Twisted Higgs Bundles, Topological Recursion, and Quantum Curves

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Abstract

The focus of this thesis is on the interplay between Higgs bundles and topological recursion. Our interests lie in the relationship between quantum curves and the quantization of Hitchin spectral curves, and also the relationship between Eynard-Orantin differentials and the geometry of the Hitchin moduli space.

We give an overview of existing results in the literature on quantum curves, covering the necessary material to construct a quantum curve from a meromorphic $SL(2,\mathbb{C})$-Hitchin spectral curve. Starting from the quantum curve, we offer a new perspective on the quantization that includes the spectral correspondence and $\mathbb{C}^*$-action. We view the quantization as a procedure that happens on the spectral curve, rather than the base. This idea frames quantization around the tautological section, rather than the Higgs field.

Previous works relating meromorphic Higgs bundles to topological recursion have considered non-singular models to allow the recursion to be done on a smooth Riemann surface. In this thesis, we start from an $\mathcal{L}$-twisted Higgs bundle. By studying the deformation theory of the $\mathcal{L}$-twisted moduli space, we interpret $\mathcal{L}$ as meromorphic data on a subbundle of an ordinary Higgs bundle. We encode this meromorphic data as a b-structure on the base Riemann surface and spectral curve. We then propose a so-called twisted recursion on the spectral curve, where the Eynard-Orantin differentials live in the twisted cotangent bundle. We show that the $g = 0$ twisted Eynard-Orantin differentials compute the Taylor expansion of the period matrix of a Hitchin spectral curve, mirroring a result for ordinary Higgs bundles and topological recursion. In particular, this shows that the geometry of the spectral curve is independent of the ambient space in which it resides.

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Contents

Select Notation

X	Riemann surface
$\mathcal{E}$	holomorphic vector bundle
$\mathcal{L}$	holomorphic line bundle
K_X	canonical bundle on X
$H^i(X, \bullet)$	i-th Čech cohomology
$\mathcal{O}(d)$	line bundle of degree d on $\mathbb{P}^1$
$B(z_1, z_2)$	Bergman kernel
τ_{ij}	period matrix
$\mathrm{Tot}(\mathcal{E})$	total space of the vector bundle $\mathcal{E}$
$\mathcal{M}_X^{\mathcal{L}}(r, d)$	moduli space of rank r degree d $\mathcal{L}$-twisted Higgs bundles
$\mathcal{H}$	Hitchin map
$\mathcal{B}$	Hitchin base
η	tautological section
S	spectral curve
K_p	recursion kernel at ramification point p
$W_{g,n}$	Eynard-Orantin differentials
$\mathcal{B}_{eff}$	effective Hitchin base
$\widetilde{\mathcal{B}}$	image of $\mathcal{B}_{eff}$ in $\mathcal{B}$
$\widehat{B}(z_1, z_2)$	symmetrized Bergman kernel

1 Introduction

1.1 Higgs bundles

Discovered by Hitchin [64, 65], Higgs bundles have proven to be a powerful tool and rich subject of research. A *Higgs bundle* is a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic vector bundle with a one-form valued endomorphism $\phi \in H^0(\mathrm{End}\mathcal{E} \otimes K)$, called a *Higgs field*. Appearing initially as solutions to a dimensionally-reduced version of the self-dual Yang-Mills equations, they can be generalized by allowing ϕ to have poles (*cf.* [16, 34, 87]) or by allowing ϕ to take values in another holomorphic line bundle $\mathcal{L}$ (*cf.* [81, 83]). Under sufficient stability conditions, we can consider the moduli space of Higgs bundles

$$\mathcal{M}_X^{\mathcal{L}} = \frac{\left\{\text{stable } \mathcal{L}\text{-twisted Higgs bundles}\right\}}{\text{conjugation}}.$$

The moduli space of Higgs bundles $\mathcal{M}_X^{\mathcal{L}}$ contains rich geometry, as it is a completely integrable Hamiltonian system, and admits a hyperkähler structure in the ordinary ($\mathcal{L} = K$) setting.

1.2 Topological recursion and quantum curves

Topological recursion, originally introduced in [26, 42, 44], is a recursive formula that associates to a spectral curve S, which is a complex algebraic curve arising as the spectrum of a matrix-valued function with additional conditions, a family of multi-differentials $W_{g,n}$. These *Eynard-Orantin differentials* are built out of canonical geometric data of the spectral curve and its embedding into the cotangent bundle of a base curve K_X by

- $W_{0,1}$ is a meromorphic 1-form on S (typically chosen to be the tautological section of K_X); and
- $W_{0,2}(z_1, z_2) = B(z_1, z_2)$,

with the remaining terms of $2g - 2 + n \geq 0$, being defined recursively by

$$W_{g,n+1}(z_0, \mathbf{z}) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z) \left[W_{g-1,n+2}(z, \sigma_p(z), \mathbf{z}) + \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{z}}}' W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \right]$$

where the prime signifies summation excluding the cases (g_1, I) or $(g_2, J) = (0, 0)$.

Topological recursion was developed in the context of matrix models and random matrix theory, but has since shown close ties to problems and fundamental structures in enumerative geometry. From this recursion procedure, it is possible to recover many known invariants: Weil-Peterson volumes (Mirzakhani's recursion) [43, 76], Hurwitz numbers [21, 41, 58], Virasoro constraints for two-dimensional gravity [29], and Tutte's enumeration of maps [1, 37]. There are conjectures that topological recursion is also related to knot invariants [18, 28]. As such, topological recursion

sheds light on a myriad of deep and mysterious connections between topology, algebraic and differential geometry, representation theory, combinatorics, and physics.

Based on intuition arising from matrix model theory [10, 11], there is a connection between topological recursion, and the theory of PDEs and quantization through a conjecture that topological recursion is able to reconstruct the Wentzel-Kramers-Brillouin (WKB) solution for Schrödinger-type differential equations called *quantum curves*. This quantum curve is thought of as a quantization of the spectral curve, using the usual map sending coordinates (x, y) to operators $(x, \hbar\frac{d}{dx})$. Its WKB solution ψ is constructed using the $W_{g,n}$ arising from topological recursion on that spectral curve by the equation

$$\psi(z;\beta) = \exp\Big[\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_\beta^z\cdots\int_\beta^z\Big(W_{g,n}(z_1,...,z_n) - \delta_{g,0}\delta_{n,0}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big)\Big],$$

where β is a simple pole of x. This was verified for a small class of genus-zero spectral curves in various contexts in [11,20,32,34,36] and for a larger breadth in [19]. In general, the quantum curve is not a straightforward quantization and can have $\hbar$ correction terms.

Theorem 1.2.1 (Bouchard-Eynard, [19])**.** *If S is an admissible spectral curve, then $\psi(z;\beta)$ satisfies the following differential equation:*

$$\begin{aligned}\Big[D_1D_2\ldots D_{r-1}\frac{p_0(x)}{x^{\lfloor\alpha_r\rfloor}}D_r + D_1D_2\ldots D_{r-2}\frac{p_1(x)}{x^{\lfloor\alpha_{r-1}\rfloor}}D_{r-1}\\ + \cdots + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}}D_1 + \frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}} - \hbar C_1D_1D_2\ldots D_{r-2}\frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-2}\rfloor}}\\ - \hbar C_2D_1D_2\ldots D_{r-3}\frac{x^{\lfloor\alpha_{r-2}\rfloor}}{x^{\lfloor\alpha_{r-3}\rfloor}} - \cdots - \hbar C_{r-1}\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}}\Big]\psi(z;\beta) = 0\end{aligned} \tag{1.2.1}$$

1.3 Connections between Higgs bundles and topological recursion

From the Higgs field, we can produce a *Hitchin spectral curve* by looking at the zero locus of its characteristic equation (when interpreted correctly). When this spectral curve satisfies the correct properties, it becomes a candidate curve on which to apply topological recursion. Recent work has been done to understand the relationship between topological recursion and Higgs bundles. Dumitrescu-Mulase [32–35] have looked at generalizing topological recursion to a larger class of Hitchin spectral curves and the quantization of these spectral curves, while Baraglia-Huang [6], Bertola-Korotkin [12] and Chaimanwong et al. [25] have investigated the relationship between topological recursion on Hitchin spectral curves and geometric properties of the moduli space of Higgs bundles.

1.3.1 Quantum curves

In ordinary topological recursion, the spectral curve is a local covering of $\mathbb{P}^1$. Hitchin spectral curves are globally defined objects. A suitable definition of a quantum curve for a Hitchin spectral curve needs to include the global structure. In this setting, Dumitrescu-Mulase [35] define the quantum curve of a Higgs bundle on a Riemann surface X as a Rees $\mathcal{D}$-module whose semi-classic limit is the Hitchin spectral curve. This produces a collection of differential operators on a cover of X.

If we restrict to $SL(2,\mathbb{C})$-Higgs bundles (possibly meromorphic), we can view the quantization of Hitchin spectral curves in two ways. If we equip X with a projective coordinate system and spin structure $K^{\frac{1}{2}}$, we can produce from a Hitchin section-type Higgs bundle $(\mathcal{E},\phi)$ an oper $(\mathcal{E},\nabla)$. This oper gives rise to a Rees $\mathcal{D}$-module, which is, in particular, a quantum curve. On the other hand, starting from a spectral curve S (or non-singular model $\widetilde{S}$, if necessary), we can define a global topological recursion. These two viewpoints are related by the following theorem.

Theorem 1.3.1 (WKB analysis for $SL(2,\mathbb{C})$-quantum curves, [35])**.** *The PDE topological recursion* (5.3.5) *with an appropriate choice of initial data provides an all-order WKB analysis for the generator* (5.3.6) *of the Rees $\mathcal{D}_X$-module $E(q)$ on a small neighbourhood in X of each zero or pole of q of odd order, i.e. we can use the PDE topological recursion to construct a solution to*

$$P_\alpha(x_\alpha,\hbar)\psi_\alpha(x_\alpha,\hbar)=\left[\left(\hbar\frac{d}{dx_\alpha}\right)^2-q_\alpha\right]\psi_\alpha(x_\alpha,\hbar)=0 \tag{1.3.1}$$

of the form

$$\psi_\alpha(x_\alpha,\hbar)=\exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x_\alpha)\right). \tag{1.3.2}$$

1.3.2 Geometry of Hitchin moduli spaces

The complex integrable system structure of the ordinary Hitchin moduli space $\mathcal{M}_X$ gives rise to a special Kähler structure on the Hitchin base. This special Kähler structure can be written in terms of the period matrix of the spectral curve at the point in the Hitchin base. The special Kähler metric combines with a metric along the fibres of the moduli space to produce the semi-flat metric on the regular locus of $\mathcal{M}_X$. This can be thought of as an approximation of the complete hyperkähler metric. Baraglia-Huang [6] show that the Taylor series expansion of the period matrix (and by extension, information about the hyperkähler metric) about a point in the base can be computed using the $g=0$ Eynard-Orantin differentials on the Hitchin spectral curve associated to that point.

Theorem 1.3.2 (Baraglia-Huang, [6])**.**

$$\partial_{i_1}\partial_{i_2}\dots\partial_{i_{m-2}}\tau_{i_{m-1}i_m}=-\left(\frac{i}{2\pi}\right)^{m-1}\int_{p_{i_1}\in b_{i_1}}\cdots\int_{p_m\in b_{i_m}}W_{0,m}(p_1,...,p_m) \tag{1.3.3}$$

1.4 Results

1.4.1 Quantization of Hitchin spectral curves

We consider the quantization of $SL(2,\mathbb{C})$-co-Higgs bundles on $\mathbb{P}^1$, with the Higgs field having the form

$$\phi=\begin{bmatrix}0 & \alpha\\ 1 & 0\end{bmatrix}:\mathcal{E}\to\mathcal{E}\otimes\mathcal{O}(2), \tag{1.4.1}$$

where $\alpha\in H^0(\mathrm{End}E\otimes\mathcal{O}(4))$ is a quadratic vector field.

