STUDY OF HIGH-N MODES IN TOKAMAKS USING A HIGH SPEED NONLOCAL GYROKINETIC MODEL

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Abstract

Gyrokinetic theory has been used to derive a system of integral equations which nonlocally describe low frequency, short wavelength modes in a plasma of axisymmetric toroidal geometry with low-$\beta$ and circular nonconcentric flux surfaces with small Shafranov shift. The eigenmode equations contain the two potential approximation in $\varphi$ and $A_\parallel$ with full finite Larmor radius and trapped electron effects in the collisionless limit. The analysis makes use of the so-called "ballooning formalism" to lowest order in $1/n$ which yields a radially local calculation for the eigenfrequencies and the eigenfunctions. This representation, in conjunction with an efficient numerical algorithm, allows the eigenfrequencies to be computed with sufficient accuracy and high speed for arbitrary high-$n$ modes in the drift and shear-Alfvén branches. This is the main accomplishment of this work. Test cases using artificial and actual tokamak experimental discharge parameters for the collisionless-trapped-electron, ion-temperature-gradient and ballooning modes have been benchmarked with the premium, comprehensive kinetic formulation of Rewoldt exhibiting favourable results.
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Chapter 1

Introduction

1.1 Preliminary Perspective

The advancement of civilization in the past century has greatly relied on, and is due in large measure to energy. Sufficient energy resources are reflected in a nation’s ability to power its industries, supply food for its citizens and create a general environment of economic well being. Controlled thermonuclear fusion is emerging as a promising candidate to meet the world’s long term energy needs and offers significant advantages that are well known. However, the path towards this ambitious realization has been hampered by a myriad of technological and theoretical difficulties which have proven to be quite formidable to resolve. This is to be expected since the ultimate goal amounts to no less than containing a small star in a cold, terrestrial type environment. The sun and the stars are able to sustain their thermonuclear reactions as a result of their inherent and immense gravitational forces which both confine and heat their plasmas to the required densities and temperatures. Since it is currently impossible to replicate the gravitational forces needed to contain man-made plasmas on earth, a
different avenue to confinement must be employed.

The basic device used to contain a hot dense plasma is a "magnetic bottle" which can be of the form of a torus with closed magnetic field lines or open-ended linear "mirror machines". To date, the toroidal geometry concept in the tokamak configuration has yielded the most scientific progress and has been instituted as the main vehicle for high temperature research by the majority of the world fusion program. Although the tokamak has shown to be the leading candidate in approaching ignition regimes, it does not necessarily lend itself as the most expeditious solution. Other magnetic fusion reactor concepts are beginning to show much promise as well, most notably the stellerator [1][2], the spherical torus [3][4] and the reversed field pinch [5][6]. This thesis will consider only the conventional tokamak concept for studying plasma phenomena related to fusion, specifically the virulent, limiting processes called plasma instabilities that can terminate or degrade the tokamak discharge.

Instabilities are a natural response to the forces of constraint imposed upon the plasma. They manifest themselves in the form of wave perturbations that transport energy and plasma constituents away from the core and outward towards its periphery. The result is a loss of plasma current and energy and an immediate deterioration in plasma confinement. Hence, a tokamak plasma can only reach thermonuclear conditions if it exists in a certain discharge parameter regime. One of the main purposes of tokamak plasma theory is to elucidate and determine the limits of this operational regime where instabilities do not exist or are at least benign. Plasma modelling, born out of stability studies, usually addresses only very specific classes of instabilities or modes rather than purporting to simulate all or many modes in a pandemic fashion. This in turn reflects the complexity of the plasma, the reason being the vast range of length and timescales that characterize each class of instability.
The main purpose of this thesis is to develop a comprehensive, non-local eigenmode kinetic model that can recover two general classes of high toroidal mode number instabilities that are indigenous to tokamaks, namely the shear-Alfvén and drift mode branches. Of these, three particular modes will be selected for study and benchmarking; the ballooning mode, the ion temperature gradient (ITG) mode and the collisionless trapped electron (CTE) mode. The major accomplishment of this work is that these modes can be recovered with significant speed and sufficient accuracy through the combined utilization of the so-called ballooning representation [7][8][9] and efficient algorithmic manipulation of the integral equation nature of the formulations, while still considering realistic discharge conditions. This allows the model to be employed without the use of a supercomputer, as is usually the case with analyses of this type, and enables small research institutions with limited resources the opportunity to pursue studies in an area otherwise unafforded to them. To ascertain consistency of the results and to benchmark the code, comparisons will be made with the model of Rewoldt [10][11], which is considered to be one of the most comprehensive eigenmode analyses of this kind to date, in recovering some results from actual Tokamak experiments [12][13][14][15]. The remainder of this introductory chapter will provide a brief synopsis of the instabilities to be studied and some of the early and current models used to describe them. The chapter then concludes with a summary of the major findings.

1.2 Review of Tokamak Instabilities

The tokamak is essentially a device that contains a toroidally circulating plasma current which can be induced by a transformer (Ohmic drive) and/or formed by the
bootstrap effect [16]. The plasma itself forms a single turn secondary winding, the primary being wound on the transformer core. Closed poloidal magnetic surfaces are produced by the current flow and surround the plasma thereby serving to confine it. A strong toroidal magnetic field is needed to stabilize the plasma torus and is generated by a toroidal field coil system. Fig. 1.1 illustrates a basic tokamak configuration. Additional vertical and horizontal magnetic field systems may also be used to control the position of the plasma within the containment vessel. Ohmic heating due to induction can heat the plasma to a temperature of approximately $T \approx 1 - 3$ keV but, since plasma resistivity decreases with temperature, becomes ineffective beyond this range. Supplementary heating in the form of neutral beam or ion cyclotron resonance frequency heating is then required.

The combination of the above fields can generate a plasma equilibrium but only tenuously. The delicate balance of magnetic forces in harmony with those of the plasma may become unstable if the system is not in a perfect thermodynamic equilibrium state. Wave excitation is possible which serves to lower the amount of free energy available in the system and allows the plasma to achieve a more true thermodynamic equilibrium at the expense of degrading particle and energy confinement. The type of free energy available usually manifests itself in the form of two broad instability classifications, namely current-driven modes and pressure-driven modes. The instabilities to be investigated in this work are of the pressure-driven type and have been chosen for two reasons. The first is that in the linear regime, when the toroidal mode number $n$ becomes large, these instabilities generally exhibit a strong ballooning type character which allows the equations that describe them to transform into one dimensional form (in coordinate space) through use of the aforementioned ballooning representation. The resulting reduced equations are then solved along the
Figure 1.1: a) Schematic of a basic tokamak system and b) a closer view of the corresponding magnetic field geometry (figures adapted from R. O. Dendy [17] p. 32).
equilibrium magnetic field lines on a single, chosen magnetic surface to lowest order
in 1/n with appropriate boundary conditions. This transformation retains most of
the essential features of these modes and allows the problem to be greatly simplified
which thus enables ordinary personal computers (i.e. Pentium systems) or work sta-
tions (i.e. Alpha or Sunsparc systems) to perform the simulation calculations with
reasonable speed and accuracy. Secondly, low frequency drift-type electromagnetic
microinstabilities driven by ion temperature gradient and/or trapped-particle dynam-
ics have continued to be strongly supported candidates to account for the anomalous
particle and thermal transport observed in all tokamak plasmas [18][19][20][21][22].
These modes, in association with ballooning instabilities, represent in large part the
most serious of obstacles for efficient plasma confinement and deserve further study.

1.2.1 Interchange and Ballooning Instabilities

Historically, the first paper to address the problem of the interchange instability
was that of Suydam [23], who obtained the stability criterion for a cylindrical, cur-
rent carrying plasma column for flute perturbations parallel to magnetic field lines.
Rosenbluth et al. [24] also investigated these modes and extended the work to include
finite Larmor radius effects. Later. Furth, Kileen, Rosenbluth and Coppi [25] took
the same basic model, included the effect of curvature of the field lines by a variable
gravitational force, and discovered the ballooning instability. The ballooning mode
is analogous to the aneurisms which develop at weak spots in a pressurized elastic
container. In tokamak plasmas, these modes manifest themselves by the interaction
of the plasma pressure gradient with local regions of unfavourable magnetic curva-
ture which cause the plasma to bulge or “balloon” out in these regions. Accessing
the so called “second stability regime” [26][27][28][29] in (s, \(\alpha\)) space where s is the
shear parameter and $\alpha$ the normalized pressure gradient is seen as a promising way of circumventing this problem but to date has not seen experimental fruition [30][31][32].

**Interchange Instabilities**

The interchange instability forms as a consequence of the plasma pressure gradient combining with adverse magnetic field curvature. It is similar in nature to the Rayleigh-Taylor instability in which a heavy fluid (plasma) is supported by a lighter fluid (magnetic field) in the presence of a gravitational force (magnetic field line curvature). Consider conditions restricted to ideal MHD motion in which the plasma is perfectly conducting. The magnetic flux is frozen into the plasma and electromagnetic and fluid perturbations are not independent, but completely described by the displacement of fluid volume elements. Thus, any motion of the plasma consists of an “interchange” of nearby volume elements or flux tubes, which carry their magnetic fields along with them. Instability occurs when the relative sign of the magnetic field line curvature and plasma pressure gradient are considered. If the field lines are convex towards the plasma, their tension tends to make them shorter and collapse inward. The plasma pressure, however, has a natural tendency to expand outward. An interchange of a flux tube at higher pressure will occur with a flux tube at lower pressure. Only when the field lines are concave to the plasma, is the system stable to interchange perturbations.

This instability is also known as a flute instability due to the fact that the resultant interchanged fluid elements produce surfaces of constant density which resemble fluted Greek columns that extend in the direction of the magnetic field. This fluting or rippling of the plasma boundary is characterized by the electrostatic potential $\varphi$ of the induced electric field that results from charge separation of the ions and electrons.
Figure 1.2: Electron, ion and $E \times B$ drifts in the interchange instability.

Particles travelling along the curved magnetic field lines encounter a centrifugal or effective gravitational force $g$ directed along the radius of curvature. Ions and electrons drift in opposite directions to each other, due to the acceleration, with drift velocities

$$V_i = \frac{M g \times B}{e B^2}, \quad V_e = -\frac{m g \times B}{e B^2}$$  \hspace{1cm} (1.1)

where $B$ is the magnetic field, $m$ and $M$ are the electron and ion masses respectively and $e$ is the electron/ion charge. If the plasma boundary is displaced by an interchange of volume elements, a charge separation will result due to the opposite ion and electron drifts thus producing an electric field. The resultant $E \times B$ drift enhances the original perturbation with more volume elements being interchanged as seen in Fig. 1.2. The direction of the magnetic field curvature with respect to the pressure gradient determines whether the perturbation will grow or be damped.

A simple electrostatic MHD model can be given to illustrate and qualitatively approximate the growth rate of the interchange instability. Consider a perfectly conducting plasma within an arbitrary magnetic field. The linearized continuity equation
of the ion guiding-centres without the ion temperature perturbation is

\[
\left( \frac{\partial}{\partial t} + \mathbf{V}_{di} \cdot \nabla \right) n_i + \nabla \cdot \left( n_0 \mathbf{v}_E \right) + n_0 \nabla \cdot \mathbf{v}_{pi} = 0 \tag{1.2}
\]

where \( n_i \) and \( n_0 \) are the perturbed and nonperturbed ion densities, and \( \mathbf{V}_{di}, \mathbf{v}_E \) and \( \mathbf{v}_{pi} \) are the averaged ion magnetic, \( \mathbf{E} \times \mathbf{B} \) and ion polarization drift velocities respectively, given as:

\[
\mathbf{V}_{di} = \frac{2cT_i}{eB^2} \mathbf{B} \times \nabla B, \tag{1.3}
\]

\[
\mathbf{v}_E = c \frac{\mathbf{B} \times \nabla \varphi}{B^2} \left( 1 - k_r^2 \rho_t^2 \right) \tag{1.4}
\]

and

\[
\mathbf{v}_{pi} = - \frac{e}{\rho_i \Omega_i} \left( \frac{\partial}{\partial t} + \mathbf{V}_{di} \cdot \nabla \right) \nabla \varphi \tag{1.5}
\]

where \( \Omega_i \) is the ion cyclotron frequency and \( \nabla \perp \) is the perpendicular gradient with respect to the equilibrium magnetic field. Details of a proper analysis with the ion temperature perturbation may be found in refs. [21] or [33]. It is noted that the ion diamagnetic drift velocity \( \mathbf{V}_{di} \) does not appear in the convective derivative because it is not a guiding-centre drift. By substituting \( \mathbf{V}_{di}, \mathbf{v}_E \) and \( \mathbf{v}_{pi} \) into Eq. (1.2), Fourier transforming components that vary as \( \exp \left( \mathbf{k} \cdot \mathbf{x} - \omega t \right) \) and simplifying, one can obtain the perturbed ion density,

\[
n_i = \left[ \frac{\omega_{ie} - \omega_{de}}{\omega + \omega_{di}} - \frac{\omega + \omega_{ei}}{\omega + \omega_{di}} \left( k_r \rho_i \right)^2 \right] \frac{e\varphi}{T_e} n_0, \tag{1.6}
\]

where \( \rho_i = \sqrt{T_e/M/\Omega_i} \) is the ion Larmor radius with electron temperature, \( \omega_{de(i)} = k_r \mathbf{V}_{de(i)} \) and \( \omega_{ei} = k_r \mathbf{V}_{ei} \) are the electron (ion) magnetic and diamagnetic drift frequencies respectively, with \( \mathbf{V}_{ei} \) given as

\[
\mathbf{V}_{ei} = \mp \frac{cT_e(i)}{eB^2} \mathbf{B} \times \nabla \ln n_0 \tag{1.7}
\]
where the minus (plus) sign corresponds to electrons (ions) respectively. The electron continuity equation is similar to that of the ions, with the exception of the omission of the inertia term and is given in linearized form as

\[
\left( \frac{\partial}{\partial t} + \mathbf{V}_{de} \cdot \nabla \right) n_e + \nabla \cdot (n_0 \mathbf{V}_E) = 0.
\]

(1.8)

where \( n_e \) and \( n_0 \) are the perturbed and nonperturbed electron densities. In a similar fashion, one can readily obtain the perturbed electron density,

\[
n_e = \frac{\omega_{se} - \omega_{de} e \varphi}{\omega - \omega_{de} T_e} n_0,
\]

(1.9)

where we assume a shearless magnetic field for which the variation along the magnetic field is negligibly small. \( \mathbf{B} \cdot \nabla \varphi \simeq 0 \) or \( k|| \simeq 0 \) (\( k|| \) is the wavenumber parallel to \( \mathbf{B} \)). Since the plasma is ideally conducting, it retains a high degree of charge neutrality (\( n_i \simeq n_e \)). Thus equating Eq. (1.6) with Eq. (1.9) yields the following electrostatic dispersion relation

\[
\omega^2 = -2 \frac{(\omega_s - \omega_d) \omega_d}{(k_\perp \rho)^2}
\]

(1.10)

where it is assumed that the particle species have near equal temperatures, \( T_i \simeq T_e \). Instability is characterized by the sign of \( \omega^2 \). from which, \( \omega^2 > 0 \) indicates stability and \( \omega^2 < 0 \) instability. From our simple system it is clear that \( \omega^2 \) is always real and hence \( \omega \) itself is either real (oscillating or stable system) or purely complex (growing or unstable system). For instability to result, the diamagnetic and magnetic drifts must occur in the same direction or else \( \omega_s \omega_d \) will be negative. This corresponds to the pressure and magnetic gradients orientated in the same direction since \( \omega_s \propto \nabla p \) and \( \omega_d \propto \nabla B \). Hence the growth rate of the interchange instability is approximately

\[
\gamma \equiv \text{Im} (\omega) \sim \frac{\sqrt{\omega_s \omega_d}}{k_\perp \rho} \sim \frac{c_s}{\sqrt{L_n R}}
\]

(1.11)
where \( c_s \equiv \sqrt{T_e / M} \) is the acoustic speed. \( L_n \equiv - (d \ln n_0 / dr)^{-1} \) is the density gradient scale length of the plasma and \( R \) is the radius of curvature of the magnetic field lines. Note that in deriving the result of Eq. (1.11), long cross-field wavelengths \( (k_- \rho)^2 \ll 1 \) and large mode frequency \( |\omega| \gg \omega_e, \omega_d \) have been assumed.

**Ballooning Instabilities**

The ballooning mode is also driven by the pressure gradient and is a variation of the interchange instability. It primarily occurs in tori or multidimensional confinement systems and imposes a constraint on \( \beta \), the ratio of plasma pressure to magnetic pressure, \( \beta = 8 \pi p / B^2 \) where \( p \) is the plasma pressure. This constraint is considered important since it is economically desirable to maximize the fusion fuel density while attempting to minimize a costly magnetic field which is also technologically limited. Tokamak scaling studies [34][35] have shown that reactor performance depends sensitively on \( \beta \) and that a minimum value in the range of 5% - 10% is required for satisfactory operation. In order for a fusion reactor to be economically feasible, \( \beta \) values well in excess of this must be achieved, and hence the attention that this mode has received over the years is well warranted.

The ballooning instability is considered to be one of the most detrimental in terms of plasma confinement since the amplitude of the perturbation is approximately constant along the equilibrium magnetic field line, \( i.e. \) flute-like. This is undesirable since stabilization can only be effected through line bending which is usually inherent in a perturbation which varies along a field line. Ballooning modes in which line bending occurs are a form of destabilized shear-Alfvén mode and are analogous to transverse waves propagating on a string under tension. Heuristically, this mode can
be illustrated by the following equation
\[ \omega^2 = k_{||}^2 (r, \theta) V_A^2 - \frac{\omega_i \omega_d}{(k_{\perp} \rho)^2}. \] (1.12)

Here \( k_{||}^2 (r, \theta) \) is the local value of the component of the mode wave vector along the equilibrium magnetic field and is a function of the minor radius \( r \) and the poloidal angle \( \theta \). \( V_A = \sqrt{B_0^2/4\pi \rho_0} \) is the Alfvén speed where \( \rho_0 \) is the plasma density, and the last term represents the interchange effect as seen before in Eq. (1.11). The shear term \( (k_{||} V_A)^2 \) arises from the restoring force due to the tension in the magnetic field lines and depends on \( r \) because the direction of the magnetic field changes with it (magnetic shear). It also depends on \( \theta \) because the relative orientation of the field lines with respect to the pressure gradient is a function of \( \theta \) so that energy is to be gained by the plasma perturbation if the amplitude of the mode is localized on the outside \( (\theta = 0) \) of the torus. For modes so localized, the interchange term prevails and the following condition
\[ \frac{\omega_i \omega_d}{(k_{\perp} \rho)^2} > (k_{||} V_A)^2 \] (1.13)
ocurs which results in a growing instability. The shear term provides a measure of stabilization since it imposes a threshold on the magnitude of the interchange term in order for instability to occur. This threshold condition can also be used to crudely approximate a value for critical \( \beta \), above which ballooning appears. The parallel wave number \( k_{||} \) can be approximately given by
\[ k_{||} \simeq \frac{2\pi}{L_c} = \frac{1}{qR} \] (1.14)
where \( L_c = 2\pi qR \) is the connection length for a closed field line. Using this expression for \( k_{||} \) and using the condition given by the inequality of (1.13) yields the critical \( \beta \) value
\[ \beta_c = \frac{1}{q^2 \frac{L_p}{R}} \] (1.15)
Figure 1.3: A typical ballooning displacement in which plasma motion occurs across the field lines (figure adapted from R. B. White [36] p. 123).

where $L_p$ is the scale length of the pressure gradient. Compressional plasma perturbations, which can occur either parallel or perpendicular to the field lines, are also possible and are referred to as magnetosonic or compressional Alfvén modes. These modes only become significant when $\beta$ becomes large.

The term “ballooning”, as so designated by the plasma physics literature due to its spatial appearance, arises from the fact that in multidimensional geometries such as a tokamak, the curvature of the magnetic field often alternates between regions which are “favourable” and “unfavourable”. “Favourable” regions refer to the inner side of the plasma torus where the curvature of the magnetic field is concave towards the plasma whereas “unfavourable” is just the opposite; the field lines are convex towards the plasma. Thus a perturbation which varies gradually along a field line allows the mode to be concentrated or to “balloon” out on the unfavourable side as seen in Fig. 1.3. This results in an increase of the pressure-driven, destabilizing contribution to
the amount of free energy available to the plasma. The favourable curvature region provides a conduit from which electrons can rush quickly along the magnetic field lines to charge neutralize the ions since interchanges are suppressed here. The situation in the unfavourable curvature region is, however, reversed. The magnetic field gradient and curvature induced drifts of the electrons and ions cause charge separation which cannot be shorted out and leads to \( \mathbf{E} \times \mathbf{B} \) drift and instability. If the plasma is highly collisional, the electron motion is impeded and the charge separation is sustained which effectively disconnects the good and bad curvature regions from each other. This is analogous to the way a large resistor in an electrical circuit causes a capacitor in series to discharge very slowly. If the plasma resistivity is small, there will exist a large parallel current which tries to short out any excess charge. However, since the plasma is highly conducting, a large inductance will result which will oppose any rapid increase in magnetic field energy by producing a back emf. Once again, the electron parallel motion is hindered and the good and bad curvature regions are isolated. This is similar to the way a large inductor will inhibit the discharge of a capacitor in series. Hence two basic types of ballooning modes are characterized on the basis of whether the electron motion is impeded resistively or inductively. The former are basically electrostatic and the latter electromagnetic. However, since resistive ballooning modes require an unrealistically low plasma temperature, they are not expected to be significant [37].

It is of interest to determine the robustness of the ballooning mode in terms of its stability properties when other effects such as trapped electrons are considered. Trapped electrons are known to weakly stabilize this mode [14][38][39] as they reduce the destabilizing circulating-electron, parallel current perturbation. This effect has been implemented as a generalization of the model presented in this thesis where
damping of the ballooning mode has also been observed. As will be seen later, trapped electrons play an important role in other low-frequency modes such as drift wave instabilities where these particles can cause instability and coupling among some of the various drift branches.

1.2.2 Drift Type Instabilities

Drift wave instability studies date back to the early sixties in which long wavelength modes due to temperature gradients and current along field lines were first discovered by Rudakov and Sagdeev [40] and Kadomtsev [41]. The investigation of these drift waves in the short wavelength limit by Galeev [42], Mikhailovskii and Rudakov [43], and Kadomtsev and Timofev [44] revealed that such waves were unstable even in the absence of temperature gradients and parallel current, provided that only a density gradient was present. Due to the pervasive nature of this mode, the term “universal” has also been adopted in the nomenclature to describe it. Efforts to determine the extent of this mode indicated that damping could be effected by magnetic shear when considering plane slab geometry [45][46]. In toroidal geometry, a branch of the drift mode exists for which shear damping can be suppressed by the curvature of the magnetic field lines [47][48]. When the curvature is large enough, the mode is marginally stable and can be destabilized by dissipative effects such as collisions, resonant electrons or trapped electrons.

In contrast to interchange modes, drift waves have a small but finite component \( \mathbf{k}_1 \) along \( \mathbf{B} \). The constant density surfaces thus resemble fluted columns akin to the interchange mode but with a slight helical twist. A simple model based on the equations of motion for the particle guiding-centres can be used to illustrate the mechanism of the basic mode in the absence of magnetic field curvature and gradient.
Drift waves are primarily electrostatic and the dispersion relation for collisionless, long cross-field modes, \((k_\perp \rho)^2 \ll 1\), can be derived from the following lowest order linearized equations:

\[
\frac{\partial}{\partial t} n_i + \nabla \cdot (n_0 v_E) + n_0 \nabla \cdot v_{\parallel i} = 0,
\]

\[
\frac{\partial}{\partial t} v_{\parallel i} = -\frac{e}{M} \nabla \parallel \varphi (1.16)
\]
and

\[
n_e = \frac{e \varphi}{T_e} n_0. (1.18)
\]

Eq. (1.16) is the guiding-centre equation for the ions with the inclusion of the perturbed parallel ion velocity \(v_{\parallel i}\) and Eq. (1.17) is the component of the ion equation of motion parallel to the magnetic field. This indicates that only parallel compressibility enters into the analysis. Perpendicular compressibility would enter if the effects of magnetic field line curvature were included. Assuming isothermal conditions, as in the case of the interchange mode, allows the electrons to quickly establish parallel equilibration in comparison to the slow parallel ion dynamics thus establishing the ordering, \(k_{\parallel} v_{ti} \ll \omega \ll k_{\parallel} v_{te}\), where \(v_{ti}\) and \(v_{te}\) are the ion and electron thermal velocities respectively. The adiabaticity of the electrons can be then be adequately described by the Boltzmann distribution given by Eq. (1.18). The parallel gradient is taken along the unperturbed field line, \(\nabla \parallel = b \cdot \nabla\), with \(b \equiv B_0 / B_0\).

The perturbed parallel ion velocity \(v_{\parallel i}\) can be found from Eq. (1.17) which is then substituted into Eq. (1.16) to yield the perturbed ion density \(n_i\). From quasineutrality, \(n_i = n_e\), the dispersion relation is immediately given as

\[
\omega^2 - \omega \omega_{ce} - c_s^2 k_{\parallel}^2 = 0. \quad (1.19)
\]

When \(k_{\parallel}\) becomes large, such that \(c_s^2 k_{\parallel}^2 \gg (\omega_{ce})^2\), the two branches of the solution
turn into the ordinary ion acoustic wave,

$$\omega = c_s k_\parallel.$$  \hfill (1.20)

The upper branch departs from the acoustic mode and approaches $\omega_{ae}$ in the limit $k_\parallel \to 0$ where the waves are propagating nearly perpendicular to $\mathbf{B}$. The simple drift wave branch then presents itself as

$$\omega = \omega_{ae}. \hfill (1.21)$$

The prevalence of perturbations which scale as $\omega \sim \omega_{ae}$ in both experiments and theoretical calculations is a major reason for studying drift ordered evolution. Note that this simple result can also be derived by the hydrodynamic response nature of the ions which reflects perpendicular convection in $v_E$.

$$\frac{d}{dt} n_i \simeq 0. \hfill (1.22)$$

Solving for $n_i$ and equating with $n_e$ through quasineutrality yields the expected result.

The drift waves considered thus far are purely oscillatory. The charge separation that occurs between electrons and ions due to ion inertia is neutralized by rapid electron motion along the lines of magnetic force. However, if some of the electrons are moving slowly along the magnetic field line, specifically those satisfying the condition $\omega \simeq k_\parallel v_{te}$, resonance with the wave can occur. This resonant contribution leads to instability of “electron” drift waves which results from a nonadiabatic electron deviation from the Boltzmann distribution as

$$n_e = (1 - i \delta) \frac{e_0}{T_e^2} n_0 \hfill (1.23)$$

where $\delta$ is the nonadiabatic electron response and is the term responsible for instability or damping. In addition to resonant electrons, it can also arise from the
aforementioned dissipational effects such as collisional resistance and trapped electrons.

Ion drift waves are also possible if the mode frequency becomes comparable to the ion transit frequency, $\omega \sim k_i v_{ti}$. Although it is known that resonant ion effects tend to stabilize or even dampen these waves, an exception occurs if the resonance takes place in the presence of an unusually strong ion temperature gradient $\eta_i$, defined as

$$\eta_i = \frac{d \ln T_i}{d \ln n_0}. \quad (1.24)$$

This particular form of instability is known as the ion-mixing or ion temperature gradient (ITG) mode and arises out of the ion acoustic branch of the drift wave, the details of which are discussed in the next section.

