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"Notes on Russell's Theory of Number"

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by

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FOREWORD

This paper treats the development of the real number system. As the title suggests, it is based on the theory of number as presented by Bertrand Russell in his two works, the "Introduction to Mathematical Philosophy" and the "Principles of Mathematics". My chief aim has been to reduce the concept of 'number' to such logical concepts as 'class' and 'relations'. The first part of this paper deals with these concepts and the latter parts with their applications to 'number'. Regarding the operations between numbers, much is left undone. I merely offer the essential definitions. Certain refinements of these operations, such as the associative and distributive laws of algebra, are omitted. These omissions are not due to the fact that such laws are unimportant or that they cannot be derived from 'number' as defined in this paper, but to the fact that I discuss here only the essential features of the number system and not the various laws which may be deduced from these.

References are given that the reader may amplify these notes should he so desire, and a bibliography is appended. I wish to express my appreciation of the assistance given to me by Dr. G.H. Ling in the presentation of this thesis.

H.M.

PART I

CLASSES AND RELATIONS.

There are three main schools of thought in mathematics; Formalistic, Intuitionist, and Logistic[1], but we shall be concerned only with the last of these. Briefly stated, the thesis of the Logistic school is that pure mathematics is a branch of logic. Russell is regarded as being the chief exponent of this school and his views as contained in the "Introduction to Mathematical Philosophy" are expressed in what follows.

There are two methods of mathematical investigation. The first method is constructive. We adopt a set of premises as for example the natural numbers 1,2,3..... and deduce results which necessarily follow. In fact, all traditional pure mathematics can be derived completely from the natural numbers by using propositions of logic concerning these natural numbers[2]. That is to say; if we take the natural numbers 1,2,3..... as our starting point and subject this initial premise to a series of logical deductions, the whole of pure mathematics can be made to follow by implication. This type of mathematical investigation proceeds from fundamental principles to those more complex, the reasoning is in general deductive, and the results are justified only if the principles themselves are justified. There is another method of mathematical investigation which is analytical[3] and this is the method with which Russell is prim-

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1. See Black "Nature of Mathematics" p.7.
2. Russell "Introduction to Mathematical Philosophy" p.4.
3. Russell "Introduction to Mathematical Philosophy" p.1.

arily concerned. We seek to establish such concepts as 'class', 'relation' and 'order', out of which our former starting point, the natural numbers must follow as a logical consequence, and it is the purpose of Part I of this paper to discuss this latter method.

Reference has been made above to 'logical deductions', but no attempt will be made in this paper to investigate the principles of logic as this is the duty of the philosopher. We are concerned with the applications of these principles rather than with the principles themselves. Russell states[1] that, "By the help of ten principles of deduction and ten other premises of a general logical nature, all mathematics can be strictly and formally deduced; and all the entities that occur in mathematics can be defined in terms of the above twenty premises". For our purposes the important part of the above statement is that all the entities occurring in mathematics can be defined in terms of the above twenty premises of logic. In what follows, a discussion of 'class', 'relation', and 'order', will be given. If as has been suggested, these concepts can be made to depend upon logic alone and our number system can be deduced (logically) from these concepts, then it must follow that pure mathematics can be made to rest entirely upon the above principles of logic[2].

The following discussions of class, relation, and order, are to be treated as outlines of these concepts rather than

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1. Russell "Principles of Mathematics." p.4.
2. "Logic is the youth of mathematics and mathematics is the manhood of logic" ---"Introduction to Mathematical Philosophy" p.194.

as exhaustive studies. Examples will be given where possible, and specific references will be made to the works of Russell wherein these concepts are treated fully.

CLASSES

In the preface of his "Principles of Mathematics", Russell states, "In the case of classes I must confess I have failed to perceive any concept fulfilling the conditions requisite for the notion of Class", while in his Introduction to Mathematical Philosophy[1] he states, "A class may be defined in two ways...". These statements may at first appear to be contradictory. That is: In the former statement he implies that a class is a primitive notion which cannot be defined in terms of any other concept while in the latter statement he proceeds to define it. I will first discuss these two statements and show that they are not necessarily contradictory.

The important word in the above statements is 'define'. When something is defined, it may be done in two ways. (a) The concept to be defined may be replaced by another concept, the meaning of which is understood, and the properties of which may be identical with those of the former concept. For example; we shall, in a later section, define 'number' in terms of 'class'. Here the concept of number will evolve from the concept of class.

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1. Russell "Introduction to Mathematical Philosophy". p. 12.

(b) Or the concept in question may be defined by what properties it has rather than what it is. In elementary text books[1] in Euclidian geometry the question is raised, "What is a point?" The author then proceeds to say that a point has position but no area etc. Strictly speaking the question has not been answered. Such a definition does not tell us what a point is, nor does it replace the concept 'point' by a more primitive concept 'fulfilling the conditions requisite for the notion of point'. If, however, we are willing to accept the properties of a point as a definition of that concept, we may say that it has been defined by its characteristic properties.

We can apply this latter method of definition to classes. If I say, "All the people in Canada" I am referring to a collection (class) of individuals all having the property that their position is within certain definite geographical limits, and this is one of their defining properties. Again, if I say "All isosceles triangles", I have in mind a collection of three sided figures of any dimensions provided that two sides of each are equal in length. As another example, consider the class "All the perfect squares". This is a class of numbers[2] each member of which is the 'corresponding'[3] square of some cardinal number and this is the defining property of the class. Analytic geometry provides many illustrations of the

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1. See Hall and Stevens "Plane Geometry" p. 26
2. Classes do not presuppose numbers. This is merely an illustration of class by characteristic properties. "Principles of Mathematics" p. 69.
3. 'Correspondence' will be discussed in the section on relations.

class concept. For example, consider the equation $x^2+y^2=a^2$ where 'a' is any positive real number. To every such number there corresponds a circle, and a class of concentric circles is defined by this relation.

The above examples ought to illustrate how we may define classes of objects by making use of characteristics common to each member of the class, and we now give the following definition of classes in general.

If a statement (or proposition)[1] is made concerning a term x, then all of the terms x for which this statement is true constitute a class[2]. When a class of objects is so defined we say that it is defined 'intensionally'. That this definition will serve to define the given examples of classes is easily shown. Take for example, the first one. "All the people in Canada". Consider the statement or proposition. "x is an individual living in Canada". If we substitute for x, the name of some individual this statement may be true or it may be false. It certainly will be true if we actually replace x by the name of any individual who does live in Canada, and it will be false for all others. That is: Every term x satisfying the above statement is a member of the class in question; and conversely, every member x of such a class must make the statement true.

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1. See "Principles of Mathematics" p. 12.
2. Russell defines class as: "All the terms satisfying some <u>propositional function</u>". p. 20. "Principles of Mathematics" The above definition avoids the use of the term 'propositional function', but makes use of its properties.

There is another method[1] of defining certain classes. We may enumerate their members. This method, however, is inadaquate when the class is infinite[2]. For example we could define the class "All the people in Canada" by enumerating all of its members (if we had sufficient time). We could not however, enumerate all of the perfect squares. When a class is defined by the enumeration of its members it is said to be defined 'extensionally'.