From the viewpoint of topological recursion, we can produce a quantization using the data of the spectral curve, *i.e.* the quantum curve. We use this as motivation to interpret the quantization of Higgs bundles as a procedure on

Hitchin spectral curves. We view the tautological section η as both a classical and quantum object. As a classical object, it acts by multiplication on sections of the spectral line bundle $\mathcal{Q}$. As a quantum object, we view it as the differential operator $\hbar\frac{d}{dx}$. By studying the spectral correspondence and $\mathbb{C}^*$-action in the classical picture, we produce quantum analogues.

	Classical (in ϕ)	Classical (in η)	Quantum
action	multiplication ($\mathcal{O}_X$-linear)	multiplication ($\mathcal{O}_S$-linear)	momentum ($\mathbb{C}$-linear)
operator	Higgs field ϕ	tautological section η	momentum $\hbar\frac{d}{dx}$
sheaf	$\mathcal{O}(\mathcal{E})$	$\mathcal{O}(\mathcal{Q})$	$L^2(\mathcal{Q})$
spectrum	$\eta^2-\alpha=0$	$\eta^2-\alpha=0$	$\hbar^2\frac{d^2}{dx^2}-\alpha=0$
$\mathbb{C}^*$-fixed point	$\phi=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$\alpha=0$	$\psi\in L^2(\mathcal{Q})$ with $\hbar^2\frac{d^2}{dx^2}\psi=0$

1.4.2 Geometry of $\mathcal{L}$-twisted Hitchin spectral curves

Notable changes occur when considering $\mathcal{L}$-twisted Higgs bundles in lieu of ordinary Higgs bundles. In the twisted moduli space, the Hitchin base and the fibres of the moduli space no longer have the same dimension. This means we cannot view the Hitchin base as the space of deformation of the spectral curve as was done in the $\mathcal{L}=K$ setting. In the ordinary setting, the tautological section η (which is used to define Hitchin spectral curves) is related to the canonical symplectic structure on the cotangent bundle, however in the twisted setting, there is no longer a canonical symplectic structure, and so η is viewed purely as an algebraic object valued in $\pi^*\mathcal{L}$. The lack of a canonical symplectic structure also removes a "nice" canonical coordinate system in which to work.

We study the hypercohomology of stable $\mathcal{L}$-twisted Higgs bundles $(\mathcal{E},\phi)$ on a Riemann surface X with spectral curve S given by the two double complexes

$$D=(\delta,\wedge\phi),$$
$$D'=(\wedge\phi,\delta),$$

where $\wedge\phi$ is the differential coming from the Higgs field, and δ is the Čech differential. We find a suitable expression for the tangent space of the fibres,

$$\begin{aligned}T_{(\mathcal{E},\phi)}\mathcal{M}_X^{\mathcal{L}}(r,d)&\cong T_{\mathcal{Q}}Jac(S)\times T_{h(\mathcal{E},\phi)}\mathcal{B}\\&=H^1\left(\bigoplus_{i=0}^{r-1}\mathcal{L}^{-i}\right)\times\mathcal{B}.\end{aligned}$$

We then define a so-called effective base, which is dual to $T_{\mathcal{Q}}Jac(S)$.

Definition 1.4.1. *We call $\mathcal{B}_{eff}:=H^0\left(\bigoplus_{i=0}^{r-1}\mathcal{L}^i\otimes K\right)$ the **effective Hitchin base**.*

Choosing a section $s\in H^0(X,K^*\otimes\mathcal{L})$, we reframe our $\mathcal{L}$-twisted Higgs bundle as a $K(Z)$-valued Higgs bundle, where Z is the zero-divisor of s. This imposes the structure of b-geometry onto X and S, and thus a log-symplectic structure. In this b-geometric picture, we define the twisted Eynard-Orantin invariants (for suitably defined $\widehat{B}$ and modified K_p below).

Definition 1.4.2. *The* $\mathcal{L}$***-twisted Eynard-Orantin differentials*** $W_{g,n}$ *are meromorphic sections of the* n*-th exterior tensor product* $K_S(Z)^{\boxtimes n}$*, i.e. multi-b-differentials, defined as follows:*

The initial conditions of the recursion are given by:

$$W_{0,1}(z) = y(z)\frac{dx(z)}{x(z)} \tag{1.4.2}$$

$$W_{0,2}(z_1, z_2) = \widehat{B}(z_1, z_2). \tag{1.4.3}$$

For all $g, n \in \mathbb{N}$ *and* $2g - 2 + n \geq 0$*, define* $W_{g,n}$ *recursively by*

$$W_{g,n+1}(z_0, \boldsymbol{z}) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z) \left[W_{g-1,n+2}(z, \sigma_p(z), \boldsymbol{z}) + \sum_{\substack{g_1+g_2=g \\ I \cup J = \boldsymbol{z}}}' W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \right] \tag{1.4.4}$$

where the prime signifies summation excluding the cases (g_1, I) *or* $(g_2, J) = (0, 0)$*.*

We argue that local computations of the twisted Eynard-Orantin differentials mirror the $\mathcal{L} = K$ setting, so the twisted differentials satisfy the same properties as the ordinary ones. In particular, they also satisfy a variational formula.

Theorem 1.4.3 (Variational Formula for twisted-E-O invariants)**.** *For* $g + k > 1$*,*

$$\delta_i W_{g,k}(p_1, ..., p_k) = -\frac{1}{2\pi i} \int_{p \in b_i} W_{g,k+1}(p, p_1, ..., p_k), \tag{1.4.5}$$

where the cycle b_i *is chosen so that it contains no ramification points.*

Using this variational formula, we prove that the $g = 0$ twisted Eynard-Orantin differentials compute the Taylor expansion of the period matrix of S in the image of the effective Hitchin base.

Theorem 1.4.4.

$$\partial_{i_1} \partial_{i_2} \ldots \partial_{i_{m-2}} \tau_{i_{m-1} i_m} = -\left(\frac{i}{2\pi}\right)^{m-1} \int_{p_{i_1} \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_{0,m}(p_1, ..., p_m) \tag{1.4.6}$$

1.5 Overview

In this thesis, we will begin by reviewing the foundational theory required to engage in the later chapters. In *Chapter 2*, we review the theory of Riemann surfaces and algebraic curves over the complex numbers, with emphasis on the properties of holomorphic vector bundles. In *Chapter 3*, we review basic properties of Higgs bundles. We approach this topic in both the usual case and the $\mathcal{L}$-twisted case. We then introduce the Eynard-Orantin topological recursion in *Chapter 4*, and explicitly produce a quantum curve for the Airy spectral curve. This sets the stage for the general theorem relating spectral curves to WKB solutions in the kernel of the associated quantum curve. In *Chapter 5*, we highlight some of the work of Dumitrescu-Mulase on topological recursion for Hitchin spectral curves and their relationship to deformations of complex structures on bundles supported on the projective line. We use this as a starting point to incorporate and interpret the spectral correspondence and $\mathbb{C}^*$-action in the quantum curve framework. In *Chapter 6*, we study the deformation of the $\mathcal{L}$-twisted moduli space, and define a "twisted" version of topological

recursion, showing that the relationship between topological recursion and the geometry of the spectral curves does not depend on the space in which the spectral curve lives. Finally, in *Chapter 7*, we discuss the further aims of the thesis work, which includes a further dive into the geometry of the $\mathcal{L}$-twisted setting, the development of an invariant recursion that exploits tautological geometric features of spectral curves to produce recursion kernels, the inclusion of the $\mathcal{L}$-twisted picture from the viewpoint of quantum Airy structures (in the sense of Kontsevich-Soibelman [70]) and questions related to condensed matter physics, namely the theory of topological materials.

2 Holomorphic vector bundles on Riemann surfaces

Riemann surfaces are the main setting of the following exposition. We will be concerned chiefly with structures on, and equations written in, holomorphic vector bundles on such surfaces. In this chapter we recall some necessary definitions, state some useful theorems, and prove a few facts that will be used throughout the thesis. A standard textbook on Riemann surfaces will contain the majority of information and proofs of the theorems. For the purpose of this exposition, [47] and [67] were used. For the most part, we will assume the reader is aware of basic definitions surrounding complex manifolds, smooth bundles on such manifolds, and sheaves over such manifolds. Differential-geometric notions such as one-forms and vector fields are also assumed.

2.1 Basic notions

Recall that a Riemann surface X is a one-dimensional complex manifold. For us, X will always be smooth, compact, and connected. Equivalently, X can be regarded as a non-singular, connected, projective algebraic curve over $\mathbb{C}$. We will tend to prefer the Riemann surface point of view, but we will use the words surface and curve interchangeably when there is no confusion. To a Riemann surface X we can associate a fundamental topological, numerical invariant other than its dimension. This is the *genus* $g \in \mathbb{N}$, which is roughly speaking the number of "holes" in X. This is a complete *topological* invariant, meaning that, up to homeomorphism, X is classified by g. Below, we give an alternative definition for g that is more rigorous.

A rank m holomorphic vector bundle $\mathcal{E}$ on X is a holomorphic structure, meaning an integrable $\overline{\partial}$-operator on a smooth rank $2m$ real bundle E on X. Here, the operator is a $\mathbb{C}$-linear, $\mathcal{O}_X$-Leibnizian functional

$$\overline{\partial}_E : \Gamma(E) \to \Gamma(E \otimes \Omega^{0,1}(X)),$$

where $\mathcal{O}_X$ is the sheaf of $\mathbb{C}$-valued functions on X, $\Gamma(E)$ is the infinite-dimensional affine space of smooth sections of E, and $\Omega^{0,1}(X)$ is the bundle of anti-holomorphic one-forms on X with regards to the integrable complex structure with which X comes equipped as a complex manifold. The integrability of the holomorphic structure is precisely $\overline{\partial}_E^2 = 0$, which can be thought of as a "flatness" condition. Roughly speaking the structure picks out which sections are "holomorphic": these are the sections in $\ker(\overline{\partial}_E)$.

Note that the number m did not enter the discussion of the $\overline{\partial}$-operators definition of a holomorphic bundle. The number m is built into the definition of a real bundle, as half the dimension of its fibre, which is isomorphic as a vector space to $\mathbb{R}^{2m}$. Alternatively, we can pose a definition for $\mathcal{E}$ in which the emergence of the fibres is made plain. For this, we suppose that X is decorated by a covering of open sets $\{U_\alpha\}$. Then we can describe $\mathcal{E}$ as a complex manifold with a holomorphic projection $\pi : \mathcal{E} \to X$ that satisfies the following properties:

1. for all $z \in X$, $\pi^{-1}(z)$ is an m-dimensional complex vector space,
2. all $z \in X$ have a neighbourhood U such that

$$\phi_{U_\alpha} : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^m$$

   is a homeomorphism. The map ϕ_{U_α} is called a *trivialization*.
3. $\phi_{U_\beta} \circ \phi_{U_\alpha}^{-1} : (z, w) \mapsto (z, g_{\alpha\beta}(z)w)$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$ is holomorphic.

The map $g_{\alpha\beta}$ is called a *transition function*. We will mostly prefer the open covering point of view in what follows (except that the $\overline{\partial}$ point of view will be used later to define the Hitchin equations).

