Fixed Point Theorems, Coincidence Point Theorems and Their Applications

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Abstract

This study aims to illuminate a general framework for fixed point and coincidence point theorems. Our theorems work with functions defined on ball spaces $(X, \mathcal{B})$. This notion provides the minimal structure that is needed to express the basic assumptions which are used in the proofs of such theorems when they are concerned with functions that are contractive in some way. We present a general fixed point theorem which can be seen as the underlying principle of proof for fixed point theorems of Banach and of Prieß-Crampe and Ribenboim. Also we study two types of general coincidence point theorems and their applications to metric spaces (Theorem due to K. Goebel) and ultametric spaces (Theorem due to Prieß-Crampe and Ribenboim). Further, we find an alternative approach to coincidence point theorems. We introduce a general B_x theorem which does not deal with obtaining a coincidence point for two functions f, g directly, but allows a variety of applications. Then we present two coincidence point theorems as its applications. Finally, we introduce three different coincidence point theorems for ultrametric spaces. These theorems are: a special case of one of the general B_x theorem's applications, a coincidence point theorem due to Prieß-Crampe and Ribenboim, and an ultrametric version of Goebel's theorem. We study the logical relation between these theorems.

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In the Name of Allah the Beneficent the Merciful

Dedicated to

my husband *Mohamed*, my sweethearts *Ahmed* and *Arwa*,

and my parents *Abdalla* and *Soad*

Contents

Chapter 1

Introduction

Coincidence point theorems concern two functions f, g from a set X into another set Y that, under certain conditions, admit a coincidence point. A *coincidence point* is an element $x \in X$ such that its images under the functions f, g are the same; in other words, $fx = gx$. Fixed point theorems consider one function f from a set X into itself and give conditions for the existence of a *fixed point*, that is, an element $x \in X$ such that $fx = x$. Fixed point theorems can be considered as special cases of coincidence point theorems where $X = Y$ and the second function g is the identity.

The existence of a fixed point of a function defined on some space X expresses some kind of completeness of this space. The classical Banach Fixed Point Theorem requires the completeness of the metric space on which the contractive function is defined. A similar result for ultrametric spaces was obtained by S. Prieß-Crampe (see [9]) and then generalized by S. Prieß-Crampe and P. Ribenboim (see [10], and [11]). In this case the required completeness property is called *spherical completeness*. The classical Brouwer Fixed Point Theorem requires compactness of a topological space, which is equivalent to an analogue of spherical completeness, now applied to the collection of all nonempty closed subsets of this space.

Recently Franz-Viktor Kuhlmann and Katarzyna Kuhlmann [4], [5], [6], and [7] have developed a general framework for fixed point theorems which work with contractive functions. This general framework considers functions defined on ball spaces that are spherically complete.

The main focus of this thesis is to extract a general principle of proof that works with fixed point theorems and coincidence point theorems. In this thesis, I include some of the work done by Franz-Viktor Kuhlmann and Katarzyna Kuhlmann [4], [5], [6], and [7] in Chapter 4, and apply the general fixed point theorems to metric and ultrametric spaces.

The second and third chapters are devoted to definitions, tools, and background material that are needed in the later chapters. We introduce some concepts about ordered sets in Chapter 2. Also, we discuss metric, ultrametric, and topological spaces. Chapter 3 consists of two sections. In the first section, we introduce ball spaces and their classification, and we take a closer look at the properties of functions on ball spaces in the second section. Throughout these two chapters, we give examples of the concepts for more explanation.

In the first section of Chapter 4, we introduce a general fixed point theorem that works with functions f that are defined on spherically complete ball spaces, and the balls B in the space are f-closed, i.e., $f(B) \subseteq B$. We give the proof of this theorem and use it to find a fixed point for a function that is defined on a ball space whose elements are f-contracting (see Definition 3.2.2). After that we apply the general fixed point theorem to metric spaces (Banach's Theorem), and ultametric spaces (Theorem of Prieß-Crampe and Ribenboim). In the second section, we present a B_x-type fixed point theorem. In this theorem, a function on a ball space is needed to be self-contractive (see Definition 3.2.5) in order to have a fixed point on the space. As in the first section, we give the application of the B_x-type fixed point theorem to metric and ultrametric spaces to prove the theorems of Banach and of Prieß-Crampe and Ribenboim.

Chapter 5 deals with obtaining a coincidence point for two functions. We present two types of general coincidence point theorems. We apply the General Coincidence Point Theorem I to metric spaces to prove a theorem of K. Goebel, and the General Coincidence Point Theorem II to ultrametric spaces to prove a theorem due to Prieß-Crampe and Ribenboim (cf. [12]).

In the last chapter, we introduce an alternative approach to coincidence point theorems. We work with functions that take every element in a set X to a ball B_x in a ball space $(Z, \mathcal{B})$. We first prove a general theorem that is not itself a coincidence point theorem, but very flexible in its applications. The key idea is to use an arbitrary assertion $P(x)$ on the elements $x \in X$. In Section 6.2 we present two basic applications of this general theorem, considering functions $f, g : X \to Y$. In the first application, we take X to be a ball space and use a function from X to its balls, taking the assertion $P(x)$ to say that $f(B_x) \cap g(B_x) \neq \emptyset$. In the second application, we take Y to be a ball space and use a function from X to the balls in Y, taking the assertion $P(x)$ to say that $fx, gx \in B_x$. In Section 6.3, we introduce another B_x-type coincidence point theorem for ultrametric spaces which is a special case of our last application, and we compare it with the coincidence point theorem of Prieß-Crampe and Ribenboim and the ultrametric version of Goebel's theorem.

Chapter 2

Metric, Ultrametric, and Topological Spaces

In this chapter we will introduce the necessary concepts in metric, ultrametric, and topological spaces that are needed for the following chapters. Our aim is to give information as needed without going into too much detail. For more details about these spaces, we refer the reader to [2], [8], [13].

In our work, we will deal with ordered sets, in particular sets that are totally ordered.

Definition 2.0.1 (Ordered set)**.** *Let S be a set and $\preceq$ a relation on S. The pair $(S, \preceq)$ is called a **partially ordered set** if the following conditions hold:*

1) Reflexivity: $a \preceq a$ for all $a \in S$.

2) Antisymmetry: $a \preceq b$ and $b \preceq a$ implies $a = b$.

3) Transitivity: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$.

*The pair $(S, \preceq)$ is called a **totally (linearly) ordered set** if it is a partially ordered set and satisfies the additional condition:*

4) Comparability: for any $a, b \in S$, $a \preceq b$ or $b \preceq a$.

Definition 2.0.2. *Take a, b in a (totally or partially) ordered set $(S, \preceq)$ with $a \preceq b$. Then the following subsets are called* **intervals***:*

$(a, b) := \{c \in S \mid a \prec c \prec b\}$, $[a, b] := \{c \in S \mid a \preceq c \preceq b\}$, as well as

$\{c \in S \mid a \preceq c \prec b\}, \{c \in S \mid a \prec c \preceq b\}$, $\{c \in S \mid a \prec c\}, \{c \in S \mid a \preceq c\}$, $\{c \in S \mid c \prec b\}, \{c \in S \mid c \preceq b\}$, and S itself.

Intervals of the form (a, b) are called **open bounded***, and intervals of the form $[a, b]$ are called* **closed bounded***.*

Definition 2.0.3. *An ordered set $(X, <)$ is called:*

*1) a **discretely ordered set** if it satisfies that for all $z \in X$ there exist $x, y \in X$ for which $x < y$ such that $\{z\} = (x, y)$.*

*2) a **densely ordered set** if for all $x, y \in X$ for which $x < y$, there is some $z \in X$ such that $x < z < y$.*

Definition 2.0.4. *Let T be a subset of an ordered set S. Then T is called:*

1. *a **initial segment** if $a \in T$ and $c \prec a$ implies $c \in T$, and*

2. *a **final segment** if $a \in T$ and $a \prec c$ implies $c \in T$.*

Definition 2.0.5. *A* ***cut*** *in a totally ordered set* $(S,<)$ *is a pair* $\mathcal{C}=(D,E)$ *with* D,E *subsets of* S *such that* $D\cap E=\emptyset$, $D\cup E=S$, *and* $d<e$ *for all* $d\in D$, $e\in E$.

Example 2.0.6. *The set* $\mathbb{R}$ *of real numbers with the usual ordering is totally ordered. It is* ***cut complete****, that is, for every cut* (D,E) *with* D *and* E *nonempty, either* D *has a largest or* E *has a smallest element. Therefore, every initial and every final segment is an interval (however, not bounded). But this is not in general true in every ordered set.*

Example 2.0.7. *The rational function field* $\mathbb{R}(X)$ *can be ordered such that* X *becomes a positive infinitesimal and* $1/X$ *is infinitely large. The set of all elements that are larger than every element of* $\mathbb{R}$ *is a final segment, but it is not an interval as it has no infimum.*

Example 2.0.8. *The set of all subsets of a fixed set is partially ordered by inclusion.*

Definition 2.0.9. *A* **nest** *of subsets of a given set is a nonempty collection of subsets which is totally ordered by inclusion.*

Now we will introduce the spaces which our work is based on.

Definition 2.0.10. *Let* X *be a nonempty set. A function* $d: X\times X\to\mathbb{R}$ *is called a* ***metric*** *if the following conditions are satisfied for all* $x,y,z\in X$:

(M1) $d(x,y)=0$ *if and only if* $x=y$.

(M2) $d(x,y)\geq 0$.

(M3) $d(x,y)=d(y,x)$.

(M4) $d(x,z)\leq d(x,y)+d(y,z)$ *(triangle inequality).*

The pair (X,d) is called a ***metric space****.*

Definition 2.0.11. *Let* (X,d) *be a metric space and*

$$r\in\mathbb{R}^{\geq 0}:=\{r\in\mathbb{R}\mid r\geq 0\}.$$

An ***open metric ball*** $B_r^\circ(x)$ *is defined by*

$$B_r^\circ(x):=\{y\in X\mid d(x,y)<r\},$$

and a ***closed metric ball*** $B_r(x)$ *by*

$$B_r(x):=\{y\in X\mid d(x,y)\leq r\}.$$

Definition 2.0.12. *A metric space* (X,d) *is said to be*

1. ***complete*** *if every Cauchy sequence in* X *converges in* X, *and*

2. ***compact*** *if it is compact as a topological space, where the open metric balls are taken as a basis of the topology (see Definition 2.0.20).*

Example 2.0.13. *The set* $\mathbb{R}$ *of real numbers with* $d(a,b) := |a-b|$ *is a metric space.*

Definition 2.0.14. *Let* X *be a nonempty set.* (X,d) *is called an* ***ultrametric space*** *if* $d : X \times X \to \Gamma$, *where* Γ *is a totally ordered set with a minimal element 0, satisfies the following properties:*
(U1) $d(x,y) = 0$ *if and only if* $x = y$,
(U2) $d(x,y) = d(y,x)$, *and*
(U3) $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ *(ultrametric triangle law).*

Remark 2.0.15. *A metric space is an ultrametric space if its metric satisfies condition (U3). An ultrametric space is a metric space if and only if* $\Gamma \subseteq \mathbb{R}^{\geq 0}$.

