut in an ordered abelian group \(G\). Then the following are equivalent:

1. \(S\) has a greatest element or \(T\) has a smallest element.
2. The center of \((S, T)\) belongs to \(G\) and \((S, T)\) is not invariant under translation by any \(\delta \in G, \delta \neq 0\).

**Remarks.**

1. The above result is already known for Hahn groups [CG]. But in [CG] a cut which is not invariant under translation by any \(\delta \in G, \delta \neq 0\) is called a Dedekindian cut.

2. Let \(H_\theta = \{ \gamma \in H_G : w(\gamma - \theta) \notin U \}\). Then \(H_\theta\) is the translation of the convex subgroup \(H_0 = \{ \gamma \in H_G : w(\gamma) \notin U \}\) of \(H_G\) by \(\theta\). \(H_\theta\) helps us to have a picture of the cut symbols in mind:

**Case i.** Suppose the cut symbol corresponding to \((S, T)\) is of the form \((U, \theta)\). Then, \(G \cap H_\theta = \emptyset\). Using the forms of \(S\) and \(T\) as given in Theorem 2.2.6, we have the following diagram (note that the diagram actually shows the subset \(S \cup \{\theta\} \cup T\) of \(H_G\)).

```
Diagram corresponding to \((U, \theta)\) __________________________ S -> \(\theta\) ---- \(\theta\) ---- T
```

**Case ii.** Suppose that the cut symbol associated to \((S, T)\) is of the form \((U, \theta, +)\). Then, using the form of \(S\) and \(T\) as in Theorem 2.2.6, we have \(G \cap H_\theta = S \cap H_\theta\). Therefore, we obtain the following diagram:

```
Diagram corresponding to \((U, \theta, +)\) __________________________ S --> \(\theta\) ---- \(\theta\) ---- T
```

**Case iii.** Suppose that the cut symbol associated to \((S, T)\) is of the form \((U, \theta, -)\). Then \(G \cap H_\theta = T \cap H_\theta\). Therefore similar to the previous case we obtain the following diagram:

```
Diagram corresponding to \((U, \theta, -)\) __________________________ S --> \(\theta\) ---- \(\theta\) ---- T
```
2.4 Filling the cuts of an ordered abelian group

Suppose \((S, T)\) is a cut in a divisible ordered abelian group \(G\). Let \(\mathcal{H}_G = \mathcal{H}_{i \in I} \mathbb{R}\) be the Hahn group associated to \(G\) and let \(\eta \in \mathcal{H}_G\). We say that \(\eta\) "fills" the cut \((S, T)\) if \(S \leq \eta \leq T\) (that is, \(\alpha \leq \eta \leq \beta\) for all \(\alpha \in S, \beta \in T\)). If for every cut \((S, T)\) in \(G\) with non-empty circle there exists \(\eta \in G\) which fills it then \(G\) is said to be "cut complete". It is a well-known result that the only cut complete divisible ordered abelian group is \(\mathbb{R}\). However, for an arbitrary divisible ordered abelian group \(G\) it is possible to build a larger divisible ordered abelian group extension \(\tilde{G}\) of \(G\) such that each cut of \(G\) is filled in \(\tilde{G}\). Our purpose in this section is to find such an extension. In fact, we will build \(\tilde{G}\) in such a way that any cut in \(G\) is "filled" in \(\tilde{G}\) "in the strong sense": For each cut \((S, T)\) in \(G\) there will be an \(\eta \in \tilde{G}\) such that \(S < \eta < T\). So we come to some kind of completion of \(G\). For other ways of completing an ordered abelian group see for example \([CG, Sc]\).

Let \(\mathcal{C}\) be a family of cuts of \(G\). If \(\mathcal{C}\) contains just one cut \((S, T)\), then it is easy to see that there is a divisible ordered abelian group \(\tilde{G} = G \oplus \tau \mathbb{Q}\) containing \(G\) which satisfies \(S < \tau < T\). It is possible, and we are going, to find an explicit form of \(\tilde{G}\) in the general case.

Remarks. (1) Suppose that \(\hat{I}\) is an ordered set containing the value set \(I\) of the natural valuation on \(G\). Then the Hahn group \(\hat{\mathcal{H}} = \mathcal{H}_{i \in I} \mathbb{R}\) contains \(\mathcal{H} = \mathcal{H}_{i \in I} \mathbb{R}\). That is, we can identify \(\alpha = (\alpha_i)_{i \in I} \in \mathcal{H}\) with \(\alpha' = (\alpha'_i)_{i \in I} \in \hat{\mathcal{H}}\) where \(\alpha'_i = \alpha_i\) if \(i \in I\), and \(\alpha'_i = 0\) otherwise. In this way, we obtain a bigger ordered abelian group \(\tilde{G} = \hat{\mathcal{H}}\) which contains (the image of) \(G\).

(2) Suppose \((S, T)\) is a cut in \(G\) and, \(U \subseteq I, \theta \in \mathcal{H}\) are respectively the center and the circle of the cut \((S, T)\). Moreover, suppose that \(\hat{I}, \hat{\mathcal{H}}\) are as in part (1) of the remarks. Let \(\tau \in \hat{\mathcal{H}}\) be such that \(S < \tau < T\), then \(\tau_i = \theta_i\), for all \(i \in U\) except possibly for the last element of \(U\). Reason. Assume that \(i \in U\) is not the last element of \(U\). Then there exist \(\alpha \in S \subseteq \hat{\mathcal{H}}, \beta \in T \subseteq \hat{\mathcal{H}}\) such that \(\theta_j = \alpha_j = \beta_j\), for all \(j \in I, j \leq i\). On the other hand, we have that \(\alpha < \tau < \beta\); therefore, \(\tau_j = \alpha_j = \beta_j\), for all \(j \in \hat{I}, j \leq i\). Thus \(\tau_j = \theta_j\), for all \(j \in \hat{I}, j \leq i\); and we are done.
Before discussing filling the cuts belonging to \( C \), we discuss filling the cuts of an ordered set \( I \). Suppose that \( I \) is a family of cuts of \( I \). Let \( (U, V) \in I \). Then we can enlarge \( I \) by inserting an element \( k_U \) between \( U \) and \( V \). This is possible, we just define \( i < k_U < j \), for all \( i \in U, j \in V \). Now let \( \tilde{I} \) be the set consisting of all the elements of \( I \) and all \( k_U \) with \( (U, V) \in I \). We order \( \tilde{I} \) in the following way: For any \( i \in I \) and lower cuts \( U, U' \subseteq I \) we define (1) \( i < k_U \) if and only if \( i \in U \), (2) \( k_{U'} < k_U \) if and only if \( U' \nsubseteq U \). In this way we obtain an ordered set \( \tilde{I} \supseteq I \) with an ordering which extends that of \( I \). To simplify notations, we denote \( k_U \) simply by \( U \).

**Example 2.4.1** Let \( I \) be the value set of the natural valuation on the ordered abelian group \( \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \) (ordered lexicographically). We may identify \( I \) with \( \{1, 2, 3\} \). Let \( I \) be the family of all the cuts of \( I \). Then \( \tilde{I} = \{0, 1, \{1\}, 2, \{1, 2\}, 3, \{1, 2, 3\}\} \).

**Remark.** If the cardinality of \( I \) is finite then \( |\tilde{I}| = 2|I| + 1 \). If the cardinality of \( I \) is infinite then all we can say is that \( |\tilde{I}| \geq |I| \).

**Example 2.4.2** (1) Suppose that \( I = \mathbb{Q} \). Each non-empty lower cut \( U \neq \mathbb{Q} \) in \( \mathbb{Q} \) has a least upper bound \( r \) in \( \mathbb{R} \). Now corresponding to each \( r \in \mathbb{Q} \) there are exactly two cuts in \( I \), and corresponding to an irrational number \( r \in \mathbb{R} \) there exists a unique cut in \( I \). Therefore, we have \( |\tilde{I}| = 2|I| \). (2) Let \( I = \mathbb{R} \). Then corresponding to each real number \( r \) there are exactly two cuts in \( \mathbb{R} \); therefore, \( |\tilde{I}| = |I| \).

**Proposition 2.4.1** Suppose \( C \) is a family of cuts in \( G \). Then we can find an ordered set \( \tilde{I} \supseteq I \) so that for each element \((S, T)\) of \( C \) there exists an element \( \tau_S \in \mathbb{R} = \mathcal{H}_{i \in I} \mathbb{R} \) such that \( S < \tau_S < T \).

**Proof.** Let

\[
\tilde{I} = I \cup I_1, \text{ where}
\]

\[
I_1 = \{U : U \text{ is the circle of a cut in } C \text{ having a cut symbol of the form } (U, \theta, \pm)\}
\]

It is clear that \( \tilde{I} \) is an ordered set. Suppose \((S, T) \in C \). Depending on the cut symbol of \((S, T)\), there are three cases to consider: (1) If the cut symbol is of the form \((U, \theta)\), let \( \tau_S = \theta \). Then it is obvious that \( S < \tau_S < T \). (2) If the cut symbol is
of the form \((U, \theta, +)\) then let \(\tau_S = \theta + 1_U\), where \(1_U \in \overline{H}\) and has all its coordinates equal to 0 except for the \(U\)-th one where the coordinate is 1. To prove that \(\tau_S < T\) let \(\gamma \in T\), then by Theorem 2.2.6 we have \(\theta < \gamma\) and \(w(\gamma - \theta) \in U\). Therefore, there exists \(i \in U\) such that \(\theta_j = \gamma_j\) for all \(j \in I, j < i\), and \(\theta_i < \gamma_i\). Then \((\tau_S)_j = \theta_j = \gamma_j\) for all \(j \in I, j < i\), and \((\tau_S)_i = \theta_i < \gamma_i\). Thus \(\tau_S < \gamma\). To prove \(S < \tau_S\), let \(\gamma \in S\). Then again by Theorem 2.2.6 we have either \(\gamma < \theta\) or \(w(\gamma - \theta) \notin U\). If \(\gamma < \theta\), then \(\gamma < \theta < \tau_S\) and we are done. So assume that \(w(\gamma - \theta) \notin U\). Then \((\tau_S)_j = \theta_j = \gamma_j\) for all \(j \in I, j < U\). Moreover, \((\tau_S)_U = (\theta + 1_U)_U = 1 > 0 = \theta_U = \gamma_U\). Thus \(\gamma < \tau_S\).

(3) If the cut symbol is of the form \((U, \theta, -)\), let \(\tau_S = \theta - 1_U\), where \(1_U\) is as defined in part (2). Then we can employ a similar reasoning as in part (2) to obtain the result. QED

Corollary 2.4.2 If \(G\) is a (not necessarily divisible) ordered abelian group, then there is an ordered abelian group extension \(\hat{G}\) of \(G\) so that for every cut \((S, T)\) of \(G\) there exists \(\tau_S \in \hat{G}\) such that \(S < \tau_S < T\).

Proof. First we extend \(G\) to a divisible ordered abelian group \(G_1\). Then we let \(C\) be the set of all cuts in \(G_1\). Every cut \((S, T)\) in \(G\) extends to a cut \((S_1, T_1)\) in \(G_1\), e.g., let \(S_1 = \{\gamma_1 \in G_1 : \gamma_1 \leq \gamma, \text{for some } \gamma \in S\}, T_1 = G_1 \setminus S_1\). Now use the above proposition to get an extension \(\hat{G} = \hat{H} = \mathcal{H}_i \in \mathcal{I} \mathbb{R}\), where \(\mathcal{I}\) is the ordered set corresponding to \(C\) as in the above proposition. QED

Remark. The above result is interesting and has applications, see for example [Sc, Sz]. In the latter reference, a model theoretic proof is supplied and interest is shown in finding a purely algebraic proof (the statement after Corollary 2.2). Meanwhile, the constructive proof given here paves the way for our later results (see for example Proposition 3.2.3).
Chapter 3

Cuts in Real Closed Fields

In this chapter we are going to study the orderings on the field $R(y_1, \ldots, y_n)$ and more generally on the ring $R[y_1, \ldots, y_n]$, where $R$ is a real closed field. We will begin our study by the correspondence between the orderings on $R(y)$ and the cuts in $R$. Then we will see how a cut in the field $R$ can be characterized by a suitable cut symbol and as a consequence then by an element of a suitable maximally complete field $\kappa^\alpha(G^\alpha)$. The notion of a cut symbol will then be employed to show that there is a correspondence between the orderings on the ring $R[y_1, \ldots, y_n]$ and an $n$-tuple $(\phi_1, \ldots, \phi_n)$, where $\phi_i$, $1 \leq i \leq n$, is a suitable cut symbol or an element of a suitable real closed field. Finally, we will define immediate transcendental, residue transcendental, and value transcendental cuts and obtain a version of the Abhyankar inequality which is related to our discussion.

3.1 Introduction

The study of the orderings on the field of rational functions $R(y_1, \ldots, y_n)$ can be done inductively: Suppose that $\geq$ is an ordering on $R(y_1, \ldots, y_{n-1})$. Let $R$ be the real closure of $R(y_1, \ldots, y_{n-1})$ with the ordering $\geq$. Then any ordering on $R(y_n)$ restricts naturally to an ordering on $R(y_1, \ldots, y_n)$. Conversely, suppose that we know an ordering $\geq^*$ on $R(y_1, \ldots, y_n)$. This ordering restricts to an ordering on $R(y_1, \ldots, y_{n-1})$ with the real closure $R$. But the real closure $R^*$ of the ordering $\geq^*$ on $R(y_1, \ldots, y_n)$ contains $R(y_n)$. Thus we have an ordering on $R(y_n)$. 
Therefore, our study is reduced to the study of the orderings on the rational function field $R(y)$, where $R$ is a real closed field. If $A$ is a subset of an ordered field $(F, \geq)$ and $x \in F$ then we write $A > x$ (resp., $A < x$) if for all $y \in A$ we have $y > x$ (resp., $y < x$). There is a nice relationship between the orderings on $R(y)$ and the cuts in $R$ (see [Gi]):

**Proposition 3.1.1** Suppose that $R$ is a real closed field. Then there is a one-to-one correspondence between the orderings on the field of rational functions $R(y)$ and the cuts of $R$.

**Proof.** Suppose that $>$ is an ordering on $R(y)$. Then there is a unique cut $(A, B)$ in $R$ such that $A < y < B$. Actually, $A$ is defined by $A = \{ r \in R : r < y \}$ and $B$ by $B = \{ r \in R : r > y \}$. Conversely, suppose that $(A, B)$ is a cut in $R$. We want to define an ordering on $R(y)$ such that $A < y < B$. Let $f(y) \in R[y], f \neq 0$ be a monic polynomial. So $f(y)$ can be written as a product of a finite number of linear factors $y - r_i, r_i \in R$ and a finite number of monic and irreducible quadratic polynomials over $R$. We say that $f > 0$ in $R(y)$ if and only if the number of the roots $r_i$ of $f$ (counting multiplicities) which belong to $B$ is even. If $g \in R(y), g \neq 0$ then there exist $a \in R, a \neq 0, f_1, f_2 \in R[y], f_1, f_2$ monic polynomials, such that $g = af_1/f_2$. We say $g > 0$ if either $a > 0$ and $f_1f_2 > 0$, or $a < 0$ and $f_1f_2 < 0$.

Now suppose that $f, g \in R(y), f > 0, g > 0$. Then it is obvious that $fg > 0$. To see that we have also $f + g > 0$, let $f = af_1/f_2, g = bg_1/g_2$, where $a, b \in R$, and $f_1, f_2, g_1, g_2 \in R[y]$ are monic polynomials. We have $f = (af_1g_2)/(f_2g_2)$ and $g = (bg_1f_2)/(f_2g_2)$. Therefore, by multiplying by a suitable linear factor if necessary, we may assume that $f_2g_2, af_1g_2$, and $bg_1f_2$ are all positive. Then we just need to show that if $h_j \in R[y], h_j > 0, j = 1, 2$, then $h_1 + h_2 > 0$. First note that we have the following obvious fact:

**Fact.** Suppose that $h \in R[y]$. Then:

1. If there exists $c \in A$ which is larger than all of the roots of $h$ which belong to $A$ and $h(c) > 0$ in $R$ then $h > 0$ in $R(y)$. Conversely, if $h > 0$ in $R(y)$ and $A$ does not have a largest element then there exists $c \in A$ which is greater than the
maximum of the roots of $h$ which belong to $A$ and $h(c') > 0$ for any $c' \in A, c' \geq c$.

(2) If there exists $d \in B$ which is smaller than all of the roots of $h$ which belong to $B$ and $h(d) > 0$ in $R$ then $h > 0$ in $R(y)$. Conversely, if $h > 0$ in $R(y)$ and $B$ does not have a smallest element then there exists $d \in B$ which is smaller than the minimum of the roots of $h$ which belong to $B$ and $h(d') > 0$ for any $d' \in B, d' \leq d$.

Now either $A$ does not have a largest element or $B$ does not have a smallest element. Assume that $A$ does not have a largest element. Since $h_j > 0, j = 1, 2$, part (1) of the above fact implies that there exist $c_{ij}, j = 1, 2$, such that $c_1$ (resp., $c_2$) is greater than the maximum of the roots of $h_1$ (resp., $h_2$) and $h_1(c'_1) > 0$ (resp., $h_2(c'_2) > 0$) for any $c'_1 \in A, c'_1 \geq c_1$ (resp., $c'_2 \in A, c'_2 \geq c_1$). Let $c = \text{Max}\{c_1, c_2\}$. Then $c \in A$ is greater than all of the roots of $h_j, j = 1, 2$, which are in $A$. So $h_j(c) > 0, j = 1, 2$. Therefore, $h_1(c) + h_2(c) > 0$. Thus by fact (1), $h_1 + h_2 > 0$ in $R(y)$. A similar argument can be used for the case when $B$ does not have a smallest element.

To conclude our proof, suppose that $>'$ is another ordering on $R(y)$ such that $A < y < B$. This means that for all $a \in A$ and $b \in B$ we have that $y - a >' 0$ and $y - b <' 0$. We just need to prove that if a monic polynomial $h \in R[y]$ is given then $h$ is positive with respect to the ordering $>'$ if and only if it is positive with respect to the ordering $>$. We can write

$$h(y) = \prod_{i=1}^{k}(y - r_i) \prod_{j=1}^{l}(y^2 + s_jy + t_j),$$

where $r_i, 1 \leq i \leq k$ are the roots of $h$ in $R$ and $y^2 + s_jy + t_j, 1 \leq j \leq l$ are non-reducible quadratics in $R(y)$. These quadratics are positive in any ordering of $R(y)$. So $h >' 0$ if and only if the number of the roots $r_i$ of $h$ which belong to $B$ is even. Therefore $h(y) >' 0$ if and only if $h(y) > 0$. Thus the ordering $>$ on $R(y)$ obtained above is unique and we are done. QED

Remarks. (1) If $F$ is an ordered field and $(A, B)$ is a cut in $F$ then there may be many orderings on $F(y)$ such that $A < y < B$. In fact, if $R$ denotes the real closure of $F$ and $M = \{x \in R : A < x < B\}$ then (i) If $M = \emptyset$ then there is just one such ordering. (ii) If $M$ is a singleton then there are two such orderings. (iii) If
$M$ contains more than one element then there are infinitely many such orderings (see [Gi]).

(2) Proposition 3.1.1 also follows from Proposition 3.2.3 in the next section.

3.2 Cuts in real closed fields

We now return to the problem of characterizing the cuts in an arbitrary real closed field $R$. We keep the notation introduced earlier, i.e., $v$ is the natural valuation associated to the unique ordering on $R$ with the value group $G$ and the residue field $\kappa$. We also fix a "proper" embedding $R \hookrightarrow \kappa((G))$ and identify $R$ with its image (for the definition of proper see the remark on page 22). Let $(A, B)$ be a cut in $R$ and let $S = \{v(b - a) : a \in A, b \in B\}$ be the associated lower cut in $G$.

**Proposition 3.2.1** There is a unique element $p = \sum_{\gamma \in G} p_\gamma \gamma \in \mathbb{R}((G))$ such that

1. If $a \in A, b \in B$, then $v(p - a) \geq v(a - b)$ and $v(p - b) \geq v(a - b)$.
2. $p_\gamma = 0$ if $\gamma \notin S$.
3. If $S$ has a largest element $\theta$, then for any $a \in A, b \in B$, if $v(a - b) = \theta$, then $a_\theta \leq p_\theta \leq b_\theta$.

**Proof.** This is clear. Actually, this is just Proposition 2.2.3 applied to the additive group $(R, +)$ and the Hahn product $(\mathbb{R}((G)), +)$. QED

**Remark.** Note that the coefficients $p_\gamma, \gamma \in S$, except possibly the last one $p_\theta$, belong to $\kappa$.

We continue to apply the results of chapter 2; we see that the cuts $(A, B)$ in $R$ are classified by $R$-cut symbols $(S, p), (S, p, +)$ and $(S, p, -)$, where $p$ is an element of $\mathbb{R}((G))$ and $S$ is a lower cut in $G$ satisfying:

1. $p_\gamma = 0$ if $\gamma \notin S$, and
2. For all $\gamma \in S$, there exists $c \in R$ such that $v(c - p) \geq \gamma$.

Recall that the cut symbol $(S, p)$ corresponds to the case when for all elements $c \in R$ we have $v(c - p) \in S$. The cut symbol $(S, p, +)$ is defined if there exists an element $c \in A$ such that $v(c - p) \notin S$ (in which case all such elements $c$ with $v(c - p) \notin S$
are in $A$). The cut symbol $(S, p, -)$ is defined if there exists an element $c \in B$ such that $v(c - p) \notin S$ (in which case all such elements $c$ with $v(c - p) \notin S$ are in $B$). $p$ is called the center and $S$ the circle of the cut $(A, B)$.

**Proposition 3.2.2** Cuts in $R$ are in a natural one-to-one correspondence with the $R$-cut symbols defined above.

**Proof.** Just apply Theorem 2.2.6. QED

**Example 3.2.1** Let $\kappa \subseteq \mathbb{R}$ be real closed, and let $R$ be a real closed field such that $\kappa(x) \subseteq R \subseteq \mathcal{P}_\kappa(x)$ (= the field of Puiseux series with coefficients in $\kappa$). We want to understand the cuts in $R$. Let $(A, B)$ be a cut in $R$ with the associated circle $S \subseteq \mathbb{Q}$ and center $p$. Our argument is based on the form of $p$:

**Case i.** Suppose $p$ is a finite sum, i.e., $p$ is of the form $p = \sum_{i=1}^{n} p_i x^{r_i}$, $n \geq 1$ an integer, $r_i$ a rational number, $p_i \in \mathbb{R}, 1 \leq i \leq n$. Each coefficient $p_i$, except possibly the last one $p_n$, belongs to $\kappa$.

**Subcase i.1.** $p_n \notin \kappa$. In this case, it is obvious that there exists no $c \in R$ with $v(p - c) \notin S$. Therefore, the cut symbol is of the form $(S, p)$ and by Theorem 2.2.6, the form of the cut $(A, B)$ is uniquely determined by $p$. Moreover, $S = (-\infty, r_n] \cap \mathbb{Q}$ in this case.

**Subcase i.2.** $p_n \in \kappa$. In this case, $p \in R$. Depending on whether $p \in A$ or $p \in B$, the cut symbol will be of the form $(S, p, +)$ or $(S, p, -)$ respectively. In either case the cut $(A, B)$ and the circle $S$ are not uniquely determined by $p$ (see the next example).