**Ion Temperature Gradient Mode**

The ion temperature gradient mode arises primarily from an increase in $\eta_i$ [49] due to auxiliary heating in which steep ion temperature profiles are produced [50][51][52] or when the ion density profile broadens [53][54][55][56][57]. Confinement degradation of the plasma then occurs in the form of anomalous electron and ion thermal transport, initially noted by Coppi, Rewoldt and Schepp [58]. Much progress has been made in ITG-based transport models to date which have claimed to predict successfully the plasma thermal transport over a wide range of parameters [59][60][61]. The reduction of this transport would help pave the way towards the achievement of a fusion reactor and hence, tokamak configurations that are free of this instability are of great interest. Many such configurations to achieve ITG-mode stabilization have been proposed recently, including plasmas with sheared rotation [60][62], $\nabla B$ drift reduction [19] and noncircular cross section geometry [63].
The ITG mode is inherently electrostatic in nature and is associated with the ion acoustic branch of drift-waves as mentioned previously. It takes its free energy from the ion pressure gradient and becomes destabilized when the \( \eta_i \) parameter approaches a critical value (estimated to be between 1 and 2) \([64][65]\) in the presence of parallel ion motion (slab ITG) \([66][67][68][69]\) or poloidal magnetic drifts due to toroidicity (toroidal ITG) \([70][71][72]\). Both of these destabilizing mechanisms can be studied through use of fluid theory which describes all of the essential features of this mode. As in the case of the basic drift-wave, the electrons establish a near Boltzmann response as given by Eq. (1.18). The ions respond in the long wavelength limit \( (k_i^2 \rho_i^2 \ll 1) \) with oscillations in the ion density \( n_i \), the parallel ion velocity \( v_{ii} \) and the ion pressure \( p_i = n_i T_i \) for finite ion temperature gradient \( (\eta_i \gg 1) \) \([33]\) as given by the following linearized fluid equations:

\[
\frac{\partial}{\partial t} n_i + \nabla \cdot (n_0 \mathbf{v}_E) + \nabla \cdot (n_0 \mathbf{v}_{st}) + n_0 \nabla \cdot \mathbf{v}_{ii} = 0. \tag{1.25}
\]

\[
M n_0 \frac{\partial}{\partial t} v_{ii} = -\nabla \cdot (n_0 e \varphi + p_i) \tag{1.26}
\]

and

\[
\frac{\partial}{\partial t} p_i + \nabla \cdot (p_0 \mathbf{v}_E) + p_0 \nabla \cdot \mathbf{v}_{ii} = 0 \tag{1.27}
\]

where \( \mathbf{v}_{st} \) is the perturbed ion diamagnetic frequency given by

\[
\mathbf{v}_{st} = \frac{c}{n_0 e B^2} \mathbf{B} \times \nabla \mathbf{p}_i. \tag{1.28}
\]

After some algebra, and utilizing the condition of quasineutrality, the dispersion relation of the system is found to be

\[
1 + \frac{\omega_{st}}{\omega} - \frac{\omega_{di}}{\omega} + \frac{\omega_{spi} \omega_{di}}{\omega^2} - \frac{k_i^2 c_s^2}{\omega^2} \left( 1 - \frac{\omega_{spi}}{\omega} \right) = 0 \tag{1.29}
\]

where \( \omega_{spi} \equiv \omega_{st}(1 + \eta_i) \) is the ion pressure gradient drift frequency. Eq. (1.29) agrees with the result of ref. \([73]\) in the limit \( T_e = T_i \) which has been assumed for simplicity.
The driving mechanisms for the instability are clearest in two limiting cases: the slab limit and the toroidal limit. In both cases, a finite ion temperature is present ($\eta_i \gg 1$) along with a very small density gradient ($\omega \gg \omega_{ci}$) which produces the ordering, $\omega_{ci} \ll \omega \ll \omega_{spi}$. In the slab limit, the toroidal drift frequency $\omega_{di} \to 0$ and along with the aforementioned ordering condition yields

$$\omega^3 + k_{||}^2 c_s^2 \omega_{spi} = 0. \quad (1.30)$$

One of the solutions of this equation has $\text{Im}(\omega) > 0$ and thus the growth rate is found to be

$$\gamma = \left( \frac{3\sqrt{3}}{8} k_{||}^2 c_s^2 \omega_{spi} \right)^{1/3} \approx \left( k_{||}^2 c_s^2 \omega_{spi} \right)^{1/3}. \quad (1.31)$$

The energy for the ITG slab mode arises in the case of variable temperature or pressure. In an inhomogeneous plasma, $E \times B$ convection leads to a transfer of the perpendicular pressure gradient into an increase in the pressure term of the parallel ion motion. In other words, convection leads to a continuous flow of heat from a region with a high unperturbed temperature into a region where the temperature is rising on account of the compression due to the ion acoustic wave which leads to wave growth [40][49].

In the toroidal limit, where $k_{||} c_s < \omega_{di}$, the toroidal drift frequency drives the instability and Eq. (1.29) becomes

$$\omega^2 \approx -\omega_{spi} \omega_{di}. \quad (1.32)$$

which thus gives the maximum growth rate as

$$\gamma \approx \frac{k_{||} \rho_s u_{ti}}{\sqrt{L_{pi} R}} \quad (1.33)$$

where $L_{pi} \equiv -(d \ln p_i / dr)^{-1}$ is the ion pressure gradient scale length and $k_{||}$ is the poloidal wave number. As can be seen from Eq. (1.32), the ITG mode in the toroidal
limit is an electrostatic type interchange mode as it closely resembles the result of Eq. (1.11). The convective plasma motions allow for the interchange of high and low pressure fluid elements and due to the so-called “unfavourable” toroidal curvature \( \omega_{pi}\omega_{di} > 0 \), the unstable convection occurs on the outside of the torus where it is amplified by the ion temperature gradient.

Amplification of the growth rate can also occur if the dynamics of trapped particles, particularly electrons, are considered [12][74][75]. In this case, the trapped electron (TE) instability couples with the ITG mode to form a hybrid or coupled mode in which the critical \( \eta_i \) threshold disappears and the mode frequency reverses from the ion to electron diamagnetic direction. The mechanisms of trapped particle and TE modes are addressed in the next section.

Trapped Particle Modes

Since the toroidal magnetic field in a tokamak varies as \( 1/R \), the field on the outer side of the torus is smaller than that on the inner side. Hence a magnetic well or trap is formed and particles satisfying the condition \( \epsilon v_\perp^2 > v_i^2 \) (where \( \epsilon \) is the inverse aspect ratio of the torus, \( \epsilon = r/R_0, r \) is the minor radius and \( R_0 \) is the major radius) undergo a magnetic mirror reflection as they move into a region of higher magnetic field. If collisions are absent, the particles are trapped in the low field region, undergoing repeated reflections as they bounce back and forth between orbit reflection points. The orbit of the drifts resemble “bananas” and hence are referred to as banana orbits in the literature. Particles with smaller magnetic moments, able to traverse the strong field regions, are called untrapped, transit or circulating particles.

In a tokamak's closed toroidal geometry, both transit and trapped particles are present and both react differently to low-frequency perturbations. The trapped
particles's response is nonadiabatic due to their inability to establish a Boltzmann distribution in response to the perturbed potential \( \varphi \). The instability that subsequently results from the magnetic trapping of particles under these conditions was first underscored by Kadomtsev and Pogutse in 1966 [76] and since then the investigation of trapped-particle modes has been a very active area of theoretical research [20][75][77][78][79]. These modes comprise one of the principal categories of drift instabilities and can be classified very broadly as trapped-electron modes and trapped-ion modes.

Trapped-electron modes require that the effective collision frequency of the trapped electrons be small enough to allow the execution of many bounce orbits before being collisionally removed from the local magnetic well. Thus, the trapped electrons are in the banana regime where the collisionality parameter, \( \nu_e^* \equiv \nu_{\text{eff},e}/\omega_{be} < 1 \) (\( \nu_{\text{eff},e} = \nu_e/\epsilon \) is the “effective” collision frequency where \( \nu_e \) is the Spitzer collision frequency for 90°-scattering and \( \omega_{be} = \sqrt{2e/k_T}v_{te} \) is the trapped electron bounce frequency) and the usual drift ordering, \( \omega_{ti}, \omega_{bi} < \omega < \omega_{te}, \omega_{be} \) is in place. Trapped-ion modes require both \( \nu_e^* < 1 \) and \( \nu_i^* < 1 \), and fall into the range \( \omega < \omega_{ti}, \omega_{bi} \). At very high temperatures, collisional effects become negligible and both trapped ions and electrons reside in the banana regime where they evolve into collisionless trapped-particle modes. Older tokamaks such as the Russian T-11 [80] and the French TFR [81] were among the very first to approach low collisionalities through neutral beam injection (NBI) and ion cyclotron resonance heating (ICRH). Plasmas approaching thermonuclear conditions are usually collisionless in nature and are routinely achieved in the confinement region of modern tokamaks such as JT-60U (Japan Tokamak) [82], JET (Joint European Tokamak) [83] and TFTR (Tokamak Fusion Test Reactor) where, in particular, electron collisionalities have approached \( \nu_e^* \sim 0.02 \) within the
vicinity of the core region [84]. In this thesis, trapped-ion effects are omitted be-
cause they violate some of the restrictions imposed by the ballooning representation
formalism. Only collisionless trapped-electron dynamics will be considered.

The existence of trapped electrons leads to the possible appearance of an instabil-
ity of the flute type. The trapped electrons located in a given flux tube between orbit
reflection points are completely isolated from the other regions of the plasma where
they undergo an unfavourable magnetic drift. A small flute perturbation will lead
to a charge separation which reinforces the initial perturbation. Since the flutes are
immers in a plasma containing untrapped particles, the potential associated with
the trapped electrons can be reduced but cannot be eliminated completely.

To estimate heuristically the growth rate of this CTE mode, the frequency range
of consideration is taken to be

\[ \omega_{dz}, \omega_{de}, \omega_{di} \ll \omega < \omega_{be}, \omega_{le} \]  

(1.34)

with the drift ordering, \( \omega \sim \omega_{se} \). The response of the trapped electrons is similar
to that of an anisotropic longitudinally cold distribution with density \( n_{Te} = \sqrt{\epsilon} n_0 \)
whereas the circulating electrons are that of a hot isotropic population and described
by the Boltzmann distribution with density \( n_{Ce} = n_0 - n_{Te} \simeq n_0 \) with \( \epsilon \ll 1 \).
The subscripts \( C \) and \( T \) denote circulating and trapped particles respectively. The
linearized continuity equation of the guiding centres of the trapped electrons is

\[ \frac{\partial \tilde{n}_{Te}}{\partial t} + \nabla \cdot (\tilde{n}_{Te} \mathbf{v}_{de} + n_{Te} \mathbf{v}_E) = 0 \]  

(1.35)

where \( \tilde{n}_{Te} \) is the perturbed trapped electron density and all other quantities have been
previously defined. Fourier transforming Eq. (1.35) yields the perturbed trapped
electron density,

\[ \tilde{n}_{Te} = \sqrt{\epsilon} \frac{\omega_{se}}{\omega - \omega_{de}} \frac{e^\phi}{T_e} n_0 \]  

(1.36)
where $\omega_d/\omega_e \sim \epsilon$. Further simplifying the result of Eq. (1.36) by noting that $\omega_{de} < \omega$ and combining with the adiabatic contribution of the untrapped electrons,

$$n_{Ce} = \frac{e^2 }{m_e} n_0,$$  

(1.37)

gives the total perturbed electron density as

$$n_e = \left[ 1 + \sqrt{\frac{\omega_{de}}{\omega}} \left( 1 + \frac{\omega_{de}}{\omega} \right) \right] \frac{e^2}{m_e} n_0.$$  

(1.38)

The ion response can be represented by a single cold population with a negligible trapped ion influence as dictated by the drift ordering in (1.34). The perturbed ion density for long wavelengths ($k^2 \rho_i^2 \ll 1$) in the limit $\omega_{di} < \omega$ with $\eta_i \simeq 0$ is thus given to lowest order as

$$n_i = -\frac{\omega_{di} e^2}{\omega} n_0.$$  

(1.39)

Noting that $\omega_{ti} = -(T_i/T_e) \omega_{te}$ and assuming $T_e \simeq T_i$ yields the approximate growth rate.

$$\gamma = \sqrt{\omega_{te} \omega_{de}} = \epsilon^{1/4} \frac{k_e \rho_i}{\sqrt{L_n} \rho_i}.$$  

(1.40)

The combination of the trapped electrons oscillating within the unfavourable curvature region in conjunction with the inwardly directed radial density gradient gives rise to a flute or interchange type mode. This can also be seen from the close resemblance of the above result, which is proportional to the square root of the fraction of trapped particles, with that given in Eq. (1.11). Eq. (1.40) is also essentially identically to the form of the general trapped-particle instability given by Kadomtsev and Pogutse [76] and was also recovered by Coppi and Rewoldt [85] and Coppi and Pegoraro [86] in the fluid limit specifically as a trapped-electron mode. A more complete analysis to higher order and with finite Larmor radius effects may be found in ref. [87].

The elementary derivations given here and in the previous sections serve only to give an heuristic guide to the concepts and physical mechanisms that drive these
collisionless nonresonant electrostatic modes. Many essential features such as the magnetic shearing of the lines of force, particle resonances in the form of Landau damping or growth, and electromagnetic effects have been omitted. Nonetheless, these simple models provide an essential prelude for the more comprehensive kinetic analysis to follow that will collectively describe all three of these instabilities (ballooning, ITG and CTE modes) in the kinetic limit using the gyrokinetic ordering without specifically tailoring assumptions to fit each case.

1.3 Early and Present Theoretical Models

No theoretical model today can reproduce the entire range of experimental results for all time and length scales at any spatial location in a tokamak. The apparent limitation lies in the problem of unifying disparate phenomena that may vary greatly with position and time. This is quite evident when one, for example, considers the theoretically accordant conditions in the core confinement area as compared to the theoretically adverse conditions at the plasma surface where the magnetic field geometry, equilibrium and impurity ionization conditions may be difficult to describe [88][89]. To complicate matters further, advances in realistic theoretical modelling are roughly in stride with advances in computer technology which places restrictions on the level of generalization that a theory may entail and, for the moment, appears to be the only way to expedite progress in the field [90]. Hence analyses are generally undertaken to describe only one or two specific aspects of the plasma in a tokamak device which usually results in a large number of simplified and uniquely tailored models. The most frequently and widely used techniques for the stability study of shear-Alfvén and drift-type modes are essentially classified into fluid and kinetic models.
1.3.1 Fluid Models

In a fluid description of plasma phenomena, moments of the assumed distribution function are taken in accordance with more complicated relations that may describe, for example, the evolution of density, pressure, momentum or energy flux of the ions or electrons. Conventional wisdom in the past dictated that problems which inherently involve wave-particle interactions and/or finite Larmor radius (FLR) effects, often referred to as kinetic effects, could not be adequately addressed with fluid techniques, since these processes require detailed information concerning the distribution function. However, as has been recently shown [91][92][93][94][95], many kinetic effects can be included in purely fluid theories when derived with such results in mind.

When moments are taken of the gyrokinetic equation with the inclusion of kinetic effects which are missing from traditional fluid equations, then the fluid description is referred to as a “gyrofluid” equation [92][96][97][98]. This type of model incorporates sufficiently accurate representations of kinetic effects to simulate the broad range of instabilities which are covered by the gyrokinetic ordering. The strength of this type of model is that the reduced dimensionality of the coordinate space variables allows for faster and more efficient numerical simulation as compared to using a gyrokinetic model. Furthermore, many authors [70][99][100] often utilize these types of theories, especially in nonlinear analyses, because of the relative simplicity and occasional insights afforded by the reduced dynamics of the equations. However, since the gyrofluid equations are often derived from the gyrokinetic equation, gyrokinetic eigenmode and particle simulations are more fundamental and potentially more accurate. In addition, fluid approaches are usually derived with the intent to study tokamak turbulence and transport effects since the fluid equations express fundamental conservation laws
which the turbulence must satisfy (conservation of particles, parallel momentum, parallel and perpendicular energy and more if higher moments are retained). Since it is the goal of this work to develop a realistic and comprehensive tokamak discharge model that is able to resolve the stability properties of plasma phenomena with an optimum balance of accuracy and speed, attention will be concentrated more on kinetic approaches when comparing results with other models in the field.

1.3.2 Kinetic Models

The methodology of the kinetic approach can be generally classified into two broad categories: eigenmode and particle (also referred to as particle-in-cell (PIC)) methods. Both techniques, most notably the former, often adopt the gyrokinetic equation framework [101][102][103][104][105][106] which has been successful in producing advanced kinetic algorithms. It is most often applied for the study of plasma instabilities in the linear limit whereas the nonlinear approach is usually used to study transport phenomena or turbulence saturation levels [107][108]. In essence, the gyrokinetic equation is a gyrophase-averaged kinetic equation that allows for spatial variations of field perturbations on the length scale of the Larmor radius $\rho$. The validity of the gyrokinetic formalism depends on an ordering scheme in which temporal variations are slower than the cyclotron frequency ($\omega/\Omega \ll 1$), spatial variations in the equilibrium quantities are weaker than those in the perturbing electromagnetic fields ($L_\nu \gg L$) and the characteristic wavelengths parallel to the equilibrium magnetic field are long compared to the wavelengths perpendicular to the magnetic field ($k_\parallel/k_\perp \ll 1$).
Eigenmode Methods

Eigenmode analyses adopt the gyrokinetic equation to render a problem which, in its most general form, is two-dimensional (in $r$ and $\theta$) by nature and involves a complicated orbit integral along the field line. When coupled with the quasineutrality condition and Ampère’s law, the system of equations is closed and an integral equation is yielded that determines the variation of the scalar and vector potentials ($\varphi$ and $A_x, A_\perp$ respectively) along the field at a given radial position. Since this method is able to resolve the structure of an electromagnetic eigenmode over many rational surfaces, it is also known as a global analysis. A basic example of this technique is the work of Marchand et al. [109] who used it to study low-$n$ trapped ion dynamics ($\omega \sim \omega_{te} \leq \omega_{bi}, \omega_{ti}$) in which the radial structure of the instability extends over many rational surfaces in the presence of high magnetic shear. Although this work was tailored for the electrostatic limit, it was the first to present a 2-D global analysis. In order to study the trapped-ion instability with the inclusion of finite-$\beta$ effects, Kim et al. [110] extended Marchand’s work to include electromagnetic effects.

Many 2-D models, such as those by Rewoldt et al. [111][112], are based on the work of Marchand and Kim and require an impractically large expenditure of supercomputer time to determine numerical solutions when all relevant physical features are included in the calculation. In order to reduce the problem to a more tractable form in which analytical methods or relatively simple numerical algorithms could be applied, many early models incorporated simplifying assumptions into the system of equations. The most common of these is to use the lowest order ballooning representation [7][8][9] - local to a magnetic surface and limited to perturbations with high toroidal mode numbers - which effectively reduces the eigenmode calculations from 2-D ($r, \theta$) to 1-D ($\eta$). Two-dimensional studies conducted by Tang et al. [113] and by
Frieman et al. [9] have found that this one-dimensional transformation still retains most of the essential features of these instabilities as long as the wavelengths do not get appreciably large nor the magnetic shear appreciably small. However, it is at this point where the similarities among most of the various models end and the differences begin. In most cases, a constant is used to approximate the poloidal dependence of the ion response which is responsible for linking the different $k_i$'s together in a nonlocal fashion to account for the compressional ion acoustic dynamics. This simplification results in a tremendous savings in computational time as it effectively reduces the integral equation into one of two forms: a second-order ordinary differential equation (reduced kinetic model) which can be solved by using a Runge-Kutta method in conjunction with a root finder [34][38][114][115][116] or a simple eigenvalue type problem (local or semilocal analysis) [117][118][119] that requires a root finder only. These are essentially the two most basic kinetic models used in the study of microinstabilities and may contain further simplifications with respect to various kinetic effects such as the representation of the electron response through use of an adiabatic fluid model. In this thesis, both of these simplified forms will be used to compare results with the main model in chapter four with respect to speed, accuracy and applicability.

**Particle Methods**

Particle methods, as of late, have seen promising and excellent development as a result of the remarkable increases in the speed and memory of parallel supercomputers. Although these models are primarily formulated to simulate plasma turbulence on the shortest timescales, they may also be extended to examine the stability properties of tokamak plasmas occurring over larger timescales.
Particle simulations written primarily to model tokamak plasmas were, until recently, dominated by three codes, associated with the names Okuda, Lee and LeBrun. The Okuda model [120][121][122] was perhaps the first particle method devised to deal with tokamak plasma problems and described the ion dynamics by the Lorentz force law and the electrons by a drift approximation. The particle techniques used were relatively simple and crude resulting in a high level of particle noise from the small time and length scales that the simulation employed. By switching to a gyrokinetic model, a substantial reduction in noise was achieved since the particle gyromotion is averaged while keeping effects due to the radius of the gyro-orbits. Lee [105] was the first to pursue this avenue and used it to model electrostatic drift instabilities in the slab. shearless case. Later, Lee generalized his electrostatic model towards a fully electromagnetic gyrokinetic particle code, with toroidal and general geometry considerations incorporated [123].

The particle code of LeBrun [124] who referred to his model as the toroidal particle code (TPC), is much closer in spirit to Okuda's code than Lee's. TPC has been noted for its accurate use of a toroidal geometry which can also be extended to model Cartesian geometry. Since its inception, the code seems to have been in a continuous state of development, the latest offering the option of describing the particle species in the full kinetic, drift kinetic and Boltzmann limits. Each of these descriptions can be modelled using a standard PIC treatment or a Lagrangian Vlasov ($\delta f$ method [125][126]) representation in addition to an optional Monte Carlo collision operator. Using a Lorentz description for the ion species and a drift kinetic description for the electrons, the code has been used to generate drift results [127] that are in agreement with linear theory, notably a toroidicity-induced mode in the absence of temperature gradients and a transition from an electron-dominated to an ion-dominated drift-wave
behaviour as the ion temperature gradient $\eta_i$ is increased. The latest version of this code [128] incorporates the gyrokinetic equation in a general metric (specifically the toroidal geometry of a tokamak with a circular or noncircular cross-section) that is capable of examining the global and semi-global structures of driftwave instabilities.

The initial, successful application of particle methods by Okuda, Lee and LeBrun has led to a plethora of many PIC simulation studies by other workers in the field, such as Parker et al. [107][125], Dimits et al. [129] and Cohen et al. [130]. Since all of these models are based on the gyrokinetic formalism, they share the common trait of not being constrained to the cyclotron time scale (due to averaging) and thus the simulations can employ a larger timestep that is more appropriate for the diamagnetic drift frequency and hence amenable to the study of drift-type instabilities. Although this may relieve some of the computational pressure on the time evolution process of resolving the plasma mode structure, the spatial extent of the system may be very large - $10^4 - 10^6$ Debye lengths in size and consist of several million electrons and ions. Thus, only parallel computers with hundreds to thousands of processors and sufficient memory are able to carry out these types of simulations such as Caltech's Intel Delta Touchstone machine or the IBM 3090-600J supercomputer at the Cornell National Supercomputer Facility [90]. However, even with access to powerful computers, the codes may take a substantial period of time to yield useful results. For example, the electrostatic particle model of Lee, using a CRAY-2 computer, resolved the structure of low frequency fluctuations, relevant to parameters describing the Caltech tokamak [131], in approximately 50 hours [113].
1.3.3 Compendium

The preceding discussions on the gyrokinetic 2-D eigenmode and PIC methods have underscored advantages which are quite obvious. Both methods allow for a comprehensive stability study that is able to resolve large-scale kinetic behaviour extending over many rational surfaces. Secondly, when considering complicated kinetic effects such as trapped particles, analyses that are multi-dimensional become necessary when the mode to be resolved resides in the very long wavelength regime where the one-dimensional ballooning representation analysis breaks down [132]. The disadvantages of the above models are also quite obvious. A high material cost is inextricably associated with the study, in the guise of expensive computer resources that renders itself as the only means possible for the calculations to be effected. This limitation is aptly lamented in the paper by Z. Sedláček titled, “Plasma Physics on the TI-85 Calculator or Down with Supercomputers” [133] which lightly satirizes mathematical and numerical ingenuity over brute force, supercomputer-only type calculations, in its study of plasma oscillations. Also, significant advances in the generalization of PIC methods usually occur in tandem with developments in computer technology as current models push even the largest available computers to their limits.

In light of this significant material drawback, it is thus desirable to devise a model possessing the following characteristics in studying shear-Alfvén and drift branch instabilities; sufficient speed and accuracy in the calculation of the mode structures via modest computational resources, and realism and comprehensiveness of the tokamak simulation by retaining all pertinent kinetic effects. With these considerations in mind, a 1-D eigenmode, integral equation analysis based on the ballooning mode formalism presents itself as an expedient solution. Care must be taken to insure that the finite shear and long-wavelength restrictions are abided by, as they are the only
drawbacks of the method. The very well known model of Rewoldt was one of the first to incorporate a comprehensive eigenmode analysis of this type and will be used as the main source of comparison in confirming results from this thesis.

1.4 Summary of Findings

A nonlocal gyrokinetic model has been formulated to describe low frequency, short wavelength modes in a plasma of axisymmetrical toroidal geometry with low-\(\beta\) and circular nonconcentric flux surfaces with small Shafranov shift. The eigenmode equations contain the two potential approximation in \(\phi\) and \(A_\parallel\) with full finite Larmor radius and trapped electron effects in the collisionless limit. The analysis incorporates the so-called “ballooning formalism” to lowest order in \(1/n\) which yields a radially local calculation for the eigenfrequencies and the eigenfunctions. This representation, in conjunction with an efficient numerical algorithm, allows the eigenfrequencies to be computed with sufficient accuracy and high speed for arbitrary high-\(n\) modes in the drift and shear-Alfvén branches. This finding, which may allow for eigenmode computations in real time, is the major accomplishment of this work. Test cases using artificial and actual tokamak experimental discharge parameters for the CTE, ITG and ballooning modes have been benchmarked with the comprehensive kinetic formulation of Rewoldt exhibiting favourable results.
Chapter 2

Formulation of the Key Equations

The fundamental equations presented in this chapter will be based on kinetic theory, from which, the gyrokinetic equation is used to shoulder the burden of the analysis. The gyrokinetic equation was first introduced in the classic work by Rutherford and Freeman [134] and Taylor and Hastie [135] and was restricted to the electrostatic case with the magnetic shear treated as weak. Their formulations were successfully applied to multipole systems without toroidal fields but were inadequate when sheared toroidal fields were included in the analysis. The basic difficulty in addressing such sheared fields is that of reconciling the doubly-periodic nature of perturbations in a torus with a basic property of low frequency eigenmodes; namely that they have long wavelength parallel to, but short wavelength perpendicular to the equilibrium magnetic field $B_0$ (i.e. $k \parallel / k_\perp \ll 1$). Attempts by Connor and Hastie [136] and by Coppi and Rewoldt [137] to reconcile these properties were only partially successful since they led to the introduction of an arbitrary phase function into the eikonal representation without establishing an equation determining this arbitrary function.
It was not until the virtually concurrent presentation of work by a series of authors (Lee and Van Dam [138], Glasser et al. [139] and Connor, Hastie and Taylor [34][140]) that the wavelength ordering problem could be reconciled. The method they developed, known in the literature as the ballooning representation, is used primarily for the study of shear-Alfvén and drift-type instabilities. In this thesis, the ballooning representation is applied to the gyrokinetic equation with electromagnetic effects included with the purpose of determining the nonadiabatic response of the perturbed distribution function. The ensuing result is subsequently combined with the quasi-neutrality condition and Ampère's law to close the system of equations. The results are then benchmarked with the work of Rewoldt and other appropriate models in the plasma physics literature.

2.1 Equations of Motion

2.1.1 Time and Scale Length Ordering

The gyrokinetic equation is a gyrophase-averaged kinetic equation that allows for sharp spatial variation of field perturbations. It is most often applied in the linear limit to the study of plasma instabilities whose wavelengths are of the order of the Larmor or gyroradius \( \rho \). The relevance of this consideration is dictated by the parameter \( \delta = \rho / L \) (where \( L \) is the scale length characterizing the plasma), for which, a value of \( \delta \sim 1 \) indicates that the plasma is unmagnetized and unsuitable for even a simplified kinetic description. Although physically, \( \delta \) represents the ratio of gyroradius to scale length, mathematically, it also represents the order of various terms.
The derivation begins by distinguishing between perturbed and unperturbed quantities as follows:

\[ \Psi = \Psi_0 + \Psi_1 \]  

(2.1)

where \( \Psi \) represents an arbitrary field or distribution function and

\[ \frac{\Psi_1}{\Psi_0} \ll 1. \]  

(2.2)

The quantity \( \Psi_0 \) is usually associated with equilibrium and varies slowly enough such that

\[ \frac{\partial \Psi_0}{\partial t} = 0. \]  

(2.3)

As mentioned previously, the basic gyrokinetic ordering assumes sharp, spatial variations in perturbed quantities whereas equilibrium quantities are allowed to evolve slowly over long scale lengths.

\[ \nabla \Psi_0 \sim \frac{\Psi_0}{L}, \quad \nabla \Psi_1 \sim \frac{\Psi_1}{\rho} \]  

(2.4)

with \( \delta = \rho/L \ll 1 \). If one assumes that the perturbed variations occur over distances associated with the parallel and perpendicular scale lengths with respect to the equilibrium magnetic field \( B_0 \), then

\[ \nabla_\| \Psi_1 \sim \frac{\Psi_1}{L_\|}, \quad \nabla_\perp \Psi_1 \sim \frac{\Psi_1}{L_\perp} = k_\perp \Psi_1 \]  

(2.5)

where \( L \sim L_\| \gg L_\perp \sim \rho \) and hence \( k_\|/k_\perp \sim \delta \). The conditions outlined in (2.5) restrict the perpendicular scale lengths to be short and the parallel scale lengths to be long which is consistent with the structure of the most dangerous form of interchange mode in which the amplitude of the perturbation is approximately constant along the direction of the confining equilibrium magnetic field line.