The balls (open and closed) in an ultrametric space are defined in the same way as the balls in a metric space, but now with radii in Γ:

Definition 2.0.16. *Let* (X,d) *be an ultrametric space and* $\gamma \in \Gamma$.
If $\gamma > 0$, *an* ***open ball*** *in* X *is defined by*

$$B^{\circ}_{\gamma}(x) := \{y \in X \mid d(x,y) < \gamma\},$$

and a ***closed ball*** *in* X *is the set*

$$B_{\gamma}(x) := \{y \in X \mid d(x,y) \leq \gamma\}.$$

We obtain the ultrametric ball space $(X, \mathcal{B})$ by taking $\mathcal{B}$ to be the set of all balls

$$B(x,y) := \{z \in X \mid d(x,z) \leq d(x,y)\} = B_{d(x,y)}(x).$$

This is the smallest ultrametric ball that contains both x and y.

Example 2.0.17. *In every valued field, the valuation induces an underlying ultrametric. For example, take the p-adic valuation* v_p *on* $\mathbb{Q}$. *Then* $d_p(a,b) := p^{-v_p(a-b)}$ *(with* $d_p(a,a) := 0$*) is an ultrametric on* $\mathbb{Q}$. *Its set of values is a subset of* $\mathbb{R}^{\geq 0}$, *so it is also a metric.*
The valuation ring $\mathcal{O}_p := \{a \in \mathbb{Q} \mid v_p(a) \geq 0\}$ *is a closed ultrametric ball (and so are all of its scaled translates* $c\mathcal{O}_p + d$ *for* $c, d \in \mathbb{Q}$*). The valuation ideal* $\mathcal{O}_p := \{a \in \mathbb{Q} \mid v_p(a) > 0\}$ *is an open ultrametric ball (and so are all of its scaled translates* $c\mathcal{O}_p + d$ *for* $c, d \in \mathbb{Q}$ *with* $c \neq 0$*).*

The balls in ultrametric spaces have important properties which we will use to prove some theorems in our paper. These properties can be proved easily from the definition of ultrametric spaces.

properties 2.0.18. *Let* $w, x, y, z \in X$ *and* $\gamma, \delta \in \Gamma$.

1. *Every point inside a ball is its center; i.e, if* $y \in B_{\gamma}(x)$, *then* $B_{\gamma}(x) = B_{\gamma}(y)$.

2. *If* $B_{\gamma}(x) \bigcap B_{\delta}(z) \neq \emptyset$, *then either* $B_{\gamma}(x) \subseteq B_{\delta}(z)$ *or* $B_{\delta}(z) \subseteq B_{\gamma}(x)$. *Moreover, if* $\gamma \leq \delta$, *then* $B_{\gamma}(x) \subseteq B_{\delta}(z)$.

3. *If* $B_\gamma(x) \subsetneqq B_\delta(z)$, *then* $\gamma < \delta$.

4. $B(w,z) \subseteq B(x,y)$ *if and only if* $z \in B(x,y)$ *and* $d(w,z) \le d(x,y)$.

5. *If* $d(x,y) \le d(y,z)$, *then* $B(x,y) \subseteq B(y,z)$.

6. *If* $d(x,y) < d(y,z)$, *then* $z \notin B(x,y)$ *and* $B(x,y) \subsetneqq B(y,z)$.

Proof. 1. Suppose $y \in B_\gamma(x)$. Then $d(x,y) \le \gamma$. Take any $z \in B_\gamma(x)$, so $d(z,x) \le \gamma$. Then we have

$$d(z,y) \le \max\{d(z,x), d(x,y)\} \le \gamma.$$

Therefore $z \in B_\gamma(y)$, and $B_\gamma(x) \subseteq B_\gamma(y)$.

Now take any element $w \in B_\gamma(y)$, so $d(w,y) \le \gamma$. Then

$$d(w,x) \le \max\{d(w,y), d(y,x)\} \le \gamma.$$

Hence $w \in B_\gamma(x)$, and $B_\gamma(y) \subseteq B_\gamma(x)$. So $B_\gamma(x) = B_\gamma(y)$.

2. Suppose that $B_\gamma(x) \bigcap B_\delta(z) \neq \emptyset$. This means there is $y \in B_\gamma(x) \bigcap B_\delta(z)$. So we have $d(y,x) \le \gamma$, and $d(y,z) \le \delta$.

   First suppose that $\gamma \le \delta$, and let $w \in B_\gamma(x)$. Then by definition we have $d(w,x) \le \gamma \le \delta$, and since $d(y,x) \le \gamma \le \delta$, we get $d(y,w) \le \delta$. Also $d(y,z) \le \delta$ and $d(y,w) \le \delta$ imply $d(w,z) \le \delta$. This means that $w \in B_\delta(z)$. Thus $B_\gamma(x) \subseteq B_\delta(z)$.

   Now suppose that $\gamma \ge \delta$, and let $v \in B_\delta(z)$. Then by definition we have $d(v,z) \le \delta \le \gamma$, and since $d(y,z) \le \delta \le \gamma$, we get $d(y,v) \le \gamma$. Also $d(y,x) \le \gamma$ and $d(y,v) \le \gamma$ imply $d(v,x) \le \gamma$ which means that $v \in B_\gamma(x)$. Therefore $B_\delta(z) \subseteq B_\gamma(x)$.

3. Assume that $B_\gamma(x) \subsetneqq B_\delta(z)$. This means that there is $y \in B_\delta(z)$ but $y \notin B_\gamma(x)$. So $d(y,x) > \gamma$. Also $d(y,x) \le \delta$ since $B_\delta(y) = B_\delta(z)$ by property 1. Thus $\gamma < b\delta$.

4. First suppose that $B(w,z) \subseteq B(x,y)$. Then $w, z \in B(x,y)$. We have

$$d(w,z) \le \max\{d(w,x), d(x,z)\} \le d(x,y).$$

   Now suppose that $z \in B(x,y)$ and $d(w,z) \le d(x,y)$. Take an element $v \in B(w,z)$. Then since $z \in B(x,y)$ we have

$$d(v,x) \le \max\{d(v,z), d(z,x)\} \le \max\{d(w,z), d(x,y)\} = d(x,y).$$

   Thus $v \in B(x,y)$, and hence $B(w,z) \subseteq B(x,y)$.

5. Assume that $d(x,y) \le d(y,z)$. Since y is a common member of both balls $B(x,y), B(y,z)$, we can apply property 4, and we get $B(x,y) \subseteq B(y,z)$.

6. Assume that $d(x,y) < d(y,z)$. Then by property 5, $B(x,y) \subseteq B(y,z)$. We have

$$d(x,z) \leq \max\{d(x,y),(y,z)\} = d(y,z) > d(x,y).$$

So $z \notin B(x,y)$. Therefore $B(x,y) \subsetneq B(y,z)$.

□

Definition 2.0.19. *An ultrametric space (X,d) is called **spherically complete** if the intersection of every nest of closed ultramtric balls is nonempty.*

Definition 2.0.20. *Let X be a set and τ a collection of subsets of X satisfying the following properties:*

T1) $\emptyset \in \tau$ *and* $X \in \tau$.

T2) *If* $S_i \in \tau$ *for all* $i \in I$, *then* $\bigcup_{i\in I} S_i \in \tau$.

T3) *If* $S_j \in \tau$ *for* $j \in \{1,...,n\}$, *then* $\bigcap_{j=1}^{n} S_j \in \tau$.

*Then τ is called a **topology** on X and (X,τ) is a **topological space**, and we say that a subset U of X is an **open set** in X if $U \in \tau$.*

Definition 2.0.21. *Let (X,τ) be a topological space. A subset A of X is said to be **closed** if its complement $X\backslash A$ is open.*

Remark 2.0.22. *The collection of all unions of arbitrary sets of open balls in a metric space forms a topology.*

Theorem 2.0.23. *In a topological space, the intersection of any collection of closed sets is closed.*

Proof. Take a topological space (X,τ), and let $X_i \subseteq X$ be a closed set for each $i \in I$. Then by *de Morgan's Law*, we have

$$X\backslash \bigcap_{i\in I} X_i = \bigcup_{i\in I}(X\backslash X_i).$$

Since the sets X_i are closed for all $i \in I$, their complements are open. Also the union of open sets is open, so $A = \bigcup_{i\in I}(X\backslash X_i)$ is an open set. Thus by definition of closed set, $\bigcap_{i\in I} X_i$ is closed. □

Definition 2.0.24. *A topological space (X,τ) is called **compact** if every open covering U of X contains a finite subcollection that also covers X.*

Theorem 2.0.25. *Let X be a compact space. Let $(C_i)_{i\in I}$ be a nest of nonempty closed sets of X. Then $\bigcap_{i\in I} C_i \neq \emptyset$.*

Proof. Assume that $\bigcap_{i\in I} C_i = \emptyset$, and let $U_i = X\backslash C_i$ for all $i \in I$. These sets are open since their complements are closed. Note that the family of open sets $(U_i)_{i\in I}$ covers X since

$$X = X\backslash \bigcap_{i\in I} C_i = \bigcup_{i\in I}(X\backslash C_i) = \bigcup_{i\in I} U_i.$$

Since X is compact, it has a finite subcover. We have that the set $(C_i)_{i\in I} = (X\backslash U_i)_{i\in I}$ is totally ordered by inclusion, so we can extract a finite subcover of X as $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k$. Then U_k must be equal to X. Thus $C_k = X\backslash U_k = \emptyset$, contradicting non-emptiness of the closed sets in the nest $(C_i)_{i\in I}$. Therefore our assumption cannot be true, so $\bigcap_{i\in I} C_i \neq \emptyset$. □

Chapter 3

Ball Spaces

In this chapter we will introduce a generalized completeness property by defining so-called *spherically complete ball spaces*. We will show a classification of ball spaces based on their completeness properties and illustrate this classification by examples. In the second part we will look at the properties of functions defined on ball spaces.

