A STUDY OF THE HAMBURGER MOMENT PROBLEM ON THE REAL LINE

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Abstract

This thesis contains an exposition of the Hamburger moment problem. The Hamburger moment problem is an interesting question in analysis that deals with finding the existence of a Borel measure representing a given positive semi-definite linear functional. We begin our exposition by constructing orthogonal polynomials associated with a positive definite sequence. Then we discuss the interlacing property of the zeros of these orthogonal polynomials. We proceed by finding a solution to the truncated Hamburger moment problem and then extend the found solution to the complete Hamburger moment problem. After obtaining a solution to the Hamburger moment problem, we address the problem of determinacy of the moment problem. Finally, we discuss a result that proves the density of polynomials with complex coefficients under the assumption that the Carleman's condition is satisfied.

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Dedication

With sincere gratitude, I dedicate this work to Almighty God, The Alpha and Omega, who has always been my guidance right from birth. I also dedicate this work to my late Supervisor; Professor Murray Marshall. He gave me the opportunity to study this area of Mathematics. May His gentle soul rest in perfect peace.

Contents

Chapter 1

INTRODUCTION

1.1 Brief Historical Review

" The moment problem is a classical question in analysis, remarkable not only for its own elegance, but also for the extraordinary range of theoretical and applied subjects which it has illuminated " [23]. The classical moment problem is connected with a large number of areas in mathematics such as function theory, spectral representation of operator theory, approximation theory, the interpolation problem for functions of a complex variable and integral equations [23]. Its relevance to physics and statistics has been evident in the prediction of stochastic process, in probability, in approximation and numerical methods, in electrical and mechanical inverse problems and the design of algorithms for simulating physical systems [23]. For example, in probability, the question can occur to determine the existence of a probability distribution satisfying some conditions on its known moments. In mathematical physics, there is also a problem of determining if a spectral measure of a random operator is absolutely continuous with respect to the density of it state [32].

The term moment problem was first used in the work of Stieljtes when he published his work [33] in 1894 about the analytic behaviour of continued fractions. His beautiful work on continued fractions led him to a problem which he later named moment problem. Stieltjes discovered the equivalence between integrals and continued fractions in [33] even though Laguerre was the first to discuss continued fractions and integrals [21]. Stieltjes developed the moment problem on the positive real axis $[0, \infty)$ and solved the existence and uniqueness part of the moment problem. He was able to achieve this by developing what we now regard as the Stieltjes integral.

Stieltjes chose the name moment from mechanics. He interpreted a measure μ as a distribution of mass along the interval $[0, \infty)$ such that the total mass on the interval $[0, \infty)$ can be written in the form

$$\int_0^\infty d\mu(x),$$

and that the integral $\displaystyle\int_0^\infty x\, d\mu(x)$ represents the statistical moment of mass distribution. This motivated Stieltjes to define

$$\int_0^\infty x^n d\mu(x)$$

as the n-th moment and his main question was whether the distribution of mass can be ascertained from a knowledge of all the known moments [31]. After the brilliant work done by Stieltjes, the moment problem was considered again by Hamburger and Hausdorff.

Hausdorff continued the work of Stieltjes by studying the moment problem on a finite closed interval. He solved the moment problem based on a closed interval $[0, 1]$ and published his result in [16]. He proved that there exists a measure μ when the moment problem is restricted to a finite interval $[0, 1]$. Hausdorff contribution to moment problem is widely regarded as the Hausdorff moment problem. More historical details about the Hausdorff moment problem can be found in [21].

Hamburger in 1920 was interested in moment problem and he continued with the work done by Stieljtes [15]. He was the first person to consider the moment problem as a theory of its own as the role of $(0, \infty]$ in Stieltjes moment problem was replaced by the real line in Hamburger's work. The moment problem is usually referred to Hamburger due to his extensive discussion of this problem.

Our main motivation for studying the Hamburger moment problem is based on finding the required conditions for the existence of a measure when a sequence of real numbers $\{s_n\}_{n=0}^\infty$ is given. There are special kinds of sequences of real numbers that generate positive definite Hankel matrices. These particular kinds of sequences of real numbers are usually called positive definite sequences. These Hankel matrices are crucial in constructing orthogonal polynomials $\{p_n\}_{n=0}^\infty$ corresponding to the given positive definite real sequence $\{s_n\}_{n=0}^\infty$. We realised that a solution to the truncated moment problem is given by a measure whose support is the finite set of zeros of orthogonal polynomials of the first kind. It turns out that the orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ are related by a three term recurrence relation. Later on in this thesis, we will discuss how to extend our solution from the truncated moment problem to the full Hamburger moment problem using some limiting tools. Due to the extensive work of Hamburger on moment problem, we shall concentrate only on the Hamburger moment problem in this thesis.

Suppose that there exists a solution μ to the moment problem, we would also like to determine to what extent is the solution μ unique. There may be only one solution to the moment problem which we refer to as the determinate moment problem or we could have more than one solution which we refer to as the indeterminate problem. This serves as another motivation for exploring the Hamburger moment problem. Thus, the question to find the required conditions that makes the moment problem determinate or indeterminate arose. In 1926, Carleman was able to come up with a result in [9] that answers the determinacy question of the moment problem. Given a sequence of real numbers $\{s_n\}_{n=0}^{\infty}$, Carleman proved that if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty,$$

then the moment problem is determinate. Carleman's result on the determinacy of the moment problem will be reviewed later in this thesis.

1.2 Thesis Outline

There are three theorems to be discussed in this thesis namely: Theorem 3.5, Theorem 3.7 and Theorem 3.11. Theorem 3.5 is based on finding a solution to the Hamburger moment problem and Theorem 3.7 discusses the uniqueness of the solution. Theorem 3.11 is a result that is based on the Carleman's condition which involves showing the density of polynomials with complex coefficients.

The organization of this thesis is as follows. Useful concepts and terminologies in measure theory are briefly introduced in chapter 2. In chapter 3, we begin with the statement of the Hamburger moment problem. Then we proceed with a brief discussion of the three main theorems of this thesis. In chapter 4, we begin with a positive definite sequence and explain the construction of the orthogonal polynomials of the first and second respectively. Lastly, we present the proofs of our main results in chapter 5.

Chapter 2

PRELIMINARIES

We use the following usual notations: $\mathbb{N}, \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural numbers, ring of integers, the field of rationals, the field of real numbers and the field of complex numbers respectively. Let $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ denote the non-negative elements of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively. We denote the univariate polynomial ring by $\mathbb{R}[x]$.

We will review some basic measure theory concepts and also explain the relationship between a Borel measure and a distribution function.

Definition 2.1. Let X be a non-empty set. Then a σ-algebra $\mathcal{H}$ is a family of subsets of X such that the following properties hold:

(1) $X \in \mathcal{H}$ and $\emptyset \in \mathcal{H}$.

(2) If $A \in \mathcal{H}$, then $A^{\complement} \in \mathcal{H}$, where $A^{\complement}$ denotes the set complement of A.

(3) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{H}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{H}$.

The pair $(X, \mathcal{H})$ is called a measurable space and the sets in $\mathcal{H}$ are called measurable sets.

Consider the collection $\mathcal{O}$ of all open sets of $\mathbb{R}$. Then it follows that $\mathcal{O}$ is not a σ-algebra of subsets of $\mathbb{R}$. That is, if $A \in \mathcal{O}$, then by definition $A^{\complement}$ is a closed set and so $A^{\complement} \notin \mathcal{O}$. However, we know that $\sigma(\mathcal{O})$ which is the σ-algebra generated by $\mathcal{O}$ exists and satisfies $\mathcal{O} \subset \sigma(\mathcal{O}) \subset 2^{\mathbb{R}}$ [11]. Therefore, it is natural to give the following definition.

Definition 2.2. The Borel σ-algebra of $\mathbb{R}$ denoted as $\mathbb{B}$ is the σ-algebra generated by the open sets of $\mathbb{R}$. The elements of the Borel σ-algebra are called the Borel sets.

Definition 2.3. Let $(X, \mathcal{H})$ be a measurable space. A measure on $(X, \mathcal{H})$ is a function $\mu : \mathcal{H} \to [0, \infty]$ such that

(1) $\mu(\emptyset) = 0$.

(2) For any sequence of mutually disjoint countable sets $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{H}$, we have that

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Example 2.4. Let X be any set and let $\mathcal{H} = 2^X$. The measure $\delta_x : \mathcal{H} \to \{0, 1\}$ with

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called a Dirac measure.

Definition 2.5. A Borel measure μ is a measure that is defined on the Borel σ-algebra of $\mathbb{R}$.

Remark 2.6. If $\mu(\mathbb{R}) = 1$, then measure μ is called a probability measure.

Definition 2.7. A triple $(X, \mathcal{H}, \mu)$ is called a measure space. It is simply a measurable space equipped with a measure.

Definition 2.8. Let $(X, \mathcal{H}, \mu)$ be a measure space. A measure μ is called finite if $\mu(X) < \infty$.

The following definition identifies the kind of functions that are ideal for integration.

Definition 2.9. Let $(X, \mathcal{H})$ and $(Y, \mathcal{G})$ be measurable spaces. A function $f : X \to Y$ is said to be measurable if $f^{-1}(B) \in \mathcal{H}$ for every $B \in \mathcal{G}$.

Definition 2.10. Let $(X, \mathcal{H})$ be a measurable space. For any set $A \subset X$, the characteristic function χ_A of A is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The characteristic function will play an important role in our definition of an integral.

Definition 2.11. Let $(X, \mathcal{H})$ be a measurable space. A simple function $f : X \to \mathbb{R}$ is a function of the form

$$f = \sum_{k=1}^{n} a_k \chi_{A_k}, \tag{2.1}$$

where $a_1, a_2, \cdots, a_n \in \mathbb{R}$ and $A_1, A_2, \cdots, A_n \in \mathcal{H}$. We refer to Equation 2.1 as the standard representation of f when the constants a_n are distinct and the sets A_n are disjoint.

Definition 2.12. Let $(X, \mathcal{H}, \mu)$ be a measure space. Let g be a measurable simple function on X of the form

$$g = \sum_{k=1}^{n} a_k \chi_{A_k}.$$

Then the integral of g over X with respect to a measure μ is defined as

$$\int_X g d\mu = \sum_{k=1}^{n} a_k \mu(A_k).$$

Proposition 2.13. [11, p. 49] Let g and h be simple measurable functions on a measure space $(X, \mathcal{H}, \mu)$. If $0 \leq g \leq h$, then

$$\int_X g d\mu \leq \int_X h d\mu.$$

Proof. Assume that $g \leq h$. Define $\phi = h - g$ on X. Then by linearity, we have that

$$\int_X h d\mu - \int_X g d\mu = \int_X (h - g) d\mu = \int_X \phi d\mu \geq 0.$$

Since the non-negative simple function ϕ has a non-negative integral, we have that

$$\int_X g d\mu \leq \int_X h d\mu.$$

$\square$

Definition 2.14. Let $(X, \mathcal{H}, \mu)$ be a measure space and let f be a measurable function on $(X, \mathcal{H}, \mu)$. We define the Lebesgue integral of f over X as

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : 0 \leq g \leq f, g \text{ is simple} \right\}.$$

Note that by Proposition 2.13, the two definitions 2.12 and 2.14 agree when f is a simple function, as the family of simple functions over which the supremum is taken include itself.

We shall give the following definition in order to describe the relationship between a distribution function and a Borel measure.

Definition 2.15. If $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$, then f is said to be:

(1) increasing if $f(x) \leq f(y)$ whenever $x \leq y$ where $x, y \in X$;

(2) strictly increasing if $f(x) < f(y)$ whenever $x < y$ where $x, y \in X$;

(3) decreasing if $f(y) \leq f(x)$ whenever $x \leq y$ where $x, y \in X$;

(4) strictly decreasing if $f(y) < f(x)$ whenever $x < y$ where $x, y \in X$;

(5) monotone if the function is either decreasing or increasing;

(6) right continuous if $f(a^+) = \lim_{x \to a^+} f(x) = f(a)$ for all $a \in X$;

(7) left continuous if $f(a^-) = \lim_{x \to a^-} f(x) = f(a)$ for all $a \in X$.

Remark 2.16. For the purpose of this thesis, the description of Borel measures on the real line given in Theorem 1.16 of Folland [11] is very important. It states that if $f : \mathbb{R} \to \mathbb{R}$ is any right continuous, increasing function, then there is a unique Borel measure μ_f such that

$$\mu_f((a, b]) = f(b) - f(a)$$

for all $a, b \in \mathbb{R}$. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$f(x) = \begin{cases} \mu(0, x] & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu(0, x] & \text{if } x < 0, \end{cases}$$

then f is a right continuous, increasing function and $\mu = \mu_f$. We refer to a right continuous, increasing function as a distribution function.

From now on, we will avoid confusion between distribution functions and measures by writing

$$\int df$$

instead of

$$\int d\mu_f.$$

Theorem 2.17. [29, p. 96] The set of discontinuities of an increasing function f is at most countable.

Proof. Let f be an increasing function and let $D(f) = \{x \in \mathbb{R} : f(x^+) > f(x^-)\}$ be the set of points at which f is discontinuous. For each $x \in D(f)$, we can choose a rational number r_x such that $f(x^-) < r_x < f(x^+)$. Let us define a function $g : D(f) \to \mathbb{Q}$ such that $x \longmapsto r_x$. Since f is increasing, we see that if $x \neq y$ then $r_x \neq r_y$. Therefore, g is a one-to-one function from $D(f)$ to $\mathbb{Q}$. Since $\mathbb{Q}$ is countable and g is one-to-one, then we conclude that the set of point at which f is discontintinuous is at most countable. $\square$

Definition 2.18. A sequence of functions $\{f_n\}_{n=0}^{\infty}$ on $\mathbb{R}$ is said to converge to a limiting function f if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

at every point $x \in \mathbb{R}$ where f is continuous.

The following two theorems will be used in proving Theorem 3.5. We will only prove Helly's second theorem because it will give us a clearer meaning of what it means to do integration with respect to a distribution function.