Of these two methods of defining a class the intensional method is most appropriate for our purposes. It is not always necessary, for our requirements, to list or enumerate each term of a class. It is, however, essential that we have a 'test' for a given class such that if anyone should propose a term, this test will positively include the term in the class or positively exclude it. This test is supplied by the 'proposition' mentioned in the intensional definition of class. There is a special type of class called the null-class. This may be defined as the class having no members[3].

We have made reference above to finite and infinite classes. The distinction between these will be made after we have discussed relations in the next section.

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1. Objection may be taken to giving two definitions of the same thing. However, this second definition can be reduced to the former definition. For the consistency of these definitions see "Introduction to Mathematical Philosophy." p. 12.
2. Infinite classes will be discussed in the section on relations.
3. Russell "Principles of Mathematics" p. 73.

RELATIONS

The Concept of Relation : The concept of relation is fundamental and no attempt will be made here to define it[1]. However, as in the case of classes, a great deal may be known about relations without a definition.

If any pair of unfamiliar objects were placed before us and we were asked to describe it, I doubt if we would be able to give a description without in some way or other setting up a relation or comparison involving the two objects. At some point in the description we would have to distinguish one from the other. We might refer to one object as being 'to the left of' the other, 'larger than' the other, 'darker than' the other, or 'nearer than' the other. All these are merely relations between the two objects which may occur to us as we described them. In fact, language itself is merely a means of relating objects to words[2].

Notation: If a relation exists between two terms x and y, we shall use Russell's notation xRy to imply this relationship. For example, suppose R means 'to the left of', and x and y are points on a line. Then xRy would mean the point x is to the left of the point y. Or suppose R is the relation 'parent', and x and y are individuals, then xRy would mean x is a parent of y. Among the types of relations with which we are most familiar are the relations 'less than' or 'greater

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1. For a discussion of the definition of relations, see "Principles of Mathematics" p. 95.
2. This is suggested in "Functional Thinking" H. R. Hamley.

than'. If R is taken to mean greater than, and x and y are numbers, then xRy means x is greater than y. All such expressions xRy will be called 'relational propositions'[1].

Converse Relation: If x is related to y by a certain relation R, then y is related to x by a relation which we will denote by $\bar{R}$, and the relation $\bar{R}$ is said to be the converse of R. That is: If xRy is true then y$\bar{R}$x is also true, the relations R and $\bar{R}$ are said to differ in sense.

Uses of Relations in Mathematics: In mathematics, an importand use of relations is for 'ordering' classes of objects. In considering a class of objects it is a mistake to say that they have some natural order, or that some particular order is an inherent characteristic of the class itself[2]. The members of a class are capable of having several orders and we cannot say that any one of these is more natural than the others. Furthermore, an 'order' can only occur when a certain type of relation exists between the members themselves. When an order obtains between the members of a class we must be able to say of any two members, that one 'precedes' and the other 'follows'. When a class has this property it is said to be an ordered class (or ordered aggregate), and the relation which exists between the members is said to serial.

It is only by using certain types of relations that we can order classes. The following is a classification of relations and from these various types of relations we shall

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1. "Principles of Mathematics" p. 95.
2. "Introduction to Mathematical Philosophy" Russell. p. 30.

select certain combinations which will have the above property.

Asymmetrical Relations: If a relation R exists between two terms x and y, but a different relation exists between y and x, then the relation R is said to be asymmetrical. Symbolically this may be expressed as follows; If xRy is true then yRx must be false[1]. One of the simplest examples of this type of relation is the relation 'less than' between numbers. Thus if $x<y$ is true then $y<x$ is false. Or if x and y are distinct points on a line, and R is the relation 'to the left of', then xRy and yRx cannot both be true. Examples of asymmetrical relations which are not commonly used in mathematics can easily be found, as for example the relation 'parent'. In this case xRy would mean x is a parent of y. Obviously yRx cannot also be true.

Symmetrical relations; Relations which do not have the above property are called symmetrical relations. An example of such a relation is the relation 'unequal'. Thus if x and y are numbers and R means 'unequal', then xRy and yRx are both true.

It is necessary here, to introduce three new terms in connection with relations. These are domain, converse domain and field[2]. The domain of a relation is the class of terms each member of which has the given relation to something or other, while the converse domain is the class of terms to which

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1. "Principles of Mathematics" p. 218.
2. "Principles of Mathematics" p. 97. Also "Introduction to Mathematical Philosophy" p. 32.

something or other has the given relation. The field of a relation is the class composed of all the terms of the domain and converse domain. For example, let R be the relation 'owner' and consider the relational proposition xRy. Here, x represents any one who owns something and y represents anything which is owned. Hence the class of all such x's is the domain while the class of y's is the converse domain; and the field is the class consisting of both of these classes.

Transitive Relations: Suppose a term x is related to a term y by a certain relation R, and y is related to z by the same relation R. Then if X is related to z by this relation, R is said to be transitive. In other words, if xRy and yRz to-gether always imply xRz then R is transitive. When xRy and yRz always exclude xRz then R is said to be intransitive. For example, consider the relation 'ancestor'. xRy and yRz mean x is an ancestor of y, and y is an ancestor of z. It is obvious that xRz is true, hence the relation is transitive. The relation 'unequal' is an example of a relation which is not transitive, for if x, y and z are numbers and R means 'unequal' then xRy and yRz do not necessarily imply xRz, for x and z may be the same number.

Connected Relations: If x and y are any two terms in the field of a relation R, and either one of the relational propositions xRy or yRx is true, then R is said to be a connected relation[1].

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1. Russell "Introduction to Mathematical Philosophy" p.32.

Having so classified relations, we can construct or invent certain relations having some of the above properties. The purpose of this as we have already pointed out, is, to establish an order among the members of unordered classes.

<u>Serial Relations</u>[1]: Let R be a relation which is <u>asymmetrical</u>, <u>transitive,</u> and <u>connected</u>. Then by the above definitions R must have the following properties. If x, y, and z are members of the field of R, then: (1) either xRy or yRx is true. (2) xRy and yRx cannot both be true. (3) xRy and yRz together imply xRz. When R has these properties, it is said to be a <u>serial</u> relation. As an illustration of the uses of this type of relation, consider the following problem. Suppose we were given a class of objects a,b,c, and a serial relation R whose field includes the members of this class and we were to order them in accordence with this relation. We could proceed as follows:

Either aRb or bRa is true, but both are not true.
" bRc " cRb " " " " " " "
" aRc " cRa " " " " " " "

Suppose the relational propositions bRa, bRc, and aRc are true, while the others are false. Then if we regard bRc as meaning b 'precedes' c, and write it as "b,c" and use this notation for the other relational propositions, the order as determined by R is b,a,c. Moreover, if we write these in any other order,

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1. Russell "Introduction to Mathematical Philosophy" p. 42

at least one of the above relational propositions would be contradicted. For example if we expressed the order as a,c,b, this would imply that aRc, aRb, and cRb were true, but the last two are false since the relation R is asymmetrical. Similarly any other order or 'arrangement' would lead to a contradiction.

PART II

NUMBER.