It is also necessary to insist on the time scale ordering, \( \omega/\Omega = \delta \ll 1 \), where \( \omega \) is the characteristic frequency of the perturbations. From the point of view of a gyrating
particle, this slowness requirement is also similar to the spatial variation requirement noted earlier. Since $d/dt$ gives the time derivative at the particle's changing position, the requirement $k_\parallel \ll k_\perp$ already implies that the convective part of the total derivative satisfies the slowness requisite, since $v \cdot \nabla \ll v/\rho \sim \Omega$. Specifically, this means that the $B$ field does not change much during a single gyro-orbit, either due to its intrinsic time variation or due to the particle's motion and hence the gyro-orbits can be viewed as essentially being unperturbed. This is an important consideration for the task at hand since it provides a welcome simplification in ignoring the time variation of the magnetic moment.

### 2.1.2 General Solution of Vlasov Equation

As mentioned beforehand, the gyrokinetic equation is used to determine the first order nonadiabatic response of the perturbed distribution function, $f_1$, which in turn is computed via the Vlasov Equation. If one assumes that nuclear, atomic and collisional processes are absent, then the particles of each plasma species are conserved and hence each distribution function must be constant along particle trajectories,

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{e}{m} \left[ E(x, t) + \frac{1}{c} v \times B(x, t) \right] \cdot \frac{\partial f}{\partial v} = 0 \quad (2.6)$$

and the values $e$ and $m$ represent the charge and mass of each particle respectively (particle species labels are suppressed for the moment). Eq. (2.6) can be solved linearly by assuming that the perturbed quantities are small and, in accordance with the condition $k_\parallel/k_\perp \ll 1$, vary according to $e^{i(k_\perp x - \omega t)}$. This form describes perturbations which vary rapidly across the field on the scale of the Larmor radius and dominate the dynamics of the system. The perturbed quantities are also to be expanded in a
simple power series in $\delta$. Thus

$$f(x, v, t) = f_0(v) + f_1(x, v, t).$$  \hfill (2.7)

$$E(x, t) = E_1(x, t) = -\nabla \varphi_1 - \frac{1}{c} \frac{\partial A_1}{\partial t}$$  \hfill (2.8)

and

$$B(x, t) = B_0 + B_1(x, t) = B_0 + \nabla \times A_1.$$  \hfill (2.9)

The equilibrium distribution function, $f_0$, is assumed to be isotropic. The lowest order Vlasov equation in linearized form is then given as

$$\frac{i}{\Omega} k_\perp \cdot \mathbf{v} f_1 - \frac{i e}{\Omega m} \varphi_1 k_\perp \frac{\partial f_0}{\partial \mathbf{v}} + (\mathbf{v} \times \mathbf{b}) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = 0.$$  \hfill (2.10)

In order to evaluate $\frac{\partial f_1}{\partial \mathbf{v}}$, we employ the guiding centre coordinate system where it is more convenient to express this operator in terms of the gyrophase variable $\gamma$. As can be seen from Fig. 2.1 the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ form an orthogonal triad with the normalized equilibrium magnetic field, $\mathbf{b} \equiv B_0 / B_0$, and $X$ is the position of the guiding centre ($X = x - \rho$) with the gyroradius $\rho$ given as

$$\rho = \rho (\mathbf{e}_1 \sin \gamma + \mathbf{e}_2 \cos \gamma).$$  \hfill (2.11)

The velocity $\mathbf{v}$ and wavenumber $\mathbf{k}$ are defined respectively as

$$\mathbf{v} \equiv b \mathbf{v}_\parallel + \mathbf{v}_\perp$$  \hfill (2.12)

where

$$\mathbf{v}_\perp = u_\perp (\mathbf{e}_1 \cos \gamma - \mathbf{e}_2 \sin \gamma)$$  \hfill (2.13)

and

$$\mathbf{k} \equiv b k_\parallel + k_\perp \mathbf{e}_1.$$  \hfill (2.14)
Figure 2.1: a) Guiding-centre coordinate system and b) gyrokinetic coordinate system. The two systems are linked by $X = x - \rho$ (figures adapted from Hazeltine and Meiss [141] p. 34).
The partial derivative $\partial/\partial \nu$ can be expressed in terms of $\gamma$ as

$$\frac{\partial}{\partial \nu} = \frac{\partial \gamma}{\partial \nu} \frac{\partial}{\partial \gamma}$$

(2.15)

which gives

$$\frac{\partial}{\partial \nu} = -\frac{\hat{\rho}}{v_\perp} \frac{\partial}{\partial \gamma}$$

(2.16)

where $\hat{\rho}$ is the unit gyroradius vector. Hence Eq. (2.10) reduces to

$$\frac{\partial f_1}{\partial \gamma} - i k_\perp \rho \cos \gamma f_1 = -i \frac{e}{m \Omega} \phi_1 \hat{k}_\perp \cdot \frac{\partial f_0}{\partial \nu}$$

(2.17)

which is a simple first order differential equation. Since the plasmas being dealt with are toroidally axisymmetric, the lowest order equilibrium distribution function is roughly a local isotropic Maxwellian distribution,

$$f_0 = F_M = n_0 (m/2\pi T)^{-3/2} \exp \left(-m v^2 / 2T\right).$$

(2.18)

Integrating Eq. (2.17) then yields, after some algebra, the general solution of the Vlasov equation to lowest order,

$$f_1 = - \frac{e \phi_1}{T} f_0 + h \exp (i k_\perp \cdot \rho).$$

(2.19)

The second term in Eq. (2.19) represents the nonadiabatic response, $h \equiv h(x, \nu, t)$, and is yet to be determined. An eikonal form can be used to represent $e^{i k_\perp \cdot \rho}$ where the rapid spatial dependence can be averaged over a gyrophase orbit to leave a function that varies smoothly. The details of this procedure are outlined in the next section.

The response $h$ can be computed by seeking the next order correction to the lower-order quantities using the gyrokinetic equation.

### 2.1.3 Gyrokinetic Equation

The gyrokinetic equation is in essence a useful approximation to the well known kinetic equation. The kinetic equation given in Eq. (2.6) is the electromagnetic version of
the general form, given as
\[ \frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h + a \frac{\partial h}{\partial v} = 0. \quad (2.20) \]

Instead of Cartesian phase-space coordinates, it is much more convenient to be able
to generalize to arbitrary coordinates. The left hand side of Eq. (2.20) measures the
total change in \( h \) along a particle trajectory as a scalar difference
\[ h [\mathbf{x} (t + \Delta t), \mathbf{v} (t + \Delta t), t + \Delta t] - h [\mathbf{x} (t), \mathbf{v} (t), t]. \quad (2.21) \]

This expression may also be written in terms of non-Cartesian coordinates, \( z_i : i = 1, ..., 6 \), in the form
\[ h [z_i (t + \Delta t), t] - h [z_i, t] \]

where the variable \( z_i \) can be any smooth function of \( \mathbf{x} \) and \( \mathbf{v} \) on condition that the
transformation \((\mathbf{x}, \mathbf{v}) \rightarrow (z_1, ..., z_6)\) be nonsingular. Thus the kinetic equation may
be presented as
\[ \frac{\partial h}{\partial t} + \sum_i \frac{dz_i}{dt} \frac{\partial h}{\partial z_i} = 0. \quad (2.22) \]

A relatively straightforward derivation of the gyrokinetic equation can be obtained
by incorporating the guiding centre coordinates \((\mathbf{X}, \mu, \varepsilon, \gamma)\) into Eq. (2.22), where \( \mu \)
is the magnetic moment \((\mu = mv_\perp^2/2B_0)\) and \( \varepsilon \) is the total guiding centre energy
\( \left( \varepsilon = mv_\parallel^2/2 + e\varphi + \mu B \right)\). The result of Eq. (2.22) then becomes
\[ \frac{\partial h}{\partial t} + \frac{\partial \mathbf{X}}{\partial t} \cdot \frac{\partial h}{\partial \mathbf{X}} + \frac{\partial \mu}{\partial t} \frac{\partial h}{\partial \mu} + \frac{\partial \varepsilon}{\partial t} \frac{\partial h}{\partial \varepsilon} + \frac{\partial \gamma}{\partial t} \frac{\partial h}{\partial \gamma} = 0. \quad (2.23) \]

By averaging over the gyrophase, the \( \gamma \)-dependence of \( h \) can be eliminated which makes the kinetic description of the magnetized plasma motions much simpler. When
the orbit of a gyrating particle is averaged in this way, what survives is a drifting
magnetic dipole or a guiding centre. The distribution of guiding centres is denoted
by \( \overline{h} \) where the overbar refers to the gyrophase average. The gyrophase average is explicitly defined as
\[
\overline{\Theta} = \langle \Theta \rangle \equiv \int \frac{d\gamma}{2\pi} \Theta(x, \mu, \varepsilon, \gamma)
\] (2.24)
where the expression for \( h \) can be written in terms of a slowly varying part, \( \overline{h} \), and a rapidly varying, gyrophase dependent part, \( \tilde{h} \), in the form
\[
h = \overline{h} + \tilde{h}.
\] (2.25)

To lowest order, \( \tilde{h} \) is negligible and after gyro-averaging, Eq. (2.23) becomes
\[
\frac{\partial \overline{h}}{\partial t} + \left( \frac{dX}{dt} \right) \cdot \frac{\partial \overline{h}}{\partial X} + \left( \frac{\partial \mu}{\partial t} \right) \frac{\partial \overline{h}}{\partial \mu} + \left( \frac{\partial \varepsilon}{\partial t} \right) \frac{\partial \overline{h}}{\partial \varepsilon} = 0.
\] (2.26)

The result of Eq. (2.26) is a sufficiently accurate gyrokinetic equation for most applications of interest. Note that \( \overline{h} \) varies only on a slow spatial scale length; it is independent of \( \gamma \) at fixed \( X \). This is the essential feature of the gyrokinetic equation, which is able to describe this slow response in the presence of sharply varying electromagnetic fields.

It now becomes a tedious exercise to evaluate the various coefficients, the averages of which will be determined shortly thereafter. Starting with the guiding centre velocity, it can be seen that
\[
\frac{dX}{dt} = \nu \cdot \nabla X + \frac{dv}{dt} \cdot \frac{\partial X}{\partial \nu}
\] (2.27)
where after some vector algebra the convective term is found to be
\[
\nu \cdot \nabla X = \nu + \frac{1}{\Omega} \left[ \nu \times (\nu \cdot \nabla) b - \frac{1}{B} \nu \times (\nu \cdot \nabla B) \right],
\] (2.28)
\[
\frac{dv}{dt} \cdot \frac{\partial X}{\partial \nu} = V_E - v_\perp - \frac{1}{B} \nu \times (\nu \times B_1)
\] (2.29)
with
\[
V_E = c \frac{E \times b}{B_0} = V_{IE}.
\] (2.30)
The details of the calculations pertaining to Eqs. (2.28) and (2.29) are listed in Appendix A. It is convenient to express Eq. (2.27) in terms of perturbed ($V_1$) and nonperturbed ($V_0$) components as

$$\frac{dX}{dt} = V_0 + V_1$$  \hspace{1cm} (2.31)

where

$$V_0 = b v_{\parallel} + v \times (v \cdot \nabla) \left( \frac{b}{\Omega} \right) + \rho \left( \frac{v \cdot \nabla}{B_0} \right) B_0$$  \hspace{1cm} (2.32)

and

$$V_1 = V_1 \epsilon - v \frac{b \cdot B_1}{B_0} + B_1 \frac{v_{\parallel}}{B_0}.$$  \hspace{1cm} (2.33)

Attention is now focused on the time rate of change of the magnetic moment. Since it was previously assumed that $\omega/\Omega \ll 1$ and $k_{\parallel}/k_\perp \ll 1$ (slow time varying magnetic field), the magnetic moment is conserved and thus the following well known result presents itself

$$\frac{d\mu}{dt} = 0.$$  \hspace{1cm} (2.34)

In order to determine the last coefficient, the time rate of change of the guiding centre energy, it is noted that the Lorentz equation of motion,

$$\frac{d}{dt} v = \frac{e}{m} \left( E + \frac{1}{c} v \times B \right),$$  \hspace{1cm} (2.35)

can also be expressed in the form

$$\frac{d}{dt} (b v_{\parallel} + v_\perp) = \frac{e}{m} E - \Omega v_\perp \tilde{\rho}.$$  \hspace{1cm} (2.36)

Multiplying this equation by $v$ immediately yields the desired result;

$$\frac{d\varepsilon}{dt} = e \frac{\partial \varphi}{\partial t} - \frac{\bf{v} \cdot \partial \bf{A}}{c}.$$  \hspace{1cm} (2.37)

The present task at hand is to determine the gyrophase averages of (2.31) and (2.37) via Eq. (2.24). The nonperturbed part of the guiding centre velocity, given by
(2.32), is considered first. Since the equilibrium quantities vary slowly, the distinction between $X$ and $x$ can be ignored as well as second order corrections in $\delta$. The average is computed to yield

$$\langle V_0 \rangle = b v_\| + \frac{1}{\Omega} b \times \left( \frac{e}{m} \nabla B_0 + v_\| b \cdot \nabla b \right). \quad (2.38)$$

For the perturbed guiding centre velocity, the result is found to be

$$\langle V_1 \rangle = \frac{c}{B_0} \langle E_1 \rangle \times b + \frac{v_\|}{B_0} \langle B_{1 \perp} \rangle - \frac{1}{B_0} \langle B_{1 \parallel} v_\perp \rangle \quad (2.39)$$

where the electromagnetic fields must remain inside the averaging brackets because of their rapid variation on the short spatial scale. Averaging over the time rate of change of the guiding centre energy yields

$$\left\langle \frac{d\varepsilon_1}{dt} \right\rangle = e \left[ \left\langle \frac{\partial \varphi_1}{\partial t} \right\rangle - \frac{1}{c} \left\langle \nabla \cdot \frac{\partial A_1}{\partial t} \right\rangle \right] \quad (2.40)$$

where the unperturbed energy change vanishes as expected.

Before incorporating the results of (2.38)-(2.40) into the gyrokinetic equation given in (2.26), it is convenient to linearize it first. The result, with overbars suppressed, is

$$\frac{\partial h_1}{\partial t} + \langle V_0 \rangle \cdot \frac{\partial h_1}{\partial X} + \langle V_1 \rangle \cdot \frac{\partial h_0}{\partial X} + \left\langle \frac{d\varepsilon_1}{dt} \right\rangle \frac{\partial h_0}{\partial \varepsilon} = 0. \quad (2.41)$$

The evaluation of the gyrophase averages becomes problematic since we must take into consideration the short spatial scale nature of the electromagnetic fields. The difficulty arises when attempting to evaluate the average of the perturbed fields at fixed $X$, as, for example, in the case for the perturbed electric field,

$$\langle E_1 \rangle = \oint \frac{d\gamma}{2\pi} E_1 (X + \rho). \quad (2.42)$$

In this particular instance the distinction between $X$ and $x$ cannot be ignored since the field varies rapidly over $\rho$ and hence the gyroaverage at fixed $X$ cannot be a
local function of position. One way to circumvent this problem is to use the eikonal representation \[142\] to treat the rapid spatial dependence. In this form, the rapid variation of each perturbed field can be expressed as

\[\Psi_1(x) = \psi_0 e^{ik_\perp x} = \psi(x) e^{ik_\perp \rho}\]  

(2.43)

where

\[\psi(x) \equiv \psi_0 e^{ik_\perp x}\]  

(2.44)

and both the envelope \(\psi(x)\) and perpendicular wave number \(k_\perp\) vary slowly. Note that Eq. (2.43) is not expressed in terms of the conventional representation of the eikonal \(k \cdot x \rightarrow S(x)\) but possesses the necessary conditions that are characteristic of it. \(i.e. \ k_\perp \equiv \nabla S(x) \sim \nabla \psi \sim O(1)\). Thus the gyroaverage becomes

\[\langle \Psi_1(x) \rangle = \psi \langle e^{ik_\perp \cdot \rho} \rangle.\]  

(2.45)

This average can be computed explicitly through use of the identity

\[J_n(z) = \int \frac{d\gamma}{2\pi} e^{iz \sin \gamma - n\gamma}\]  

(2.46)

where \(J_n\) is the Bessel function. The average of (2.45) is thus found to be

\[\langle e^{ik_\perp \cdot \rho} \rangle = J_0(k_\perp \rho).\]  

(2.47)

The other terms in Eq. (2.39) can be evaluated in the same manner to yield

\[\langle \hat{\nabla}_\perp e^{ik_\perp \cdot \rho} \rangle = iJ_1(k_\perp \rho) \frac{k_\perp \times b}{k_\perp}\]  

(2.48)

and

\[\langle \hat{p} e^{ik_\perp \cdot \rho} \rangle = iJ_1(k_\perp \rho) \frac{k_\perp}{k_\perp}.\]  

(2.49)

Substituting these expressions into Eqs. (2.39) and (2.40) yields respectively

\[\langle V_1 \rangle = \frac{i}{B_0} J_0(k_\perp \rho) \left( \phi_1 - \frac{1}{c} A_{1||} \right) k_\perp \times b - \frac{i}{B_0} J_1(k_\perp \rho) B_{1||} \frac{k_\perp \times b}{k_\perp}\]  

(2.50)
and

$$\left< \frac{d\varepsilon_{\perp}}{dt} \right> = e \frac{\partial}{\partial t} \left[ J_0 (k_{\perp} \rho) \left( \varphi - \frac{v_{\parallel}}{c} A_{\parallel} \right) + J_1 (k_{\perp} \rho) \frac{v_{\parallel}}{c k_{\perp}} B_{\parallel} \right]. \tag{2.51}$$

Substituting Eqs. (2.50) and (2.51) into Eq. (2.41) yields a useful approximation to the kinetic equation based on the eikonal representation, where the first order nonadiabatic response denoted by \( h_1 = h_{11} \), varies on the slow spatial scale. By dropping the subscript 1 on all perturbed quantities, the gyrokinetic equation presents itself as

$$\frac{\partial h}{\partial t} + \left[ v_{\parallel} + \frac{1}{\Omega} b \times \left( \frac{\mu}{m} \mathbf{\nabla} B_0 + v_{\parallel}^2 b \cdot \mathbf{\nabla} b \right) \right] \cdot \mathbf{\nabla} h +$$

$$e \left[ \frac{\partial f_0}{\partial \varepsilon_{\perp}} \frac{\partial}{\partial t} + \frac{ic}{eB_0} k \times b \cdot \mathbf{\nabla} f_0 \right] \left[ J_0 (k_{\perp} \rho) \left( \varphi - \frac{v_{\parallel}}{c} A_{\parallel} \right) + J_1 (k_{\perp} \rho) \frac{v_{\parallel}}{c k_{\perp}} B_{\parallel} \right] = 0. \tag{2.52}$$

We further seek to simplify Eq. (2.52) by showing that compressional effects associated with \( B_0 \) may be neglected by using a simple MHD model to justify our assertion. In an inhomogeneous plasma, unstable modes present themselves as coupled combinations of various waves. However, for a low-\( \beta \) plasma with time and spatial orderings scaling as \( \omega/\Omega, k_{\parallel}/k_{\perp} \ll 1 \), these modes become insignificant compared to the shear Alfvén wave which is the most unstable. Consider the well known dispersion relation for MHD waves (Cross [143])

$$\left[ \omega^2 - k_{\perp}^2 V_A^2 \right] \left[ \omega^4 - \omega^2 k_{\perp}^2 \left( c_s^2 + V_A^2 \right) + k_{\perp}^2 k_{\parallel}^2 c_s^2 V_A^2 \right] = \left( \frac{\omega}{\Omega} \right)^2 k_{\perp}^2 k_{\parallel}^2 V_A^4 \left( \omega^2 - k_{\perp}^2 V_A^2 \right) \tag{2.53}$$

where \( c_s = \sqrt{T_e/M} \) is the acoustic speed and \( V_A = \sqrt{B_0^2/4\pi \rho_0} \) the Alfvén speed. The low-\( \beta \) assumption implies that \( c_s^2 \ll V_A^2 \) since \( (c_s/V_A)^2 = \beta/2 \) where \( \beta = 8\pi n_0 T / B_0^2 \).

Hence, in conjunction with the scale orderings, Eq. (2.53) simplifies to

$$\left( \frac{\omega^2}{k_{\parallel}^2 V_A^2} - 1 \right) \left( \frac{\omega^2}{k_{\perp}^2 V_A^2} - 1 + \frac{k_{\parallel}^2 c_s^2}{\omega^2} \right) = 0. \tag{2.54}$$
which finally leads to the shear Alfvén result

$$\omega^2 = k_\parallel^2 V_A^2.$$  \hfill (2.55)

The apparent absence of compressional effects justifies our previous claim.

The final form of the gyrokinetic equation in simplified form is thus,

$$\frac{\partial h}{\partial t} + [v_\parallel + V_d] \cdot \nabla h + e \left( \frac{\partial f_0}{\partial \varepsilon} \frac{\partial}{\partial t} + \frac{ic}{eB_0} k \times b \cdot \nabla f_0 \right) \left( \varphi - \frac{v_{\parallel}}{c} A_\parallel \right) J_0(k_\perp \rho) = 0$$  \hfill (2.56)

where the magnetic drift velocity approximation,

$$V_d = \frac{1}{\Omega} b \times \left( \frac{\mu}{m} \nabla B_0 + v_\parallel^2 b \cdot \nabla b \right) \simeq \frac{cm}{eB_0^2} \left( \frac{1}{2} v_\perp^2 + v_{\parallel}^2 \right) B_0 \times \nabla B_0,$$  \hfill (2.57)

applies for the low-\(\beta\) assumption which causes the curvature of the magnetic field to be nearly identical to that of the magnetic field gradient, \(b \cdot \nabla b \simeq \nabla B_0 / B_0\). The form of Eq. (2.56) is equivalent to those of Hazeltine et al. [141] and Catto et al. [144].

Eq. (2.56) in its present form is complete yet intractable for determining \(h\) when considering tokamak geometry. The problem lies in resolving the inherent, doubly periodic nature of the perturbed quantities in combination with the assumed \(k_\parallel / k_\perp\) ordering. One way around this problem is to assume a “ballooning representation” form for these quantities and then to subsequently transform them to “ballooning space” in order for appropriate boundary conditions to be imposed on the system. The following section will address these issues.
Figure 2.2: Toroidal coordinate system \((r, \theta, \phi)\) where \(R_0\) is the major radius and \(r\) is the minor radius of the flux surface (figure adapted from R. O. Dendy [17] p. 32).

2.2 Ballooning Representation and Transformation

2.2.1 Ballooning Mode Representation

Before proceeding directly with the details of this important mathematical representation, it is necessary to first become familiar with the internal topological structure of the tokamak and the parameters that describe it. The geometry of the tokamak is represented by the toroidal coordinate system \((r, \theta, \phi)\) as shown in Fig. 2.2. As mentioned beforehand, the current flow produces helically shaped magnetic field lines that trace out magnetic flux surfaces in the form of nested tori. A significant geometrical
simplification can be implemented if it is assumed that the system is axisymmetric, that is, the equilibrium is independent of the toroidal angle \( \phi \). This reduces the number of coordinates from three to two since \( \phi \) becomes ignorable.

The helicity of the field lines is gauged by the safety factor \( q \), so called because of its role in minimizing current driven instabilities. In an axisymmetrical equilibrium, each magnetic field line has a value of \( q \). If the field line, at an arbitrary position in the poloidal plane, has a toroidal angle \( \phi \), it will return to that poloidal position after a change of toroidal angle \( \Delta \phi \) and hence, \( q = \Delta \phi / 2\pi \). If the equation of the field line is used with poloidal and toroidal magnetic fields (\( B_\theta \) and \( B_\phi \) respectively) specified for a given flux surface of minor radius \( r \) and major radius \( R_0 \), the safety factor then assumes the general form

\[
q(r) = \frac{1}{2\pi} \int \frac{1}{R_0} \frac{B_\phi}{B_\theta} ds
\]  

(2.58)

where the integral is evaluated over a single poloidal circuit around the selected flux surface. The tokamak equilibria assume a relatively simple form if a low-\( \beta \) plasma with circular but nonconcentric (due to Shafranov shifting) flux surfaces is considered in conjunction with a small inverse aspect ratio, \( \epsilon \ll 1 \). These assumptions help reduce the \( q \)-value equation to the cylindrical limit,

\[
q = \frac{r B_\phi}{R_0 B_\theta},
\]  

(2.59)

where \( B_\phi \) is nearly constant. Because field lines never cross, \( q \) is the same for all field lines on a magnetic surface and hence it is essentially a surface property.

The \( q \)-values corresponding to each magnetic field line can assume rational or irrational values. If \( q = m/n \), where \( m \) and \( n \) are integers, the field line joins up on itself after \( m \) toroidal and \( n \) poloidal rotations around the torus. Field lines that do not close upon themselves cover a surface ergodically and correspond to irrational
numbers. The pitch of the helical field lines is not constant across the toroidal surfaces but varies as a function of the minor radius. This radial rate of change of $q$ is known as shear and is given by $s = d \ln q / d \ln r$.

Unstable modes are also characterized by the mode numbers $m$ and $n$ and comprise a virtual infinite spectrum where they assume the form $e^{i(m\theta - n\phi)}$. Magnetic surfaces for which the magnetic field helix matches that of the mode on the rational surfaces ($k_\parallel \equiv -i b \cdot \nabla = 0$) produce ideal circumstances in which the magnetic field can transfer its potential energy directly into plasma kinetic energy to produce wave instabilities. The most unstable modes correspond to the high-$n$ type in which case the stabilizing effect of magnetic line bending is substantially minimized. Since the safety factor is approximately $q \approx 1$ at the magnetic axis and $q \approx 4 - 5$ at the periphery of the plasma, and since $k_\perp \sim m/r$ along with $k_\parallel = (m - nq) / qR$ in conjunction with $\epsilon \ll 1$, this consideration is still consistent with the previous wavelength assumption that $k_\parallel / k_\perp \ll 1$.

Due to the periodic nature of the toroidal and poloidal perturbations, the scalar and vector potentials may be constructed in terms of Fourier components. Using the scalar potential as an example, the series expansion yields

$$
\varphi (r, \theta, \phi) = \sum_{m=-\infty}^{\infty} \varphi_m (r) e^{i(m\theta - n\phi)}.
$$

Because of the assumed axisymmetry, the toroidal $n$ modes are separable and need only be considered on an individual basis. The same cannot be said of the poloidal Fourier $m$-components which may couple significantly as a result of the curvilinear nature of the differential operators and the equilibrium magnetic field, whose strength and direction both vary with the poloidal angle $\theta$. It is the very nature of this mode coupling that leads to the ballooning representation; the toroidal curvature induces plasma perturbations, which are centred on neighbouring rational surfaces,
to superimpose, thereby creating a radially extended structure that "balloons" on the
unfavourable curvature side of the torus. If the toroidal coupling is strong, then the
Fourier modes can be considered to be degenerate. In the presence of weak toroidal
curvature (weak coupling), the perturbations assume the form of single, well localized
Fourier modes on each rational surface where each mode views other adjacent modes
as weak sidebands that are virtually isolated and noninteracting. The effect of varying
the diamagnetic frequency, as examined by Connor and Taylor [145], has also been
shown to influence the strength of mode coupling.