**Case ii.** Suppose that $p$ is an infinite sum. $p \in \kappa((\mathbb{Q}))$ is easily seen to be a pseudo-limit of a pseudo-convergent sequence which is cofinal in $A$. Since condition (2.2.8) must hold, the ordinal number of $\text{Supp}(p)$ is at most $\omega$. Thus $p$ should have the form $p = \sum_{i=1}^{\infty} p_i x^{r_i}$, where $p_i \in \kappa$ for all $i \in \mathbb{N}$, and $\{r_i\}_{i \in \mathbb{N}}$ is an increasing sequence of rational numbers; moreover, $x^{r_i} \in R$, for all $i \in \mathbb{N}$. It is not difficult to see that $p \in R$ if and only if the cut symbol is of the form $(S, p, +)$ or $(S, p, -)$ with $S = \mathbb{Q}$. Actually, if $p \in R$, then the cut symbol has one of these forms. Conversely, assume $p \notin R$. If for some $c \in R$ we have that $v(p - c) \notin S$, then $p$ is an initial segment of $c$. But this can happen only if $c = p$. Now, we also have:
Subcase iii. If \( p \in A \) then \( S = Q \), \( A = \{ c \in R : c \leq p \} \), and \( B = \{ c \in R : c > p \} \).

Subcase ii. If \( p \in B \) then \( S = Q \), \( A = \{ c \in R : c < p \} \), and \( B = \{ c \in R : c \geq p \} \).

Subcase ii. If \( p \notin R \) then \( S = (-\infty, s) \cap Q \), where \( s = \sup \{ r_i \}_{i \in \mathbb{N}} \) (\( s \) might be \( \infty \)) and the corresponding cut symbol is \((S, p)\).

**Example 3.2.2** Suppose \( r \in \mathbb{R} \setminus \mathbb{Q} \). Let \((A, B)\) be a cut in \( \mathcal{P}_R(x) \) defined by \( A = \{ c : c < x^r \} \) and \( B = \{ c : c > x^r \} \) (note that \( c < x^r \) and \( c > x^r \) are well-defined as \( \mathcal{P}_R(x) \subset \mathbb{R}((\mathbb{Q})) \subset \mathbb{R}((\mathbb{R})) \)). Let \( c = \sum_{i=1}^{\infty} c_i x^{\gamma_i} \). Suppose \( c \neq 0 \). We may assume that the first term \( c_1 x^{\gamma_1} \) of \( c \) is non-zero. Therefore, \( c \in A \) if and only if \( c_1 < 0 \) or, \( c_1 > 0 \) and \( \gamma_1 > r \). Moreover, \( c \in B \) if and only if \( c_1 > 0 \) and \( \gamma_1 < r \). Thus, by the definition of the lower cut \( S \subseteq Q \), we have \( S = (-\infty, r) \cap Q \). Now let \( \ell < \ell' \) be two rational numbers less than \( r \). Then \( x^{\ell'} \in B \) and \( -x^{\ell} \in A \); therefore, the coefficient \( d_\ell \) of \( x^\ell \) in the representation \( \sum d_\ell x^\ell \) of \( p \) is zero. Thus \( p = 0 \) and the cut symbol corresponding to \((A, B)\) is \( ((-\infty, r) \cap Q, 0, +) \).

For the next result, let \( R \) be a real closed field having \( G \) and \( \kappa \) as its value group and residue field respectively. Moreover, let \( C \) be the family of all the cuts in the divisible ordered abelian group \( G \), \( I \) the value set of the natural valuation on \( G \), \( \hat{I} \supseteq I \) the corresponding ordered set as in the proof of Proposition 2.4.1, and \( G \subseteq \mathfrak{H} \subseteq \hat{\mathfrak{H}} \), where \( \mathfrak{H} \) is the Hahn product of \( I \) copies of \( \mathbb{R} \) and \( \hat{\mathfrak{H}} \) is the Hahn product of \( \hat{I} \) copies of \( \mathbb{R} \). Finally, suppose that \( \iota : R \hookrightarrow \kappa((G)) \) is a proper embedding.

**Proposition 3.2.3** For each \( R \)-cut symbol \( \alpha \), we have a naturally defined archimedean real closed field \( \kappa^\alpha \subseteq \mathbb{R} \) extending \( \kappa \), and a naturally defined divisible ordered abelian group \( G^\alpha \subseteq \hat{\mathfrak{H}} \) extending \( G \), and a naturally defined element \( \phi^\alpha \in \kappa^\alpha((G^\alpha)) \) such that if \((A, B)\) is the cut in \( R \) corresponding to \( \alpha \) then \( A < \phi^\alpha < B \). In other words, there exists an embedding \( \iota^\alpha : R(y) \hookrightarrow \kappa^\alpha((G^\alpha)) \) which extends \( \iota \) and maps \( y \) to \( \phi^\alpha \) such that the ordering on the field of rational fractions \( R(y) \) induced by the cut \((A, B)\) is the same as the ordering on \( R(y) \) induced by \( \iota^\alpha \). Moreover, if \( R^\alpha \) denotes the real closure of \( R(y) \) at the induced ordering then the value group and residue field of \( R^\alpha \) at the natural valuation are \( G^\alpha \) and \( \kappa^\alpha \) respectively. In other
words, \( \kappa^\alpha((G^\alpha)) \) is a maximal immediate extension of \( R^\alpha \). Furthermore, \( \iota^\alpha \) is a proper embedding.

Proof. To the cut \((A, B)\) in \( R \) there corresponds, via \( \iota \), the cut \((A', B')\) in \( R' = \iota(R) \), where \( A' = \iota(A), B' = \iota(B) \). The residue field of \( R' \) is \( \kappa \). Moreover, since \( \iota \) is proper, \( R' \) has value group \( G \). Now there are various cases to consider:

Case 1. The cut symbol \( \alpha \) is of the form \((S, p)\), where \( p = \sum_{\gamma \in G} p_\gamma x^\gamma \). In this case, \( p \) is transcendental over \( R' \). There are two subcases:

Subcase 1a. \( S \) does not have a last element. Hence all the coefficients \( p_\gamma \) of the terms in \( p \) belong to \( \kappa \). We choose \( \kappa^\alpha = \kappa, G^\alpha = G, \) and \( \phi^\alpha = p \). Then the result is obvious by part (I) of the proof of Theorem 2.2.6. Note that the value group of \( R'(p) \) is the divisible abelian group \( G \). Therefore, the value group of the real closure of \( R'(p) \), and hence that of \( R^\alpha \) is \( G^\alpha = G \). Moreover, the residue field of \( R^\alpha \) is clearly \( \kappa = \kappa^\alpha \).

Subcase 1b. \( S \) has a last element \( \theta \). So we can write \( p = \sum_{\gamma \leq \theta} p_\gamma x^\gamma \), where \( p_\gamma \in \kappa, \gamma < \theta, p_\theta \in R \). We saw previously, in part (III) of the remarks on page 37, that \( p_\theta \notin \kappa \). Write \( p = q + p_\theta x^\theta \). Now we take \( \kappa^\alpha \) to be the real closure of \( \kappa(p_\theta) \), \( G^\alpha = G \), and \( \phi^\alpha = p \). So again the result follows by Theorem 2.2.6. Note that the value group of \( R'(p) \), and hence that of \( R^\alpha \), is clearly \( G^\alpha = G \). Moreover, \( p_\theta \) is in the residue field of \( R'(p) \). Reason. There exist \( c = \sum_{\gamma \in G} c_\gamma x^\gamma \in R', d = \sum_{\gamma \in G} d_\gamma x^\gamma \in R' \) such that \( v(c - d) = \theta \). That is \( v(p - c) = \theta \). Then \( v((p - c)/(d - c)) = 0 \). So \( (p_\theta - c_\theta)/(d_\theta - c_\theta) \) is in the residue field of \( R'(p) \). Thus \( p_\theta \) is in the residue field of \( R'(p) \). This means, by Theorem 1.2.4, that the residue field of the real closure of \( R'(p) \), and hence that of \( R^\alpha \), is equal to the real closure of \( \kappa(p_\theta) \), i.e., is equal to \( \kappa^\alpha \).

Case 2. The cut symbol \( \alpha \) is not of the form \((S, p)\).

Subcase 2a. The cut symbol \( \alpha \) is of the form \((S, p, +)\). The value group of \( R' \) is \( G \) and \( S \) is a lower cut in \( G \). By Corollary 2.4.2 (and the proof of Proposition 2.4.1), there exists \( \tau_S \) in the ordered abelian group extension \( \hat{R} \) of \( G \) such that \( S < \tau_S < G \setminus S \), and consequently, \( A' < p + x^{\tau_S} < B' \). Let \( \phi^\alpha = p + x^{\tau_S} \). Then
\( \tau_S \) belongs to the value group of \( R'(\phi^a) \). Reason. There exists \( a \in R' \) such that \( v(p - a) \notin S \). Therefore, \( \tau_S < v(p - a) \). Since \( x^{\tau_S} = (\phi^a - a) - (p - a) \) we have \( \tau_S = v(\phi^a - a) \). Thus \( \tau_S \) is in the value group of \( R'(\phi^a) \). Now if we let \( G^a \) be the divisible hull of \( G(\tau_S) \), and \( \kappa^a = \kappa \) then the result is obvious.

Subcase 2b. The cut symbol \( \alpha \) is of the form \( (S, p, -) \). In this case, we define \( \tau_S \) as in the subcase 2a. Let \( \phi^a = p - x^{\tau_S} \), \( G^a \) be the divisible hull of \( G(\tau_S) \), and \( \kappa^a = \kappa \).

Then the result follows similarly as in the previous subcase. QED

Remark. Subcase 1a in the proof of the above proposition is referred to as the immediate transcendental case, Subcase 1b as the residue transcendental case, and Subcase 2a and Subcase 2b as the value transcendental case.

Corollary 3.2.4 Suppose that \( R \) is a real closed field with value group \( G \) and residue field \( \kappa \). Moreover, let \( R_1 \) be a real closed subfield of \( R \) with value group \( G_1 \subseteq G \) and residue field \( \kappa_1 \subseteq \kappa \). Suppose that there exists a proper order preserving embedding \( \iota_1 \) of \( R_1 \) into \( \kappa_1((G_1)) \). Then there exists a proper order preserving embedding \( \iota \) of \( R \) into \( \kappa((G)) \) extending \( \iota_1 \).

Proof. Let \( A \) be the set of all pairs \( (F, \iota_F) \) such that \( F \) is a real closed subfield of \( R \), \( R_1 \subseteq F \), and \( \iota_F : F \rightarrow \kappa_F((G_F)) \) is a proper order preserving isomorphism extending \( \iota_1 \), where \( G_F, \kappa_F \) are respectively the value group and the residue field of \( F \). Using Zorn's Lemma we see that \( A \) has a maximal element \( (R^*, \iota^*) \). Now applying Proposition 3.2.3, it is not difficult to see that \( R^* = R \). QED

Remark. In the above corollary if we take \( R_1 = \kappa \), \( G_1 = 0 \), \( \kappa_1 = \kappa \), and \( \iota_1 \) to be the identity embedding then we obtain another proof for Theorem 1.4.8.

Example 3.2.3 Let \( R \) be a real closed subfield of \( \mathbb{R} \). We want to find the orderings on \( R = R(y) \). \( R \) is archimedean so \( G = \{0\} \), \( \kappa = R \). Let \( (A, B) \) be a cut in \( R \). If \( A = \emptyset \) (resp., \( B = \emptyset \)), then the associated cut symbol is \( (\emptyset, 0, -) \) (resp., \( (\emptyset, 0, +) \)). Therefore, the ordering on \( R(y) \) is induced by the embedding \( R(y) \hookrightarrow \mathbb{R}((\mathbb{Q})) \) defined by \( y \mapsto -x^{-1} \) (resp., \( y \mapsto x^{-1} \)). In this case, we say that \( y \) is infinitely small (resp., infinitely large) as compared to the elements of \( R \). If \( A \neq \emptyset \neq B \), then \( S = G = \{0\} \),
so $S$ has a last element. Therefore, $p \in \mathbb{R}$. Now if $p \notin R$ then we have the cut symbol $\{\{0\}, p\}$ and we are in the case 1b of the proof of above proposition. Therefore, the ordering on $R(y)$ is induced by the embedding $R(y) \hookrightarrow \mathbb{R}$ defined by $y \mapsto p$ (this means that the ordering on $R(y)$ is archimedean). Finally, if $p \in R$ then we have the cut symbol $\{\{0\}, p, +\}$ or $\{\{0\}, p, -\}$ according as $p \in A$ or $p \in B$. Then the ordering on $R(y)$ corresponding to these cut symbols are induced by the embedding $R(y) \hookrightarrow R(\mathbb{Q})$ defined by $y \mapsto p + x$ or $y \mapsto p - x$ respectively. Moreover, we may as well say that $y - p$ is infinitely small and positive (resp., negative) if the cut symbol is of the form $\{\{0\}, p, +\}$ (resp., $\{\{0\}, p, -\}$).

**Example 3.2.4** Suppose that $F$ is a subfield of $\mathbb{R}$. We want to find the orderings on $F(y)$. Since the real closure of any ordering on $F(y)$ contains the real closure $F^*$ of $F$, we are reduced to study the orderings on $F^*(y)$. This is done in the previous example.

**Example 3.2.5** We want to find the orderings on the field $K = \kappa(y_1, y_2)$, where $\kappa$ is a real closed subfield of $\mathbb{R}$. This is discussed in [ABR] assuming that $\kappa = \mathbb{R}$. Any ordering on $\kappa(y_1, y_2)$ restricts to an ordering on $\kappa(y_1)$. Fix an ordering on $\kappa(y_1)$. If the ordering on $F = \kappa(y_1)$ is archimedean (see Example 3.2.3) then the ordering on $\kappa(y_1, y_2) = F(y_2)$ is easily found using Example 3.2.4. So assume that the ordering on $F$ is non-archimedean. Using the results of example 3.2.3, i.e., by a suitable translation of $y_1$, replacing $y_1$ by $1/y_1$, or (and) replacing $y_1$ by $-y_1$, we may assume that the ordering on $F$ is the one which corresponds to the cut symbol $\{\{0\}, 0, +\}$ (i.e., $0 < y_1 < \varepsilon$, for all $\varepsilon \in \kappa, \varepsilon > 0$). The real closure $R$ of $F$ at this ordering is a subfield of $\mathcal{P}_\kappa(y_1)$. Let $L = \mathbb{R}((\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}))$, $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ ordered lexicographically; moreover, let $x_0 = x^{(1,0,0)}, x_1 = x^{(0,1,0)}, x_2 = x^{(0,0,1)}$. Suppose that $(A, B)$ is a cut in $R$. Then as in Proposition 3.2.3 we have the following cases:

**Case 1.** The cut symbol of $(A, B)$ is of the form $(S, p)$. Then by using the results in Example 3.2.1 we have:

**Subcase 1a.** $p \notin R$ and $p$ is an infinite sum of the form $p = \sum_{i=1}^{\infty} c_i y_i^r \in \kappa(\mathbb{Q})$. Therefore, the ordering on $K$ is obtained by the embedding $K \hookrightarrow L$ defined by
\( y_1 \mapsto x_1, \) and \( y_2 \mapsto \sum_{i=1}^{\infty} c_i x_i^\alpha. \)

**Subcase 1b.** \( p \) is a finite sum of the form \( p = \sum_{i=1}^{n} c_i y_i^\alpha, \) where \( c_i \in \kappa, 1 \leq i \leq n - 1, \) and \( c_n \in \mathbb{R} \setminus \kappa. \) So the ordering on \( K \) is obtained by the embedding \( K \hookrightarrow L \) given by \( y_1 \mapsto x_1, \) and \( y_2 \mapsto \sum_{i=1}^{n} c_i x_i^\alpha. \)

**Case 2.** The cut symbol of \((A, B)\) is not of the form \((S, p)\). Then it follows from Example 3.2.1 that \( p = \sum c_i x_i^\alpha \in R \) (\( p \) may be a finite sum or an infinite sum). So the corresponding cut symbol is \((S, p, \pm)\).

**Subcase 2a.** The cut symbol of \( S \) is of the form \((U, \theta)\). Then \( U \neq \emptyset \) (i.e., \( U = I = \{1\} \)) and \( \theta \in \mathbb{R} \setminus \mathbb{Q} \). Thus \( \tau_S = \theta \) and the ordering on \( K \) is obtained by the embedding \( K \hookrightarrow L \) given by \( y_1 \mapsto x_1 \) and \( y_2 \mapsto \sum c_i x_i^\alpha \pm x_0^\theta. \)

**Subcase 2b.** The cut symbol of \( S \) has the form \((U, \theta, \pm)\).

**Subcase 2b1.** \( U = \emptyset, \) i.e., \( S = \emptyset, \) or \( S = \mathbb{Q} \). Then \( \theta = 0 \). Therefore, \( \tau_S = \theta \pm 1_U = \pm 1_U = \pm (1, 0, 0) \) (with + if \( S = \mathbb{Q}, \) - if \( S = \emptyset \)). Then the ordering on \( K \) is obtained by the embedding \( K \hookrightarrow L \) given by \( y_1 \mapsto x_1 \) and \( y_2 \mapsto \sum c_i x_i^\alpha \pm x_0^1 \) if \( S = \mathbb{Q}, \) and \( y_2 \mapsto \sum c_i x_i^\alpha \pm x_0^{-1} \) if \( S = \emptyset \) (in the latter case, i.e., when \( S = \emptyset, p = 0 \) so \( y_2 \mapsto \pm x_0^{-1} \)).

**Subcase 2b2.** \( U \neq \emptyset \) (i.e., \( U = I = \{1\} \)). Then \( S, \mathbb{Q} \setminus S \) are nonempty and \( \theta \in \mathbb{Q} \).

So either \( \theta \in S \) or \( \theta \in \mathbb{Q} \setminus S \). Then \( \tau_S = \theta \pm 1_U, 1_U = (0, 0, 1) \) (with + if \( \theta \in S, \) - if \( \theta \notin S \)). Then the ordering on \( K \) is obtained by the embedding \( K \hookrightarrow L \) given by \( y_1 \mapsto x_1 \) and \( y_2 \mapsto \sum c_i x_i^\alpha \pm x_1^\theta x_2 \) if \( \theta \in S, \) and \( y_2 \mapsto \sum c_i x_i^\alpha \pm x_1^\theta x_2^{-1} \) if \( \theta \notin S. \)

### 3.3 Orderings on the ring \( R[y_1, \ldots, y_n] \)

In this section we discuss the orderings on the ring \( R[y_1, \ldots, y_n] \), where \( R \) is a real closed field. We will find that there is a one-to-one correspondence between the orderings of such a ring and the "order symbols" which will be defined later. Orderings on a ring \( A \), where \( A \) is a commutative ring with identity, were introduced by M. Coste and M.F. Roy (see for example [CR]). An ordering on a ring \( A \) is a subset \( P \subseteq A \) which satisfies (1) \( P + P \subseteq P, \) (2) \( PP \subseteq P, \) (3) \( P \cup -P = A, \) and (4) \( P \cap -P = \mathfrak{p}, \) where \( \mathfrak{p} \) is a prime ideal in \( A. \) \( P \) is called the positive cone, and \( \mathfrak{p} \) the
support of the corresponding ordering on $A$ ($\mathfrak{p} = Supp(P)$). If $P$ is an ordering on $A$ with support $\mathfrak{p}$ then it induces an ordering on the field of quotients of the integral domain $A/\mathfrak{p}$, which we denote by $\kappa_A(\mathfrak{p})$. $\kappa_A(\mathfrak{p})$ is called the residue field of $A$ at $\mathfrak{p}$. It is a well-known fact (see [Mr]) that:

Proposition 3.3.1 The set of orderings on $A$ is in natural one-to-one correspondence with the set of pairs $(\mathfrak{p}, \bar{P})$ where $\mathfrak{p} \subseteq A$ is a prime and $\bar{P}$ is an ordering on $\kappa_A(\mathfrak{p})$.

Actually, we have the natural ring homomorphism $A \to A/\mathfrak{p} \to \kappa_A(\mathfrak{p})$ in which the preimage of the positive cone $\bar{P}$ of $\kappa_A(\mathfrak{p})$ is the positive cone $P$ of the ordering on $A$. Moreover, $A/\mathfrak{p}$ inherits an ordering $(A/\mathfrak{p}) \cap \bar{P} = \{a + \mathfrak{p} : a \in P\}$ with support $\{0\}$.

We begin with the orderings on $R[y]$, where $R$ is a real closed field. Orderings on $\mathbb{R}[y]$ are not difficult to find (see [Mr]). But with the tools we have already provided, it is possible to classify the orderings more generally on $R[y]$. Fix a proper embedding $R \hookrightarrow \kappa((G))$, where $\kappa$ (resp., $G$) is the residue field (resp., value group) of the natural valuation on $R$. Given an ordering on $R[y]$, we obtain the following natural maps.

$$R \hookrightarrow R[y] \twoheadrightarrow R[y]/\mathfrak{p} \hookrightarrow \kappa_{R[y]}(\mathfrak{p}) \hookrightarrow R',$$

where $R'$ is the real closure of $\kappa_{R[y]}(\mathfrak{p})$. For convenience, let's make the following convention: If $\phi$ is either a cut symbol in $R$ or an element of $R$, then we call $\phi$ a simple order symbol.

Claim. The orderings on $R[y]$ are in one-to-one correspondence with the simple order symbols defined above.

Proof of the claim. Case 1. Suppose that there is an ordering $P$ on $R[y]$ with $\mathfrak{p} = Supp(P) = \{0\}$. Then $\kappa_{R[y]}(\mathfrak{p})$ is isomorphic to $R(y)$. Thus the ordering on $R[y]$ is given by that of $R(y)$. But each ordering on $R(y)$ corresponds to a cut symbol $\phi$ in $R$. Thus we have a simple order symbol $\phi$ corresponding to the order on $R[y]$. 

Conversely, if $\phi$ is a cut symbol in $R$ then it is obvious that we obtain a support zero ordering on $R[y]$.

Case 2. Suppose that there is an ordering $P$ on $R[y]$ with $p = \text{Supp}(P) \neq \{0\}$. Then there exists an irreducible monic polynomial $f \in R[y]$ such that $p = fR[y]$. Since $R$ is real closed, $f$ has degree one or two. If $\deg(f)$ is two, then $R[y]/p$, and consequently $\kappa_{R[y]}(p)$, is isomorphic to $R(\sqrt{-1})$. But there is no ordering on $R(\sqrt{-1})$. Thus $f$ has to be of degree one. Then there exists an element $r \in R$ such that $p = (y - r)R[y]$. Therefore, we have obtained a simple order symbol $r \in R$. Conversely, assume that $r \in R$ is given. Let $p = (y - r)R[y]$. Then $p$ is a maximal (thus a prime) ideal in $R[y]$. Therefore $R[y]/p$ is isomorphic to $R$. Thus we have an ordering on $R[y]$ and the claim is proved.

Suppose that $B \subseteq A$ are commutative rings with identity. Assume that there is an ordering on $A$ (resp., $B$) with positive cone $P$ (resp., $Q$) and support $p$ (resp., $q$). We say that the given ordering on $A$ extends that of $B$ if $Q = P \cap B$. In this case it is obvious that we also have $q = p \cap B$.

**Lemma 3.3.2** Let $A$ be a commutative ring with identity. Moreover, suppose that we have an ordering $P$ on $A$ with support $p$. Let $R$ be the real closure of the residue field $\kappa_A(p)$ of $A/p$ at the corresponding ordering. Then there is a one-to-one correspondence between the simple order symbols of $R$ and the orderings on $A[y]$ which extend the ordering $P$ on $A$.

**Proof.** Suppose that there is an ordering on $A[y]$ with support $q$ which extends that of $A$. Let $R'$ denote the real closure of $\kappa_{A[y]}(q)$ at the associated ordering. Then $p = q \cap A$ and we have the following commutative diagram:

$$
\begin{array}{ccc}
A & \rightarrow & A/p \\
\downarrow & & \downarrow \\
A[y] & \rightarrow & A[y]/q \\
\downarrow & & \downarrow \\
\kappa_A(p) & \rightarrow & \kappa_{A[y]}(q) \\
\downarrow & & \downarrow \\
R & \rightarrow & R'
\end{array}
$$

For brevity, let $F = \kappa_A(p)$, $F' = \kappa_{A[y]}(q)$, and let $F_1$ be the image of $F$ in $F'$. It is easily seen that $F'$ is generated over $F_1$ by $\bar{y}$, where $\bar{y}$ is the image of $y$ in $F'$.
Now there are two cases to consider:

Case 1. \( \bar{y} \) is algebraic over \( F_1 \). Then \([F' : F_1]\) is finite. Thus \( R' = R \), and \( \bar{y} \in R \) is the required simple order symbol.