3.1 Definition and classification of ball spaces

Definition 3.1.1. *A* ***ball space*** *is a pair $(X, \mathcal{B})$, where X is a nonempty set and $\mathcal{B}$ is a nonempty collection of nonempty subsets of X. The elements of $\mathcal{B}$ will be called balls.*

Note that we do not require any topology here. Our definition gives us flexibility and we can adapt it to many cases, for example:

- intervals (closed or open) in an ordered set,
- metric (closed or open) balls in a metric space,
- any kind of ultrametric balls in an ultrametric space,
- closed or open subsets of a topological space.

We can also restrict attention to any nonempty subset of a given set of balls. In particular, we will be interested in *nests of balls*.

Definition 3.1.2. *A* ***nest of balls*** *is a nonempty collection of balls which is totally ordered by inclusion.*

Lemma 3.1.3. *Let $(X, \mathcal{B})$ be a ball space. Then the set of all nests which contain a given ball $B_0 \in \mathcal{B}$ has maximal elements.*

Proof. Take $B_0 \in \mathcal{B}$. Let $\mathcal{C}$ be the set of all nests which contain B_0:

$$\mathcal{C} = \{\mathcal{N} \subseteq \mathcal{B} \mid \mathcal{N} \text{ is a nest and } B_0 \in \mathcal{N}\}.$$

This set is partially ordered by inclusion. Take any subset $\mathcal{F}$ of $\mathcal{C}$ such that $(\mathcal{F}, \subseteq)$ is a totally ordered set. Then $\bigcup_{\mathcal{N} \in \mathcal{F}} \mathcal{N}$ is a totally ordered set, so it is a nest containing B_0 since $B_0 \in \mathcal{N}$ for all $\mathcal{N} \in \mathcal{C}$. This nest is an upper bound for the chain $\mathcal{F} \subseteq \mathcal{C}$. Hence by Zorn's Lemma the set $\mathcal{C}$ contains at least one maximal element $\mathcal{N}_0$. $\square$

The basic completeness property which we will introduce now, is borrowed from ultrametric spaces.

Definition 3.1.4. *A ball space $(X, \mathcal{B})$ is called **spherically complete** if the intersection of every nest of balls is nonempty.*

Example 3.1.5. *A spherically complete ultrametric space together with the family of closed ultrametric balls is a spherically complete ball space.*

Example 3.1.6. *By Theorem 2.0.25, a compact topological space with the family of all nonempty closed subsets is a spherically complete ball space.*

Example 3.1.7. *The reals, as an ordered set, with the family of all nonempty, closed and bounded intervals is a spherically complete ball space, even though it is not compact under the derived topology.*

Now we will observe what completeness of a metric space means in the language of spherically complete ball spaces.

Theorem 3.1.8. *Take a metric space (X, d) and a set $S \subseteq \mathbb{R}^+$ which has 0 as its unique limit point. Define a ball space on X as*

$$\mathcal{B}_S := \{B_r(x) \mid x \in X, r \in S\}.$$

Then the metric space (X, d) is complete if and only if the ball space $(X, \mathcal{B}_S)$ is spherically complete.

Proof. First suppose that (X, d) is a complete metric space. Take any nest $\mathcal{N}$ of closed metric balls in $\mathcal{B}_S$. If the nest contains a smallest ball, then its intersection is nonempty; so we assume that is does not. Since S has 0 as its unique limit point, we have that S is discretely ordered, and every infinite descending chain in S can be indexed by the natural numbers. Therefore the nest $\mathcal{N}$ is of the form

$$(B_{r_n}(x_n))_{n\in\mathbb{N}},$$

where $r_n > r_{n+1}$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} r_n = 0.$$

Take $\epsilon > 0$ and $N \in \mathbb{N}$ such that $r_N < \frac{\epsilon}{2}$. Since $(B_{r_n}(x_n))_{n\in\mathbb{N}}$ is a nest, we have that the ball $B_{r_N}(x_N)$ contains x_m, x_n for every $m, n > N$. Therefore, $d(x_m, x_n) \leq 2r_N < \epsilon$. We have shown that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let y be its limit. We have to show that y lies in the intersection of all balls in $\mathcal{N}$.

Take a ball $B_{r_n}(x_n) \in \mathcal{N}$ and suppose that $y \notin B_{r_n}(x_n)$. Then $d(x_n, y) > r_n$, and we set $\varepsilon := d(x_n, y) - r_n > 0$. Since $\lim_{n\to\infty} r_n = 0$, there is $m \in \mathbb{N}$ such that $m > n$ and $d(x_m, y) < \varepsilon$. Since $\mathcal{N}$ is a nest, we have that $x_m \in B_{r_n}(x_n)$, so $d(x_n, x_m) \leq r_n$. Thus,

$$r_n + \varepsilon = d(x_n, y) \leq d(x_n, x_m) + d(x_m, y) < r_n + \varepsilon,$$

a contradiction. We have proved that $y \in B_{r_n}(x_n)$ for every $n \geq N$. Hence, y is in the intersection of the nest $\mathcal{N}$, proving that $(X, \mathcal{B}_S)$ is spherically complete.

Now assume that $(X, \mathcal{B}_S)$ is spherically complete. Take any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X. By our assumptions on S, we can choose a sequence $(s_i)_{i\in\mathbb{N}}$ in $\{s \in S \mid s < s_0\}$ such that $0 < 2s_{i+1} \leq s_i$. By induction on $i \in \mathbb{N}$ we choose an increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers such that the balls $B_i := B_{s_i}(x_{n_i})$ form a nest.

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we have that for every $i \in \mathbb{N}$ there is n_1 such that $d(x_n, x_m) < s_2$ for all $n, m > n_1$. Once we have chosen n_{i-1} , we choose $n_i > n_{i-1}$ such that $d(x_n, x_m) < s_{i+1}$ for all $n, m \geq n_i$. We show that the so obtained balls B_i form a nest. Take $i \in \mathbb{N}$ and $x \in B_{i+1} = B_{s_{i+1}}(x_{n_{i+1}})$. This means that $d(x_{n_{i+1}}, x) \leq s_{i+1}$. Since $n_i, n_{i+1} \geq n_i$, we have that $d(x_{n_i}, x_{n_{i+1}}) < s_{i+1}$. We compute:

$$\begin{aligned} d(x_{n_i}, x) &\leq d(x_{n_i}, x_{n_{i+1}}) + d(x_{n_{i+1}}, x) \\ &\leq s_{i+1} + s_{i+1} = 2s_{i+1} \leq s_i \end{aligned}$$

Thus $x \in B_i$ and hence $B_{i+1} \subseteq B_i$ for all $i \in \mathbb{N}$. The intersection of this nest $(B_i)_{i\in\mathbb{N}}$ contains some y, by our assumption. We have that $y \in B_i$ for all $i \in \mathbb{N}$, which means that $d(x_{n_i}, y) \leq s_i$. Since

$$\lim_{i\to\infty} s_i = 0,$$

we obtain that

$$\lim_{i\to\infty} x_{n_i} = y,$$

which proves that (X, d) is a complete metric space. □

Now we will introduce a classification of ball spaces. The classification is based on the behavior of the intersections of their nests.

Definition 3.1.9. *1. A ball space $(X, \mathcal{B})$ is called S_1 if the intersection of each nest of balls is nonempty. In other words, an S_1 ball space is a spherically complete ball space.*

2. A ball space $(X, \mathcal{B})$ is called S_2 if the intersection of each nest of balls contains a ball.

3. A ball space $(X, \mathcal{B})$ is called S_3 if the intersection of each nest of balls contains a largest ball.

4. A ball space $(X, \mathcal{B})$ is called S_4 if the intersection of each nest of balls is a ball.

Remark 3.1.10. *We have the following implications: $S_4 \Rightarrow S_3 \Rightarrow S_2 \Rightarrow S_1$.*

Example 3.1.11. *Consider the complete metric space $X = \mathbb{R}$ and a ball space $(X, \mathcal{B}_S)$ like in the Theorem 3.1.8. It is S_1 , but not S_2 since $\mathcal{B}_S$ does not contain singleton balls.*

Example 3.1.12. *Take a spherically complete ultrametric space* (X, d, Γ)*, where* Γ *is a densely ordered set with least element* 0*. By definition, the ball space* $(X, \mathcal{B})$*, where* $\mathcal{B} = \{B_\gamma(x) \mid \gamma \in \Gamma,\ x \in X\}$*, is spherically complete. We define general ultrametric balls as follows. Take* $x \in X$ *and a lower cut set* $L \subseteq \Gamma$*, and set*

$$B_L(x) := \{y \in X \mid d(x, y) \in L\}.$$

Observe that

$$B_L(x) = \bigcup_{\gamma \in L} B_\gamma(x).$$

Suppose that L *does not admit a supremum in* Γ*. Then the intersection of the nest*

$$\{B_\gamma(x) \mid \gamma \notin L\}$$

is $B_L(x)$*. Assume in addition that for every* $\gamma \in L$ *there is some* $x_\gamma \in X$ *such that* $d(x, x_\gamma) = \gamma$*. (This condition is always satisfied when* (X, d, Γ) *is homogeneous, that is, for every* $x \in X$ *and* $\gamma \in \Gamma$ *there is* x_γ *such that* $d(x, x_\gamma) = \gamma$*. Note that ultrametric spaces induced by valued fields are homogeneous.) Then* $B_L(x)$ *is not itself of the form* $B_\gamma(x)$ *for any* γ*, so it is not a ball in* $\mathcal{B}$*, and it does not even contain a largest ball. Indeed, if* $B_\delta(y)$ *were such a largest ball contained in* $B_L(x)$*, then* $\gamma = d(x, y) \in L$*, so* $B_\delta(y) \subseteq B_\gamma(x)$*; but* $B_\gamma(x)$ *is not the largest ball in* $B_L(x)$*. However,* $B_0(x) \subset B_L(x)$*. This shows that* $(X, \mathcal{B})$ *is* S_2 *, but not* S_3*.*

If our ultrametric space is induced by a valued field with value group $\mathbb{Q}$*, such as the algebraic closure of the field* $\mathbb{Q}_p$ *of* p*-adic numbers, then it is homogeneous, but* Γ *is not cut complete, that is, there are lower cut sets that do not admit a supremum. This shows that ultrametric spaces with the above properties exist.*