In the previous sections we have been concerned with the concepts of class, relation, and order. It will be recalled that we treated classes and relations separately, and used these two notions to discuss <u>order</u>. That is: We said that a <u>class</u> could be ordered by a certain type of <u>relation</u>. While these concepts have a far wider scope than we shall discuss here, yet they do form the basis of Russell's theory of number. I reserved Part I entirely for a discussion of these in place of introducing them only as required; and while it is true that certain refinements of these such as 'one-one' relations, similarity of classes, and correspondence, must be introduced before we can discuss number, yet these follow easily from the general properties of the concepts themselves.

One-one relations can be understood best by first considering 'one-many' and 'many-one' relations. Suppose the relation R means 'square' and x and y are real numbers. Then xRy would mean "x is the square of y". This is an example of a one-many relation since there is more than one number y whose square is x. One-many relations may be defined as relations such that xRy and x'Ry cannot both be true unless x and x' are the same term.[1] 'Many-one' relations are relations such that xRy and xRy' cannot both be true unless y and y' are the same term. If R means 'squareroot' and x and y are real

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1. Russell "Introduction to Mathematical Philosophy" p. 47.

numbers then xRy means "x is a square root of y" and such a relation is many-one since x is the square root of only one number while y has more than one square root. If a relation is such that there is only one x for which xRy is true, and <u>also</u> only one y for which xRy is true, then R is said to be a one-one[1] relation.

<u>Similarity of Classes:</u> Suppose that to each member of a class A, there corresponds one and only one member of a class B, and to each member of B there corresponds one and only one member of A; then the classes A and B are said to be in 'one-one' correspondence. Another way of stating this is to say that if there is a one-one relation which correlates the members of one class each with one member of the other class, then the classes are said to be in one-one correspondence. The following example of one-one correspondence between classes is given by E.V. Huntington[2]. "The class of soldiers in an army can be put in one-one correspondence with the class of rifles which they carry, since (as we suppose) each soldier is the owner of one and only one rifle and each rifle is the property of one and only one soldier". The one-one relation in this case would be a relation of ownership, and the two classes, soldiers and rifles would be in one-one correspondence. Single valued algebraic functions present examples of this property of classes. The function $y=2x-1$ implies a one-one

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1. Russell "Principles of Mathematics" p. 113
2. "The Continuum" p. 4.

relation between a class of x's and a class of y's. To each one of a set or class of numbers x there corresponds a number y. When there is a one-one correspondence between two classes, the classes are said to be similar. Russell's definition[1] of similar classes in terms of relation, domain, and converse domain, is as follows: "One class is said to be similar to another when there is a one-one relation of which the one class is the domain, while the other is the converse domain."

Finite and Infinite Classes: One-one correspondence of classes enables us to define finite and infinite classes, from which definitions will follow a distinction between finite and infinite numbers. In order to define these we must first state what is meant by a part (or proper part) of a class. If A is a class other than the null-class, and B is a class consisting entirely of some but not all of the members of A, then B is said to be a proper part of A, There are now two possibilities in connection with A. and B. Either A can or cannot be put into one-one correspondence with its proper part B. If it is possible to do so then A is said to be an infinite class. If A is not an infinite class then it is finite.

Finite Cardinal Numbers: In this section we shall discuss the finite cardinal numbers, and unless otherwise stated we shall refer to these as 'numbers'. In a general way numbers may be regarded as properties of classes[2], and this association of numbers with classes is entirely consistent with Russell's

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1. Russell "Introduction to Mathematical Philosophy" p. 16.
2. Russell "Principles of Mathematics" p. 113.

statement[1] that mathematics can be formally deduced as a branch of logic.

Suppose we consider a certain finite class of objects. Regardless of the individual objects themselves, this class has what we shall call a number associated with it. Now if we consider another class of different objects, but such that the two classes are similar, then the two classes have some feature or characteristic in common—namely that the members of one class can be put in one-one correspondence with the members of the other. We then say that the two classes have the same cardinal number. For example, a baseball team is an instance of what we commonly call the number nine; also the gloves they carry is another instance of the same number, since (we suppose) the class of players and the class of gloves can be put in one-one correspondence. But this is only saying that the class of players is similar to the class of gloves and that these classes have the same number. We could extend our illustration so as to include other classes of any objects whatever, but with the restriction that all such classes must be similar to either of the two given classes and hence to each other[2]. We would then have before us a set (or class) of similar classes each class having a property (similarity) common to all the members of the set. A perfectly arbitrary symbol, the symbol 9, is invented to denote this set of similar classes.

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1. Page 2 of this paper.
2. Russell "Introduction to Mathematical Philosophy" p. 16.

The foregoing illustrates the use of the properties of similar classes in defining a particular cardinal **number**, and we now give Russell's definition of number in general. "A number is a class of similar classes"[1]. To some, this definition may seem repugnant. It may appear that Russell complicates matters by defining numbers in so unusual a manner. For, according to his definition, the number 2 would be the class of all couples[2], or better still, the class of all couples is the number 2. If objection were taken to the above definition, there are two features concerning it that ought to be considered. First: Does the definition involve or imply any terms which have not already been discussed and reduced to logical concepts? The answer is no. The terms class, and similar class have been built up by using only logical principles[3]. Second: Can numbers as defined be used by the applied mathematician? In order to answer this question it would be necessary to discuss the mathematical operations addition, subtraction, multiplication, and division. These operations will be treated later but it will suffice here to state that Russell's definition of number does not in any way impair the use of numbers. If we associate numbers with classes, and can say that to a particular class there corresponds a number (the number of the class), and to a number there corresponds a set of classes, then operations between numbers can be made to depend upon operations between classes. These operations

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1. Russell "Principles of Mathematics" p. 116.
2. Couple is defined on p. 135 "Principles of Mathematics"
3. If I have failed to justify this contention it is due to the brevity of this paper rather than to any inherent weakness in the argument.

as applied to classes will be discussed later and their counterparts—operations between numbers—will follow as a necessary consequence. To conclude: Cardinal numbers are symbols corresponding to sets of similar classes. Following Russell's summary[1], we say that 0 is the class of classes whose only member is the null class. There is only one class without any members and that is the null class. Hence the set of null classes has only a single member—the null class itself, and this set we call the number 0. The number 1 is defined as follows. Consider a class K which is not the null class but such that if a term x belongs to K, the class without the x is the null class, then the set of all classes similar to K is the number.1. Similarly, this method can be extended to define the finite cardinals in general. Thus if A and B are two finite similar classes, then according to the definition of number, A and B have the same number n. Let A' be a class composed of all the terms of A and K[2]; and let B' be a class composed of all of the terms of B and a class similar to K. Then since A and B are similar, A' and B' are also similar. Now consider the set of all classes similar to A and B, and the Set of all classes similar to A' and B'. The former set is, by definition, the cardinal number n, while the latter set is defined to be the number (n+1). The number (n+1) is to be re-

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1. Russell "Principles of Mathematics" p. 128.
2. Assume A and K have no members in common and are hence exclusive classes.

garded here as a composite symbol representing the cardinal number of a set of classes rather than the arithmetic sum of n and 1. The foregoing is meant only to show that if n be any finite cardinal we can define another cardinal number (n+1) different from n by using only the notion of similar classes.