To satisfy the earlier requirement of short perpendicular wavelengths, the expansion
of \( \varphi(r, \theta, \phi) \) must extend to large \( m \) which corresponds to a large toroidal mode
number \( n \), since the \( m/n \) ratio must match the pitch \( q \) of the magnetic field lines.
The result of \( n \) being large has an important consequence: the mode rational sur-
faces become closely packed together. This can be seen by approximating the radial
dependence of \( q(r) \) as linear over a small region about a rational surface \( r_0 \). Thus,

\[
q(r) = q(r_0 + \Delta r) \approx q(r_0) + \Delta r \frac{dq}{dr}
\]

(2.61)

which gives

\[
\Delta r \approx \frac{1}{nq'}.
\]

(2.62)

As can be seen, the distance between rational surfaces, \( \Delta r \), becomes very small for
large \( n \). This is significantly smaller than the minor radius \( r \) and also much smaller
than some of the typical equilibrium scale lengths such as that for the density, \( L_n \sim r \),
the toroidal magnetic field strength, \( L_B \sim R \), or the magnetic shear, \( L_s \sim q^2/\epsilon q' \sim qR \). Hence in the limit of large \( n \), the equilibrium quantities vary slowly with \( m \),
or in other words, from one rational surface to another and thus their exists local
translational invariance among the Fourier components [8],

\[
\varphi_m(r_{nq}) \sim \varphi_{m-1}(r_{nq-1}) \sim \cdots \sim \varphi(r_0),
\]

(2.63)

where the \(r_i\) represent the radial locations of the successive rational surfaces with the change in phase between rational surfaces ignored. The invariant nature of (2.63) is akin to that of the Block wavefunctions in a lattice with a periodic potential [146] and was also noticed by Catto and Tsang in their study of trapped electron instabilities [147][148]. Translational invariance of the Fourier components makes it much more convenient to replace the radial variable \(r\) with the local dimensionless radial variable, \(\chi = nq(r) - m\), such that \(\varphi_m(r) \sim \varphi(r) \to \tilde{\varphi}(\chi)\) and hence Eq. (2.60) is modified to become

\[
\varphi(r, \theta, \phi) = \sum_{m=-\infty}^{\infty} \tilde{\varphi}(nq(r) - m) e^{i(m\theta - n\phi)}.
\]

(2.64)

Since it was previously assumed that the system be axisymmetrical, the toroidal coordinate \(\phi\) becomes ignorable which reduces the linear drift and ballooning eigenmode calculation to a two-dimensional eigenvalue problem,

\[
\mathcal{L}(r, \theta; \omega) \varphi(r, \theta) = 0.
\]

(2.65)

In general, \(\mathcal{L}\) is an integral operator in the poloidal variable \(\theta\) and radial variable \(r\) but can also reduce to a differential operator in certain limiting cases. The boundary conditions imposed on \(\varphi(r, \theta)\) must satisfy two conditions: ensure that the solution is single valued on the toroidal surface, and secondly, satisfy the periodic constraint on \(\theta\). Early attempts to solve eigenmode equations like Eq. (2.65) employed a WKB-type expansion to lowest order in \(k_\parallel/k_\perp\) which produced an aperiodic lowest-order eigenfunction in \(\theta\). Since the field lines do not close on themselves on a magnetic surface whose safety factor, \(q\), is irrational, the problem of satisfying the periodicity condition becomes untenable. The issue is resolved by introducing the ballooning
representation which represents the eigenfunction as an infinite sum of aperiodic functions that add up to give a periodic function. The representation is constructed by first considering the Poisson summation formula \[149\], which states that if \(G(u)\) is the Fourier transform of a function \(g(t)\), then

\[
\sum_{m=-\infty}^{\infty} e^{imz} g(y + md) = \frac{1}{d} \sum_{l=-\infty}^{\infty} G\left(\frac{2\pi l + x}{d}\right) e^{-\frac{i2\pi l + z}{d}y} \tag{2.66}
\]

where the Fourier transform is defined as

\[
g(t) = \int_{-\infty}^{\infty} du \ G(u) e^{iut}. \tag{2.67}
\]

For the problem at hand, transforming to the reciprocal space allows the boundary value problem of periodicity to be remedied. Let \(\mathcal{F}(k)\) be the Fourier transform of \(\mathcal{F}(z)\). Then, by applying the Poisson formula of Eq. (2.66) with \(d = -1\) to Eq. (2.60), the long sought after ballooning representation is obtained

\[
\varphi(r, \theta, \phi) = e^{-in\phi} \sum_{l=-\infty}^{\infty} \mathcal{F}(2\pi l + \eta) e^{i(2\pi l + \eta)nq} \tag{2.68}
\]

where \(\eta\), known as the extended ballooning parameter, allows \(\mathcal{F}\) to extend from \(-\infty\) to \(\infty\) since it was initially defined as following a field line. To ensure convergence of this sum, \(\mathcal{F}\) must vanish sufficiently fast as \(|\eta| \to \infty\). Note that \(\varphi(r, \theta, \phi)\) satisfies the periodic constraint stated earlier with the implementation of these boundary conditions since periodicity is still maintained in the toroidal and poloidal angles. Similar forms apply for the perturbed distribution function and the parallel and perpendicular components of the perturbed vector potential, \(A||\) and \(A_|\). In the context of WKB theory, the slowly varying amplitude \(\mathcal{F}\) appears as the coefficient of a conventional eikonal. In the case of the lowest order WKB-problem, infinite degeneracy occurs which allows for the formation of the periodic sum \[138\]. It is evident that the ballooning representation, or more specifically the Fourier or "ballooning" transform (as


it is commonly referred to in the literature), reduces the two-dimensional character of the problem to one that is one-dimensional. This is a crucial simplification in the analysis because it expedites the efficiency of the numerical process in resolving the mode structure.

The ballooning representation transforms the linear eigenmode problem of Eq. (2.65) to an associated linear problem in the reciprocal or so called “ballooning” space in \( \eta \)

\[
\mathcal{L}(r, \theta; \omega) \varphi(r, \theta) \rightarrow \hat{\mathcal{L}}(\eta; \omega, n) \hat{\varphi}(\eta, n) = 0. 
\] (2.69)

Dewar and Glasser [150] have shown that the two eigenmodes corresponding to \( \mathcal{L} \) and \( \hat{\mathcal{L}} \) have essentially the same spectrum. Characteristic of the representation is that, while \( \hat{\varphi} \) determines \( \varphi \), the series is usually not inverted; a prescription for resolving the actual ballooning mode amplitude in cyclical space is not deemed useful. However, Hazeltine [151] has derived the inversion of the formula for reasons of intrinsic mathematical interest.

To solve Eq. (2.69) to lowest order, the operator \( \hat{\mathcal{L}} \) can be expanded in powers of the ordering parameter \( \delta \), where as before, \( \delta \sim k_i/k_\perp \sim 1/n \), yielding

\[
\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 + \delta \hat{\mathcal{L}}_1 + \delta^2 \hat{\mathcal{L}}_2 + ... 
\] (2.70)

By expanding \( \hat{\varphi} \) and the eigenvalue \( \omega \) in a similar series and insisting that Eq. (2.69) be satisfied order by order produces the lowest order result

\[
\hat{\mathcal{L}}_0(\eta_l + \eta; \omega_0, n) \hat{\varphi}_0 = 0 
\] (2.71)

with \( \eta_l = 2\pi l \) and the lowest order eigenvalue has the dependence \( \omega_0 = \omega_0(\eta_l, n) \). Choosing \( \eta_l = 0 \) corresponds to retaining only perturbations that are centred at the outside of the torus. This is where poloidal localization maximizes and occurs for,
\[ l = 0, \pm 1, \ldots \text{ when considering up-down symmetric equilibria [152][153][154]. This limit is also known as the strong ballooning approximation and corresponds to strong coupling.} \]

Any principal Fourier component possessing radial width exceeding the distance separating adjacent rational surfaces will result in an overlap of all adjacent component functions. This strong coupling effect can be produced by strong shear since a large \( q' \) results in a small \( \Delta r \). Retaining only the \( l = 0 \) term is predicate upon the strong poloidal localization of \( \varphi \) about \( \eta = 0 \) and hence Eq. (2.68) reduces to

\[ \varphi (r, \theta, \phi) \approx \tilde{\varphi} (\eta) e^{i(n\eta - n\phi)}. \]  

(2.72)

In the opposite limit, where ballooning is weak, the \( \varphi_m \) components overlap only fractionally with each other resulting in the nonlocal extension of \( \varphi \) in the poloidal direction. In this case, many terms in Eq. (2.68) are necessary to describe it properly. Further complications arise if the radial structure of the eigenmode must be taken into account due to the untenability of the invariance condition. This occurs when the shear becomes weak or the toroidal mode number \( n \) too small since the distance between rational surfaces scales as \( (nq')^{-1} \). These conditions then lead to an inappropriate application of the ballooning representation.

### 2.2.2 Gyrokinetic Transformation

The ballooning representation is an important and convenient tool which facilitates the implementation of appropriate boundary conditions while reducing the dimensionality of the eigenmode problem. Before transforming the gyrokinetic result of Eq. (2.56) to ballooning space, we will first simplify it by computing some of its terms. Since it was assumed that the distribution function is approximately a local isotropic
Maxwellian, given by Eq. (2.18), we have
\[
\nabla f_0 = f_0 \left(1 + \eta \left[ \frac{K}{T} - \frac{3}{2} \right] \right) \nabla \ln n_0
\]
(2.73)
where \( \eta = (\partial \ln T) / (\partial \ln n_0) \) and
\[
\frac{\partial f_0}{\partial K} = -\frac{f_0}{T}.
\]
(2.74)
Inserting Eqs. (2.73) and (2.74) into Eq. (2.56), and Fourier transforming all perturbed quantities, yields
\[
v_{\parallel} k_{\parallel} h - (\omega - \omega_d) h + \frac{e f_0}{T} \left( \phi - \frac{v_{\parallel}}{c} A_{\parallel} \right) J_0 (k_{\perp} \rho) = 0
\]
(2.75)
where the magnetic drift frequency \( \omega_d \) is given as
\[
\omega_d = V_d \cdot k = \frac{cm}{eB_0^2} \left( \frac{1}{2} v_{\perp}^2 + v_{\parallel}^2 \right) B_0 \times \nabla B_0 \cdot k_{\perp}
\]
(2.76)
and the diamagnetic drift frequency \( \omega^T \) as
\[
\omega^T = V^T \cdot k = \frac{cT}{eB_0^2} \left(1 + \eta \left[ \frac{K}{T} - \frac{3}{2} \right] \right) \nabla \ln n_0 \times B_0 \cdot k_{\perp}.
\]
(2.77)
Note that in the derivation of Eq. (2.75), the equilibrium electric field is absent since it was assumed that the plasma was hot and hence highly conducting. The del operator \( \nabla \), in conjunction with the geometry of the equilibrium magnetic field, determines the explicit structures of \( k_{\parallel} \) and \( k_{\perp} \). For tokamaks with circular cross sections, \( \nabla \) in toroidal geometry is
\[
\nabla = e_\phi \frac{1}{R} \frac{\partial}{\partial \phi} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_r \frac{\partial}{\partial r}
\]
(2.78)
and the equilibrium magnetic field is given as
\[
B_0 = B_\phi e_\phi + B_\theta e_\theta.
\]
(2.79)
Figure 2.3: Coordinates associated with a shift (exaggerated) of the magnetic surfaces.

Thus, we have

\[ k_\parallel \equiv -ib \cdot \nabla \simeq -i \left( \frac{1}{R} \frac{\partial}{\partial \phi} + \frac{1}{qR} \frac{\partial}{\partial \theta} \right) = \frac{m - nq}{qR} \quad (2.80) \]

where the condition \( B_\phi \gg B_\theta \) is well satisfied for small \( \epsilon \) and \( q \sim 1 \) according to Eq. (2.59). The result for \( k_\perp \) is

\[ k_\perp = -i \left( e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_r \frac{\partial}{\partial r} \right) = e_\theta \frac{m}{r} + e_r \left( nq \frac{\partial q}{\partial r} + nq \frac{\partial \theta}{\partial r} \right). \quad (2.81) \]

The dependence of \( \theta \) on \( r \) is a result of our previous assumption that the circular magnetic surfaces are Shafranov shifted. Shifting is due to two effects; the pressurized plasmas' natural tendency to expand, known as the "tire tube force" and the inclination of a ring of current (toroidal plasma current) to increase its inductance, known as the "hoop force". The shift of the outermost surface with respect to the magnetic axis is represented by \( \Delta \) as illustrated in Fig 2.3 and is small compared with the minor radius, \( \Delta / r \ll 1 \). In general, the interaction of toroidal curvature with plasma pressure will distort interior surfaces away from circularity. However, in
the limit of low-$\beta$ and large aspect ratio, the inner flux surfaces will remain approximately circular [155]. A magnetic field geometry existing in this state is known as the so called "$s-\alpha$" model MHD equilibrium [34] and allows one to vary $s$ while keeping all of the other equilibrium parameters fixed, including $q$. The manner in which axisymmetric toroidal equilibria are usually obtained is by numerically evaluating an elliptic partial differential equation (Grad-Shafranov equation) which is capable of incorporating unique, cross sectional geometries such as ellipticity and triangularity [156]. The famous "Princeton Equilibrium, Stability and Transport" code (PEST) [152][157][158] is based on this approach.

The geometry of Fig. 2.3 allows $\gamma$ to be approximated as

$$\gamma \simeq -\Delta \frac{\sin \theta}{r}. \quad (2.82)$$

Differentiating Eq. (2.82) with respect to $r$ while noting that $\gamma = \theta - \theta'$ yields

$$\frac{\partial \theta}{\partial r} = -\frac{\sin \theta}{r} \frac{\partial \Delta}{\partial r} + \frac{\Delta}{r^2} \sin \theta - \frac{\Delta}{r} \cos \theta \frac{\partial \theta}{\partial r} \quad (2.83)$$

where the $\Delta/r$ terms are negligible and $\Delta'$ could be of order unity [29], hence

$$\frac{\partial \theta}{\partial r} \simeq -\alpha \frac{\sin \theta}{r} \quad (2.84)$$

with $\Delta' \simeq \alpha \equiv -q^2 R d\beta/dr$ for small shifts and $\alpha$ is known as the ballooning parameter.

We are now ready to compute the final forms of $k_\parallel$ and $k_\perp$. Recalling the local dimensionless radial variable, $\chi = nq(r) - m$, and substituting this into Eq. (2.80) for $k_\parallel$ yields

$$k_\parallel = -\frac{\chi}{qR}, \quad (2.85)$$

while $k_\perp$ becomes

$$k_\perp \simeq k_\theta [\mathbf{e}_\theta + \mathbf{e}_r (s\theta - \alpha \sin \theta)] \quad (2.86)$$
in the limit of large \( n \) where \( k_\theta \equiv m/r \). The magnetic drift frequency is also considered, which gives

\[
\omega_d = \frac{cm}{eB_0} \left( \frac{1}{2} v_r^2 + v_\theta^2 \right) B_0 \times \nabla B_0 \cdot \mathbf{k}_r
\]
\[
= \frac{cm}{eB_0 R} \left( \frac{1}{2} v_r^2 + v_\theta^2 \right) e_z \cdot [e_\theta k_\theta + e_r k_\theta (s\theta - \alpha \sin \theta)]
\]
\[
= 2\epsilon_n \omega_\star \left[ \cos \theta + (s\theta - \alpha \sin \theta) \sin \theta \right] \left( \frac{1}{2} \tilde{v}_r^2 + \tilde{v}_\theta^2 \right) (2.87)
\]

and

\[
\tilde{\omega}_d \equiv 2\epsilon_n \omega_\star \left[ \cos \theta + (s\theta - \alpha \sin \theta) \sin \theta \right] (2.88)
\]

with \( \epsilon_n = L_n/R \). \( \omega_\star = cT k_\phi / eB_0 L_n \) and \( \tilde{v}_\perp(\perp) = v_{\perp(\parallel)} / v_T \).

Now all that is left is to transform all perturbed quantities in our intermediate result of Eq. (2.75) to ballooning space, with the aid of the ballooning representation form of Eq. (2.72). The transformed quantities are as follows:

\[
h \to \tilde{h}(\eta)
\]
\[
\varphi \to \tilde{\varphi}(\eta)
\]
\[
A_{\parallel} \to \tilde{A}_{\parallel}(\eta)
\]
\[
k_{\parallel} \to -i \frac{\partial}{qR \partial \eta}
\]
\[
\theta \to \eta
\]

(2.89)

Substituting these expressions into Eq. (2.75) produces the final formulation of the gyrokinetic equation in ballooning space to lowest order,

\[
\frac{v_\parallel}{qR} \frac{\partial \tilde{h}}{\partial \eta} - i [\omega - \omega_d(\eta)] \tilde{h} + i \frac{e f_0}{T} (\omega - \omega_\star) \left[ \tilde{\varphi}(\eta) - \frac{v_\parallel}{c} \tilde{A}_{\parallel}(\eta) \right] J_0(\Lambda) = 0 (2.90)
\]

where \( \Lambda = k_\perp(\eta) \rho \).
2.3 Eigenmode Formulation

Thus far, we have essentially completed two of three analytical stages in obtaining the lowest order eigenmode equation in the collisionless limit. The first involved deriving an appropriate kinetic theory approximation that replaces the Lorentz force law with an equation for smoothed particle distributions in phase space. The second required transforming this equation into ballooning space so that acceptable boundary conditions could be applied to the response function while satisfying the \( k_{\parallel}/k_{\perp} \ll 1 \) condition which characterizes the most unstable form of wave perturbation, the shear Alfvén mode. This section concerns itself with completing the analytical treatment by combining the perturbed distribution function with the quasineutrality condition and parallel Ampère's law to form a well posed eigenmode problem in which the ballooning structure and eigenfrequency of a drift or shear-Alfvén instability may be resolved.

As the ballooning mode frequency \( \omega \) approaches the ion pressure gradient drift frequency \( \omega_{\text{pi}} \) at marginal stability [159][160], two frequency regimes accordingly present themselves: a high frequency, short wavelength regime, \( \omega > \omega_{\text{ti}} \), corresponding to \( n > r^2/Rq^2\rho_i \), and a lower frequency, longer wavelength regime, \( \omega < \omega_{\text{bi}} \), requiring \( n < r^2/Rq^2\rho_i \). For such low values of \( n \) it may be necessary to carry out the next order correction in \( 1/n \) for meaningful results to be obtained. In light of this, we must restrict ourselves to sufficiently high values of \( n \) corresponding to the ordering, \( \omega_{\text{ti}}, \omega_{\text{bi}} < \omega \). We are also interested in modes that are subject to trapped electron effects, and as such, require that \( \omega < \omega_{\text{be}}, \omega_{\text{te}} \). Trapped electron modes can only exist in this frequency range [161].

Trapped-electron effects are incorporated by assuming that the trapped electrons reside in the collisionless banana regime where the finite banana width is neglected.
For low-$\beta$ plasmas, trapped electron drift modes usually dominate the dynamics whereas at higher $\beta$, the shear-Alfvén ballooning mode dominates. Trapped ions are excluded due to the assumption, $\omega > \omega_{ni}, \omega_{pi}$, which implies that the trapped ion dynamics are primarily electrostatic and thus negligible for finite-$\beta$. Contributions from the circulating ions are retained without approximation. In association with employing the $s-\alpha$ model, all $\epsilon$ corrections will be neglected except in situations where it represents a necessary first order correction. Terms of order $\epsilon^{1/2}$, which are associated with the distinction between trapped and circulating electrons are retained.

### 2.3.1 Solution of Gyrokinetic Equation

Eq. (2.90) is a first order differential equation in $\eta$ that can be easily solved for $\hat{h}$ by integration. For the circulating particles, the boundary conditions on $\hat{h}$ are essentially that $\hat{h} \rightarrow 0$ sufficiently fast as $|\eta| \rightarrow \infty$. It is also necessary to insist that $\gamma \equiv \text{Im}(\omega) > 0$, which corresponds to unstable eigenmodes. This automatically ensures the convergence condition at one of the two extreme limits in $\eta$. Eq. (2.90) may be solved exactly by integrating along the field lines, producing two solutions, $\hat{h}^+$ and $\hat{h}^-$, corresponding respectively to particles with positive and negative values of the parallel velocity $v_\parallel$. In addition, the distinction between trapped and circulating particles is also made, with the subscript $C$ denoting circulating particles and the subscript $T$ for those that are trapped.

Integrating Eq. (2.90) yields for the circulating particles,

$$
\hat{h}^C = -i \int_{-\infty}^{\eta} d\eta' \exp\left(-i \frac{\Xi_\eta}{v_\parallel}\right) \frac{qR e f_0}{T} \frac{\omega - \omega^T}{\omega - \omega^I} \left[ \varphi - \frac{|v_\parallel|}{c} A_\parallel \right] J_0 (\Lambda')
$$

(2.91)
and

\[ \hat{h}_C = -i \int_{\eta}^{\infty} d\eta' \exp \left( i \Xi^\eta_{\eta'} \right) \frac{q R e f_0}{|v_\parallel| T} (\omega - \omega^\tau) \left[ \hat{\varphi} + \frac{|v_\parallel|}{c} \hat{A}_\parallel \right] J_0 (\Lambda') \]  

(2.92)

with \( \Lambda' = k_\perp (\eta') \rho \) and the argument of the integrating factor is

\[ \Xi^\eta_{\eta'} = \int_{\eta}^{\eta'} d\eta'' \frac{q R}{|v_\parallel|} [\omega - \omega_d (\eta'')] . \]  

(2.93)

Determining the response \( \hat{h} \) for the trapped electrons is considerably more difficult than the circulating particle case. The problem is complicated by the infinite number of turning or reflection points that mark the poloidal limits of the bounce orbit motions that resemble "bananas". A schematic of the trapped electron orbits is shown in Fig. 2.4 where the radial excursion of the electrons from the flux surface is due primarily to cross field drifts. The boundary conditions on \( \hat{h} \) are such that for an arbitrary bounce orbit with turning points, \( \eta_1 \) and \( \eta_2 \), where \( \eta_1 \leq \eta \leq \eta_2 \), we have

\[ \hat{h}^+ (\eta_1) = \hat{h}^- (\eta_1) \equiv \hat{h}_1 \]  

(2.94)

and

\[ \hat{h}^+ (\eta_2) = \hat{h}^- (\eta_2) \equiv \hat{h}_2. \]  

(2.95)

We may integrate Eq. (2.90) by assuming that typical particle excursions away from the flux surfaces along the unperturbed trajectory are small compared to the equilibrium scale length [134][135][163][164]. As a result, this produces

\[ \hat{h}^\sigma (\eta) = \exp \left( i \sigma \Xi^\eta_{\eta_0} \right) \hat{h}^\sigma (\eta_0) - i \sigma \int_{\eta_0}^{\eta} d\eta' \exp \left( i \sigma \Xi^\eta_{\eta'} \right) \Upsilon^\sigma (\eta') \]  

(2.96)

where \( \sigma = \pm 1 \) with

\[ \Upsilon^\sigma \equiv \Upsilon_\varphi + \sigma \Upsilon_\Lambda, \]  

(2.97)
Figure 2.4: a) Schematic of trapped-electron orbit and guiding-centre motion in a tokamak. b) A cross-sectional view of the torus illustrates a typical banana orbit (figures adapted from R. A. Gross [162] p. 89).
\[ \tau_\phi \equiv -\frac{qR}{|v||} \frac{e\omega_e}{T_e} (\omega - \omega_e^T) J_0 \left( \Lambda_e^* \varphi (\eta') \right) \tag{2.98} \]

and

\[ \tau_A \equiv -\frac{qR}{|v||} \frac{e\omega_e}{T_e} (\omega - \omega_e^T) J_0 \left( \Lambda_a^* \left[ -\frac{|v||}{c} A || (\eta') \right] \right). \tag{2.99} \]

In solving Eq. (2.90), the variable \( \eta_0 \) is chosen arbitrarily, with the only constraint being that it must satisfy \( \eta_1 \leq \eta_0 \leq \eta_2 \). In order to eliminate \( \eta_0 \), so that the final form of \( \hat{h}_\sigma (\eta) \) can be determined, we must construct two equations; one corresponding to \( \sigma = -1 \), \( \eta_0 = \eta_1 \) and \( \eta = \eta_2 \), and the other to \( \sigma = +1 \), \( \eta_0 = \eta_2 \) and \( \eta = \eta_1 \). The two equations are

\[ \hat{h}_-(\eta_2) = \exp \left( -i\Xi_{\eta_1}^\eta \right) \hat{h}_-(\eta_1) + i \int_{\eta_1}^{\eta_2} d\eta' \exp \left( -i\Xi_{\eta_1}^{\eta_2} \right) \frac{1}{\eta} \tag{2.100} \]

and

\[ \hat{h}_+(\eta_1) = \exp \left( i\Xi_{\eta_1}^\eta \right) \hat{h}_+(\eta_2) - i \int_{\eta_1}^{\eta_2} d\eta' \exp \left( i\Xi_{\eta_1}^{\eta_2} \right) \frac{1}{\eta} \tag{2.101} \]

We can substitute Eq. (2.100) into Eq. (2.101) through use of the turning point condition of Eq. (2.95) to give

\[ \hat{h}(\eta_1) = \frac{i \exp \left( i\Xi_{\eta_1}^{\eta_2} \right)}{1 - \exp \left( 2i\Xi_{\eta_1}^{\eta_2} \right)} \int_{\eta_1}^{\eta_2} d\eta' \left[ \exp \left( -i\Xi_{\eta'}^{\eta_2} \right) \tau_\phi (\eta') + \exp \left( i\Xi_{\eta'}^{\eta_2} \right) \tau_A (\eta') \right]. \tag{2.102} \]

Since \( \eta_0 \) is arbitrary, we set it equal to \( \eta_1 \) which allows Eq. (2.102) to be substituted into Eq. (2.96) to produce the general form of the trapped electron response function,

\[ \hat{h}_\sigma (\eta) = \frac{i \sigma \Xi_{\eta_1}^{\eta_2}}{2 \sin \left( \Xi_{\eta_1}^{\eta_2} \right)} \int_{\eta_1}^{\eta_2} d\eta' \left[ \exp \left( -i\Xi_{\eta'}^{\eta_2} \right) \tau_\phi (\eta') + \exp \left( i\Xi_{\eta'}^{\eta_2} \right) \tau_A (\eta') \right] - i \sigma \int_{\eta_1}^{\eta} d\eta' \exp \left( i\sigma \Xi_{\eta'}^{\eta_2} \right) \tau_\sigma (\eta'). \tag{2.103} \]
Since we wish to study modes with frequencies $\omega \sim \omega_{di} \sim \omega_{de} \ll \omega_{be}$, we can make the approximation, $\Xi_{m}^{n} \sim \omega / \omega_{be} \ll 1$. Eq. (2.103) then simplifies to

$$\hat{h}^{\sigma} (\eta) \simeq \frac{1}{\Xi_{m}^{n}} \int_{\eta_{1}}^{\eta} d\eta' \left\{ \Upsilon_{\sigma} + i \left[ \Xi_{m}^{n} - H (\eta - \eta') \Xi_{m}^{n} \right] \Upsilon_{\sigma} \right\} + O \left( \frac{\omega}{\omega_{be}} \right)$$  \hspace{1cm} (2.104)

where $H (\eta - \eta')$ is the Heaviside function defined as

$$H (\eta - \eta') = \begin{cases} 1, & \eta - \eta' > 0 \\ 0, & \eta - \eta' \leq 0 \end{cases} \hspace{1cm} (2.105)$$

Also note that in obtaining Eq. (2.104), $\int_{\eta_{1}}^{\eta} d\eta' \Upsilon_{\sigma} \sim O (\omega / \omega_{be})$ and $\int_{\eta_{1}}^{\eta} d\eta' \Upsilon_{\sigma} \sim O (1)$.