Example 3.1.13. *Consider the set* $X = \{0, 1\} \times \mathbb{N}$ *with the lexicographic order. (This is the sum* $\mathbb{N} + \mathbb{N}$ *in the sense of ordered sets.) Take* $\mathcal{B}$ *to be the set of all closed bounded intervals with end points in* $\{0, 1\} \times 2\mathbb{N}$*. Take a nest* $\mathcal{N}$ *in* $\mathcal{B}$*. If there is an interval* $I \in \mathcal{N}$ *with both endpoints in* $\{0\} \times 2\mathbb{N}$ *or both endpoints in* $\{1\} \times 2\mathbb{N}$*, then the intersection of* $\mathcal{N}$ *is equal to* $\bigcap\{J \in \mathcal{N} \mid J \subseteq I\}$*. The set* $\{J \in \mathcal{N} \mid J \subseteq I\}$ *is finite, so the intersection of this set is just the smallest interval in this set. Now assume that every interval in* $\mathcal{N}$ *is of the form*

$$[(0, 2a), (1, 2b)] \quad \textit{with } a, b \in \mathbb{N}$$

and the nest $\mathcal{N}$ *is not finite. Then the intersection of the nest* $\mathcal{N}$ *is equal to the set*

$$\{(1, 1), (1, 2), \ldots, (1, 2b_0)\},$$

where b_0 *is a smallest natural number with the property that there is* $a \in \mathbb{N}$ *such that* $[(0, 2a), (1, 2b_0)] \in \mathcal{N}$*. Note that this intersection is not a ball in* $\mathcal{B}$*, but it contains the largest ball* $[(1, 2), (1, 2b_0)]$*. Therefore, the space* $(X, \mathcal{B})$ *is* S_3*, but not* S_4*.*

Example 3.1.14. *Every ball space with a finite number of balls is* S_4 *.*

Example 3.1.15. *A compact topological space with the set of all nonempty closed subsets is* S_4 *as we have seen in Theorem 2.0.23.*

Example 3.1.16. *It is known that if (X, d, Γ) is spherically complete, then also the ball space*

$$(X, \{B_L(x) \mid L \text{ lower cut set in } \Gamma \text{ and } x \in X\})$$

is spherically complete (cf. [3]). The intersection of every nest of general ultrametric balls is again a general ultrametric ball, if nonempty. So this ball space is S_4 .

3.2 Functions on ball spaces

We will take a closer look at the properties of functions on ball spaces. We will also introduce some additional properties that functions defined on a ball space can have. We start with the following proposition which shows when the property of spherical completeness is preserved under a function between ball spaces.

Proposition 3.2.1. *Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be ball spaces and $f : X_1 \to X_2$ a function. Suppose that the preimage of every ball in $(X_2, \mathcal{B}_2)$ is a ball in $(X_1, \mathcal{B}_1)$. If $\mathcal{N}$ is a nest of balls in $(X_2, \mathcal{B}_2)$, then the preimages of the balls in the nest $\mathcal{N}$ form a nest of balls in $(X_1, \mathcal{B}_1)$. If $(X_1, \mathcal{B}_1)$ is spherically complete, then also $(X_2, \mathcal{B}_2)$ is spherically complete.*

Proof. Let $\mathcal{N}$ be a nest of balls in $(X_2, \mathcal{B}_2)$. Then the preimage of every ball in $\mathcal{N}$ is a ball in $(X_1, \mathcal{B}_1)$ by assumption. Since the balls in $\mathcal{N}$ are totally ordered by inclusion, their preimages are also totally ordered by inclusion, so they form a nest of balls in $\mathcal{B}_1$.

Now suppose that $(X_1, \mathcal{B}_1)$ is spherically complete. Take a nest $\mathcal{N}$ in $(X_2, \mathcal{B}_2)$ and the collection $\mathcal{N}' = \{f^{-1}(B) \mid B \in \mathcal{N}\}$ of preimages in X_1 , which is a nest in $\mathcal{B}_1$. Note that $f(\bigcap \mathcal{N}') \subseteq \bigcap_{B \in \mathcal{N}} f(f^{-1}(B)) \subseteq \bigcap \mathcal{N}$. The intersection of $\mathcal{N}'$ is nonempty by assumption, so it contains some $x \in X_1$. Then $f(x) \in \bigcap \mathcal{N}$. This proves that $(X_2, \mathcal{B}_2)$ is spherically complete. $\square$

Now we will consider the functions defined on a ball space with the values in the same space and define some properties a ball may have with respect to such functions, as well as properties of the functions themselves. We will use them in the next sections.

Definition 3.2.2. *Take a ball space $(X, \mathcal{B})$ and a function $f : X \to X$. We say that a ball $B \in \mathcal{B}$ is:*

1. *f**-closed** if $f(B) \subseteq B$,*
2. *f**-contracting** if it is f-closed, and $f(B) \subsetneq B$ unless B is a singleton.*

Definition 3.2.3. *The function $f : X \to X$ on a ball space $(X, \mathcal{B})$ is called* ***strongly contracting on orbits*** *if there is a function*

$$X \ni x \longmapsto B_x \in \mathcal{B}$$

such that the following conditions hold for all $x \in X$:

(S1) $x \in B_x$,

(S2) $B_{fx} \subseteq B_x$, *and if $x \neq fx$, then $B_{f^i x} \subsetneq B_x$ for some $i \geq 1$.*

Definition 3.2.4. *Take a function $f : X \to X$ on a ball space $(X, \mathcal{B})$ which is strongly contracting on orbits. A nest of balls $\mathcal{N} \subset \mathcal{B}$ is called an f**-nest** if $\mathcal{N} = \{B_x \mid x \in S\}$ for some set $S \subseteq X$ which is closed under f.*

Definition 3.2.5. *The function f is called **self-contractive** if it is strongly contracting on orbits and satisfies:*

(S3) *if $\mathcal{N}$ is an f-nest and if $z \in \bigcap \mathcal{N}$, then $B_z \subseteq \bigcap \mathcal{N}$.*

Chapter 4

Fixed Point Theorems

We say that a function f from a set X into itself has a fixed point if there is $x \in X$ such that $fx = x$. Fixed point theorems try to find proper conditions on the function f and the space X to obtain a fixed point. In this chapter we will study the function f in two cases. First we will define the function f on a ball space $(X, \mathcal{B})$ and put some conditions on balls in $\mathcal{B}$ to obtain the fixed point. Thereafter we will consider the function f on a ball space $(X, \mathcal{B})$ to be *strongly contracting on orbits* (see Definition 3.2.3) and find sufficient conditions to ensure the existence of a fixed point. In both cases, we will apply the general theorems in metric and ultrametric space to prove Banach's Fixed Point Theorem and a theorem of S. Prieß-Crampe and P. Ribenboim.

4.1 General fixed point theorems

Theorem 4.1.1 (General Fixed Point Theorem)**.** *Take a ball space $(X, \mathcal{B})$ and a function $f : X \to X$ such that the following conditions are satisfied:*
(GF1) *every ball $B \in \mathcal{B}$ is f-closed (see Definition 3.2.2),*
(GF2) *every non-singleton ball $B \in \mathcal{B}$ properly contains some $B' \in \mathcal{B}$,*
(GF3) *the intersection of every nest of balls in $\mathcal{B}$ is a singleton or contains some $B \in \mathcal{B}$.*
Then f has a fixed point in every ball.

Proof. Take any ball $B_0 \in \mathcal{B}$. Then by Lemma 3.1.3, there is a maximal nest $\mathcal{N}$ containing B_0. Since every ball in $\mathcal{B}$ is f-closed,

$$f(\bigcap \mathcal{N}) \subseteq \bigcap_{B \in \mathcal{N}} f(B) \subseteq \bigcap_{B \in \mathcal{N}} B = \bigcap \mathcal{N},$$

that is, $\bigcap \mathcal{N}$ is f-closed. By condition (GF3), $\bigcap \mathcal{N}$ is a singleton or contains a ball $B \in \mathcal{B}$. If $\bigcap \mathcal{N}$ is a singleton, we obtain a fixed point. So suppose that $\mathcal{N}$ is not a singleton. Then it contains a ball $B \in \mathcal{B}$. If B is not a singleton, then by (GF2), it properly contains a ball $B' \in \mathcal{B}$. But $\mathcal{N} \bigcup \{B'\}$ is then a nest that properly contains $\mathcal{N}$, and this contradicts the maximality of $\mathcal{N}$. Thus B must be a singleton $\{x\}$, and x is a fixed point since $B = \{x\}$ is f-closed by (GF1). Also $x \in B_0$ since B is contained in B_0 because B_0 is a ball in the nest $\mathcal{N}$. Hence f has a fixed point in every ball in $\mathcal{B}$. □

Theorem 4.1.2. *Let f be a function on a ball space $(X, \mathcal{B})$ such that the following conditions are satisfied:*
(F1) *there is at least one f-contracting ball (see Definition 3.2.2) in the ball space,*
(F2) *for every f-contracting ball $B \in \mathcal{B}$ there is an f-contracting ball in the image $f(B)$,*
(F3) *the intersection of every nest of f-contracting balls contains an f-contracting ball.*
Then f admits a fixed point.

Proof. Define $\mathcal{B}' := \{B \in \mathcal{B} \mid B \text{ is an } f\text{-contracting ball}\}$. By (F1) $\mathcal{B}' \neq \emptyset$, so $(X, \mathcal{B}')$ is a ball space. Every f-contracting ball is in particular f-closed, so (GF1) in Theorem 4.1.1 holds (for $\mathcal{B}'$ in place of $\mathcal{B}$). Take any non-singleton ball $B \in \mathcal{B}'$. Then by (F2) there is an f-contracting ball B' in the image $f(B)$. Hence $B' \subseteq f(B) \subsetneq B$ which proves that (GF2) holds. Now to show that also (GF3) holds, take a nest of f-contracting balls $\mathcal{N}$. Then by (F3), there is $B_0 \in \mathcal{B}'$ such that $B_0 \subseteq \bigcap \mathcal{N}$. Thus (GF3) holds. So now we can apply the General Fixed Point Theorem to $(X, \mathcal{B}')$ to obtain a fixed point x_0 in every ball $B \in \mathcal{B}'$. $\square$

In the following subsections we will apply the General Fixed Point Theorem 4.1.1 to prove Banach's Fixed Point Theorem and a fixed point theorem due to Prieß-Crampe and Ribenboim.