PART III

OPERATIONS BETWEEN CARDINALS

Addition: Up to the present we have been concerned with defining the cardinal numbers 1,2,3,.. These symbols, it will be recalled, were used for denoting classes. However, in practice we use these symbols for other purposes, one of which is for the common arithmetic processes. The cardinal numbers 1,2,3,.. as we have defined them are merely names for sets of similar classes of objects, and this is the only use to which we can put them up to now. We have not yet defined a number system but merely a collection of symbols. A number system can be defined as a set of elements or symbols and a law for combining these symbols[1]. The law of combination, or operation, which we now develope, is the operation of addition of cardinals.

Addition as we shall regard it here, is essentially an operation between classes, and we shall secure a definition of addition of cardinal numbers by first investigating addition of classes. Russell defines addition as follows:[2] "If u and v are classes, their 'logical sum' is the class to which belongs every term which either belongs to u or belongs to v". By this definition it is seen that when two classes are combined by the above rule, a third class is defined.

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1. This is suggested by Huntington in the "Continuum"
2. Russell "Principles of Mathematics" p. 117.

This third class may be regarded as a sum-class, and is said to be the sum of the two given classes, and the law of combination or operation on the two given classes, is the operation of addition. In other words the above operation would be as follows. Let A and B be two finite[1] classes which have no member in common. Suppose C is a class defined as follows: Every term of A and of B is a term of C; while every term of C is a term either of A or of B. The class C is then said to be the sum of the classes A and B. We may now extend this definition to addition of cardinal numbers. Every finite class has a cardinal number and every finite cardinal number is the number of some class. If N_1, N_2, N_3 are the cardinal numbers of the above classes A, B, and C respectively, then N_3 is said to be the numerical sum of N_1 and N_2, and we use the notation $N_1 + N_2 = N_3$ to imply this relation. The above definition of addition of cardinals is dependent upon addition of classes, and it is due to this fact that the commutative law is implied. For in the above discussion of the classes A and B, no references were made to the order in which the terms of A and B were to be combined. It is immaterial whether we say that C contains every term of A and of B, or whether we say C contains every term of B and of A. Hence since N_1 is the cardinal number of A and N_2 the cardinal number of B, we have $N_1 + N_2 = N_3$ or $N_2 + N_1 = N_3$. A case which requires special attention is the case where N_1 and N_2 are the same number. For example, suppose the class A has the cardinal number 1, and B also has this

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1. This restriction is not necessary, but only finite numbers are to be considered here. See Section 112. Russell "Principles of Mathematics".

cardinal number (A and B are exclusive classes); then 1+1 is defined to be the cardinal number of a class D, where the class D is defined to be the logical sum of A and B. Addition as defined can be extended to any number of cardinals. Thus $N_1+N_2+N_3$ means $M+N_3$ where $N_1+N_2=M$ and since M is a cardinal number the operation may be repeated.

<u>Subtraction:</u>[1]

Subtraction, the inverse operation of addition, may be defined as in the case of addition by first defining subtraction of classes. Suppose every term of a class B is a term of another class A. Define C as the class consisting of every term of A which is not a term of B. Then if N_1, N_2, N_3 are the cardinal numbers of A, B, and C respectively, the relation existing between N_1, N_2, and N_3 may be expressed as $N_1-N_2=N_3$, and N_3 is the number obtained by <u>subtracting</u> N_2 from N_1. In case C as defined above is the null class, then its cardinal number is 0, and A and B represent the same class. Then also N_1 and N_2 are two symbols denoting the same cardinal number, and we express this by the relations $N_1-N_2=0$ or by $N_1=N_2$. This definition of subtraction is restricted to the case where every term of B is a term of A, but this restriction can be partially removed in the following manner. Suppose B' is any class which is similar to a proper part[2]

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1. Russell refers to subtraction as the inverse operation of addition. The definition given here is my own, but I have deduced it in a manner similar to that in which addition was deduced.
2. See p. 15 of this paper.

of A or to A itself; then B' is similar to the class B above, and hence has the same cardinal number N_2. As before, C can be defined in terms of A and B and hence in terms of A and B', and the above definition of subtraction applies to the numbers corresponding to A, B', and C. It is to be noted that the above operation is not applicable to all numbers. The obvious restriction is that B' must be a class similar to a proper part of A or to A itself.

Multiplication: As in the case of addition and subtraction, multiplication of numbers is defined by first considering multiplication of classes. The following definition of multiplication will serve for any finite number of numbers[1]. Let A and B be two finite exclusive classes. Suppose C is a class of classes each of whose terms is itself a class consisting of one term of A and one term of B. Then C is called the Multiplicative class of A and B. If N_1, N_2, N_3 are the cardinal numbers corresponding to A, B, and C Respectively, then N_3 is defined to be the 'product' of N_1 and N_2, and we express this relation as $N_1 \times N_2 = N_3$. Since the multiplactive class C ✓✓ does not depend upon the order in which we choose terms from A and B, we may say that C is the multiplicative class of B and A and hence $N_2 \times N_1 = N_3$. Thus the commutative law of multiplication is implied in the definition of multiplication itself. In case A is the null class, then C can have no terms

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1. Russell "Principles of Mathematics" p. 119, Section 115.

and hence is also the null class, and we have $0 \times N_2 = 0$. Similarly if B is the null class $N_1 \times 0 = 0$. This definition of a product applies to any finite number of numbers, but fails if we wish to define the product of an infinite number of 'factors'[1]. It also applies to exponentiation if we define $(a)^b$ to mean the product of b factors, each factor corresponding to one of b similar classes, and 'a' being the cardinal number of <u>each</u> of these classes.

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1. For a definition of such a product see p. 119 Section 115 Russell "Principles of Mathematics."

PART IV

Inequalities and the Ordinal Character of Number.

We have defined classes, relations, and cardinal numbers; and shown the dependence of cardinal number upon the concept of class, In fact, cardinals are classes of similar classes, and when we use for example the symbol 2, we are merely generalizing, and using a convenient method of referring to certain collections of objects. The important point is that 2 (used as a cardinal) denotes a collection of some objects or other, and nothing else.

However, we abuse the symbols 1,2,3... by making them serve more purposes than the purpose mentioned above. For example, if a group of soldiers were lined up before us, and we regarded them as a collection of objects, then they would instance some cardinal number. Suppose the cardinal number of their class is 12. But if they are given the command to 'number from the right', each soldier uses one of the above symbols 1,2,3,...12 to denote his position in the group, and not to describe classes of objects. These symbols, used in the latter sense, are used to 'label' individual objects and when so used, we say that they are used for 'counting', and they represent the 'ordinal' numbers. This double usage of the above symbols is well expressed by E.G. Phillips[1] who says: "The distinction between a cardinal and an ordinal number is

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1. "Analysis". p. 9. -- E.G Phillips

rendered difficult by the fact that each finite positive integer is made to serve two distinct purposes; it may be used to count, when it is acting in the ordinal sense, and it may be used to number when it is acting as a cardinal number. Symbolically there is no distinction whatever between a cardinal and an ordinal number, but logically there is a fundamental difference between them." In a general way this distinction might be expressed as follows: Cardinal numbers denote classes without any reference to order among the elements of each class, while ordinal numbers are 'relation-numbers' and imply an order between the individual objects.