Theorem 2.19. [10, Theorem 2.2] (**Helly's first theorem**). Every sequence $\{f_n\}$ of uniformly bounded increasing functions defined on $\mathbb{R}$ contains a subsequence $\{f_{nk}\}$ which converges on $\mathbb{R}$ to an increasing bounded function f.

Theorem 2.20. [10, Theorem 2.3] (**Helly's second theorem**). Let $\{f_n\}$ be a uniformly bounded sequence of increasing functions defined on an interval $[a, b]$ and let it converge on $[a, b]$ to a function f. Then for every continuous function h on $[a, b]$,

$$\lim_{n\to\infty} \int_a^b h df_n = \int_a^b h df.$$

Proof. By definition, we have that there exists a $K > 0$ such that

$$0 \leq f_n(b) - f_n(a) \leq K, \forall n \in \mathbb{N}$$

because $\{f_n\}$ is uniformly bounded. Since f is the limiting function of the uniformly bounded sequence $\{f_n\}$, then it follows that

$$0 \leq f(b) - f(a) \leq K.$$

That is, f is bounded by K. Let $\epsilon > 0$ be given. Since h is real and continuous on $[a, b]$, then it follows that h is uniformly continuous on $[a, b]$. So there is a partition $p_\epsilon = \{x_0, x_1, x_2, \cdots, x_m\}$ of interval $[a, b]$ such that

$$|h(x^*) - h(x^{**})| < \epsilon$$

for $x^*, x^{**} \in [x_{k-1}, x_k], \quad 1 \leq k \leq m$. Now we choose $y_k^* \in [x_{k-1}, x_k]$ and write

$$\Delta_k f = f(x_k) - f(x_{k-1})$$

and

$$\Delta_k f_n = f_n(x_k) - f_n(x_{k-1}).$$

By the mean value theorem for integrals, we have that

$$\int_{x_{k-1}}^{x_k} hdf - h(y_k^*)\Delta_k f = [h(y_k^{**}) - h(y_k^*)]\Delta_k f$$

for some $y_k^{**} \in [x_{k-1}, x_k]$. Summing over k from 1 to m, we obtain

$$\left| \int_a^b hdf - \sum_{k=1}^m h(y_k^*)\Delta_k f \right| \leq \sum_{k=1}^m |h(y_k^{**}) - h(y_k^*)| \, \Delta_k f$$

$$< \epsilon \sum_{k=1}^m \Delta_k f \leq \epsilon K.$$

Similarly, it follows that

$$\left| \int_a^b hdf_n - \sum_{k=1}^m h(y_k^*)\Delta_k f_n \right| < \epsilon K.$$

Thus,

$$\left| \int_a^b hdf - \int_a^b hdf_n \right| \leq \left| \int_a^b hdf - \sum_{k=1}^m h(y_k^*)\Delta_k f \right| + \left| \sum_{k=1}^m h(y_k^*) \left[\Delta_k f - \Delta_k f_n\right] \right|$$

$$+ \left| \int_a^b hdf_n - \sum_{k=1}^m h(y_k^*)\Delta_k f_n \right|.$$

Therefore,

$$\left| \int_a^b hdf - \int_a^b hdf_n \right| < 2\epsilon K + \sum_{k=1}^m |h(y_k^*)| \, |\Delta_k f - \Delta_k f_n| \, .$$

By keeping our partition p_ϵ fixed, we have that

$$\lim_{n\to\infty} \Delta_k(f - f_n) = 0.$$

Hence,

$$\limsup_{n\to\infty} \left| \int_a^b h df - \int_a^b h df_n \right| \leq 2\epsilon K$$

and the desired conclusion follows. $\square$

Chapter 3

STATEMENT OF THE MOMENT PROBLEM

Let $\{s_n\}_{n=0}^{\infty}$ be a given sequence of real numbers. The classical Hamburger's moment problem [15] asks:

Problem 3.1. Find necessary and sufficient conditions for the existence of a Borel measure μ on $\mathbb{R}$ such that

$$s_n = \int_{\mathbb{R}} x^n d\mu(x) \quad \forall\; n \geq 0.$$

Remark 3.2. The given sequence $\{s_n\}_{n=0}^{\infty}$ yields a unique linear functional L on $\mathbb{R}[x]$, the vector space of polynomials in the variable x with real coefficients. Namely, one defines $L(x^n) = s_n$ for $n \geq 0$ and then extends this definition linearly to the whole $\mathbb{R}[x]$. Conversely, to each linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, one obtains the corresponding real sequence $\{s_n\}_{n=0}^{\infty}$ from the evaluation of L on the basis $\{1, x, x^2, \cdots\}$, i.e. $s_n := L(x^n), \quad \forall\; n \geq 0$. Moreover, the given sequence $\{s_n\}_{n=0}^{\infty}$ is positive definite if and only if the corresponding linear functional L is positive definite.

In view of such a correspondence, the Hamburger moment problem amounts to asking that for a given linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, when is there a Borel measure μ on $\mathbb{R}$ such that

$$L(f) = \int_{\mathbb{R}} f(x) d\mu(x) \quad \forall f \in \; \mathbb{R}[x]. \tag{3.1}$$

Here the polynomial $f(x)$ is viewed a continuous function on $\mathbb{R}$.

We observe that if the linear functional L is indeed realized by a Borel measure μ in the sense of Equation 3.1, then we must have

$$L(f^2) = \int_{\mathbb{R}} [f(x)]^2 d\mu \geq 0$$

for any non-negative polynomial $f \in \mathbb{R}[x]$.

Theorem 3.3. [10, p. 15] If f is a non-negative polynomial on $\mathbb{R}$, then f can be written as

$$f = A^2 + B^2$$

for some polynomials $A, B \in \mathbb{R}[x]$.

Proof. Let f be a non-negative polynomial on $\mathbb{R}$. Then f can be decomposed into product of linear factors of the form

$$f = d \prod_{k=1}^{m} (x - \alpha_k - i\beta_k)(x - \alpha_k + i\beta_k)$$

where $d > 0$, $i = \sqrt{-1}$ and $\alpha_k, \beta_k \in \mathbb{R}$. Selecting part of the factors of f, we can write

$$\prod_{k=1}^{m} (x - \alpha_k - i\beta_k) = A(x) + iB(x)$$

and

$$\prod_{k=1}^{m} (x - \alpha_k + i\beta_k) = A(x) - iB(x)$$

where $A(x), B(x) \in \mathbb{R}[x]$. Therefore,

$$f = d[A^2(x) + B^2(x)].$$

□

Definition 3.4. A linear functional $L : \mathbb{R}[x] \to \mathbb{R}$ is said to be positive semi-definite if $L(f) \geq 0$ for all non-negative polynomials f on $\mathbb{R}$. If $L(f) > 0$ for all non-negative polynomials f on $\mathbb{R}$ such that $f \neq 0$, then we say that L is a positive definite linear functional. In this case, without loss of generality, we might assume that $L(1) = 1$.

We observe above that the semi-definite positivity of L is necessary for solving the Hamburger's moment problem. The result below shows that it is sufficient as well.

Theorem 3.5. [1, Theorem 2.1.1] A linear functional $L : \mathbb{R}[x] \to \mathbb{R}$ is realized by a positive Borel measure μ on $\mathbb{R}$ if and only if L is positive semi-definite.

The measure μ above is called a solution of the moment problem relative to the linear functional L (or with respect to the given sequence $\{s_n\}_{n=0}^{\infty}$). Our next course of investigation is concerned with the uniqueness of the solution μ.

Definition 3.6. The moment problem relative to a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$ is said to be determinate if it has a unique solution. Otherwise, it is said to be indeterminate.

We have the following result regarding the determinacy of moment problem. Denote by $\mathbb{C}[x]$ the vector space of polynomials with complex coefficients and also note that any linear functional L on $\mathbb{R}[x]$ extends uniquely (still denoted by L) on $\mathbb{C}[x]$ by the formula:

$$L(\sum z_n x^n) = L(\sum \Re z_n x^n) + iL(\sum \Im z_n x^n) \tag{3.2}$$

where $\Re$ and $\Im$ denotes the real and imaginary parts respectively.

Theorem 3.7. [25, Theorem 1.1] The moment problem relative to a positive semi-definite linear functional L is determinate if and only if there exists a sequence of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ in $\mathbb{C}[x]$ such that $f_n(i) = 1$ for all $n \geq 1$ and $L(|f_n|^2) \to 0$ as $n \to \infty$ where $|f_n|^2 = f_n \cdot \overline{f_n}$.

Finally, we will consider a problem of approximating integrable functions as an application of moment problem as this is closely related to the Carleman's condition on the determinacy of moment problem.

Definition 3.8. For $p \geq 1$, we define the vector space

$$L^p(\mu) = \{f : \mathbb{R} \to \mathbb{C} |\ f \text{ is measurable and } ||f||_p < \infty\}$$

where

$$||f||_p = \left[\int_{\mathbb{R}} |f|^p d\mu\right]^{\frac{1}{p}}.$$

Remark 3.9. Let μ be a probability measure. Note that if $1 \leq p \leq q < \infty$, then

$$L^q(\mu) \subseteq L^p(\mu) \subseteq L^1(\mu).$$

Indeed for $f \in L^q(\mu)$ and using the Hölder's inequality on $\displaystyle\int_{\mathbb{R}} |f^p(x)| d\mu(x)$, we have that

$$\int_{\mathbb{R}} |f^p(x)| d\mu(x) \leq \left(\int_{\mathbb{R}} |f^p(x)|^{\frac{q}{p}} d\mu(x) \right)^{\frac{p}{q}} \left(\int_{\mathbb{R}} |1|^{\frac{p-q}{p}} d\mu(x) \right)^{\frac{p}{p-q}}$$

$$= \left(\int_{\mathbb{R}} |f^p(x)|^{\frac{q}{p}} d\mu(x) \right)^{\frac{p}{q}} = \left(\int_{\mathbb{R}} |f(x)|^q d\mu(x) \right)^{\frac{p}{q}}.$$

Hence, we have that

$$||f||_p \leq ||f||_q,$$

and this implies that

$$f \in L^q(\mu) \subseteq L^p(\mu).$$

Moreover, it is well known that $L^q(\mu)$ is dense in $L^p(\mu)$ (See [4, p. 112]).

Theorem 3.10. [4, Theorem 2.9.5] (**Carleman's condition**). Let $\{s_n\}_{n=1}^{\infty}$ be a moment sequence. If

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty,$$

then the moment problem $s_n = \displaystyle\int_{-\infty}^{\infty} x^n d\mu(x)$ is determinate.

Theorem 3.11. [6, Theorem 3] If $L : \mathbb{R}[x] \to \mathbb{R}$ satisfies the Carleman condition, then not only is the measure μ unique but also $\mathbb{C}[x]$ is dense in $L^p(\mu)$ for all real $p \geq 1$.

The detailed proofs of Theorem 3.5, Theorem 3.7, Theorem 3.10 and Theorem 3.11 will be presented in chapter 5 of this thesis.

Chapter 4

POSITIVE DEFINITE SEQUENCES

This goal of this section is to present detailed explanations and results that are needed to prove Theorem 3.5. We will begin with the constructions of orthogonal polynomials in this section under the assumption that $\{s_n\}_{n=0}^{\infty}$ is a positive definite real sequence. At the end of the proof of Theorem 3.5 in chapter 5, we will consider the case in which of the $\{s_n\}_{n=0}^{\infty}$ is positive semi-definite.

Definition 4.1. Given a sequence of real numbers $\{s_n\}_{n=0}^{\infty}$, we form for all $n \geq 0$ the Hankel matrices

$$H_n = (s_{i+j})_{0 \leq i,j \leq n},$$

which are matrices of size $(n+1)$ by $(n+1)$. Written explicitly, we get

$$H_n = \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ s_n & s_{n+1} & \cdots & s_{2n} \end{pmatrix}.$$

Definition 4.2. A sequence $\{s_n\}_{n=0}^{\infty}$ of real numbers is called positive semi-definite if $\forall n \geq 0$

$$\sum_{i,j=0}^{n} s_{i+j} c_i c_j \geq 0 \tag{4.1}$$

for all $(c_0, c_1, c_2, \cdots, c_n) \in \mathbb{R}^{n+1}$. Similarly, a sequence $\{s_n\}_{n=0}^{\infty}$ of real numbers is called positive definite if for $\forall n \geq 0$ and for all $(c_0, c_1, \cdots, c_n) \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\sum_{i,j=0}^{n} s_{i+j} c_i c_j > 0.$$

The positive definite property of this sequence $\{s_n\}_{n=0}^{\infty}$ is equivalent to the fact each associated Hankel matrix H_n,

$$H_n = (s_{i+j})_{0 \leq i,j \leq n}$$

is positive definite for all $n \geq 0$. From elementary linear algebra, it follows that the determinant of the Hankel matrix denoted by D_n is positive. Thus, the positive definite property of the sequence $\{s_n\}_{n=0}^{\infty}$ allow us to define an inner product on the vector space $\mathbb{R}[x]$ as follows.

Lemma 4.3. Suppose that $\{s_n\}_{n=0}^{\infty}$ is a positive definite sequence and let $L : \mathbb{R}[x] \to \mathbb{R}$ be its corresponding positive definite linear functional such that $L(x^n) = s_n$ for all $n \geq 0$. Then

$$\langle p, q \rangle := L(pq), \quad p, q \in \mathbb{R}[x],$$

is an inner product $\langle ., . \rangle$ on $\mathbb{R}[x]$.

Proof.

(i) Scalar multiplication: $\langle \alpha p, q \rangle = L(\alpha pq) = \alpha L(pq) = \alpha \langle p, q \rangle$ for all $p, q \in \mathbb{R}[x]$ and $\alpha \in \mathbb{R}$.

(ii) Symmetric property: $\langle p, q \rangle = L(pq) = L(qp) = \langle q, p \rangle$ for all $p, q \in \mathbb{R}[x]$.