4.1.1 Application to metric spaces

Definition 4.1.3. *Let (X, d) be a metric space. A function $f : X \to X$ is said to be a* ***contracting*** *if there is a positive real number $c < 1$ such that $d(fx, fy) \leq cd(x, y)$ for all $x, y \in X$.*

Theorem 4.1.4 (Banach's Fixed Point Theorem)**.** *Every contracting function on a complete metric space (X, d) has a unique fixed point.*

Proof. Let f be a contracting function on a complete metric space (X, d). Then for every $x \in X$ we have:

$$d(fx, f^2x) \leq cd(x, fx). \tag{4.1.1}$$

Consequently,

$$\begin{aligned} d(x, f^i x) &\leq d(x, fx) + d(fx, f^2x) + ... + (f^{i-1}, f^i x) \\ &\leq d(x, fx) + cd(x, fx) + ... + c^{i-1}d(x, fx) \\ &\leq d(x, fx)(1 + c + c^2 + ... + c^{i-1}) \\ &\leq d(x, fx)\sum_{i=0}^{\infty} c^i = d(x, fx)\frac{1}{1-c} \end{aligned} \tag{4.1.2}$$

where we were able to use the value of the geometric series since $0 < c < 1$. We fix $x \in X$, set $d := d(x, fx)$ and define:

$$B_i := B_{\frac{c^i d}{1-c}}(f^i x) = \left\{ y \in X \mid d(y, f^i x) \leq \frac{c^i d}{1-c} \right\}$$

for all $i \in \mathbb{N}$. We wish to show that each B_i is f-closed. Take $fz \in f(B_i)$ with $z \in B_i$. By inequality 4.1.1, and since f is contracting, we obtain:

$$\begin{aligned} d(fz, f^i x) &\leq d(fz, f^{i+1}x) + d(f^{i+1}x, f^i x) \\ &\leq cd(z, f^i x) + d(f^i x, f^{i+1}x) \\ &\leq c\frac{c^i d}{1-c} + c^i d = \frac{c^i d}{1-c}. \end{aligned}$$

This shows that $fz \in B_i$, so $f(B_i) \subseteq B_i$. Thus each B_i is f-closed and (GF1) holds.

We wish to show that each B_i properly contains some ball in $\mathcal{B}$. Since $c < 1$, there is some $k \geq 1$ such that

$$\frac{c^k}{1-c} < \frac{1}{2}.$$

Then

$$\frac{d(f^{i+k}x, f^{i+k+1}x)}{1-c} \leq \frac{c^k}{1-c} d(f^i x, f^{i+1}x) < \frac{1}{2} d(f^i x, f^{i+1}x).$$

This shows that $f^i x$ and $f^{i+1}x$ cannot both lie in B_{i+k}. Therefore $B_{i+k} \subsetneq B_i$, and (GF2) holds.

Set $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$. Then the set $S = \{\frac{c^i d}{1-c} \mid i \in \mathbb{N}\}$ of radii of the balls B_i has 0 as its unique limit point. Since the metric space X is complete, Theorem 3.1.8 shows that the ball space $(X, \mathcal{B})$ is spherically complete. Take a nest $\mathcal{N} \subseteq \mathcal{B}$. Then $\bigcap \mathcal{N} \neq \emptyset$. If $\mathcal{N}$ contains a smallest ball, then $\bigcap \mathcal{N}$ is equal to this ball. Suppose that $\mathcal{N}$ does not contain a smallest ball. Then it contains balls of arbitrarily small radius. If $y, z \in \bigcap \mathcal{N}$, then $d(y, z) \leq \frac{c^i d}{1-c}$ for all $i \in \mathbb{N}$. Therefore $d(y, z) = 0$, that is, $y = z$, which shows that $\bigcap \mathcal{N}$ is a singleton. We have proved that (GF3) holds.

Now we can apply the General Fixed Point Theorem to obtain a fixed point $z \in X$.

Proof of the uniqueness: Assume that $y \in X$ is also a fixed point. Since f is contracting,

$$d(y, z) = d(fy, fz) \leq cd(y, z).$$

Since $0 < c < 1$, this can only be true if $d(y, z) = 0$, i.e., $y = z$. Hence the contracting function f has a unique fixed point z in the complete metric space (X, d). □

4.1.2 Application to ultrametric spaces

Theorem 4.1.5 (S. Prieß-Crampe, P. Ribenboim)**.** *Take a function f on a spherically complete ultrametric space (X, d) such that for all $x, y \in X$:*

1) $d(fx, fy) \leq d(x, y)$,

2) $d(fx, f^2x) < d(x, fx)$ if $x \neq fx$.

Then f has a fixed point.

Proof. For $x \in X$, let B_x be the ball defined as

$$B_x := \{y \in X \mid d(y, x) \leq d(x, fx)\}.$$

Consider the ball space $(X, \mathcal{B})$, where $\mathcal{B} = \{B_x \mid x \in X\}$. In order to show that (GF1) holds, we show that $f(B_x) \subseteq B_x$ for all $B_x \in \mathcal{B}$. Take $fy \in f(B_x)$ with $y \in B_x$ which means that $d(y,x) \leq d(x,fx)$. By the ultrametric triangle law, we have

$$d(fy,x) \leq \max\{d(fy,fx), d(fx,x)\}.$$

By assumption 1) we obtain

$$d(fy,x) \leq \max\{d(y,x), d(fx,x)\} = d(x,fx)$$

since $y \in B_x$. Hence $fy \in B_x$, so $f(B_x) \subseteq B_x$ for all $B_x \in \mathcal{B}$, and (GF1) holds.

Now we wish to show that every non-singleton $B_x \in \mathcal{B}$ properly contains some ball in $\mathcal{B}$. Note that for all $x \in X$, $d(f^2x, fx) < d(fx, x)$. By part 6 of 2.0.18, $x \notin B_{fx}$ and $B_{fx} \subsetneqq B_x$. Hence (GF2) holds.
Take a nest $\mathcal{N}$ of balls $(B_{x_i})_{i \in I}$. Since (X,d) is spherically complete, $\bigcap \mathcal{N} \neq \emptyset$. Suppose that $\bigcap \mathcal{N}$ is not a singleton since otherwise (GF3) already holds. So there is $y, z \in \bigcap \mathcal{N}$ such that $y \neq z$. This means that $y, z \in B_{x_i}$ for every $B_{x_i} \in \mathcal{N}$. Then we have

$$d(y,z) \leq \max\{d(y,x_i), d(x_i,z)\} = d(x_i, fx_i).$$

By part 4 of 2.0.18, it shows that $B_z \subseteq B_{x_i}$ for every $B_{x_i} \in \mathcal{N}$. Thus $B_z \subseteq \bigcap \mathcal{N}$, and (GF3) holds. Now we can apply Theorem 4.1.1 to obtain a fixed point. □

4.2 B_x-type fixed point theorems

Theorem 4.2.1. *Take a function f on a ball space $(X, \mathcal{B})$ which is strongly contracting on orbits (see Definition 3.2.3). If for every f-nest $\mathcal{N}$ (see Definition 3.2.4) in $(X, \mathcal{B})$, there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$, then f has a fixed point.*

Proof. Let f be a function on a ball space $(X, \mathcal{B})$ which is strongly contracting on orbits. Then for every $x \in X$, the set $\{B_{f^i x} \mid i \geq 0\}$ is an f-nest. Hence the set of all f-nests is nonempty. It is partially ordered by inclusion since the union over an ascending chain of f-nests is again an f-nest. So by Zorn's Lemma, there is a maximal f-nest $\mathcal{N}$. By the assumption of the theorem, there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$. Suppose that $z \neq fz$. Then by (S2), we have that $B_{f^i z} \subsetneqq B_z$ for some $i \geq 1$. Since $B_{f^i z} \subsetneqq \bigcap \mathcal{N}$ for some $i \geq 1$, the set $\mathcal{N} \bigcup \{B_{f^k z} \mid k \geq i\}$ is an f-nest which properly contains $\mathcal{N}$. But this contradicts the maximality of $\mathcal{N}$. Therefore z is a fixed point. □

We can rewrite Theorem 4.2.1 in the following way by using the definition of *self-contractive function* (see Definition 3.2.5).

Theorem 4.2.2. *Every self-contractive function on a spherically complete ball space has a fixed point.*

The ball space in this theorem must be spherically complete to obtain the assumption of Theorem 4.2.1.

Now we will show how to use Theorem 4.2.2 to prove fixed point theorems in metric and ultrametric spaces.

4.2.1 Application to metric spaces

Theorem 4.2.3 (Banach's Fixed Point Theorem)**.** *Every contracting function on a complete metric space (X, d) has a unique fixed point.*

Proof. Let f be a contracting function on a complete metric space (X, d). Then by inequality 4.1.2, for $x \in X$ we have

$$d(x, f^i x) \leq \frac{d(x, fx)}{1-c}.$$

For $x \in X$ and $i \geq 0$, let

$$B_{f^i x} := \{y \in X \mid d(y, f^i x) \leq \frac{c^i}{1-c} d(x, fx)\}.$$

Consider the ball space $(X, \mathcal{B})$, where $\mathcal{B} = \{B_{f^i x} \mid i \geq 0\}$. We wish to prove that f is self-contractive. By definition of the balls, $x \in B_x$ so (S1) holds. To prove that (S2) holds, take any element $y \in B_{fx}$. Then by the fact that f is contracting we have:

$$\begin{aligned} d(x, y) &\leq d(x, fx) + d(fx, y) \\ &\leq d(x, fx) + \frac{c}{1-c} d(x, fx) = \frac{d(x, fx)}{1-c}. \end{aligned}$$

So $y \in B_x$. Therefore $B_{fx} \subseteq B_x$.

We have that $f^i x \in B_x$ for all $i \geq 0$ since $d(x, f^i x) \leq \frac{d(x,fx)}{1-c}$. Since $c < 1$, there is some $i \in \mathbb{N}$ such that

$$\frac{c^i}{1-c} < \frac{1}{2}.$$

Then by inequality 4.1.1, we obtain

$$\frac{d(f^i x, f^{i+1} x)}{1-c} \leq \frac{c^i}{1-c} d(x, fx) < \frac{1}{2} d(x, fx).$$

Therefore x and fx cannot both be in $B_{f^i x}$ which means that $B_{f^i x} \subsetneq B_x$. Hence (S2) holds. Therefore f is strongly contracting on orbits.

As we show in the proof of Theorem 4.1.4 the set $S = \{\frac{c^i}{1-c} d(x, fx) \mid i \geq 0\}$ of radii of the balls $B_{f^i x}$ has 0 as its unique limit point. Since the metric space X is complete, by Theorem 3.1.8 the ball space $(X, \mathcal{B})$ is spherically complete. Take an f-nest $\mathcal{N}$. Then $\bigcap \mathcal{N} \neq \emptyset$. So there is some $z \in X$ such that $z \in \bigcap \mathcal{N}$. We wish to show that $B_z \subseteq \bigcap \mathcal{N}$. Take any $B_x \in \mathcal{N}$, then $B_{f^i x} \in \mathcal{N}$ for all $i > 0$ since $\mathcal{N}$ is an f-nest. Using that $z \in \bigcap \mathcal{N} \subseteq B_{f^i x}$ for all i, and the fact that f is contracting, we compute:

$$\begin{aligned} d(z, fz) &\leq d(z, f^i x) + d(f^i x, fz) \\ &\leq d(z, f^i x) + c d(f^{i-1} x, z) \\ &\leq \frac{c^i}{1-c} d(x, fx) + \frac{c^{i-1}}{1-c} c d(x, fx) = 2 \frac{c^i}{1-c} d(x, fx). \end{aligned}$$

Since $c^i \to 0$ as $i \to \infty$, we get $d(fz, z) = 0$. Thus $B_z = \{z\} \subseteq \bigcap \mathcal{N}$. This shows that f is self-contractive. Now we can apply Theorem 4.2.2 to obtain a fixed point.