### 2.3.2 Eigenmode Equations

For low-$\beta$ axisymmetric tokamak systems, in the limit of sufficiently long wavelengths ($k \lambda_D \ll 1$ (where $\lambda_D$ is the Debye length of the plasma), the quasi-neutrality condition,

$$\sum_{j} \int d\nu \vec{f}_{j} (\eta) = 0,$$  \hspace{1cm} (2.106)

can be applied to generate one set of equations which governs the perturbed electrostatic potential $\varphi$. The plasma is considered to be neutral enough so that the perturbed ion and electron densities are nearly equal and yet not so neutral that all of the electromagnetic forces of interest disappear. The system of equations is closed by incorporating the component of Ampère's law along the equilibrium magnetic field,

$$k_{\perp}^{2} \vec{A}_{\parallel} (\eta) = (4\pi / c) \sum_{j} e_{j} \int d\nu v_{\parallel} \vec{f}_{j},$$  \hspace{1cm} (2.107)

with the displacement current ignored. We now restore the particle species labels, indicated as such by the subscript, $j = i, e$. 
After gyroaveraging and ballooning transforming, the perturbed distribution function, Eq. (2.19), becomes

$$\tilde{f}_j = -\frac{e_j f_{0j}}{T_j} \tilde{\varphi} (\eta) + \hat{h}_j (\eta) J_0 (\Lambda_j),$$

(2.108)

which is substituted into Eqs. (2.106) and (2.107) to yield respectively,

$$\sum_j \left( -\frac{e_j n_{0j}}{T_j} \tilde{\varphi} (\eta) + \int d\nu J_0 (\Lambda_j) \left[ g_{Cj}^+ + g_{Tj}^+ \right] \right) = 0$$

(2.109)

and

$$k_+^2 A_j (\eta) = \frac{4\pi}{c} \sum_j e_j \left( \int d\nu |v_\parallel| J_0 (\Lambda_j) \left[ g_{Cj}^- + g_{Tj}^- \right] \right)$$

(2.110)

with the total nonadiabatic response for circulating (trapped) particles given as

$$g_{C}^{\sigma (T)j} (\eta') = \frac{\hat{h}_{C}^{\sigma (T)j} (\eta') + \sigma \hat{h}_{C}^{\sigma (T)j} (\eta')}{2}. \hspace{1cm} (2.111)$$

We now seek to simplify some of the terms presented in both of these equations. We consider first the case of the ions for analysis.

For the circulating ions, the $g_{C}^{\sigma}$ term can be simplified through use of the sign or signum function which automatically accounts for the sign of $v_\parallel$. Thus using Eqs. (2.91) and (2.92) we have

$$g_{C}^{\sigma} = -i \int_{-\infty}^{\infty} d\eta' \exp \left[ -i \Xi_{\eta'} \text{sgn} (\eta - \eta') \right] \frac{e f_{0i}}{|v_\parallel| T_i} \left( \omega - \omega_{i}^* \right) \times$$

$$\left\{ \frac{1}{c} |v_\parallel| \left[ \text{sgn} (\eta - \eta') \right] \delta^{\sigma - \sigma'} \tilde{\varphi} (\eta') - \frac{1}{c} \left[ \text{sgn} (\eta - \eta') \right] \delta^{\pi - \pi'} A_{\parallel} (\eta') \right\} J_0 (\Lambda_i^*) \hspace{1cm} (2.112)$$

where $\delta_{mn}$ is the Kronecker delta defined by

$$\delta_{mn} = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases} \hspace{1cm} (2.113)$$
and the signum function, \( \text{sgn}(\eta - \eta') \), is defined by

\[
\text{sgn}(\eta - \eta') = \begin{cases} 
+1 & \text{if } \eta - \eta' \geq 0 \\
-1 & \text{if } \eta - \eta' < 0 
\end{cases}
\]  

(2.114)

For trapped ions, we require \( \hat{h}_{Ti}^+ (\eta') = \hat{h}_{Ti}^- (\eta') = 0 \), and thus

\[
g_{Ti}^\sigma = 0.
\]  

(2.115)

We now proceed with the electron case, beginning first with the determination of \( g_{Ce}^\sigma \) for the circulating electrons. Once again using Eqs. (2.91) and (2.92) we find

\[
g_{Ce}^\sigma = i \int_{-\infty}^{\infty} d\eta' \exp \left[ -i \Xi_{en}^* \text{sgn} (\eta - \eta') \right] \frac{qR \ e f_{\text{be}}}{|v||T_e} (\omega - \omega_{Te}) \times \left\{ [\text{sgn} (\eta - \eta')]^{6+\sigma} \hat{\varphi} (\eta') - \frac{|v||}{c} [\text{sgn} (\eta - \eta')]^{6+\sigma} \hat{A}_\parallel (\eta') \right\} J_0 (\Lambda'_e).
\]  

(2.116)

For the trapped electrons, using Eq. (2.104), we have for \( g_{Te}^\sigma \),

\[
g_{Te}^\sigma = \frac{1}{\Xi_{enT_e}} \int_{\eta_1}^{\eta_2} d\eta' \left\{ \mathcal{T}_\varphi + i \left[ \Xi_{en}^m - H (\eta - \eta') \Xi_{enT_e}^m \right] \mathcal{T}_\lambda \right\} + O \left( \frac{\omega}{\omega_{be}} \right)
\]

\[
= -\frac{1}{\Xi_{enT_e}} \int_{\eta_1}^{\eta_2} d\eta' \frac{qR \ e f_{\text{be}}}{T_e} (\omega - \omega_{Te}) \left\{ [\text{sgn} (\eta - \eta')]^{6+\sigma} \hat{\varphi} (\eta') - \frac{i}{\Xi_{enT_e}} \left[ \Xi_{en}^m - H (\eta - \eta') \Xi_{enT_e}^m \right] \frac{|v||}{c} [\text{sgn} (\eta - \eta')]^{6+\sigma} \hat{A}_\parallel (\eta') \right\} J_0 (\Lambda'_e) + O \left( \frac{\omega}{\omega_{be}} \right).
\]  

(2.117)

We now compute the moments of the total nonadiabatic response functions. When an analytical result is not possible, the result will be retooled into a form that is
conducive for numerical analysis. Beginning once again with the ions, we find that

\[
\int dv \left| v_\perp^{\delta -1^*} \right| J_0 (\Lambda_i) g^{\sigma}_{C_i} =
\]

\[
- i \pi \int d\eta' \int_{-\infty}^{\infty} dv_\perp \int_{0}^{\infty} dv_\parallel \left| v_\perp^{\delta -1^*} \right| \exp \left[ -i \frac{\eta'}{c} \text{sgn} (\eta - \eta') \right] \frac{q R \, e \, f_{0k}}{T_i} (\omega - \omega_{*i}) \times
\]

\[
\left\{ \left[ \text{sgn} (\eta - \eta') \right]^{\delta -1^*} \tilde{\varphi} (\eta') \left[ \text{sgn} (\eta - \eta') \right]^{\delta -1^*} \tilde{A}_i (\eta') \right\} J_0 (\Lambda_i) J_0 (\Lambda'_i). \quad (2.118)
\]

The integration of Eq. (2.118) is greatly simplified if we ignore the \( v_\parallel \) modulation along the unperturbed particle orbit due to the equilibrium magnetic field. This effect produces corrections of order \( \delta \) except for the barely circulating electrons. Thus, in accordance with neglecting trapped ions, we may assume constant values for \( v_\parallel \) and \( v_\perp \) along the unperturbed orbits. Thus, the \( v_\perp \) integration can be carried out analytically to produce

\[
\int dv \left| v_\perp^{\delta -1^*} \right| J_0 (\Lambda_i) g^{\sigma}_{C_i} =
\]

\[
- i \frac{q R}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\eta' \int_{0}^{\infty} d\tilde{v}_i \exp \left( i \frac{q R}{v_\perp} \omega (\eta - \eta') - \tilde{v}_i^2 (\Gamma - \Gamma') \right) \frac{\text{sgn} (\eta - \eta') - \tilde{v}_i^2}{4a |v_\perp|} \times
\]

\[
\left\{ \omega - \omega_{*i} \left[ 1 + \eta_i \left( \tilde{v}_i^2 - \frac{3}{2} \right) + \frac{\eta_i}{2a} \left( 1 - \frac{b_i^2 + b_i + \sqrt{b_i^2 b_i + 4a I_0}}{4a I_0} \right) \right] \right\} \left| v_\perp^{\delta -1^*} \right| \times
\]

\[
\left\{ \left[ \text{sgn} (\eta - \eta') \right]^{\delta -1^*} \tilde{\varphi} (\eta') \left[ \text{sgn} (\eta - \eta') \right]^{\delta -1^*} \tilde{A}_i (\eta') \right\} I_{0e} e^{-(b_i + b_i)/8a}
\]

(2.119)

with

\[
a = \frac{1}{2} \left( 1 + i \frac{q R (\Gamma - \Gamma')}{2 |v_\parallel|} \text{sgn} (\eta - \eta') \right)
\]

(2.120)
\[ \Gamma = \int_{\eta} \tilde{\omega}_{dt} (\eta') \, d\eta' \] 
\[ b_i = k^2_\perp (\eta) \rho_i^2 \] 
\[ \tilde{v}_{||} = \frac{v_{||}}{v_{ta}} \] 

and

\[ I_{0(1)} = I_{0(1)} \left( \frac{\sqrt{b_i b_t}}{4a} \right) \] 

where \( I_{0(1)} \) are the modified Bessel functions of the first kind of the zeroeth and first orders respectively.

Before we proceed directly into the computation of the electron moments we must first distinguish between the circulating and trapped electron regions. This is conveniently done using the pitch-angle variable, \( \lambda \), defined as

\[ \lambda \equiv \frac{\mu B_0}{E} \equiv \frac{v^2}{u^2} x(\eta) \] 

where \( x(\eta) \) is defined in terms of the toroidal vacuum magnetic field \( B(\eta) \) by

\[ x(\eta) \equiv \frac{B_0}{B(\eta)} = 1 + \epsilon \cos \eta. \]

The pitch angle variable provides the following decomposition of velocity space:

\[ 0 \leq \lambda \leq x_m : \text{circulating region} \] 
\[ x_m < \lambda \leq x(\eta) : \text{trapped region} \]

where \( x_m \) is the minimum of \( x(\eta) \) on a flux surface for \( 0 \leq \eta \leq 2\pi \). The velocity coordinates, \( v \), correspondingly transform to

\[ v_{||} = v \sqrt{1 - \frac{\lambda}{x(\eta)}}, \]
\[ v_\perp = v \sqrt{\frac{2\lambda}{x(\eta)}} \]  
(2.130)

and

\[ \int d\nu = \pi \int_0^\infty d\nu \int_{C(T)} d\lambda \left( x(\eta) \sqrt{1 - \frac{\lambda}{x(\eta)}} \right)^{-1} \]  
(2.131)

where \( C(T) \) refers to the choice of integration limits corresponding to Eqs. (2.127) and (2.128) respectively. Separate trapped particle regions exist every \( 2\pi \) in \( \eta \) and end at bounce points corresponding to \( \nu_\parallel = 0 \). The location of the bounce point, \( \eta_b \), then becomes dependent on \( \lambda \) as

\[ \eta_b = \arccos \left( \frac{\lambda - 1}{\epsilon} \right). \]  
(2.132)

Thus, for an arbitrary trapped particle region, the corresponding bounce points, \( \eta_1 \) and \( \eta_2 \) are

\[ \eta_1 (\lambda, N) = 2\pi N - \eta_b \quad \eta_2 (\lambda, N) = 2\pi N + \eta_b, \]  
(2.133)

where \( N \) is chosen such that \( \eta_1 \leq \eta \leq \eta_2 \).

We are now able to compute the moments for the electron nonadiabatic response function in the velocity space coordinates \( \nu \) and \( \lambda \). The moment for the circulating electrons is found to be

\[ \int d\nu \left| \nu_\parallel^{\delta - 1\nu} \right| J_0 \left( \Lambda_e \right) g^\nu_{C_e} = \]

\[
\frac{i\pi}{\sqrt{1 - \frac{\lambda}{x(\eta)}}} \left( \omega - \omega_{e}^T \right) \left\{ \frac{e \delta_{\nu_\parallel}}{T_e} \frac{\epsilon \delta_{\nu_\parallel} \overline{\varphi}(\eta') - \nu}{c} \sqrt{1 - \frac{\lambda}{x(\eta)}} \times \left[ \text{sgn}(\eta - \eta') \right]^{\delta - 1\nu} \right. \]

\[
\left. \times \left[ \text{sgn}(\eta - \eta') \right]^{\delta - 1\nu} \hat{A}_\parallel(\eta') \right\} \]  
(2.134)
where we have made the approximation \( J_0 (\Lambda_e) , J_0 (\Lambda'_e) \simeq 1 \), since \( \Lambda_e , \Lambda'_e \ll 1 \). For the trapped electrons, the moment is

\[
\int dv |v_1^{\delta-\iota\sigma} | J_0 (\Lambda_e) g_{Te}^e = \\
-\pi \sum_{N=-\infty}^{\infty} \int d\lambda \int_0^\infty d\nu \nu^2 \int_{\eta_1 (\lambda,N)}^{\eta_2 (\lambda,N)} \frac{d\eta}{x (\eta)} \left[ u \sqrt{1 - \frac{\lambda}{x (\eta)}} \right]^{\delta-\iota \sigma - 1} \frac{qR}{\sqrt{1 - \frac{\lambda}{x (\eta)}}} \times \\
\frac{e f_{He} (\omega - \omega_{Te})}{T_e} \left\{ \Xi_{\eta N} (\eta - \eta') \Xi_{\eta N} (\eta') - \frac{v}{c} \sqrt{1 - \frac{\lambda}{x (\eta)}} \right\} \\
\left[ \Xi_{\eta N} (\eta - \eta') \Xi_{\eta N} (\eta') \right] \times \left[ \text{sgn} (\eta - \eta') \right]^{\delta-\iota \sigma} \tilde{A}_{\parallel} (\eta') \right\} \tag{2.135}
\]

where \( \Xi_{\eta N} (\eta - \eta') \Xi_{\eta N} (\eta') \) can be approximated to give

\[
\Xi_{\eta N} (\eta - \eta') \simeq \omega - (\omega_{de}) . \tag{2.136}
\]

with

\[
\langle \omega_{de} \rangle = \omega_{se} v_n^2 \left\{ \frac{2E}{K} - 1 + 4s \left[ \frac{E}{K} + \chi^2 - 1 \right] + \frac{4}{3} \alpha \left[ (1 - 2\chi^2) E + \chi^2 - 1 \right] \right\} , \tag{2.137}
\]

\[
\chi^2 = \frac{1}{2} \left( 1 + \frac{1 - \lambda}{\epsilon} \right) \tag{2.138}
\]

and

\[
K = K (\chi^2) , \quad E = E (\chi^2) \tag{2.139}
\]

are the complete elliptic integrals of the first and second kind respectively. Eq. (2.137) is well known [62][165] and the details for its calculations can be found in ref. [161]. One of the reasons for retaining the pitch angle dependence in this manner is that it becomes important for the trapped electron response if one considers advanced tokamak regimes in the second stability region or with reversed magnetic shear [166].
In particular, $\lambda$ dictates the fraction of trapped electrons that have favourable toroidal precession drifts which may stabilize trapped electron modes [165].

This completes the last of the three stages in the eigenmode formulation for high-$n$ drift and shear-Alfvén modes. The velocity moments, in conjunction with the quasineutrality condition and parallel Ampère's law, provide a complete description for these types of instabilities within the assumptions that we have previously specified. Our next concern now is to numerically resolve the spatial and eigenfrequency structures of these modes by rearranging the system of equations into integral equation form.
Chapter 3

Numerical Representation and Analysis

The eigenmode equation inherently represents an eigenvalue problem. The eigenfunctions, \( \varphi(\eta) \) and \( A(\eta) \), jointly correspond to an eigenvalue \( \omega \) from which the frequency \( (\omega_r \equiv \text{Re}(\omega)) \) and stability \( (\gamma \equiv \text{Im}(\omega)) \) can then be determined. Since the modes that we are studying are of the order of the diamagnetic frequency, the eigenvalue will be normalized in terms of the electron diamagnetic frequency. That is,

\[
\hat{\omega} = \frac{\omega_r}{\omega_{se}} + i\frac{\gamma}{\omega_{se}}.
\]  

(3.1)

Obtaining an eigenmode solution involves two steps: employing the method of Fredholm, and secondly, determining \( \hat{\omega} \) and thus the eigenfunctions using a root finder. Solutions can then be used to locate instabilities \( (\text{Im}(\hat{\omega}) > 0) \) for a given set of discharge parameters. We begin the numerical treatment by rearranging the quasineutrality and Ampère's law equations into linear, homogeneous Fredholm equations of the second kind.
3.1 Fredholm Equation

Collecting coefficients for $\tilde{\varphi} (\eta')$ and $\tilde{A}_i (\eta')$ through use of Eqs. (2.119), (2.134) and (2.135), enables Eqs. (2.109) and (2.110) to explicitly reveal their true integral equation form. After a lot of tedious algebra we find that they are respectively,

$$\tilde{\varphi} (\eta) = \frac{r}{1 + r} \int_{-\infty}^{\infty} d\eta' \left[ \Pi^{11} (\eta, \eta', \tilde{\omega}) \tilde{\varphi} (\eta') - \Pi^{12} (\eta, \eta', \tilde{\omega}) \tilde{A}_i (\eta') \right]$$

(3.2)

and

$$\tilde{A}_i (\eta) = \frac{2\beta_i}{b_i (\eta)} \int_{-\infty}^{\infty} d\eta' \left[ \Pi^{21} (\eta, \eta', \tilde{\omega}) \tilde{\varphi} (\eta') - \Pi^{22} (\eta, \eta', \tilde{\omega}) \tilde{A}_i (\eta') \right]$$

(3.3)

where $\beta_i \equiv 8\pi n_0 T_i / B^2$ is the ion beta, $r \equiv T_e / T_i$ is the species temperature ratio and the kernel, $\Pi^{mn} (\eta, \eta', \tilde{\omega})$, is given as

$$\Pi^{mn} (\eta, \eta', \tilde{\omega}) =$$

$$- \frac{i}{\sqrt{\pi}} \frac{\sqrt{b_0}}{2\epsilon_n} q \int_{-\infty}^{\infty} d\tilde{\omega}_{\parallel} \frac{\exp \left( i \frac{\sqrt{b_0}}{2\epsilon_n} q \left[ \tilde{\omega} \tau (\eta - \eta') + 2\epsilon_n \tilde{\omega}_{\parallel} G (\eta, \eta') \right] \text{sgn} (\eta - \eta') \right)}{2a |\tilde{\omega}_{\parallel}|} \times$$

$$\exp \left( -\tilde{\omega}_{\parallel}^2 \right) \left\{ \tilde{\omega} \tau + 1 + \eta_i \left[ \tilde{\omega}_{\parallel}^2 - \frac{3}{2} + \frac{1}{2a} \left( 1 - \frac{b_i' + b_i}{8a} + \frac{\sqrt{b_i' b_i} I_1}{4a I_0} \right) \right] \right\} \times$$

$$\left[ \tilde{\omega}_{\parallel} \text{sgn} (\eta - \eta') \right]^{m+n-2} I_0 \left( \frac{\sqrt{b_i' b_i}}{4a} \right) \exp \left( - \left( b_i' + b_i \right) / 8a \right) +$$

$$\frac{i}{\sqrt{\pi}} \frac{\sqrt{b_0}}{2\epsilon_n} q \int_{0}^{\infty} d\tilde{\omega}_{\parallel} \tau^3 \left[ \int_{0}^{1-e} \frac{d\lambda}{x(\eta)} \left[ \tilde{\omega} \sqrt{1 - \frac{\lambda}{x(\eta)}} \text{sgn} (\eta - \eta') \right]^{m+n-4} \right] \times$$

$$\exp \left[ -i \frac{\sqrt{b_0}}{2\epsilon_n} q \sqrt{\frac{m}{M}} \frac{\tilde{\omega} (\eta - \eta') - 2\epsilon_n \tilde{\omega} \tilde{\omega} G (\eta, \eta')}{\tilde{\omega} \sqrt{1 - \frac{\lambda}{x(\eta)}}} \right] \times$$

$$\left( \tilde{\omega} - 1 - \eta_i \left[ \tilde{\omega}^2 - \frac{3}{2} \right] \right) \left[ (\delta_m 1 \delta_{n1} + \delta_m 2 \delta_{n1}) + (\delta_m 1 \delta_{n2} + \delta_m 2 \delta_{n2}) \sqrt{\frac{m}{M}} \right] +$$
\[
\sum_{n = -\infty}^{\infty} \int_{x = -\infty}^{x(n)} \frac{d\lambda}{x(\eta)} \frac{\hat{\omega} - \eta_n [\hat{v}^2 - \frac{3}{2}]}{\hat{\omega} - \langle \omega_{de} \rangle} \left[ \hat{v} \sqrt{1 - \frac{\lambda}{x(\eta)} \text{sgn} (\eta - \eta')} \right]^{m+n-4} \times \\
\left\{ \left( \delta_{m1} \delta_{n1} + \delta_{m2} \delta_{n2} \right) + i \left( \delta_{m1} \delta_{n2} + \delta_{m2} \delta_{n1} \right) \sqrt{\frac{m}{M}} \frac{\sqrt{b_{0i}}}{2 \epsilon_n} q \sqrt{\frac{m}{M}} \times \\
\{ \hat{\omega} [\eta_2 (\lambda, N) - \eta'] - 2\epsilon_n \hat{v}^2 G (\eta_2 (\lambda, N), \eta') \} - H (\eta - \eta') (\hat{\omega} - \langle \omega_{de} \rangle) \right\} \times \\
\exp (-\hat{v}^2)
\] (3.4)

where \( q \) is positive and infinitesimally small. The other quantities are written as

\[
G (\eta, \eta') = \sin \eta - \sin \eta' + s (\sin \eta - \sin \eta' - \eta \cos \eta + \eta' \cos \eta') - \\
\frac{\alpha}{2} \left( \eta - \eta' - \frac{\sin 2\eta - \sin 2\eta'}{2} \right)
\] (3.5)

and

\[
a = \frac{1}{2} \left( 1 - i \frac{\sqrt{b_{0i}}}{2 |\hat{v}|} G (\eta, \eta') \text{sgn} (\eta - \eta') \right)
\] (3.6)

with \( b_{0i} = k_0^2 \rho_i^2 \), \( \epsilon_n = L_n / R \) and \( (v_{hi} / c) \hat{A}_{ii} (\eta') \rightarrow \hat{A}_{ii} (\eta') \) is to be assumed henceforth.

All other quantities are as previously defined. It is also noted that an integration interchange in \( \eta' \) and \( \lambda \) is implied for the trapped electron term.

The linear system defined by Eqs. (3.2)-(3.4) is solved for \( \hat{\omega}, \hat{\varphi} (\eta) \) and \( \hat{A}_{ii} (\eta) \) using “the method of solution of Fredholm” [167]. To illustrate this method, we consider the following integral equation example.

\[
\varphi (s) = \int_a^b dt K(s, t, \lambda) \varphi (t),
\] (3.7)

where \( \lambda \) is the eigenvalue. Dividing the integration range into \( n \) equal parts yields the following approximation

\[
\int_a^b dt K(s, t, \lambda) \varphi (t) \simeq h \sum_{k=1}^n K(s, t_k, \lambda) \varphi (t_k)
\] (3.8)
with \( h = (b - a) / n \) and

\[
s_1 = t_1 = a, \quad s_2 = t_2 = a + h, \quad \ldots \quad s_n = t_n = a + (n - 1) h.
\]

Eq. (3.7) then becomes

\[
\varphi (s) \simeq h \sum_{k=1}^{n} K(s, t_k, \lambda) \varphi (t_k)
\]

which must hold for all values of \( s \) in the interval \((a, b)\). In particular, this equation is satisfied at the \( n \) points of division \( s_j, j = 1, \ldots, n \). This leads to the system of equations

\[
\varphi (s_j) - h \sum_{j=1}^{n} K(s_j, t_k, \lambda) \varphi (t_k) = 0, \quad j = 1, \ldots, n.
\]

The values of \( \varphi (s_j) \) obtained by solving this algebraic system are approximate solutions of the integral equation (3.7) at the points \( s_1, s_2, \ldots, s_n \). We can treat these solutions \( \varphi (s_j) \) as ordinates and, by interpolation, construct the approximate solution \( \varphi (s) \). The corresponding eigenvalue \( \lambda \) is determined from the resolvent determinant of the algebraic system (3.11) which is given as

\[
D_n(\omega) = \begin{vmatrix}
1 - hK_{11} & -hK_{12} & \cdots & -hK_{1n} \\
-hK_{21} & 1 - hK_{22} & \cdots & -hK_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-hK_{n1} & -hK_{n2} & \cdots & 1 - hK_{nn}
\end{vmatrix}
\]

and for simplicity we have made the abbreviation, \( K_{jk} = K(s_j, t_k, \lambda) \). Setting this determinant equal to zero yields the approximate eigenvalue.

The same procedure is adopted to solve our integral equation system, yet we must adopt a more efficient way of computing the \( \eta' \) integration since the kernel we are dealing with also involves two velocity integrations. By employing Gaussian
quadrature rules whenever possible, we can reduce the computational effort when evaluating these integrals while still maintaining good accuracy. For the $\eta'$ integral, we opt to use the Laguerre formula [168], given as

$$
\int_0^\infty dx e^{-x} f(x) = \sum_{j=1}^{\infty} w_j f(x_j) + \frac{(n!)^2}{(2n)!} f^{(2n)}(\xi), \quad 0 < \xi < \infty
$$

(3.13)

where the abscissa $x_j$ are the $j^{th}$ zeroes of the Laguerre polynomials $L_n(x)$ and the weighting coefficient is written as

$$
w_j = \frac{x_j}{(n + 1)^2 [L_{n+1}(x_j)]^2}.
$$

(3.14)

The $L_n(x)$ can be determined through use of the following recurrence relations:

$$
L_0(x) = 1
$$

$$
L_1(x) = 1 - x
$$

$$
L_{n+1}(x) = (2n + 1 - x) L_n(x) - n^2 L_{n-1}(x).
$$

(3.15)

To accommodate our system, we use Eq. (3.13) to tailor the $\eta'$ integration in the following manner,

$$
\int_{-\infty}^{\infty} d\eta' f(\eta') \simeq \sum_{j=-\infty}^{-1} w_j e^{-\eta_j} f(\eta_j^+) + \sum_{j=1}^{\infty} w_j e^{\eta_j} f(\eta_j^-),
$$

(3.16)

where we omit the correction term in Eq. (3.13), since it is virtually impossible and impractical to compute. Although it is realized that this method is usually only applied when $e^{\eta'} f(\eta')$ exhibits polynomial-like behaviour, we can loosely justify its use since the system of equations is nearly diagonal (rapid $\eta'$ integral convergence) and the arguments of all oscillating functions in (3.4) vary approximately as $\eta'$, i.e. oscillation frequencies are low. Of course, there is no clear-cut rule for selecting a quadrature method since such rules are subject to indeterminacy but we have found that for
sufficiently large \( n \), the Gauss-Laguerre method is just as good as any composite formula such as Simpson's rule, with sufficiently accurate results computed in just a fraction of the time.

The only real difficulty in applying this method, lies in generating the \( n \) zeroes of \( L_n(\eta') \) which can be computed using a predictor-corrector type approach. Using approximations for the various zeroes given by Stroud and Secrest [169], the order of the polynomial can be computed to very high values, \( n \approx 960 \), using quadruple precision. The accuracy of the abscissa and weights are then checked by computing known integrals. The details of the zero-finding procedure are found in Appendix B.

Employing the representation of Eq. (3.16) allows the linear eigenmode system to be approximated as

\[
\begin{bmatrix}
\hat{\varphi}_{-n} \\
\vdots \\
\hat{\varphi}_{n} \\
\hat{A}_{n}\hat{\varphi}_{-n} \\
\vdots \\
\hat{A}_{n}\hat{\varphi}_{n}
\end{bmatrix} M(\bar{\omega}) = 0 \tag{3.17}
\]
where

\[
M(\tilde{\omega}) \equiv \begin{bmatrix}
1 - cw_1 \Pi^{11}_{n-n-n} & \cdots & -cw_n \Pi^{11}_{n-n-n} & cw_1 \Pi^{12}_{n-n-n} & \cdots & cw_n \Pi^{12}_{n-n-n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-cw_1 \Pi^{11}_{n-n-n} & \cdots & 1 - cw_n \Pi^{11}_{n-n-n} & cw_1 \Pi^{12}_{n-n-n} & \cdots & cw_n \Pi^{12}_{n-n-n} \\
-d_1 w_1 \Pi^{21}_{n-n-n} & \cdots & -d_1 w_n \Pi^{21}_{n-n-n} & 1 + d_1 w_1 \Pi^{22}_{n-n-n} & \cdots & d_1 w_n \Pi^{22}_{n-n-n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-d_n w_1 \Pi^{21}_{n-n-n} & \cdots & -d_n w_n \Pi^{21}_{n-n-n} & d_n w_1 \Pi^{22}_{n-n-n} & \cdots & 1 + d_n w_n \Pi^{22}_{n-n-n}
\end{bmatrix}
\]

with \( c = \tau / (1 + \tau) \), \( d_j = 2\beta_j / b_i(\eta_j) \) and \( \Pi^{im}(\eta_j, \eta'_k, \tilde{\omega}) = \Pi^{im}_{jk} \). Since the eigenfunctions converge rapidly for moderate values of \( \eta' \), we need not extend \( j \) all the way to \( n \). Instead, we can truncate the upper limit of the index to some acceptable value \( L^{(La)} \) (usually \( L^{(La)} \approx 70 - 80 \)), for which \( \tilde{\omega}, \tilde{\psi} \) and \( \tilde{\theta}_\parallel \) are still independent. We also note that for equilibrium geometry with up-down symmetry, symmetric eigenfunctions exist, e.g., \( \tilde{\psi}(\eta) = \tilde{\psi}(-\eta) \). This implies that \( \Pi^{im}(\eta, \eta', \tilde{\omega}) = \Pi^{im}(-\eta, -\eta', \tilde{\omega}) \), and thus the computational effort is cut in half since only \( \eta \leq 0 \) need be simulated.