In order to deal more fully with this ordinal character of number we shall first have to discuss the notions 'successor' and 'hereditary property'. These will be discussed in the next section on inequalities.

Inequalities: Inequalities of cardinal numbers are based on inequalities of the classes with which the cardinals are associated. When two cardinal numbers are unequal, we shall say that one number is 'less than' or 'greater than' the other; hence our procedure will be to specify what we mean when we say that a number N_1 is less than another number N_2, or N_2 is greater than N_1.

Suppose A is any class to which a term x does not belong, and B is the class composed of all the terms of A together with the term x, but no other terms. Then if N_1 and N_2 are the cardinal numbers of the classes A and B respectively, N_2

is said to be the successor of N_1[1]. Since we have defined addition of classes, and noting that the term x is a class whose cardinal number is 1, the above definition of successor amounts to the following definition: N+1 is the successor of N[2]. But this states a relation between N and N+1 and we may express this relationship for any two cardinals n and n+1 by the notation nR(n+1) which means (n+1) is the successor of n, or n is the immediate predecessor of (n+1). Now (n+1) is a cardinal number hence it must also have a successor which we may call n+2, then since the same relationship holds between n+1 and n+2 as between n and n+1, we have (n+1)R(n+2). Since any finite cardinal number always has a successor, this process can be continued indefinitely and we will now use the cardinal symbols to express this fact. The set of symbols 1,2,3,..n (n+1)... will imply that n+1 is the successor of n for any n in the set, and we shall call this set, the natural number series.

In order to define inequality between any two numbers of the set, we shall first have to discuss 'properties' of numbers, and in particular 'hereditary properties' in the natural number series. To assert that numbers have certain properties might seem to be a vague assertion since we are merely referring to a set of symbols 1,2,3,..., but these symbols represent

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1. Russell "Introduction to Mathematical Philosophy" p. 23
2. Russell "Introduction to Mathematical Philosophy" p. 21
also Phillips "Analysis" p. 9.

sets of classes by definition, and classes do have properties--their defining properties. In what follows, we shall not be concerned with enumerating all the properties of numbers or of their corresponding classes, as such a task would be impossible; but we shall be concerned with certain properties and the numbers which have these properties. If I state that m has every property that n has, I am not required to enumerate all of the properties of either. What I do state, is that if any one should name a property of n, then m will also have this property. Let us now consider the natural number series, and apply to this the foregoing discussion.

Suppose that whenever n has a certain property P, n+1 the successor of n, also has this property; then P is said to be 'hereditary' in the natural number series[1]. In other words, since it belongs to n+1, it also belongs to n+2 the successor of n+1, and hence to all the numbers that 'follow'. Consider the three numbers m, m+1, n, noting that the second is the successor of the first, and suppose n possesses every hereditary property possessed by m+1. Then n is said to be 'greater than m or m is said to be 'less than' n[2].

Let us now examine the relationship expressed between two numbers m and n when we say that m is less than n. If we call this relationship R, then mRn means 'm is less than n' or 'n is greater than m'. We shall now show that R is an

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1. Russell "Introduction to Mathematical Philosophy" p.21
2. Russell "Introduction to Mathematical Philosophy" p.35

asymmetrical relation. Suppose R were a symmetrical relation; then we could say that n is greater than m and m is greater than n. Consider the particular case when n is the successor of m. The relationship mR(m+1) is true while (m+1)Rm is not, since the successor of m+1 is not m. Hence the relationship 'less than' is asymmetrical. The transitivity of R follows immediately from definition; for if m,n, and p are three distinct numbers such that n is greater than m, p is greater than n, then p possesses every hereditary property of n while n possesses every hereditary property of m. Hence p possesses every hereditary property of m, and is, by definition, greater than m. The relationship is also <u>connected</u> since it has for its field, the cardinal numbers provided m and n are not the same number.

We have now established that the relationship 'less than' has the three defining properties of a serial relation[1]. It is then the type of relation which 'orders' an unordered class of terms. The importance of the above truth lies in the fact that we can <u>define</u> the so-called natural order of the cardinal symbols. If two distinct numbers m and n are proposed, then either m is greater than n, (m>n) or n is greater than m (n>m). Common usage enables us to express this order in two ways:

(a) By writing m and n such that n follows m to imply that m<n.

(b) By <u>counting</u>. The double purpose of the natural numbers

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1. See p. 11 of this paper.

1,2,3,..n is now apparent. If we consider, for example, the symbol 3, we may have in mind its <u>cardinal</u> properties, and when used in this sense 3 denotes a class or sets of classes. But 3 may also bring to our attention the number of numbers in the natural series which has <u>preceded</u> it. If we have before us a set of objects and we associate each of these objects with one of the numbers in the natural series 1,2,3,..., then we have assigned an order to the set since the symbols which <u>name</u> the individual objects are themselves an ordered set; and a relation which is similar[1] to the relation between the symbols, exists between the objects. Used in this sense, the symbols 1,2,3,... imply relationships and not classes, and when so used we refer to them as the ordinal numbers.

1. 'Similar relations' are discussed in Russell's "Introduction to Mathematical Philosophy". p.53

PART V

Integers, Rational, and Real Numbers.

We have seen that numbers serve two distinct purposes-- to denote classes (cardinal), and to denote relationships[1] (ordinal). Confining ourselves to the former use, we have also seen that we can carry out certain operations with some of these numbers. The uses of these operations need not be exphasized here since if we apply mathematics to every-day problems, the above operations are now indispensable to us.

However, if we confine ourselves to the cardinal numbers and attempt to carry out these operations on all such numbers, our attempts will fail. It is certainly true that for any two cardinals m and n, there is a number p such that $m+n=p$, but it is not true that a cardinal number r exists such that $m-n=r$ for all such numbers m and n. Similarly if we define the 'quotient' of m and n to be q where $m=n\times q$, there may or there may not be a cardinal number q which satisfies this condition. Since the cardinals will not permit the general use of these operations we have two alternatives. Either we must restrict the operations to such numbers as apply, and regard such an operation as $m-n$ (where $m<n$) as an impossibility; or we may retain the generality of the operation and attempt to re-define the symbols in such a manner as to ensure this generality. Mathematicians have chosen the latter course, and we have only to consider the results obtained, to justify this

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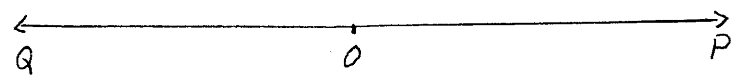
1. Russell implies this property by referring to ordinals as 'relation numbers'. "Introduction to Mathematical Philosophy" p. 63.

choice. For example, suppose a new symbol (+1) is introduced, and associated with a 'distance' OP extending to the right of a point O (Figure 1) and the symbol (-1) represents an equal distance OQ but extending to the left of O.



(Fig. 1)

Suppose also that we can combine these symbols by some rule (operation) in such a manner as to give rise to a new symbol (+n); then this new symbol may represent a definite property in some problem. In other words, physical problems may sometimes be solved symbolically by first associating symbols with the physical properties of the given problem; then operating with the symbols, and finally interpreting the symbolic result of such operations. It would seem apparent that a restricted set of symbols would naturally restrict our ability to deal with physical problems and hence the desirability of adopting a more comprehensive set of symbols would be justified. But care must be taken if we introduce new symbols, since we must not impair the logical foundation upon which mathematics has been built. An ideal state of affairs would be attained if we could define our number system logically, define operations between these numbers, and at the same time be assured that we had sufficient symbols in our number system and sufficient operations between these symbols, to solve the physical problems demanded of the mathematician.