To prove the uniqueness, see the proof of the uniqueness in Theorem 4.1.4. □

4.2.2 Application to ultrametric spaces

Theorem 4.2.4 (S. Prieß-Crampe, P. Ribenboim)**.** *Take a function f on a spherically complete ultrametric space (X, d) such that for all $x, y \in X$:*
1) $d(fx, fy) \leq d(x, y)$,
2) $d(fx, f^2x) < d(x, fx)$ if $x \neq fx$.
Then f has a fixed point.

Proof. For $x \in X$, let B_x be the ball defined as

$$B_x := B(x, fx) = \{y \in X \mid d(x, y) \leq d(x, fx)\}.$$

Consider the ball space $(X, \mathcal{B})$, where $\mathcal{B} = \{B_x \mid x \in X\}$. We want to prove that f is a self-contractive function to apply Theorem 4.2.2. We observe that $x \in B_x$, so (S1) holds.

Note that $d(fx, f^2x) < d(x, fx)$, So by part 6 of 2.0.18,

$$B_{fx} = B(fx, f^2x) \subsetneq B(x, fx) = B_x.$$

Hence (S2) holds.

Now take an f-nest $\mathcal{N}$ of balls $(B_{x_i})_{i \in I}$. Since (X, d) is spherically complete, $\bigcap \mathcal{N} \neq \emptyset$. So there is $z \in X$ such that $z \in \bigcap \mathcal{N} \subseteq B_{x_i}$ for every $B_{x_i} \in \mathcal{N}$.
By 1) and since $z \in B_{x_i}$, we have that:

$$d(fx_i, fz) \leq d(x_i, z) \leq d(x_i, fx_i).$$

By the ultrametric triangle law and the foregoing inequalities,

$$d(z, fz) \leq \max\{d(z, x_i), d(x_i, fx_i), d(fx_i, fz)\} = d(x_i, fx_i).$$

So by part 4 of 2.0.18, $B_z \subseteq B_{x_i}$ for every $B_{x_i} \in \mathcal{N}$. Therefore $B_z \subseteq \bigcap \mathcal{N}$, and (S3) holds. So we can apply Theorem 4.2.2 to obtain a fixed point. □

Chapter 5

Basic Coincidence Point Theorems

In this chapter, we will introduce general coincidence point theorems and apply them in metric and ultrametric spaces to prove coincidence point theorems due to K. Goebel and to Prieß-Crampe and Ribenboim.

5.1 Basic coincidence point theorems for ball spaces

Theorem 5.1.1 (Coincidence Point Theorem I). *Let $(X, \mathcal{B})$ be a spherically complete ball space and $f, g : X \to Y$ functions satisfying the following conditions:*
(CT1) *for every $B \in \mathcal{B}$, $f(B) \subseteq g(B)$,*
(CT2) *for every nest of balls $\mathcal{N}$, either $\bigcap_{B\in\mathcal{N}} g(B)$ is a singleton or there is $B' \in \mathcal{B}$ such that $B' \subsetneqq \bigcap \mathcal{N}$.*
Then every ball in $\mathcal{B}$ contains some $x \in X$ such that $fx = gx$.

The condition that $(X, \mathcal{B})$ be spherically complete can be dropped if for every $B \in \mathcal{B}$, $B = g^{-1}(g(B))$.

Proof. Take any $B_0 \in \mathcal{B}$. The set of all nests of balls containing B_0 is partially ordered by inclusion, and the union over a linearly ordered set of such nests is again a nest containing B_0. Hence by Zorn's Lemma there is a maximal nest $\mathcal{N}_0$ containing B_0. Suppose that $\bigcap_{B\in\mathcal{N}_0} g(B)$ is not a singleton. Then by (CT2), there is $B' \in \mathcal{B}$ such that $B' \subsetneqq \bigcap \mathcal{N}_0$. But then $\mathcal{N}_0 \cup \{B'\}$ would be a nest of balls containing B_0 and larger than $\mathcal{N}_0$, which contradicts its maximality. Therefore $\bigcap_{B\in\mathcal{N}_0} g(B)$ must be a singleton, say $\{y\}$ for some $y \in Y$.
Using (CT1),

$$f(\bigcap \mathcal{N}_0) \subseteq \bigcap_{B\in\mathcal{N}_0} f(B) \subseteq \bigcap_{B\in\mathcal{N}_0} g(B) = \{y\}.$$

Therefore $fx = y = gx$ for every $x \in \bigcap \mathcal{N}_0 \subset B_0$. If $(X, \mathcal{B})$ is spherically complete, then $\bigcap \mathcal{N}_0 \neq \emptyset$ and there is at least one such x. If on the other hand $B = g^{-1}(g(B))$ for all $B \in \mathcal{B}$, then all preimages of y are contained in every $B \in \mathcal{N}_0$ and thus again, $\bigcap \mathcal{N}_0 \neq \emptyset$. □

Remark 5.1.2. *If we take $X = Y$ and g to be the identity function in Theorem 5.1.1, we obtain Theorem 4.1.1.*

Now by taking the ball space $(X, \mathcal{B})$ to be S_2, the conditions needed in our coincidence theorem can be made nicely symmetric. We obtain the following theorem:

Theorem 5.1.3 (Coincidence Point Theorem II)**.** *Let $(X, \mathcal{B})$ be an S_2 ball space and $f, g : X \to Y$ functions satisfying the following conditions:*

(CS1) *for every $B \in \mathcal{B}$, $f(B) \cap g(B) \neq \emptyset$,*

(CS2) *for every $B \in \mathcal{B}$, either $f(B)$ is a singleton or $g(B)$ is a singleton or there is $B' \in \mathcal{B}$ such that $B' \subsetneqq B$.*

Then every ball in $\mathcal{B}$ contains some $x \in X$ such that $fx = gx$.

Proof. As before, there is a maximal nest $\mathcal{N}_0$ containing a ball B_0. Since $(X, \mathcal{B})$ is an S_2 ball space, the intersection of $\mathcal{N}_0$ contains a ball B. By (CS2) we have that $f(B)$ or $g(B)$ is a singleton $\{y\}$ for some $y \in Y$ since the existence of a ball $B' \in \mathcal{B}$ with $B' \subsetneqq B$ would contradict the maximality of the nest $\mathcal{N}_0$. Now by (CS1) we get $f(B) \cap g(B) = \{y\}$, so for some $x \in B \subset B_0$, we get $fx = y = gx$. □

5.2 Applications

In this section, we will apply Coincidence Point Theorem I to prove a theorem due to K. Goebel, and Coincidence Point Theorem II to prove a theorem due to Prieß-Crampe and Ribenboim.

5.2.1 Application to metric spaces

The following is a coincidence point theorem for metric spaces proved by K. Goebel in [1].

Theorem 5.2.1 (K. Goebel)**.** *Let X be an arbitrary set and (Y, d) a metric space, and take functions $f, g : X \to Y$ such that:*

(G1) *$f(X) \subseteq g(X)$,*

(G2) *$g(X)$ is a complete metric space,*

(G3) *there is a positive real number $c < 1$ such that $d(fx, fy) \leq cd(gx, gy)$ for all $x, y \in X$.*

Then there exists $x \in X$ such that $fx = gx$.

Proof. Take $x_0 \in X$. By (G1) there is some $x_1 \in X$ such that $fx_0 = gx_1$. If $x_0 = x_1$, we get a coincidence point and we are done, so we assume that $x_0 \neq x_1$. Define

$$d := d(fx_0, gx_0) = d(gx_1, gx_0). \tag{5.2.1}$$

Consider a sequence $(x_i)_{i \in \mathbb{N}}$ in X such that $gx_i = fx_{i-1}$ for all $i \in \mathbb{N}$. By condition (G3) we have:

$$d(gx_{i+1}, gx_i) = d(fx_i, fx_{i-1}) \leq cd(gx_i, gx_{i-1}).$$

By induction on I, we thus obtain for all $i \in \mathbb{N}$:

$$d(gx_{i+1}, gx_i) \leq c^i d.$$

For $i \in \mathbb{N}$ we consider the closed metric balls

$$A_i := A_{\frac{c^i d}{1-c}}(gx_i) = \left\{ gy \mid d(gx_i, gy) \leq \frac{c^i d}{1-c} \right\}$$

in $g(X)$. The radii of these balls form a set of positive real numbers with 0 as its only limit point. By (G2) the metric space $(g(X), d)$ is complete, therefore by Theorem 3.1.8 the ball space $(g(X), \{A_i \mid i \in \mathbb{N}\})$ is spherically complete.

Define a ball space $(X, \mathcal{B})$ on X by taking as balls the preimages

$$B_i = g^{-1}(A_i)$$

for $i \in \mathbb{N}$. We will show that this ball space satisfies the conditions of Theorem 5.1.1.

Take $z \in B_i$. Then $gz \in A_i$, and therefore $d(gz, gx_i) \leq \frac{c^i d}{1-c}$. By condition (G1), $fz = gz'$ for some $z' \in X$. Then we have, using (G3):

$$\begin{aligned} d(gz', gx_i) &= d(fz, fx_{i-1}) \leq cd(gz, gx_{i-1}) \leq c[d(gz, gx_i) + d(gx_i, gx_{i-1})] \\ &\leq c\left[\frac{c^i d}{1-c} + c^{i-1}d\right] = \frac{c^i d}{1-c}. \end{aligned}$$

This shows that $fz = gz' \in A_i$ and therefore $f(B_i) \subset g(B_i)$. We have proved that condition (CT1) of Theorem 5.1.1 is satisfied.

To show that also the second condition of Theorem 5.1.1 holds, we first show that the balls B_i, $i \in \mathbb{N}$, form a nest. Take $z \in B_{i+1}$. Then $gz \in A_{i+1}$, thus $d(gz, gx_{i+1}) \leq \frac{c^{i+1}d}{1-c}$. We have:

$$d(gz, gx_i) \leq d(gz, gx_{i+1}) + d(gx_{i+1}, gx_i) \leq \frac{c^{i+1}d}{1-c} + c^i d = \frac{c^i d}{1-c}.$$

Therefore $gz \in A_i$, which implies that $z \in B_i$, showing that $B_{i+1} \subseteq B_i$.