### 3.2 Eigenvalue and Eigenfunction Determination

Solving the characteristic equation, \( \det M(\tilde{\omega}) = 0 \), basically consists of finding the zeroes of the analytical function \( \det M(\tilde{\omega}) \) in the complex frequency plane. The zeroes can be found using the Muller’s method algorithm [170] which is quite sufficient to fill this role. The solution process begins with an approximation \( z_0 \) for \( \tilde{\omega} \), provided from some other formalism or source, which is then perturbed to give the necessary
three starting values \( z_0, z_{-1} \) and \( z_{-2} \) so that the function

\[
g(z_i) = \det[M(\tilde{\omega})]
\]

(3.19)
can be interpolated to form a quadratic polynomial \( P \). Of the (normally) two zeroes of \( P \), the one closest to \( z_0 \) is selected as the new approximation \( z_1 \) and the process is then continued with \((z_1, z_0, z_{-1})\). A zero is accepted as an eigenvalue, \( \tilde{\omega} = z_I \), pending three conditions: when \( g(z_I) \leq \epsilon_g \) after \( I \) iterations where \( \epsilon_g \) is the desired precision; \( z_I \) does not change much during successive iterations, \( z_I - z_{I-1} \leq \epsilon_z \), where \( \epsilon_z \) is the desired precision; and \( z_I \) is independent of \( L^{(L)} \) to some acceptable accuracy.

Once the eigenvalue is obtained, it becomes a simple matter to compute the eigenfunctions. The linear system of Eq. (3.17) is solved for \( \tilde{\phi}(\eta) \) and \( \tilde{A}(\eta) \) by decomposing \( M(\tilde{\omega}) \) into LU form with the aid of Crout's algorithm. By assuming that \( \tilde{\phi}_1 = 1 \), and using backsubstitution, the eigenfunctions are then immediately obtained.

### 3.3 Representation of the Particle Moments

#### 3.3.1 Circulating Ions

We will first analyse the basic form of the circulating ion moment pertaining to the scalar potential in the quasineutrality condition \((m = 1, n = 1)\) since it contains a singularity at \( \tilde{v}_1 = 0 \). It can be written in basic form as

\[
I_{C1} = \int_0^\infty d\tilde{v}_i \frac{f(\tilde{v}_i)}{\tilde{v}_i} \exp\left(-\tilde{v}_i^2\right),
\]

(3.20)

and is a natural candidate for Gauss-Hermite quadrature due to the \( e^{-\tilde{v}_i^2} \) weighting. Because Gauss quadrature is of the open type (both ends of the integration interval are omitted from the evaluation) the singularity is handled without much difficulty.
The same method is applied to the other ion moments as well. The Gauss-Hermite quadrature formula [168] for one-half the standard interval is given as

$$
\int_0^\infty dx e^{-x^2} f(x) \simeq \sum_{j=1}^{n} w_j f(x_j), \quad w_j = \frac{2^{n-1}n!\sqrt{\pi}}{n^2[H_{n-1}(x_j)]^2}.
$$

(3.21)

The abscissa $x_j$ are the $j^{th}$ zeroes of the Hermite polynomials $H_n(x)$, which can be determined from the following recurrence relations:

$$
H_0(x) = 1 \\
H_1(x) = 2x \\
H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).
$$

(3.22)

Like the Gauss-Laguerre case, the only difficulty in applying the quadrature method is the determination of the zeroes and weights due to the high order of the polynomial that is needed for sufficient accuracy. The zeroes are calculated starting with the largest, and since they are symmetric about $\bar{\nu}_1 = 0$, only the nonnegative zeroes are computed. For our purposes, Hermite polynomials of order $n \simeq 200$ are needed, with the truncation point occurring at about $L^{(He)} \simeq 25$ which corresponds to accuracy of approximately $\lesssim 1\%$. The details of the zero-finding procedure are outlined in Appendix B.

### 3.3.2 Circulating and Trapped Electrons

For the circulating electrons, the moment consists of two nested integrals resembling that of

$$
I_{Ce} = \int_0^{\infty} d\bar{\nu} \int_0^{1-\epsilon} d\lambda f(\bar{\nu}, \lambda) \exp(-\bar{\nu}^2).
$$

(3.23)

The evaluation of the integral with respect to $\bar{\nu}$ is straightforward if one uses the Gauss-Hermite quadrature formula as in the circulating ion case. For the integral
with respect to the pitch angle variable \( \lambda \), however, it is more convenient to modify it first before attempting to evaluate it. Noting the variable transformation,

\[
\int_0^a dx f(x) = a \int_0^1 dt f(at),
\]

(3.24)

uniquely tailors the integral into a form that is ideally suited for Gauss-Legendre quadrature, which is given as [168]

\[
\int_0^1 dx f(x) = \sum_{j=1}^n w_j f(x_j), \quad w_j = \frac{2}{(1-x_j^2)[P'_n(x_j)]^2}
\]

(3.25)

for the upper region of the standard interval. The \( P'_n(x_j) \) are determined from the following recurrence relations:

\[
P_0(x) = 1
\]

\[
P_1(x) = x
\]

\[
(x^2 - 1) P'_n(x) = nx P_n(x) - n P_{n-1}(x).
\]

(3.26)

For the integral at hand, Legendre polynomials of order \( n \approx 25 \) are sufficient to yield satisfactory results and are easy to compute.

The integrals for the trapped electrons are problematic. The basic form is given as

\[
I_{Te} = \sum_{N=\eta \langle \lambda, N \rangle}^\infty \int_{\eta_{1-\epsilon}}^\infty \int_0^\infty \int_{\eta_{1-\epsilon}}^\infty d\lambda d\bar{v} \int_{\eta_{1-\epsilon}}^\infty d\eta' f(\eta', \lambda, \bar{v}) \exp \left( -\bar{v}^2 \right)
\]

(3.27)

where the velocity integration, as usual, can be computed using a Gauss-Hermite rule. The difficulty arises as a result of the bounce point dependence on \( \lambda \), as dictated by Eqs. (2.132) and (2.133), which conflicts with the use of Gauss-Laguerre quadrature. Specifically, the positions of the \( \eta' \) abscissa are determined and set by the Gauss method which must be adhered to in order for the eigenmode system to be solved.
However, the pitch angle dependence places restrictions on \( \eta' \) in the form of bounce points that limits the range of the \( \eta' \) integration. Using a variable transformation to relieve the \( \eta' \) dependence on \( \lambda \) exasperates the problem because of the integral equation nature of the system, i.e. \( \mathcal{G} \) and \( \mathcal{A}_l \) must remain explicitly dependent on \( \eta' \). The only recourse is to assume that each bounce point corresponds to each of the Gauss-Laguerre abscissa within the trapped electron region, which in turn determines each \( \lambda \). The \( \lambda \) integration can then be evaluated through a simple composite formula such as the trapezoid rule. The \( \eta' \) integration must also be evaluated using a trapezoidal rule because the integration bounds in \( \eta' \) are finite and do not approach \( \infty \) which is a requirement for the use of Gauss-Laguerre quadrature. It has also been noted \textit{a posteriori} that it is not necessary to extend \( N \) to integer values beyond \( \pm 1 \) because the eigenfunctions converge rapidly. With these considerations in mind, Eq. (3.27) can then be numerically approximated as

\[
I_{Te} = \frac{h_j}{2} \sum_{N=-1}^{1} \sum_{k=j_N+1}^{j_{N-1}} h_k \sum_{i=1}^{L_{He}^{(He)}} \sum_{i=j_{N-1}}^{j} \left[ f(\eta'_{k-1}, \lambda_i, \tilde{u}_i) + f(\eta'_k, \lambda_i, \tilde{u}_i) \right] (3.28)
\]

where the \( j_N, j_{N+1} \) Gauss-Laguerre abscissa correspond approximately to the bounce points \( \pi (2N - 1) \) and \( \pi (2N + 1) \) respectively with \( j_N \leq j \leq j_{N+1} \), the indices \( j \) and \( k \) correspond to \( \eta_j \) and \( \eta'_k \) respectively, the index \( i \) and the interval width \( h_j = \lambda_j - \lambda_{j-1} \) corresponds to the trapezoid rule for the \( \lambda \) integration, the interval width \( h_k = \eta'_k - \eta'_{k-1} \) pertains to the trapezoid rule for the \( \eta' \) integration, and finally, the index \( l \), the weight function \( w_i^{(He)} \) and the truncation point \( L_{He}^{(He)} \) correspond to the Gauss-Hermite quadrature rule.
3.4 Numerical Accuracy, Convergence and Speed

The code implementing the preceding algorithms, henceforth referred to as the non-local gyrokinetic code, is programmed in Fortran-90 and run on a DEC AlphaStation 21164a machine. In order to ascertain the performance of the Gauss-Laguerre method in terms of speed and accuracy, a comparison will be made with the standard rectangular method. By restricting the codes to run in the electrostatic limit with adiabatic electrons allows us to better isolate the numerical process without complicating the analysis with the intricate electromagnetic terms which also consumes a considerable amount of time when the composite case is considered. An electromagnetic comparison will be presented shortly. The parameters found in the electrostatic integral equation paper by Dong et al. [119] for the kinetic ITG mode are used as our test parameters and are as follows: $\eta = 1$, $s = q = r = 1$, $k_d \rho_s = 0.45$, $\epsilon_n = 0.25$ and $\tilde{\omega} = -0.607 + i0.258$ which is used as the root finder starting point. For both quadrature methods we choose the maximum value of $\eta$ to be sufficiently large ($\eta_{\text{max}} \simeq 15$) so that $\tilde{\omega}$ is independent of it within acceptable limits. The velocity integration for both methods employs a Hermite polynomial of order $n = 200$ with the abscissa truncation point occurring at $L^{(He)} = 25$ which corresponds approximately to $\tilde{\nu}_{\text{limax}} \simeq 4$. The integral equation results for the rectangular quadrature method are shown in Table 3.1 and as can be seen, the eigenvalue begins to saturate for large values of $n$ which agrees with the result from ref. [119] to within 2.5% for $\omega_r/\omega_{se}$ and approximately 2% for $\gamma/\omega_{se}$. For each of the given $n$ values, the eigenvalue approaches four digit accuracy in two iterations. Note also that the matrix filling and determinant computation times increase geometrically with $n$. When $n$ is doubled, $t_{\text{matrix}}$ increases about four fold while $t_{\text{det}}$ increases roughly ten fold.

The results for the Gauss method are given in Table 3.2. For high Laguerre
<table>
<thead>
<tr>
<th>$n$</th>
<th>$h$</th>
<th>$\tilde{\omega}$</th>
<th>$t_{\text{matrix}}$ (min.)</th>
<th>$t_{\text{det}}$ (min.)</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.1500</td>
<td>$-0.6232 + i0.2485$</td>
<td>0.06</td>
<td>0.01</td>
<td>2</td>
</tr>
<tr>
<td>200</td>
<td>0.0750</td>
<td>$-0.6228 + i0.2601$</td>
<td>0.24</td>
<td>0.10</td>
<td>2</td>
</tr>
<tr>
<td>300</td>
<td>0.0500</td>
<td>$-0.6226 + i0.2623$</td>
<td>0.66</td>
<td>0.45</td>
<td>2</td>
</tr>
<tr>
<td>400</td>
<td>0.0375</td>
<td>$-0.6226 + i0.2631$</td>
<td>0.94</td>
<td>1.18</td>
<td>2</td>
</tr>
<tr>
<td>500</td>
<td>0.0300</td>
<td>$-0.6226 + i0.2634$</td>
<td>1.48</td>
<td>2.78</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3.1: Observed eigenvalues $\tilde{\omega}$ generated by the rectangular composite rule as a function of varying interval width $h$ and interval number $n$ for the parameters $\eta_i = 1$, $s = q = \tau = 1$, $k_0\rho_s = 0.45$ and $\epsilon_n = 0.25$ with $\tilde{\omega} = -0.607 + i0.258$ used as the starting guess from Dong et al. The quantities $t_{\text{matrix}}$ and $t_{\text{det}}$ represent the times taken to fill the matrix and to compute the determinant respectively over the course of $I$ root-finding iterations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L \rightarrow \eta_{\text{max}} \approx 15$</th>
<th>$\tilde{\omega}$</th>
<th>$t_{\text{matrix}}$ (min.)</th>
<th>$t_{\text{det}}$ (min.)</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>50</td>
<td>$-0.6209 + i0.2488$</td>
<td>0.044</td>
<td>0.002</td>
<td>3</td>
</tr>
<tr>
<td>600</td>
<td>61</td>
<td>$-0.6215 + i0.2538$</td>
<td>0.066</td>
<td>0.003</td>
<td>3</td>
</tr>
<tr>
<td>800</td>
<td>70</td>
<td>$-0.6218 + i0.2563$</td>
<td>0.088</td>
<td>0.004</td>
<td>3</td>
</tr>
<tr>
<td>900</td>
<td>74</td>
<td>$-0.6218 + i0.2572$</td>
<td>0.097</td>
<td>0.005</td>
<td>3</td>
</tr>
<tr>
<td>960</td>
<td>77</td>
<td>$-0.6219 + i0.2576$</td>
<td>0.105</td>
<td>0.006</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 3.2: Variation of eigenvalues $\tilde{\omega}$ with respect to Laguerre polynomial number $n$ for the same parameters as in Table 3.1. The maximum abscissa number $L$ is truncated to correspond approximately to a maximum $\eta$ value of about 15.
polynomial number $n$, the eigenvalue begins to saturate akin to the rectangular case, albeit requiring an extra iteration. The eigenvalue agrees with the rectangular and Dong result to within 0.1% and 2.4% respectively for $\omega_r/\omega_e$, and 2.2% and 0.2% respectively for $\gamma/\omega_e$. Hence both methods produce more or less the same accuracy with the Laguerre method producing results on average at approximately 10–20 times the speed of the composite method on a per iteration basis. The results are even more dramatic when the electromagnetic case is considered since the matrix swells to four times the size of the electrostatic case which causes the determinant computation time to increase sharply. It is also of interest to note that the present code is also more efficient than that of Dong who reported eigenvalues being calculated at the rate of 3 min/eigenvalue on a CRAY-2 via a combined rectangular/Gauss method and who also reported a doubling of speed over the Romanelli code [115]. To ascertain the computational performances of the various machines with respect to each other, such as the CRAY-2 relative to the DEC 21164a, all the machines cited in this work are listed in Appendix C and rated in peak MFLOPS for comparison.

The scalar eigenfunction $\varphi(\eta)$ corresponding to the Laguerre polynomial of order $n = 960$ in Table 3.2, is presented in Fig. 3.1. Upon examination of $\varphi$, we see that it is well localized, $|\eta| \lesssim 6$, and that the real and complex components, represented by $\text{Re}(\varphi)$ and $\text{Im}(\varphi)$ respectively, converge rapidly to zero as expected.

Application of the Laguerre algorithm to the complete integral equation code with electromagnetic and trapped electron components is a straightforward extension of the electrostatic case. Since we have incorporated certain approximations into the model with respect to the trapped electrons, it is more sensible to relegate numerical testing of the full code in benchmarking trials against that of the Rewoldt model.
Figure 3.1: Scalar eigenfunction $\phi$ corresponding to the Laguerre polynomial $n = 960$ for the parameters given in Table 3.1 for the nonlocal model in the electrostatic limit Eq. (3.17). The solid line corresponds to real part of $\phi$ whereas the dotted line corresponds to the complex part.
Chapter 4

Benchmarking with Other Models

The results of the application of the nonlocal gyrokinetic code are now presented in conjunction with some simplified kinetic model calculations and the comprehensive kinetic analysis by Rewoldt. In general, the simplified kinetic models may provide a faster means of predicting the analytical trends of the modes under study. However, because they are approximations of the integral equation approach, it is desirable to explore the extent of their accuracy and applicability. To keep the physics as consistent as possible, these models will also contain terms related to the trapping of electrons and compressional ion dynamics. The basic models that we will study consist of the semilocal kinetic model, the reduced kinetic or shooting code model and the ideal MHD model.

4.1 Semilocal Kinetic Model

If the eigenfunction possesses a moderate to strong ballooning structure, then we may employ the semilocal kinetic model, initially developed by Hirose [171][172]. This theory is based strongly on the local kinetic model with the sole exception being
the incorporation of \( k_\parallel, k_L \) and \( \omega_d \) through use of an averaging procedure. As a result of this key improvement, the ballooning nature of interchange type instabilities can be better represented.

The basic starting point that is common to this theory and to that of the reduced kinetic model is that the distribution function used in both cases is the same. It can be derived by applying the result from (2.89),

\[
\frac{1}{qR} \frac{\partial}{\partial \eta} = i k_\parallel, \tag{4.1}
\]

to Eq. (2.90) and then substituting the resulting \( \hat{\alpha} \) into Eq. (2.108) which immediately yields

\[
\hat{f}_j = -\frac{e_j f_0_j}{T_j} \hat{\varphi} (\eta) + \frac{\omega - \omega_{je}}{\omega - \omega_{de} (\eta) - k_\parallel v_\parallel} J_0^2 (\Lambda_j) \left[ \hat{\varphi} (\eta) - \frac{v_\parallel}{c} \hat{A}_\parallel (\eta) \right] \frac{e_j f_0_j}{T_j}. \tag{4.2}
\]

From the charge neutrality condition and Ampère’s law, the following dispersion equation is obtained after some algebra,

\[
\left[ k_\perp^2 (\eta) + 2 \left( \frac{\omega_{pe}}{c} \right)^2 F_{Ce} + 2 \left( \frac{\omega_{pi}}{c} \right)^2 F_{Pi} \right] \left[ F_{Ce0} + F_{Te0} - 1 + \tau (1 - F_{e0}) \right] = 2 \left( \frac{\omega_{pe}}{c} \right)^2 \left( F_{Ce1} + \sqrt{\frac{T_m}{M}} F_{Pi1} \right)^2 \tag{4.3}
\]

where \( \omega_{pe(i)} = [4\pi n_{e(i)} e^2 / m_{e(i)}]^{1/2} \) is the electron (ion) plasma frequency and the moments are defined by

\[
F_{in} = \int d\nu \frac{\omega + \omega_{ve}^T}{\omega + \omega_{de} - k_\parallel v_\parallel} \left( \frac{v_\parallel}{v_{\parallel e}} \right)^n J_0^2 (\Lambda_i) f_{\nu e} (v^2), \tag{4.4}
\]

\[
F_{Cen} = \int_C d\nu \frac{\omega - \omega_{ve}^T}{\omega - \omega_{de} - k_\parallel v_\parallel} \left( \frac{v_\parallel}{v_{\parallel e}} \right)^n f_{\nu e} (v^2), \tag{4.5}
\]

\[
F_{Te0} = \int_T d\nu \frac{\omega - \omega_{ve}^T}{\omega - \omega_{de} f_{\nu e} (v^2)}.
\]
with the labels $C$ and $T$ referring to the circulating and trapped regions in the electron velocity space. The integrals are computed by restricting the velocity space integration to $v_\parallel < \sqrt{\epsilon v_\perp}$ for the trapped electrons and $v_\parallel > \sqrt{\epsilon v_\perp}$ for the electrons that are circulating.

The parameters $k_\parallel, k_\perp$ and $\omega_d$ can be estimated by averaging them over an assumed trial eigenfunction of the form, $\varphi(\eta) = (1 + \cos \eta) / \sqrt{3\pi}$ where $\int_{-\pi}^{\pi} \varphi^2(\eta) \, d\eta = 1$. The norms then become

$$\langle k_\perp^2 \rangle = k_\theta^2 \int_{-\pi}^{\pi} d\eta \left[ 1 + (s\eta - \alpha \sin \eta)^2 \right] \varphi^2$$

$$= k_\theta^2 \left[ 1 + \frac{1}{3} s^2 \left( \pi^2 - \frac{15}{2} \right) - \frac{10}{9} s\alpha + \frac{5}{12} \alpha^2 \right] . \quad (4.6)$$

$$\langle k_\parallel^2 \rangle = \frac{1}{(qR)^2} \int_{-\pi}^{\pi} d\eta \varphi \frac{d\varphi}{d\eta^2}$$

$$= \frac{k_c^2}{3} . \quad (4.7)$$

where $k_c = 1/qR$ is known as the connection length and

$$\langle \omega_d \rangle = 2\epsilon_n \omega_* \int_{-\pi}^{\pi} d\eta \left[ \cos \eta + (s\eta - \alpha \sin \eta) \sin \eta \right] \varphi^2$$

$$= 2\epsilon_n \omega_* \left( \frac{2}{3} + \frac{5}{9} s - \frac{5}{12} \alpha \right) . \quad (4.8)$$

Inserting these norms into the dispersion relation of Eq. (4.3) allows it to be easily solved for $\omega$ by employing the Muller's method root finder. The velocity integrations are computed using Simpson's rule.

Although the well known local kinetic result can be obtained in the limit $\eta = 0$, we will not consider its use here since it is too crude for our purposes.
4.2 Reduced Kinetic Model

The collisionless reduced kinetic model is closer in spirit to the nonlocal gyrokinetic model since it retains a dependence on the ballooning variable $\eta$. It was first derived by Connor *et al.* [34] in the ideal MHD limit but was later more rigorously analyzed by Cheng [38] with finite Larmor radius (FLR) and other kinetic effects. It is also known as the kinetic ballooning mode equation and, as such, is primarily used to study ballooning instabilities in the high-$n$ limit. We begin the derivation of the reduced kinetic model by considering Eq. (4.2), from which we can determine the form of the ion distribution function

$$
\tilde{f}_i = -\frac{e f_{0i}}{T_i} \tilde{\varphi} (\eta) + \frac{\omega + \omega_{Ti} T_i}{\omega + \omega_{di} (\eta) - k_{||} v_{||}} J_0^2 (\Lambda_i) \left[ \tilde{\varphi} (\eta) - \delta \omega_{\ast i} \frac{\tilde{A}_{\parallel} (\eta)}{k_{||} c} \right] \frac{e f_{0i}}{T_i} \quad (4.9)
$$

where

$$
\delta = \frac{k_{||} v_{|| i}}{\omega_{\ast i}} = \frac{L_n}{(k_{\theta} \rho_i)} \sim \frac{\sqrt{\langle k_{\parallel}^2 \rangle}}{k_c} \frac{\epsilon_n}{q (k_{\theta} \rho_i)} \quad (4.10)
$$

with

$$
\langle k_{\parallel}^2 \rangle = -\frac{1}{(qR)^2} \int \frac{d^2 \tilde{\varphi}}{d \eta^2} d\eta = k_c^2 \int \frac{|d \tilde{\varphi}|^2}{d \eta} \, d\eta. \quad (4.11)
$$

The ion parallel current perturbation is then

$$
J_{\parallel i} = e \int d v_{\parallel} \tilde{f}_i
$$

$$
= \frac{e^2 n_0}{k_{||} T_i} \left( I_{\ast i} \delta \omega_{\ast i} \tilde{\varphi} - I_{\ast 2} \delta^2 \omega_{\ast i}^2 \frac{\tilde{A}_{\parallel}}{k_{||} c} \right) \quad (4.12)
$$

where

$$
I_{\ast i} = \int d v \frac{\omega + \omega_{Ti} T_i}{\omega + \omega_{di} - k_{||} v_{||} v_{ti}} \frac{v_{|| i}}{v_{ti}} J_0^2 (\Lambda_i) f_{0i} (v^2), \quad (4.13)
$$

$$
I_{\ast 2} = \int d v \frac{\omega + \omega_{Ti} T_i}{\omega + \omega_{di} - k_{||} v_{||} v_{ti}} \left( \frac{v_{|| i}}{v_{ti}} \right)^2 J_0^2 (\Lambda_i) f_{0i} (v^2). \quad (4.14)
$$
For the electrons, the electron transit frequency sets the upper limit for the mode frequency, $\omega \ll k_{||} v_{te}$. Thus, the electron distribution function for the circulating electrons can be expanded in the following fashion,

$$\tilde{f}_{Ce} \approx \frac{e f_{0e}}{T_e} \tilde{\varphi} + \frac{\omega - \omega_{eT}}{k_{||} v_{||}} \left[ \tilde{\varphi} + \frac{\omega - \omega_{de}}{k_{||} v_{||}} \tilde{\varphi} - \frac{v_{||}}{c} \tilde{A}_{||} - (\omega - \omega_{de}) \frac{\tilde{A}_{||}}{c k_{||}} \right] \frac{e f_{0e}}{T_e}.$$  \hspace{1cm} (4.15)

Using Eq. (4.15), the circulating electron parallel current perturbation is then

$$J_{||Ce} \approx -\frac{e^2}{k_{||} T_e} \int d\nu \left[ (\omega - \omega_{eT}) \tilde{\varphi} - (\omega - \omega_{eT}) (\omega - \omega_{de}) \frac{\tilde{A}_{||}}{c k_{||}} \right] f_{0e}$$ \hspace{1cm} (4.16)

where it has been noted that $J_{||}^{2} (\Lambda_{e}) \approx 1$.