Integers: Our immediate requirement is a number system which will permit the unrestricted use of the operations of addition and subtraction. In order to achieve this we propose a new set of symbols ...-3,-2,-1,0,+1,+2,+3,....which we shall call integers. Integers must not be regarded as an extension of the cardinals, hence +m is not to be confused with the cardinal number m. m is a class of classes, while +m is something entirely different[1]. The following is Russell's definition of the integers +m and -m. If m and n are finite cardinals (inductive numbers), then +m is the relation of (n+m) to n, and -m is the relation of n to (n+m). Hence +m and -m are not classes but relations. In particular they are converse relations[2]. Addition of integers can easily be defined in terms of addition of cardinals. Thus if m,n, and p are cardinal numbers and $n+m=p$, then for their corresponding integers, $(+m)+(+n)=(+p)$. In the case of subtraction, suppose m, n, and d are cardinals such that $m-n=d$, then $(+m)-(+n)=(+d)$ defines the corresponding operation between the integers (+m), (+n), and (+p). If $n>m$ then no such cardinal d exists, but there is a number d_1 such that $n-m=d_1$; hence the operation of subtraction of the integer (+n) from the integer (+m) may be expressed as $(+m)-(+n)=(-d_1)$[3]. The importance of the above discussion lies in the fact that we have given a meaning to the expression $(+m)-(+n)$ when the cardinal number n is greater than the cardinal number (m).

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1. Russell "Introduction to Mathematical Philosophy" p. 64.
2. See p. 8 of this paper.
3. Addition and subtraction of integers are not discussed in Russell's "Introduction to Mathematical Philosophy", but these operations as defined here, are, I believe, consistent with his theory.

Rational Numbers: The operation of multiplication of cardinal numbers can be carried out when any two numbers m and n are given. In other words, a cardinal number p exists such that $m \times n = p$. But for any two numbers m and n, there may or there may not be a cardinal number 'q' such that $m = n \times q$. If such a number q exists, it is said to be the quotient of m divided by n, and we shall employ the notation $\frac{m}{n} = q$ to express this operation. Thus $\frac{m}{n}$ has a meaning to us if and only if a cardinal q exists, and what we now wish to do is to assign a meaning to $\frac{m}{n}$ regardless of the existence of q. Suppose m and n are any two cardinal numbers, and that x and y are two cardinal numbers chosen such that $nx = my$. When m, n, x, and y are so defined, Russell defines the fraction $\frac{m}{n}$ to be that relation which holds between x and y [1]. If neither m nor n are 0, this relation is one-one. For suppose it is a one-many relation; then there is another number y' such that x bears the same relation to y' as it does to y. In particular, we could say that since $nx = my$, then $nx = my'$ and so $my = my'$ which is impossible unless y and y' are the same. A similar contradiction would arise if the relation were assumed to be many-one. A special case to be considered is when m is 0 but n is not. Then $\frac{0}{n}$ would be the relation between x and y when $nx = 0y$. This condition requires x to be the cardinal 0, but does not restrict y. Hence the relation of x to y is one-many. Similarly if n is 0 but m is not, then $0.x = my$ restricts y to the cardinal 0 but

1. Russell "Introduction to Mathematical Philosophy". p.64

imposes no restrictions on x, and the relation is many-one.[1] In the discussion of rational numbers in his "Introduction to Mathematical Philosophy", Russell does not discuss the relation (if such a relation exists) Expressed by $\frac{0}{0}$. Obviously this relation is the relation between x and y when $x.0=y.0$; but this condition is satisfied by <u>any</u> two numbers hence it is merely the relation of any number <u>to</u> any number.

By the above definition we have created a new set of symbols. These symbols are of the form $\frac{m}{n}$ where m and n are cardinals; they do not denote classes, and so far we have not devised any means of combining them by operations such as those which were used to combine cardinals. All we can say so far is that $\frac{m}{n}$ is a symbol which denotes a <u>relation</u> between <u>card</u>inals and hence between classes. However, the important feature to realize is that Russel defines ratios <u>logically</u>. The terms used in the definition were 'cardinal number' and 'relation', and these themselves have been previously defined in terms of logical concepts.

But even though these new symbols will permit of a logical definition, the question of their adaptability to every day problems will naturally arise. The usefulness of cardinals is suggested by their own definition since they symbolize classes of objects; and such classes do enter into these

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1. Russell "Introduction to Mathematical Philosophy" p.65
The relation $\frac{m}{0}$ is always the same for any m. Russell calls this the "infinity of rationals". In what follows we shall, unless otherwise stated, exclude this case. Hence in any fraction of the form $\frac{m}{n}$ it will be understood that n is not 0.

problems. Do relations as exhibited by ratios also enter into these problems? To answer this question would require a discussion of such concepts as 'quantity' and 'measurement' which is beyond the scope of this paper[1]. However, when we consider for example that the ratio $\frac{m}{l}$ states a relation between the cardinals x and y whenever $l.x=my$ it is evident that <u>some</u> relation exists between the cardinals l and m. But to say that the cardinal 'l' bears some relation to the cardinal m is to say that their corresponding <u>classes</u> are related; and in particular, if these classes have members in common, we are in reality discussing a relation of the <u>part</u> to the <u>whole</u>. It is probable that fractions were first introduced to denote such relations between <u>quantities</u>; and the above trivial example is given only to suggest (not to prove) how this property might be deduced from the logical definition of ratio.

<u>Operations between Ratios</u>: If $\frac{m}{n}$ and $\frac{p}{q}$ are ratios and the cardinal number mq is less than the cardinal np, then we shall say that $\frac{m}{n}$ is less than $\frac{p}{q}$ ($\frac{m}{n}<\frac{p}{q}$) and $\frac{p}{q}$ is greater than $\frac{m}{n}$. Equality between ratios is defined as follows: If $mq=np$ then the above ratios are equal. From the above, we can deduce two important properties of the rational number system. (1) If we are given any two ratios $\frac{m}{n}$ and $\frac{p}{q}$ which are not equal, then there is a ratio $\frac{r}{s}$ such that $\frac{m}{n}<\frac{r}{s}<\frac{p}{q}$. For if $\frac{m}{n}$ and $\frac{p}{q}$ are unequal then either $\frac{m}{n}<\frac{p}{q}$ or $\frac{m}{n}>\frac{p}{q}$. Suppose the former

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1. Russell discusses these concepts and their relation to 'number' in his "Principles of Mathematics" Chapters XIX to XXI

is true. Consider the ratio $\frac{m+p}{n+q}$, and compare this with the ratios $\frac{m}{n}$ and $\frac{p}{q}$; .m(n+q) is less than n(m+p) since mq is less than np and we have $\frac{m}{n} < \frac{m+p}{n+q}$. Similarly it may be shown that $\frac{m+p}{n+q} < \frac{p}{q}$. Let the cardinals r and s be such that r=m+p and s=n+q, then $\frac{m}{n} < \frac{r}{s} < \frac{p}{q}$. (2) The relations 'less than' and 'greater than' between ratios are serial relations. This follows from the fact that the relation 'less than' between cardinals is asymmetrical, transitive, and connected.