(iii) Linearity: $\langle p + h, q \rangle = L(pq + hq) = L(pq) + L(hq) = \langle p, q \rangle + \langle h, q \rangle$ for all $p, q, h \in \mathbb{R}[x]$.

(iv) Positive definite property: Let $p \in \mathbb{R}[x]$. Observe that the associated semi-norm of the polynomial p is

$$||p||^2 = \langle p, p \rangle = \sum_{i,j=0}^{n} s_{i+j} c_i c_j.$$

Since $||p||^2$ is non-negative by definition, then it follows $\langle p, p \rangle \geq 0$ for all $p \in \mathbb{R}[x]$.

□

Definition 4.4. A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is said to be orthogonal with respect to a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, if for all non-negative n and m,

(1) $p_n(x)$ is a polynomial of degree n,

(2) $L(p_n(x)p_m(x)) = 0$ for $m \neq n$,

(3) $L(p_n^2(x)) = 1$.

4.1 Construction of Orthogonal Polynomials

Proposition 4.5. [1, p. 3] Let $\{s_n\}_{n=0}^{\infty}$ be a given positive definite real sequence. Then there exists a sequence of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ corresponding to the positive definite sequence $\{s_n\}_{n=0}^{\infty}$ such that the degree of polynomial p_n is n. Also, the leading coefficient of p_n is positive for all $n \geq 0$.

Proof. Starting from a positive definite sequence $\{s_n\}_{n=0}^{\infty}$, we use the Gram-Schmidt orthogonalizing process to construct the corresponding sequence of orthogonal polynomials denoted by $\{p_n\}_{n=0}^{\infty}$. Also, we normalize our sequence $\{s_n\}_{n=0}^{\infty}$ by fixing $s_0 = 1$ because it will help simplify our subsequent formulas. For example, it implies that $p_0(x) = 1$, where $p_0(x)$ is the first of the orthogonal polynomials to be derived. Since $\{s_n\}_{n=0}^{\infty}$ is positive definite, we recall that the positive definite linear functional L induces an inner product $\langle ., . \rangle$ on $\mathbb{R}[x]$ with

$$\langle p, q \rangle := L(pq)$$

for all $p, q \in \mathbb{R}[x]$. Thus, we apply the Gram-Schimdt orthogonalization process on the standard basis for a vector space of polynomials $\{1, x, x^2, \cdots\}$ such that

$$s_{i+j} = \left\langle x^i, x^j \right\rangle, \quad \forall\, i, j \geq 0,$$

and we obtain the corresponding Gram matrix of order $n+1$ also denoted by H_n,

$$H_n = \begin{pmatrix} \langle x^0, x^0 \rangle & \langle x^0, x \rangle & \cdots & \langle x^0, x^n \rangle \\ \langle x, x^0 \rangle & \langle x, x \rangle & \cdots & \langle x, x^n \rangle \\ \langle x^2, x^0 \rangle & \langle x^2, x \rangle & \cdots & \langle x^2, x^n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x^{n-1}, x^0 \rangle & \langle x^{n-1}, x \rangle & \cdots & \langle x^{n-1}, x^n \rangle \\ \langle x^n, x^0 \rangle & \langle x^n, x \rangle & \cdots & \langle x^n, x^n \rangle \end{pmatrix}.$$

Note that the matrix H_n obtained is non-singular. Namely, it is well known that the determinant of H_n denoted by $D_n \neq 0$ if and only if the system of polynomials $\{1, x, x^2, \cdots, x^n\}$ is linearly independent. This is true since $\{1, x, x^2, \cdots, x^n, \cdots\}$ is a standard basis for a vector space of polynomials $\mathbb{R}[x]$. Hence, we have that $D_n = |H_n| > 0$ for $n \geq 0$ because the given sequence $\{s_n\}_{n=0}^{\infty}$ is positive definite (See [30, p. 365] for more details). As a result of the application of the Gram-Schmidt process, we obtain a sequence of real polynomials expressed in terms of determinant, namely

$$p_0(x) = 1,$$

$$p_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix} \quad \text{for} \quad n \geq 1,$$

where

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}.$$

Expanding the determinants above, we get

$$p_n(x) = \sqrt{\frac{D_{n-1}}{D_n}} x^n + R_{n-1}(x), \; n \geq 0 \tag{4.2}$$

where $R_{n-1}(x)$ is a polynomial of degree $n-1$. For $n \geq 1$, the polynomial

$$p_n(x) = \sqrt{\frac{D_{n-1}}{D_n}} x^n + R_{n-1}(x)$$

is a polynomial of degree n with positive leading coefficient since

$$\sqrt{\frac{D_{n-1}}{D_n}} > 0.$$

In order to verify that these polynomials $\{p_n(x)\}_{n=0}^{\infty}$ are truly orthogonal, we first write the polynomial $p_n(x)x^m$ as

$$p_n(x)x^m = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ x^m & x^{m+1} & \cdots & x^{m+n} \end{vmatrix}.$$

Then we apply a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$ on $p_n x^m$, so that

$$L(p_n x^m) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ s_m & s_{m+1} & \cdots & s_{m+n} \end{vmatrix}.$$

Expanding the above determinant for the case when $m < n$ yields two identical rows which implies that the determinant for the case when $m < n$ is equal to zero. Therefore,

$$L(p_n x^m) = \langle p_n, x^m \rangle = \begin{cases} \sqrt{\frac{D_n}{D_{n-1}}} & \text{for } m = n, \\ 0 & \text{for } m = 0, 1, 2, \cdots, n-1. \end{cases} \tag{4.3}$$

Multiplying both sides of Equation 4.2 by polynomial $p_m(x)$ now gives

$$p_n(x)p_m(x) = \sqrt{\frac{D_{m-1}}{D_n}} x^m p_n(x) + R_{m-1}(x)p_n(x). \tag{4.4}$$

Therefore,

$$L(p_n(x)p_m(x)) = \sqrt{\frac{D_{m-1}}{D_m}} L(x^m p_n) + L(R_{m-1}(x)p_n(x)). \tag{4.5}$$

By substituting the identity derived in Equation 4.3 into Equation 4.5 gives

$$L(p_n(x)p_m(x)) = \sqrt{\frac{D_{m-1}}{D_m}} \sqrt{\frac{D_n}{D_{n-1}}} = \begin{cases} 1 & for\, m = n, \\ 0 & for\ m \neq n. \end{cases}$$

□

4.2 Polynomials of the First kind

Definition 4.6. The orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ derived in Equation 4.2 are called polynomials of the first kind associated with the positive definite linear functional $L : \mathbb{R}[x] \to \mathbb{R}$.

Remark 4.7. Note that the sequence of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ also span the vector space of polynomials $\mathbb{R}[x]$. This follows from the Gram-Schmidt process since the span of the standard basis for $\mathbb{R}[x]$ is the same as the span of the orthogonal polynomials $\{p_0), p_1, \cdots, p_n, \cdots\}$.

Due to the correspondence between a positive definite sequence $\{s_n\}_{n=0}^{\infty}$ and positive definite linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, we will refer to the sequence $\{p_n\}_{n=0}^{\infty}$ of orthogonal polynomials associated with a positive definite linear functional $L : \mathbb{R}[x] \to \mathbb{R}$ as orthogonal polynomials associated to a positive definite sequence $\{s_n\}_{n=0}^{\infty}$ interchangeably. One of the important characteristics of orthogonal polynomials is the fact that any three consecutive polynomials are related by a recurrence relation. To see this, we discuss the following result.

Theorem 4.8. [10, p. 18] The orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy the following three term recurrence relation:

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x), \quad n \geq 0 \tag{4.6}$$

where $b_{-1} = p_{-1} = 0$, $b_n = \langle xp_n, p_{n+1} \rangle = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n} > 0$ and $a_n = \langle xp_n, p_n \rangle$.

Proof. Since the degree of the polynomial $xp_n(x)$ is $n+1$, then we can write

$$xp_n(x) = \sum_{k=0}^{n+1} \langle xp_n, p_k \rangle \, p_k = \sum_{k=0}^{n-2} \langle xp_n, p_k \rangle \, p_k +$$
$$\langle xp_n, p_{n-1} \rangle \, p_{n-1} + \langle xp_n, p_n \rangle \, p_n + \langle xp_n, p_{n+1} \rangle \, p_{n+1}.$$

Note that for $k < n-1$, the degree of the polynomial $xp_k(x) < n$. So,

$$\langle xp_n, p_k \rangle = 0$$

for $k \leq n-2$. Hence,

$$xp_n(x) = \langle xp_n, p_{n-1}\rangle\, p_{n-1} + \langle xp_n, p_n\rangle\, p_n + \langle xp_n, p_{n+1}\rangle\, p_{n+1},\ \ n \geq 0.$$

The sequences $b_n = \langle xp_n, p_{n+1}\rangle$ and $a_n = \langle xp_n, p_n\rangle$ are real since p_n has real coefficients. Let us now show that b_n is positive. Using Equation 4.2, the polynomial $xp_n(x)$ can be written as

$$xp_n(x) = \sqrt{\frac{D_{n-1}}{D_n}}x^{n+1} + R_n(x),$$

where $R_n(x)$ is a polynomial of degree n. Similarly from Equation 4.2, we obtain that

$$p_{n+1}(x) = \sqrt{\frac{D_n}{D_{n+1}}}x^{n+1} + R_n(x).$$

Therefore,

$$xp_n(x) = \sqrt{\frac{D_{n-1}}{D_n}}x^{n+1} + R_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1}p_{n-1}(x). \tag{4.7}$$

Substituting Equation 4.2 into Equation 4.7 yields

$$\sqrt{\frac{D_{n-1}}{D_n}}x^{n+1} + R_n(x) = b_{n-1}p_{n-1} + a_n p_n + b_n\left[\sqrt{\frac{D_n}{D_{n+1}}}x^{n+1}\right] + b_n R_n(x). \tag{4.8}$$

Then by comparing coefficients of x^{n+1}, we get

$$\sqrt{\frac{D_{n-1}}{D_n}} = b_n\left[\sqrt{\frac{D_n}{D_{n+1}}}\ \right],$$

which gives

$$b_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}. \tag{4.9}$$

Therefore, b_n is positive since $D_n > 0$ for $n \geq 0$.

□

Theorem 4.9. [10, p. 23] Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of orthogonal polynomials associated with a given positive definite sequence $\{s_n\}_{n=0}^{\infty}$. Then for any $x, y \in \mathbb{R}$ such that $x \neq y$, we have

$$(x-y)\sum_{k=0}^{n} p_k(x)p_k(y) = b_n\left[p_n(y)p_{n+1}(x) - p_n(x)p_{n+1}(y)\right]. \tag{4.10}$$

and

$$\sum_{k=0}^{n}\left[p_k(x)^2\right] = b_n\left[p_{n+1}^{'}(x)p_n(x) - p_n^{'}(x)p_{n+1}(x)\right] \tag{4.11}$$

where $p_n^{'}$ denotes the derivative of p_n.

Proof. Let $p_n(x)$ and $p_n(y)$ be orthogonal polynomial that satisfy the three term recurrence relation. This implies that

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1}p_{n-1}(x), \tag{4.12}$$

$$yp_n(y) = b_n p_{n+1}(y) + a_n p_n(y) + b_{n-1}p_{n-1}(y). \tag{4.13}$$

Multiplying Equation 4.12 by $p_n(y)$ and Equation 4.13 by $p_n(x)$, we get

$$xp_n(x)p_n(y) = b_n p_{n+1}(x)p_n(y) + a_n p_n(x)p_n(y) + b_{n-1}p_{n-1}(x)p_n(y) \tag{4.14}$$

and

$$yp_n(y)p_n(x) = b_n p_{n+1}(y)p_n(x) + a_n p_n(y) + b_{n-1}p_{n-1}(y)p_n(x). \tag{4.15}$$

If we subtract Equation 4.15 from Equation 4.14, we get

$$b_{n-1}\left[p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)\right] + b_n\left[p_n(y)p_{n+1}(x) - p_n(x)p_{n+1}(y)\right] = (x-y)p_n(x)p_n(y).$$

Summing the above identities for $1 \le k \le n$ yields Equation (4.10). By letting $x \to y$, Equation 4.10 becomes

$$\sum_{k=0}^{n}\left[p_k(x)^2\right] = b_n\left[p_{n+1}^{'}(x)p_n(x) - p_n^{'}(x)p_{n+1}(x)\right].$$

In particular, if $p_{n+1}(x) = 0$, then Equation 4.11 yields

$$\sum_{k=0}^{n} p_k(x)^2 = b_n p_n(x)p_{n+1}^{'}(x). \tag{4.16}$$

□

The next theorem is a basic fact about zeros of orthogonal polynomials.

Proposition 4.10. [1, p. 9] The orthogonal polynomial p_n has simple and real zeros.

Proof. To prove by contradiction, we assume that either $\lambda \in \mathbb{R}$ with multiplicity greater than one or $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a zero. Note that in the last case $\overline{\lambda}$ is also zero. Define

$$K(x) = \frac{p_n(x)}{|x-\lambda|^2}.$$

The Fundamental Theroem of Algebra gives that the denominator divides the numerator regardless whether $\lambda, \overline{\lambda} \in \mathbb{C} \setminus \mathbb{R}$ are zeros or $\lambda \in \mathbb{R}$ is a zero of multiplicity more than one. Observe that $K(x)$ is a polynomial of degree $n-2$. Again by the Fundamental Theorem of Algebra, we can write

$$p_n(x) = (x-\lambda)R_{n-1}(x)$$

such that the degree of $R_{n-1}(x) = n-1$. Since the degree of $K(x)$ is less than $p_n(x)$, then by our orthogonality relation explained earlier in Equation 4.3 , we have that

$$0 = \langle p_n, K \rangle = L\left[p_n(x)K(x)\right].$$

It follows from the extension of L to $\mathbb{C}[x]$ presented in Equation 3.2 that

$$0 = \langle p_n, K \rangle = L\left[p_n(x)K(x)\right] = L\left[(x-\lambda)R_{n-1}K(x)\right] = \langle R_{n-1}, R_{n-1} \rangle = ||R_{n-1}||^2,$$

but $||R_{n-1}||^2$ cannot be zero because R_{n-1} is not a zero polynomial by our definition. Therefore, our assumption that p_n has a pair of complex conjugate zeros is false. Hence, all zeros of $p_n(x)$ must be real and simple. □

Theorem 4.11. [10, Theorem 5.3] Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials associated to a positive definite sequence $\{s_n\}_{n=0}^{\infty}$. The zeros of $p_n(x)$ and $p_{n+1}(x)$ mutually separate each other. That is,

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \quad i = 1, 2, \cdots, n,$$

where $x_{n+1,i}$ and $x_{n,i}$ are the zeros of $p_{n+1}(x)$ and $p_n(x)$ respectively.