For the trapped electrons, the distribution function can be approximated from the general form of the non-adiabatic trapped electron response function, Eq. (2.103), in combination with Eq. (2.111). This approximation allows the response to be greatly simplified by relieving it of its integral equation character. Using an approach based on the work of Cheng, we find the result to be

$$g_{Te} \approx \frac{1}{\Xi_{\eta}} \int d\eta' \left\{ \Psi_{\varphi} + i \frac{\Xi_{\eta} + \Xi_{\eta'}}{2} \Psi_{A} \right\} - \frac{i}{2} \left( \int d\eta' \Psi_{A} + \int d\eta \Psi_{A} \right) + O \left( \frac{\omega}{\omega_{be}} \right).$$ \hspace{1cm} (4.17)

We now define an averaging procedure similar to that of (2.24). Let

$$\langle \Theta \rangle \equiv \frac{\int_{\eta_{m}}^{\eta_{n}} d\eta \Theta}{\int_{\eta_{m}}^{\eta_{n}} d\eta},$$ \hspace{1cm} (4.18)

then Eq. (4.17) can be written as
\[ g_{Te} \approx -\frac{e f_{oe}}{T_e} \frac{\omega - \omega_{se}}{\omega - \langle \omega_{de} \rangle} \langle \phi + \frac{i |v||}{2c} (\omega - \langle \omega_{de} \rangle) \tilde{A}_{||} \int_{n_1}^{n_2} d\eta \frac{q R}{v||} \text{sgn} (\eta - \eta') \rangle + \]

\[ \frac{e f_{oe}}{T_e} \frac{\omega - \omega_{se}}{\omega - \langle \omega_{de} \rangle} \frac{i |v||}{2c} \tilde{A}_{||} \int_{n_1}^{n_2} d\eta \frac{q R}{v||} \text{sgn} (\eta - \eta') + O \left( \frac{\omega}{\omega_{be}} \right) . \quad (4.19) \]

We can further simplify the above result by making the common approximation

\[ \langle \phi + \frac{i |v||}{2c} (\omega - \langle \omega_{de} \rangle) \tilde{A}_{||} \int_{n_1}^{n_2} d\eta \frac{q R}{v||} \text{sgn} (\eta - \eta') \rangle \]

\[ \approx \phi + \frac{i |v||}{2c} (\omega - \langle \omega_{de} \rangle) \tilde{A}_{||} \int_{n_1}^{n_2} d\eta \frac{q R}{v||} \text{sgn} (\eta - \eta') , \quad (4.20) \]

which neglects ballooning effects along the field lines. This allows Eq. (4.19) to finally yield

\[ g_{Te} \approx -\frac{e f_{oe}}{T_e} \frac{\omega - \omega_{se}}{\omega - \langle \omega_{de} \rangle} \phi + O \left( \frac{\omega}{\omega_{be}} \right) \quad (4.21) \]

where \( \langle \omega_{de} \rangle \) is given by Eq. (2.137). In order to determine the parallel current perturbation for the trapped electrons, we must retain the \( O(\omega/\omega_{be}) \) contribution to the trapped electron response function. A more convenient way of obtaining \( J_{\parallel Te} \) is to take the moment of the gyrokinetic equation, Eq. (2.90). We thus have

\[ \int d\mathbf{v} v_{||} g_{Te} = \frac{1}{k_||} \int d\mathbf{v} \left[ (\omega - \omega_{de}) g_{Te} + (\omega - \omega_{se}) \frac{e f_{oe}}{T_e} \phi \right] \quad (4.22) \]

where use of Eq. (4.1) has been made and hence

\[ J_{\parallel Te} \approx -\frac{e^2}{k_|| T_e} \int d\mathbf{v} \frac{\omega_{de} - \langle \omega_{de} \rangle}{\omega - \langle \omega_{de} \rangle} f_{oe} \phi . \quad (4.23) \]

Substituting Eqs. (4.9), (4.15) and (4.21) into the quasineutrality condition, Eq. (2.106), produces, after some algebra,

\[ [1 - r (I_m - 1) - T_{||}] \phi = (C_1 - r I_{||} \delta \omega_{se}) \frac{\tilde{A}_{||}}{c k_||} \quad (4.24) \]
where

\[ I_{i0} = \int dv \frac{\omega + \omega_e^T}{\omega + \omega_{di} - k_{||} v_{||}} J_0^2 (\Lambda_1) f_{0e} (v^2), \quad (4.25) \]

\[ C_1 = \int_C dv \frac{\omega - \omega_e^T}{\omega - \langle \omega_{de} \rangle} f_{0e} (v^2), \quad (4.26) \]

\[ T_1 = \int_T dv \frac{\omega - \omega_e^T}{\omega - \langle \omega_{de} \rangle} f_{0e} (v^2), \quad (4.27) \]

Substituting the total perturbed parallel current, \( J_{||} = J_{||i} + J_{||Ce} + J_{||Te}, \) into the parallel Ampère's law, Eq. (2.107), combined with the result of Eq. (4.24) yields

\[ k_{||} k_{\perp}^2 \tilde{\varphi} = -\frac{k_{De}^2}{c^2} \left[ \frac{C_1 - \tau I_{11} \delta \omega_{ei}}{1 - \tau (I_{10} - 1) - T_1} \right] \left( C_1 + T_2 - \tau I_{11} \delta \omega_{ei} \right) - C_2 + \tau I_{12} \delta^2 \omega_{ei}^2 \tilde{\varphi} \quad (4.28) \]

where

\[ C_2 = \int_C dv (\omega - \omega_e^T) (\omega - \omega_{de}) f_{0e} (v^2), \quad (4.29) \]

\[ T_2 = \int_T dv (\omega - \omega_e^T) \frac{\omega_{de} - \langle \omega_{de} \rangle}{\omega - \langle \omega_{de} \rangle} f_{0e} (v^2) \quad (4.30) \]

and \( k_{De} = \sqrt{\frac{4\pi e^2 n_0}{T_e}} \) is the Debye wavelength for the electrons. The final form of the reduced kinetic equation is obtained by utilizing the forms for \( k_{\perp} \) and \( k_{||} \) from Eqs. (2.86) and (2.89) respectively. Substituting these into Eq. (4.28) and simplifying gives

\[ \frac{d}{d\eta} \left\{ \left[ 1 + (s\eta - \alpha \sin \eta) \right] \frac{d\tilde{\varphi}}{d\eta} \right\} - \frac{\alpha}{2\epsilon_n \omega_{ei}^2 \left[ 1 + \eta_e + (1 + \eta_i) / \tau \right]} \times \\
\left\{ \frac{C_1 - \tau I_{11} \delta \omega_{ei}}{1 - \tau (I_{10} - 1) - T_1} \left( C_1 + T_2 - \tau I_{11} \delta \omega_{ei} \right) - C_2 + \tau I_{12} \delta^2 \omega_{ei}^2 \right\} \tilde{\varphi} = 0. \quad (4.31) \]

Solving Eq. (4.31) consists of a two step process in which a shooting code, such as the Runge-Kutta method [173], is used in conjunction with a root finder such as Muller's method. The boundary conditions imposed on \( \tilde{\varphi} (\eta) \) are simply that \( |\tilde{\varphi} (\eta)| \to 0 \) as \( |\eta| \to \infty \), similar to that of the nonlocal gyrokinetic model. The choice of even parity
solutions \((d\tilde{\psi}/d\eta = 0\) at \(\eta = 0\)) completes the boundary condition requirements. The shooting distance must be chosen sufficiently large so that the eigenvalue is independent of it; usually \(\eta_{\text{max}} \simeq 15 - 20\) is sufficient.

To carry out the velocity space integrations of the electron terms, we shall use the pitch angle dependence previously defined by Eqs. (2.125)-(2.131). Using this dependence, the circulating electron integrals may be evaluated analytically to yield

\[
C_1 = (\omega - \omega_{se}) \left[ 1 - \left(1 - \frac{1 - \epsilon}{1 + \epsilon \cos \eta} \right)^{1/2} \right] \tag{4.32}
\]

and

\[
C_2 = \frac{1}{4} (\omega - \tilde{\omega}_{de}) [\omega - \omega_{se} (1 + \eta_e)] \left[ 4 - \left(4 - \frac{1 - \epsilon}{1 + \epsilon \cos \eta} \right) \left(1 - \frac{1 - \epsilon}{1 + \epsilon \cos \eta} \right)^{1/2} \right] \tag{4.33}
\]

The trapped electron integrals can be evaluated partially to give

\[
T_1 = \frac{4}{\sqrt{n}} \int_0^\infty dv \frac{\omega - \omega_{se}^T}{\omega - \epsilon_n \omega_{se} v^2} v^2 e^{-v^2} \left(1 - \frac{1 - \epsilon}{1 + \epsilon \cos \eta} \right)^{1/2} \tag{4.34}
\]

and

\[
T_2 = \frac{4\epsilon_n}{\sqrt{n}} \int_0^\infty dv \frac{\omega_{se}^T}{\omega - \epsilon_n \omega_{se} v^2} v^4 e^{-v^2} \{\cos \eta + \sin \eta (s \eta - \alpha \sin \eta)\} \times
\]

\[
\left[ \left\{ 1 + \frac{1}{3} \left(1 - \frac{1 - \epsilon}{1 + \epsilon \cos \eta} \right)^3 \right\} - 1 \right] \left(1 - \frac{1 - \epsilon}{1 + \epsilon \cos \eta} \right)^{1/2} \tag{4.35}
\]

where we have assumed that \((\omega_{de}) \simeq \omega_{se} \epsilon_n v^2\) for the trapped electrons and the velocity integrations over \(v\) are evaluated through use of the Gauss-Hermite quadrature method.

It is also of interest to note that Eq. (4.31) reduces to the ideal MHD ballooning mode equation by neglecting the trapped electron effects and assuming the limits
\[ \omega_{d(t)}, \omega_{w(t)} \ll |\omega|, k_{\perp} \rho_i \ll 1 \] and \( \eta_{e(t)} = 0 \). The equation then reveals itself as

\[
\frac{d}{d\eta} \left\{ 1 + (s\eta - \alpha \sin \eta)^2 \right\} \frac{d\phi}{d\eta} + \alpha \left[ \cos \eta + \sin \eta \left( s\eta - \alpha \sin \eta \right) \right] \phi +
\]

\[
\omega^2 \left( \frac{qR}{V_A} \right)^2 \left[ 1 + (s\eta - \alpha \sin \eta)^2 \right] \phi = 0 \tag{4.36}
\]

which will be used for the comparison of results concerning the ballooning mode.

### 4.3 Model of Rewoldt

Rewoldt and his co-workers at the Plasma Physics Laboratory at Princeton University have developed an electromagnetic gyrokinetic eigenmode model for the study of high-toroidal mode number linear instabilities for tokamaks \([10]\) suitable for the study of modes in the shear-Alfvén and drift branches. This model was the first to incorporate extensive generalizations into an eigenmode analysis with axisymmetrical toroidal geometry and was followed by many other integral-equation type works such as that of Romanelli \([115]\), Dong et al. \([119]\), Kim et al. \([174]\) and Kotschenreuther et al. \([13]\). Linsker’s integral-equation model \([175]\) preceded that of Rewoldt’s but is relatively crude in the respect that it incorporates cylindrical geometry in the electrostatic limit without the inclusion of magnetic drift effects. The Rewoldt model is far too complicated to be given explicitly here, but we shall give an abridged version of some of the pertinent details and the numerical procedure used to solve the equations.

The Rewoldt model is interfaced with a general numerical tokamak MHD equilibrium that allows for non-circular magnetic surface considerations but can also employ the simpler \(s - \alpha\) model MHD equilibrium. The \(s - \alpha\) version will be used to facilitate comparisons for all cases. Second, the calculation is fully electromagnetic,
retaining the perturbed magnetic vector potential components parallel and perpendicular to the equilibrium magnetic field in addition to the perturbed scalar or electrostatic potential. A general form of the perturbed distribution function is used which is valid for mode frequencies of the order of or below the particle bounce or transit frequencies as well as above them. This implies that low-\( n \) modes may be analyzed in addition to the high-\( n \) types. Finite gyroradius and finite banana width effects are also included to all orders. Multiple ion species have also been taken into account as well as collisionality which is represented by an improved Krook model collisional operator. All important kinetic effects, including the effects of trapped electrons and trapped ions, are retained without approximation in the calculation. The aforementioned calculations and effects are embodied into a numerical framework known as the FULL code and run on a CRAY-T3E machine.

Exact solutions for \( \hat{f}_j \) are based on the gyrokinetic equation and obtained for both trapped and untrapped regions in terms of \( \hat{\varphi}, \hat{A}_\parallel \), and \( \hat{A}_\perp \). These solutions are then used in the quasineutrality condition as defined by Eq. (2.106). However, instead of using Ampère's law, the gyrokinetic moment equation is used. The resulting system of three linear, homogeneous, non-Hermitian integral equations is solved by the Ritz method, where the unknown scalar and vector potentials are decomposed in terms of chosen basis functions as

\[
\hat{\varphi}(\eta) = \sum_{l=0}^{L-1} \hat{\varphi}_l h_l(\eta) 
\]  

(4.37)

where

\[
h_l(\eta) = H_l(\Delta^{1/2} \eta) \exp\left(-\Delta \eta^2/2\right) M_l^{1/2},
\]  

(4.38)

\( H_l \) is the Hermite polynomial of order \( l \), \( M_l \equiv (\pi/\Delta)^{1/2} 2^l l! \), and \( \Delta > 0 \) is an
adjustable real parameter. Then, the operator,

$$\frac{T_e}{e^2 n_e} \int_{-\infty}^{\infty} d\eta h_{l'}(\eta), \quad 0 \leq l' \leq L - 1,$$

(4.39)

is applied to each of the three equations. These operations have the effect of converting the integral equations into a single large matrix equation

$$M(\omega)\begin{pmatrix} \widehat{\varphi}_1 \\ \vdots \\ \widehat{A}_{l1} \\ \vdots \\ \widehat{A}_{\perp 1} \end{pmatrix} = \sum_s Z_s \begin{pmatrix} M_{ll}^{m_1 m_1} & M_{l'l}^{m_1 m_2} & M_{l'l}^{m_1 m_3} \\ M_{l'l}^{m_2 m_1} & M_{l'l}^{m_2 m_2} & M_{l'l}^{m_2 m_3} \\ M_{l'l}^{m_3 m_1} & M_{l'l}^{m_3 m_2} & M_{l'l}^{m_3 m_3} \end{pmatrix} \begin{pmatrix} \widehat{\varphi}_1 \\ \vdots \\ \widehat{A}_{l1} \\ \vdots \\ \widehat{A}_{\perp 1} \end{pmatrix} = 0.$$  

(4.40)

Here, $M(\omega)$ is a $3 \times 3$ block matrix, with each block being an $L \times L$ matrix, where $L$ is the number of basis functions that are kept, $Z_s \equiv e_s / |e|$ is the charge number, and the complete and exact form of $M_{ll}^{m'm}(\omega)$ is given in ref. [10]. The matrix $M(\omega)$ has $3L$ eigenvalues $\lambda_i$ ($i = 1, \ldots, 3L$) and $3L$ corresponding eigenvectors. Eq. (4.40) can be satisfied only if det($M$) = 0, and this can happen only if at least one of the $\lambda_i$'s is zero, since det($M$) = \[ \prod_{i=1}^{3L} \lambda_i. \] Each of the $\lambda_i$'s is a function of $\omega$. Thus, the eigenvalue condition on $\omega$ is $\lambda_i(\omega) = 0$ for a single $\lambda_i$ corresponding to a chosen eigenvector (and eigenfunction). The eigenvalue $\omega$ can be found by standard root-finding methods, such as the method of false position. The number of basis functions, $L$, are chosen sufficiently large so that the eigenvalue is independent of $L$ to some acceptable accuracy, and is typically taken as $L = 32$. 
4.4 Results of Stability Analysis

In this section, results of the collisionless, nonlocal gyrokinetic code are applied with those of the semilocal, reduced kinetic and Rewoldt codes, to the operating conditions of a number of actual and artificial tokamak experiments. As mentioned earlier, the dominant type of short-wavelength instabilities at low $\beta$ are the drift-type modes comprised of the trapped-electron and $\eta_i$ instabilities while the dominant type at high $\beta$ is the kinetic ballooning mode, associated with the shear-Alfvén wave. The comparisons here deal with both regimes and consider the issues with respect to: (a) the relative magnitude of the maximum growth rates of the dominant modes as a function of $\beta$, $\eta_i$ and $k_\theta \rho_i$; (b) the speed and accuracy of the models under study compared to the Rewoldt model; and (c) characteristic eigenfunctions of the various spectra. When making our comparisons, we must be careful in considering only discharge conditions that satisfy the criteria of short wavelengths or moderate to high toroidal numbers, $n \gtrsim 10$ [113], and to insure that the shear parameter is also sufficiently large, $s \gtrsim 0.2$.

4.4.1 Drift-Type Modes

We shall first benchmark a trapped electron and $\eta_i$ hybrid mode with that of Rewoldt by considering a hypothetical design for the compact ignition tokamak (CIT)[12][177]. The local parameters on the chosen magnetic surface around the minor half radius are: $R_0 = 1.8\text{m}$, $B_0 = 10\text{T}$, $\beta = 0.166\%$, $q = 1.17$, $s = 0.456$, $\epsilon = 0.154$, $\epsilon_n = 0.25$, $M = 2.5\text{amu}$ (an equal mixture of deuterium and tritium), $\tau = 1.1$, $k_\theta \rho_i = 0.4$ or $n = 42$, $\eta_e = 0.955 \eta_i$, $T_e = 18.9\text{keV}$ and $T_i = 19.4\text{keV}$. The discharge is also characterized by the collision frequency $\nu_e^* = 0.0035$, yet this value is so low that we
may deem it to be negligible. Results are presented as $\eta_i$ and $\eta_e$ are varied in terms of
the normalized, linearized growth rate $\gamma/\omega_{ke}$ and frequency $\omega_r/\omega_{ke}$ as shown in Fig.
4.1 with the Rewoldt data taken from [12]. It can be clearly seen from the growth
rate that the instability has two regimes corresponding to the $\eta_i$ and trapped electron
modes. For $\eta_i \gtrsim 1.4$, the growth rate increases monotonically, and almost linearly
with $\eta_i$. In this regime the fluid $\eta_i$ mechanism destabilizes this toroidal mode yet, as
$\eta_i$ decreases from $\eta_i \approx 1.4$, the growth rate increases again slightly, before falling off.
Here, the trapped-electron magnetic drift frequency mechanism is dominant. It is this
mode that is responsible for the usual collisionless trapped-electron mode. In this case,
a single root of the mode equation combines the usual collisionless trapped-electron
mode and the usual $\eta_i$ mode into a hybrid mode. The nonlocal model also predicts
a similar trend and is able to recover most of the salient features of this coupled
mode. The semilocal model likewise predicts the correct trend of this instability and
is quite similar to that of the nonlocal result specifically with respect to the frequency.
Typical eigenfunctions $\mathcal{G}(\eta)$ and $\mathcal{A}_\parallel(\eta)$ for the nonlocal model are presented in Figs.
4.2 and 4.3 with respect to the ballooning angle $\eta$ corresponding to the cases $\eta_i = 0$
(collisionless trapped electron mode) and $\eta_i = 3.0$ ($\eta_i$ mode) respectively. In both
instances, the scalar and parallel vector potentials obey the boundary conditions
imposed earlier in §2.3.1, namely that $\mathcal{G} \to 0$ and $\mathcal{A}_\parallel \to 0$ (since $\mathcal{h} \to 0$) sufficiently
fast as $|\eta| \to \infty$. Note also that $\mathcal{G} = 1$ at $\eta = 0$ as required previously in §3.2. The
scalar potentials are similar to those of Rewoldt (data taken from reference [12] where
the parallel vector potentials were not given) with respect to a similar hybrid mode
for a TFTR discharge [178] with $\eta_i = 0$ and $\eta_i = 3.0$ as illustrated in Fig. 4.4.

We shall now examine the dependence of the growth rate and frequency on $k_\parallel \rho_i$
for an $\eta_i$-mode based on an artificial test case in the collisionless, electrostatic limit
Figure 4.1: (a) Growth rates $\gamma/\omega_\infty$ and (b) real frequencies $\omega_r/\omega_\infty$ versus $\eta_i$ for the nonlocal model Eq. (3.17), the semilocal model Eq. (4.3) and Rewoldt's model Eq. (4.40) with respect to the CIT case. Here $\eta_e/\eta_i = 0.955$. 
Figure 4.2: Real and imaginary components corresponding to the nonlocal model Eq. (3.17) for (a) the scalar eigenfunction $\tilde{\varphi}(\eta)$ and (b) the parallel vector potential $\tilde{A}_\parallel(\eta)$ versus the ballooning coordinate $\eta$ for the CIT case with $\eta_s = 0$ and $\eta_e = 0$.

Figure 4.3: Real and imaginary components corresponding to the nonlocal model Eq. (3.17) for (a) the scalar eigenfunction $\tilde{\varphi}(\eta)$ and (b) the parallel vector potential $\tilde{A}_\parallel(\eta)$ versus the ballooning coordinate $\eta$ for the CIT case with $\eta_s = 3.0$ and $\eta_e = 2.87$. 
Figure 4.4: Typical scalar eigenfunctions versus $\theta$, the ballooning coordinate, corresponding to the Rewoldt model Eq. (4.40) for (a) a collisionless trapped electron mode ($\eta_i = 0$) and (b) an $\eta_i$-mode ($\eta_i = 3.0$) pertaining to a TFTR discharge.

[13]. The parameters used are: $q = 1.5$, $s = 1.0$, $\beta = 0$, $\epsilon_n = 1/3$, $\eta_i = \eta_e = 3.0$, $T_i = T_e = 3.0\text{keV}$, $\epsilon = 1/6$, $M = 2.0\text{amu}$, $R = 2.4\text{m}$ and $B_0 = 5\text{T}$. For $k_\parallel \rho_i = 0.1$ the corresponding toroidal mode number is about $n = 20$ indicating that the nonlocal model can be applied with confidence in this low, finite Larmor radius region. The linear results are presented in Fig. 4.5 where it can be seen that the eigenvalue results for $\gamma/\omega_{ce}$ and $\omega_r/\omega_{ce}$ agree reasonably well over the range $0.2 \leq k_\parallel \rho_i \leq 0.7$. As the FLR parameter becomes smaller the growth rate differences between the two models becomes more pronounced which is due to the high-$n$ toroidal number requirement breaking down since the distance between rational surfaces is no longer less than that of the radial scale of the equilibrium variations. The semilocal model is able to recover the growth rate qualitatively and the frequency is found to be closely consistent with the other models for larger $k_\parallel \rho_i$ values.

For our last drift mode comparison case, we consider the collisionless trapped electron mode using data from the Doublet-III device [14][179]. The discharge parameters are: $\epsilon_0 = 0.16$, $T_e = 0.36\text{keV}$, $T_i = 0.37\text{keV}$, $q = 1.14$, $M = 1\text{amu}$, $s = 0.4$, $\eta_i = 0.87$, 

Figure 4.5: (a) Growth rates $\gamma/\omega_{se}$ and (b) real frequencies $\omega_r/\omega_{se}$ versus $k_g\rho_i$ from the nonlocal model Eq. (3.17), the semilocal model Eq. (4.3) and the Rewoldt model Eq. (4.40) for the artificial test case parameters in the collisionless electrostatic limit.
\[\eta_e = 0.93, \varepsilon_n = 0.0688, k_\theta \rho_i = 0.28\] and \(n = 20\). The corresponding results for the eigenvalues are shown in Fig. 4.6 Although the results among the three models differ considerably, the nonlocal and semilocal results are still able to indicate the extent and general trend of the instability for both the mode frequency and growth rate.

### 4.4.2 Ballooning Mode

Before proceeding directly into an analysis of the ballooning mode with respect to pertinent tokamak discharge parameters, we will first make a comparison of the reduced kinetic and ideal MHD models with those of Cheng [38] to insure that the codes are working correctly. Cheng's reduced kinetic model was formulated specifically to study high-\(n\) collisionless ballooning modes and contains the same assumptions used to derive Eq. (4.31). It does not include compressional acoustic effects, yet does contain the magnetosonic perturbation. Fig. 4.7 shows the eigenvalue results with respect to the ballooning parameter \(\alpha\) for the parameters: \(\varepsilon = 0.1, M = 1\)amu, \(s = 0.6, q = 1.414, \varepsilon_n = 0.1, \tau = \eta_e = \eta_i = 1\) and \(k_\theta \rho_i = 0.32\). Shown plotted are the reduced kinetic model, Eq. (4.31) (without compressional acoustic effects), Cheng's kinetic model (data taken from ref. [38]), the ideal MHD model, Eq. (4.36), the ideal MHD model from ref. [38] and lastly the nonlocal model. Agreement is good between the two corresponding sets of MHD and reduced kinetic models for both frequencies and growth rates and the real frequency for the MHD limit is simply \(\omega_r = 0\). The reduced kinetic models also agree well with the maximum growth rate as determined by the nonlocal model which, in turn, also predicts broader \(\alpha\) thresholds and a notable decrease in the mode frequency. These comparisons serve to indicate that all the codes are performing consistently.

We now seek to benchmark the nonlocal and simplified kinetic codes with that of
Figure 4.6: (a) Linear growth rates $\gamma/\omega_{se}$ and (b) real frequencies $\omega_r/\omega_{se}$ versus $\beta$ for the collisionless-trapped-electron mode as indicated by the nonlocal model Eq. (3.17), the semilocal model Eq. (4.3), and the Rewoldt model Eq. (4.40). Results are given for the discharge parameters of the Doublet-III device.
Figure 4.7: Dependence of the (a) linear growth rates and (b) real frequencies with respect to $\alpha$ for the parameters: $\epsilon = 0.1$, $M = 1$amu, $s = 0.6$, $q = 1.414$, $\epsilon_n = 0.1$, $\tau = \eta_e = \eta_i = 1$ and $k_0\rho_i = 0.32$. The plots displayed are for the nonlocal model Eq. (3.17), the reduced kinetic model Eq. (4.31), Cheng’s kinetic model from ref. [38], the ideal MHD model Eq. (4.36) and Cheng’s ideal MHD result from ref. [38].
the Rewoldt model for two tokamak experiments beginning first with the Doublet-III device whose discharge parameters were used previously for the study of the collisionless-trapped-electron mode. The local discharge conditions are the same as before except that the inverse aspect ratio is now set to $\epsilon = 0$. The calculations yield the linear growth rates and frequencies as displayed in Fig. 4.8 versus $\beta$ with the compressional acoustic effects restored for the reduced kinetic model. The MHD, nonlocal and Rewoldt (data taken from ref. [14]) results are in rough agreement when considering the first critical beta value which is found to be approximately $\beta_{c1} \approx 0.01$. The Rewoldt and nonlocal models also predict a larger second critical beta value which the reduced kinetic and semilocal models underestimate. There is, however, a large discrepancy concerning the peak growth rate of Rewoldt's result which is much larger relative to the other kinetic models. This is quite puzzling since a full comprehensive kinetic analysis of this sort should predict growth rates much lower than this since it contains such stabilizing effects as trapped electrons and compressional effects associated with the ions and $B_\parallel$. The MHD calculation, which contains no stabilizing terms, is nearly comparable in amplitude. Upon examination of the paper where the data were obtained, we are fortunate to find also included in the same figure, plots of the ideal MHD and Cheng's kinetic model. A scan of the figure from Rewoldt's paper [14] is shown in Fig. 4.9. Curve (a) in this figure represents the ideal MHD estimate, curve (b) is calculated from the Cheng kinetic model and curve (c) is calculated from the Rewoldt model. As is evident, both the MHD and Cheng amplitudes are also larger than the growth rates shown in Fig. 4.8(a) by the same margin as the Rewoldt result, approximately 50%. The origin of this discrepancy is unknown but the $\beta$ thresholds predicted by these three models are nearly identical (no shifts) to those shown in Fig. 4.8(a) indicating that
Figure 4.8: (a) Linear growth rates and (b) real frequencies plotted with respect to \( \beta \) for the Doublet-III discharge parameters corresponding to the nonlocal model Eq. (3.17), the semilocal model Eq. (4.3), the reduced kinetic model Eq. (4.31), the ideal MHD model Eq. (4.36) and the Rewoldt model Eq. (4.40).
the parameters used in the current analysis are consistent. The real frequencies shown in Fig. 4.8(b) appear to be less discordant: both the nonlocal and Rewoldt frequencies agree satisfactorily within the most unstable range of $\beta$. Also included in the figure is the analytical estimate of the real frequency for unstable kinetic ballooning modes, $\omega_r \sim \omega_{e\pi}/2$ with $\omega_{e\pi} \equiv \omega_{\ast \ast} (1 + \eta) [154][160][180][181]$. Both comprehensive kinetic models agree satisfactorily well with this frequency over the most unstable range of $\beta$ as well. Typical scalar and parallel vector eigenfunctions of a ballooning mode appear in Fig. 4.10 for the parameters pertaining to the Doublet-
Figure 4.10: Real and imaginary components corresponding to the nonlocal model Eq. (3.17) for (a) the scalar eigenfunction $\hat{\mathcal{S}}(\eta)$ and (b) the parallel vector potential $\hat{\mathcal{A}}_\parallel(\eta)$ versus the ballooning coordinate $\eta$ for the Doublet-III case with $\beta = 0.02$.

III device with $\beta = 0.02$. As expected, both are well localized and obey the boundary value restrictions mentioned previously. Typical ballooning mode eigenfunctions from Rewoldt for an ISX-B (Impurity Study Experiment) [14][182] case are shown in Fig. 4.11 for comparison.

For our second ballooning mode benchmark we will consider a so-called “H-mode” discharge where the confinement properties are significantly improved over those of the conventional “L-mode” discharge. The parameters that are characteristic of this superior confinement state pertain to the PDX (Poloidal Divertor Experiment) device [14][183] and are given as: $\epsilon = 0.15$, $T_e = 0.78$keV, $T_i = 1.05$keV, $q = 1.27$, $s = 0.71$, $\eta_e = 3.3$, $\eta_i = 3.7$, $\epsilon_n = 0.285$, $k_{\parallel}\rho_i = 0.30$ and $n = 20$. The growth rates depicted in Fig. 4.12(a) for the most part are qualitatively similar to those from the Doublet-III case, with the larger than expected peak growth rate displayed by the Rewoldt model recurring once again. A welcome and anticipated quantitative
Figure 4.11: Typical Rewoldt eigenfunctions corresponding to Eq. (4.40) for a ballooning mode. Displayed are the (a) scalar $\hat{\mathcal{F}}$ and (b) parallel vector $\hat{A}_\parallel$ eigenfunctions with respect to the ballooning coordinate $\theta$ for the parameters corresponding to the ISX-B device.

variation is given by the real frequency shown in Fig. 4.12(b) which is much greater than that of the Doublet-III case since the ion temperature gradient parameter has now increased from $\eta_i = 0.87$ to $\eta_i = 3.7$. The known tendency for the real frequency of kinetic toroidal instabilities to be shifted more strongly in the ion diamagnetic direction when $\eta_i$ is larger [184] is evident here and is also expected based upon the simple analytical estimate $\omega_r \sim \omega_{ci} (1 + \eta_i) / 2$ as noted previously. Both nonlocal and Rewoldt frequencies agree reasonably well for most of the unstable $\beta$ range.