Throughout, H^i refers to the i-th Čech cohomology of a sheaf on the specified Riemann surface. Given a sheaf $\mathcal{S}$ of vector spaces on X and a locally finite covering $\{U_\alpha\}_{\alpha \in A}$ by open sets we can define the Čech complex

$$C(U_\alpha, \mathcal{S}) = C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} \dots \tag{2.1.1}$$

as follows. The n-th cochain group is defined as

$$C^n = \prod_{(\alpha_0, \dots, \alpha_n \in A^{n+1})} \mathcal{S}(U_{\alpha_0} \cap \dots \cap U_{\alpha_n}). \tag{2.1.2}$$

Elements of C^0 look like $\{f_\alpha\} \in \mathcal{S}(U_\alpha)$, elements of C^1 look like $\{f_{\alpha,\beta}\} \in \mathcal{S}(U_\alpha \cap U_\beta)$ and so on. The boundary operator $\delta_n : C^n \to C^{n+1}$ is defined by

$$(\delta_n f)_{\alpha_0 \dots \alpha_n} = \sum_i (-1)^i f_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_n}|_{U_{\alpha_0} \cap \dots \cap U_{\alpha_{n+1}}}. \tag{2.1.3}$$

The map δ_0 looks like $\{f_\alpha\} \mapsto \{f_\alpha|_{U_\alpha \cap U_\beta} - f_\beta|_{U_\alpha \cap U_\beta} \in \mathcal{S}(U_\alpha \cap U_\beta)\}$, the map δ_1 looks like $\{f_{\alpha\beta}\} \mapsto \{f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} \in \mathcal{S}(U_\alpha \cap U_\beta \cap U_\gamma)\}$, continuing in a similar fashion as an alternating sum. Due to this alternating sum, we have $\delta^2 = 0$. With that, the Čech cohomology groups are given by

$$H^n(X, \mathcal{S}) = \frac{\ker(\delta_n)}{\operatorname{im}(\delta_{n-1})}. \tag{2.1.4}$$

In particular, $H^0(X, \mathcal{S})$ is the vector space of global sections of the sheaf. When $\mathcal{S}$ is a holomorphic vector bundle on X, which is equivalent to $\mathcal{S}$ being locally free, $H^0(X, \mathcal{S})$ is the vector space of global holomorphic sections. By $h^n(X, \mathcal{S})$ we mean the complex dimension of the corresponding cohomology.

A holomorphic line bundle, denoted $\mathcal{L}$, is the special case of a rank 1 holomorphic vector bundle. Isomorphism classes of line bundles on a Riemann surface are classified by the group cohomology $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic functions on X. (The Čech cohomology for a sheaf of groups is defined in the same way as for a sheaf of vector spaces, except that δ_n is defined using the group operations.) This group sits in the short exact sequence

$$0 \to \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \to H^1(X, \mathcal{O}^*) \to \mathbb{Z} \to 0, \tag{2.1.5}$$

known as the Euler exact sequence. We can assign to a line bundle $\mathcal{L}$ an integer invariant called the *degree* of $\mathcal{L}$, denoted $\deg \mathcal{L}$, which is the image of $[\mathcal{L}] \in H^1(X, \mathcal{O}^*)$ inside of $\mathbb{Z}$. This definition does not always allow the degree to be easily computed. A more practical way to understand the degree is that it is the number of zeros of a generic

section $s \in H^0(X, \mathcal{L})$. From a rank m vector bundle $\mathcal{E}$, we can produce a line bundle $\det(\mathcal{E}) = \bigwedge^m \mathcal{E}$, which is the line bundle whose transition functions are $\det(g_{\alpha\beta})$. The notion of degree can be extended to vector bundles by defining the *degree* of $\mathcal{E}$ by $\deg(\mathcal{E}) \coloneqq \deg(\det(\mathcal{E}))$.

A *(integral) divisor* D on X is a finite linear combination of points in X with integer coefficients. The *degree* of a divisor is the sum of its coefficients. On a compact Riemann surface the set of all divisors is the free abelian group on points of X under addition. To a divisor D we can associate a line bundle $\mathcal{L}_D$ whose generic sections have zeros at points of D with positive coefficients or poles at points of D with negative coefficients.

The holomorphic cotangent bundle $\Omega^{1,0}(X)$ of X, also referred to as the *canonical bundle*, is an important line bundle that will be used throughout. We will use the notation $K_X \coloneqq \Omega^{1,0}(X)$ when we need to be clear about the base Riemann surface, and K when there is no ambiguity.

Definition 2.1.1. *Let X be a compact Riemann surface. The **genus** of X is the number $g \coloneqq h^0(X, K)$.*

In other words, we define the genus to be the maximum number of linearly independent holomorphic one-forms on X. Meromorphic sections of K, that is, one-forms that are holomorphic away from a divisor of points are familiar from complex analysis, as per:

Theorem 2.1.2 (Residue Theorem)**.** *If X is a compact Riemann surface, $a_1, ..., a_n \in X$ distinct points, then for all holomorphic one-forms ω on $X \backslash \{a_1, ..., a_n\}$ we have*

$$\sum_{k=1}^{n} \mathrm{Res}_{a_k}(\omega) = 0. \tag{2.1.6}$$

We can equip a holomorphic vector bundle $\mathcal{E}$ on X with

2.2 Covering spaces

Consider a non-constant holomorphic map between two compact Riemann surfaces $f : X \to Y$. At each point $x \in X$, the *branching degree* of f at x, denoted $v(f, x)$, is defined to be the multiplicity with which f takes the value $f(x)$ at x. The *branching order* of f at x is

$$b(f, x) \coloneqq (v(f, x) - 1), \tag{2.2.1}$$

and the *branching order* of f is

$$b(f) \coloneqq \sum_x b(f, x). \tag{2.2.2}$$

Because X is compact, $v(f, x) \neq 1$ for only finitely many terms. The sum then only contains finitely many terms, meaning that $b(f)$ is well-defined. The branching order is counting the number of times different sheets are coming together, including the multiplicity of the sheets. Points on X where the sheets come together, and are therefore not locally invertible, are called *branch points* or *ramification points*; their images in Y are called *branch values* or *ramification values*. The *ramification divisor* of f is $R = \sum_x b(f, x)[x]$.

We can now relate the genus of the two Riemann surfaces using the following classical theorem:

Theorem 2.2.1 (Riemann-Hurwitz Formula)**.** *If $f : X \to Y$ is an n-sheeted holomorphic branched cover of compact Riemann surfaces X and Y with branching order b, and g_X and g_Y denote the genus of X and Y, then:*

$$g_X = \frac{b}{2} + n(g_Y - 1) + 1. \tag{2.2.3}$$

2.3 Properties of holomorphic bundles

The Riemann sphere $\mathbb{P}^1$ is a Riemann surface that will be of particular interest in the proceeding work. We prove a few useful facts about bundles on $\mathbb{P}^1$ that will be used later on.

Proposition 2.3.1.

$$h^0(\mathbb{P}^1, K) = 0$$

Proof. Let $U_0 = \mathbb{P}^1 \backslash \{\infty\}$, $U_1 = \mathbb{P}^1 \backslash \{0\}$. A section of K looks like $f(z)dz$ on U_0 and $f(\tilde{z})d\tilde{z}$ on U_1, where f_0, f_1 are holomorphic on $\mathbb{C}$. The two one-forms need to agree on the overlap $U_0 \cap U_1$, where the coordinate changes as $\tilde{z} = z^{-1}$, and so

$$d\tilde{z} = -z^{-2}dz. \tag{2.3.1}$$

Equality of the one-forms means that we have

$$f_0(z)dz = -z^{-2}f_1(z^{-1})dz. \tag{2.3.2}$$

Expanding each side into a power series

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n dz &= -z^{-2}\sum_{n=0}^{\infty} b_n z^{-n} dz \\ &= -\sum_{n=0}^{\infty} b_m z^{-(n+2)} dz \\ &= -\sum_{n=2}^{\infty} b_{m-2} z^{-n} dz.\end{aligned}$$

Comparing coefficients of powers of z it is clear that $a_n = b_n = 0$ for all n because the left-hand side only has positive powers of z and the right-hand side only has negative powers of z. Therefore there are no non-zero global sections of K. □

Following Definition (2.1.1), Proposition (2.3.1) says that $\mathbb{P}^1$ is genus zero, which is the expected result when thinking about the genus as the number of "holes".

We are also interested in line bundles on $\mathbb{P}^1$ that are not K. We construct a line bundle on $\mathbb{P}^1$ as follows. Consider $\mathbb{P}^1$ with U_0 and U_1 as in the proof above. Define a line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$ by its transition function on the overlap $U_0 \cap U_1 = \mathbb{C}^*$,

$$g_{01} = z^n. \tag{2.3.3}$$

A section of this line bundle is given by $s_0,\ s_1$ on $U_0,\ U_1$. The sections are related by

$$s_0(z) = z^n s_1(\tilde{z}) \tag{2.3.4}$$

on the overlap. Expanding each side as a power series and relating z to $\tilde{z}$ by $\tilde{z} = z^{-1}$ yields

$$\begin{aligned}\sum_{k=0}^{\infty} a_k z^k &= z^n \sum_{k=0}^{\infty} b_k z^{-k} \\ &= \sum_{k=0}^{\infty} b_k z^{n-k}.\end{aligned}$$

Equating coefficients of power of z yields a polynomial of the form

$$s(z) = \sum_{k=0}^{n} a_k z^k. \tag{2.3.5}$$

This means that sections of $\mathcal{O}(n)$ are given by degree n polynomials.

Thinking about K in the context of these $\mathcal{O}(n)$ line bundles, we see from comparing Equation (2.3.2) and Equation (2.3.4) that as line bundles $K = \mathcal{O}(-2)$. It turns out to be true that all vector bundles on $\mathbb{P}^1$ are related to $\mathcal{O}(n)$-type bundles. This fact is the Birkhoff-Grothendieck theorem, which is stated and proved below.

Theorem 2.3.2 (Birkhoff-Grothendieck Theorem)**.** *If $\mathcal{E}$ is a rank m holomorphic vector bundle over $\mathbb{P}^1$, then:*

$$\mathcal{E} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_m), \tag{2.3.6}$$

for some numbers $a_i \in \mathbb{Z}$, unique up to permutation.

Remark. There are modern, relatively algebraic proofs of this theorem that utilize properties of exact sequences of vector bundles. A proof of this form can be found in [67]. Instead, we employ a rather elementary proof, found in [63], utilizing only ideas from linear algebra.

Proof. Let $\mathcal{E}$ be a rank m holomorphic vector bundle over $\mathbb{P}^1$. Covering $\mathbb{P}^1$ with the usual charts U_0 and U_1, we have that the change of coordinates on the intersection is $\tilde{z} = z^{-1}$, and that, up to isomorphism, $\mathcal{E}|_{U_i} \cong U_i \times \mathbb{C}^m$ with transition function $g_{01} : U_0 \cap U_1 \to GL(m, \mathbb{C})$ a polynomial matrix. This means that the two pieces of $\mathcal{E}$ can be glued together as

$$(z, w) \mapsto (z^{-1}, g_{01}(z, z^{-1})w). \tag{2.3.7}$$

Because $g_{01}(z, z^{-1})$ is an invertible matrix for all $z \neq 0$, $z^{-1} \neq 0$, we must have that

$$\det(g_{01}(z, z^{-1})) = z^n, \tag{2.3.8}$$

for some $n \in \mathbb{Z}$. A vector bundle automorphism of $U_0 \times \mathbb{C}^m$ is a map of the form $(z, w) \mapsto (z, A(z)w)$ where $A(z) \in GL(m, \mathbb{C})$ is a polynomial matrix. Similarly, an automorphism of $U_1 \times \mathbb{C}^m$ is given by an invertible matrix $B(z^{-1})$. This means that trivializations of $\mathcal{E}|_{U_i}$ are given by such vector bundle automorphisms. It follows from this idea that isomorphism classes of rank m vector bundles on $\mathbb{P}^1$ are given by equivalence classes of invertible matrices $g_{01}(z, z^{-1})$ with $\det(g_{01}) = z^n$ for some $n \in \mathbb{Z}$. The equivalence is given by $g_{01}(z, z^{-1}) \sim \tilde{g}_{01}(z, z^{-1})$ *iff* there exists invertible polynomial matrices $A(z), B(z^{-1})$ with constant determinant such that

$$\tilde{g}_{01}(z, z^{-1}) = B(z^{-1}) g_{01}(z, z^{-1}) A(z). \tag{2.3.9}$$

The vector bundle $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_m)$ is defined by the transition function $g_{01}(z, z^{-1}) = \operatorname{diag}(z^{a_1}, \ldots, z^{a_m})$. This means that it is sufficient to show there are polynomial matrices $A(z)$ and $B(z^{-1})$ such that

$$B(z^{-1}) g_{01}(z, z^{-1}) A(z) = \operatorname{diag}(z^{a_1}, \ldots, z^{a_m}), \tag{2.3.10}$$

and the a_i are uniquely determined by g_{01}.