Proof. Let $x_{n+1,i}$ and $x_{n+1,i+1}$ be two consecutive zeros of the polynomial $p_{n+1}(x)$. Using Equation 4.16, we have that

$$b_n\left[p_{n+1}^{'}(x_{n+1,i})p_n(x_{n+1,i})\right] = \sum_{k=0}^{n}\left[p_k^2(x_{n+1,i})\right] > 0$$

and

$$b_n\left[p_{n+1}^{'}(x_{n+1,i+1})p_n(x_{n+1,i+1})\right] = \sum_{k=0}^{n}\left[p_k^2(x_{n+1,i+1})^2\right] > 0.$$

Since all $n+1$ zeros of $p_{n+1}(x)$ are real and simple, then $p_{n+1}^{'}(x_{n+1,i})$ and $p_{n+1}^{'}(x_{n+1,i+1})$ are of different signs. Therefore, $p_n(x_{n+1,i})$ and $p_n(x_{n+1,i+1})$ will have opposite signs and $p_n(x)$ has a zero in each of the intervals $(x_{n+1,i}, x_{n+1,i+1})$ for $i = 1, 2, \cdots, n$. $\square$

4.3 Polynomials of the Second kind

Let $r(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial of degree $n \geq 1$. Then for $x \neq y$, the quotient $\frac{r(x)-r(y)}{x-y}$ can be expressed in the form

$$\frac{r(x)-r(y)}{x-y} = \sum_{k=0}^{n} a_k \frac{x^k - y^k}{x-y} = \sum_{k=0}^{n}\sum_{i=0}^{k-1} a_k y^{k-i} x^i.$$

By fixing $x \in \mathbb{R}$, the polynomial $\frac{r(x)-r(y)}{x-y}$ can be viewed as a polynomial in variable y only. Therefore, for any given linear functional L on $\mathbb{R}[y]$, the expression

$$q(x) = L\left[\frac{r(x)-r(y)}{x-y}\right]$$

defines a polynomial in variable x. Hence, it is natural to give the following definition.

Definition 4.12. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials associated with a linear functional $L : \mathbb{R}[y] \to \mathbb{R}$. Then for $x \neq y$ and $n \geq 1$,

$$q_n(x) = L\left[\frac{p_n(x)-p_n(y)}{x-y}\right] \tag{4.17}$$

is a polynomial of degree $n-1$ in variable x. We also define $q_0(x) = 0$. The polynomials $\{q_n\}_{n=0}^{\infty}$ will be referred to as the polynomials of the second kind.

Proposition 4.13. [1, p. 8] The sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ defined in Equation 4.17 satisfies the three term recurrence relation as in Equation 4.6.

Proof. Let $\{p_n\}_{n=0}^{\infty}$ be orthogonal polynomials that satisfy the three term recurrence relation as in Equation 4.6. Then for $x \neq y$,

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x) \tag{4.18}$$

and

$$yp_n(y) = b_n p_{n+1}(y) + a_n p_n(y) + b_{n-1} p_{n-1}(y). \tag{4.19}$$

Subtracting Equation 4.19 from Equation 4.18, we get

$$xp_n(x) - yp_n(y) = b_n[p_{n+1}(x) - p_{n+1}(y)] + a_n[p_n(x) - p_n(y)] + b_{n-1}[p_{n-1}(x) - p_{n-1}(y)].$$

Dividing both sides of the above equation by $x - y$ yields

$$\frac{xp_n(x) - yp_n(y)}{x - y} = b_n \frac{p_{n+1}(x) - p_{n+1}(y)}{x - y} + a_n \frac{p_n(x) - p_n(y)}{x - y} + b_{n-1} \frac{p_{n-1}(x) - p_{n-1}(y)}{x - y}.$$

Therefore,

$$b_n \frac{p_{n+1}(x) - p_{n+1}(y)}{x - y} + a_n \frac{p_n(x) - p_n(y)}{x - y} + b_{n-1} \frac{p_{n-1}(x) - p_{n-1}(y)}{x - y} = x \frac{p_n(x) - p_n(y)}{x - y} + p_n(y).$$

If $n \geq 1$, by applying the linear functional L and using the fact that $L(p_n) = \langle p_n, 1 \rangle = 0$, we obtain

$$b_n L\left[\frac{p_{n+1}(x) - p_{n+1}(y)}{x - y}\right] + a_n L\left[\frac{p_n(x) - p_n(y)}{x - y}\right] + b_{n-1} L\left[\frac{p_{n-1}(x) - p_{n-1}(y)}{x - y}\right]$$
$$= xL\left[\frac{p_n(x) - p_n(y)}{x - y}\right].$$

Using Definition 4.12, we obtain that

$$xq_n(x) = b_n q_{n+1}(x) + a_n q_n(x) + b_{n-1} q_{n-1}(x), \ n \geq 1,$$

which shows that the sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ satisfies the same recurrence relation discussed in Theorem 4.8. □

Corollary 4.14. [1, p. 9] Let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be the sequences of polynomials of the first and second kind respectively. Then for any $x \in \mathbb{R}$,

$$p_n(x)q_{n+1}(x) - p_{n+1}(x)q_n(x) = \frac{1}{b_n} \tag{4.20}$$

where

$$b_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad n \geq 1.$$

Proof. Using Equation 4.10, we have that

$$b_n p_n(y)\left[\frac{p_{n+1}(x) - p_{n+1}(y)}{x-y}\right] - b_n p_{n+1}(y)\left[\frac{p_n(x) - p_n(y)}{x-y}\right] = \sum_{k=0}^{n} p_k(x)p_k(y).$$

Applying the linear functional L with respect to the variable y, we get

$$b_n p_n(y)L\left[\frac{p_{n+1}(x) - p_{n+1}(y)}{x-y}\right] - b_n p_{n+1}(y)L\left[\frac{p_n(x) - p_n(y)}{x-y}\right] = L\left[\sum_{k=0}^{n} p_k(x)p_k(y)\right].$$

Thus,

$$b_n p_n(y)q_{n+1}(x) - b_n p_{n+1}(y)q_n(x) = \sum_{k=0}^{n} p_k(x)L\left[p_k(y)\right].$$

By simplifying the above terms, we get that

$$b_n p_n(y)q_{n+1}(x) - b_n p_{n+1}(y)q_n(x) = 1.$$

Taking the limit of both sides of the above equation as $y \to x$ gives

$$\lim_{y\to x}\left[p_n(y)q_{n+1}(x) - p_{n+1}(y)q_n(x)\right] = \frac{1}{b_n}.$$

Hence,

$$p_n(x)q_{n+1}(x) - p_{n+1}(x)q_n(x) = \frac{1}{b_n}, \quad n \geq 1.$$

□

Theorem 4.15. [4, p. 85] Any two zeros of the polynomial of the first kind p_n are separated by a zero of the polynomial of the second kind q_n.

Proof. Recall that

$$p_n(x)q_{n+1}(x) - p_{n+1}(x)q_n(x) = \frac{1}{b_n}.$$

Suppose that x_k is a zero of the polynomial $p_n(x)$. Then

$$-p_{n+1}(x_k)q_n(x_k) = \frac{1}{b_n},$$

so $q_n(x_k)$ has the opposite sign of $p_{n+1}(x_k)$. Suppose that x_a, x_b are two consecutive zeros of the polynomial $p_n(x)$, we know from Theorem 4.11 that $p_{n+1}(x_a)$ and $p_{n+1}(x_b)$ have opposite signs by the interlacing property of zeros. Therefore, $q_n(x_a)$ and $q_n(x_b)$ would have opposite signs which implies that $q_n(x)$ has a zero in between (x_a, x_b). □

We would like to recall the definition of the Lagrange interpolating polynomial. This interpolating polynomial will play a key role in finding a solution to the Hamburger moment problem.

Definition 4.16. Let $x_1, x_2, x_3, \cdots, x_n$ be a set of n different real numbers arranged in increasing order

$$x_1 < x_2 < \cdots < x_n$$

and let f be a function such that $y_i = f(x_i)$. The Lagrange interpolation polynomial corresponding to $x_1, x_2, x_3, \cdots, x_n$ and $y_1, y_2, y_3, \cdots, y_n$ denoted by Q_{n-1} provides a solution to the problem of constructing a polynomial of degree at most $n-1$ with respect to the points (x_i, y_i) for $1 \leq i \leq n$. This polynomial is given by

$$Q_{n-1}(x) = \sum_{i=1}^{n} \frac{p(x)y_i}{p^{'}(x_i)(x - x_i)},$$

where

$$p(x) = (x - x_1)(x - x_2)(x - x_3)\cdots(x - x_n)$$

and $p^{'}(x_i)$ denotes the derivative of $p(x)$ at x_i.

Remark 4.17. If we examine part of the summand of the Lagrange polynomial defined above such that

$$l_k(x) = \frac{p(x)}{(x - x_k)p'(x_k)}. \tag{4.21}$$

Then,

$$l_j(x_k) = \delta_{jk}$$

and

$$Q_{n-1}(x_k) = y_k.$$

Also, observe that for two polynomials $H_n(x)$ and $G_n(x)$ of degree $n-1$ such that for $i = 1, \cdots, n$,

$$H_n(x_i) = G_n(x_i),$$

then the polynomial

$$H_n(x) - G_n(x)$$

would have the property that

$$H_n(x_i) - G_n(x_i) = 0.$$

Therefore, $H_n(x) - G_n(x)$ has n zeros whereas the degree of this polynomial is $n-1$. Since a polynomial of degree $n-1$ can only have $n-1$ zeros unless it is identically zero. Then, we have that $H_n(x) - G_n(x) = 0$ which implies that $H_n(x) = G_n(x)$. Thus, the Lagrange polynomial Q_{n-1} is in fact unique by definition.

Chapter 5

MAIN RESULTS

In this chapter, we will give detailed proofs of our three main questions namely: Theorem 3.5, Theorem 3.7 and Theorem 3.11. Before we begin presenting our proofs, we would like to emphasize again that that proofs given in this thesis are not new. We remind us that finding a solution to the truncated moment problem is the first important part in proving Theorem 3.5. Thus, we begin with the definition of a truncated moment problem.

Definition 5.1. Let a positive semi-definite sequence $\{s_n\}_{n=0}^{\infty}$ be given. The truncated moment problem of order $2n-1$ consists of finding a Borel measure μ on $\mathbb{R}$ satisfying

$$s_k = \int_{\mathbb{R}} x^k d\mu(x) \quad \text{for} \quad 0 \leq k \leq 2n-1.$$

5.1 Hamburger Moment Problem

We shall use the zeros of the orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ associated to a positive definite sequence constructed in the previous chapter of this thesis to provide a solution to the Hamburger moment problem. Note that for the proof of the converse part of Theorem 3.5, i.e. the part to show the existence of a Borel measure, we start with the positive definite assumption, and at the end of the proof, we shall consider the remaining part of the positive semi-definite condition of the Hamburger moment problem.

Proof of Theorem 3.5. We will now prove the first part of the Hamburger moment problem. Suppose that μ is a Borel measure representing the linear functional L. Then we have that

$$L(f^2) = \int_{\mathbb{R}} \left[\sum_{i=0}^{n} a_i x^i \right]^2 d\mu \geq 0$$

for non-negative polynomial

$$f = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x].$$

By Theorem 3.3, it follows that L is positive semi-definite.