Case 2. \( \bar{y} \) is transcendental over \( F_1 \). We have \( R(\bar{y}) \subseteq R' \). Therefore, \( R(\bar{y}) \) inherits an ordering from that of \( R' \). Thus we obtain a cut symbol of \( R \).

Conversely, assume that we have a simple order symbol \( \phi \) in \( R \). We want to find an ordering on \( A[y] \) which extends that of \( A \) and corresponds to \( \phi \).

Case i. \( \phi = b \) is an element of \( R \). Then \( b \) is algebraic over \( F \). Let \( f(x) \) be a minimal polynomial of \( b \) over \( F \). We may assume, for later use in the proof, that the coefficients of \( f \) are in \( A/\mathfrak{p} \). Let \( (f) \) denote the principal (prime) ideal in \( F[x] \) generated by \( f \).

Now, we can define an ordering on \( A[y] \). Let \( H = A/\mathfrak{p} \subseteq F \). There is a natural epimorphism \( \Psi : A[y] \to H[b] \subseteq F(b) \subseteq R \) which extends the natural epimorphism \( A \to H \) and \( \Psi(y) = b \). Let \( \mathfrak{q} \) be the kernel of this epimorphism. Thus the ordering on \( R \) induces an ordering on \( A[y] \) (an element of \( A[y] \) belongs to the positive cone of the induced ordering if and only if its image is non-negative in \( R \)). Then \( A[y]/\mathfrak{q} \) is order isomorphic to \( H[b] \). Therefore, the quotient field of \( A[y]/\mathfrak{q} \), i.e., \( \kappa_{A[y]}(\mathfrak{q}) = F' \), is isomorphic to the quotient field of \( H[b] \). But the quotient field of \( H[b] \) is \( F'(b) \). Thus \( F' \) is order isomorphic to \( F(b) \) and we are done.

Case ii. \( \phi \) is a cut symbol in \( R \). Corresponding to \( \phi \), there is an order on \( R(x) \). This order gives an order on \( F[x] \) with support \( \{0\} \) and hence, a support \( \{0\} \) ordering on \( (A/\mathfrak{p})[x] \). Now if we use the ring homomorphism \( A[y] \to (A/\mathfrak{p})[x] \) then we get an ordering on \( A[y] \) with support \( \mathfrak{q} \) which is the kernel of this homomorphism. Therefore, using the order isomorphism between \( A[y]/\mathfrak{q} \) and \( (A/\mathfrak{p})[x] \), we see that \( F' \) is order isomorphic to the field of quotients of \( (A/\mathfrak{p})[x] \) and hence to \( F(x) \). Thus the proof of case ii and that of the lemma is complete. QED

Remarks. (1) \( \mathfrak{q} \) in the above lemma contains \( \mathfrak{q}' = p[y] = \{\sum_{i=0}^{n} a_i y^i : n \) a non-negative integer, \( a_i \in \mathfrak{p}, 0 \leq i \leq n\} \). If \( \phi \) is a cut symbol in \( R \) as in the case ii of proof of the above lemma then \( \mathfrak{q}' = \mathfrak{q} \). On the other hand, if we are not in the case ii, i.e., \( \phi \) is an element of \( R \), then \( \mathfrak{q}' \subseteq \mathfrak{q} \).
(2) If $F, F'$ are as defined in the above lemma and $\phi$ is a simple order symbol in the real closure $R$ of $F$, then using the proof of the above lemma together with Proposition 3.2.3, we see that $F'$ is order isomorphic to $F(\phi)$.

Suppose that $R$ is a real closed field. Fix a proper embedding $R \hookrightarrow \kappa((G))$, where $G$ (resp., $\kappa$) is the value group (resp., the residue field) of the natural valuation on $R$. Moreover, fix an embedding $G \hookrightarrow \mathbb{H}$, where $\mathbb{H}$ is the Hahn product of $I$ copies of $\mathbb{R}$ and $I$ is the rank of $G$. Suppose that $P$ is an ordering on $A = R[y_1, \ldots, y_n]$. We want to define recursively an order symbol $(\phi_1, \ldots, \phi_n)$ which corresponds to $P$. The ordering $P$ on $A$ restricts, for each $1 \leq j \leq n$, to an ordering on $A_j = R[y_1, \cdots, y_j]$. Therefore by Lemma 3.3.2 and Proposition 3.2.3, for each $j = 0, \cdots, n-1$ we have the following commutative diagram:

\[
\begin{array}{c}
A_j \rightarrow R_j \hookrightarrow \kappa_j((G_j)) \hookrightarrow \mathbb{R}((\mathbb{H}_j)) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
A_{j+1} \rightarrow R_{j+1} \hookrightarrow \kappa_{j+1}((G_{j+1})) \hookrightarrow \mathbb{R}((\mathbb{H}_{j+1}))
\end{array}
\]

where $R_j$ is the real closure of the field of quotients of the residue field of $A_j$ corresponding to the restricted ordering on $A_j$, $G_j$ (resp., $\kappa_j$) the value group (resp., residue field) of the natural valuation on $R_j$, and $\mathbb{H}_j = \mathcal{H}_{i \in I_j} \mathbb{R}$ in which $I_j$ is the value set of the natural valuation on $G_j$ (note that $A_0 = R_0 = R, G_0 = G$, and $\kappa_0 = \kappa$). Moreover, by Proposition 3.2.3, we obtain a simple order symbol $\phi_j \in \mathbb{R}((\mathbb{H}_j))$ for each $j = 1, \cdots, n$. Note that the way $I_j$ is defined implies that either $I_{j+1} = I_j$ or $I_{j+1}$ has just one more element than $I_j$.

We are interested to make arrangements so that all the $\phi_j$'s, $1 \leq j \leq n$, belong canonically to a power series field $K$ which does not depend on a particular ordering $P$ (also see section 5 in [KMZ]). Let $J_0 = I$. For each $1 \leq j \leq n$ define $J_j$ inductively by $J_j = \hat{J}_{j-1}$ (for the definition of $\hat{J}_{j-1}$ see Proposition 2.4.1). Let $\mathbb{K}_j = \mathcal{H}_{i \in J_j} \mathbb{R}, 0 \leq j \leq n$ and $\mathcal{K} = \mathbb{K}_n$. Now in order to define $\phi_1, \cdots, \phi_n$ in a canonical way, we try to prove that there exists a canonical embedding $\mathbb{R}((\mathbb{H}_j)) \hookrightarrow \mathbb{R}((\mathbb{K}_j))$ for each $0 \leq j \leq n$. Obviously it is enough to show that for each $j, 0 \leq j \leq n$, there
exists a canonical embedding \( E_j \hookrightarrow K_j \). Hence we just need to prove that there exists a canonical embedding \( I_j \hookrightarrow J_j \) for each \( j, 0 \leq j \leq n \). Before showing this, we note that with its proof we will be done. Actually, we would have then proved that each \( \phi_j, 1 \leq i \leq n \) belongs canonically to \( K_j \) and hence to \( K_n \). Therefore, it is enough to prove:

Claim. For each \( j, 0 \leq j \leq n \) there exists a canonical embedding \( I_j \hookrightarrow J_j \).

Proof of the claim. Since \( I_0 = J_0 = I \), the claim is true when \( j = 0 \). Assume the claim for \( j, 0 \leq j < n \), i.e., assume that there exists a canonical embedding of \( I_j \) into \( J_j \). We need to find a canonical embedding of \( I_{j+1} \) into \( J_{j+1} \). It is enough to find the image of the element \( s \) in \( I_{j+1} = I_j \cup \{ s \} \). This means that \( s \) defines a cut \( \alpha \) in \( I_j \). Therefore, there is an element \( t \in J_{j+1} \) filling \( \alpha \). Then there exists a smallest integer \( k, 0 \leq k \leq j + 1 \) so that there is \( u \in J_k \setminus I_j \) which fills \( \alpha \).

Subclaim. There is a unique \( u \in J_k \) which fills \( \alpha \).

Proof of the subclaim. If not then there are \( u_1, u_2 \in J_k, u_1 < u_2 \) which fill \( J_k \). It is easily seen that \( k > 1 \). Since \( J_k = \hat{J}_{k-1} \), there exists an element \( v \in J_{k-1} \) such that \( u_1 \leq v \leq u_2 \) and \( v \) fills \( \alpha \). But this contradicts our assumption on \( k \) and proves the subclaim.

Now we take the image of \( s \) to be \( u \in J_k \subseteq J_{j+1} \). Therefore, we have a well-defined map \( \psi_{j+1} \) from \( I_{j+1} \) into \( J_{j+1} \). It is not difficult to see that this map is also order preserving. Thus \( \psi_{j+1} \) defines an embedding of \( I_{j+1} \) into \( J_{j+1} \) and we are done.

**Proposition 3.3.3** There is a one-to-one correspondence between the orderings on \( R[y_1, \ldots, y_n] \) and the order symbols \( (\phi_1, \ldots, \phi_n) \) defined on page 58.

**Proof.** Suppose that \( P \) is an ordering on \( A = R[y_1, \ldots, y_n] \). Then as we saw above, there exists an order symbol \( (\phi_1, \ldots, \phi_n) \) corresponding to it. For the converse, let's see what we mean by the given order symbol \( (\phi_1, \ldots, \phi_n) \). We mean, actually, that for each \( 1 \leq i \leq n \) the simple order symbol \( \phi_i \) corresponds, as in Lemma 3.3.2, to an ordering on \( A_i = A_{i-1}[y_i] \) that extends that of \( A_{i-1} \) (\( A_0 = R \) by convention). So the result is obvious. QED
For further discussion in the next section, we make some more conventions. Let $R_0 = R$, $\kappa_0 = \kappa$, and $G_0 = G$. If we denote the residue field (resp., value group) of $R_j$, $0 \leq j \leq n$ by $\kappa_j$ (resp., $G_j$), then $\kappa_{j+1} = \kappa_j^\alpha$ (resp., $G_{j+1} = G_j^\alpha$), where $\alpha$ is either a cut symbol or an element of $R_j$ as used in the $j$-th step of the construction of the order symbol $(\phi_1, ..., \phi_n)$. In this case we also define $\phi_{j+1} = \phi^\alpha$. We note that if $\alpha = p$ is an element of $R_j$ then $\kappa_j^p = \kappa_j$, $G_j^p = G_j$. Finally, let $p_0 = \{0\}$, and for each $l$, $l \geq 0$, denote the support of the ordering on $R[y_1, ..., y_l]$ by $p_l$. Then we have $\{0\} \subseteq p_1 \subseteq \ldots \subseteq p_n$.

Remarks. (1) The proof of Lemma 3.3.2 gives us a constructive way to find the order symbol which corresponds to a given ordering; conversely, it can also be used to find the ordering which corresponds to an order symbol.

(2) If $\mathbb{R} \subseteq R$, then $\kappa_i = \mathbb{R}$, for all $i \geq 0$.

(3) Suppose that $R$ is a real closed field and $(\phi_1, ..., \phi_n)$ is an order symbol. Then for each integer $1 \leq i \leq n$, the field of quotients of $R[y_1, \dots, y_l]/p_i$ is isomorphic to the ordered field $R(\phi_1, ..., \phi_i)$. Moreover, for each $1 \leq i \leq n$ the field of quotients of $R[y_1, \cdots, y_l]/p_i$ is embeddable in $\kappa_i((G_i))$.

3.4 Defect of a cut

Suppose $R$ is a real closed field and $\alpha$ is an $R$-cut symbol. Moreover, suppose that we are in the situation of Proposition 3.2.3, i.e., $\alpha$ is a cut symbol corresponding to a cut in $R$, $G$ and $G^\alpha$ (resp., $\kappa$ and $\kappa^\alpha$) the value groups (resp., residue fields) of the natural valuation on $R$ and $R^\alpha$ respectively ($R^\alpha$ defined as in Proposition 3.2.3).

Recalling the remark following the same Proposition, we have:

**Definition 3.4.1** (1) We say that $\alpha$ is immediate transcendental if $G^\alpha = G$ and $\kappa^\alpha = \kappa$. This occurs if $\alpha = (S, p)$ and all the coefficients of the terms in $p$ are in $\kappa$.

(2) We say that $\alpha$ is residue transcendental if $G^\alpha = G$ and $\text{trdeg}(\kappa^\alpha : \kappa) = 1$. This occurs if $\alpha = (S, p)$ and the coefficient of the last non-zero term of $p$ does not belong to $\kappa$. 
(3) We say \( \alpha \) is value transcendental if \( \dim_\mathbb{Q}(G^\alpha / G) = 1 \) and \( \kappa^\alpha = \kappa \). This occurs if \( \alpha = (S, p, +) \) or \( \alpha = (S, p, -) \).

Note: It is obvious that \( \alpha \) is an immediate transcendental cut if and only if \( R^\alpha \) is an immediate extension of \( R \).

Let \( P \) be an ordering on the ring \( R[y_1, \ldots, y_n] \). Moreover, let \( R_j, G_j, \kappa_j, 0 \leq j \leq n \) be as defined on pages 58 and 59. The definition of \( R_j \) as well as the construction in Lemma 3.3.2 show that for each \( j, 0 \leq j \leq n - 1 \), we have that \( \text{trdeg}(R_{j+1} : R_j) \) is either 0 or 1. It is 0 if and only if the simple order symbol \( \phi_{j+1} = \phi^\alpha \) is an element of \( R_j \). Now depending on the type of the simple order symbol \( \phi^\alpha \), there are two cases to consider. (1) If \( \phi^\alpha = p \) is an element of \( R_j \), then \( R_{j+1} = R_j, \kappa_{j+1} = \kappa_j, \) and \( G_{j+1} = G_j \). (2) If \( \alpha \) is a cut symbol then \( \text{trdeg}(R_{j+1} : R_j) = 1 \). In either case, for \( 0 \leq j \leq n - 1 \) one has

\[
\text{trdeg}(R_{j+1} : R_j) \geq \dim_\mathbb{Q}(G_{j+1}/G_j) + \text{trdeg}(\kappa_{j+1} : \kappa_j) \tag{3.4.10}
\]

Moreover, we have strict inequality if and only if \( \alpha \) is an immediate transcendental cut. Now let \( j \) in (3.4.10) run from 0 through \( n - 1 \) and sum up the corresponding inequalities to obtain

\[
\text{trdeg}(R_n : R_0) \geq \dim_\mathbb{Q}(G_n/G) + \text{trdeg}(\kappa_n : \kappa) \tag{3.4.11}
\]

This is a well-known inequality in valuation theory which is sometimes referred to as the Abhyankar Inequality. \( \dim_\mathbb{Q}(G_n/G) \) (resp., \( \text{trdeg}(\kappa_n : \kappa) \)) tells us the number of times a value transcendental cut (resp., a residue transcendental cut) is used in the recursive construction. The difference

\[
d = \text{trdeg}(R_n : R_0) - \dim_\mathbb{Q}(G_n/G) - \text{trdeg}(\kappa_n : \kappa)
\]

which we refer to as the defect of the ordering, tells us the number of times an immediate transcendental cut is used in the recursive construction. Thus the defect is 0 if and only if whenever there is an increase in \( \text{trdeg}(R_{j+1} : R_0) \) for some \( 0 \leq j < n \), then there is an increase in either the \( \mathbb{Q} \)-dimension of \( G_{j+1}/G \), or in \( \text{trdeg}(\kappa_{j+1} : \kappa) \).
If \( \text{trdeg}(R_n : R_0) = m, 0 \leq m \leq n \), then in our recursive construction of the \( n \) simple order symbols, there have been \( m \) cut symbols. Therefore, \( m \) counts the number of \( j, 0 \leq j < n \) such that \( p_{j+1} = p_j[y_{j+1}] \), where \( p_j \) is the support of the restricted ordering on \( A_j = R[y_1, ..., y_j] \) (see part (1) of the remarks after the proof of Lemma 3.3.2). Moreover, the integer \( n - m \) counts the number of times that we have used field elements. Thus \( n - \text{trdeg}(R_n : R_0) \) is equal to the number of prime ideals \( p_j \) in the sequence \( \{0\} \subseteq p_1 \subseteq ... \subseteq p_n \) such that \( p_{j+1} \supseteq p_j[y_{j+1}] \). For example, if \( p_n = \{0\} \) then \( \text{trdeg}(R_n : R_0) = n \).
Chapter 4

Description of Value Groups and Real Places

Suppose that $F$ is an ordered field with the real closure $R \subseteq \kappa((G))$, where $G$ and $\kappa$ are respectively the value group and the residue field of the natural valuation on $R$. Moreover, suppose that $\phi$ is a simple order symbol of $R$. Our first aim in this chapter is to find the relation between the value group of $F(\phi)$ and that of $F$. This is easy to find in some special cases. For example, when $F = R$; or when $F = \mathbb{R}(x)$, where $R \subseteq \mathcal{P}_R(x)$ ($\mathcal{P}_R(x)$ is the field of Puiseux series with coefficients in $\mathbb{R}$), and $\phi$ has just a few terms. In these examples we find out that the value group of $F(\phi)$ is generated over that of $F$ by the exponents $\gamma$ such that $x^7$ appears in $\phi$. Our main attempt is to show that this result holds in general (see Theorem 4.2.1). Then we will be able to conclude that if $(\phi_1, \cdots, \phi_n)$ is an order symbol which corresponds to an ordering on $F[y_1, \cdots, y_n]$ then the value group of $F(\phi_1, \cdots, \phi_n)$ is generated over that of $F$ by all the exponents $\gamma$ such that $x^7$ appears in some $\phi_i, 1 \leq i \leq n$ (Corollary 4.2.2). This result enables us to characterize all the orderings on $R[y_1, \cdots, y_n]$ which have the same associated real place as the one which has the order symbol $(\phi_1, \cdots, \phi_n)$ (Theorem 4.3.2). The final goal in this chapter is to characterize all possible value groups of $F(\phi)$ when the value group $V$ of $F$ is known (see Theorem 4.4.1, also Theorem 4.4.2). In our procedure we need a result which is closely related to the results of Kaplansky on maximally complete fields [Ka]. So we first obtain this result (Theorem 4.1.7).
4.1 Kaplansky's Theorem

In this section we will quote some of the definitions and results from Kaplansky's paper. Assume that $\Gamma$ is an ordered abelian group, $K$ a field, and $v : K \to \Gamma \cup \{\infty\}$ a valuation defined on $K$. Also assume that $\{a_{\rho}\}$ is a pseudo-convergent sequence in $K$.

**Theorem and Definition 4.1.1** Suppose $\{a_{\rho}\}$ is a pseudo-convergent sequence in $K$ and, $f$ is a polynomial in $K[x]$. Then we have either

$$v(f(a_{\rho})) = v(f(a_{\sigma})) \quad \text{for sufficiently large } \rho \text{ and } \sigma \quad (4.1.12)$$

or,

$$v(f(a_{\rho})) < v(f(a_{\sigma})) \quad \text{for sufficiently large } \rho \text{ and } \sigma, \rho < \sigma \quad (4.1.13)$$

If condition $(4.1.12)$ holds for every $f \in K[x]$, then we say $\{a_{\rho}\}$ is of transcendental type. On the other hand, if condition $(4.1.13)$ holds for at least one $f \in K[x]$, then $\{a_{\rho}\}$ is called of algebraic type.

**Remark 1.** Suppose $\{a_{\rho}\}$ is of algebraic type. Among all polynomials in $K[x]$ with degree $\geq 1$ which satisfy $(4.1.13)$, one can pick up one $f$ with the smallest degree. This polynomial $f$ is called a *minimal polynomial* of the sequence $\{a_{\rho}\}$.

**Remark 2.** Recall that a pseudo-limit of a pseudo-convergent sequence is also referred to simply as a "limit".

Now we quote some well known results.

**Lemma 4.1.2 (Ostrowski [Os])** Let $\Gamma$ be an ordered abelian group. Choose the elements $\beta_1, \ldots, \beta_m$ in $\Gamma$. Let $\{\gamma_{\rho}\}$ be a well-ordered monotone increasing set of elements of $\Gamma$, without a last element. Furthermore, let $t_1, \ldots, t_m$ be distinct positive integers. Then there exists an ordinal $\mu$ and an integer $k \ (1 \leq k \leq m)$ such that

$$\beta_i + t_i \gamma_{\rho} > \beta_k + t_k \gamma_{\rho} \quad (4.1.14)$$

for all $i \neq k$ and $\rho > \mu$. 
Theorem 4.1.3 (Kaplansky [Ka]) Suppose that \( \{a_p\} \) is a pseudo-convergent sequence of transcendental type in \( K \), without a limit in \( K \). Then there exists an immediate transcendental extension \( K(z) \) of \( K \) such that \( z \) is a limit of \( \{a_p\} \). The extension of the valuation \( v \) to \( K(z) \) can be specifically defined as follows: for any polynomial \( h(z) \) with coefficients in \( K \) we define \( v(h(z)) \) to be the fixed value which \( v(h(a_p)) \) ultimately assumes.

Conversely, if \( K(u) \) is any extension of \( K \), with a valuation which is an extension of \( v \) such that \( u \) is a limit of \( \{a_p\} \), then \( u \mapsto z \) defines a value preserving isomorphism over \( K \) between \( K(u) \) and \( K(z) \).

Remark. Note that in the second assertion of Theorem 4.1.3, \( u \) must necessarily be transcendental over \( K \).

Theorem 4.1.4 (Kaplansky [Ka]) Suppose that \( \{a_p\} \) is a pseudo-convergent sequence of algebraic type in \( K \), without a limit in \( K \). Then there exists an immediate algebraic extension \( K(z) \) of \( K \) such that \( z \) is a limit of \( \{a_p\} \). The valuation of \( K(z) \) can be specifically defined as follows: Let \( q(x) \) be a minimal polynomial of \( \{a_p\} \), and let \( z \) be a root of \( q(x) = 0 \). For any polynomial \( f(z) \) of degree less than \( n = \text{deg}(q(x)) \), define \( v(f(z)) \) to be the fixed value which \( v(f(a_p)) \) ultimately assumes.

Conversely, if \( u \) is a root of \( q(x) = 0 \), and if \( K(u) \) has a valuation which is an extension of \( v \) such that \( u \) is a limit of \( \{a_p\} \), then \( u \mapsto z \) defines a value preserving isomorphism over \( K \) between \( K(u) \) and \( K(z) \).

We still need two more results as stated in Kaplansky's paper [Ka]. But let's first introduce some notations. Suppose that \( \{a_p\} \) is of algebraic type in \( K \) and, \( q(x) \in K[x] \) is a minimal polynomial of \( \{a_p\} \). Let \( q_i(x) \) be defined according to the expansion \( q(x + y) = \sum_{i=0}^{n} q_i(x)y^i \), where \( n \) is the degree of \( q \). We can choose \( \rho \) so large that \( v(q_1(a_p)), v(q_2(a_p)), \ldots \) do not depend on \( \rho \). In this case, we denote \( v(q_i(a_p)) \) by \( \beta_i \). If \( x \) is a limit of the sequence \( \{a_p\} \) then let \( \gamma_p \) denote \( v(x - a_p) \). In what follows in this section, \( p \) is defined as follows: If the residue field of \( K \) is not 0 we take \( p \) to be the characteristic of this residue field, otherwise we take \( p = 1 \).
Lemma 4.1.5 Let \( i = p^r \), \( j = p^s \) with \( r > 1 \). Moreover, suppose that \((r, p) = 1\), then
\[
\beta_i + i\gamma < \beta_j + j\gamma
\] (4.1.15)
for sufficiently large \( \rho \).

Proof. This is just Lemma 7 in [Ka].

The following lemma is an immediate consequence of Lemma 8 and Lemma 9 in [Ka].