Before proving (2.3.10), we adopt the notation $D(a_1, \ldots, a_m) = \mathrm{diag}(z^{a_1}, \ldots, z^{a_m})$, and will suppose that

$$a_1 \geq a_2 \geq \cdots \geq a_m. \tag{2.3.11}$$

We first prove uniqueness. Suppose that there are two matrices $D(a_1, \ldots, a_m)$ and $D(a'_1, \ldots, a'_m)$ that are equivalent to g_{01}. This means that $D(a_1, \ldots, a_m)$ and $D(a'_1, \ldots, a'_m)$ are equivalent, meaning there are matrices $A(z)$ and $B(z^{-1})$ such that

$$B(z^{-1})D(a_1, \ldots, a_m) = D(a'_1, \ldots, a'_m)A(z). \tag{2.3.12}$$

For a matrix A, denote by $A^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ the minor obtained by taking the determinant of the submatrix of A obtained by removing all rows and columns indexed in $\{1, \ldots, m\} \backslash \{i_1, \ldots, i_k\}$ and $\{1, \ldots, m\} \backslash \{j_1, \ldots, j_k\}$ respectively. For two matrices A and B, the product has minors given by

$$(AB)^{i_1 \ldots i_k}_{j_1 \ldots j_k} = \sum_{r_1 < \cdots < r_k} A^{i_1 \ldots i_k}_{r_1 \ldots r_k} B^{r_1 \ldots r_k}_{j_1 \ldots j_k}. \tag{2.3.13}$$

We want to apply (2.3.13) to (2.3.12). As D is a diagonal matrix, the only term in the sum of that would be non-zero is a term of the form $D^{i_1, \ldots, i_k}_{i_1, \ldots, i_k}$, as anything else would produce a row of zeroes. Applied to the left-hand side of (2.3.12),

$$\begin{aligned}
(B(z^{-1})D(a_1, \ldots, a_m))^{1,2,\ldots,k}_{i_1, \ldots, i_k} &= \sum_{r_1 < \cdots < r_k} B^{1,2,\ldots k}_{r_1 \ldots r_k}(z^{-1}) D^{r_1 \ldots r_k}_{i_1 \ldots i_k}(a_1, \ldots, a_m) \\
&= B^{1,2,\ldots k}_{r_1 \ldots r_k}(z^{-1}) D^{i_1 \ldots i_k}_{i_1 \ldots i_k}(a_1, \ldots, a_m) \\
&= B^{1,2,\ldots k}_{i_1 \ldots i_k}(z^{-1}) z^{a_{i_1} + \cdots + a_{i_k}}.
\end{aligned} \tag{2.3.14}$$

A similar computation can be done for the right-hand side. Putting them together we have

$$B^{1,2,\ldots k}_{i_1 \ldots i_k}(z^{-1}) z^{a_{i_1} + \cdots + a_{i_k}} = z^{a'_1 + \cdots + a'_2} A^{1,2,\ldots k}_{i_1 \ldots i_k}(z) \tag{2.3.15}$$

for all $i_1 < \cdots < i_k$.

Because $A(z)$ is an invertible matrix, there must be some $i_1, \ldots, i_k$ such that

$$A^{1,2,\ldots k}_{i_1 \ldots i_k}(z) \neq 0. \tag{2.3.16}$$

The left-hand side of (2.3.15) has degree $\leq a_{i_1} + \cdots + a_{i_k}$, while the right-hand side has degree $\geq a'_1 + \cdots + a'_k$. Equality of both sides means that

$$a_{i_1} + \cdots + a_{i_k} \geq a'_1 + \cdots + a'_k. \tag{2.3.17}$$

By our choice ordering, (2.3.11), we have then $a_1 + \cdots + a_k \geq a'_1 + \cdots + a'_k$ for all k. If we multiply (2.3.12) by A^{-1} on the right and B^{-1} on the left we can repeat this argument and get $a'_1 + \cdots + a'_k \geq a_1 + \cdots + a_k$ for all k. Together these results yield that for all $i = 1, ..., m$, $a_i = a'_i$, proving the uniqueness of $D(a_1, \ldots, a_m)$.

We now prove the existence. We can multiply $g_{01}(z, z^{-1})$ with by a suitable power z^k to obtain a matrix $G(z)$ that contains no negative powers of z. Utilizing column operations, in the form of a matrix A(z), we can make $G_{11} \neq 0$

and $G_{1i} = 0$ for $i = 2, \ldots, m$.

Because $\det(g_{01}) = z^n$, we must have that $G_{11} = z^{n_1}$ for some positive integer n_1. Let G_2 denote the lower-right $(m-1) \times (m-1)$ submatrix of G. We can prove existence by induction on the size of the matrix. The base case $m = 1$ is trivially true. In this fashion we supposed that there are matrices $A_2(z)$ and $B_2(z^{-1})$ such that

$$C(z) = \begin{bmatrix} 1 & 0 \\ 0 & B_2 \end{bmatrix} G \begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} z^{n_1} & 0 & \ldots & 0 \\ c_2 & z^{n_2} & & 0 \\ \vdots & 0 & \ddots & \\ c_m & & & z^{n_m} \end{bmatrix}, \tag{2.3.18}$$

where $n_1, \ldots, n_m$ are positive integers (n_1 the same as above), and $c_i = c_i(z, z^{-1})$ are polynomials. By performing row operations using a suitable matrix $B(z^{-1})$, it is possible to use the first row to make the c_i into a polynomial of only z.

Consider all polynomial matrices of the form (2.3.18) up to conjugation of eigenvalues. We can choose one for which k_1 is maximal because $k_2, \ldots, k_m \geq 0$, by construction, and $k_1 \leq \deg(\det G(z))$. We claim that $k_1 \geq k_i$ for $i = 2, \ldots, m$. Suppose that $k_1 < k_i$ for some i. A suitable row operation, again a matrix $B(z^{-1})$, lets us subtract a multiple of the first row from the i-th row to yield a matrix with $c_i = z^{k_1+1}c(z)$. Interchanging the first and i-th row we have a matrix $G(z)$ with greatest common divisor in the first row as $z^{k_1'}$, where $k_1' \geq k_1 + 1 > k_1$. If we repeat the procedure above to this new matrix we arrive at a contradiction because we assumed that k_1 was maximal. This means that we have $k_1 \geq k_i$ for $i = 2, \ldots, m$. Using column operations of the form $A(z)$ we can make it so that $\deg(c_i) \leq k_i$ by subtracting the 2-nd,...,m-th column from the first one. Because $k_1 \geq k_i \geq \deg(c_i)$, we can use row operations $B(z^{-1})$ to find a suitable multiple of s^{k_1} that equals c_i, and subtraction to make $c_2 = \cdots = c_m = 0$. This proves that there are integers $k_1, \ldots, k_m$, and matrices $A(z)$ and $B(z^{-1})$, obtained by the column and row operations from above, such that

$$B(z^{-1})z^n g_{01}(z, z^{-1})A(z) = \mathrm{diag}(z^{k_1}, \ldots, z^{k_m}). \tag{2.3.19}$$

Multiplying by z^{-n} yields (2.3.10), with $a_i = k_i - n$, and completes the proof. □

We highlight here a few properties of holomorphic bundles in general that will be of use later on:

Let $\mathcal{E}$, $\tilde{\mathcal{E}}$ be holomorphic vector bundles on X with transition functions $g_{\alpha\beta}(\mathcal{E})$ and $g_{\alpha\beta}(\tilde{\mathcal{E}})$, sections $s \in H^0(X, \mathcal{E})$ and $\tilde{s} \in H^0(X, \tilde{\mathcal{E}})$, and $f : \tilde{X} \to X$ a holomorphic map.

1. Define the *dual bundle* $\mathcal{E}^{-1} := \mathcal{E}^*$. It has transition functions $g_{\alpha\beta}(\mathcal{E}^*) = g_{\alpha\beta}(\mathcal{E})^{-1}$. The dual bundle has degree $\deg(\mathcal{E}^*) = -\deg(\mathcal{E})$.
2. Define the tensor product $\mathcal{E} \otimes \tilde{\mathcal{E}}$ by its transition functions $g_{\alpha\beta}(\mathcal{E} \otimes \tilde{\mathcal{E}}) = g_{\alpha\beta}(\mathcal{E}) \otimes g_{\alpha\beta}(\tilde{\mathcal{E}})$. A section of $\mathcal{E} \otimes \tilde{\mathcal{E}}$ looks like $s \otimes \tilde{s}$. The tensor product has degree
$$\deg(\mathcal{E} \otimes \tilde{\mathcal{E}}) = \deg(\mathcal{E})\mathrm{rank}(\tilde{\mathcal{E}}) + \mathrm{rank}(\mathcal{E})\deg(\tilde{\mathcal{E}}). \tag{2.3.20}$$
3. We can form the homomorphism bundle $Hom(\mathcal{E}, \tilde{\mathcal{E}}) \cong \mathcal{E}^* \otimes \tilde{\mathcal{E}}$. Holomorphic homomorphisms between vector bundles are holomorphic sections of $\mathcal{E}^* \otimes \tilde{\mathcal{E}}$.

4. Define the pullback of $\mathcal{E}$ on X by

$$f^*\mathcal{E} \coloneqq \{(x,q) \in \mathcal{E} \times \tilde{X} : \pi(x) = f(q)\}. \tag{2.3.21}$$

It has transition functions $g_{\alpha\beta} \circ f$. A section of $f^*\mathcal{E}$ is a holomorphic map $s : \tilde{X} \to \mathcal{E}$ such that $\pi \circ s = f$.

2.4 Finiteness theorems

There are two classic "finiteness" theorems that will prove to be useful when dealing with vector bundles on Riemann surfaces:

Theorem 2.4.1 (Serre Duality)**.** *If $\mathcal{E}$ is a holomorphic vector bundle on a compact Riemann surface X, then:*

$$H^1(X,\mathcal{E}) \cong H^0(X, K \otimes \mathcal{E}^*)^*. \tag{2.4.1}$$

Theorem 2.4.2 (Riemann-Roch Theorem)**.** *If $\mathcal{E}$ is a holomorphic vector bundle over a compact Riemann surface X of genus g, then:*

$$\dim H^0(X,\mathcal{E}) - \dim H^1(X,\mathcal{E}) = \deg \mathcal{E} + \mathrm{rk}\mathcal{E}(1-g). \tag{2.4.2}$$

2.5 Direct image sheaves

In later chapters, we will be interested in the situation where we push forward a line bundle on a ramified r-sheeted cover to the base Riemann surface. In other words, let $f : Y \to X$ be a holomorphic map between Riemann surfaces X and Y with $\deg f = r$, and let $\mathcal{L}$ be a holomorphic line bundle on Y. We want to understand the object we that get on X by pushing forward $\mathcal{L}$ via the covering map f by studying the direct image sheaf.