Conversely, we consider the linear functional L to be positive definite and let $\{s_n\}_{n=0}^{\infty}$ be the corresponding positive definite real sequence (See Remark 3.2). We first will solve the truncated moment problem. In chapter 4, we constructed a sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ associated with a positive definite sequence $\{s_n\}_{n=0}^{\infty}$ and in Proposition 4.10, we proved that $p_n(x)$ has real and simple zeros. Denote by $x_{n,1}, x_{n,2}, \cdots, x_{n,n}$ the zeros of $p_n(x)$ such that $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$. Let R_{2n-1} be an arbitrary polynomial of degree at most $2n-1$ and construct the Lagrange interpolation polynomial Q_{n-1} which corresponds to the zeros $x_{n,k}$ and $y_k = R_{2n-1}(x_{n,k})$ for $1 \leq k \leq n$. Using Definition 4.16, we have

$$Q_{n-1}(x) = \sum_{k=1}^{n} \frac{p_n(x)}{(x - x_{n,k})p_n'(x_{n,k})} R_{2n-1}(x_{n,k}).$$

Thus the polynomial

$$H(x) = R_{2n-1}(x) - Q_{n-1}(x)$$

is a polynomial of degree at most $2n-1$ which vanishes at the zeros $x_{n,k}$ for $1 \leq k \leq n$. That is,

$$H(x) = \mathcal{R}(x)p_n(x)$$

where $\mathcal{R}(x)$ is a polynomial of degree at most $n-1$. Thus, the polynomial $R_{2n-1}(x)$ can be written as

$$R_{2n-1}(x) = p_n(x)\mathcal{R}(x) + Q_{n-1}(x) \tag{5.1}$$

where $Q_{n-1}(x)$ is the Lagrange polynomial. Rewriting the polynomial $R_{2n-1}(x)$ yields

$$R_{2n-1}(x) = p_n(x)\mathcal{R}(x) + \sum_{k=1}^{n} \frac{p_n(x)}{(x - x_{n,k})p_n'(x_{n,k})} R_{2n-1}(x_{n,k}). \tag{5.2}$$

By applying the linear functional L on both sides of Equation 5.2, we obtain that

$$L(R_{2n-1}(x)) = L(p_n(x)\mathcal{R}(x)) + \sum_{k=1}^{n} L\left[\frac{p_n(x)}{x - x_{n,k}}\right] \frac{R_{2n-1}(x_{n,k})}{p_n'(x_{n,k})}.$$

Since the degree of $\mathcal{R}(x)$ is at most $n-1$, then by the orthogonality relation proved in 4.3,

$$L(R_{2n-1}(x)) = \sum_{k=1}^{n} L\left[\frac{p_n(x)}{x - x_{n,k}}\right] \frac{R_{2n-1}(x_{n,k})}{p_n'(x_{n,k})}.$$

Since $x_{n,k}$ is a zero of the polynomial $p_n(x)$, then

$$L(R_{2n-1}(x)) = \sum_{k=1}^{n} L\left[\frac{p_n(x) - p_n(x_{n,k})}{x - x_{n,k}}\right] \frac{R_{2n-1}(x_{n,k})}{p_n'(x_{n,k})}.$$

Using Definition 4.12, we rewrite the above identity to obtain

$$L(R_{2n-1}(x)) = \sum_{k=1}^{n} \frac{q_n(x_{n,k})}{p_n'(x_{n,k})} R_{2n-1}(x_{n,k}). \tag{5.3}$$

Let $a_{nk} = \dfrac{q_n(x_{n,k})}{p_n'(x_{n,k})}$. Then Equation 5.3 becomes

$$L(R_{2n-1}(x)) = \sum_{k=1}^{n} a_{nk} R_{2n-1}(x_{n,k}). \tag{5.4}$$

Note that for $k = 1, 2, \cdots, n$, the numbers a_{nk} are all positive. Indeed, by Equation 4.21, if we insert

$$R_{2n-1}(x) = l_m^2(x) = \left[\frac{p_n(x)}{(x - x_{n,m})p_n'(x_{n,m})}\right]^2$$

into Equation 5.4, and we obtain that

$$L(l_m^2(x)) = \sum_{k=1}^{n} a_{nk} l_m^2(x_{n,k}) = \sum_{k=1}^{n} a_{nk}\delta_{km} = a_{nm},$$

so that $a_{nm} > 0$ since L is positive definite. Also, if we choose $R_{2n-1} = 1$, then Equation 5.4 becomes

$$L(1) = \sum_{k=1}^{n} a_{nk} = 1.$$

Moreover, Equation 5.4 can be expressed as

$$L(x^i) = \sum_{k=1}^{n} a_{nk} x_{n,k}^i \quad \text{for} \quad 0 \le i \le 2n-1. \tag{5.5}$$

We now need to provide a connection between the finite sum in Equation 5.5 and our desired integral representation. To do this, we make use of the positive numbers a_{nk} and define a piecewise function μ_n such that

$$\mu_n(x) = \begin{cases} 0 & \text{if } x < x_{n,1} \\ a_{n1} + a_{n2} + a_{n3} + \cdots + a_{nk} & \text{if } x_{n,k} \leq x < x_{n,k+1},\ 1 \leq k < n-1 \\ 1 & \text{if } x \geq x_{n,n}. \end{cases}$$

Observe that the function μ_n defined above has three properties. First, it is increasing on $\mathbb{R}$ for all x_{ni}, i.e., for real zeros x_{n1}, x_{n2} such that $x_{n1} < x_{n2}$, it follows that

$$\mu_1(x) = a_{n1} < a_{n1} + a_{n2} = \mu_2(x).$$

Secondly, the function μ_n is right continuous at all zeros $x_{n1} < x_{n2} < \cdots < x_{nn}$ since

$$\lim_{x \to x_{n,k}^+} \mu_n(x) = \sum_{k=1}^{n} a_{nk} = \mu_n(x_{n,k})$$

for $1 \leq k \leq n$. Lastly, μ_n is uniformly bounded because

$$|\mu_n(x)| \leq 1$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Thus the function μ_n that we have constructed is a increasing, right continuous and bounded function. Since the function μ_n is discontinuous at zeros $x_{n,k}$, then the jumps at $x_{n,k}$ are precisely a_{nk}'s. Therefore, we have that

$$a_{nk} = \mu_n(x_{n,k}^+) - \mu_n(x_{n,k}^-) > 0, \quad k = 1, 2, \cdots, n. \tag{5.6}$$

Substituting Equation 5.6 into Equation 5.5, we get

$$L(x^i) = s_i = \sum_{k=1}^{n} \left[\mu_n(x_{n,k}^+) - \mu_n(x_{n,k}^-)\right] x_{n,k}^j, \quad \text{for} \quad 0 \leq j \leq 2n-1. \tag{5.7}$$

Therefore, Equation 5.7 takes on the form

$$L(x^i) = s_i = \int_{\mathbb{R}} x^i d\mu_n(x) \quad \text{for} \quad 0 \leq i \leq 2n-1. \tag{5.8}$$

where μ_n denotes our solution to the truncated moment problem of order $2n-1$. From now on, we will use distribution function and measure freely so that we avoid ambiguity (See Remark 2.16).

Helly's first and second theorems will play a role in extending our solution from the truncated moment problem to the full moment problem. These two theorems will also be used to prove our final argument for the positive semi-definite case. We shall now extend the solution μ_n to the complete moment problem. Take the sequence $\{\mu_n\}$ of uniformly bounded sequence of non-decreasing functions that we constructed. Then by Helly's first theorem (Theorem 2.19), the sequence $\{\mu_n\}$ contains a subsequence $\{\mu_{n_j}\}$ which converges to a bounded, non-decreasing function μ. In other words,

$$\lim_{j\to\infty} \mu_{n_j}(x) = \mu(x).$$

From Equation 5.8, we can write that

$$L(x^i) = s_i = \int_{\mathbb{R}} x^i d\mu_{n_j}(x),\ n_j \geq \frac{i+1}{2}.$$

Using Helly's second theorem (Theorem 2.20), we can then say that for a real interval $[A, B]$ that

$$\int_A^B x^i d\mu(x) = \lim_{j\to\infty} \int_A^B x^i d\mu_{nj}(x).$$

Assume that $A < 0$, $B > 0$ and $n_j > i + 1$. Then

$$\int_{-\infty}^{\infty} x^k d\mu_{nj}(x) = \int_A^B x^k d\mu_{nj}(x) + \int_{-\infty}^A x^k d\mu_{nj}(x) + \int_B^{\infty} x^k d\mu_{nj}(x)$$

and

$$\left| s_i - \int_A^B x^i d\mu(x) \right| = \left| \int_{\mathbb{R}} x^i d\mu_{n_j}(x) - \int_A^B x^i d\mu(x) \right|.$$

This implies that

$$\left| s_i - \int_A^B x^i d\mu(x) \right| = \left| \int_A^B x^i d\mu_{nj}(x) + \int_{-\infty}^A x^i d\mu_{nj}(x) + \int_B^{\infty} x^i d\mu_{nj}(x) - \int_A^B x^i d\mu(x) \right|$$

$$\leq \left| \int_{-\infty}^A x^i d\mu_{nj}(x) \right| + \left| \int_B^{\infty} x^i d\mu_{nj}(x) \right| + \left| \int_A^B x^i d\mu_{nj}(x) - \int_A^B x^i d\mu(x) \right|.$$

But

$$\left| \int_B^{\infty} x^i d\mu_{nj}(x) \right| = \left| \int_B^{\infty} \frac{x^{2i+2}}{x^{i+2}} d\mu_{nj}(x) \right| \leq \int_B^{\infty} \left| \frac{x^{2i+2}}{x^{i+2}} \right| d\mu_{nj}(x) \leq \frac{1}{B^{i+2}} \left| \int_B^{\infty} x^{2i+2} d\mu_{nj}(x) \right|$$

$$\leq \frac{1}{B^{i+2}} s_{2i+2}.$$

Similarly,

$$\left|\int_{-\infty}^{A} x^i d\mu_{nj}(x)\right| = \left|\int_{-\infty}^{A} \frac{x^{2i+2}}{x^{i+2}} d\mu_{nj}(x)\right| \leq \frac{1}{|A|^{i+2}} \left|\int_{-\infty}^{A} x^{2i+2} d\mu_{nj}(x)\right| \leq \left|\frac{1}{A^{i+2}}\right| s_{2i+2}.$$

Therefore

$$\left|s_i - \int_A^B x^i d\mu(x)\right| \leq \left[\frac{1}{B^{i+2}} + \left|\frac{1}{A^{i+2}}\right|\right] s_{2i+2} + \left|\int_A^B x^i d\mu_{nj}(x) - \int_A^B x^i d\mu(x)\right|.$$

Define

$$\epsilon = \left[\frac{1}{B^{i+2}} + \left|\frac{1}{A^{i+2}}\right|\right] s_{2i+2}$$

and taking the limit as $j \to \infty$, then

$$\left|s_i - \int_A^B x^i d\mu(x)\right| \leq \epsilon. \tag{5.9}$$

Letting $A \to -\infty$ and $B \to \infty$, we obtain that as $\epsilon \to 0$,

$$L(x^i) = s_i = \int_{-\infty}^{\infty} x^i d\mu(x), \quad i \geq 0.$$

We will now prove our final argument for the positive semi-definite case. Let $L : \mathbb{R}[x] \to \mathbb{R}$ be a given positive semi-definite linear functional. Fix L^* to be some positive definite linear functional with $L^*(1) = 1$. For $\epsilon > 0$, define

$$L_\epsilon = L + \epsilon L^*.$$

Observe that since $\epsilon > 0$ and L^* is positive definite, then L_ϵ is positive definite. Let μ_ϵ be a corresponding distribution function on $\mathbb{R}$. Choose $\epsilon = \frac{1}{n}$. By applying Helly's first and second theorems on the sequence $\left\{\mu_{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$, we get the corresponding limiting function μ. Since

$$L_\epsilon = L + \frac{1}{n}L^* \to L$$

as $n \to \infty$, μ is the corresponding measure representing the positive semi-definite linear functional L. $\square$

Part (c) of the following result will be used in proving Theorem 3.7. We will only prove the case (c) implies (d) of Theorem 5.2 because the proof of the remaining parts require more technical tools that are beyond the scope of this thesis. We therefore refer the reader to Akhiezer [1] and Berg [4] for more details.

Theorem 5.2. [4, Theorem 2.7.13] Let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be the sequences of orthogonal polynomials of the first kind and second kind respectively (See Definition 4.6 and Definition 4.12). Then the following conditions are equivalent:

(a.) The moment problem relative to the linear functional $L : \mathbb{R}[x] \to \mathbb{R}$ is indeterminate.

(b.) There exists a point $x_0 \in \mathbb{R}$ such that $\sum_{n\in\mathbb{N}} \left(p_n^2(x_0) + q_n^2(x_0)\right) < \infty$.

(c.) There exists a point $z_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $\sum_{n\in\mathbb{N}} |p_n(z_0)^2| < \infty$ and $\sum_{n\in\mathbb{N}} |q_n(z_0)^2| < \infty$.

(d.) The series $\sum_{n\in\mathbb{N}} |q_n(z)^2|$ and $\sum_{n\in\mathbb{N}} |p_n(z)^2|$ converge uniformly on open disks.

Proof. Suppose there exist a point $y \in \mathbb{C}\backslash\mathbb{R}$ such that $\sum_{n\in\mathbb{N}} |p_n(y)^2| < \infty$ and $\sum_{n\in\mathbb{N}} |q_n(y)^2| < \infty$. We want to show that the series $\sum_{n\in\mathbb{N}} |p_n(x)^2|$ converges uniformly on open disks. Let f be a polynomial of degree n, then there are complex constants a_k such that

$$f(x) = \sum_{k=0}^{n} a_k p_k(x). \tag{5.10}$$

Observe that if we multiply Equation 5.10 by $p_m(x)$ such that $0 \leq m \leq n$ and applying the linear functional L, we obtain that

$$L\left(f(x)p_n(x)\right) = a_m L\left(p_m(x)^2\right).$$

Thus

$$a_n = L\left(f(x)p_n(x)\right). \tag{5.11}$$

Recall that for $n \geq 1$, the quotient

$$\frac{p_n(x) - p_n(y)}{x - y}$$

is a polynomial of degree $n-1$ in variable x. Thus, it can be written as

$$\frac{p_n(x) - p_n(y)}{x - y} = \sum_{k=0}^{n-1} a_{nk} p_k(x). \tag{5.12}$$

Simplifying the above equation, we have that

$$p_n(x) - p_n(y) = (x - y) \sum_{k=0}^{n-1} a_{nk} p_k(x). \tag{5.13}$$

Using Equation 5.11, it follows that

$$a_{nk} = L\left(\frac{p_n(x) - p_n(y)}{x - y} p_k(x)\right).$$

Note that the coefficients a_{nk} can be calculated as follows. Observe that

$$a_{nk} = L\left(\frac{p_n(x) - p_n(y)}{x - y} p_k(x)\right)$$

$$= L\left(\frac{p_n(x) - p_n(y)}{x - y} p_k(x) + \frac{p_n(x) - p_n(y)}{x - y} p_k(y) - \frac{p_n(x) - p_n(y)}{x - y} p_k(y)\right).$$

By simplifying the above equation, we obtain

$$a_{nk} = L\left(\frac{p_n(x) - p_n(y)}{x - y} p_k(x)\right)$$

$$= L\left(\frac{p_n(x) - p_n(y)}{x - y} p_k(x) + \frac{p_n(x) - p_n(y)}{x - y} p_k(y) - \frac{p_n(x) - p_n(y)}{x - y} p_k(y)\right)$$

$$= p_k(y) L\left(\frac{p_n(x) - p_n(y)}{x - y}\right) + L\left(\frac{p_k(x) - p_k(y)}{x - y}\left[p_n(x) - p_n(y)\right]\right).$$