Take a nest $\mathcal{N}$ of balls in $(X, \mathcal{B})$. It is of course a subnest of the nest $\{B_i \mid i \in \mathbb{N}\}$. Then $\{g(B) \mid B \in \mathcal{N}\}$ is a nest in $(g(X), \{A_i \mid i \in \mathbb{N}\})$. Since this ball space is spherically complete, $\bigcap_{B\in\mathcal{N}} g(B)$ is nonempty. Assume that $\bigcap_{B\in\mathcal{N}} g(B)$ is not a singleton, i.e., there are z_1, z_2 in X such that $gz_1, gz_2 \in \bigcap_{B\in\mathcal{N}} g(B)$, $gz_1 \neq gz_2$. Since $g^{-1}(g(B)) = B$, we obtain that z_1, $z_2 \in B$ for every $B \in \mathcal{N}$, thus z_1, $z_2 \in \bigcap_{B\in\mathcal{N}} B$. Since $0 < c < 1$, there is $k \in \mathbb{N}$ such that

$$d(gz_1, gz_2) > 2\frac{c^k d}{1-c}. \tag{5.2.2}$$

This implies that gz_1 and gz_2 cannot be both contained in A_k , so z_1 and z_2 cannot be both contained in B_k . But as the former are both contained in $g(B)$ for every ball $B \in \mathcal{N}$, we find that the nest $\{g(B) \mid B \in \mathcal{N}\}$ can only contain balls A_i with radii larger than that of A_k , which means that $i < k$. For these A_i we have that $B_k = g^{-1}(A_k) \subseteq g^{-1}(A_i) = B_i$, so $B_k \subseteq \bigcap \mathcal{N}$. Since z_1 and z_2 are both contained in $\bigcap \mathcal{N}$, but not in B_k , we find that $B_k \subsetneq \bigcap \mathcal{N}$. This shows that also condition (CT2) of Theorem 5.1.1 holds.

Now we can apply Theorem 5.1.1 to obtain a coincidence point. □

5.2.2 Application to ultrametric spaces

Theorem 5.2.2 (S. Prieß-Crampe, P. Ribenboim)**.** *Let (X, d) be an ultrametric space, and take functions $f, g : X \to X$. Assume that:*

(PR1) *$(g(X), d)$ is spherically complete,*

(PR2) *$f(X) \subseteq g(X)$,*

(PR3) *if $gx \neq fx = gy$, then $d(fx, fy) < d(fx, gx)$,*

(PR4) *if $d(gx, gy) \leq d(gx, fx)$, then $d(gy, fy) \leq d(gx, fx)$.*

Then there is $x \in X$ such that $fx = gx$.

Proof. By condition (PR2) we can consider the ultrametric ball

$$B_{d(fx,gx)}(gx) = \{gy \mid d(gy, gx) \leq d(fx, gx)\}$$

in (X, d) for every $x \in X$. Then we define (not necessarily ultrametric) balls on X as:

$$B_x := g^{-1}(B_{d(fx,gx)}(gx)) = \{y \in X \mid d(gy, gx) \leq d(fx, gx)\}$$

and set

$$\mathcal{B} := \{B_x \mid x \in X\}.$$

We have that $B_x \neq \emptyset$ since $x \in B_x$, so $(X, \mathcal{B})$ is a ball space. To prove that it is S_2, we will show first that $B_y \subseteq B_x$ for every $y \in B_x$. Take $y \in B_x$ and $z \in B_y$. That means that $d(gx, gy) \leq d(fx, gx)$ and $d(gy, gz) \leq d(fy, gy)$. By condition (PR4) we have $d(gy, fy) \leq d(gx, fx)$, thus by the ultrametric triangle law $d(gx, gz) \leq d(gx, fx)$, so $z \in B_x$.

Consider a nest of balls $(B_{x_i})_{i \in I}$ in $(X, \mathcal{B})$. Then $(g(B_{x_i}))_{i \in I}$ is a nest of closed ultrametric balls in $(g(X), d)$. Since $(g(X), d)$ is spherically complete by (PR1), there is gz in the intersection of the nest $(g(B_{x_i}))_{i \in I}$ for some $z \in X$. But then $z \in B_{x_i}$ and $B_z \subseteq B_{x_i}$ for every $i \in I$, which shows that B_z is contained in the intersection of the nest $(B_{x_i})_{i \in I}$. We have now proved that $(X, \mathcal{B})$ is an S_2 ball space.

By Condition (PR2), for every $x \in X$ there is $y \in X$ such that $gy = fx \in B_{d(fx,gx)}(gx)$. Then also $y \in B_x$, so $fx = gy \in f(B_x) \cap g(B_x)$ and condition (CS1) of Theorem 5.1.3 is satisfied.

Take a ball $B_x \in \mathcal{B}$. If $g(B_x) = B_{d(fx,gx)}(gx)$ is not a singleton, then $fx \neq gx$. In this case, $fx = gy$ for some $y \neq x$. We have that $y \in B_x$, so $B_y \subseteq B_x$. By using condition (PR3) we obtain:

$$d(gy, gx) = d(fx, gx) > d(fx, fy) = d(gy, fy).$$

This shows that $x \notin B_y$ and therefore, $B_y \subsetneqq B_x$. Hence, condition (CS2) is also satisfied.

Now we can apply Theorem 5.1.3 to obtain a coincidence point. $\square$

Chapter 6

B_x type Coincidence Point Theorems

In this chapter we will introduce a general B_x type theorem that is not itself a coincidence point theorem, but allows high flexibility in its applications. In the second section we will derive from the general B_x type theorem coincidence point theorems for two distinct cases which are based on either the domain or codomain of the functions under consideration being chosen to be a ball space. In the last section we will introduce three different types of coincidence point theorems for ultrametric spaces and find the logical relation between them.

6.1 A general B_x type theorem

Theorem 6.1.1 (Basic Theorem)**.** *Take a set X and a ball space Z. Let $P(x)$ be any assertion about the element $x \in X$. Assume that there is a function*

$$X \ni x \longmapsto B_x \in \mathcal{B}(Z)$$

such that:

($*$) if $(B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(Z)$ and $P(x_i)$ holds for all $i \in I$, then there exists some $y \in X$ such that $P(y)$ holds and B_y is a singleton or $B_y \subsetneqq \bigcap_{i \in I} B_{x_i}$.

Then for every $x_0 \in X$ such that $P(x_0)$ holds, there is $z_0 \in X$ such that $P(z_0)$ holds and B_{z_0} is a singleton contained in B_{x_0}.

The condition ($*$) can be broken down into two conditions:

($*1$) If B_x is not a singleton and $P(x)$ holds, then there exists $y \in X$ such that $B_y \subsetneqq B_x$ and $P(y)$ holds.

($*2$) If $(B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(Z)$ and $P(x_i)$ holds for all $i \in I$, then there exists some $y \in X$ such that $P(y)$ holds and $B_y \subseteq \bigcap_{i \in I} B_{x_i}$.

In applications, condition ($*$) is often checked by checking the two cases ($*1$) and ($*2$) separately.

Proof. Take $x_0 \in X$ such that $P(x_0)$ holds. Then the set $S = \{B_x \subseteq B_{x_0} \mid x \in X \text{ and } P(x) \text{ holds}\}$ contains B_{x_0} and is thus nonempty. By ($*2$), it is downward inductively ordered by inclusion. Hence by Zorn's Lemma, there is a minimal element B_{z_0} in S. Suppose that B_{z_0} is not a singleton. Then by ($*1$) there exists $y \in X$ such that $B_y \subsetneqq B_{z_0}$ and $P(y)$ holds. Thus $B_y \in S$ which contradicts the minimality of B_{z_0}. Therefore B_{z_0} must be a singleton. Since $B_{z_0} \in S$, $P(z_0)$ must hold. $\square$

6.2 B_x type coincidence point theorems for ball spaces

We take two sets X and Y and functions $f, g : X \to Y$. In the first application of Theorem 6.1.1, we consider the set X to be a ball space, and take the assertion $P(x)$ to say that $f(B_x) \bigcap g(B_x) \neq \emptyset$.

Theorem 6.2.1. *Take a ball space X, a set Y, and functions $f, g : X \to Y$. Assume that there is a function*

$$X \ni x \longmapsto B_x \in \mathcal{B}(X)$$

such that

(A1) *if B_x is not a singleton and $f(B_x) \bigcap g(B_x) \neq \emptyset$, then there is $y \in X$ such that $B_y \subsetneqq B_x$ and $f(B_y) \bigcap g(B_y) \neq \emptyset$,*

(A2) *if $(B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(X)$ such that $f(B_{x_i}) \bigcap g(B_{x_i}) \neq \emptyset$ for all $i \in I$, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i \in I} B_{x_i}$ and $f(B_y) \bigcap g(B_y) \neq \emptyset$.*

Then for every $x_0 \in X$ such that $f(B_{x_0}) \cap g(B_{x_0}) \neq \emptyset$, there is $z \in B_{x_0}$ such that $fz = gz$.

Proof. Apply Theorem 6.1.1 by setting $Z = X$ and take $P(x)$ to be the assertion that $f(B_x) \bigcap g(B_x) \neq \emptyset$. By the theorem, there is $z_0 \in X$ such that B_{z_0} is a singleton contained in B_{x_0} and $f(B_{z_0}) \bigcap g(B_{z_0}) \neq \emptyset$. Since B_{z_0} is a singleton, say $B_{z_0} = \{z\}$ with $z \in X$, it follows that $\emptyset \neq f(B_{z_0}) \bigcap g(B_{z_0}) = \{fz\} \bigcap \{gz\}$, hence $fz = gz$. Since $B_{z_0} \subseteq B_{x_0}$, we have that $z \in B_{x_0}$. □

Corollary 6.2.2. *Take a ball space X, a set Y, and functions $f, g : X \to Y$. Assume that there is a function*

$$X \ni x \longmapsto B_x \in \mathcal{B}(X)$$

such that $f(B_x) \bigcap g(B_x) \neq \emptyset$ for all $x \in X$ and the following conditions are satisfied:

(A1′) *if B_x is not a singleton, then there is $y \in X$ such that $B_y \subsetneqq B_x$,*

(A2′) *if $(B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(X)$, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i \in I} B_{x_i}$.*

Then there is some $z \in X$ such that $fz = gz$.

Now in the next application of Theorem 6.1.1, we consider the set Y to be a ball space and take the assertion $P(x)$ to say that $fx, gx \in B_x$.