Definition 2.5.1. *Let $\mathcal{S}$ a sheaf on Y. The **direct image sheaf** $f_*\mathcal{S}$ on X is defined by*

$$(f_*\mathcal{S})(U) \coloneqq \mathcal{S}(f^{-1}(U)),$$

on each open set U on X.

For our purposes, we will mainly be interested in the setting where $\mathcal{S} = \mathcal{L}$. Following from the definition, we have the following facts about the direct image sheaf:

(1) $H^0(X, f_*L) = H^0(Y, L)$,

(2) For V a holomorphic vector bundle on X,

$$f_*\mathcal{O}(L \otimes f^*V) \cong \mathcal{O}(f_*\mathcal{L} \otimes V).$$

We will show the push forward of $\mathcal{L}$ is a vector bundle on X, and compute its rank and degree in terms of f, $\mathcal{L}$, and the genera of X and Y. To begin we prove that the pushfoward is in fact a vector bundle, and compute its rank.

Proposition 2.5.2. *If $f : Y \to X$ a holomorphic map between Riemann surfaces, $\mathcal{L}$ is a holomorphic line bundle on Y, then f_*L is a rank r holomorphic vector bundle on X, where $r = \deg f$.*

Proof. To prove this we need to show that $f_*\mathcal{O}(L)$ is a locally-free sheaf, i.e. we need to show that for each point $x \in X$, there is a neighbourhood U of x such that

$$f_*\mathcal{O}(L)(U) \cong \bigoplus_{i=1}^{r} \mathcal{O}(U).$$

There are two cases that we need to be considered: x is a regular (unbranched) value, and x is a branch value.

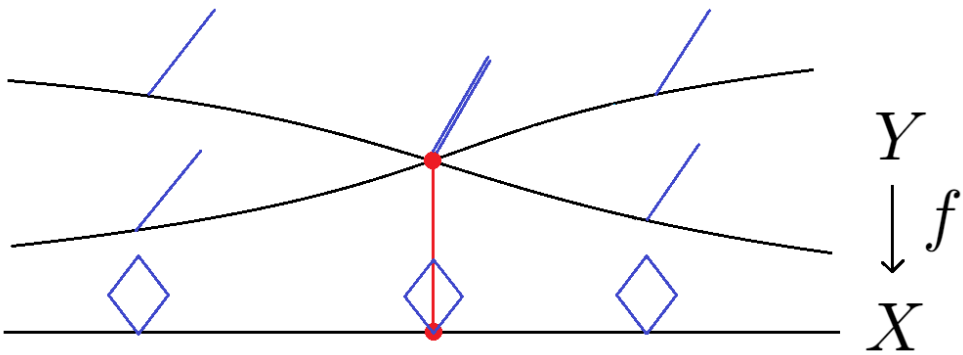

Figure 2.1: A line bundle $\mathcal{L}$ on Y being pushed forward to a vector bundle on X.

Suppose x is a regular value. In this case, the preimage $f^{-1}(x)$ consists of r distinct points, and $f' \neq 0$ at all of the points in this preimage. This means that there are r disjoint open sets $U_i \subset Y$, one around in each point in the preimage, so that $f^{-1}(U) = \bigoplus_{i=1}^{r} U_i$. On each U_i, f is a holomorphic diffeomorphism , and thus

$$f_*\mathcal{O}(L)(U) = \mathcal{O}(L)(f^{-1}(U)) = \bigoplus_{i=1}^{r} \mathcal{O}(U_i).$$

Suppose x is a branch value. In this case, the preimage $f^{-1}(x)$ consists of less than r distinct points, as some number of sheets are coming together. Around a branch point, there is a number k, and local neighbourhoods $U \subset Y$, $x \in V \subset X$, where f looks like that map $z \to z^k$, and is a $w = z^k$ local coordinate on U. A section of L over V is a local holomorphic function whose Taylor expansion around 0 can be written as $h(z) = \sum_{i=0}^{\infty} a_i z^i$. To understand how this section gets pushed forward to U, we need to write $h(z)$ in terms of the local coordinate w. We do this as follows:

$$\begin{aligned}
h(z) &= \sum_{i=0}^{\infty} a_i z^i \\
&= \sum_{p=0}^{k-1} \sum_{q=0}^{\infty} a_{qk+p} z^{qk+p} \\
&= \sum_{p=0}^{k-1} \sum_{q=0}^{\infty} a_{qk+p} z^p z^{qk} \\
&= \sum_{p=0}^{k-1} z^p h_q(z^k) \\
&= h_0(w) + z h_1(w) + \cdots + z^{k-1} h_{k-1}(w),
\end{aligned}$$

where $h_q(w) = \sum_{q=0}^{\infty} a_{qk+p} w^q$. From this last expression, we see that a local section on U is a combination of k local holomorphic functions in w. The total multiplicity of branch points satisfies

$$\sum_{z \in f^{-1}(x)} k_z = \deg(f) = r,$$

which means that by taking the direct sum across all points in $f^{-1}(x)$, we have

$$f_* \mathcal{O}(L)(U) = \bigoplus_{z \in f^{-1}(x)} (\oplus_{i=1}^{k_z} \mathcal{O}(U)) = \bigoplus_{i=1}^{r} \mathcal{O}(U).$$

□

Now that we have shown that $E = f_* \mathcal{L}$ is a vector bundle, we compute its degree. Let g_X and g_Y be the genera for X and Y respectively.

Proposition 2.5.3.

$$\deg E = \deg \mathcal{L} + (1 - g_Y) - \deg f(1 - g_X) \tag{2.5.1}$$

Proof. We first recall the following fact about vector bundles. If V is a vector bundle on X, then for a sufficiently large integer n,

$$H^0(X, V \otimes (M^*)^n) = 0,$$

where M is a choice of ample line bundle on X with $\deg M = 0$.

We apply this to the vector bundle $V^* \otimes K_X$, and use Serre duality to find that

$$H^1(X, V \otimes M^n)^* \cong H^0(X, V^* \otimes K_X \otimes (M^*)^n) = 0.$$

This means that for large enough n, we have

$$H^1(Y, \mathcal{L} \otimes f^* M^n) = 0,$$

$$H^1(X, f_* \mathcal{L} \otimes M^n) = 0.$$

We can plug this into a Riemann-Roch computation to get

$$\dim H^0(Y, \mathcal{L} \otimes f^* M^n) = \deg \mathcal{L} + rn + (1 - g_Y)$$

$$\dim H^0(X, f_* \mathcal{L} \otimes M^n) = \deg f_* \mathcal{L} + rn + r(1 - g)$$

Using facts (1) and (2) about the direct image sheaf, these dimensions must equal, and so we have

$$\deg E = \deg \mathcal{L} + (1 - g_Y) - \deg f(1 - g_X).$$

□

2.6 The prime form and fundamental differentials

In order to define and study topological recursion in later chapters, we will need to make use of certain fundamental differentials on a Riemann surface and their properties. Such objects are studied in depth in [46, 78].

Let X be a Riemann surface of genus g. Choose a symplectic basis $\langle A_1, ..., A_g, B_1, ..., B_g \rangle$ for $H_1(X, \mathbb{Z})$[1]. Let $v_1, \ldots, v_g$ be a basis of holomorphic differentials, normalized by

$$\int_{A_j} v_i = \delta_{ij}. \tag{2.6.1}$$

With respect to the symplectic basis, the period matrix τ of X is given by

$$\tau_{ij} = \int_{B_j} v_i. \tag{2.6.2}$$

We can identify the Jacobian of X as $Jac(X) = Pic^0(X)$, which is isomorphic to $Pic^{g-1}(X)$. The *theta divisor* Θ of $Pic^{g-1}(X)$ is defined by

$$\Theta = \{\mathcal{L} \in Pic^{g-1}(X) | \dim H^1(X, \mathcal{L}) > 0\}.$$

Consider the diagram

$$\begin{tikzcd}
 & Jac(X) & \\
 & X \times X \arrow[u, "\delta"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & X
\end{tikzcd}$$

where π_j denotes projection onto the jth component, and

$$\delta : X \times X \ni (p, q) \longmapsto p - q \in Jac(X).$$

Definition 2.6.1. *The **prime form** $E(z_1, z_2)$ is defined as a holomorphic section*

$$E(p, q) \in H^0\left(X \times X, \pi_1(K)^{-\frac{1}{2}} \otimes \pi_2(K)^{-\frac{1}{2}} \otimes \delta^*(\Theta)\right)$$

where we choose Riemann's spin structure (or the Szegö kernel) $K^{\frac{1}{2}}$, which has a unique global section up to the constant multiplication ([46] Theorem 1.1).

The prime form satisfies the following properties:

- $E(p, q)$ vanishes to first order along the diagonal $\Delta \in X \times X$, and is otherwise nonzero.
- $E(p, q) = -E(q, p)$.
- Let z be a local coordinate on X. This means that $dz(p)$ gives a local trivialization of K around p. At a point q near to p, $\delta^*(\Theta)$ is also trivialized around $(p, q) \in X \times X$ with local expression

$$E(z(p), z(q)) = \frac{z(p) - z(q)}{\sqrt{dz(p)} \cdot \sqrt{dz(q)}} \left(1 + O((z(p) - z(q))^2)\right). \tag{2.6.3}$$

[1] [19] call a Riemann surface together with such a symplectic basis a **Torelli surface**.

Definition 2.6.2. *The* ***fundamental normalized differential of the second kind*** *(or* ***Bergman kernel****)*

$$B(z_1, z_2) = d_1 d_2 \log E(z_1, z_2) \in H^0(X \times X, \pi_1^*(K) \otimes \pi_2^*(K) \otimes \mathcal{O}(2\Delta)) \tag{2.6.4}$$

is the unique bi-linear meromorphic differential on $X \times X$ that satisfies the following:

- *It is symmetric: $B(z_1, z_2) = B(z_2, z_1)$.*
- *It is holomorphic everywhere except for a double pole at $z_1 = z_2$ and can be expanded in a neighbourhood of the diagonal as*

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + O(1) dz_1 dz_2. \tag{2.6.5}$$

- *It is normalized on A-cycles:*

$$\oint_{z_1 \in A_i} B(z_1, z_2) = 0 \tag{2.6.6}$$

  for $i = 1, ..., g$.

- *The integrals along B-cycles are given by:*

$$\oint_{z_1 \in B_j} B(z_1, z_2) = 2\pi i v_j(z_2) \tag{2.6.7}$$

  for $i = 1, ..., g$.

Definition 2.6.3. *The* ***normalized differential of the third kind*** *(or* ***normalized Cauchy kernel****)*

$$\omega^{a-b}(z) = d_z \log \frac{E(a, z)}{E(b, z)} = \int_b^a B(t, z) \tag{2.6.8}$$

is the unique meromorphic differential on X satisfying;

- *It is holomorphic except for $z = a$ and $z = b$.*
- *It is normalized on A-cycles:*

$$\oint_{z \in A_i} \omega(z) = 0 \tag{2.6.9}$$

  for $i = 1, ..., g$.

- *It has a simple pole at $z = a$ with residue 1 and at $z = b$ with residue -1.*

3 Higgs bundles

3.1 Hitchin equations

Given their name by Simpson [93], the objects we now call "Higgs bundles" were first introduced by Hitchin in [64,65] as solutions of the self-dual dimensionally-reduced Yang-Mills equations, called the *Hitchin equations*, on a compact Riemann surface X. They play a key role in mathematical physics as a bridge between gauge theory and integrable systems. Our interest in studying Higgs bundles comes from their relation to spectral curves. A Higgs bundle is a holomorphic vector bundle with a K-valued endomorphism. Locally, this endomorphism looks like a matrix, so we can look at its characteristic equation. This generates a spectral curve from a Higgs bundle.