Thus, using Definition 4.12, it follows that

$$a_{nk} = p_k(y) q_n(y) - q_k(y) p_n(y). \tag{5.14}$$

In the above calculation, we have used the fact that $p_n(x)$ is orthogonal to the polynomial $\dfrac{p_k(x) - p_y(y)}{x - y}$ of degree less than n. Using the fact that

$$(a + b)^2 \leq 2(a^2 + b^2)$$

and

$$(a - b)^2 \leq 2(a^2 + b^2),$$

Equation 5.14 becomes

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |a_{nk}|^2 \leq 2 \sum_{n=1}^{\infty} |q_n(y)|^2 \sum_{k=0}^{n-1} |p_k(y)|^2 + 2 \sum_{n=1}^{\infty} |p_n(y)|^2 \sum_{k=0}^{n-1} |q_k(y)|^2$$

$$< 4 \sum_{n=0}^{\infty} |q_n(y)|^2 \sum_{k=0}^{\infty} |p_k(y)|^2 < \infty.$$

Let $\epsilon > 0$ and let $R > 0$. We shall now show that the series $\sum_{k=0}^{\infty} |p_n(x)|^2 < \infty$ converges uniformly for $|x-y| < R$. Choose $n_0 = n_0(\epsilon, R)$, so that

$$\sum_{n=n_0}^{\infty} |p_n(x)|^2 \leq \frac{\epsilon}{2}$$

and

$$\sum_{n=n_0}^{\infty} \sum_{k=0}^{\infty} |a_{nk}|^2 \leq \frac{\epsilon}{2R^2}.$$

Note that for $N \geq n_0$ and $|x-y| < R$, we then get that by Equation 5.13

$$\sum_{n=n_0}^{N} |p_n(x)|^2 \leq 2 \sum_{n=n_0}^{N} |p_n(y)|^2 + 2|x-y|^2 \sum_{n=n_0}^{N} \left| \sum_{k=0}^{n-1} a_{nk} p_k(x) \right|^2 \tag{5.15}$$

$$\leq \epsilon + 2R^2 \sum_{n=n_0}^{N} \left(\sum_{k=0}^{n-1} |a_{nk}|^2 \sum_{k=0}^{n-1} |p_k(x)|^2 \right) \leq \epsilon + \epsilon \sum_{k=0}^{N} |p_k(x)|^2.$$

Hence,

$$(1-\epsilon) \sum_{n=n_0}^{N} |p_n(x)|^2 \leq \epsilon + \epsilon \sum_{k=0}^{n_0-1} |p_k(x)|^2. \tag{5.16}$$

We now choose $\epsilon = \frac{1}{2}$ and for $N \geq n_1 = n_0(\frac{1}{2}, R)$ and $|x-y| < R$, we get that

$$\sum_{n=n_1}^{N} |p_n(x)|^2 \leq 1 + \sum_{k=0}^{n_1-1} |p_k(x)|^2. \tag{5.17}$$

This shows that the series $\sum_{n=0}^{\infty} |p_n(x)|^2$ is pointwise convergent for $|x-y| < R$. We will now prove uniform convergence. Define

$$K = \sup \left\{ \sum_{k=0}^{n_1-1} |p_k(x)|^2 : |x-y| < R \right\}$$

and by continuity of the orthogonal polynomials, we have that $K < \infty$. This show that

$$\sum_{n=0}^{\infty} |p_n(x)|^2 = \sum_{n=0}^{n_1-1} |p_n(x)|^2 + \sum_{n=n_1}^{\infty} |p_n(x)|^2 \leq 1 + 2K \tag{5.18}$$

for $|x-y| < R$. From Equation 5.16, we obtain that for $N \geq n_0$, $|x-y| < R$,

$$\sum_{n=n_0}^{N} |p_n(x)|^2 \leq \frac{\epsilon}{1-\epsilon} \left(1 + \sum_{k=0}^{n_0-1} |p_n(x)|^2 \right) \leq \frac{\epsilon}{1-\epsilon}(2+2K), \tag{5.19}$$

which shows that $\sum_{n=0}^{\infty} |p_n(x)|^2$ converges uniformly for $|x-y| < R$.

Also, the series $\sum_{n=0}^{\infty} |q_n(x)|^2$ has the same property and can be proved analogously using the formula

$$q_n(x) = q_n(y) + (x-y)\sum_{k=0}^{n-1} a_{nk}q_k(x).$$

□

Theorem 5.3. [4, Theorem 2.9.2] Let b_n be as defined in Theorem 4.8. If $\sum_{n=1}^{\infty} b_n = \infty$, then the moment problem relative to a linear functional L is determinate.

Proof. Using Equation 4.20, we have that for any $x \in \mathbb{R}$,

$$\frac{1}{b_n} = p_n(x)q_{n+1}(x) - p_{n+1}(x)q_n(x).$$

In particular if $x = 0$, we still have

$$\frac{1}{b_n} = p_n(0)q_{n+1}(0) - p_{n+1}(0)q_n(0).$$

Recall that for real numbers a, b, c, d, it is true that

$$a^2 + b^2 + c^2 + d^2 \geq 2ab + 2cd.$$

Thus, it follows that

$$\frac{1}{b_n} = p_n(0)q_{n+1}(0) - p_{n+1}(0)q_n(0) \leq \frac{1}{2}\left(p_n^2(0) + p_{n+1}^2(0) + q_{n+1}^2(0) + q_n^2(0)\right). \tag{5.20}$$

Hence, by summing the above terms for $n \geq 0$, we get

$$\sum_{n=0}^{\infty} \frac{1}{b_n} \leq \sum_{n=0}^{\infty} \left(p_n^2(0) + q_n^2(0)\right),$$

and our desired result follows from Theorem 5.2. □

Lemma 5.4. Let μ be a finite measure. Then for $p \geq 1$, we have that $\mathbb{C}[x] \subseteq L^p(\mu)$.

Proof. Let M_n be the space of all polynomials of degree at most n and let $\{1, x, x^2, \cdots, x^n\}$ be its usual basis. By the Gram-Schimdt orthogonalization processs, we recall from Remark 4.7 that the span$\{p_0, p_1, p_2, \cdots, p_n\}$ = span$\{1, x, x^2, \cdots, x^n\}$ where $p_0, p_1, p_2, \cdots, p_n$ are orthogonal polynomials of the first kind. Observe that

$$1 = L(p_n^2) = \int_{\mathbb{R}} |p_n^2(x)| d\mu(x).$$

Thus the polynomial $p_n \in L^2(\mu) \subseteq L^1(\mu)$ since

$$\int_{\mathbb{R}} |p_n^2(x)| d\mu(x) < \infty.$$

Since the space of all polynomials with complex coefficients $\mathbb{C}[x]$ is generated by p_n, then it follows that $\mathbb{C}[x] \subseteq L^1(\mu)$. Let $m \in \mathbb{N}$ and let $f \in \mathbb{C}[x]$. Then $f^m \in \mathbb{C}[x]$. This implies that the polynomial $f^m \in L^1(\mu)$. By Definition 3.8, it follows that

$$\int_{\mathbb{R}} |f^m(x)| d\mu(x) < \infty. \tag{5.21}$$

Re-writing the above equation, we have that

$$\int_{\mathbb{R}} |f(x)|^m d\mu(x) < \infty. \tag{5.22}$$

Thus, by Equation 5.22, we have that $f \in L^m(\mu)$. Hence, $\mathbb{C}[x] \subseteq L^m(\mu)$. For $1 \leq p < \infty$, let $m = [p]$ such that $m \leq p < m + 1$. Therefore, by Remark 3.9, we have that

$$\mathbb{C}[x] \subseteq L^{m+1} \subseteq L^p(\mu).$$

□

Theorem 5.5. [1, p. 43-45] Let μ be a solution of a determinate moment problem. Then the set of all polynomials with complex coefficients $\mathbb{C}[x]$ is dense in $L^2(\mu)$.

Proof. Suppose that μ is a solution to the determinate moment problem. Before we begin with the proof of this theorem, we need to introduce some extra tools that were not mentioned earlier in this thesis. Let μ be a Borel measure on $\mathbb{R}$. For any $y \in \mathbb{C} \backslash \mathbb{R}$, the Stieltjes transform of μ is defined by

$$w_\mu : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}, \quad w_\mu : y \mapsto \int_{\mathbb{R}} \frac{1}{x-y} d\mu(x).$$

Also, for $f \in L^2(\mu)$, the Fourier coefficients with respect to the orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ is defined by

$$c_k = \int_{\mathbb{R}} f(x) p_k(x) d\mu(x), \quad k \geq 0.$$

For $y \in \mathbb{C} \setminus \mathbb{R}$, we define the function

$$f_k(x) = \frac{1}{(x-y)^{k+1}}, \quad k \geq 0.$$

Since

$$M = \inf\{|x-y| : x \in \mathbb{R}\} > 0,$$

then

$$f_k \leq \frac{1}{M^{k+1}}$$

so that $f_k \in L^2(\mu)$ as μ is a finite measure. In particular, for $k = 0$, the function

$$f_0(x) = \frac{1}{(x-y)} \in L^2(\mu)$$

and we find that

$$\begin{aligned} c_k = \int_{\mathbb{R}} \frac{1}{(x-y)} p_k(x) d\mu(x) &= \int_{\mathbb{R}} \frac{p_k(x) + p_k(y) - p_k(y)}{(x-y)} d\mu(x) \\ &= \int_{\mathbb{R}} \frac{p_k(x) - p_k(y)}{(x-y)} d\mu(x) + p_k(y) \int_{\mathbb{R}} \frac{1}{(x-y)} d\mu(x). \end{aligned}$$

Putting

$$w_\mu(y) = \int_{\mathbb{R}} \frac{1}{x-y} d\mu(x)$$

in the above equation and using Definition 4.12, we have that

$$\int_{\mathbb{R}} \frac{1}{(x-y)} p_k(x) d\mu(x) = q_k(y) + w_\mu(y) p_k(y). \tag{5.23}$$

Also,

$$\int_{\mathbb{R}} \left| \frac{1}{x-y} \right|^2 d\mu(x) = \frac{1}{y-\overline{y}} \int_{\mathbb{R}} \left[\frac{1}{(x-y)} - \frac{1}{(x-\overline{y})} \right] d\mu(x)$$

$$\int_{\mathbb{R}} \left| \frac{1}{x-y} \right|^2 d\mu(x) = \frac{w_\mu(y) - \overline{w_\mu(y)}}{y-\overline{y}}. \tag{5.24}$$

Using the definition of the determinacy of moment problem in [1, Definition 2.3.3, Theorem 2.3.3], there exists a $y_0 \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\frac{w_\mu(y_0) - \overline{w_\mu(y_0)}}{y_0 - \overline{y_0}} = \sum_{k=0}^{\infty} |q_k(y_0) + w_\mu(y_0) p_k(y_0)|^2 . \tag{5.25}$$

Observe that from Equation 5.23, we have that

$$\sum_{k=0}^{\infty} |q_k(y_0) + w_\mu(y_0) p_k(y_0)|^2 = \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}} \frac{1}{x-y_0} p_k(x) d\mu(x) \right|^2 .$$

Using Equation 5.24 and Equation 5.25, we have

$$\sum_{k=0}^{\infty} |q_k(y_0) + w_\mu(y_0)p_k(y_0)|^2 = \frac{w_\mu(y_0) - \overline{w_\mu(y)}}{y_0 - \overline{y_0}} = \left[\int_{\mathbb{R}} \left|\frac{1}{x-y_0}\right|^2 d\mu(x)\right],$$

which implies that

$$\sum_{k=0}^{\infty} |q_k(y_0) + w_\mu(y_0)p_k(y_0)|^2 = \int_{\mathbb{R}} \left|\frac{1}{x-y_0}\right|^2 d\mu(x).$$

Therefore, by the Parseval equality, the function f_0 can be approximated in $L^2(\mu)$ by $\sum_{k=0}^{n} c_k p_k$, where $c_k = q_k(y_0) + w_\mu(y_0)p_k(y_0)$ (See [1, p. 38]) for more details). Note that the same reason also applies to the conjugate function $\overline{f_0(x)}$. We will now prove by induction that for $y_0 \in \mathbb{C} \setminus \mathbb{R}$, the function

$$f_k(x) = \frac{1}{(x-y_0)^{k+1}}$$

can be approximated in $L^2(\mu)$ to any degree of accuracy by a polynomial for all $k \geq 0$. Note that for a given $\epsilon > 0$, there exists a $P \in \mathbb{C}[x]$ such that

$$||f_k(x) - P||_2 \leq \epsilon\,|N|$$

where $N = \text{Im } y_0$. Dividing the polynomial P by $x - y_0$, we get

$$P(x) = (x-y_0)Q(x) + a$$

where Q is another polynomial and $a \in \mathbb{C}$. Observe that

$$||f_{k+1}(x) - af_0(x) - Q(x)|\,|_2^2 = \int_{\mathbb{R}} \left|\frac{1}{(x-y_0)^{k+2}} - \frac{a}{x-y_0} - Q(x)\right|^2 d\mu(x)$$

$$= \int_{\mathbb{R}} \left|\frac{1}{(x-y_0)^2}\right| \left|\frac{1}{(x-y_0)^k} - a - (x-y_0)Q(x)\right|^2 d\mu(x).$$

Thus,

$$||f_{k+1}(x) - af_0(x) - Q(x)||_2^2 \leq \frac{1}{|N|^2}||f_k(x) - P||_2 \leq \epsilon^2.$$

Note that similar argument also works for the conjugate of $f_{k+1}(x)$. We will now argue by contrapositivity. Suppose that $\mathbb{C}[x]$ is not dense in $L^2(\mu)$. This means that there exists a linear functional ϕ_0 in $L^2(\mu)$ which is zero on $\mathbb{C}[x]$ but not identically zero. In other words, by the Riesz representation theorem ([29], Theorem 6.16), there exists a unique $g \in L^2(\mu)$ such that the functional ϕ_0 has the form

$$\phi_0(f) = \int_{\mathbb{R}} f(x)g(x)d\mu(x)$$

for every $f \in L^2(\mu)$ and $||\phi_0|| = ||g||_2$. Since ϕ_0 is not identically zero, we have that

$$\int_{\mathbb{R}} |g(x)|^2 d\mu \neq 0$$

but

$$\int_{\mathbb{R}} g(x)x^n d\mu = 0, \quad k \geq 0$$

by our assumption. Using the fact that the functions $f_{k+1}(x)$ and its conjugate $\overline{f_{k+1}}(x)$ can be arbitrarily approximated by a polynomial in $L^2(\mu)$, it follows that

$$\int_{\mathbb{R}} g(x)f_{k+1}(x)d\mu = \int_{\mathbb{R}} \frac{g(x)}{(x-y_0)^{k+1}}(x)d\mu = 0, \quad \forall k \geq 0.$$