We can now deduce the following property of ratios. If $\frac{m}{n}$, $\frac{r}{s}$, and $\frac{p}{q}$ are any three ratios we shall say that $\frac{r}{s}$ lies between the other two if $\frac{m}{n} < \frac{r}{s} < \frac{p}{q}$ or if $\frac{m}{n} > \frac{r}{s} > \frac{p}{q}$. Since $\frac{m}{n}$ and $\frac{p}{q}$ were any unequal ratios it follows that there are always ratios between any two given ratios and no two of these are consecutive, since if they were property (1) would be denied. Furthermore, if we consider the class of all unequal ratios and we order this class by the serial relation 'less than', we have before us an infinite ordered class of terms; each term being of the form $\frac{m}{n}$ where m and n are cardinals. Such an ordered class of terms we shall call a series and since there are always terms between any two given terms we shall call the series compact[1]. Hence we can now refer to the ordered rational numbers as a compact series.

Addition: If $\frac{m}{n}$ and $\frac{p}{q}$ are ratios then their sum is defined to be $\frac{np+mq}{nq}$ which is itself a ratio since np+mq and nq are cardinals. Hence addition is always possible within the

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1. Russell "Introduction to Mathematical Philosophy". p.66

rational number system, and we shall express this operation as $\frac{m}{n} + \frac{p}{q} = \frac{np+mq}{nq}$.

<u>Subtraction</u>[1]: We shall say that the difference between $\frac{m}{n}$ and $\frac{p}{q}$ is $\frac{mq-np}{nq}$. This difference represents a rational number if and only if the cardinal mq is greater than or equal to np. This operation is expressed by $\frac{m}{n} - \frac{p}{q} = \frac{mq-np}{nq}$.

<u>Multiplication:</u> We shall say that the product of $\frac{m}{n}$ and $\frac{p}{q}$ is $\frac{mp}{nq}$ and express this operation by the relation $\frac{m}{n} \times \frac{p}{q} = \frac{mp}{nq}$.

<u>Division</u>: The quotient of $\frac{m}{n}$ by $\frac{p}{q}$ (where p is not 0) may be defined to be $\frac{mq}{np}$ which is itself a ratio since mq and np are cardinal numbers.

From these definitions it is seen that the operations of addition, multiplication, and division can be carried out between any two ratios (with the one exception that p must not be 0 in the case of division). The operation of subtraction alone is restricted since mq-np is a cardinal only if mq is greater than or equal to np. Furthermore, the commutative law of addition and multiplication between ratios follows immediately from the commutative law of addition and multiplication of cardinals.

In order to define subtraction between any two ratios it is necessary to assign a <u>meaning</u> to $\frac{mq-np}{nq}$ when np is greater than mq. This, Russell does in a manner analogous to the manner in which he defined negative integers[2]. Thus the <u>positive</u>

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1. The operations of subtraction, multiplication, and division are not discussed in Russell's "Introduction". The definitions given here, are, I believe, consistent with his theory of rationals.
2. Russell "Introduction to Mathematical Philosophy" p.66

ratio $(+\frac{p}{q})$ is the <u>relation</u> of $(\frac{m}{n}+\frac{p}{q})$ to $\frac{m}{n}$ while the <u>negative</u> ratio $(-\frac{p}{q})$ is the relation of $\frac{m}{n}$ to $(\frac{m}{n}+\frac{p}{q})$.

It has now been established that the four <u>rational</u> operations—addition, multiplication, subtraction, and division can be carried out between any two numbers $\frac{m}{n}$ and $\frac{p}{q}$ in the rational number series. Hence since operations between any two members of this series have been defined, we can refer to this series as a <u>number system</u>.

Thus to sum up: The rational number system is an infinite collection of terms each term of which is of the form $(+\frac{m}{n})$ or $(-\frac{m}{n})$. These terms may be ordered by the serial relation 'less than', and when so ordered, they form a series. Since there are terms of this series between any two given terms, the series is <u>compact</u>, and no two terms are <u>consecutive</u>. Finally we may say that the four rational operations are possible between any two members of the system[1].

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1. An additional property possessed by the rational number system is that it constitutes <u>a closed set</u> under each of the four rational operations. That is: If any two rationals are combined by one of these operations, the result of such a combination is itself a number of the set of rationals; and under these circumstances we say that the set is <u>closed</u> (one exception is that the divisor must not be 0). When the set is closed for <u>all</u> of the rational operations we say that the set is a number <u>field</u> (see Dickson "Modern Algebraic Theories" p. 150). It follows that the set of rational numbers is a field. This is not true, however, of all sets of numbers; for if we consider the set of positive and negative integers, this set is closed under the operations of addition, subtraction, and multiplication, but is not closed under division.

The Real Number System.

We have, so far, discussed four operations between numbers. These operations were first introduced as laws for combining classes; they were then applied to the symbols representing classes (i.e. the cardinals), and in this manner we defined laws for combining these cardinal numbers. It was found however, that in order to retain the generality of the inverse operations (subtraction and division) we had to revise our number system, and the system which finally evolved was the rational number system. Now if we introduce new operations; these may, or they may not, apply to all rational numbers. The operation of exponentiation involves no serious difficulty; for if m is any number there is a number M such that $m^2 = M$. However, the inverse operation, extraction of roots, is limited to certain rationals. If given a cardinal number m there may or there may not be a number n such that $n^2 = m$. Thus we see again that it is an inverse operation which presents difficulties when we attempt to ensure the free use of such operations within our number system. It is easily shown[1] that no number M other than a square number (2) has a square root within the rational number system. Here, as before, the mathematician has two courses to follow. He may restrict such an operation to those numbers as have rational square roots; or he may seek a number system whose

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1. E.G. Phillips "Analysis" p.15
2. If there is an integer m such that $m^2 = M$

elements will admit of a logical definition, and such that the proposed operation will be possible within the new number system. A simple example will perhaps suffice to show the advisability of the latter course.

In any number system, I think it is safe to say that geometric interpretations ought to be considered[1]. By choosing a suitable unit of measurement, we can associate with each rational number a line segment provided the line segment has the two properties --length and direction. The converse however, is not true. There is no rational number which denotes the length of the diagonal of a unit square. We can say the symbol $\sqrt{2}$ represents this length, and that such a symbol is an irrational number, but this is by no means gibing a logical definition of irrational numbers in general.

Russell arrives at a definition of such irrational numbers not by extending the rational system so as to include the irrationals, but by defining a new number system which he calls the real number system. The symbols used in the rational system are also used in the real number system, but although these symbols are retained, they are logically different. The following is an outline of Russell's theory of real numbers.

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1. Burkhardt in his "Theory of Functions of a complex Variable" points out this connection between numbers and their geometric counterpart. p. 3 Section 2.

We have seen that the rational number system is a compact series, and hence the class of all such rationals is an infinite ordered class. The relation which has been used to order this class is the serial relation 'less than'. We shall refer to this as the relation P. We require the two following definitions in order to discuss real numbers.