Lemma 4.1.6 Suppose that \( \{a_\rho\} \) is a pseudo-convergent sequence of algebraic type having \( q \) as its minimal polynomial. If \( x \) is a limit of \( \{a_\rho\} \), then there exists an integer \( h \), which is a power of \( p \), such that for all \( \rho \):
\[
\nu(q(x)) > \beta_h + h\gamma
\] (4.1.16)

Definition 4.1.1 Let \( \kappa \) denote the residue field of the valuation \( \nu \) on \( K \), and let \( p \) be the characteristic of \( \kappa \) if this characteristic is not 0, and \( p = 1 \) otherwise. We say that \((K, \nu)\) satisfies Kaplansky's "hypothesis A" if the following two conditions hold:

(1) Any equation of the form
\[
x^{p^n} + a_1x^{p^{n-1}} + \ldots + a_{n-1}x^p + a_nx + a_{n+1} = 0
\] (4.1.17)
with coefficients in \( \kappa \) has a root in \( \kappa \).

(2) The value group \( \Gamma \) satisfies \( \Gamma = p\Gamma \).

Remark. Note that when the characteristic of \( \kappa \) is 0, i.e., when \( p = 1 \), then hypothesis A automatically holds.

Theorem 4.1.7 Let \((M, \nu)\) be a maximally complete field satisfying hypothesis A. Moreover, let \( L \) be a subfield of \( M \).

(1) Suppose that \( \{a_\rho\} \) is a pseudo-convergent sequence in \( L \) which is algebraic in \( L \) with a minimal polynomial \( f \) over \( L \). Then \( \{a_\rho\} \) has a pseudo-limit \( a \in M \) such that \( f(a) = 0 \).
(2) Suppose that \( L \) also satisfies hypothesis A and \( M' \) is an immediate extension of \( L \). Then there exists an \( L \)-embedding \( \tau : M' \to M \) preserving the valuation.

**Proof.** Before starting the proof, we note that the argument in the proof of part (1) is similar to the one given in [Ka], but we have modified it a little to meet our assumptions. In fact, in the mentioned reference, \( M \) is required to be an immediate extension of \( L \), and \( L \) is required to satisfy hypothesis A. Of course, this implies that \( M \) also satisfies hypothesis A.

Let \( n \) denote the degree of \( f \). To prove part (1), we need the following:

**Claim 1.** If for some limit \( t \in M \) of \( \{a_{\rho}\} \) we have \( v(f(t)) = \alpha \), then we can obtain a better approximation \( v(f(t^*)) > \alpha \), where \( t^* \in M \) is a limit of \( \{a_{\rho}\} \) such that

\[
v(t^* - t) = \max_{i = \rho^*} (\alpha - \beta_i) / i,
\]

where \( \beta_i \) is defined as on page 64 and \( i \) ranges as indicated over the powers of \( p \) (\( 1 \leq i \leq n \)) if \( p \neq 1 \), and \( i = 1 \) if \( p = 1 \).

**Proof of the claim 1.** Define \( \delta \) as follows:

\[
\delta = \max (\alpha - \beta_i) / i,
\]

the range of \( i \) being the powers of \( p \), as it will be throughout the proof (see the definition of \( p \) on pages 64 and 65 and note that if \( p = 1 \) then \( \delta = \alpha - \beta_1 \)). Now by Lemma 4.1.6 there exists a power \( h \) of \( p \) such that \( \alpha = v(f(y)) > \beta_h + h\gamma_p \). Thus \((\alpha - \beta_h) / h > \gamma_p \). Therefore,

\[
\delta > \gamma_p \tag{4.1.19}
\]

for all \( \rho \). The value group \( \Gamma \) satisfies condition (2) of hypothesis A. So there exists \( k \in M \) such that \( v(k) = \delta \). Letting \( z \in M \) and using Taylor's theorem [Al], we obtain

\[
\frac{f(t + kz)}{f(t)} = \sum_{j=0}^{n} k^j z^j \frac{f_j(t)}{f(t)} \tag{4.1.20}
\]

In the polynomial (4.1.20), the coefficient of \( z^j \) has the value \( j\delta + \beta_j - \alpha \). If \( j \) is a power of \( p \), then we have

\[
j\delta + \beta_j - \alpha \geq 0 \tag{4.1.21}
\]
by (4.1.18) and (4.1.19). On the other hand, if \( j \) is not a power of \( p \) then let \( i \) be the highest power of \( p \) dividing \( j \). Then Lemma 4.1.5 tells us that \( j\gamma_p + \beta_j - \alpha > i\gamma_p + \beta_i - \alpha \), for sufficiently large \( \rho \). Therefore, by (4.1.21) we have

\[
 j \delta + \beta_j - \alpha > 0
\]  

(4.1.22)

Taking these facts together, we observe that if we replace each coefficient of the polynomial in (4.1.20) by its residue class, then we obtain a polynomial, say \( \bar{F}(z) \), with coefficients in the residue field \( \kappa \) of \( K \) and of the same type as used in part (1) of hypothesis A (if \( p = 1 \) then \( \bar{F}(z) = z + 1 \)). Hence, \( \kappa \) contains a root \( \bar{z}_1 \) of \( \bar{F}(z) = 0 \). If \( z_1 \in M \) is any representative of the residue class \( \bar{z}_1 \), then we have

\[
v[f(t + kz_1)/f(t)] > 0 \text{ or } v(f(t + kz_1)) > \alpha.
\]

Also, by the choice of \( k \), \( v(kz_1) = \delta \). By (4.1.19), \( kz_1 \) is in the breadth of \( \{a_\rho\} \), whence by Lemma 1.3.3, \( t + kz_1 \) is a limit of \( \{a_\rho\} \). Now with the choice of \( t^* = t + kz_1 \) the proof of the claim is complete.

**Claim 2.** There exists a transfinite set of elements \( \{t_\mu\} \) of \( M \) such that

1. each \( t_\mu \) is a limit of \( \{a_\rho\} \);
2. if \( v(f(t_\mu)) = \alpha_\mu \), then \( \alpha_\mu < \alpha_\nu \) for \( \mu < \nu \);
3. \( v(t_\nu - t_\mu) = \max(\alpha_\mu - \beta_i)/i \) (\( \mu < \nu \)), the range of \( i \) again being the powers of \( p \).

Let us first observe that with claim 2 the proof of the first part of the theorem is completed. In fact, the cardinality of the transfinite set in the claim 2 can not exceed that of \( M \). Therefore, this set should end up with an element \( a = t_\zeta \in M \), which is a limit of \( \{a_\rho\} \), and moreover \( v(f(t_\zeta)) = \infty \). Then \( f(a) = 0 \).

**Proof of claim 2.** The proof is independent of hypothesis A and can be found in the proof of Theorem 5 in [Ka].

(2) Without loss of generality, we may assume that \( M' \) is a maximal immediate extension of \( L \). Let \( \mathcal{L} \) be the set of all pairs \( (L^*, \tau_{L^*}) \), where \( L^* \) is an extension of \( L \) such that \( L^* \subseteq M' \) and \( \tau_{L^*} : L^* \rightarrow M \) is an \( L \)-embedding which preserves the valuation. This set is non-empty and moreover, it is ordered in the usual way: If \( (L^*_1, \tau_{L^*_1}) \in \mathcal{L}, i = 1, 2 \), then we define \( (L^*_1, \tau_{L^*_1}) \prec (L^*_2, \tau_{L^*_2}) \) if and only if \( L^*_1 \subseteq L^*_2 \) and \( \tau_{L^*_2} \) is an extension of \( \tau_{L^*_1} \). It is obvious that every chain in \( \mathcal{L} \) has an upper
bound in \( L \). Therefore, by Zorn's Lemma, \( L \) has a maximal element \((M^*, \tau_{M^*})\). Thus the proof is complete if we show that \( M^* = M' \). Assume, on the contrary, that \( M^* \not\subseteq M' \). Then, by Theorem 1.4.5, there exists a pseudo-convergent sequence \( \{a_i\} \) in \( M^* \) without any limit in \( M^* \) but having a limit in \( M' \). So the image of \( \{a_i\} \) in \( M \) under \( \tau_{M^*} \) is also pseudo-convergent.

**Case 1.** \( \{a_i\} \) is of transcendental type in \( M^* \). Let \( a \in M' \) be a limit of \( \{a_i\} \). \( M^*(a) \) is an immediate extension of \( L \). Therefore, \( a \) is transcendental over \( M^* \) (see the remark after Theorem 4.1.3). Moreover, \( \{\tau_{M^*}(a_i)\} \) is of transcendental type in \( \tau_{M^*}(M^*) \subseteq M \) and hence has a limit \( c \in M \) which is transcendental over \( \tau_{M^*}(M^*) \). Therefore, we can extend \( \tau_{M^*} \) to an \( L \)-embedding \( \tau^*: M^*(a) \hookrightarrow M \) by sending \( a \) to \( c \). As \( M^*(a) \subseteq M' \) is a valued field, \( \tau^* \) induces a valuation on its image \( \tau^*(M^*(a)) \).

Now by Theorem 4.1.3, this induced valuation on \( \tau^*(M^*(a)) \) is the same as the valuation \( v \) restricted to this subfield of \( M \). Thus \( \tau_{M^*} = \tau^* \) also preserves the valuation. But this contradicts the maximality of \((M^*, \tau_{M^*})\).

**Case 2.** \( \{a_i\} \) is of algebraic type in \( M^* \). \( M' \) satisfies hypothesis A; moreover, it is assumed to be a maximally complete field. On the other hand, we have \( M^* \subseteq M' \). Therefore by part (1) of our theorem, \( \{a_i\} \) has a limit \( a \in M' \) such that \( f(a) = 0 \), where \( f = \sum_{i=0}^{n} c_i x^i \in M^*[x] \) is a minimal polynomial of \( \{a_i\} \). It is easily seen that \( f \) is irreducible in \( M^*[x] \). Corresponding to \( \{a_i\} \) and \( f \) respectively, we obtain via \( \tau_{M^*} \), the pseudo-convergent sequence \( \{\tau_{M^*}(a_i)\} \) in \( \tau_{M^*}(M^*) \) and the polynomial \( g = \sum_{i=0}^{n} \tau_{M^*}(c_i) X^i \in \tau_{M^*}(M^*)[X] \). It is obvious that \( \{\tau_{M^*}(a_i)\} \) is of algebraic type and has \( g \) as a minimal polynomial. Then by Theorem 4.1.7, there exists \( b \in M \) such that \( g(b) = 0 \). The polynomial \( g \) is irreducible in \( \tau_{M^*}(M^*) \) and has the same degree as \( f \). Therefore, by sending \( a \) to \( b \), we obtain an \( L \)-embedding \( \tau^*: M^*(a) \hookrightarrow M \).

Then employing a similar method as in case 1 and using Theorem 4.1.4, we see that \( \tau_{M^*} = \tau^* \) also preserves the valuation. But this contradicts the maximality of \((M^*, \tau_{M^*})\). Thus \( M^* = M' \) and we are done. QED
4.2 Description of value groups

We are going to do the inductive step of finding the value group of an ordering on \( R[y_1, \ldots, y_n] \). We return to the set-up considered in section 3.2. Moreover, we assume that \( F \) is an ordered field having \( R \) as its real closure. Also, we allow \( \alpha \) to be either an \( R \)-cut symbol or an element of \( R \). If \( \alpha \) is an \( R \)-cut symbol we define \( G^\alpha, \kappa^\alpha, \phi^\alpha, \) and \( R^\alpha \) as in Proposition 3.2.3. If \( \alpha = p \) is an element of \( R \), we define \( G^\alpha = G, \kappa^\alpha = \kappa, \phi^\alpha = p, \) and \( R^\alpha = R \). Denote by \( V \) the value group of \( F \) and by \( V^\alpha \) the value group of \( F(\phi^\alpha) \).

**Theorem 4.2.1** Suppose \( F \subseteq \kappa((V)) \). Then the exponents \( \gamma \) such that \( x^\gamma \) appears in the power series expansion of \( \phi^\alpha \) are in \( V^\alpha \). Equivalently, \( F(\phi^\alpha) \subseteq \kappa^\alpha((V^\alpha)) \). Equivalently, \( V^\alpha \) is generated over \( V \) by the exponents \( \gamma \) such that \( x^\gamma \) appears in the power series expansion of \( \phi^\alpha \).

**Remarks.** (1) Let \( W \) be the abelian group generated over \( V \) by the exponents appearing in \( \phi^\alpha \). The first assertion in the above theorem states that every exponent appearing in \( \phi^\alpha \) is an element of \( V^\alpha \). This is obviously equivalent to \( W \subseteq V^\alpha \). The third assertion in the above theorem states that \( W = V^\alpha \). The fact that \( W \subseteq V^\alpha \) and \( W = V^\alpha \) are equivalent can be seen from the inclusion \( F(\phi^\alpha) \subseteq \mathbb{R}((W)) \) which is true in general.

(2) \( \kappa \) is the residue field of the real closure \( R \) of \( F \). Moreover, \( \kappa^\alpha \) is the residue field of the real closure \( R^\alpha \) of \( F(\phi^\alpha) \). By construction, the coefficients of \( \phi^\alpha \) are in \( \kappa^\alpha \). On the other hand, \( F \) and \( F(\phi^\alpha) \) also have residue fields and these are generally smaller. We denote by \( \kappa_0 \) (resp., \( \kappa_3 \)) the residue field of \( F \) (resp., \( F(\phi^\alpha) \)). It is not in general true that the coefficients of \( \phi^\alpha \) are in \( \kappa_3 \), even if we assume \( F \subseteq \kappa_0((V)) \). Although this is a shortcoming of the theory, it does not come into play if we restrict ourselves to the case where \( \mathbb{R} \subseteq F \). Note. In section 3.3, \( \kappa_0 \) was defined to be the residue field of \( R_0 = R \). In this chapter, unless otherwise stated, \( \kappa_0 \) will be defined as above.

Before starting the proof to the theorem, let us give an example which illustrates part (2) of the above remark.
Example 4.2.1 Let \( k \) denote the field of real algebraic numbers (i.e., \( k = \) the real closure of \( \mathbb{Q} \) in \( \mathbb{R} \)). In this example, for any field \( L \), we denote by \( \kappa_L \) its residue field.  

Case 1. Immediate transcendental case. Consider the ordered field \( F = \mathbb{Q}(x) \subseteq k((\mathbb{Q})) \). Let \( R \subseteq k((\mathbb{Q})) \) be the real closure of \( F \), and let \( \alpha \) be an \( R \)-cut symbol defined by \( \phi^\alpha = \sqrt{2x^{1/2} + \sum_{n=1}^{\infty} x^n/n!} \). \( \phi^\alpha \not\in R \). Reason. We know that \( \sqrt{2x^{1/2}} \in R \). So if \( \phi^\alpha \in R \) then \( \sum_{n=1}^{\infty} x^n/n! \in R \). But \( \sum_{n=1}^{\infty} x^n/n! \) is transcendental over \( R \) (because \( e^x = \sum_{n=0}^{\infty} x^n/n! \) is transcendental over \( \mathbb{Q}(x) \)). Therefore, \( \phi^\alpha \not\in R \). Then \( \phi^\alpha \) corresponds to an immediate transcendental cut in \( R \) (see Example 3.2.1). We want to show that \( \sqrt{2} \) does not belong to the residue field of \( F(\phi^\alpha) \). It is enough to show that the residue field of \( L_1 = L(\sqrt{2x^{1/2}}) \), where \( L = F(\sum_{n=1}^{\infty} x^n/n!) \), is equal to \( \mathbb{Q} \). Since \( 1/2 \in v(L_1) \), we have \([v(L_1) : v(L)] = 2 \). On the other hand, as \( \sqrt{2x^{1/2}} \) is the root of the equation \( X^2 - 2x = 0 \), we have that \([L_1 : L] = 2 \). Therefore, by using the fundamental inequality, we have \([\kappa_{L_1} : \kappa_L] = 1 \). Thus \( \kappa_{L_1} = \kappa_L = \mathbb{Q} \).

Remark. A similar argument as above works for the algebraic case too. Actually, one just needs to show for \( \phi = \sqrt{2x^{1/2}} \) - which is algebraic over the same ordered field \( F \) as defined above - we have that \( \sqrt{2} \) does not belong to the residue field of \( F(\phi) \). An obvious modification of the reasoning in case 1 (by choosing \( L \) to be just \( F \)) gives us this result.

Case 2. Residue transcendental case. Suppose \( F, R \) are defined as in case 1. Let \( \alpha \) be an \( R \)-cut symbol defined by \( \phi = \phi^\alpha = x + \pi x^{3/2} \). We want to show that \( \pi \) does not belong to the residue field of \( F(\phi) \). Let \( L = S(x) \), where \( S = \mathbb{Q}(\pi^2) \). Claim. \( \kappa_{L(\phi)} = S \). Proof of the claim. As \( (\phi - x)/x = \pi x^{1/2} \), we have \([v(L(\phi)) : v(L)] = 2 \). On the other hand, since \( (\phi - x)^2 = \pi^2 x^2 \), we have \([L(\phi) : L] = 2 \). Therefore, by the fundamental inequality we obtain \([\kappa_{L(\phi)} : \kappa_L] = 1 \). Thus \( \kappa_{L(\phi)} = \kappa_L = \mathbb{S} \) and the claim is proved. Now, as \( \pi \) is transcendental over \( \mathbb{Q} \), we have \( \pi \not\in \mathbb{Q}(\pi^2) = S = \kappa_{L(\phi)} \). But \( \kappa_{F(\phi)} \subseteq \kappa_{L(\phi)} \). Hence \( \pi \not\in \kappa_{F(\phi)} \) and we are done.

Case 3. Value transcendental case. Suppose \( F, R \) are defined as in case 1. Let \( \alpha \) be an \( R \)-cut symbol defined by \( \phi = \phi^\alpha = \sqrt{2x^{1/2}} + x^\tau \). We will show that \( \sqrt{2} \) does not belong to the residue field of \( F(\phi) \). Let \( \tau \) be defined by \( \tau = (\phi^2 - 2x)/\phi \). An easy calculation shows that \( v(\tau) = \pi \). Therefore, \( v(F(\tau)) = \mathbb{Z} \oplus \pi \mathbb{Z} \). On the other hand,
$1/2 \in v(F(\phi))$; therefore, $v(F(\phi)) = (1/2)\mathbb{Z} \oplus \pi\mathbb{Z}$. Hence, $[v(F(\phi)) : v(F(\tau))] = 2$. 

Since $\phi$ satisfies: $\phi^2 - \tau\phi - 2\pi = 0$, we have $[F(\phi) : F(\tau)] = 2$. So by using the fundamental inequality, we obtain $[\kappa_{F(\phi)} : \kappa_{F(\tau)}] = 1$. That is $\kappa_{F(\phi)} = \kappa_{F(\tau)}$. On the other hand it is not difficult to see, by considering the form of an arbitrary element of $F(\tau)$ having value zero, that $\kappa_{F(\tau)} = \mathbb{Q}$. Thus $\kappa_{F(\phi)} = \mathbb{Q}$.

**Proof of Theorem 4.2.1.** We have $F \subseteq \kappa((V)) \subseteq \kappa^a((G^a))$ and $\phi^a \in \kappa^a((G^a))$.

Take $M^a$ to be a maximal immediate extension of $F(\phi^a)$ in $\kappa^a((G^a))$. Moreover, let $M$ be a maximal immediate extension of $F$ in $M^a$. By part (2) of Theorem 4.1.7, $M^a$ and $M$ are maximally complete. Take $H$ to be a maximal immediate algebraic extension of $F$ in $M$. By Theorem 1.5.3, $H$ is Henselian.

**Claim 1.** $H \subseteq \mathbb{R}((V))$.

**Proof of claim 1.** By Theorem 4.1.7, there exists an $F$-embedding $\tau : M \to \kappa((V))$ preserving the valuation. Let $H' = \tau(H)$. $H'$ is an algebraic extension of $F$ in $\kappa((V))$. As $R$ is the algebraic closure of $F$ in $\kappa^a((G^a))$, we have that $H, H' \subseteq R$. 

Since $\tau|_H : H \to H'$ preserves the valuation, it also preserves the ordering. **Reason:** Let $a \in H, a \neq 0$. Since $H$ is an immediate extension of $F$, there exists $b \in F$ such that

$$v(b - a) > v(a) \quad (4.2.23)$$

Thus $a$ and $b$ have the same sign. Moreover, equation (4.2.23) implies that $v(b - \tau(a)) > v(\tau(a))$. So $\tau(a)$ and $b$ have also the same sign. Therefore, $\tau(a)$ and $a$ have the same sign too. Now, by Proposition 1.1.6, $\tau|_H$ extends to an automorphism of $R$. Since the only $F$-automorphism of $R$ is the identity map (by Corollary 1.1.7), we have $H' = H$. Thus $H \subseteq \kappa((V)) \subseteq \mathbb{R}((V))$, and claim 1 is proved.

Let $c \in M^a$ be algebraic over $F$.

**Claim 2.** There exists $d \in H(c)$ such that the ramification index of $H(d)$ over $H$ and the residue degree of $H(c)$ over $H(d)$ are both equal to 1.

**Proof of claim 2.** Denote by $\kappa_H$ (resp., $\kappa_{H(c)}$) the residue field of $H$ (resp., $H(c)$). Let

$$\ell = [\kappa_{H(c)} : \kappa_H] \quad (4.2.24)$$

Then by the primitive element theorem (see for example [La]), there exists an element $\bar{d} \in \kappa_H(c)$ such that

$$\kappa_H(\bar{d}) = \kappa_H(c)$$  \hspace{1cm} (4.2.25)

Let $\bar{f}$ be the monic irreducible polynomial which has degree equal to $\ell$, has its coefficients in $\kappa_H$, and has $\bar{d}$ as a root. Then corresponding to $\bar{f}$, we can choose a monic polynomial $f$ with degree equal to $\ell$ and with coefficients in the valuation ring $B_H$ of $H$, and by Hensel's Lemma this has a root $d$ in the valuation ring $B_{H(c)}$ of $H(c)$ (where $d$ corresponds to $\bar{d}$ in the natural homomorphism $B_{H(c)} \rightarrow \kappa_{H(c)}$).

Thus

$$[H(d) : H] \leq \ell$$  \hspace{1cm} (4.2.26)

Let $\kappa_{H(d)}$ denote the residue field of $H(d)$. There is a natural embedding $\kappa_H(\bar{d}) \hookrightarrow \kappa_{H(d)}$ over $\kappa_H$. Then using (4.2.24) and (4.2.25), we obtain

$$[\kappa_{H(d)} : \kappa_H] \geq [\kappa_H(\bar{d}) : \kappa_H] = \ell$$  \hspace{1cm} (4.2.27)

The inequalities (4.2.26) and (4.2.27) along with the fundamental inequality imply that

$$[v(H(d)) : v(H)] = 1$$  \hspace{1cm} (4.2.28)

Therefore, the ramification index of $H(d)$ over $H$ is 1. Moreover, we obtain

$$[\kappa_{H(d)} : \kappa_H] = \ell$$  \hspace{1cm} (4.2.29)

Now using the identity $[\kappa_{H(c)} : \kappa_H] = [\kappa_{H(c)} : \kappa_{H(d)}][\kappa_{H(d)} : \kappa_H]$ and the relations (4.2.24) and (4.2.29), we obtain $[\kappa_{H(c)} : \kappa_{H(d)}] = 1$. Thus the residue degree of $H(c)$ over $H(d)$ is 1 and claim 2 is proved.

Claim 3. $c \in \mathbb{R}((V^\alpha \cap G))$.

Proof of claim 3. $c \in R \subseteq \mathbb{R}((G))$, so we need only verify that $c \in \mathbb{R}((V^\alpha))$.

Let the polynomial $f$ and the element $d$ be as defined in the proof of claim 2. By claim 1, the coefficients of $f$ are in $\mathbb{R}((V))$. We have $d \in \mathbb{R}((V))$. Reason:

By Hensel's Lemma applied to $\mathbb{R}((V))$, $F$ has a root $d^r \in \mathbb{R}((V))$ with $d^r = d$.