Similarly,

$$\int_{\mathbb{R}} g(x)\overline{f_{k+1}}(x)d\mu = \int_{\mathbb{R}} \frac{g(x)}{(x-\overline{y_0})^{k+1}}(x)d\mu = 0, \quad \forall k \geq 0.$$

Consider the Stieltjes transform

$$w_{g\mu}(z) = \int_{\mathbb{R}} \frac{g(x)}{x-z} d\mu(x)$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. We see that the k-th derivative denoted by

$$w_{g\mu}(z)^{(k)} = k! \int_{\mathbb{R}} \frac{g(x)}{(x-z)^{k+1}} d\mu(x) = 0$$

at the point $z = y_0$ for all $k \geq 0$. Since the Stieltjes transform $w_\mu(z)$ is analytic at all points z in the upper half plane of $\mathbb{C} \setminus \mathbb{R}$ and that $w_{g\mu}(z)^{(k)} = 0$ at the point $z = y_0$ for $k \geq 0$. Then it follows that $w_{g\mu}(z)$ is identically zero. By the Perron-Stieltjes inversion formula [4, Theorem 2.7.10], we have that $g = 0$, μ-almost everywhere which contradicts the fact that

$$\int_{\mathbb{R}} |g(x)|^2 d\mu \neq 0.$$

Therefore, $\mathbb{C}[x]$ is indeed dense in $L^2(\mu)$ if μ is determinate. $\square$

Theorem 5.6. [25, Corollary 3.3] For $1 \leq p < \infty$, the following statements are equivalent:

(a) $\mathbb{C}[x]$ is dense in $L^p(\mu)$.

(b) There exists a sequence of polynomial $\{g_n\}_{n=0}^{\infty} \in \mathbb{C}[x]$ such that $g_n(i) = 1$ and

$$\left\| g_n(x) - \frac{1}{x-i} \right\|_p \to 0$$

as $n \to \infty$.

(c) There exists a sequence of polynomial $\{f_n\}_{n=0}^{\infty} \in \mathbb{C}[x]$ such that $f_n(i) = 1$ and

$$\left\| \frac{f_n(x)}{x-i} \right\|_p \to 0$$

as $n \to \infty$.

Proof. The proof of this result was given by Marshall [25] using the application of localization to the multivariate moment problem. For more details, see [25, Corollary 3.3]. □

5.2 Determinacy of Moment problem

Proof of Theorem 3.7. Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials in $\mathbb{C}[x]$ such that $f_n(i) = 1$ for all $n \geq 1$ and $L(|f_n|^2) \to 0$ as $n \to \infty$. By Remark 4.7, we write the polynomial $f_n(x)$ of degree n as a linear combination of orthogonal polynomials. In other words, f_n can be represented in the form

$$f_n(x) = \sum_{k=0}^{n} a_k p_k(x),$$

where $a_k \in \mathbb{C}$ and $p_k(x)$ are the orthogonal polynomials of the first kind. Thus, we have that

$$f_n(i) = \sum_{k=0}^{n} a_k p_k(i) = 1$$

where i is a complex number. Simplifying the above term and applying the linear functional L, we derive that

$$L(|f_n|^2) = \sum_{k=0}^{n} |a_k|^2. \tag{5.26}$$

Since, $f_n(i) = 1$, we use the Cauchy inequality on the complex sequences $\{a_k\}_{k=0}^{\infty}$ and $\{p_k(i)\}_{k=0}^{\infty}$ and obtain that

$$1 = \left| \sum_{k=0}^{n} a_k p_k(i) \right|^2 \leq \left(\sum_{k=0}^{n} |a_k|^2 \right) \left(\sum_{k=0}^{n} |p_k(i)|^2 \right) = L(|f_n|^2) \left(\sum_{k=0}^{n} |p_k(i)|^2 \right) \tag{5.27}$$

Define

$$\left(\sum_{k=0}^{n} |p_k(i)|^2\right) = K_n(i).$$

Thus, we have that

$$L(|f_n|^2) \geq \frac{1}{K_n(i)}.$$

In the case of equality in Equation 5.27, which is only attained only if the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{p_k(i)\}_{k=0}^{\infty}$ are proportional, i.e., when

$$a_k = \lambda\overline{p_k(i)},$$

for $0 \leq k \leq n$, we find that

$$\begin{aligned} 1 &= f_n(i) \\ &= \sum_{k=0}^{n} a_k p_k(i) \\ &= \sum_{k=0}^{n} \lambda\overline{p_k(i)}p_k(i) \\ &= \lambda\sum_{k=0}^{n} |p_k(i)|^2 . \end{aligned}$$

Thus,

$$\lambda = \frac{1}{\sum_{k=0}^{n} |p_k(i)|^2} = \frac{1}{K_n(i)}$$

and Equation 5.26 changes to

$$L(|f_n|^2) \geq \frac{1}{\sum_{k=0}^{n} |p_k(i)|^2}. \tag{5.28}$$

Taking the limit as $n \to \infty$ of the above equation, we obtain that

$$\sum_{k=0}^{\infty} |p_k(i)|^2 \geq \frac{1}{\lim_{n\to\infty} L(|f_n|^2)}. \tag{5.29}$$

Since $\lim_{n\to\infty} L(|f_n|^2) = 0$, then it follows that

$$\sum_{k=0}^{\infty} |p_k(i)|^2 = \infty. \tag{5.30}$$

Therefore, by Theorem 5.2, the moment problem relative to a linear functional L is determinate.

Conversely, assume that the moment problem relative to the linear functional L is determinate. Then by Theorem 5.5, we have that $\mathbb{C}[x]$ is dense in $L^2(\mu)$. Thus, by Theorem 5.6, there exists a sequence of polynomial $\{f_n\}_{n=0}^{\infty} \in \mathbb{C}[x]$ such that $f_n(i) = 1$ and $\left|\left|\frac{f_n}{x-i}\right|\right|_2 \to 0$ as $n \to \infty$. Since we assumed that our moment problem relative to the linear functional L is determinate, then the measure μ must have a compact support (See [4, Theorem 2.1.7], [18, Corollary 1.38]). Observe that we can write $|f_n|^2$ as

$$|f_n^2(x)| = \left|\frac{f_n(x)}{(x-i)}\right|^2 \times |x-i|^2 .$$

Hence,

$$L(|f_n|^2) = \int_{\mathbb{R}} |f_n(x)|^2 d\mu(x) = \int_{\mathbb{R}} \left|\frac{f_n(x)}{x-i}\right|^2 \times |x-i|^2 \, d\mu(x).$$

Since the measure μ has compact support, then it follows that the $|x-i|^2$ is bounded on the support of μ because it is a continuous function. Since

$$|x-i|^2 \leq K$$

we have that

$$L(|f_n^2|) \leq K \int_{\mathbb{R}} \left|\frac{f_n(x)}{(x-i)}\right|^2 d\mu(x)$$

where

$$K = \sup\left\{|x-i|^2 : x \in \text{ support of } \mu\right\}.$$

Hence, our assumption that

$$\left(\int_{\mathbb{R}} \left|\frac{f_n(x)}{x-i}\right|^2 d\mu(x)\right) \to 0$$

implies that $L(|f_n|^2) \to 0$ as $n \to \infty$ and our desired result follows. □

5.3 Density of Polynomials

This section is aimed to briefly explain the role played by the theory of quasi-analytic functions in the study of the determinacy of the moment problem. In particular, the quasi-analytic criterion is crucial in determining when a measure μ is unique. We begin this section by defining log-convex sequences and explain how they are associated to the concept of quasi-analyticity. Also, we state the Denjoy-Carleman theorem, an important result that explains how the Carleman's condition is connected to the theory of quasi-analytic classes. Finally, we shall use the Denjoy-Carleman theorem to prove that the set of polynomials with complex coefficients is dense in $L^p(\mu)$ for any $p \geq 1$, when the Carleman's condition is satisfied.

Definition 5.7. A sequence of positive real numbers $\{m_n\}_{n=0}^{\infty}$ is said to be log-convex if and only if $m_n^2 \leq m_{n-1}m_{n+1}$ for all $n \in \mathbb{N}$.

Definition 5.8. Given a log-convex sequence of real numbers $\{m_n\}_{n=0}^{\infty}$, we define the class $C\{m_n\}$ as the set of all functions that are infinitely differentiable such that $\left|f^{(n)}(x)\right| \leq KA^n m_n$ for $x \in \mathbb{R}, n \geq 0$ where $K > 0, A > 0$ depends on f, and $f^{(n)}$ denotes the n-th derivative of f.

Definition 5.9. A function $f \in C\{m_n\}$ is quasi-analytic if the condition $f^{(n)}(0) = 0, n \geq 0$ implies that f is identically zero on $\mathbb{R}$.

Theorem 5.10. [19, Theorem 1.7] (**Denjoy-Carleman**). The class $C\{m_n\}$ is quasi-analytic if and only if $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[n]{m_n}} = \infty$ for all $n \in \mathbb{N}$.

We will now present the proof of the Carleman's condition, a direct consequence of Denjoy-Carleman theorem. Before proving the Carleman's condition, we need to present the proof of the Carleman's inequality because it is useful.

Lemma 5.11. [1, p. 86] (**Carleman's Inequality**) Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers. Then

$$\sum_{n=1}^{\infty} \sqrt[n]{(u_1 u_2 \cdots u_n)} \leq \mathbf{e} \sum_{n=1}^{\infty} u_n.$$

Proof. We define the numbers $x_1, x_2, \cdots, x_n, \cdots$, by the equation

$$x_n = n\left(1 + \frac{1}{n}\right)^n = n\left(\frac{1+n}{n}\right)^n = \frac{(n+1)^n}{n^{n-1}} \quad n \geq 1$$

so that

$$x_n < \mathbf{e} \cdot n$$

and

$$(x_1 \cdot x_2 \cdot x_3 \cdots x_n) = (n+1)^n.$$

Observe that

$$\sqrt[n]{(u_1 u_2 \cdots u_n)} = \sqrt[n]{(u_1, u_2 \cdots u_n)\,\frac{(n+1)^n}{(n+1)^n}}$$

$$= \frac{1}{n+1}\sqrt[n]{(u_1, u_2 \cdots u_n)\,(n+1)^n}$$

$$= \frac{1}{n+1}\sqrt[n]{(u_1, u_2 \cdots u_n)\,(x_1 x_2 x_3 \cdots x_n)}$$

$$= \frac{1}{n+1}\sqrt[n]{(x_1 u_1) \cdots (x_n u_n)}.$$

Using the arithmetic-geometric mean inequality we obtain that

$$\sqrt[n]{(u_1 u_2 \cdots u_n)} \leq \frac{1}{n(n+1)}\left[u_1 x_1 + u_2 x_2 + \cdots + u_n x_1\right].$$

Thus, it follows that

$$\sum_{n=1}^{\infty} \sqrt[n]{(u_1, u_2 \cdots u_n)} \leq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} u_k x_k = \sum_{k=1}^{\infty} u_k x_k \sum_{n=k}^{\infty} \frac{1}{n(n+1)}. \tag{5.31}$$

Note that

$$\sum_{n=k}^{\infty} \frac{1}{n(n+1)} = \sum_{n=k}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{k}.$$

Observe that Equation 5.31 becomes

$$\sum_{n=1}^{\infty} \sqrt[n]{(u_1, u_2 \cdots u_n)} \leq \sum_{k=1}^{\infty} \frac{u_k x_k}{k} < \mathbf{e} \sum_{k=1}^{\infty} u_k$$

Hence,

$$\sum_{n=1}^{\infty} \sqrt[n]{(u_1, u_2 \cdots u_n)} \leq \mathbf{e} \sum_{k=1}^{\infty} u_k.$$

□

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Proof of Theorem 3.10. It follows from the three term recurrence relation (Equation 4.6) that

$$p_n(x) = \frac{1}{(b_0 b_1 \cdots b_{n-1})} x^n + R_{n-1}(x),$$

where $R_{n-1}(x)$ is a polynomial of degree $n-1$. This can be re-written as

$$(b_0 b_1 \cdots b_{n-1}) p_n(x) = x^n + Q_{n-1}(x) \tag{5.32}$$

where $Q_{n-1}(x)$ is a polynomial of degree $n-1$. Multiplying both sides of Equation 5.32 by a polynomial $p_n(x)$ and passing integral with respect to a measure μ yields

$$\int_{-\infty}^{\infty} (b_0 b_1 \cdots b_{n-1}) p_n(x) p_n(x) d\mu(x) = \int_{-\infty}^{\infty} x^n p_n(x) d\mu(x) + \int_{-\infty}^{\infty} Q_{n-1}(x) p_n(x) d\mu(x).$$

By our orthogonality property, we know that

$$\int_{-\infty}^{\infty} Q_{n-1}(x) p_n(x) d\mu(x) = 0,$$

so

$$\int_{-\infty}^{\infty} (b_0 b_1 \cdots b_{n-1}) p_n(x) p_n(x) d\mu(x) = \int_{-\infty}^{\infty} x^n p_n(x) d\mu(x). \tag{5.33}$$

By Cauchy-Schwartz inequality for integrals, Equation 5.33 becomes

$$\int_{-\infty}^{\infty} (b_0 b_1 \cdots b_{n-1}) p_n(x) p_n(x) d\mu(x) = \int_{-\infty}^{\infty} x^n p_n(x) d\mu(x)$$

$$\leq \sqrt{\int_{-\infty}^{\infty} x^{2n} d\mu} \cdot \left(\int_{-\infty}^{\infty} p_n(x)^2 d\mu(x) \right)^{\frac{1}{2}}.$$