Maximum: If 'A' is a class, ordered by the serial relation P, and x is a term of A such that xPy is not true for any y which is a member of A, then x is said to be a maximum of A with respect to P[1]. If for example, the class to be ordered is the class of all ratios less than $\frac{1}{1}$, then $\frac{1}{1}$ would be the maximum of this class since it bears the relation 'less than' to no member of the class, but is itself a member of the class.

Minimum: The minimum of a class with respect to P is its maximum with respect to $\overline{P}$. In the above example there is no minimum, but if we consider the class of ratios equal to and greater than $\frac{1}{1}$, then $\frac{1}{1}$ is its minimum.

We shall use these two terms maximum and minimum to define first a particular 'irrational number' $\sqrt{2}$, and then to define any real number.

Consider the ordered class of all ratios, and suppose that we divide these into two class (sections), in such a manner that one class is composed of all the ratios whose squares are less than $\frac{2}{1}$, and the other class is composed of all the

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1. Russell "Introduction to Mathematical Philosophy". p. 70.

ratios whose squares are greater than $\frac{2}{1}$. We shall call the former class a lower section and the latter, an upper section. Logically there are four possible cases to consider. (1) There may be a maximum to the lower section and a minimum to the upper section. (2) There may be a maximum to the lower section but no minimum to the upper section. (3) There may be no maximum to the lower section but a minimum to the upper. (4) There may be neither a maximum to the lower nor a minimum to the upper.

If the lower section has a maximum ratio r, then by definition $r^2 < \frac{2}{1}$ but the square of any ratio <u>greater</u> than r would be equal to or greater than $\frac{2}{1}$. Since no rational number whose square is equal to 2 exists, it follows that the square of any number greater than r is greater than $\frac{2}{1}$. But this is not true since we can find rational numbers (by the arithmetic rule) whose squares lie between r^2 and $\frac{2}{1}$. Hence no such maximum can exist. A similar contradiction arises if we assume that the upper section has a minimum, hence case (4) obtains and the sections have neither maximum nor minimum. We then define the real irrational number $\sqrt{2}$ to be the class of all ratios whose squares are less than the rational number $\frac{2}{1}$.

We now treat the general case and give Russell's definition of any real number.

Let the ordered class of <u>all</u> ratios be divided into two classes--a lower class C_L and an upper class C_u such that every member of C_L is less than every member of C_u. Then logically there are four possibilities: (1) C_L may have a maximum and C_u a minimum. (2) C_L may have a maximum but C_u no minimum. (3) C_L may have no maximum but C_u may have a minimum. (4) C_L may have no maximum and C_u no minimum. The first of these can be eliminated since it implies the existence of two ratios r_1 and r_2 such that r_1 is a member of C_L while every other member of C_L is less than r_1. Similarly r_2 would be a member of C_u while every other member of C_u would be greater than r_2. Then since $r_1 < r_2$ there would be a ratio between r_1 and r_2 which would be a member of neither class. Thus (1) leads to a contradiction. Before discussing the three remaining cases we shall introduce the term 'upper boundary'. If C_L has a maximum, we shall call this maximum the upper boundary of C_L. If C_L has no such maximum but C_u has a minimum, we shall call this minimum the upper boundary of C_L. In other words, the upper boundary of C_L is either the maximum of CL or the minimum of C_u.[1] There can be no ambiguity since we have shown that both of these cannot exist. From this it follows that if C_L has no maximum and C_u no minimum, then no upper boundary exists.

Cases (2), (3), and (4) can now be reduced to two and only two possibilities; Either CL has an upper boundary, or it has not. If cases (2) or (3) obtain, then it has such a boundary while if case (4) obtains, it has not. If the

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1. Russell "Introduction to Mathematical Philosophy" p. 70-71

lower section C_L has an upper boundary r we shall call the ordered class of all ratios less than r a segment, and r will be the boundary of this segment, while if no such boundary r exists, then the lower section C_L is itself a segment. We can now give the following definitions:

1. A rational real number is a segment of the series of ratios which has a boundary, and this boundary is the rational number to which it corresponds.
2. An irrational number is a segment of the series of ratios which has no boundary [1].

If we regard the real number system as the totality of all such numbers as defined in 1. and 2. above, then we can say that a real number is a segment of the series of ratios.

This definition of real numbers may appear to be unduly complicated, but it is logically sound. By this definition, the real number $\sqrt{2}$ is the ordered class of all ratios whose squares are less than the rational number $\frac{2}{1}$; and the real number 1, is the ordered class of all ratios less than $\frac{1}{1}$. These classes do exist hence Russell's definition of real number, even though complicated, has the merit of employing no terms whose existence can be questioned.

The operations which were used to combine rational numbers can be applied to real numbers, hence none of the useful properties of 'number' have been lost by this definition. In each

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1. Russell "Introduction to Mathematical Philosophy" p. 72.

case, the operation in question can be reduced to the corresponding operation between rational numbers as the following definition of addition will show.

If U and V are any two real numbers, then they are segments of ratios. Let r_1 be any ratio of the class U and r_2 be any ratio of the class V. Form the sum r_1+r_2 as defined by addition of rationals. The class of sums found by choosing r_1 and r_2 in all possible ways from U and V is itself a segment of ratios, and hence determines a real number S. Then S is the sum of U and V.[1]

An important result of the above definition of real number is the principle of continuity[2]. The following example illustrates this principle. It has been shown that all ratios can be divided into two ordered classes accordingly as their squares are less than or greater than 2. There is no rational number <u>between</u> these two classes since these classes consist of <u>all</u> ratios; and when this situation arises we say we have a 'Dedekind Cut' or an 'irrational section'. But if we now consider the <u>real</u> number $\sqrt{2}$ as previously defined, it will be seen that this number corresponds to the irrational section. Thus even though there is no rational number between the two classes there is always a <u>real</u> number having this property. Another way of illustrating this principle is as follows: Consider any segment of <u>real</u> numbers. Each real

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1. Russell " Introduction to Mathematical Philosophy" p. 73
2. See "Analysis" by E. G. Phillips. p. 29. This has been termed the "<u>axiom</u> of continuity" owing to a different treatment of <u>irra</u>tionals. See Russell "Principles of Mathematics" p. 279.

number of the segment is, by definition, a segment of ratios. Hence a segment of real numbers defines a segment of ratios which is, by definition a real number. In other words every segment of the class of real numbers determines a real number. This was not true of the rational number system since the lower class in the above example defined no rational number.

Thus to conclude: Real numbers have been defined by using only the concepts of class, relation, and order, together with certain terms which themselves evolve from these concepts. That is to say; we constructed a chain of definitions, each link of this chain being a particular type of number defined in terms of the preceding type. We started with classes of elements and relations, and from these we deduced a definition of cardinal number. We then defined positive and negative integers by using the definition of cardinal number together with a relation ($+m$ being the relation of $n+m$ to n). Rational numbers evolved from cardinals by defining a rational number in terms of the previously defined cardinals and a relation R, (R being the relation of x to y when $xn=my$). Finally, real numbers were defined in terms of rationals, together with such concepts as 'segment', 'boundary' and 'series', and these latter terms were deduced from the properties of ordered classes.

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