By the uniqueness assertion of Hensel's Lemma applied to $\mathbb{R}((G^\alpha))$ (see part (2)
of the remarks after Theorem 1.5.2), $d = d' \in \mathcal{R}(V)$. Therefore using claim 1, we have $H(d) \subseteq \mathcal{R}(V))$. Now, suppose $\gamma$ is in the value group of $H(c)$ and $y \in H(c)$ is chosen such that $v(y) = \gamma$. Replacing $y$ by $-y$ if necessary, we can assume that $y > 0$. There exists a natural number $n$ such that $ny \in v(H(d))$. Moreover, $\kappa_{H(d)} = \kappa_{H(c)}$. Therefore, there exists $z \in H(d)$ such that $z$ and $y^n$ have the same leading term. Thus $v(y^n - z) > v(y^n)$. Let $s = (z - y^n)/y^n$. Then $s \in H(c)$ and $v(s) > 0$. We have $z = y^n(1 + s)$. Since $H(c)$ is Henselian, there exists an element $(1 + s)^{1/n} \in H(c)$ (this element is unique if $n$ is odd. If $n$ is even then there are two such elements. In this case, we choose the positive one). Thus $z^{1/n} = y(1 + s)^{1/n} \in H(c)$, and $v(z^{1/n}) = v(y) = \gamma$. On the other hand, $z \in H(d) \subseteq \mathcal{R}(V))$. Therefore, we can write $z = ax^n(1 + t)$, for some $t \in \mathcal{R}(V)$ of positive value and $a \in \mathcal{R}, a > 0$. Thus $z^{1/n} = a^{1/n}x^{\gamma}(1 + t)^{1/n} \in \mathcal{R}(V^a))$.

**Reason.** Using binomial expansion for $(1 + t)^{1/n}$ (or applying Hensel's Lemma to the polynomial $X^n - (1 + t)$), we see that $(1 + t)^{1/n} \in \mathcal{R}(V)) \subseteq \mathcal{R}(V^a))$. Moreover, $H(c) \subseteq M^a$; therefore, $\gamma \in V^a$. Now, applying this to a set of generators $\gamma_1, ..., \gamma_k$ of the value group of $H(c)$ modulo $V$ (= $v(H(d))$, we obtain elements $z_i \in H(d)$ with $z_i^{1/n_i} \in \mathcal{R}(V^a)$, $i = 1, ..., k$. Then $H(c) = H(d, z_1^{1/n_1}, ..., z_k^{1/n_k})$. **Reason.** $\tilde{H} = H(d, z_1^{1/n_1}, ..., z_k^{1/n_k})$ is Henselian. Let $\kappa_{\tilde{H}}$ denote the residue field of $\tilde{H}$. Then the residue degree of $H(c)$ over that of $\tilde{H}$ is 1, i.e., $[\kappa_{H(c)} : \kappa_{\tilde{H}}] = 1$. Therefore, Theorem 1.5.5 gives us $[H(c) : \tilde{H}] = [v(H(c)) : v(\tilde{H})] = 1$. That is $H(c) = \tilde{H}$. But this implies that $H(c) \subseteq \mathcal{R}(V^a))$ and claim 3 is thus proved.

Claim 3 actually proves the theorem in the algebraic case. In fact, if $\alpha = p$ is an element of $R$, then $\phi^\alpha = p \in M^\alpha$ is algebraic over $F$ so by applying claim 3 to $c = \phi^\alpha, F(\phi^\alpha) \subseteq \mathcal{R}(V^a))$. Now, suppose that $\alpha$ is a cut symbol in $R$. There are three cases to consider:

1. **Immediate transcendental case.** $\alpha = (S, p)$, and the coefficients of all the terms in $p$ belong to $\kappa$. Then $\kappa^\alpha = \kappa, G^\alpha = G$, and $\phi^\alpha = p$.

2. **Residue transcendental case.** $\alpha = (S, p)$, and there exists a term in $p$ with the coefficient $p_0 \notin \kappa$. Then $\kappa^\alpha = \kappa(p_0)$ in $\mathcal{R}, G^\alpha = G$, and $\phi^\alpha = p$. 
(3) **Value transcendental case.** \( \alpha = (S, p, \pm) \). Then \( \kappa^\alpha = \kappa \), \( G^\alpha \) is the divisible hull of \( G(\tau_S) \), and \( \phi^\alpha = p \pm x^\tau_S \).

**Claim 4.** In case (1), for each \( \gamma \in S \), there exists \( c \in M^\alpha \) algebraic over \( F \) with \( v(\phi^\alpha - c) > \gamma \). In case (2), there exists \( c \in M^\alpha \) algebraic over \( F \) such that \( v(\phi^\alpha - c) = \theta \). In case (3), there exists \( c \in M^\alpha \) algebraic over \( F \) such that \( v(\phi^\alpha - c) = \tau_S \).

Before proving claim 4, let us see that with the proof of it the theorem is actually proved. In fact, we want to show that \( \phi^\alpha \in \mathbb{R}((V^\alpha)) \). By claim 3, \( c \in \mathbb{R}((V^\alpha)) \) for any \( c \in M^\alpha \) algebraic over \( F \). But the statement in the claim 4 states that each initial segment (except the last one in cases 2 and 3) of \( \phi^\alpha \) is the same as an initial segment of a suitable \( c \in M^\alpha \) and thus belongs to \( \mathbb{R}((V^\alpha)) \). Moreover, in case 2 (resp., case 3), \( v(\phi^\alpha - c) = \theta \) (resp., \( v(\phi^\alpha - c) = \tau_S \)) will also imply that \( \theta \) (resp., \( \tau_S \)) belongs to \( V^\alpha \).

**Proof of claim 4.** We build recursively a transfinite sequence \( (c_k)_{k < \lambda} \), where \( \lambda \) is an ordinal number, of elements of \( M^\alpha \) with each \( c_k \) algebraic over \( F \) and with \( \{v(\phi^\alpha - c_k)\} \) increasing (i.e., \( v(\phi^\alpha - c_l) < v(\phi^\alpha - c_k) \) if \( l < k \)). This sequence will have the property that for each \( \gamma \in S \) (except for the last element of \( S \)), \( v(\phi^\alpha - c_k) > \gamma \) for \( k \) sufficiently large. Moreover, in case 1 the sequence \( (c_k) \) will have no last element. Furthermore, in case 2 (resp., case 3), \( (c_k) \) will have a last element \( c_{k_i} \) such that \( v(\phi^\alpha - c_{k_i}) = \theta \) (resp., \( v(\phi^\alpha - c_{k_i}) = \tau_S \)).

We take \( c_1 = 0 \). Suppose that the sequence \( (c_j) \) has been constructed for \( j < i \). Moreover, suppose that in our inductive procedure we obtain \( c_j \) such that \( v(\phi^\alpha - c_j) = \gamma \notin G \). Then we can not be in case 1 or case 2 and, if we are in case 3 then \( \gamma = \tau_S \); therefore, \( c_j \) is the last element of the sequence \( (c_i) \) and we are done. So assume that \( v(\phi^\alpha - c_j) = \gamma \in G \). Then \( \gamma \in S \). **Reason.** This is easy to see in cases 2 and 3. To show that it also holds in case 1, assume on the contrary that \( \gamma \notin S \). This means that \( v(p - c_j) = \gamma \notin S \). That is \( \alpha \) is not of the form \((S, p)\) which is a contradiction.

Depending on \( i \) being a successor ordinal or not, the construction of \( c_i \) will be done in two parts.
(I) Suppose that $i$ is a successor ordinal. Let $i = j + 1$. If we are in case 2 and $\gamma = \theta$, then $c_j$ is the last element of $(c_i)$ and we are done. So assume that this is not the case. Let $s = \phi^\alpha - c_j = ax^\gamma + \cdots, a \in \kappa, a \neq 0$. Since $G$ is the divisible hull of $V$, there exists a natural number $n$ such that $n\gamma \in V$. Therefore, we have an element $t \in F$ which is of the form $t = bx^{n\gamma} + \cdots, b \neq 0$. As $s$ and $t$ belong to $M^\alpha$, we have $s^n/t \in M^\alpha$. Since $v(s^n/t) = 0$, the image of $s^n/n$, i.e., $a^n/b$, is in the residue field of $M^\alpha$. Moreover, $a^n/b \in \kappa$ and, $\kappa$ is algebraic over the residue field $\kappa_0$ of $F$. Therefore, there exists a monic irreducible polynomial $\tilde{h}(x) \in \kappa_0[x]$ having $a^n/b$ as a simple root. Corresponding to $\tilde{h}$, there exists a monic polynomial $h$ with coefficients in the valuation ring of $F$. By Hensel's Lemma, $h$ has a root $u$ in the valuation ring of $M^\alpha$ such that $v((s^n/t) - u) > 0$. So $v(s^n - tu) > v(s) = v(s^n)$.

Therefore, $v(1 - (tu/s^n)) > 0$. Let $-1 + (tu/s^n) = q$. Then $q \in M^\alpha$, and $v(q) > 0$. Moreover, we have $tu = s^n(1 + q)$. Now, there is an element $(1 + q)^{1/n} \in M^\alpha$. Furthermore, $tu$ is algebraic over $F$ (since $t \in F$, and $u \in M^\alpha$ is a root of $h$). On the other hand, $(1 + q)^{1/n} = (s(1+q)^{1/n})^n$. Therefore, $(1 + q)^{1/n}$ is algebraic over $F$. Now let $c_i = c_j + s(1 + q)^{1/n}$. Then $c_i \in M^\alpha$ is algebraic over $F$, and

$$
v(\phi^\alpha - c_i) = v(\phi^\alpha - c_j - s(1 + q)^{1/n}) = v(s - s(1 + q)^{1/n}) = vs + v(1 - (1 + q)^{1/n}) = v(\phi^\alpha - c_j) + v(1 - (1 + q)^{1/n}) \quad (4.2.30)
$$

But

$$1 - (1 + q)^{1/n} = -q(\frac{1}{n} + \frac{n - 1}{2n^2}q + \cdots) \quad (4.2.31)$$

Therefore, $v(1 - (1 + q)^{1/n}) = v(q) > 0$. This together with (4.2.30) imply that $v(\phi^\alpha - c_i) > v(\phi^\alpha - c_j)$, and we are done.

(II) Suppose that $i$ is a limit ordinal. First note that if we are in case 1 and for each $\gamma \in S$ there exists $j < i$ with $v(p - c_j) > \gamma$ then we are done. So assume that we are not in this case. Then

Subclaim. There exists $c \in R$ such that $v(\phi^\alpha - c) > v(\phi^\alpha - c_j)$, for all $j < i$. 

Proof of the subclaim. Suppose first that we are in the case 1. Then, by the assumption we just made, there exists \( \gamma \in S \) such that \( v(p - c_j) \leq \gamma \), for all \( j < i \).

Now by the construction of \( p \), there exists \( c \in R \) such that \( v(p - c) > \gamma \). Then \( v(\phi^\alpha - c_j) \leq \gamma < v(\phi^\alpha - c) \) for all \( j < i \), and we are done. If we are in case 2 then write \( \phi^\alpha = p = c + px^\alpha \), where \( c \in R \). Then \( v(\phi^\alpha - c) = \theta > v(\phi^\alpha - c_j) \), for all \( j < i \). If we are in case 3, choose \( c \in R \) such that \( v(c - p) \notin S \), then we have \( v(\phi^\alpha - c) = \tau_S > v(\phi^\alpha - c_j) \in S \) for all \( j < i \), and the subclaim is thus proved.

Let \( c \) be as in the above subclaim and let \( f \) be the minimal polynomial of \( c \) over \( F \). Denote by \( n \) the degree of \( f \). \((c_j)_{j<i}\) is a pseudo-convergent sequence with \( c \) as a limit. Reason. First note that \((c_j)_{j<i}\) is pseudo-convergent: Let \( j_1 < j_2 < j_3 < i \).

Then
\[
v(c_{j_2} - c_{j_1}) = v((\phi^\alpha - c_{j_1}) - (\phi^\alpha - c_{j_2})) \\
= v(\phi^\alpha - c_{j_1}) \\
< v(\phi^\alpha - c_{j_2}) \\
v((\phi^\alpha - c_{j_2}) - (\phi^\alpha - c_{j_3})) \\
v(c_{j_3} - c_{j_2})
\] (4.2.32)

Moreover, for \( j < i \) we have
\[
v(c - c_j) = v((\phi^\alpha - c_j) - (\phi^\alpha - c)) \\
v(\phi^\alpha - c_j) \\
v((\phi^\alpha - c_j) - (\phi^\alpha - c_{j+1})) \\
v(c_{j+1} - c_j)
\]

Therefore, \( c \) is a limit of \((c_j)_{j<i}\).

Now consider the Taylor expansion
\[
f(c_j) = f(c) + \sum_{m=1}^{n} \frac{f^{(m)}(c)}{m!} (c_j - c)^m = \sum_{m=1}^{n} \frac{f^{(m)}(c)}{m!} (c_j - c)^m
\] (4.2.33)

By Lemma 4.1.6, there exists a positive integer \( k \), \( 1 \leq k \leq n \), such that
\[
v(\frac{f^{(m)}(c)}{m!} (c_j - c)^m) = v(\frac{f^{(m)}(c)}{m!}) + mu(c_j - c) > \\
v(\frac{f^{(k)}(c)}{k!}) + ku(c_j - c) = v(\frac{f^{(k)}(c)}{k!} (c_j - c)^k),
\] (4.2.34)
for all $1 \leq m \leq n, m \neq k$, and for all $j < i$ sufficiently large. But (4.2.33) and (4.2.34) imply that

$$v(f(c_j)) = v\left(\frac{f^{(k)}(c)}{k!}\right) + kv(c_j - c)$$

(4.2.35)

So if $j < j' < i$ are sufficiently large then $v(f(c_j)) < v(f(c_{j'}))$. Therefore, the pseudo-convergent sequence $(c_j)_{j<i}$ is algebraic. Let $g$ be a minimal polynomial of $(c_j)_{j<i}$ in $F(c_j|j < i)$. Then by Theorem 4.1.7 there exists some limit, say $c_i$, of $(c_j)_{j<i}$ in $M^a$ such that $g(c_i) = 0$. Since $c_i$ is algebraic over $F(c_j|j < i)$, it is also algebraic over $F$. Moreover, by the definition of $c_i$ as the limit of $c_j$, for each $j < i$ we have

$$v(c_i - c_j) = v(c_{j+1} - c_j)$$

$$= v((\phi^a - c_j) - (\phi^a - c_{j+1}))$$

$$= v(\phi^a - c_j)$$

Therefore,

$$v(\phi^a - c_i) = v((\phi^a - c_j) + (c_j - c_i))$$

$$\geq v(\phi^a - c_j),$$

for all $j < i$. This together with the fact that $\{v(\phi^a - c_j)\}_{j<i}$ is increasing imply that $v(\phi^a - c_i) > v(\phi^a - c_j)$ for all $j < i$, and we are done. QED

Let $R$ be a real closed field. Fix an ordering on the ring $R[y_1, \cdots, y_n]$. Corresponding to this ordering we have an order symbol $(\phi_1, \cdots, \phi_n)$. For each natural number $j, 1 \leq j \leq n$, let $R_j$ be defined as on page 57 (actually, $R_j$ is the real closure of the field $R(\phi_1, \cdots, \phi_j)$). Moreover, let $G_j$ and $\kappa_j$ be respectively the value group and the residue field of $R_j$. Furthermore, let $V_j$ denote the value group of $F(\phi_1, ..., \phi_j)$.

**Corollary 4.2.2** Suppose that $F$ is an ordered subfield of $R$ having $R$ as its real closure. Denote by $V \subseteq G$ the value group of $F$. Assume that the embedding $R \subseteq \kappa((G))$ is chosen so that $F \subseteq \kappa((V))$. Moreover, let $(\phi_1, ..., \phi_n)$ be an order
symbol. Then $V_n$ is a subgroup of $G_n$ generated over $V$ by all $\gamma$ such that $x^\gamma$ appears in the power series expansion of at least one of the $\phi_i, i = 1, \ldots, n$. Equivalently, $F(\phi_1, \ldots, \phi_n) \subseteq \kappa_n((V_n))$.

**Proof.** We use induction on $n$. By Theorem 4.2.1, this is true if $n = 1$. So assume the result for $l, 1 \leq l < n$. Let $\phi^\alpha = \phi_{l+1}$. Now, again by using Theorem 4.2.1, we have $F(\phi_1, \ldots, \phi_{l+1}) = (F(\phi_1, \ldots, \phi_l))(\phi_{l+1}) \subseteq \kappa_l^l(V_{l+1}^\alpha) = \kappa_{l+1}(V_{l+1})$. QED

**Example 4.2.2** Consider the ordering on $\mathbb{R}(y_1, y_2)$ which is given by the order symbol $(\phi_1, \phi_2)$ where

$$\phi_1 = x, \quad \phi_2 = x + x^{l+\frac{1}{2}} + x^{l+\frac{1}{3}} + \cdots$$

Here, $G_1 = G_2 = \mathbb{Q}, V_1 = \mathbb{Z}, V_2 = \mathbb{Q}$. Note: $\phi_1 = \phi^\alpha$ where $\alpha$ is the value transcendental cut symbol ($\{0\}, 0, +$), and $\phi_2 = \phi^\alpha$ where $\alpha$ is the immediate transcendental cut symbol ($\mathbb{Q}, \phi_2$).

### 4.3 Order symbols and real places

Suppose now that $F = R$, i.e., $F = R$ is real closed. Fix the positive integer $n$. As before, let $G_n$ be the divisible hull of the value group $V_n$ of $F(\phi_1, \ldots, \phi_n)$.

**Claim.** The group $V_n/2V_n$ is finite.

**Proof of the claim.** $G_n/G$ is a vector space over $\mathbb{Q}$. By Corollary 4.2.2, it is obvious that $\dim_\mathbb{Q}(G_n/G)$ is equal to the number of value transcendental cut symbols (chapter 3, page 60) so, in particular, $\dim_\mathbb{Q}(G_n/G) \leq n$. On the other hand, $V_n/2V_n$ is a vector space over $\mathbb{Z}_2$. Now if we prove that $\dim_\mathbb{Z}_2(V_n/2V_n) \leq n$ then we have $|V_n/2V_n| \leq 2^n$ and the claim is proved. Therefore, it is enough to show that $\dim_\mathbb{Z}_2(V_n/2V_n) \leq \dim_\mathbb{Q}(G_n/G)$. To prove this inequality, it suffices to show that if $\alpha_1, \cdots, \alpha_l \in V_n$ are such that $\alpha_i + 2V_n, 1 \leq i \leq l$, are linearly independent in the $\mathbb{Z}_2$ vector space $V_n/2V_n$ then $\alpha_i + G, 1 \leq i \leq l$, are linearly independent in the $\mathbb{Q}$ vector space $G_n/G$. Assume, on the contrary, that there are rational numbers $r_1, \cdots, r_l$, not all zero, such that $\sum^l_{i=1} r_i \alpha_i \in G = V$. Clearing denominators, we can assume that $r_i \in \mathbb{Z}, 1 \leq i \leq n$. Using the fact that $G$ is divisible and dividing by a suitable
non-negative power of 2, we can assume that one of \( r_i \)'s, say \( r_j \), is an odd integer. Then we have \( \sum_{i=1}^l r_i \alpha_i \in G = 2G = 2V \subseteq 2V_n \) and hence \( \sum_{i=1}^l \delta_i (\alpha_i + 2V_n) = 2V_n \), where \( \delta_i = 0 \) or 1, for \( 1 \leq i \leq l \), and \( \delta_j = 1 \). But this contradicts the assumption that \( \alpha_i + 2V_n, 1 \leq i \leq l \), are linearly independent.

A 2-character of \( V_n \) is defined to be a group homomorphism \( \sigma : V_n \rightarrow \{-1, 1\} \). Every 2-character \( \sigma \) of \( V_n \) induces an automorphism of the field \( \kappa_n((V_n)) \), where \( \kappa_n \) is the residue field of \( R(\phi_1, ..., \phi_n) \). Actually, the map \( \iota_\sigma \) defined by \( f = \sum a_\gamma x^\gamma \mapsto f_\sigma = \sum a_\gamma \sigma(\gamma) x^\gamma \), for each \( f \in \kappa_n((V_n)) \), is easily seen to be an automorphism of the field \( \kappa_n((V_n)) \). Note that \( \iota_\sigma \) is identity on \( \kappa_n \). The number of 2-characters of \( V_n \) is equal to \( |V_n/2V_n| \). \textit{Reason.} This is well-known. Actually, we saw that \( |V_n/2V_n| = 2^k \), for some \( 0 \leq k \leq n \). Furthermore, every homomorphism \( \tau : V_n/2V_n \rightarrow \{-1, 1\} \), which is in fact an element of the dual space of \( V_n/2V_n \), is obviously in one-to-one correspondence with a 2-character \( \sigma \) on \( V_n \). Therefore, the set of 2-characters on \( V_n \) and the set \( V_n/2V_n \) have the same cardinality, namely \( 2^k \).

\textit{Remark.} Suppose that \( F = R \) is real closed. Then the ordering on \( F(\phi_1, ..., \phi_n) \) has an associated real place \( \tau \). In fact, as \( F(\phi_1, ..., \phi_n) \subseteq \mathbb{R}((G_n)) \), \( \tau \) is the restriction to \( F(\phi_1, ..., \phi_n) \) of the natural real place on \( \mathbb{R}((G_n)) \). The real place \( \tau \) has \( B = \{ f \in F(\phi_1, ..., \phi_n) : \tau(f) \neq \infty \} = \{ f \in F(\phi_1, ..., \phi_n) : v(f) \geq 0 \} \) as its valuation ring and \( M = \{ f \in F(\phi_1, ..., \phi_n) : \tau(f) = 0 \} = \{ f \in F(\phi_1, ..., \phi_n) : v(f) > 0 \} \) as the corresponding maximal ideal, where \( v \) is the natural valuation on \( \mathbb{R}((G_n)) \).

In order to state our next result, we need the following theorem which is a part of Theorem 1.3.2 in [Mr] (see also [La1]).

\textbf{Theorem 4.3.1 (Baer-Krull)} Let \( F \) be a field and let \( \tau : F \rightarrow \mathbb{R} \cup \{ \infty \} \) be a real place. Then the set of orderings associated to \( \tau \) is finite iff \( v(F)/2v(F) \) is finite and, if this is the case, then \( |v(F)/2v(F)| = \) the number of orderings associated to \( \tau \).

Suppose that \( \sigma \) is a 2-character of \( V_n \) and \( (\phi_1, ..., \phi_n) \) is an order symbol. Then by \( \phi_{i\sigma}, 1 \leq i \leq n \), we mean just \( (\phi_i)_\sigma \) as defined above.

\textbf{Theorem 4.3.2} Suppose \( F = R \) is real closed and \( (\phi_1, ..., \phi_n) \) is an order symbol. Then, for each 2-character \( \sigma \) of \( V_n \), \( (\phi_{1\sigma}, ..., \phi_{n\sigma}) \) is an order symbol. The map
\(\sigma \mapsto (\phi_{1\sigma}, \ldots, \phi_{n\sigma})\) is injective. The orderings corresponding to the order symbols 
\((\phi_{1\sigma}, \ldots, \phi_{n\sigma})\), \(\sigma\) a 2-character of \(V_n\), all belong to the same real place and are the full equivalence class of orderings belonging to this real place.

**Proof.** Suppose that \(\sigma\) is a 2-character of \(V_n\). We proceed by induction to show that \((\phi_{1\sigma}, \ldots, \phi_{n\sigma})\) is an order symbol. Let \(\phi_1\) be a simple order symbol. Then corresponding to \(\phi_1\) we have an ordering on \(R[y_1]\). Recall that \(\sigma\) is identity on \(\kappa_n \supseteq \kappa_1\). Therefore, if \(\phi_1\) is algebraic, immediate transcendental, or residue transcendental then we have \(\phi_{1\sigma} = \phi_1\), hence \(\phi_{1\sigma}\) is an order symbol. So we just need to consider the case where \(\phi_1\) is value transcendental, i.e., \(\phi_1 = p \pm x^{\tau_{1\sigma}}\). In this case, the ordering on \(R[y_1]\) corresponding to \(\phi_1\) has \(p_1 = \{0\}\) as its support; therefore, \(\phi_{1\sigma} = p \mp x^{\tau_{1\sigma}}\) gives us also an ordering on \(R[y_1]\) with support \(p_1 = \{0\}\). That is, \(\phi_{1\sigma}\) is an order symbol.