Definition 3.1.1. *Let $\mathcal{E}$ be a Hermitian vector bundle over a Riemann surface X with unitary connection A, and $\phi : \mathcal{E} \to \mathcal{E} \otimes K$ be a smooth linear map. The* ***Hitchin equations*** *are*

$$\begin{aligned} F_0(A) + \phi \wedge \phi^* &= 0 \\ \overline{\partial}_A \phi &= 0, \end{aligned}$$

where $F_0(A)$ is the trace-free curvature of A, and $\overline{\partial}_A : \mathcal{C}^\infty(\mathcal{E}) \to \Omega^{0,1}(\mathcal{E})$ is the holomorphic structure on $\mathcal{E}$ obtained by taking the $(0,1)$-part of A.

Definition 3.1.2. *(cf. [65]) A* ***Higgs bundle*** *is a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic vector bundle on a Riemann surface X and $\phi : \mathcal{E} \to \mathcal{E} \otimes K$ is a global holomorphic map. The map ϕ is called a* ***Higgs field****. The rank and degree of $(\mathcal{E}, \phi)$ are the rank and degree of $\mathcal{E}$ respectively.*

There is a striking similarity between a Higgs bundle $(\mathcal{E}, \phi)$ and a solution (A, ϕ) of the Hitchin equations. A natural question is to ask which Higgs bundles correspond to solutions of the Hitchin equations. The answer is precisely the *stable* Higgs bundles.

Definition 3.1.3. *Let $\mathcal{E}$ be a holomorphic vector bundle on a Riemann surface X. The* ***slope*** *of $\mathcal{E}$ is*

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}. \tag{3.1.1}$$

Definition 3.1.4. *A Higgs bundle $(\mathcal{E}, \phi)$ is* ***stable*** *if*

$$\mu(\mathcal{U}) < \mu(\mathcal{E}) \tag{3.1.2}$$

for all subbundles $0 \subsetneq \mathcal{U} \subsetneq \mathcal{E}$ satisfying $\phi(\mathcal{U}) \subseteq \mathcal{U} \otimes K$, and ***semi-stable*** *if equality is permitted in the slope condition.*

The equivalence between stable Higgs bundles and solutions to the Hitchin equations is an instance of the Hitchin-Kobayashi theorem (Theorem 4.3 of [65]). It depends on Uhlenbeck-Yau weak compactness arguments. As these

would require a significant detour to motivate, we omit the proof.

Before we can make sense of the relationship between Higgs bundles and solutions to the Hitchin equations, we first want to know what kinds of Riemann surfaces have stable Higgs bundles. For genus 0 Riemann surfaces, i.e. $X = \mathbb{P}^1$, we have the following result:

Proposition 3.1.5. *All rank $r \geq 2$ Higgs bundles $(\mathcal{E}, \phi)$ on $\mathbb{P}^1$ are unstable.*

Proof. Because we are dealing with $\mathbb{P}^1$ we have that $K = \mathcal{O}(-2)$, and by the Birkhoff-Grothendieck Theorem 2.3.2, we have that $\mathcal{E}$ can be written as $\mathcal{E} = \oplus_{i=1}^{r} \mathcal{O}(a_i)$, for some a_i.

The Higgs bundle ϕ is given by a matrix with components

$$(\phi)_{ij} = \phi_{ij} : \mathcal{O}(a_i) \to \mathcal{O}(a_j) \otimes \mathcal{O}(-2). \tag{3.1.3}$$

Let $a \coloneqq max_i\{a_i\}$. For every i we have

$$H^0\left(\mathcal{O}(a)^* \otimes \mathcal{O}(a_i) \otimes \mathcal{O}(-2)\right) = H^0(\mathcal{O}(a_i - a - 2)). \tag{3.1.4}$$

By the choice of a we have

$$\begin{aligned} a \geq a_i \ &\Rightarrow a_i - a \leq 0 \\ &\Rightarrow a_i - a - 2 \leq -2, \end{aligned}$$

and so

$$\dim H^0\left(\mathcal{O}(a)^* \otimes \mathcal{O}(a_i) \otimes \mathcal{O}(-2)\right) = 0 \tag{3.1.5}$$

for all i. This means that the column corresponding to a is a column of zeros and thus

$$\phi : \mathcal{O}(a) \to \mathcal{O}(a_i) \otimes \mathcal{O}(-2) \tag{3.1.6}$$

is the zero map. This shows that $\mathcal{O}(a)$ is a ϕ-invariant subbundle of $\mathcal{E}$ with $\deg(\mathcal{O}(a)) = a$ and $\mathrm{rk}(\mathcal{O}(a)) = 1$. Comparing the slopes yields

$$\begin{aligned} \mu(\mathcal{E}) &= \frac{\sum_{i=1}^{r} a_i}{r} \\ &\leq \frac{\sum_{i=1}^{r} a}{r} \\ &= \frac{ra}{r} \\ &= a \\ &= \mu(\mathcal{O}(a)) \end{aligned}$$

meaning that $(\mathcal{E}, \phi)$ is not stable. □

A similar result can be proved for genus 1 Riemann surfaces. This means the only Riemann surfaces that admit stable Higgs bundles have genus $g \geq 2$. The simplest examples of such Higgs bundles are given when $\phi = 0$. In this setting, the Hitchin equations reduce to $F(A) = 0$, whose solutions are flat unitary connections. These objects are

already well studied, and a famous result of Narasimhan-Seshadri relates flat unitary connections to stable holomorphic bundles [80]. In particular, the moduli space of stable holomorphic bundles has positive dimension $3g-3$ when the rank is 2 and $g > 1$. The Hitchin-Kobayashi correspondence can be regarded as a generalization of Narasimhan-Seshadri to the non-unitary case. The non-unitary case arises precisely when $\phi \neq 0$. An example in rank 2 of a stable Higgs bundle with $\phi \neq 0$ is given by $\mathcal{E} = K \oplus \mathcal{O}$ with Higgs field given by

$$\phi = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}. \tag{3.1.7}$$

The 1 can be interpreted as the identity endomorphism as the bottom left entry of ϕ is a map

$$K \to \mathcal{O} \otimes K = K.$$

The section α is a non-zero element of $H^0(X, \mathcal{O} \otimes K \otimes K) = H^0(X, K^2)$, called a *quadratic differential*. The form of ϕ prevents the existence of a proper invariant subbundle $\mathcal{U}$, and therefore the Higgs bundle is automatically stable. Note that $\alpha \neq 0$ necessitates that $\deg K > 0$, which forces g to be at least 2.

The dimension of $H^0(X, K^2)$ over $\mathbb{C}$ is $3g-3$, which can be computed via Riemann-Roch and Serre duality. This entails that this example generates a $(3g-3)$-dimensional family of stable Higgs bundles, none of which are equivalent to one another as $-\alpha$ is the determinant of the Higgs field and so no two choices of α give Higgs bundles that are isomorphic under change of basis in $\mathcal{E}$. This indicates that for genus 2 or larger, the moduli space of rank 2 Higgs bundles is at least $(3g-3)$-dimensional. (In fact, it is $(6g-6)$-dimensional. The fact that this is twice $3g-3$, which is the dimension of the moduli space of stable bundles, is no coincidence. This will be borne out in calculations below.)

3.1.1 Twisted Higgs bundles

Riemann surfaces of genus 0 and 1 are of interest across of variety of problems, with $\mathbb{P}^1$ playing an important role later on in this thesis. While they do not admit stable Higgs bundles, we can modify the definition of a Higgs bundle to allow for stable bundles. There are two natural ways of doing this. We can drop the holomorphic condition on the Higgs field and allow our Higgs bundle to have a Higgs fields with poles. This leads to *meromorphic Higgs bundles*, or, with more initial data, *parabolic Higgs bundles*. More generally, we can replace K with another holomorphic line bundle $\mathcal{L}$ on X, producing *twisted Higgs bundles*. More formally:

Definition 3.1.6. *(cf. [34, 87]) Let D be a divisor on X. A* ***meromorphic Higgs bundle*** *with poles at D is a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic vector bundle on a Riemann surface X and $\phi : \mathcal{E} \to \mathcal{E} \otimes K(D)$, where $K(D) = K \otimes D$.*

Definition 3.1.7. *(cf. [81, 83]) Let $\mathcal{L}$ be a holomorphic line bundle on a Riemann surface X. An* ***$\mathcal{L}$-twisted Higgs bundle*** *on X is a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic vector bundle, and $\phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$.*

Remark 3.1.8. When working with $\mathcal{L}$-twisted Higgs bundles, we will only be considering the case where $\deg \mathcal{L} > \deg K$.

Later in the thesis, we will be particularly interested in twisted Higgs bundle with $\mathcal{L} = K^*$.

Definition 3.1.9. *(cf. [82, 83]) A* ***co-Higgs bundle*** *is a K^*-twisted Higgs bundle.*

Such Higgs bundles come up in the study of generalized complex geometry as co-Higgs bundles [84], in the study of quiver varieties [88, 89], in the study of monodromy of the Hitchin map [7], and in the study of representations of fundamental groups of compact Kähler manifolds [54].

A natural question to ask is if there is an analogous set of Hitchin equations (Definition 3.1.1) associated to an $\mathcal{L}$-twisted Higgs bundle. We can suitably modify the Hitchin equations in the following way. The second equation,

$$\overline{\partial}_A \phi = 0$$

says that the Higgs field needs to be holomorphic with respect to the connection A, a condition that can be considered with no necessary change. The first equation,

$$F_0(A) + \phi \wedge \phi^* = 0,$$

makes sense because both summands are endomorphism-valued two-forms. In the twisted setting, ϕ is a section of $End(\mathcal{E}) \otimes \mathcal{L}$, so in order to make sense of the first equation, we need to choose a section $s \in H^0(K \otimes \mathcal{L}^*)$, with which to multiply ϕ. Like this, we can modify the first equation as

$$F_0(A) + s\phi \wedge (s\phi)^* = F_0(A) + |s|^2 \phi \wedge \phi^* = 0.$$

We can also easily modify our notion of stability to this setting.

Definition 3.1.10. *An $\mathcal{L}$-twisted Higgs bundle $(\mathcal{E}, \phi)$ is **stable** if*

$$\mu(\mathcal{U}) < \mu(\mathcal{E}) \tag{3.1.8}$$

*for all subbundles $0 \subsetneq \mathcal{U} \subsetneq \mathcal{E}$ satisfying $\phi(\mathcal{U}) \subseteq \mathcal{U} \otimes \mathcal{L}$, and **semi-stable** if equality is permitted in the slope condition.*

3.2 Moduli space of Higgs bundles

For this section we follow the treatment in [86]. The correspondence between solutions to the Hitchin equations and ordinary Higgs bundles descends to the level of moduli spaces, where we have an equivalence between two moduli spaces. The space of solutions yields a gauge-theoretic moduli space given by the space of solutions (A, ϕ) taken up to gauge equivalence. It has the structure of a smooth, non-compact manifold, which can be interpreted as an infinite-dimensional hyperkähler quotient. This endows the moduli space with a hyperkähler structure. On the Higgs bundle side, the algebro-geometric moduli space is formed by quotienting the space of stable pairs $(\mathcal{E}, \phi)$ by the conjugation action of holomorphic automorphisms of $\mathcal{E}$. This quotient has the structure of a non-singular, quasi-projective variety and can be interpreted as a geometric-invariant theory quotient, with stability condition given by the stability for Higgs bundles as above. This equivalence extends to the meromorphic and $\mathcal{L}$-twisted setting; however, we lose some properties in the interim, such as the hyperkälher structure.