But

$$\left(\int_{-\infty}^{\infty} p_n(x)^2 d\mu(x) \right)^{\frac{1}{2}} = 1.$$

This implies that

$$(b_0 b_1 \cdots b_{n-1}) \int_{-\infty}^{\infty} p_n(x) p_n(x) d\mu(x) \leq \sqrt{\int_{-\infty}^{\infty} x^{2n} d\mu}.$$

Hence,

$$(b_0 b_1 \cdots b_{n-1}) \leq \sqrt{s_{2n}}. \tag{5.34}$$

Rewriting Equation 5.34, we get

$$\sqrt[n]{(b_0 b_1 \cdots b_{n-1})} \leq \sqrt[2n]{s_{2n}}.$$

Therefore, by Carleman inequality, we obtain that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} \leq \sum_{n=1}^{\infty} \sqrt[n]{\frac{1}{b_0 b_1 \cdots b_{n-1}}} \leq \mathbf{e} \sum_{n=1}^{\infty} \frac{1}{b_n}.$$

Thus, it follows that if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{1}{b_n} = \infty,$$

and by Theorem 5.3, the moment problem is determinate. □

Proof of Theorem 3.11. A subspace F of a vector space $L^p(\mu)$ is dense in $L^p(\mu)$ if and only if every continuous functional ϕ on $L^p(\mu)$ vanishing on the subspace F must be identically zero. For $1 \leq p < \infty$, it is known that the dual space of $L^p(\mu)$ can be identified with $L^q(\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. By the Riesz representation theorem ([29], Theorem 6.16), we have that any continuous linear functional ϕ on $L^p(\mu)$ has the form

$$\phi(f) = \int_{\mathbb{R}} f(x) g(x) d\mu(x), \ f \in L^p(\mu),$$

for a uniquely determined $g \in L^q(\mu)$. Since the subspace F in our case is $\mathbb{C}[x]$, then ϕ vanishes on $\mathbb{C}[x]$ precisely if

$$\phi(x^n) = \int_{\mathbb{R}} x^n g(x) d\mu(x) = 0, \quad n \geq 0. \tag{5.35}$$

Therefore, showing that $g = 0$ μ-almost everywhere yields our desired result.

Let μ be a probability measure so that $s_0 = 1$. Recall that for $n \geq 1$, we define $s_n = \int_{\mathbb{R}} x^n d\mu(x)$. Thus,

$$\sqrt[np]{s_{np}} = \left(\int_{\mathbb{R}} |x^{np}| d\mu(x) \right)^{\frac{1}{np}} = \left(\int_{\mathbb{R}} |x|^{np} d\mu(x) \right)^{\frac{1}{np}} = ||x||_{np}.$$

By Remark 3.9 it follows that the sequence $\{\sqrt[np]{s_{np}}\}$ is increasing for $n \geq 1$. Using the definition of Fourier transform in [11], the Fourier transform of the measure $g(x)d\mu(x)$ is defined as

$$\mathcal{F} := \mathcal{F}(w) = \int_{\mathbb{R}} e^{-iwx} g(x) d\mu(x), \quad w \in \mathbb{R}.$$

The importance of the Fourier transform in this proof is its uniqueness property in determining an intregable function. That is, if function $g \in L^1(\mu)$ and

$$\mathcal{F}(w) = \int_{\mathbb{R}} e^{-iwx} g(x) d\mu(x) = 0, \forall w \in \mathbb{R},$$

then $g = 0$ almost everywhere [11]. Note that the Fourier transform $\mathcal{F}$ is infinitely differentiable with

$$\mathcal{F}^{(n)}(w) = \int_{\mathbb{R}} (-ix)^n e^{-iwx} g(x) d\mu(x) = i^n \int_{\mathbb{R}} (-x)^n e^{-iwx} g(x) d\mu(x). \tag{5.36}$$

Observe that

$$\mathcal{F}^{(n)}(0) = i^n \int_{\mathbb{R}} (-x)^n e^{-iwx} g(x) d\mu(x),$$

and by Equation 5.35, we have that $\mathcal{F}^{(n)}(0) = 0$. Using the well known Hölder's inequality on Equation 5.36, we obtain that

$$\left|\mathcal{F}^{(n)}(w)\right| \leq \left(\int_{\mathbb{R}} |x^{np}| d\mu\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g^q(x)| d\mu\right)^{\frac{1}{q}}.$$

Thus,

$$\left|\mathcal{F}^{(n)}(w)\right| \leq \sqrt[p]{s_{np}} ||g||_q. \tag{5.37}$$

Denoting $m_n = \sqrt[p]{s_{np}}$ and applying the Cauchy-Schwartz inequality on m_n gives

$$m_n^2 = (\sqrt[p]{s_{np}})^2 = \left(\int_{\mathbb{R}} x^{np} d\mu(x)\right)^2 = \left(\int_{\mathbb{R}} x^{\frac{np+p+np-p}{2}} d\mu(x)\right)^2$$

$$\leq \left(\int_{\mathbb{R}} x^{np+p} d\mu(x)\right) \left(\int_{\mathbb{R}} x^{np-p} d\mu(x)\right) = m_{n+1} m_{n-1}.$$

This implies that

$$m_n^2 \leq m_{n+1} m_{n-1}.$$

Therefore, the sequence $m_n = \sqrt[p]{s_{np}}$ is log-convex. Since Equation 5.37 satisfies Definition 5.8, then $\mathcal{F} \in \mathcal{C}\{m_n\}$. We claim that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{m_n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[np]{s_{np}}} = \infty, \tag{5.38}$$

so the class $\mathcal{C}\{m_n\}$ is quasi-analytic. Hence, we have that $\mathcal{F} \equiv 0$ which implies that $g = 0$ μ-almost everywhere. For simplicity, let

$$a_n = (s_{np})^{\frac{-1}{np}} = \frac{1}{\sqrt[np]{s_{np}}}$$

and let $p = 2r$. Since $\{\sqrt[np]{s_{np}}\}_{n=1}^{\infty}$ is increasing then we have that $a_1 \geq a_2 \geq . \cdots \geq a_n \geq \cdots$. By splitting the sum of the sequence a_n into r sums with the terms

$$a_{nr}, a_{nr+1}, \cdots, a_{nr+r-2}, a_{nr+r-1}$$

gives

$$\infty = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \sum_{k=0}^{r-1} a_{nr+k} \leq \sum_{k=0}^{r} a_k + r \sum_{n=1}^{\infty} a_{nr}. \tag{5.39}$$

Hence, we can say that

$$\sum_{n=1}^{\infty} a_{nr} = \infty.$$

However

$$\infty = \sum_{n=1}^{\infty} a_{nr} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[2nr]{(s_{2nr})}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[np]{(s_{np})}} \tag{5.40}$$

which proves our claim in Equation 5.38.

□

5.4 Summary and Future Research

Our main focus in this thesis was to give a detailed solution to the Hamburger moment problem. We began by deriving two kinds of polynomials namely: the orthogonal polynomials of the first and second kind. Then we showed that these two kinds of polynomials have simple zeros and their zeros have interlacing property. It turned out that the existence of a solution to the truncated moment problem follows from Lagrange interpolation at the zeros of the orthogonal polynomials of the first kind. Then we extended this solution to the full moment problem using Helly's first and second theorems. After a solution was found to the Hamburger moment problem, the determinacy of moment problem was discussed. Finally, we proved the density of polynomials with complex coefficients under the assumption that the Carleman's condition is satisfied. Also, researchers who are interested in studying moment problem in one dimension can refer to this thesis to have a better understanding of the subject, as relevant results in the standard reference [1] have been simplified and explained here.

The one dimensional moment problem is an old problem that has been thoroughly investigated by various authors but the multivariate moment problem is a more recent question. Let $\mathbb{R}[X] = \mathbb{R}[x_1, x_2, \cdots, x_n]$ denote the polynomial ring in n variables $x_1, x_2, \cdots, x_n$. The multivariate moment problem states that given a positive semi-definite linear functional $L : \mathbb{R}[X] \to \mathbb{R}$, there exist a positive Borel measure μ on $\mathbb{R}^n$ such that

$$L(f) = \int f d\mu, \forall f \in \mathbb{R}[X].$$

The multivariate moment problem has been solved but there appears to be few articles dealing with the moment problem in infinitely many variables. The general case was examined and proved in [14]. Moment problem in infinitely many variables has also been investigated in [2]. The specific case in which the linear functional L is continuous with respect to a sub-multiplicative semi-norm has been treated in [13]. There are interesting open problems and questions in the infinite dimensional moment problem. We will briefly mention few open problems that were selected from [20].

Question 5.12. [20, Question 2.6] Given a topological unital commutative R-algebra (A, τ) and a $2d$-power module M of A, what are the weakest assumptions on an M-positive linear functional such that it admits a X_M-representing measure?

Question 5.13. [20, Question 2.7] Is there any criterion for the non-negativity of a linear functional on an arbitrary $2d$-power module with $d \geq 2$?

We implore interested readers to check [20] for more open questions on infinite dimensional moment problem.

References

[1] N. Akhiezer, *The classical moment problem and some related questions in analysis*, Hafner, New York, (1965).

[2] D. Alpay, P. E. Jorgensen, and D. P. Kimsey, *Moment problems in an infinite number of variables*, Infinite Dimensional Analysis and Quantum Probability, 18 (2015), no. 4, 1550024, 14 pp.

[3] C. Berg, *The multidimensional moment problem and semigroups*, Proc. Symp. Appl. Math, Vol. 37, (1987), pp. 110–124.

[4] ——, *Moment problems and orthogonal polynomials*, Lecture Notes, Department of Mathematics, University of Copenhagen, (1994).

[5] ——, *Moment problems and polynomial approximation*, Annales de la Faculté des Sciences de Toulouse: Mathématiques, Vol. 5, (1996), pp. 9–32.

[6] C. Berg and J. Christensen, *Exposants critiques dans le problème des moments*, C. R. Acad. Sci. Paris, 296 (1983), pp. 661–663.

[7] T. M. Bisgaard and Z. Sasvári, *Characteristic functions and Moment sequences: Positive Definiteness in Probability*, Nova Publishers, (2000).

[8] T. Carleman, *Sur le problème des moments*, C. R. Acad. Sci. Paris, 174 (1922), pp. 1680–1682.

[9] T. Carleman, *Les fonctions quasi analytiques*, The work of D.E. Men'shov in the theory of analytic functions, 99 (1926).

[10] T. S. Chihara, *An introduction to orthogonal polynomials*, Courier Corporation, (2011).

[11] G. B. Folland, *Real analysis: Modern techniques and their applications*, John Wiley & Sons, (2013).

[12] S. H. Friedberg, A. J. Insel, and L. E. Spence, *Linear algebra*, Pearson Higher Ed., (2014).

[13] M. Ghasemi, M. Infusino, S. Kuhlmann, and M. Marshall, *Moment problem for symmetric algebras of locally convex spaces*, arXiv:1507.06781, (2015).

[14] M. Ghasemi, S. Kuhlmann, and M. Marshall, *Moment problem in infinitely many variables*, Israel Journal of Mathematics, 212 (2016), pp. 989–1012.

[15] H. Hamburger, *Über eine Erweiterung des stieltjesschen momentenproblems*, Mathematische Annalen, 81 (1920), pp. 235–319.

[16] F. Hausdorff, *Summationsmethoden und Momentfolgen. I*, Mathematische Zeitschrift, 9 (1921), pp. 74–109.

[17] E. Helly, *Über mengen konvexer körper mit gemeinschaftlichen punkte.*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 32 (1923), pp. 175–176.

[18] A. Hora and N. Obata, *Quantum probability and orthogonal polynomials*, Quantum Probability and Spectral Analysis of Graphs, (2007), pp. 1–63.

[19] M. Infusino, *Quasi-analyticity and determinacy of the full moment problem from finite to infinite dimensions*, Stochastic and Infinite Dimensional Analysis, Springer, (2016), pp. 161–194.

[20] M. Infusino and S. Kuhlmann, *Infinite dimensional moment problem: open questions and applications*, to appear in AMS Contemporary Mathematics (CONM); Proceedings of the Conference on Ordered Algebraic Structures and Related Topics, (2017).

[21] T. H. Kjeldsen, *The early history of the moment problem*, Historia Mathematica, 20 (1993), pp. 19–44.

[22] H. Landau, *The classical moment problem: Hilbertian proofs*, Journal of Functional Analysis, 38 (1980), pp. 255–272.

[23] ——, *Classical background of the moment problem*, Moments in Mathematics, (1987), pp. 1–15.

[24] M. Marshall, *Positive polynomials and sums of squares*, No. 146, American Mathematical Soc., 2008.

[25] ——, *Application of localization to the multivariate moment problem*, Math. Scand, 115 (2014), pp. 269–286.

[26] D. S. Mitrinovic and P. M. Vasic, *Analytic inequalities*, Vol. 1, Springer, (1970).

[27] T. S. Motzkin, *The arithmetic-geometric inequality*, Proc. Sympos., pp. 205–224.

[28] M. Riesz, *Sur le problème des moments.*, Ark. Mat. Fys, 16 (1923), pp. 1–52.

[29] W. Rudin, *Real and complex analysis*, Tata McGraw-Hill Education, 1987.

[30] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Vol. 265, Springer Science & Amp; Business Media, (2012).

[31] J. A. Shohat and J. D. Tamarkin, *The problem of moments*, No. 1, American Mathematical Soc., (1943).

[32] O. Knill, *On Hausdorff's moment problem in higher dimensions*, Technical report, Department of Mathematics, Harvard University, (1997).

[33] T.-J. Stieltjes, *Recherches sur les fractions continues*, in Annales de la Faculté des sciences de Toulouse: Mathématiques, Vol. 8, (1894), pp. 1–22.

[34] G. Szego, *Orthogonal polynomials*, Amer. Math. Soc, Providence, R. I., (1959), pp. 100–284.