In brief, the benchmarking results of the nonlocal model with the Rewoldt code are favourable for both the drift and shear-Alfvén branch modes. It was not anticipated that the results would be in excellent quantitative agreement due to some of the assumptions used and the different numerical algorithms employed in both works but they do agree qualitatively. The numerical processes alone can produce significant disparities when attempting to solve such complex problems. An example of this can
Figure 4.12: (a) Linear growth rate and (b) real frequencies for the nonlocal model Eq. (3.17), the semilocal model Eq. (4.3), the reduced kinetic model Eq. (4.31), the ideal MHD model Eq. (4.36) and the Rewoldt model Eq. (4.40) with respect to $\beta$ for the parameters corresponding to the PDX device.
be found in the work of Kotschenreuther et al. [13] who made a comparison study of two completely different solution methods for evaluating the drift-mode. One code is that of Rewoldt, which was described in detail earlier, and the other is based on an initial value method. Although many of the comparisons were good, differences of up to 50% in some cases were also noted.
A high speed, nonlocal kinetic eigenmode equation is derived to lowest order in $1/n$ in the collisionless limit. The ballooning formalism is employed to reduce the dimensionality of the problem (from 2-D to 1-D) and it is assumed that the plasma is axisymmetrical with large aspect ratio and low-$\beta$. These assumptions permit the problem to be simplified further by retaining only two potentials ($\varphi$ and $A_\parallel$) and allowing the flux surfaces to be circular and nonconcentric with small Shafranov shift ("$s - \alpha$" model equilibrium). The combination of the perturbed distribution function with the quasineutrality condition and parallel Ampère’s law, yields a well defined eigenmode problem in which the ballooning structure and eigenfrequency of a drift or shear-Alfvén instability may be computed. The resulting system of two linear, homogeneous Fredholm equations can then be solved for the eigenvalue $\omega$ and the eigenfunctions $\varphi$ and $A_\parallel$ by using “the method of solution of Fredholm”. By employing the ballooning mode representation, in conjunction with an efficient numerical
algorithm based on the Gauss-Laguerre quadrature rule, eigenvalues are produced rapidly at an average rate of approximately 6 – 8 sec./eigenvalue without trapped electrons and approximately 1 min/eigenvalue with trapped electrons. The results have been found to agree satisfactorily with the comprehensive analysis of Rewoldt for the CTE and ITG modes in the drift branch for both frequencies and growth rates. Although the frequencies for the ballooning mode cases agreed well with each other for the most unstable regions of \( \beta \), the growth rates did not. Using the ideal MHD model and the kinetic model of Cheng as a reference gauge, it was found that the Rewoldt growth rates were larger with respect to the nonlocal model by approximately 50% for both the Doublet-III and PDX cases. The origin of the discrepancy is unknown, yet if it were taken into account, the nonlocal results would agree quite well with those of the Rewoldt model.

Also included in the benchmarking analysis were the semilocal and reduced kinetic models which can produce results much faster than the integral equation approach. The intention of including these simplified kinetic models was to explore their accuracy and applicability for the instabilities under investigation. The semilocal model is the fastest among the codes considered with eigenvalues being computed at the rate of approximately 0.5 – 1 sec/eigenvalue on a DEC AlphaStation 21164a. The agreement between the semilocal and nonlocal models with respect to the frequencies for the drift-branch results was surprisingly close given the crudeness of the semilocal model itself. The good qualitative agreement with the Rewoldt results for the CTE and ITG case modes imply that this method may be used with confidence when applied to drift-branch instabilities. Semilocal results for the ballooning mode were, however, more sporadic with growth rates being either over-estimated or under-estimated and the frequencies consistently overestimated. One problem with this method is that the
validity of the solutions are impossible to check since there are no boundary conditions to be satisfied. One must ensure that the discharge parameters being considered are well within the range of the assumptions used in its formulation. The ballooning mode results for the reduced kinetic model exhibited better accuracy with respect to the thresholds in $\beta$, yet the growth rates and frequencies were consistently over-estimated. The eigenvalues were found to be computed at the rate of about $2 - 3$ sec/eigenvalue and were also computed on the Alpha 21164a. Unlike the semilocal method, this method produces scalar eigenfunctions that are subject to similar boundary conditions as the nonlocal model and can be checked to insure that the solution is acceptable.

5.1 Suggestions for Future Work

It is desirable to generalize the current model to include a variety of other kinetic and geometrical effects to enable the code to become more realistic, especially when making quantitative comparisons with experimental observations or to provide reliable benchmarks for other codes. We have identified four areas of generalization that we would like to pursue in the future, namely impurity contribution effects, shaped discharge profiles, effects due to alpha particles and velocity sheared flows.

5.1.1 Impurity Contribution Effects

Impurity modes have found to be driven unstable in plasmas where the impurity density profile is outwardly peaked while the hydrogenic ion and electron densities are inwardly peaked, just as it is at the boundary of tokamak plasmas even without the ion temperature gradient [185]. Recently, the impurity mode has been considered to account for the energy transport in the plasma periphery and the isotope effect on the
energy confinement in tokamak plasmas [119][186][187][188]. In most of these early studies of the impurity mode, however, the ion temperature gradient is neglected. Based on the fact that fusion plasmas are not pure element plasmas composed of electrons and a single species of ion only but of a mixture of different ionic elements, the effects of a second ion species (impurity) on the ITG mode have been studied in detail in recent years [189][190]. It has also been found that impurity ions may have a stabilizing or destabilizing effect on the ITG mode depending on whether the density profile of the impurity ions is outwardly peaked or inwardly peaked. The approach to implement impurity ions into nonlocal kinetic analysis is straightforward and would be a logical extension of the present work.

5.1.2 Shaped Discharges

Stability to kinetic high-$n$ instabilities is an important element of tokamak design and operation. Most notably, as mentioned beforehand, the high-$n$ ballooning modes play a crucial role in limiting the plasma $\beta$ value in tokamaks. Among the various factors that influence the stability of these modes, the effect of the shape of the plasma boundary has received much attention in past studies [191][192][193][194][195]. Kwon and Hender [191] have examined the role of the plasma shape in gaining accessibility to the second ballooning stability regime. In the Doublet-III-D experiment [192], a record high $\beta$ of 11% has been achieved by taking a D-shaped plasma boundary with elongation $\kappa_a = 2.34$. The effect of the triangularity of the plasma boundary on the $\beta$ limit set by ideal ballooning modes has also been studied in refs. [193][194][195] for typically high values of boundary elongation ($\kappa_a \geq 1.9$). In general, the effect of triangularity is found to be stabilizing. A simple scaling law for the maximum $\beta$ for marginal stability to high-$n$ ballooning modes has been obtained in terms of
elongation $\kappa$, $q$ and aspect ratio $R/a$. Some scaling laws incorporating triangularity
$\delta_a$ have been obtained in refs. [196][197] where they are usually valid over a limited
parameter range. Nevertheless they emphasize the fact that the plasma geometry
plays an important role in determining the maximum achievable $\beta$, and much further
work needs to be done to elucidate the complex dependencies on these factors. Recent
work done by Agarwal et al. [198] and Miller et al. [199] have allowed tokamak
equilibrium models, local to a flux surface, to be implemented into the ballooning
representation form of the gyrokinetic equation in place of the shifted, circular $s-\alpha$
model without too much difficulty. The advantage of these models is that they are
simple and computationally efficient with reasonable accuracy when compared to
direct numerical solutions of the Grad-Shafranov equation.

5.1.3 Alpha Particles

Hot alpha particles and thermalized helium ash particles in tokamaks can have sig-
nificant effects on high-$n$ instabilities such as the trapped-electron mode and the
ballooning mode. In particular, the effects can be stabilizing, destabilizing, or neg-
ligible, depending on the parameters involved [200]. In high-temperature tokamaks,
capable of producing significant numbers of hot alpha particles, the predominant in-
teraction of the mode with the alpha particles is through resonances of various sorts.
Broadly speaking, this depends principally on the relative magnitudes of the mode
frequency $\omega$ and the precessional drift frequency $\omega_{d\alpha}$ of the hot alphas. For example,
with respect to high-$n$ ballooning modes, the condition $\omega \approx \omega_d$ (corresponding to
moderately energetic particles), causes resonant coupling between the particles and
the mode thus causing destabilization. For $\omega_{d\alpha} \gg \omega$ (corresponding to very energetic
particles in the several-MeV range), the particles move too fast to interact with the
mode and their influence is hence stabilizing [201]. Studies by Connor et al. [202] have also shown that this condition may also lead to direct access between the first and second stability regimes. For trapped alphas, stabilization will occur for mode frequencies that are well off resonance with the trapped alpha precession frequency, \( \omega \ll \Omega_{da} \), but will become destabilizing when the drift precession resonates with the mode frequency, \( \omega \simeq \Omega_{da} \) [203]. However, for such low frequency modes in this regime, \( n < r^2/Rq^2 \rho_i \), it may be necessary to carry out the next order correction in \( 1/n \) in order to obtain meaningful eigensolutions.

5.1.4 Velocity Shear Flows

Sheared rotation dynamics are widely believed to have significant influence on experimentally observed confinement transitions in major tokamak experiments such as TFTR [204], with regard to high-\( n \) toroidal drift and shear-Alfvén modes. The high-\( n \) toroidal drift modes, destabilized by the combined effects of ion temperature gradients and trapped particles in toroidal geometry can be strongly affected by radially-sheared toroidal and poloidal plasma rotation [205]. Previous work has shown that the ITG mode is destabilized by parallel sheared flow while perpendicular sheared flow is stabilizing [206]. Recent work has also shown that high-\( n \) ballooning modes are subject to stabilization by sheared toroidal rotation [207],[208]. Although it has been pointed out that the ballooning representation breaks down for substantial values of the Mach number \( M = \Omega R / (2T)^{1/2} \) [209],[210], it has also been indicated that this representation is still usable if the Mach number is small [209].
Appendix A

Calculation of \( \mathbf{v} \cdot \nabla \mathbf{X} \) and \( \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{v}} \)

Here we present the details for the calculations of Eqs. (2.28) and (2.29). Beginning with Eq. (2.28) we have

\[
\mathbf{X} = \mathbf{x} - \frac{1}{\Omega} \mathbf{b} \times \mathbf{v}
\]  

(A.1)

which can also be written in tensor form as

\[
X_i = x_i - \frac{1}{\Omega} \varepsilon_{ijk} b_j v_k
\]  

(A.2)

where \( \varepsilon_{ijk} \) is the Levi Civita tensor. The gradient of \( \mathbf{X} \) can be written in the form

\[
\frac{\partial X_i}{\partial x_j} = \delta_{ij} - \varepsilon_{ijk} \left( \frac{v_k}{\Omega} \frac{\partial b_j}{\partial x_j} - \frac{b_j v_k}{\Omega^2} \frac{\partial \Omega}{\partial x_j} \right).
\]  

(A.3)

which is used in the following product to yield

\[
v_j \frac{\partial X_i}{\partial x_j} = v_j \delta_{ij} - \varepsilon_{ijk} \left( \frac{v_j v_k}{\Omega} \frac{\partial b_j}{\partial x_j} - \frac{b_j v_k}{\Omega^2} \frac{\partial \Omega}{\partial x_j} \right)
\]  

\[
= v_j - \varepsilon_{ijk} \frac{1}{\Omega} \frac{\partial b_j}{\partial x_j} v_j + \varepsilon_{ijk} \frac{1}{\Omega^2} (b_j v_k) v_j \frac{\partial \Omega}{\partial x_j}
\]  

(A.4)

which then produces the desired result in vector form

\[
\mathbf{v} \cdot \nabla \mathbf{X} = \mathbf{v} + \frac{1}{\Omega} \left[ (\mathbf{v} \times \nabla) \mathbf{b} \cdot \mathbf{v} + \frac{1}{B} (\mathbf{b} \times \mathbf{v}) \mathbf{v} \cdot \nabla B \right].
\]  

(A.5)

For Eq. (2.29), we start by noting that the Lorentz law, Eq. (2.35), can be written as

\[
\frac{dv_j}{dt} = \frac{e}{m} E_j - \Omega v_j \rho_j + \frac{e}{mc} \varepsilon_{jki} v_k B_{ii}
\]  

(A.6)
and the velocity derivative of Eq. (A.2) as

\[
\frac{\partial X_i}{\partial v_j} = \frac{1}{\Omega} \varepsilon_{ikj} b_k. \tag{A.7}
\]

Taking the dot product of Eq. (A.6) with that of Eq. (A.7) gives

\[
\frac{dv_j}{dt} \frac{\partial X_i}{\partial v_j} = \frac{e}{m \Omega} \varepsilon_{ijk} E_j b_k - \varepsilon_{ijk} v_\perp \hat{p}_j b_k + \frac{e}{mc \Omega} \varepsilon_{ikj} \varepsilon_{jkl} v_k B_{1l} b_k
\]

\[
= \frac{c}{B} E \times b - v_\perp \hat{p} \times b - \frac{1}{B} b \times (v \times B_1) \tag{A.8}
\]

which then yields the expected result

\[
\frac{dv}{dt} \cdot \frac{\partial X}{\partial v} = V_E - v_\perp - \frac{1}{B} b \times (v \times B_1). \tag{A.9}
\]
Appendix B

Approximations for zeroes

Here we list the approximations of the zeroes used in the Laguerre and Hermite Gaussian quadrature methods. As mentioned previously, the zeroes of the Laguerre and Hermite polynomials are computed using a predictor-corrector type method. For the Gauss-Laguerre method, the process begins by approximating the zeroes $x_j$ starting with the smallest. The approximations used are given by Stroud [169]

\[ x_1 \simeq \frac{3}{1 + 2.4n}, \quad (c1) \]

\[ x_2 - x_1 \simeq \frac{15}{1 + 2.5n} \quad (c2) \]

and

\[ \frac{x_{j+2} - x_{j+1}}{x_{j+1} - x_j} \simeq \frac{1 + 2.55j}{1.9j}, \quad j = 1, 2, \ldots, n - 2. \quad (c3) \]

The approximate zeroes for the Hermite method are calculated starting with the largest and since they are symmetric about $x = 0$, only the nonnegative zeroes are computed. The approximation used for the largest zero is given by Szegö [212]

\[ x_n \simeq (2n + 1)^{1/2} - 1.85575 (2n + 1)^{-1/6}. \quad (c4) \]

Similar approximations are given by Stroud for the second, third, fourth and all remaining zeroes,

\[ x_{n-1} \simeq x_n - \frac{1.14n^{0.426}}{x_n}, \quad (c5) \]

\[ x_{n-2} \simeq 1.86x_{n-1} - 0.86x_n, \quad (c6) \]

\[ x_{n-3} \simeq 1.91x_{n-2} - 0.91x_{n-1}. \quad (c7) \]

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and
\[ x_j = 2x_{j-1} - x, \quad 1 \leq j < n - 3. \]  
(8)

By utilizing these approximations for both quadrature methods as a starting point, the zeroes converge rapidly and in successive order without skips when the root finder (Muller’s method) is applied directly to $L_n(x)$ and $H_n(x)$. Quadruple precision has been used to insure accuracy ($\leq 2\%$) which has been checked by computing known integrals such as the plasma dispersion function.
Appendix C

Machine Performance Figures

Hitherto, comparisons of computers, and in particular supercomputers, have most often quoted peak computational performance in MFLOPS (Mega-Floating OPerations per Second). This gauge is commonly used (mainly in mainframes and workstations) for scientific and engineering tasks because of its efficiency and effectiveness in terms of simplicity and indicative abilities [213]. Listed below are the performance measures (given in MFLOPS) of all of the machines that have been cited in this work:

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Peak Capacity (MFLOPS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cray Research Inc. Cray-1(^1)</td>
<td>160(^*)</td>
</tr>
<tr>
<td>Cray Research Inc. Cray-2(^1)</td>
<td>1900(^*)</td>
</tr>
<tr>
<td>Cray Research Inc. T3E(^1)</td>
<td>2400000(^*)</td>
</tr>
<tr>
<td>DEC AlphaStation 21164a(^2) at 400MHz</td>
<td>454.7(^*)</td>
</tr>
</tbody>
</table>

\(^1\)Data taken from CRI’s website: www.cray.com
\(^2\)Data taken from Compaq Computer Corp.’s website: www.compaq.com
\(^*\)A performance of 1720 MFLOPS has been measured during a calculation of a 512 × 512 matrix multiplication at NASA Ames.
\(^\dagger\)Figure based on the Linpack 1000 × 1000 matrix benchmark.
\(^\dagger\)*Benchmark used for CRI’s figures was not given.
Appendix D

List of Symbols

Comments:

1. The most commonly used symbols appearing in the plasma physics literature have been used to represent physical quantities whenever possible. Occasionally, however, a symbol will have multiple definitions. The context under which the symbol is used should make it clear as to which definition is relevant since the various uses do not overlap.

2. Vector quantities are indicated by bold face.

3. Perpendicular \( \perp \) and parallel \( \parallel \) subscripts refer to such orientation with respect to the equilibrium magnetic field \( \mathbf{B}_0 \).

4. Perturbed quantities are indicated by a variety of different subscripts including \( 1, \pm \) and \( \downarrow \). Subscripts denoting perturbed quantities with respect to distribution functions \( f \) and particle densities \( n \) are usually denoted by \( 1 \) or \( \pm \) as is commonplace in the literature unless designated otherwise. Nonperturbed or equilibrium quantities are usually denoted by \( 0 \).

5. Quantities that have been gyrophase averaged are represented by an overbar whereas quantities that vary rapidly with the gyrophase angle are represented by a tilde. Quantities that have been transformed into ballooning space, or have been normalized by a thermal velocity or the electron diamagnetic frequency, are represented by the caret symbol.

6. Not included in the list are quantities that are used only briefly to assist in formulating some of the equations.
Operators

\( \nabla_\perp, \nabla_\parallel \) Perpendicular and parallel components of \( \nabla \) operator

\( k_\parallel \) Parallel differential operator: \( \frac{\mathcal{L}}{qR \partial_n} \)

\( \mathcal{L}, \tilde{\mathcal{L}} \) General integral operator in \( \theta \) space and ballooning space

\( \langle \ldots \rangle \) Velocity average over Maxwellian distribution

\( \langle \ldots \rangle \) Ballooning space average over eigenfunction

\( \langle \ldots \rangle \) Gyrophase average over gyroangle \( \gamma \)

Latin

\( A_\parallel, A_\perp \) Parallel and perpendicular components of the vector potential A

\( \tilde{A}_\parallel \) Parallel vector potential transformed to ballooning space

\( B, B_0, B_1 \) Arbitrary, equilibrium and perturbed magnetic fields

\( B_\parallel, B_\perp \) Parallel and perpendicular perturbed components of the magnetic field B

\( B_\phi, B_\theta \) Toroidal and poloidal magnetic fields

\( C_1, C_2 \) Integrals associated with circulating electrons for the reduced kinetic model

\( E, E_1 \) Arbitrary and perturbed electric fields

\( E \) Complete elliptical function of the second kind
\( F_{in}, F_{Cen}, F_{Ten} \) Ion, circulating electron and trapped electron moments of order \( n \), associated with the semilocal model

\( F_M \) Maxwellian distribution

\( H \) Heaviside function

\( H_n \) Hermite polynomial of order \( n \)

\( I_n \) Modified Bessel function of the first kind of order \( n \)

\( I_{in} \) Ion moments associated with the reduced kinetic model

\( J_{||}, J_{||C_e}, J_{||Te} \) Parallel ion, circulating electron and trapped electron current densities

\( J_n \) Bessel function of the first kind of order \( n \)

\( K \) Complete elliptical function of the second kind

\( K \) Kinetic energy

\( L_\varphi, L_n, L_B, L_s, L_{pi}, L \) Equilibrium, density, magnetic field, shear, ion pressure and plasma gradient scale lengths

\( L_n \) Laguerre polynomial of order \( n \)

\( M \) Nonlocal and Rewoldt representation of eigenmode system in matrix form

\( M \) Ion mass

\( N \) Integer specifying trapped electron region

\( P_n \) Legendre polynomial of order \( n \)
$R, R_0$  Radius of curvature of field lines and major radius of plasma torus

$T_i, T_e$  Ion and electron equilibrium temperatures

$T_1, T_2$  Integrals associated with trapped electrons for the reduced kinetic model

$V_0, V_1$  Nonperturbed and perturbed components of the guiding centre velocity

$V_i, V_e$  $F \times B$ drift for ions and electrons

$V_{di}, V_{de}$  Average ion magnetic drift for ions and electrons

$V_{xi}, V_{xe}$  Ion and electron diamagnetic drift velocities

$V^T$  Diamagnetic velocity as a function of $\bar{v}_\parallel$ and $\bar{v}_\perp$

$V_A$  Alfvén speed

$X$  Position of guiding centre

$a$  Minor radius of tokamak

$b$  Normalized equilibrium magnetic field: $B_0/B_0$

$b_{0i}$  Finite Larmor radius parameter: $k_0^2 \rho_i^2$

$b_i$  $k^2 (\eta) \rho_i^2$

$c$  Speed of light

$c_s$  Acoustic speed

$e$  Charge of electron
\( e_j \) \hspace{1cm} \text{Electronic charge with sign dependence on species type}

\( f_{0i}, f_{0e} \) \hspace{1cm} \text{Maxwellian ion and electron distribution functions}

\( f_0, f_1 \) \hspace{1cm} \text{Maxwellian and perturbed distribution functions}

\( \hat{f}_i, \hat{f}_e \) \hspace{1cm} \text{Perturbed ion and electron distribution functions transformed to ballooning space}

\( g \) \hspace{1cm} \text{Effective gravity due to curvature}

\( g_{C_i}, g_{C_e}, g_{Te} \) \hspace{1cm} \text{Nonadiabatic response functions for circulating ions and circulating and trapped electrons}

\( \hat{h}_g \) \hspace{1cm} \text{General solution of gyrokinetic equation in ballooning space}

\( \hat{h}_{C_i}, \hat{h}_{C_e}, \hat{h}_{Te} \) \hspace{1cm} \text{Solution of gyrokinetic equation in ballooning space for circulating ions and circulating and trapped electrons}

\( k, k_\perp \) \hspace{1cm} \text{Arbitrary and perpendicular wavenumber vectors}

\( k_c \) \hspace{1cm} \text{Connection length wavenumber: } \frac{1}{qR} \text{ (Here, } q \text{ is the safety factor)}

\( k_{Di}, k_{De} \) \hspace{1cm} \text{Debye wavenumbers for ions and electrons}

\( k_o \) \hspace{1cm} \text{Poloidal wavenumber}

\( m \) \hspace{1cm} \text{Poloidal mode number}

\( m \) \hspace{1cm} \text{Electron mass}

\( n \) \hspace{1cm} \text{Toroidal mode number}

\( n_0, n_i, n_e \) \hspace{1cm} \text{Equilibrium, ion perturbed and electron perturbed densities}

\( n_{Ce}, n_{Te} \) \hspace{1cm} \text{Equilibrium circulating and trapped electron densities}
\( \tilde{n}_{Ce}, \tilde{n}_{Te} \)  Perturbed circulating and trapped electron densities

\( p_0, p_i \)  Equilibrium and perturbed ion pressures

\( q, q' \)  Safety factor and \( dq/dr \)

\( r \)  Minor radius coordinate of a plasma torus centred at \( R_0 \)

\( r' \)  Minor radius coordinate of an arbitrary, shifted plasma torus

\( r_0 \)  Radial location of a rational surface

\( s \)  Magnetic shear parameter

\( \text{sgn} \)  Signum or sign function

\( t \)  Time

\( v \)  Velocity

\( v_z \)  \( E \times B \) drift

\( v_{\text{pi}} \)  Ion polarization drift

\( u_1, u_{\perp} \)  Parallel and perpendicular components of velocity

\( \tilde{u}_1, \tilde{u}_{\perp} \)  Parallel and perpendicular components of velocity

normalized by the thermal velocity

\( v_{\parallel i} \)  Perturbed parallel ion velocity

\( v_{Ti}, v_{Te} \)  Ion and electron thermal velocities

\( x \)  Spatial position

\( \mathbf{x(\eta)} \)  \( 1 + \epsilon \cos \eta \)

\( x_m \)  Minimum of \( x(\eta) \): \( 1 - \epsilon \)
Greek

$\Delta$ Shafranov shift parameter

$\Lambda_i, \Lambda_e$ $k_\rho_i, k_\rho_e$

$\Xi_{\eta'}$ Argument of the integrating factor in $\tilde{h}\eta$

$\Pi^{kl}$ Kernal of the matrix equation $M$

$\Upsilon^\sigma, \Upsilon_{\varphi}, \Upsilon_A$ Variables used to condense terms in $\tilde{h}\sigma$

$\hat{\varphi}$ Fourier transform of $\varphi$

$\Psi, \Psi_0, \Psi_1$ Arbitrary, nonperturbed and perturbed fields or distribution functions

$\Omega_i, \Omega_e$ Ion and electron cyclotron frequencies

$\alpha$ Ballooning parameter

$\beta$ Ratio of plasma pressure to magnetic pressure

$\beta_c$ Critical $\beta$ above which ballooning effects occur

$\beta_i$ Ratio of ion plasma pressure to magnetic pressure

$\gamma$ Gyrophase angle

$\gamma$ Unnormalized growth rate: $\text{Im}(\omega)$

$\gamma$ Angle between $r$ and $r'$

$\delta$ Ordering or smallness parameter

$\delta_{kl}$ Kronecker delta

$\epsilon$ Inverse aspect ratio
\( \epsilon_n \)  Density gradient inverse aspect ratio \\
\( \varepsilon \)  Total guiding centre energy \\
\( \eta \)  Poloidal coordinate in ballooning space \\
\( \eta_0 \)  Arbitrary ballooning coordinate satisfying \( \eta_1 \leq \eta_0 \leq \eta_2 \) \\
\( \eta_b \)  Arbitrary bounce point \\
\( \eta_1, \eta_2 \)  \( 2\pi N - \eta_b, 2\pi N + \eta_b \) \\
\( \eta_i, \eta_e \)  Ion and electron temperature gradients \\
\( \theta \)  Poloidal angle \\
\( \kappa^2 \)  Pitch angle variable for trapped electrons: \( \frac{1}{2} \left( 1 + \frac{1-\lambda}{\epsilon} \right) \) \\
\( \lambda \)  Standard pitch angle variable \\
\( \lambda_D \)  Debye length of plasma \\
\( \mu \)  Magnetic moment \\
\( \nu^*_i, \nu^*_e \)  Collisionality parameter for ions and electrons \\
\( \nu_{\text{eff},e} \)  Effective collision frequency for electrons \\
\( \rho_i, \rho_e \)  Ion and electron Larmor radius \\
\( \rho_s \)  Ion Larmor radius with electron temperature \\
\( \sigma \)  Indicates sign of \( \tilde{E}_q \) \\
\( \tau \)  Ratio of electron to ion equilibrium temperature \\
\( \varphi \)  Scalar potential \\
\( \tilde{\varphi} \)  Scalar potential transformed to ballooning space
\( \varphi_m \)  
Poloidal Fourier component of \( \varphi \)

\( \tilde{\varphi} \)  
Poloidal Fourier component as a function of \( \chi \)

\( \phi \)  
Toroidal coordinate

\( \chi \)  
Local dimensionless radial coordinate

\( \omega \)  
Unnormalized eigenvalue or mode frequency

\( \hat{\omega} \)  
Eigenvalue normalized by \( \omega_{se} \)

\( \omega_r \)  
Unnormalized real frequency: \( \text{Re}(\omega) \)

\( \omega_A \)  
Alvén drift frequency

\( \omega_{bi}, \omega_{be} \)  
Ion and electron bounce frequencies

\( \omega_{di}, \omega_{de} \)  
Ion and electron magnetic drift frequencies

\( \hat{\omega}_{di}, \hat{\omega}_{de} \)  
Velocity averaged ion and electron magnetic drift frequencies

\( \omega_{se}, \omega_{es} \)  
Ion and electron diamagnetic drift frequency

\( \omega^r \)  
Diamagnetic drift frequency as a function of \( \hat{v}\parallel \) and \( \hat{v}\perp \)

\( \omega_{pi} \)  
Ion pressure gradient drift frequency
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