The main object of interest for us is the Higgs bundle moduli space, although when necessary, we will appeal to this correspondence to benefit from properties of the gauge-theoretic interpretation. We will start by considering the larger class of $\mathcal{L}$-twisted Higgs bundles, and study some properties of the $\mathcal{L} = K$ case in a later subsection. More formally, we define the moduli space of $\mathcal{L}$-twisted Higgs bundles:

Definition 3.2.1. *Fix r, d coprime. The* ***moduli space of rank r, degree d $\mathcal{L}$-twisted Higgs bundles****, $\mathcal{M}_X^{\mathcal{L}}(r,d)$, is defined by the quotient*

$$\mathcal{M}_X^{\mathcal{L}}(r,d) = \frac{\{\text{stable rank } r, \text{ degree } d\ \mathcal{L}\text{-twisted Higgs bundles } (\mathcal{E},\phi)\}}{\sim}, \tag{3.2.1}$$

where the equivalence relation is given by conjugation: $(\mathcal{E},\phi) \sim (\mathcal{E}',\phi')$ iff there exists an invertible bundle map $\psi : \mathcal{E} \to \mathcal{E}'$ such that $\phi' = \psi^{-1}\phi\psi$.

Remark 3.2.2. The coprime condition guarantees that $\mathcal{M}_X^{\mathcal{L}}(r,d)$ is a smooth manifold.

Remark. We will often simplify this notation by writing $\mathcal{M}_X^{\mathcal{L}}$, or when working in the $\mathcal{L} = K$ setting by writing $\mathcal{M}_X$.

An important tool for studying $\mathcal{M}_X^{\mathcal{L}}(r,d)$ is the *Hitchin map*:

Definition 3.2.3. *The* ***Hitchin map***

$$\mathcal{H} : \mathcal{M}_X^{\mathcal{L}}(r,d) \to \mathcal{B} = \bigoplus_{i=1}^{r} H^0(X, \mathcal{L}^i) \tag{3.2.2}$$

is defined by sending an isomorphism class of a Higgs bundle $(\mathcal{E},\phi)$ to the r-tuple of coefficients of the characteristic polynomial of ϕ. The affine space $\mathcal{B}$ is called the ***Hitchin base****.*

Definition 3.2.4. *The* ***nilpotent cone*** *is the fiber of the Hitchin map above $0 \in \mathcal{B}$, i.e. $\mathcal{H}^{-1}(0)$.*

Our first step in studying $\mathcal{M}_X^{\mathcal{L}}$ is to know its dimension. A result of Hitchin (in the $\mathcal{L} = K$ case) and Nitsure (in the general case) tells us that the dimension of the moduli space is given by:

Proposition 3.2.5. *([65, 81])*

- *For $\deg \mathcal{L} > \deg K$:* $\dim \mathcal{M}_X^{\mathcal{L}}(r,d) = r^2 \deg \mathcal{L} + 1$.
- *For $\mathcal{L} = K$:* $\dim \mathcal{M}_X(r,d) = r^2(2g-2) + 2$.

In a later subsection we will argue briefly why this is true for the $\mathcal{L} = K$ case, and we will prove the result for the $\deg \mathcal{L} > \deg K$ case in *Chapter 6* when studying the deformation theory of the $\mathcal{L}$-twisted moduli space.

3.2.1 Spectral correspondence

We want to understand now how $\mathcal{M}_X^{\mathcal{L}}$ looks. We will do this by studying the fibres of the Hitchin map. A result of Hitchin [65] and Nitsure [81] shows that the Hitchin map is proper, meaning that $\mathcal{M}_X^{\mathcal{L}}$ is fibred by compact subvarieties, called *Hitchin fibres*, which can be shown to be tori.

Given a line bundle $\mathcal{L}$ on a Riemann surface X, the *tautological section* η is defined as follows. Let (x,y) be local coordinates on $\mathrm{Tot}(\mathcal{L})$, where x is the base coordinate and y is the fiber coordinate of $\mathrm{Tot}(\mathcal{L})$, with projection $\pi : \mathcal{L} \to X$. The pullback bundle $\pi^*\mathcal{L}$ has attached to a point $(x, y(x))$ a copy of the fibre $\mathcal{L}_x$. This bundle has a natural section, η, defined by $\eta(x, y(x)) = y(x) \in (\pi^*\mathcal{L})_y = \mathcal{L}_x$.

Definition 3.2.6. *Let $a = (a_1, \ldots, a_r) \in \mathcal{B}$. The* ***spectral curve*** *S_a associated to a is the zero locus of the polynomial*

$$\det(\eta - \pi^*\phi) = \eta^r + a_1\eta^{r-1} + \cdots + a_r. \tag{3.2.3}$$

Remark. When $(\mathcal{E}, \phi)$ is a Higgs bundle such that $\mathcal{H}([\mathcal{E}, \phi]) = a$, then we will say that the spectral curve is associated to $(\mathcal{E}, \phi)$.

It can be shown that the Hitchin fibres are precisely Jacobians of the respective spectral curves, by identifying eigenspaces of a given ϕ with holomorphic line bundles on the spectral curve. In other words, the Hitchin fibration is globally a torus fibration and, in fact, is a completely integrable Hamiltonian system whose Poisson structure is induced from the open dense embedding of the cotangent bundle of the moduli space of bundles and whose Hamiltonians are the real and imaginary components of the Hitchin map [64]. This integrable system has attracted immense interest in both mathematics and physics, as every classical integrable system is believed to be an instance, limit, or modification of the Hitchin system (*cf.* for instance [31]). Rather than do justice to integrability in the framework of Higgs bundles, we will just argue that the fibres are in fact Jacobians of the respective spectral curves.

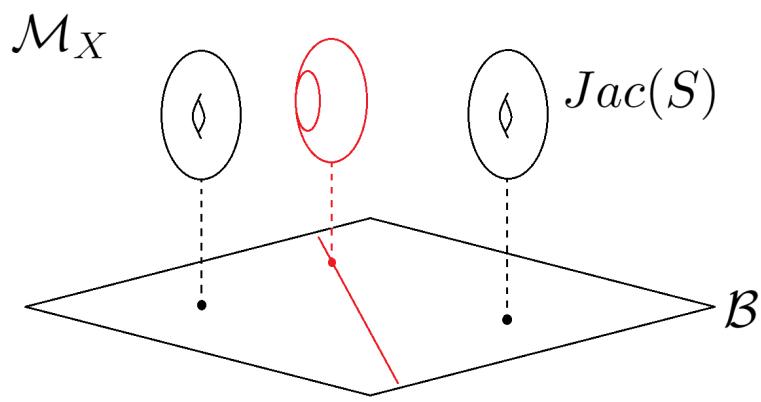

Figure 3.1: Generic fibres of $\mathcal{M}_X^L$ are tori. There is a locus of degenerate singular tori.

For a generic choice of $a \in \mathcal{B}$, the spectral curve S_a is an $r:1$ branched cover of X living inside of $\mathrm{Tot}(\mathcal{L})$. We can restrict the bundle projection map $\pi : \mathrm{Tot}(\mathcal{L}) \to X$ to S_a to get a map $\pi : S_a \to X$. Let $\mathcal{Q}$ be a line bundle on S_a. On this line bundle, the tautological section can be seen as acting by

$$\begin{aligned} \eta|_{S_a} :& \mathcal{Q} \to \mathcal{Q} \otimes \pi^* \mathcal{L} \\ & s \mapsto s \cdot y. \end{aligned}$$

By Proposition 2.5.2, the direct image $\mathcal{E} = \pi_* \mathcal{Q}$ is a rank r vector bundle over X. The tautological section pushes forward to a linear map $\mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$, an $\mathcal{L}$-twisted Higgs field for $\mathcal{E}$. Thus, from the data of a line bundle $\mathcal{Q}$ on S_a, we can produce an $\mathcal{L}$-twisted Higgs bundle on X. To see this in the other direction, the Hitchin map sends a Higgs field $(\mathcal{E}, \phi)$ to an r-tuple of spectral data $a \in \mathcal{B}$. From this data, we can produce a spectral curve S_a, which, by Definition 3.2.6, is the spectrum of the ϕ, having r distinct eigenvalues generically over $x \in X$ in correspondence to S_a being a branched $r:1$ cover of X (*i.e.* branched points correspond to repeated eigenvalues). The eigenspaces of ϕ are generically 1-dimensional, and form a line bundle over S_a.

What we have shown thus far is the *spectral correspondence* [8, 30, 31, 65]: an isomorphism class of holomorphic line bundles $[\mathcal{Q}]$ on S_a is equivalent to the data of an isomorphism class of Higgs bundles $[(\mathcal{E}, \phi)]$ on X. It then follows

that a generic fiber $\mathcal{H}^{-1}(a)$ is isomorphic to the Jacobian variety of the spectral curve S_a. By Proposition 2.5.3, this is not the space of degree 0 line bundles, but rather the degree is given by

$$deg\widetilde{\mathcal{L}} = d - (1 - g_{S_a}) + r(1 - g).$$

3.2.2 $\mathbb{C}^*$-action

Our description up to this point is not enough to fully understand the topology of the moduli space because of the presence of special degenerate fibers. To continue the investigation, we appeal to the correspondence between the gauge-theoretic and algebro-geometric pictures. Studying the gauge-theoretic moduli space, we could employ Morse-Bott theory on a symplectic leaf that contains the nilpotent cone (given that the whole of the moduli space is not naturally Kähler when $\mathcal{L} \ncong K$), and justifying that the (rational) cohomology of the leaf coincides with that of the whole moduli space. Here, we must study the critical points of the height function $f(\mathcal{E}, \phi) = \frac{1}{2}||\phi||^2$, the L^2-norm on the moduli space. On the algebro-geometric side, we can employ Białynicki-Birula theory [13] and study the fixed points for the algebraic group action

$$\lambda.(\mathcal{E}, \phi) = (\mathcal{E}, \lambda \cdot \phi)$$

of $\mathbb{C}^*$. These two points of view come together by the following facts:

- the fixed points of the $\mathbb{C}^*$-action are fixed points of $U(1) \subset \mathbb{C}^*$

$$\lambda.(\mathcal{E}, \phi) = (\mathcal{E}, e^{i\lambda} \cdot \phi)$$

- the height function f is a moment map for the $U(1)$-action, and the fixed points of the $U(1)$-action are critical points of f.

What is fortunate about the Białynicki-Birula approach is that there is no need to appeal to a Kähler structure.

A theorem of Frankel [48] tells us that f is a nondegenerate perfect Morse-Bott function, and so the Poincaŕe polynomial of $\mathcal{M}_X(r, d)$ is given by

$$P[\mathcal{M}_X(r, d)](t) = \sum_{i \in I} t^{\beta(\mathcal{C}_i)} P[\mathcal{C}_i](t), \tag{3.2.4}$$

where $\mathcal{C}_i$ are the critical subvarieties of f, and $\beta(\mathcal{C}_i)$ is the Morse index of $\mathcal{C}_i$, *i.e.* the rank of the subbundle of the normal bundle on which the Hessian of f is negative definite. This tells us that the topology of the moduli space is contained in this fixed locus.

We now focus our attention on the set of fixed points of the $U(1)$-action, which we denote by $\mathcal{M}_X(r, d)^{U(1)}$. Stable Higgs bundles are fixed *iff* there exists an automorphism A_λ of $\mathcal{E}$ such that

$$A_\lambda \phi A_\lambda^{-1} = e^{i\lambda} \phi \tag{3.2.5}$$

for all $\lambda \in [0, 2\pi)$, *i.e* there is a change of basis that undoes the $U(1)$-action.

We would like to develop a better description of the fixed points. Let $(\mathcal{E}, \phi) \in \mathcal{M}_X(r,d)^{U(1)}$ with A_λ the one-parameter family of automorphisms that satisfy (3.2.5). There is a limiting endomorphism Λ that generates the family A_λ infinitesimally,

$$\Lambda \coloneqq D_\lambda(A_\lambda)|_{\lambda=0},$$

where D_λ is a suitably-defined derivative. This limiting endomorphism interacts with the Higgs field in the following way:

Lemma 3.2.7. $[\Lambda, \phi] = i\phi$.