Theorem 6.2.3. *Take a ball space Y, a set X, and functions $f, g : X \to Y$. Assume that there is a function*

$$X \ni x \longmapsto B_x \in \mathcal{B}(Y)$$

such that:

(B1) *if B_x is not a singleton and $fx, gx \in B_x$, then there is $y \in X$ such that $B_y \subsetneqq B_x$ and $fy, gy \in B_y$,*

(B2) *if $(B_{x_i})_{i \in I}$ is a nest of balls such that $fx_i, gx_i \in B_{x_i}$ for all $i \in I$, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i \in I} B_{x_i}$ and $fy, gy \in B_y$.*

If there is any $x_0 \in X$ such that $fx_0, gx_0 \in B_{x_0}$, then there is some $z \in X$ such that $fz = gz$.

Proof. Apply Theorem 6.1.1 by setting $Z = Y$ and taking $P(x)$ to be the assertion that $fx, gx \in B_x$. By the theorem, there is $z \in X$ such that B_z is a singleton contained in B_x and $fz, gz \in B_z$. Since B_z is a singleton, $fz = gz$. $\square$

Corollary 6.2.4. *Take a ball space Y, a set X, and functions $f, g : X \to Y$. Assume that there is a function*

$$X \ni x \longmapsto B_x \in \mathcal{B}(Y)$$

such that $fx, gx \in B_x$ for all $x \in X$ and the following conditions are satisfied:
(B1′) *if B_x is not a singleton, then there is $y \in X$ such that $B_y \subsetneqq B_x$,*
(B2′) *if $(B_{x_i})_{i\in I}$ is a nest of balls, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i\in I} B_{x_i}$.*
Then there is some $z \in X$ such that $fz = gz$.

6.3 B_x-type coincidence point theorems for ultrametric spaces

In this section, we will introduce three theorems for ultrametric spaces. These theorems are: a special case of Corollary 6.2.4, a coincidence point theorem due to Prieß-Crampe and Ribenboim, and the ultrametric version of Theorem 5.2.1.

Theorem 6.3.1. *Let X be a set and (Y, d) an ultrametric space, and take functions $f, g : X \to Y$. For each $x \in X$, set $B_x := B(fx, gx)$. Assume that:*
(C1) *for all $x \in X$ such that B_x is not a singleton, there is $y \in X$ such that $B_y \subsetneqq B_x$,*
(C2) *if $(B_{x_i})_{i\in I}$ is a nest of balls, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i\in I} B_{x_i}$.*
Then there is $z \in X$ such that $fz = gz$.

Theorem 6.3.1 is a special case of Corollary 6.2.4. Clearly the conditions (C1) and (C2) on the ultrametric ball space $(Y, \{B_x \mid x \in X\})$ are same as the conditions (B1′) and (B2′); further, $fx, gx \in B_x$ for each $x \in X$ by definition of the ball B_x .

We will now consider the following two theorems. The first one was proved by Prieß-Crampe and Ribenboim in [12].

Theorem 6.3.2 (S. Prieß-Crampe, P. Ribenboim)**.** *Let (X, d) be an ultrametric space and $f, g : X \to X$. Assume that:*
(PR1) *$(g(X), d)$ is spherically complete,*
(PR2) *$f(X) \subseteq g(X)$,*
(PR3) *if $gx \neq fx = gy$, then $d(fx, fy) < d(fx, gx)$,*
(PR4) *if $d(gx, gy) \leq d(gx, fx)$, then $d(gy, fy) \leq d(gx, fx)$.*
Then there is $z \in X$ such that $fz = gz$.

The second theorem is an ultrametric version of Theorem 5.2.1.

Theorem 6.3.3. *Let X be an arbitrary set and (Y, d) an ultrametric space. Take functions $f, g : X \to Y$ such that*

(GU1) *$g(X)$ is spherically complete,*

(GU2) *$f(X) \subseteq g(X)$,*

(GU3) *$d(fx, fy) \leq d(gx, gy)$ for all $x, y \in X$, and if $gx \neq gy$, then $d(fx, fy) < d(gx, gy)$.*

Then the following holds:

i) there exists $z \in X$ such that $fz = gz$,

ii) if $fz = gz$ and $gz = gx$ then also $fx = gx$, and

iii) if $fz = gz$ and $fy = gy$, then $gz = gy$.

Remark 6.3.4. *Statements ii) and iii) are immediate consequences of the hypothesis and only the existence of a coincidence point is nontrivial.*

Indeed, suppose that $fz = gz$. To show ii), take $z \neq x \in X$ such that $gz = gx$. Then by (GU3), $d(fx, fz) \leq d(gx, gz) = 0$. So $fx = fz$, and since $fz = gz$, it follows that $fx = gz = gx$.

To show iii), suppose that $y, z \in X$ are coincidence points. We want to show that their images under g are the same. We have that $d(gy, gz) = d(fy, fz) \leq d(gy, gz)$ where the equation holds since y, z are coincidence points, and the inequality holds by assumption (GU3). Again by (GU3), it follows that $d(gy, gz) = 0$, and therefore $gy = gz$.

The following lemma which exhibits the logical relations between the conditions of Theorems 6.3.1, 6.3.2 and 6.3.3.

Lemma 6.3.5. *Let X be a set, (Y, d) an ultrametric space, and $f, g : X \to Y$ functions. For each $x \in X$, set $B_x := B(fx, gx)$. Then:*

1) Condition (GU3) of Theorem 6.3.3 implies conditions (PR3) and (PR4) of Theorem 6.3.2.

2) Condition (PR3) of Theorem 6.3.2 implies:

if B_x is not a singleton, then

$$\forall x, y \in X : gy = fx \Rightarrow B_y \subsetneq B_x, \tag{6.3.1}$$

and condition (PR4) of Theorem 6.3.2 implies:

$$\forall x, y \in X : gy \in B_x \Rightarrow B_y \subseteq B_x. \tag{6.3.2}$$

3) Assume that (PR2) holds, so that we can set $Y = g(X)$ in Theorem 6.3.1. Then (6.3.1) implies condition (C1) of Theorem 6.3.1, and (6.3.2) together with (PR1) implies condition (C2) of Theorem 6.3.1.

Proof. 1) a) Assume that $gx \neq gy = fx$. Then we can apply the condition (GU3) of Theorem 6.3.3 to obtain that

$$d(fx, fy) < d(gx, gy) = d(gx, fx) = d(fx, gx),$$

which proves (PR3) of Theorem 6.3.2.

b) Assume that $d(gx, gy) \le d(gx, fx)$. Then we can apply condition (GU3) of Theorem 6.3.3 to obtain that

$$d(fx, fy) \le d(gx, gy) \le d(gx, fx),$$

so

$$d(gy, fy) \le \max\{d(gy, gx), d(gx, fx), d(fx, fy)\} = d(gx, fx),$$

which proves (PR4).

2) a) Assume that B_x is not a singleton. Then $fx \ne gx$. Assume that $gy = fx$. Then by condition (PR3) of Theorem 6.3.2, $d(fx, fy) < d(fx, gx)$. So

$$d(gy, fy) = d(fx, fy) < d(fx, gx),$$

and since $gy = fx \in B_x$, we obtain from part 6 of 2.0.18 that $B_y = B(fy, gy) \subsetneqq B(fx, gx) = B_x$.

b) Assume that $gy \in B_x$. Then $d(gx, gy) \le d(gx, fx)$ and condition (PR4) of Theorem 6.3.2 gives $d(gy, fy) \le d(gx, fx)$. By part 5 of 2.0.18, this yields that $B_y = B(gy, fy) \subseteq B(gx, fx) = B_x$.

3) a) Assume that B_x is not a singleton. Since $Y = g(X)$, there is $y \in X$ such that $fx = gy$. By 6.3.1 it follows that $B_y \subsetneqq B_x$. This proves (C1).

b) Assume that (Y, d) is spherically complete. Take a nest $\mathcal{N} = (B_{x_i})_{i \in I}$. Since (Y, d) is spherically complete, there is some $b \in \bigcap \mathcal{N}$, and since g is surjective, there is $y \in X$ such that $gy = b$. We have to show that $B_y \subseteq \bigcap \mathcal{N}$. For this, we show that $B_y \subseteq B_{x_i}$ for all $i \in I$. Since $gy \in \bigcap \mathcal{N}$, we have that $gy \in B_{x_i}$ for all $i \in I$. By 6.3.2, $B_y \subseteq B_{x_i}$ for all $i \in I$. Thus $B_y \subseteq \bigcap \mathcal{N}$, which proves (C2). □

According to this lemma, we have the following connection between the three theorems stated at the beginning of this section:

Proposition 6.3.6. *Theorem 6.3.1 implies Theorem 6.3.2, and Theorem 6.3.2 implies Theorem 6.3.3.*

In the following, we will illustrate the use of Theorem 6.3.1 by deriving Theorem 6.3.3 directly from it.

Proof. We assume that the conditions of Theorem 6.3.3 are satisfied. Then as in part 3) of Lemma 6.3.5, we can take $Y = g(X)$ in Theorem 6.3.1, and we set

$$B_x := B(fx, gx) \in \mathcal{B}(Y)$$

for every $x \in X$.

Take $x \in X$ and assume that B_x is not a singleton, i.e., $fx \ne gx$. Since $f(X) \subset g(X)$, there is $y \in X$ such that $gy = fx \ne gx$. By condition (GU3), we have:

$$d(gy, fy) = d(fx, fy) < d(gx, gy).$$

By part 6 of 2.0.18, it follows that $B_y = B(fy, gy) \subsetneqq B(fx, gx) = B_x$, which proves (C1) of Theorem 6.3.1.

Take a nest of balls $\mathcal{N} = (B_{x_i})_{i\in I}$. Since $g(X)$ is spherically complete, there is $gy \in \bigcap B_{x_i}$ for some $y \in Y$. We wish to show that $B_y \subseteq \bigcap B_{x_i}$. By the ultrametric triangle inequality, we obtain for all $i \in I$:

$$d(fy, gy) \leq \max\{d(fy, fx_i), d(fx_i, gx_i), d(gx_i, gy)\}.$$

We have that

$$d(fy, fx_i) \leq d(gy, gx_i) \leq d(fx_i, gx_i),$$

where the first inequality follows from (GU3) and the second inequality holds since $gy \in B_{x_i} = B(fx_i, gx_i)$. By part 5 of 2.0.18, it follows that $B_y = B(fy, gy) \subseteq B(fx_i, gx_i) = B_{x_i}$. Therefore $B_y \subseteq \bigcap \mathcal{N}$, which proves (C2) of Theorem 6.3.1.

Now by Theorem 6.3.1, there is $z \in X$ such that $fz = gz$. $\square$

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