Note also that \(R(\phi_{1\sigma})\) is isomorphic to \(R(\phi_1)\) under the map \(a, x^\gamma \mapsto a, \sigma(\gamma)x^\gamma\).

We saw that \(\iota_{\sigma}\) is a field isomorphism on \(\kappa_n((G_n))\). For each \(1 \leq i \leq n\), the restriction of \(\iota_{\sigma}\) to the field \(F_i = R(\phi_1, \ldots, \phi_i)\) gives us an isomorphism of the fields \(F_i\) and \(F_{i\sigma} = R(\phi_{1\sigma}, \ldots, \phi_{i\sigma})\). Now assume that \(1 \leq k < n\), is an integer such that for each integer \(j, 1 \leq j \leq k\), \((\phi_{1\sigma}, \ldots, \phi_{j\sigma})\) is an order symbol and moreover, the orderings on \(R[y_1, \ldots, y_j]\) corresponding to the order symbols \((\phi_1, \ldots, \phi_j)\) and \((\phi_{1\sigma}, \ldots, \phi_{j\sigma})\) have the same support \(p_j\). From the results of section 3.3, we know that for each \(1 \leq j \leq k\), the quotient field \(F_j'\) of \(R[y_1, \ldots, y_j]/p_j\) with the ordering which corresponds to the order symbol \((\phi_1, \ldots, \phi_j)\) is isomorphic to \(F_j\). Now to show that \((\phi_{1\sigma}, \ldots, \phi_{(k+1)\sigma})\) is an order symbol, according as \(\phi_{k+1}\) is algebraic, immediate transcendental, residue transcendental, or value transcendental over \(F_k\), there are four cases to consider. We give details of the proof for the algebraic and value transcendental cases.

(1) Suppose that \(\phi_{k+1}\) is algebraic over \(F_k\). Under the field isomorphism \(\iota_{\sigma}\), the element \(\phi_{(k+1)\sigma} \in F_{(k+1)\sigma}\) corresponds to \(\phi_{k+1} \in F_{k+1}\). Therefore, \(\phi_{(k+1)\sigma}\) is algebraic over \(F_{k\sigma}\). Hence \(\phi_{(k+1)\sigma}\) is a simple order symbol corresponding to \(F_{\sigma}\). Thus \((\phi_{1\sigma}, \ldots, \phi_{(k+1)\sigma})\) is an order symbol. To complete our induction, we need to show one more fact: We should show that the support of the orderings on \(R[y_1, \ldots, y_{k+1}]\) corresponding to the order symbols \((\phi_1, \ldots, \phi_{k+1})\) and \((\phi_{1\sigma}, \ldots, \phi_{(k+1)\sigma})\) are the
same. By the proof of Lemma 3.3.2, the support of the ordering on \( R[y_1, \ldots, y_{k+1}] \) induced by the cut symbol \((\phi_1, \cdots, \phi_{k+1})\) (resp., \((\phi_{1\sigma'}, \cdots, \phi_{(k+1)\sigma'})\)) is completely determined by the minimal polynomial of \( \phi_{k+1} \) (resp., \( \phi_{(k+1)\sigma'} \)) over \( F_k \) (resp., \( F_{k\sigma'} \)). But \( \phi_{k+1} \in F_{k+1} \) and \( \phi_{(k+1)\sigma'} \in F_{(k+1)\sigma'} \). Thus either of the two mentioned orderings has the same support \( p_{k+1} \).

(2) Now suppose that \( \phi_{k+1} \) is value transcendental over \( F_k \). Therefore, there exists \( p = \sum a_i x^i \) such that \( \phi_{k+1} = p + x^{\tau_5} \), where \( p \in \kappa_k((G_k)) \), \( G_k \) the divisible hull of \( V_k \), and \( x^{\tau_5} \in V_{k+1} \). Corresponding to the order symbol \((\phi_1, \cdots, \phi_k)\) we have an ordering on \( A_k = R[y_1, \cdots, y_k] \). Now associated to the simple order symbol \( \phi_{k+1} \), by Lemma 3.3.2 and Proposition 3.2.3, there exists a unique extension of the ordering on \( A_k \) to an ordering on \( A_{k+1} = A_k[y_{k+1}] \). Corresponding to the simple order symbol \( \phi_{k+1} \) we have the cut symbol \((S, p, +)\), where \( S \) is a lower cut in \( G_k \). Since \( \sigma \) preserves the valuation, the value group \( V_{k\sigma} \) of \( F_{k\sigma} \) is \( V_k \) (by Theorem 4.2.1 and using induction). Hence the divisible hull \( G_{k\sigma} \) of \( V_{k\sigma} \) is \( G_k \). \((S, p_{\sigma}, \sigma(\tau_5))\) is a cut symbol and defines a value transcendental cut in the real closure of \( F_{k\sigma} \). Corresponding to this cut we have the simple order symbol \( p_{\sigma} + \sigma(\tau_5) x^{\tau_5} = \phi_{(k+1)\sigma'} \). This means, by Theorem 3.3.3 and the arguments preceding it, that \((\phi_{1\sigma'}, \cdots, \phi_{(k+1)\sigma'})\) is an order symbol. Moreover, support of the ordering on \( R[y_1, \cdots, y_{k+1}] \) which corresponds to the order symbol \((\phi_1, \cdots, \phi_{k+1})\) and that of the ordering which corresponds to \((\phi_{1\sigma'}, \cdots, \phi_{(k+1)\sigma'})\) are the same and both equal to \( p_k[y_{k+1}] \) (see part (1) of remarks on page 57).

To see the injectivity of the map \( \sigma \mapsto (\phi_{1\sigma}, \cdots, \phi_{n\sigma}) \), suppose that \( \sigma, \sigma' \) are 2-characters on \( V_n \) such that \( \sigma' \neq \sigma \). Since \( F \) is real closed, the value group of \( F \) is \( G \) and \( \sigma, \sigma' \) are identity on \( v(F) \). So, by Corollary 4.2.2, there exists \( \gamma \in V_n \) such that \( \sigma'(\gamma) \neq \sigma(\gamma) \). This means that there exists \( 0 < i \leq n \) such that \( \phi_{i\sigma'} \neq \phi_{i\sigma} \). Hence \((\phi_{1\sigma'}, \cdots, \phi_{n\sigma'}) \neq (\phi_{1\sigma}, \cdots, \phi_{n\sigma}) \). That is the map \( \sigma \mapsto (\phi_{1\sigma}, \cdots, \phi_{n\sigma}) \) is injective. To prove the last assertion note first that the automorphism \( f \mapsto f_\sigma \) preserves \( v \); therefore, by the remark on page 60 the orderings corresponding to the order symbols \((\phi_{1\sigma}, \cdots, \phi_{n\sigma}) \), \( \sigma \) a 2-character of \( V_n \), all belong to the same real place. Now \( |V_n/2V_n| \) is finite and is equal to the number of 2-characters of \( V_n \). On
the other hand, the number of orderings associated to a given real place \( \alpha \) is, by Theorem 4.3.1, exactly equal to \( |V_n/2V_n| \). Thus the orderings corresponding to the order symbols \( (\phi_1, \cdots, \phi_m) \), \( \sigma \) a 2-character of \( V_n \), is the full equivalence class of the orderings associated to the real place \( \alpha \) and we are done. QED

**Example 4.3.1** Real places on \( \mathbb{R}(y_1) \). We know the orderings on \( \mathbb{R}(y_1) \) and, for any ordering we have \( V_1 = \mathbb{Z} \). Either \( \sigma(1) = 1 \) or \( \sigma(1) = -1 \). Therefore, there are exactly two 2-characters of \( V_1 \). \( \phi = r + x \) and \( \phi = r - x \) have the same associated real place. Also, \( \phi = x^{-1} \) and \( \phi = -x^{-1} \) have the same associated real place.

**Example 4.3.2** For the ordering on \( \mathbb{R}(y_1, y_2) \) with order symbol \( \phi_1 = x, \phi_2 = x + x^{1+\frac{1}{2}} + x^{1+\frac{1}{2}+\frac{1}{3}} + \cdots \) (see Example 4.2.2), \( V_2 = \mathbb{Q} \) and the only 2-character of \( V_2 \) is the trivial one.

### 4.4 Extensions with prescribed value groups

**Theorem 4.4.1** Assume the set-up in Theorem 4.2.1. Moreover, suppose that \( \kappa_0 \) (resp., \( \kappa_0^\circ \)) denotes the residue field of \( F \) (resp., \( F(\phi^\circ) \)).

1. **Algebraic case**. If \( \alpha = p \), an element of \( R \) then \( F(\phi^\circ) \) is a finite extension of \( F \) so \( V^\alpha/V \) is finite and \( \kappa_0^\circ \) is a finite extension of \( \kappa_0 \).

2. **Value transcendental case**. If \( \alpha \) is value transcendental then \( V^\alpha = \mathbb{Z}\delta \oplus W \) where \( \mathbb{Z}\delta \) is infinite cyclic, \( W \supseteq V, W/V \) is finite, and \( \kappa_0^\circ \) is a finite extension of \( \kappa_0 \).

3. **Residue transcendental case**. If \( \alpha \) is residue transcendental then \( V^\alpha/V \) is finite, \( \kappa_0^\circ \) is a finitely generated extension of \( \kappa_0 \) and \( \text{trdeg}(\kappa_0^\circ : \kappa_0) = 1 \).

4. **Immediate transcendental case**. If \( \alpha \) is immediate transcendental then \( V^\alpha/V \) is countable and torsion; moreover, \( \kappa_0^\circ \) is an algebraic extension of \( \kappa_0 \).

**Note.** Recall that for an ordered field \( F \), \( \kappa_F \) is the residue field of the natural valuation \( v \) on \( F \).

**Proof.** (1) By the fundamental inequality, we have \( [V^\alpha : V][\kappa_0^\circ : \kappa_0] \leq [F(\phi^\circ) : F] \). Therefore, both \( [V^\alpha : V] \) and \( [\kappa_0^\circ : \kappa_0] \) are finite and we are done.
(2) Suppose that \( \alpha = (S, p, \pm) \) and \( \phi^\alpha = p \pm z^\tau \). \( F(\phi^\alpha) \) is a simple extension of \( F \) and \( \tau_S \) is not in \( G \) (= divisible hull of the value group of \( F \)). On the other hand, by Theorem 4.2.1, \( \tau_S \) is in the value group of \( F(\phi^\alpha) \). So there is \( b \in F(\phi^\alpha) \) such that \( \nu(b) = \tau_S \). Therefore, by part (1) of Corollary 1.2.7 there exists an ordered abelian group extension \( W \) of \( V \), \( W/V \) finite, such that \( V^\alpha = \mathbb{Z}^\delta \oplus W \), for some \( \delta \in V^\alpha \setminus G \). Moreover, \( \kappa^\alpha_0 \) over \( \kappa_0 \) is finite by Corollary 1.2.7 (1).

(3) Suppose that \( \alpha = (S, p) \) and \( \phi^\alpha \) is of the form \( \phi^\alpha = p = p_1 + p_\infty z^\delta \), where \( p_\infty \notin \kappa \), i.e., \( p_\infty \) is transcendental over \( \kappa \) hence over \( \kappa_0 \). We know that \( p_\infty \in \kappa^\alpha_0 \); therefore, the real closure of \( \kappa^\alpha_0 \) is equal to the real closure of \( \kappa_0(p_\infty) \). This means that \( \text{trdeg}(\kappa^\alpha_0 : \kappa_0) = 1 \). Then there exists \( b \in F(\phi^\alpha) \) such that \( \nu(b) = 0 \) and the image \( \tilde{b} \) of \( b \) in \( \kappa^\alpha_0 \) is transcendental over \( \kappa_0 \). So by Corollary 1.2.7 (2), \( V^\alpha/V \) is finite; moreover, \( \kappa^\alpha_0 \) is a finite extension of \( \kappa_0(\tilde{b}) \). Therefore, \( \kappa^\alpha_0 \) is finitely generated over \( \kappa_0 \). Thus we are done with the proof of part (3).

(4) Suppose that \( \alpha = (S, p) \) is an immediate transcendental cut and \( \phi^\alpha = p \). Then \( V^\alpha \subseteq G \) so \( V^\alpha/V \) is torsion. To see that \( V^\alpha/V \) is countable, let \( p = \sum_{\gamma \in \Lambda} p_\gamma z^\gamma \), where \( \Lambda = \text{Supp}(p) \). For each \( \delta \in \Lambda \), let \( p^\delta = \sum_{\gamma \leq \delta} p_\gamma z^\gamma \); moreover, choose \( c_\delta \in R \) such that \( \nu(p_\delta - c_\delta) > \delta \). By part (1), \( V(F(c_\delta))/V \) is finite. Therefore, if we let \( W_\delta \) be the subgroup generated over \( V \) by all \( \gamma \in \Lambda \) such that \( \gamma \leq \delta \), then the group \( A_\delta = W_\delta/V \) is finite. Moreover, \( A_\delta \subseteq A_\delta \) whenever \( \delta, \delta' \in \Lambda \) and \( \delta' \leq \delta \). We have \( V^\alpha/V = \bigcup_{\delta \in \Lambda} A_\delta \). Choose \( \Lambda' \subseteq \Lambda \) such that for each \( \delta \in \Lambda \) there exists \( \delta' \in \Lambda' \) with \( A_\delta = A_{\delta'} \) and, moreover if \( \delta', \delta'' \in \Lambda' \) and \( \delta' \neq \delta'' \), then \( A_{\delta'} \neq A_{\delta''} \). Therefore, we have \( V^\alpha/V = \bigcup_{\delta \in \Lambda'} A_\delta \). \( \{|A_\delta| : \delta \in \Lambda'\} \) is a set of positive integers. Furthermore, \( |A_\delta| \neq |A_{\delta'}| \) for any \( \delta, \delta' \in \Lambda', \delta \neq \delta' \). Therefore, \( |\Lambda'| \) is a countable cardinal. Thus \( V^\alpha/V \) is countable. To prove the last assertion of part (4), note that \( \phi^\alpha \in \kappa((G)) \) so we have \( F(\phi^\alpha) \subseteq \kappa((G)) \). Thus \( \kappa^\alpha \subseteq \kappa \). This together with the fact that \( \kappa \) is the real closure of \( \kappa_0 \) proves that \( \kappa^\alpha \) is algebraic over \( \kappa_0 \). QED

Finally we show that each of cases in Theorem 4.4.1 occurs. Suppose \( F \subseteq \kappa((V)) \), \( G \) is the divisible hull of \( V \), and \( W \) a group extension of \( V \) such that \( W \subseteq G \).

**Remark.** Suppose that \( W/V \) is finite. Then \( W/V \) is naturally isomorphic to \( \mathbb{Z}_m \oplus \)
... $\oplus \mathbb{Z}_{m_k}$ for some positive integer $k$ where, for each $1 \leq i \leq k$, $m_i$ is a positive integer and, $\mathbb{Z}_{m_i} = \mathbb{Z}/m_i\mathbb{Z}$. Therefore, if $A$ is a set of generators of $W$ over $V$ then there exists a finite subset of $A$ which generates $W$ over $V$.

**Theorem 4.4.2** Suppose that $F$ is an ordered field with real closure $R$. Let $V$ (resp., $G$) be the value group of $F$ (resp. $R$) and $\kappa_0$ (resp., $\kappa$) be the residue field of $F$ (resp., $R$). Let $W \subseteq G$ be an extension of $V$.

1. **Algebraic case.** If $W/V$ is finite then there exists a simple order symbol $\phi_0$ such that $\phi_0 = p \in R$ and the value group of $F(p)$ is $W$.

2. **Value transcendental case.** If $W/V$ is finite and $\mathbb{Z}\delta \oplus W$, where $\delta \notin G$, is an ordered abelian group extension of $V$ then there exists a value transcendental cut $\alpha$ with the corresponding cut symbol $\phi_0$ such that the value group of $F(\phi_0)$ is $\mathbb{Z}\delta \oplus W$.

3. **Residue transcendental case.** If $W/V$ is finite and there exists $\tau \in R \setminus \kappa$ then there exists a residue transcendental cut $\alpha$ with the corresponding cut symbol $\phi_0$ such that the value group of $F(\phi_0)$ is $W$.

4. **Immediate transcendental case.** If $W/V$ is countable and torsion and $R$ does not contain $\kappa((W))$ then there exists an immediate transcendental cut $\alpha$ with the corresponding cut symbol $\phi_0$ such that the value group of $F(\phi_0)$ is $W$.

**Proof.** (1) **Algebraic case.** If $W/V = \{0\}$ then we are done. So assume $W/V \neq \{0\}$. Then $W/V$ is naturally isomorphic to $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ for some positive integer $k$ where, for each $1 \leq i \leq k$, $m_i$ is a positive integer and, $\mathbb{Z}_{m_i} = \mathbb{Z}/m_i\mathbb{Z}$. In this isomorphism, let $a_1, \cdots, a_k \in W$ correspond respectively to the generators $e_1 = (1, 0, \ldots, 0), \ldots, e_k = (0, \ldots, 1)$ of $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$. We have $[W : V] = m_1 \cdots m_k$.

For each $i, 1 \leq i \leq n$, choose an element $p_i \in F$ such that $v(p_i) = m_i a_i$. Then $v(F') \supseteq W$, where $F' = F(\sqrt[n]{p_1}, \cdots, \sqrt[n]{p_k})$. Using these facts along with the fundamental inequality we obtain:

\[
m_1 \cdots m_k \geq \prod_{i=1}^{k} \left[ F(\sqrt[n]{p_1}, \cdots, \sqrt[n]{p_i}) : F(\sqrt[n]{p_1}, \cdots, \sqrt[n]{p_{i-1}}) \right] = [F' : F] \geq [v(F') : V][\kappa_{F'} : \kappa_0] \geq [v(F') : V] \geq [W : V] = m_1 \cdots m_k
\]
Therefore, \([v(F') : V] = m_1 \cdots m_k = [W : V]\). So \(v(F') = W\); moreover, we also have \(k_{F'} = k_0\). Now as \(F'\) has characteristic 0, we can use the primitive element theorem [La] to pick \(p \in F'\) such that \(F' = F(p)\). That is, if we let \(\phi^\alpha = p\) then the value group of \(F(\phi^\alpha)\) is \(W\) and we are done.

(2) **Value transcendental case.** Use part (1) to choose an element \(q = \sum_{\gamma < \lambda} c_\gamma x^\gamma\) in the real closure \(R\) of \(F\) so that \(v(F(q)) = W\). By Theorem 4.2.1, the exponents \(\gamma\) such that \(x^\gamma\) appears in \(q\) generate \(W\) over \(V\). \(W/V\) is finite so there exists \(\mu \in W\) such that the exponents \(\gamma\) such that \(x^\gamma\) appears in \(q\) and \(\gamma \leq \mu\) generate \(W\) over \(V\). Replacing \(\delta\) by \(-\delta\) if necessary, we can assume \(\delta > 0\). Replacing \(\delta\) by \(\delta + \mu\) if necessary, we can assume \(\delta > \mu\). Let \(p = \sum_{\gamma < \delta} c_\gamma x^\gamma\). The set \(S = \{\gamma \in G : \gamma < \delta\}\) is a lower cut in \(G\). Now it is obvious that associated to \(S\) and \(p\) we have a cut symbol \((S, p, +)\). Corresponding to this cut we obtain \(\phi^\alpha = p + x^\delta\). Thus by Theorem 4.2.1, the value group of \(F(\phi^\alpha)\) is \(Z\delta \oplus W\) and we are done.

(3) **Residue transcendental case.** Pick \(q = \sum_{\gamma \in G} c_\gamma x^\gamma\) as in part (1). Since \(W/V\) is finite, we can find finitely many terms \(c_{\gamma_i} x^\gamma_i, 1 \leq i \leq l\), in \(q\) such that the exponents \(\gamma_1 < \cdots < \gamma_l\) generate \(W\) over \(V\). Let \(p = (\sum_{\gamma < \gamma_1} c_\gamma x^\gamma) + r x^{\gamma_1}\). The set \(S = \{\gamma \in G : \gamma \leq \gamma_1\}\) is a lower cut in \(G\). Now it is obvious that \(S\) and \(p\) give us a residue transcendental cut \((S, p)\) with the associated \(\phi^\alpha = p\). Thus by Theorem 4.2.1, the value group of \(F(\phi^\alpha)\) is \(W\) and we are done.

(4) **Immediate transcendental case.** Subcase 1. Suppose that \(W/V\) is finite. Use part (1) to pick \(q \in R\) with \(v(F(q)) = W\). Choose \(q' \in \kappa((W)) \setminus R\). There exists \(q'' \in \kappa((W)) \setminus R\) so that the exponents \(\gamma\) such that \(x^\gamma\) appears in \(q\) and \(\gamma < v(q'')\) generate \(W\) over \(V\). **Reason.** Replacing \(q'\) by \(1/q'\) if necessary, we may assume that \(v(q') \geq 0\). Choose \(\gamma_1 < \cdots < \gamma_l\) as in part (3) such that \(\gamma_i, 1 \leq i \leq l\), generate \(W\) over \(V\). Now just let \(q''\) be \(q'\) multiplied by an element of \(F\) of value \(n\gamma_l\), where \(n\) is an integer such that \(\gamma_l < n\gamma_l \in V\). Let \(r = q + q''\), and \(S \subseteq G\) be the set of those \(\gamma \in G\) such that there exists \(a \in R\) with \(v(r - a) = \gamma\). It is easy to see that \(S\) is a lower cut in \(G\). Moreover, \(S\) does not have a last element. **Reason.** Assume that \(\gamma^* = v(r - b), b \in R\) is the last element of \(S\). Then by changing the \(\gamma^*\) slot of \(b\) suitably, we obtain an element \(b^* \in R\) such that \(v(r - b^*) > \gamma^*\) which
is a contradiction. Let \( p \) be obtained from \( r \) by deleting those terms in \( r \) for which \( \gamma \notin S \).

**Claim.** For all \( a \in R \) we have \( v(p - a) \in S \).

**Proof of the claim.** Assume, on the contrary, that there exists \( a \in R \) such that \( v(p - a) \notin S \). Let \( \gamma = v(r - a) \in S \). As \( S \) is a lower cut in \( G \), we have \( v(p - a) > \gamma \). So \( v(r - p) = \min\{v(r - a), v(p - a)\} = \gamma \in S \). Therefore, by the construction of \( p \), \( r \) and \( p \) have the same \( \gamma \)-th term. On the other hand, the inequality \( v(p - a) > \gamma \) implies that \( p \) and \( a \) have the same \( \gamma \)-th term. Hence \( r \) and \( a \) have the same \( \gamma \)-th term. But this is a contradiction to \( \gamma = v(r - a) \). Thus the claim is proved.

By Theorem 2.2.6, corresponding to \( S \) and \( p \) we have a value transcendental cut symbol \((S, p)\). On the other hand, \( v(q') = v(r - q) \in S \). Therefore, by the construction of \( p \), the exponents appearing in \( p \) generate \( W \) over \( V \). Thus by Theorem 4.2.1, we have \( v(F(p)) = W \).

**Subcase 2.** Suppose that \( W/V \) is countably infinite. Assume that \( W \) is generated over \( V \) by \( \alpha_1, \alpha_2, \ldots \). Without loss of generality we can assume that \( \alpha_i > 0 \), \( i \geq 1 \). For each integer \( n \geq 1 \), let \( \beta_n = \sum_{i=1}^{n} \alpha_i \). Therefore, \( 0 < \beta_1 < \beta_2 < \cdots \), \( \{\beta_i\}_{i \geq 1} \) generate \( W \) over \( V \), and no finite subset of \( \{\beta_i\}_{i \geq 1} \) generates \( W \) over \( V \). By the proof of part (1), for each \( i \geq 1 \) there exists \( q_i \in R \) with \( v(F(q_i)) \subseteq W \) such that \( v(q_i) = \beta_i \). Let \( p_1 = q_1 \) and \( i_1 = 1 \). There exists some \( \beta_j, j \geq 2 \) such that \( \beta_j \) is not in the value group of \( F(p_1) \). Let \( i_2 \) be the smallest of these numbers \( j \) and let \( p_2 = p_1 + q_{i_2} \). In the same way, let \( i_3 \) be the smallest \( i \) such that \( \beta_{i_3} \) is not in the value group of \( F(p_2) \) and let \( p_3 = p_2 + q_{i_3} \), etc. Therefore, we obtain a sequence \( \{p_i\}_{i \geq 1} \) of the elements of \( R \). Let the lower cut \( S \subseteq G \) consist of all elements \( \gamma \) such that \( \gamma < \beta_{i_k} \) for some \( k \). Moreover, let \( p \in \kappa((G)) \) be such that (i) \( v(p - p_k) > \beta_{i_k} \) for all \( k \), and (ii) \( p \) has no term in \( x^\gamma \) if \( \gamma \notin S \). Note that the sequence \( \{\beta_{i_k}\}_{k \geq 1} \) is cofinal in \( S \), and \( S \) does not have a largest element. On the other hand, for each \( a \in R \) we have \( v(p - a) \in S \). **Reason.** Assume, on the contrary, that there exists \( a \in R \) with \( v(p - a) \notin S \). Then the exponents \( \gamma \) of any term of \( p \) is the exponent of a term of \( a \). This means, by Theorem 4.2.1, that the value group of \( F(a) \) contains \( \{\beta_i\}_{i \geq 1} \). So \( v(F(a)) \) is infinite over \( V \). But this, as \( a \in R \), contradicts part (1) of Theorem
4.4.1. Therefore, \( v(p - a) \in S \) for all \( a \in R \). Then by Theorem 2.2.6, we have a cut in \( R \) with the cut symbol \((S, p)\). Thus by Theorem 4.4.1 (4), \( v(F(p)) = W \) and we are done. QED
Chapter 5

Truncation Closed Fields

5.1 Introduction

Suppose that $L$ (resp., $H$) is a subfield (resp., subgroup) of a formal power series field $k((G))$ (resp., a Hahn product $\mathcal{H}_{i \in I}G_i$). We say that $L$ (resp., $H$) is truncation closed if for each element $x = \sum_{i < \lambda} a_i x^i$ in $L$ (resp., $(\theta_i)_{i \in I}$ in $H$) and for any $\mu < \lambda$ (resp., $j \in I$) the truncated part $\sum_{i < \mu} a_i x^i$ (resp., $(\theta_i)_{i < j}$) belongs to $L$ (resp., $H$). We saw in chapter 3 how the knowledge of truncation-closedness could be related to the study of the cuts and order in a real closed field. When we were concerned about this subject, we found a paper written by M. H. Mourguès and J. P. Ressayre [MR]. This was actually very interesting to us. Although the purpose of their paper has been something else, namely, to show that every real closed field has an integer part, the main bridge through which their result is obtained is the proof that every real closed field can be embedded in a suitable formal power series field in such a way that its image is truncation closed (Corollary 4.2 in [MR]).

The main purpose in this chapter is to prove Theorem 5.5.1 and then Corollary 5.5.2 (which is Corollary 4.2 in [MR]). Theorem 5.5.1 is proved here using Proposition 3.2.3 and the results in [MR]. The first main step is to show that if $F$ is a truncation closed subfield of $k((G))$, where $k \subseteq \mathbb{R}$ and $G$ is a divisible ordered abelian group, and $y \in k((G)) \setminus F$ such that every truncated part of $y$ belongs to $F$ then $F(y)$ is truncation-closed (Corollary 5.3.5 which is Lemma 3.4 in [MR]). The second main step will be showing that the closure of $F(y)$ in $k((G))$ remains trunca-
tion closed under certain conditions (Corollary 5.4.3 which is Lemma 3.5 in [MR]). The result then follows by an application of Zorn's Lemma. In the last section, we conclude the chapter with a discussion of truncation-closedness in ordered abelian groups. One point to be mentioned here is that the results in this chapter, except for the proofs of Lemma 5.4.1 and of the theorems in section 5.5, are taken from [MR]. However, Theorem 5.5.1 which is proved by using Proposition 3.2.3 did not appear in [MR]. Theorem 5.6.2 is similar to Theorem 5.5.1 in the ordered abelian group case.

Throughout the chapter we assume that $k$ is a real closed subfield of $\mathbb{R}$ and $G$ is a divisible ordered abelian group.

### 5.2 Henselian elements over a field

Suppose that $L$ is a subfield of $k((G))$. Moreover, suppose that $v$ also denotes the restriction of the natural valuation $\nu$ on $k((G))$ to $L$. Let $P(X) = \sum_{i=0}^{n} A_i X^i \in B_L[X]$, where $B_L$ is the valuation ring of $L$. Define $\nu(P) = \min\{\nu(A_i)\}_{i=0}^{n}$. The polynomial $P$ is said to be primitive if $\nu(P) = 0$. It is easily seen that the product of two primitive polynomials is a primitive polynomial. Recall that the image of an element $y \in k((x))$ in the residue field $k$ is denoted by $\bar{y}$. Corresponding to $P$ we define $\bar{P} = \sum_{i=0}^{n} \tilde{A}_i X^i \in k[X]$. Let $y$ be an element in the real closure $L^{\sim}$ of $L$. Then it is obvious that $y$ has a minimal polynomial $P \in B_L[X]$ which is primitive. This minimal polynomial $P$ is called a primitive minimal polynomial of $y$.

**Definition 5.2.1** Suppose that $L \subseteq k((G))$ is a field with the real closure $L^{\sim}$. Let $y \in L^{\sim}$ with $\nu(y) = 0$. Suppose that $P(X) \in B_L[X]$ is a primitive minimal polynomial of $y$. We say that $y$ satisfies "condition H" over $L$, or $y$ is a Henselian element over $L$, if $\bar{P}'(\bar{y}) \neq 0$.

Let $L$ be a subfield of $k((G))$ such that $k \subseteq L$. Moreover, suppose that $L$ contains all $\pi^\gamma$ for $\gamma \in G$ (note that this implies in particular that $\nu(L) = G$).

**Proposition 5.2.1** Suppose that the field $L$ is as above. Let $y$ be an element of the real closure $L^{\sim}$ of $L$ and $\nu(y) = 0$. Then there exist $y_1, \ldots, y_k$ in $L^{\sim}$ such
that } y = y_1 + \ldots + y_l \text{ and for each } 1 \leq i \leq l, \ y_i/x^{v(y_i)} \text{ satisfies condition } H \text{ over } L(y_1, \ldots, y_{i-1}).

\textbf{Note.} (1) The above proposition is Lemma 2.5 in [MR]. (2) If } i = 1 \text{ then by } L(y_1, \ldots, y_{i-1}) \text{ we simply mean } L.

\textbf{Proof.} Let } P \text{ be the minimal polynomial of } y \text{ over } L. \text{ Without loss of generality, we can assume that } P \text{ is primitive. If } \tilde{P}'(\tilde{y}) \neq 0 \text{ then } y \text{ satisfies condition } H \text{ over } L \text{ and the proposition holds. So assume that } \tilde{P}'(\tilde{y}) = 0. \text{ Then as } P \text{ is primitive, } \tilde{P} \text{ is not identically } 0. \text{ On the other hand, } \tilde{P}(\tilde{y}) = \tilde{P}'(\tilde{y}) = 0. \text{ So } \tilde{y} \text{ is a root of } \tilde{P} \text{ with a multiplicity } m > 1. \text{ Therefore, } \tilde{P}(\tilde{y}) = \tilde{P}'(\tilde{y}) = 0. \text{ Hence } y_1/x^{v(y_1)} = y_1 \text{ satisfies condition } H \text{ over } L.

\text{Since } P \text{ is the minimal polynomial of } y \text{ while } P^{(m-1)}(y_1) = 0 \text{ and } m > 1, \text{ we have } y_1 \neq y. \text{ On the other hand, } \tilde{y}_1 = \tilde{y} \text{ and } v(y_1) = v(y) = 0; \text{ therefore, } \infty \neq \delta = v(y - y_1) > 0. \text{ Let } z_1 = (y - y_1)/x^\delta. \text{ Thus } v(z_1) = 0 \text{ and } z_1 \text{ is a root of the polynomial } Q(X) = P(x^\delta X + y_1). \text{ Using Taylor's formula, we obtain:}

\begin{align*}
Q(X) &= P(y_1) + x^\delta P'(y_1)X + \cdots + x^{(m-1)\delta} P^{(m-1)}(y_1)X^{m-1}/(m-1)! + \\
&\quad + \cdots + x^{n\delta} P^n(y_1)X^n/n!,
\end{align*}

\text{(5.2.36)}

\text{where } n \text{ is the degree of } P. \text{ Let } Q_1(X) = Q(X)/x^{v(Q)}. \text{ Therefore, } Q_1(X) \text{ is a primitive polynomial over } L(y_1) \text{ having } z_1 \text{ as a root. Then the primitive minimal polynomial } P_1 \text{ of } z_1 \text{ over } L(y_1) \text{ divides } Q_1. \text{ To find } y_2, \text{ just replace the polynomial } P \text{ by } P_1 \text{ and } y \text{ by } z_1. \text{ Therefore if } \tilde{P}'_1(\tilde{z}_1) \neq 0, \text{ then by choosing } y_2 \text{ to be just } z_1x^\delta, \text{ we obtain } y = y_1 + y_2 \text{ and are done. Otherwise, continue the process. We need to prove that this process ends after a finite number of iterations. For this purpose we will show that the multiplicity of } \tilde{z}_1 \text{ in } Q_1, \text{ and hence in } \tilde{P}_1, \text{ is strictly less than } m.\n
\text{Let } C_i, 0 \leq i \leq n, \text{ denote the coefficient of } X^i \text{ in } Q_1. \text{ } P^{(m-1)}(y_1) = 0, \text{ so } C_{m-1} = 0.

\textbf{Claim.} For } i > m, \text{ the coefficient } \tilde{C}_i \text{ in } Q_1 \text{ is zero.}
Proof of the claim. First note that \( v(P^{(m)}(y_1)) = 0 \), since otherwise \( v(P^{(m)}(y_1)) > 0 \) implies that \( \bar{P}^{(m)}(\bar{y}_1) = 0 \) which is a contradiction. Therefore, \( v(C_m) = m\delta - v(Q) \). Hence by the definition of \( v(Q) \), we have \( v(C_m) \geq 0 \). On the other hand, \( v(C_i) \geq i\delta - v(Q) > m\delta - v(Q) = v(C_m) \) for \( i > m \). Then we have \( v(C_i) > 0 \) for \( i > m \). Therefore, \( \bar{C}_i = 0 \) for \( i > m \) and the claim is proved.

So we obtain \( \bar{Q}_1(X) = \bar{C}_0 + \cdots + \bar{C}_{m-2}X^{m-2} + \bar{C}_mX^m \). The coefficient of \( X^{m-1} \) in this polynomial is zero, so \( \bar{Q}_1(X) \) can not have a nonzero root of multiplicity \( m \). Thus the multiplicity of \( \bar{z}_1 \) in \( \bar{Q}_1 \) is less than \( m \) and we are done. \( \Box \)

### 5.3 Truncation-closedness of \( L(y) \)

By an initial segment of \( z = \sum_{i \leq \lambda} a_i x^{\gamma_i} \in k((G)) \) we mean a truncation \( u = \sum_{i < \mu} a_i x^{\gamma_i} \) of \( z \). If \( \delta \in G \) is the minimum of \( \{ \gamma_i : i \geq \mu \} \) then \( u \) is also denoted by \( \sum_{\gamma_{\delta} < \delta} a_i x^{\gamma_i} \) or simply by \( (z)_{\delta} \). For any \( \delta \in G, \sum_{\gamma_{\delta} < \delta} a_i x^{\gamma_i} \) is an initial segment of \( z \). The set of all initial segments of \( z \) is a well-ordered set in the obvious way.

**Lemma 5.3.1** Let \( s, t \in k((G)) \) and \( (st)_{\delta} \) be a strict initial segment of \( st \). Then there exists a unique strictly increasing sequence \( (\alpha_0, \ldots, \alpha_n) \) of \( \text{Supp}(s) \) and a unique strictly decreasing sequence \( (\beta_0, \ldots, \beta_n) \) of \( \text{Supp}(t) \) such that for all \( 0 \leq i \leq n, \alpha_i + \beta_i \geq \delta \), and

\[
(st)_{\delta} = t(s)_{\alpha_0} + (t)_{\beta_0}((s)_{\alpha_1} - (s)_{\alpha_0}) + \ldots + (t)_{\beta_i}((s)_{\alpha_{i+1}} - (s)_{\alpha_i}) + \ldots + (t)_{\beta_n}(s - (s)_{\alpha_n}).
\]

(5.3.37)

**Proof.** See [MR]. \( \Box \)

**Corollary 5.3.2** Let \( H \) be a subfield of \( k((G)) \). Suppose that every initial segment of \( s, t \) belongs to \( H \). Then every initial segment of \( st \) also belongs to \( H \).

**Proof.** By Lemma 5.3.1, an initial segment of \( st \) is a finite sum of the products of initial segments of \( s \) and initial segments of \( t \). Therefore, the result is immediate. \( \Box \)
Lemma 5.3.3 Suppose \( s, t \in k((G)) \). Let \( t' = (t)_{<\beta_\lambda} \) be a strict initial segment of \( t \) (\( \beta_\lambda \) is the smallest element of \( \text{Supp}(t) \setminus \text{Supp}(t') \)). Then:

1. The following set

\[
\Delta = \{ \delta \in \text{Supp}(s) + \text{Supp}(t) : \forall \beta \in \text{Supp}(t'), v(s) + \beta < \delta \} \tag{5.3.38}
\]

is a non-empty and has a smallest element \( \gamma \).

2. There exist a unique increasing sequence \((\alpha_i)_{i=1}^{n_1}\) of \( \text{Supp}(s) \) and a unique decreasing sequence \((\beta_i)_{i=1}^{n_2}\) of \( \text{Supp}(t) \) such that \( \alpha_i + \beta_i \geq \delta \), for \( 1 \leq i \leq n \), and

\[
(st)_{<\gamma} = t'(s)_{<\alpha_1} + (t)_{<\beta_1} ((s)_{<\alpha_2} - (s)_{<\alpha_1}) + \ldots + (t)_{<\beta_n} (s - (s)_{<\alpha_n}) \tag{5.3.39}
\]

Moreover, \((t)_{<\beta_i}, 1 \leq i \leq n\), are strict initial segments of \( t' \).

Proof. (1) Because \( v(s) + \beta_\lambda \in \Delta, \Delta \neq \emptyset \). Moreover, since \( \text{Supp}(s) + \text{Supp}(t) \) is well-ordered, \( \Delta \) has a smallest element \( \gamma \).

(2) Let \((\alpha_i)_{i=0}^{n_1}, (\beta_i)_{i=0}^{n_2}\) be the sequences which correspond to \( (st)_{<\gamma} \) as in Lemma 5.3.1. \( v(s) + \beta_\lambda \in \Delta \), so \( v(s) + \beta_\lambda \geq \gamma \). On the other hand, \( v(s) \) is the smallest element of \( \text{Supp}(s) \). Therefore, the definition of \( \alpha_0 \) as in Lemma 5.3.1, implies that \( \alpha_0 = v(s) \). Moreover, \( \gamma \in \Delta \), so \( v(s) + \beta < \gamma \) for all \( \beta \in \text{Supp}(t') \). So the definition of \( \beta_0 \) implies that \( \beta_0 = \beta_\lambda \). Thus \( (s)_{<\alpha_0} = 0, (t)_{<\beta_0} = (t)_{<\beta_\lambda} = t' \), and the equation (5.3.39) holds. Since \((\beta_i)_{i=0}^{n_1}\) is decreasing, the last assertion in part (2) is obvious.

QED

Corollary 5.3.4 Let \( H \) be a subfield of \( k((G)) \), \( s \in k((G)), s \neq 0 \). If every initial segment of \( s \) belongs to \( H \), then every initial segment of \( 1/s \) will also belong to \( H \).

Proof. Assume, on the contrary, that there exists a strict initial segment of \( 1/s \) which does not belong to \( H \). Let \( t' = (1/s)_{<\beta_\lambda} \) be the shortest of such segments. Applying (the proof of) part (1) of Lemma 5.3.3 to \( t = 1/s \), we obtain \( \gamma < v(s) + \beta_\lambda \). Therefore, by part (2) of the same lemma we have

\[
(s.1/s)_{<\gamma} = t'(s)_{<\alpha_1} + C \tag{5.3.40}
\]

where \( C \) is a finite sum of finite products of initial segments of \( s \) and (by part (2) of Lemma 5.3.3) strict initial segments of \( t' \). But \((s.1/s)_{<\gamma}\) is 1 or 0. Moreover,
(s)_{<a_1} \neq 0 \) (since by the proof of part (2) of Lemma 5.3.3, \((s)_{<a_1} \neq (s)_{<a_0} = 0\)). Therefore, the equation (5.3.40) implies that \(C \in H\) which is a contradiction to the definition of \(t'\). QED

**Corollary 5.3.5** Suppose \(L\) is a subfield of \(k((G))\) closed under truncation. Let \(y \in k((G))\) be such that every strict initial segment of \(y\) belongs to \(L\). Then \(L(y)\) is also closed under truncation.

**Proof.** Suppose that \(t = \sum_{i=0}^{n} a_i y^i \in L[y] \subseteq L(y)\), where \(a_i \in L, 0 \leq i \leq n\). Let \((t)_{<\delta}\) be an initial segment of \(t\). Then \((t)_{<\delta} = \sum_{i=0}^{n} (a_i y^i)_{<\delta}\). Therefore by Corollary 5.3.2 and induction on \(n\), we see that \((t)_{<\delta} \in L[y]\). Now, each element of \(L(y)\) can be represented as \(t/s\) with \(s, t \in L[y], s \neq 0\). By Corollary 5.3.4, every initial segment of \(1/s\) belongs to \(L(y)\); therefore, using Corollary 5.3.2 once more, we conclude that every initial segment of \(t/s\) belongs to \(L(y)\). QED

### 5.4 Real closure of a truncation closed field

Our aim in this section is to prove that the truncation-closedness property remains valid through taking real closure. This will be done in two steps. We first show that if \(y\) belongs to the real closure \(L^\sim\) of \(L\) and satisfies condition \(H\) on \(L\), then every truncation of \(y\) will also belong to \(L^\sim\). To show this result, we need the following lemma:

**Lemma 5.4.1** (F. Delon) Suppose that \(K\) is a subfield of \(k((G))\) such that \(k \subseteq K\) and \(v(K) = G\). Moreover, suppose that \(y \in K^\sim \setminus K\) (\(K^\sim\) is the real closure of \(K\)) and \(y\) satisfies condition \(H\) over \(K\). Let

\[
I(y, K) = \{v(y - z) : z \in K\},
\]

Then \(I(y, K)\) is a lower cut of \(G\) closed under addition.

**Proof.** It is easy to see that \(I(y, K)\) is a lower cut of \(G\): Let \(z \in K\) and \(\alpha \in G, \alpha < v(y - z)\). Then there exists \(z' \in K\) such that \(v(z') = \alpha\). Therefore, \(\alpha = v(y - (z - z')) \in I(y, K)\). In order to prove that \(I(y, K)\) is closed under
addition, we then need to show that for all \( z, z' \in K \) there exists \( z^* \in K \) such that 
\[
v(y - z) + v(y - z') \leq v(y - z^*).
\]
This is clear if \( v(y - z) \leq 0 \) (by taking \( z^* = z' \)) or 
\[
v(y - z') \leq 0 \) (by taking \( z^* = z \)). So assume that \( v(y - z) > 0, v(y - z') > 0 \). Without 
loss of generality one can assume that \( v(y - z') \leq v(y - z) \). So 
\[
v(y - z) + v(y - z') \leq 2v(y - z).
\]
Thus we just need to show that for all \( z \in K \) with \( v(y - z) > 0 \), there 
exists \( z^* \in K \) such that \( 2v(y - z) \leq v(y - z^*) \).

Let \( P \) be a primitive minimal polynomial of \( y \) over \( K \). Since \( v(y - z) > 0 \), we 
have \( \tilde{y} = \tilde{z} \). Therefore, \( \tilde{P}'(\tilde{z}) = \tilde{P}'(\tilde{y}) \neq 0 \). Thus \( v(P'(z)) = 0 \) and hence \( P'(z) \neq 0 \).

Now we use Newton’s formula to find \( z^* \):
\[
z^* = z - \frac{P(z)}{P'(z)}
\]  
(5.4.41)

Therefore,
\[
y - z^* = y - z + \frac{P(z)}{P'(z)}
\]  
(5.4.42)

Now using Taylor’s expansion for the polynomial \( P(X) \) around \( z \) gives us
\[
P(X) = \sum_{i=0}^{n} \frac{P^{(i)}(z)}{i!} (X - z)^i
\]  
(5.4.43)

Since \( v(y - z) > 0 \), we have \( v(z) = 0 \) and hence \( v(P^{(i)}(z)) \geq 0 \). Therefore, by 
substituting \( y \) for \( X \) in (5.4.43) one obtains
\[
0 = P(y) = P(z) + P'(z)(y - z) + \text{terms of value} \geq 2v(y - z)
\]  
(5.4.44)

Using the fact that \( v(P'(z)) = 0 \) and dividing both sides of (5.4.44) by \( P'(z) \), we have
\[
0 = \frac{P(z)}{P'(z)} + (y - z) + \text{terms of value} \geq 2v(y - z)
\]  
(5.4.45)

Thus if we combine (5.4.42) and (5.4.45) then
\[
v(y - z^*) \geq 2v(y - z),
\]
and we are done. QED

Before stating the next result, we need to introduce some notations. Suppose 
that \( y \) belongs to \( L^- \) and \( y' \) is an initial segment of \( y \). We denote by \( G_0 \) the convex
subgroup of $G$ generated by $\text{Supp}(y')$. If $s \in k((G))$ then we denote by $(s)_{<G_0}$ the largest initial segment of $s$ such that $\text{Supp}((s)_{<G_0}) \subseteq G_0$. It is obvious that $y'$ is an initial segment of $(y)_{<G_0}$.

Lemma 5.4.2 Suppose that $L$ is a subfield of $k((G))$ closed under truncation. Moreover, suppose that $k \subseteq L$ and $v(L) = G$. Let $y \in L^\sim$ be such that $v(y) = 0$ and $y$ satisfies the condition $H$ over $L$. Then any truncation of $y$ belongs to $L^\sim$.

Proof. Assume, on the contrary, that there are some initial segments of $y$ which do not belong to $L^\sim$. Let $y'$ be the shortest of such initial segments. There are two cases to consider.

Case 1. $y' = (y)_{<G_0}$. In this case, let $P(X) = \sum_{i=0}^{n} A_i X^i$ be a primitive minimal polynomial of $y$ over $L$ (as in Definition 5.2.1), and define the polynomial $(P)_{<G_0}$ by $(P)_{<G_0}(X) = \sum_{i=0}^{n} (A_i)_{<G_0} X^i$.

Claim 1. $(P)_{<G_0}$ is a non-constant polynomial and $(P)_{<G_0}(y') = 0$.

Proof of claim 1. Since $y$ satisfies condition $H$ and $P$ is the primitive minimal polynomial of $y$, there exists $l, 0 < l \leq n$ such that $v(A_l) = 0$. Thus $(P)_{<G_0}$ is a non-constant polynomial. Write

$$P(X) = (P)_{<G_0}(X) + R(X), \quad (5.4.46)$$

where

$$R(X) = \sum_{i=0}^{n} (A_i - (A_i)_{<G_0}) X^i = \sum_{i=0}^{n} B_i X^i \quad \text{with} \quad v(B_i) > G_0 \quad (5.4.47)$$

Hence $v(R(y)) \geq \text{Min}(v(B_i)) > G_0$. Using Taylor's expansion, we obtain

$$(P)_{<G_0}(y) = (P)_{<G_0}(y') + T, \quad \text{where} \quad T = \sum_{i=1}^{n} \frac{(P)^{(i)}}{i!} (y - y')^i \quad (5.4.48)$$

Since $y' = (y)_{<G_0}$, we have $v(y - y') > G_0$. Therefore, $v(T) > G_0$. Using (5.4.46) and (5.4.48), we obtain $0 = P(y) = (P)_{<G_0}(y) + R(y) = (P)_{<G_0}(y') + T + R(y)$ with $\text{Supp}((P)_{<G_0}(y')) \subseteq G_0$. Moreover, we have $v(T + R(y)) > G_0$. Thus $(P)_{<G_0}(y') = 0$, and claim 1 is proved.
Thus in case 1, \( y' \) is a root of the non-zero polynomial \((P)_{<G_0}\). Since \( L \) is assumed to be truncation closed, this polynomial belongs to \( L[X] \). Therefore, \( y' \in L^\sim \) which contradicts the definition of \( y' \).

**Case 2.** \( y' \neq (y)_{<G_0} \). In this case, \( y' \) is a strict initial segment of \((y)_{<G_0}\). Proving the following claim obviously concludes the lemma.

**Claim 2.** If \( y' \neq (y)_{<G_0} \) then \( y' \) is in the field generated over \( L \) by its strict initial segments.

**Proof of claim 2.** Let \( F \) be the field generated over \( L \) by the strict initial segments of \( y' \). We just need to prove that \( y' \in F \).

First, note that \( F \) is closed under truncation. **Reason.** Let \( \{y_i\}_{i \in I} \) denote the set of strict initial segments of \( y' \), where \( I \) is a well-ordered set. For each \( i \in I \), let \( F_i \) denote the field generated over \( L \) by all \( y_j, j \leq i \). It is clear that \( F = \cup_{i \in I} F_i \).

If \( i_0 \) is the smallest element of \( I \) then \( F_{i_0} = L \) is closed under truncation. Now if \( F \) is not truncation closed, then there exists a smallest \( j > i_0 \) such that \( F_j \) is not closed under truncation and \( F_{i_0} \) is truncation closed for \( i < j \). Therefore, the field \( F' = \cup_{i<j} F_i \) is closed under truncation. But \( F_j = F'(y_j) \). Then, by Corollary 5.3.5, \( F_j \) is truncation closed which is a contradiction.

Now, if \( y \in F \) then \( y' \in F \) too and we are done. So assume \( y \in F^\sim \setminus F \), where \( F^\sim \) is the real closure of \( F \). As \( L \subseteq F \) and \( y \) satisfies condition \( H \) over \( L \), \( y \) will also satisfy condition \( H \) over \( F \). By Lemma 5.4.1, \( I(y, F) \) is a lower cut of \( G \) closed under addition. Moreover, since every initial segment of \( y' \) belongs to \( F \), we have \( \text{Supp}(y') \subseteq I(y, F) \). Then, \( G_0 \subseteq I(y, F) \) (this is obvious as \( \text{Supp}(y') \geq 0 \)).

Let \( \beta \) be the smallest element of \( \text{Supp}((y)_{<G_0}) \setminus \text{Supp}(y') \). Therefore, \( \beta \in I(y, F) \).

So there exists \( z \in F \) such that \( v(y - z) = \beta \). Then

\[
v(y' - z) \geq \text{Min}(v(y - y'), v(y - z)) = \beta
\]

But \( \text{Supp}(y') < \beta \); therefore, \( v(y' - z) \geq \beta \). This implies that \( y' \) is an initial segment of \( z \). Therefore, as \( F \) is truncation closed, we have \( y' \in F \). Thus the claim, and hence the lemma, is proved. \( \text{QED} \)
Corollary 5.4.3 (F. Delon) Suppose that $L$ is a truncation closed subfield of the field $k((G))$ such that $k \subseteq L$ and $v(L) = G$. Then $L^\sim$ is also truncation closed.

Proof. Suppose that $y \in L^\sim$. Without loss of generality we can assume that $v(y) = 0$: Let $\alpha = v(y)$. So $v(y/x^\alpha) = 0$. We have $v(L) = G$, $k \subseteq L$, and $L$ is truncation closed, so $x^\alpha \in L$ and hence $y/x^\alpha \in L^\sim$. But initial segments of $y$ are in $L$ if and only if those of $y/x^\alpha$ are in $L$.

Now, if $y$ satisfies condition $H$ over $L$ then by Lemma 5.4.2 we are done. Otherwise, the hypotheses on $L$ allow one to use Proposition 5.2.1 to get $y_1, \ldots, y_\ell$ in $L^\sim$ such that $y = y_1 + \cdots + y_\ell$ and for each $1 \leq i \leq \ell$, $y_i/x^{v(y_i)}$ satisfies condition $H$ over $L(y_1, \ldots, y_{i-1})$. For each $0 \leq i \leq \ell$, let $L_i$ be the closure under truncation and field operations of $L(y_1, \ldots, y_i)$. So we are done if we show that for all $0 \leq i \leq \ell$, the field $L_i$ is a subfield of $L^\sim$. For $i = 0$, this is true by the assumption on $L_0 = L$. Let $0 \leq j < \ell$ and assume the result for all $i \leq j$. So we just need to show that $L_{j+1} \subseteq L^\sim$.

The field $L_{j+1}$ is the closure under truncation and field operations of $L_j(y_{j+1})$. Now, every truncation of $y_{j+1}/x^{v(y_{j+1})}$ and hence that of $y_{j+1}$ belongs to $L^\sim$ by Lemma 5.4.2. Let $\{s_i\}_{i \in I}$ denote the set of initial segments of $y_{j+1}$, where $I$ is a well-ordered set. The field $L^1 = L_j(s_1)$ is truncation closed by Corollary 5.3.5. Moreover, $L^1 \subseteq L^\sim$. For each $l \in I$, $l > 1$, we can define $L^1 = (\cup_{p \in I} L^p)(y_l)$. Then a transfinite induction together with Corollary 5.3.5 imply that for each $l \in I$, $L^1$ is truncation closed and $L^1 \subseteq L^\sim$. Then it is easy to see that $\cup_{I \subseteq I} L^1 = L_{j+1}$. Thus $L_{j+1} \subseteq L^\sim$ and we are done. QED

5.5 Embeddings with truncation closed images

The embedding $\iota$ on the ordered field $F$ into some power series is said to be truncation closed if $\iota(F)$ is truncation closed. In the proof of the following theorem, the value group (resp., residue field) of a real closed field $F$ is denoted by $G_F$ (resp., $\kappa_F$).
Theorem 5.5.1 Suppose that $R_1 \subseteq R$ are real closed fields having respectively value groups $G_1 \subseteq G$ and residue fields $\kappa_1 \subseteq \kappa \subseteq R$. Moreover, suppose that there exists a proper truncation closed embedding $\iota_1 : R_1 \hookrightarrow \kappa_1((G_1))$. Then there exists a proper truncation closed embedding $\iota : R \hookrightarrow \kappa((G))$ which extends $\iota_1$.

Proof. Let $\mathcal{A}$ be the set of all ordered pairs $(F, \iota_F)$ such that $F$ is a real closed subfield of $R$ and $\iota_F : F \hookrightarrow \kappa_F((G_F))$ is a proper truncation closed embedding which extends $\iota_1$.

Claim. $\kappa_F \subseteq \iota_F(F)$. Consequently, $\kappa_1 \subseteq \iota_1(R_1)$.

Proof of the claim. By the proof of Theorem 1.4.8, a real closed field $F$ has a maximal archimedean subfield $S$ and there exists an isomorphism $\lambda$ from $\kappa_F$ onto $S$. By Proposition 1.1.6, $\lambda$ is an order isomorphism. Let $\iota'_F$ denote the restriction of $\iota_F$ to $S$ and let $s \in S$. We can write $\iota'_F(s) = a + q$, where $a \in \kappa_F$ and either $q = 0$ or $v(q) > 0$. If $q \neq 0$ then as $\iota_F$ is a truncation closed embedding, there exists $s' \in F$, $s' \neq s$ such that $\iota_F(s') = a$. The element $s'$ is archimedean and $\iota_F(s - s') = q$ which is impossible as $v(q) > 0$. Therefore, $\iota'_F$ is an embedding of $S$ into $\kappa_F$. We just need to show that $\iota'_F$ is surjective, or equivalently, it is enough to show that the isomorphism $\omega = \iota'_F \lambda : \kappa_F \rightarrow \kappa_F$ is surjective. In fact we show that $\omega$ is identity. If not, then there exists $t \in \kappa_F$ such that $t' = \omega(t) \neq t$. Without loss of generality, assume that $t' > t$. Then there exists a rational number $r$ such that $t' > r > t$. Then $r = \omega(r) > \omega(t) = t'$ which is a contradiction. Thus the claim is proved.

As $(R_1, \iota_1) \in \mathcal{A}$, $\mathcal{A}$ is not empty. We order $\mathcal{A}$ as usual: If $(F, \iota_F)$ and $(F', \iota_{F'})$ are two arbitrary elements of $\mathcal{A}$, then $(F', \iota_{F'}) \prec (F, \iota_F)$ if and only if $F' \subseteq F$ and $\iota_F|_{F'} = \iota_{F'}$. Now suppose that $\{(F_i, \iota_{F_i})\}_{i \in I}$ is a chain in $\mathcal{A}$. Then by using Theorem 1.1.3, it is easy to see that $F = \cup_{i \in I} F_i$ is real closed. Therefore, $(F, \iota_F) \in \mathcal{A}$ is an upper bound for $\{(F_i, \iota_{F_i})\}_{i \in I}$, where $\iota_F$ is the embedding defined on $F$ such that $\iota_F|_{F_i} = \iota_{F_i}$ for each $i \in I$. Thus by Zorn's Lemma, $\mathcal{A}$ has a maximal element $(K, \iota_K)$.

Claim. $K = R$.

Proof of the claim. If not then choose $x \in R \setminus K$. $K(x)$ is an ordered subfield of $R$. Therefore, by using Proposition 3.2.3 we obtain a proper embedding $\iota_K^* : (K^a, \iota_K) \hookrightarrow \kappa_K((G_K^a))$ which extends $\iota_K$, where $K^a$ is the real closure of $K(x)$ in $R$ and $G_K^a$
(resp., $\kappa^a_K$) the value group (resp., residue field) of $K^a$. We are going to show that $K^* = \iota_K^a(K^a)$ is truncation closed. But $K^*$ is the real closure of $\bar{K}(\phi^a)$, where $\bar{K} = \iota_K(K)$. Now depending on whether $x$ defines an immediate transcendental cut, a residue transcendental cut, or a value transcendental cut in $R$, there are three cases to consider:

Immediate transcendental case. Suppose that $\phi^a = p = \sum_{i<\lambda} c_i \tau_i \in \kappa^a_K((G^a_K))$, where $G^a_K = G_K$ and $\kappa^a_K = \kappa_K$. Let $p_\mu = \sum_{i<\mu} c_i \tau_i$, $\mu < \lambda$, be a strict initial segment of $p$. Then by the construction of $p$, $p_\mu$ is an initial segment of an element in $\bar{K}$. Since $\bar{K}$ is truncation closed, $p_\mu \in \bar{K}$. Therefore, by Corollary 5.3.5, $\bar{K}(\phi^a)$ is truncation closed. But by Proposition 3.2.3, $\bar{K}(\phi^a) \subseteq \kappa_K((G_K))$. Therefore, $G_K \subseteq G_K$ and $\kappa_K = \kappa_K \subseteq \iota_K(K) \subseteq \bar{K}(\phi^a)$. Thus we can use Corollary 5.4.3 to see that $K^*$ is truncation closed.

Residue transcendental case. Suppose that $\phi^a = p = \sum_{i<\lambda} c_i \tau_i \in \kappa^a_K((G^a_K))$, where $c_i \in N$, for $i < \lambda$, $c_\lambda \in N \setminus N$, $\kappa^a_K$ is the real closure of $\kappa(c_\lambda)$, and $G^a_K = G_K$. As in the previous case, every strict initial segment of $p$ belongs to $\bar{K}$; therefore, $\bar{K}(\phi^a)$ is truncation closed. So $c_\lambda \in \bar{K}(\phi^a)$ and hence $\kappa^a_K \subseteq K^*$. Let $L$ be the field generated by $\bar{K}(\phi^a)$ and $\kappa^a_K$. Using Corollary 5.3.5 along with Zorn's Lemma, we see that $L$ is truncation closed. Moreover, we have $G_L = G_{K^*} = G_K$. Then $L \subseteq \kappa^a_K((G_K))$ satisfies the hypotheses of Corollary 5.4.3; therefore, the real closure of $L$, i.e., $K^*$, is truncation closed.

Value transcendental case. Suppose that $\phi^a = p = \sum_{i<\lambda} c_i \tau_i \in \kappa^a_K((G^a_K))$, where $G^a_K$ is the divisible hull of $G_K(\tau_S)$ and $\kappa^a_K = \kappa_K$. In this case, every (not necessarily strict) initial segment of $p$ is an initial segment of an element in $\bar{K}$. Therefore, every strict initial segment of $\phi^a$ belongs to $\bar{K}$. So by Corollary 5.3.5, $\bar{K}(\phi^a)$ is truncation closed. By Theorem 4.2.1, the value group of $\bar{K}(\phi^a)$ is equal to $G_K(\tau_S)$. Let $\gamma \in G_K(\tau_S)$. Since $\bar{K}(\phi^a)$ is truncation closed, $\tau^\gamma \in \bar{K}(\phi^a)$. Now let $n$ be a positive integer. Using Corollary 5.3.5, we see that the field generated by $\bar{K}(\phi^a)$ and $\tau^\gamma/n$ is truncation closed. Therefore, using Zorn's Lemma one obtains an extension field $L \subseteq K^*$ of $\bar{K}(\phi^a)$ such that $v(L)$ is equal to the divisible hull of $G_K(\tau_S)$, i.e., $v(L) = G^a_K$. Then $L \subseteq \kappa_K((G^a_K))$ satisfies the hypotheses of Corollary 5.4.3;
therefore, the real closure of $L$, i.e., $K^*$, is truncation closed.

Therefore, in all cases $K^*$ is truncation closed and $(K^*, \iota_K^*) \in \mathcal{A}$. But this contradicts the maximality of $(K, \iota_K)$. Thus we are done with the proof of the claim and hence with that of the theorem. QED

**Corollary 5.5.2** Suppose that $R$ is a real closed field. Let $G, \kappa \subseteq \mathbb{R}$ be respectively the value group and the residue field of the natural valuation $\nu$ on $R$. Then there exists an embedding $\iota : R \hookrightarrow \kappa((G))$ such that $\iota(R)$ is truncation closed.

**Proof.** By the proof of Theorem 1.4.8, there exists $\kappa_0 \subseteq R$ such that $\kappa_0$ is isomorphic to $\kappa$. This means that there exists a proper truncation closed embedding $\omega_0 : \kappa_0 \hookrightarrow \kappa((\{0\}))$. Now use Theorem 5.5.1 to extend $\omega_0$ to a (proper) truncation closed embedding $\iota : R \hookrightarrow \kappa((G))$. QED

### 5.6 Truncation-closedness in ordered abelian groups

Now we want to discuss the truncation-closedness in the case of divisible ordered abelian groups. Suppose that $G$ is a divisible ordered abelian group with the corresponding Hahn product $\mathcal{H}_{i \in I} G_i$. A subgroup $D$ of $\mathcal{H}_{i \in I} G_i$ is said to be truncation closed if for each $\{\theta_k\}_{k \in I} \subseteq D$ and each $l \in I$, the truncation $\{\theta'_k\}_{k \in I}$ belongs to $D$, where $\theta'_k = \theta_k$ if $k < l$ and $\theta'_k = 0$ if $k \geq l$. An embedding $\iota : G \hookrightarrow \mathcal{H}_{i \in I} G_i$ is called truncation closed if $\iota(G)$ is truncation closed. Hausner-Wendel in their proof of the Hahn embedding theorem have actually proved that there exists an embedding of $G$ into its associated Hahn product such that its image is truncation closed [Ha, HW, Fu]. Here we state and prove that result by our method.

Suppose that $G$ is a divisible ordered abelian group. As in section 2.1.2, let the ordered set $I$ denote the value set of the natural valuation $w$ on $G$, and for each $i \in I$, let the archimedean divisible group $G_i$ be the residue group of $G$ at $i$. Actually, $G_i = D_i/C_i$, where $D_i = \{\gamma \in G : w(\gamma) \geq i\}, C_i = \{\gamma \in G : w(\gamma) > i\}$ are $\mathbb{Q}$ vector spaces. Therefore, there exists a $\mathbb{Q}$ vector space $G'_i$ such that $D_i = G'_i \oplus C_i$. $G'_i$ is easily seen to be (order) isomorphic over $\mathbb{Q}$ to $G_i$. For any divisible subgroup $L$ of $G$ let $\mathcal{H}_{i \in I} L_i$ be its associated Hahn product, where $I_L$ is the value set of $w|_L$. 
on \( L \) (note that \( I = I_G \)). If \( \bar{L} \) is also a divisible subgroup of \( G \) such that \( \bar{L} \subseteq L \) then \( I_L \subseteq I_L \subseteq I \); moreover, for each \( i \in I_L \) we can write \( \bar{L}_i \subseteq L_i \subseteq G_i \).

**Theorem 5.6.1 (Hahn's Embedding Theorem, [Ha])** Let \( H \) be a divisible subgroup of \( G \). Suppose that \( \iota_H : H \hookrightarrow \mathcal{H}_{i \in I_H} H_i \) is a truncation closed embedding over \( \mathbb{Q} \). Then there exists a truncation closed embedding \( \iota : G \hookrightarrow \mathcal{H}_{i \in I} G_i \) over \( \mathbb{Q} \) which extends \( \iota_H \).

**Proof.** Let \( \mathcal{L} \) be the set of all ordered pairs \( (L, \iota_L) \) such that \( L \supseteq H \) is a divisible subgroup of \( G \) and \( \iota_L : L \hookrightarrow \mathcal{H}_{i \in I_L} L_i \) is a truncation closed extension over \( \mathbb{Q} \) of \( \iota_H \) which embeds \( L \) into its corresponding Hahn product. There is a natural partial order \( \leq \) defined on \( \mathcal{L} \). It is easy to check that the hypotheses of Zorn's Lemma are satisfied on \( (\mathcal{L}, \leq) \). Therefore, \( (\mathcal{L}, \leq) \) has a maximal element \((K, \iota_K)\). We just need to show that \( K = G \). Assume, on the contrary, that there exists \( \delta \in G \setminus K \). Then we have a cut \((A, B)\) in \( K \) defined by \( A = \{\gamma \in K : \gamma < \delta\}, B = \{\gamma \in K : \gamma > \delta\} \). Corresponding to this cut there is a cut \((S, T)\) in \( K' = \iota_K(K) \): \( S = \{\lambda \in K' : \lambda = \iota_K(\gamma) \text{ for some } \gamma \in A\}, T = \{\lambda \in K' : \lambda = \iota_K(\gamma) \text{ for some } \gamma \in B\} \). Associated to \((S, T)\) we obtain, as in Proposition 2.2.3, a lower cut \( U \subseteq I_K \) and an element \( \theta \in \mathcal{H}_{i \in I_K} \mathbb{R} \).

**Case 1.** The cut \((S, T)\) is of value transcendental type. Then there exists \( \lambda \in K' \) so that \( w(\lambda - \theta) \notin U \). So we can fill the cut \((S, T)\) by a suitable element \( \theta \pm 1_U \) (as in Proposition 2.4.1). Since \( \iota_K \) is truncation closed, it is easily seen that \( \theta \in K' \). Let \( \gamma_0 = \iota_K^{-1}(\theta) \). Then \( \delta - \gamma_0 \in G \) fills the cut \((A - \gamma_0, B - \gamma_0)\) in \( K \), and \( \pm 1_U \) fills the cut \((S - \theta, T - \theta)\) in \( K' \). It follows that \( U < w(\delta - \gamma_0) < I_K \setminus U \). Hence we can define \( w(1_U) = w(\delta - \gamma_0) = l \in I \setminus I_K \). Thus by sending \( \delta - \gamma_0 \) to \( \pm 1_U \) (or equivalently, \( \delta \) to \( \theta \pm 1_U \)), we obtain an extension \( \iota_K \) of \( \iota_K \) defined on the group \( \bar{K} \) which is generated over \( \mathbb{Q} \) by \( K \cup \{\delta\} \) to the Hahn product associated to the group \( \bar{K} \). It is easily checked that \( \iota_K \) is truncation closed. But this contradicts the maximality of \((K, \iota_K)\).

**Case 2.** The cut \((S, T)\) is residue transcendental. Then \( U \) has a last element \( l \) and we can write \( \theta = \theta_0 + \theta_1 \), where \( \theta \in \iota_K(K) \) and \( \theta_1 \in \mathbb{R} \setminus G_i \). As in case 1, let...
\[ \gamma_0 = \iota_K^{-1}(\theta_0). \] The element \( \delta - \gamma_0 \) belongs to \( G \); moreover, it is not difficult to check that \( \theta_i \in G_i \setminus K_i \). Thus by sending \( \delta - \gamma_0 \) to \( \theta_k \) (or equivalently, \( \delta \) to \( \theta \)), we obtain a truncation closed extension \( \iota_K \) of \( \iota_K \) defined on the group \( \bar{K} \) which is generated over \( Q \) by \( K \cup \{ \delta \} \) to the Hahn product associated to the group \( \bar{K} \). But this also contradicts the maximality of \((K, \iota_K)\).

**Case 3.** The cut \((S, T)\) is immediate transcendent. Then \( \theta \in \mathcal{H}_{i \in I_K} K_i \setminus K' \). On the other hand, every proper truncation of \( \theta \) is a truncation of an element of \( K' \) and hence belongs to \( K' \). Therefore, by sending \( \delta \) to \( \theta \), we obtain a truncation closed extension \( \iota_K \) of \( \iota_K \) defined on the group \( \bar{K} \) which is generated over \( Q \) by \( K \cup \{ \delta \} \) to \( \mathcal{H}_{i \in I_K} K_i \) which is the same as the Hahn product associated to the group \( \bar{K} \). But this again contradicts the maximality of \((K, \iota_K)\).

Thus \( K = G \) and we are done. \[ \text{QED} \]

Let \( L \) be a divisible subgroup of \( G \). Then \( L_i \subseteq L \) is isomorphic to \( L_i \), for each \( i \in I_L \) (see the beginning of this section for the definition of \( G_i' \) which is associated to the group \( G \)). So there exists an embedding \( \psi_L \) of the direct sum \( \sum_{i \in I_L} L_i \) into \( L \). With a modification of the proof of the above theorem we obtain:

**Theorem 5.6.2** Let \( H \) be a divisible subgroup of \( G \). Suppose that \( \iota_H : H \to \mathcal{H}_{i \in I_H} H_i \) is a truncation closed embedding over \( Q \) such that the composition \( \iota_H \circ \psi_H \) is the identity map, where \( \psi_H : \sum_{i \in I_H} H_i \to H \) is the embedding defined above. Then there exists a truncation closed embedding \( \iota : G \to \mathcal{H}_{i \in I} G_i \) over \( Q \) which extends \( \iota_H \); moreover, the composition \( \iota \circ \psi \) is the identity map, where \( \psi : \sum_{i \in I} G_i \to G \) is an embedding as defined above and extends \( \psi_H \).
REFERENCES