Proof. We start with (3.2.5) and apply $D_\lambda(\cdot)|_{\lambda=0}$ to both sides.
On the right-hand side, we have

$$D_\lambda(e^{i\lambda}\phi)|_{\lambda=0} = ie^{i\lambda}\phi|_{\lambda=0} = i\phi.$$

On the left-hand side, we have

$$\begin{aligned} D_\lambda(A_\lambda \phi A_\lambda^{-1}) &= D_\lambda(A_\lambda)\phi A_\lambda^{-1}|_{\lambda=0} + A_\lambda \phi D_\lambda(A_\lambda^{-1})|_{\lambda=0} \\ &= D_\lambda(A_\lambda)\phi A_\lambda^{-1}|_{\lambda=0} + A_\lambda \phi(-1) D_\lambda(A_\lambda) A_\lambda^{-2}|_{\lambda=0} \\ &= \Lambda\phi - \phi\Lambda \\ &= [\Lambda, \phi], \end{aligned}$$

where we use the definition of Λ, and the fact that A_0 is the identity map. □

Returning to the gauge-theoretic viewpoint, for a pair (A, ϕ) that satisfy the Hitchin equations, we have that the $\overline{\partial}$-operator induced by A, denoted $\overline{\partial}_A$, satisfies

$$\overline{\partial}_A \Lambda = 0.$$

In this way, we have that $\mathcal{E}$ decomposes into a direct sum of eigenspaces $\mathcal{B}_1, \ldots, \mathcal{B}_n$ of Λ,

$$\mathcal{E} = \oplus_{k=1}^n \mathcal{B}_k.$$

These eigenspaces are in fact holomorphic subbundles of $\mathcal{E}$, and the corresponding eigenvalues $s_1, \ldots, s_n$ of Λ are global holomorphic functions on X. Applying both sides of Lemma 3.2.7 to some $\mathcal{B}_k$, we find that

$$\Lambda(\phi\mathcal{B}_k) = (s_k + 1)(\phi\mathcal{B}_k).$$

The image of $\mathcal{B}_k$ under ϕ is then a subbundle of the eigenbundle for eigenvalue $s_k + i$. This means we can re-index (as needed) and group the eigenspaces into sequences with eigenvalues $s_k, s_k + i, s_k + 2i, \ldots$, terminating when the image of an eigenbundle under ϕ is zero (*i.e.* when we have reached the last eigenbundle). There cannot be multiple, disconnected sequences for a fixed point, as that would violate the stability condition. This means that for a fixed point of the $U(1)$-action, $(\mathcal{E}, \phi) \in \mathcal{M}_X(r,d)^{U(1)}$, there is a number n such that $\mathcal{E} = \oplus_{k=1}^n \mathcal{B}_k$, and

$$\mathcal{B}_1 \xrightarrow{\phi_1} \mathcal{B}_2 \otimes \mathcal{L} \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{n-1}} \mathcal{B}_n \otimes \mathcal{L}^{\otimes n-1} \xrightarrow{\phi_n} 0,$$

where $\phi_k \coloneqq \phi|_{\mathcal{B}_k}$ is not identically zero for $k < n$. A Higgs bundle satisfying this description is called a *holomorphic chain*, *cf.* [2, 3, 22, 53]. In the case when $L = K$, these can be regarded as complex variations of Hodge structures [92].

With this description, we can write a fixed point in a basis of sections where ϕ looks like:

$$\phi = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \phi_1 & 0 & \cdots & 0 & 0 \\ 0 & \phi_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \phi_{n-1} & 0 \end{pmatrix}. \tag{3.2.6}$$

This local matrix description of ϕ is nilpotent, and since every fixed point can be written in this form, they all live in the nilpotent cone (Definition 3.2.4).

Notably, not all points within the nilpotent cone are fixed points of the action. Only those that can be written in the form of (3.2.6) are fixed. Regardless, the topological information of $\mathcal{M}_X(r,d)$ is contained within the nilpotent cone.

We can also think of the fixed points of this action as $\mathcal{L}$-twisted representations of A-type quivers, with lengths and labels determined by partitions of r and d:

$$\bullet_{r_1,d_1} \to \bullet_{r_2,d_2} \to \cdots \to \bullet_{r_n,d_n}.$$

Such a representation is called a *quiver bundle*, *cf.* [56, 57, 85, 88, 90].

3.2.3 $\mathcal{L} = K$ moduli space

We wish to restrict now to the $\mathcal{L} = K$ case and take an opportunity to look specifically at the ordinary Higgs bundle moduli space, $\mathcal{M}_X(r,d)$. We will start by looking at the Hitchin base $\mathcal{B}$, aiming to show that

$$\dim \mathcal{B} = r^2(g-1) + 1. \tag{3.2.7}$$

We begin by applying Serre duality to the $h^1(K^n)$ terms.

$$\begin{aligned} h^1(K^n) &= h^0(K \otimes (K^n)^*) \\ &= h^0(K \otimes (K^*)^n) \\ &= h^0((K^*)^{n-1}) \\ &= 0 \end{aligned}$$

Because $\deg(K^n) = (2g-2)n$, we know that $(K^*)^{n-1}$ has degree $(2-2g)(n-1) < 0$, meaning that $h^1(K^n)$ vanishes.

Applying the Riemann-Roch theorem for $n > 1$:

$$\begin{aligned} h^0(K^n) &= \deg K^n + rkK^n(1-g) \\ &= (2g-2)n + (1-g) \\ &= (2n)g - 2n + 1 - g \\ &= (2n-1)g - (2n-1) \\ &= (g-1)(2n-1). \end{aligned}$$

Putting it all together we get that

$$h^0(K^n) = \begin{cases} g & n = 1 \\ (g-1)(2n-1) & n \geq 2 \end{cases}$$

Now that we know the dimension of each homology group, we can sum them together to get the dimension of the Hitchin base.

$$\begin{aligned}
\dim \mathcal{B} &= \sum_{i=1}^{r} h^0(K^i) \\
&= \sum_{i=2}^{r} h^0(K^i) + h^0(K) \\
&= \sum_{i=2}^{r} (g-1)(2i-1) + g \\
&= (g-1) \sum_{i=2}^{r} (2i-1) + g \\
&= (g-1) \left(2 \sum_{i=2}^{r} i - \sum_{i=2}^{r} 1 \right) + g \\
&= (g-1) \left(2(\frac{r(r+1)}{2} - 1) - r + 1 \right) + g \\
&= (g-1)(r^2 + r - 2 - r + 1) + g \\
&= (g-1)(r^2 - 1) + g \\
&= r^2(g-1) - (g-1) + g \\
&= r^2(g-1) + 1.
\end{aligned}$$

Using the deformation theory of sheaves, we can argue that the tangent space to the moduli space of stable bundles at any point $\mathcal{E}$ is isomorphic to $H^1(X, \mathrm{End}(\mathcal{E}))$, which by Riemann-Roch also has dimension $r^2(g-1)+1$. By Serre duality, the cotangent space is, therefore, $H^0(X, \mathrm{End}(\mathcal{E}) \otimes K)$, which is the space of possible Higgs fields for $\mathcal{E}$. In other words, $\mathcal{M}_X(r,d)$ contains the cotangent bundle to the moduli space of stable bundles. This containment is open and dense, and so the dimension of $\mathcal{M}_X(r,d)$ is $2r^2(g-1)+2$.

From this, it turns out that the dimension of $\mathcal{M}_X(r,d)$ is twice the dimension of both the Hitchin base and the moduli space of stable bundles. These two different fibrations are related by what is called "hyperkähler rotation" — we will not discuss this feature here.

Example 3.2.8. Returning to the example $\mathcal{E} = K \oplus \mathcal{O}$ with Higgs field given by

$$\phi = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}, \tag{3.2.8}$$

the Hitchin map sends ϕ to $\eta^2 - \alpha \in H^0(X, K^2)$. These Higgs bundles form the *Hitchin section*, a locus of Higgs bundles that intersects the Hitchin fibres at exactly one point each, as for every element of $\mathcal{B}$ there is only one Higgs field with the above form.

Remark 3.2.9. It is possible to reformulate Example 3.2.8 with a different Higgs bundles, $\mathcal{E} = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ with Higgs field

$$\phi = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}, \tag{3.2.9}$$

where $K^{\frac{1}{2}}$ is a choice of holomorphic square root of K (there are 2^{2g} such choices). The Higgs field has similar properties to the example above, in particular $\eta - \det \phi \in H^0(X, K^2)$. The main difference between these examples is the degree of the Higgs bundle. The degree of $\mathcal{E} = K \oplus \mathcal{O}$ is $2g - 2$, while the degree of $\mathcal{E} = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ is 0.

3.2.4 $SL(r, \mathbb{C})$-Higgs bundles

Until now, we have been considering the situation where the structure group for $\mathcal{E}$ is $GL(r, \mathbb{C})$. There is often a preference to fix the determinant of the Higgs bundle, with the convention being to either let $\det \mathcal{E} = \mathcal{P}$ for some fixed line bundle $\mathcal{P}$, or $\det \mathcal{E} = \mathcal{O}_X$ specifically. For our purposes, we will consider the latter. Fixing the determinant takes us from the $GL(r, \mathbb{C})$ setting to the $SL(r, \mathbb{C})$ setting.

Definition 3.2.10. *Let $\mathcal{L}$ be a holomorphic line bundle on a Riemann surface X. An $SL(r, \mathbb{C})$-**Higgs bundle** on X is a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic vector bundle with $\det \mathcal{E} = \mathcal{O}_X$, and $\phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$ such that $\operatorname{tr}(\phi) = 0$.*

The moduli space of $SL(r, \mathbb{C})$-Higgs bundles $\mathcal{M}_X^{\mathcal{L},0}$ is a subvariety of $\mathcal{M}_X^{\mathcal{L}}$. The restricted Hitchin map on $\mathcal{M}_X^{\mathcal{L},0}(r, d)$ has codomain $\mathcal{B}^0 := \bigoplus_{i=2}^{r} H^0(X, \mathcal{L}^i)$ since $\operatorname{tr} \phi$ belongs to $H^0(X, \mathcal{L})$. In particular, the nilpotent cones of $\mathcal{M}_X^{\mathcal{L},0}$ and $\mathcal{M}_X^{\mathcal{L}}$ are coincident, as Higgs bundles in the nilpotent cone have, by definition, trace-free Higgs fields even in the $GL(r, \mathbb{C})$ case. The dimension of $\mathcal{M}_X^{\mathcal{L},0}$ is $\deg \mathcal{L}(r^2 - 1)$.

4 Topological recursion

4.1 Matrix models

A main concern in random matrix theory is the statistical behaviour of the spectrum of a matrix, particularly in a large N limit. For most "nice" cases, the density of eigenvalues converges to a continuous density function, referred to as the *equilibrium measure*. This equilibrium measure has compact support and takes the form of a complex algebraic function. This means that to a sufficiently "nice" random matrix model, we can associate an algebraic curve S.

Two classic examples that demonstrate this are the Wigner semi-circle distribution, which is the equilibrium measure for eigenvalues of a Gaussian random matrix given by

$$\rho(x)dx = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{[-2,2]}dx, \tag{4.1.1}$$

and the Marchenko-Pastur distribution, which describes the asymptotic behaviour of $M \times N$ Gaussian random matrices given by

$$\rho(x)dx = \frac{N}{2\pi M\sigma^2}\frac{\sqrt{\frac{M}{N}\sigma^4 - \frac{M^2}{N^2}\sigma^4 - (x-\sigma^2)^2}}{x}\chi_{[a,b]}dx, \tag{4.1.2}$$

where σ^2 is the variance, and a, b are the roots of term in the square-root.

It is known (*cf.* for instance [39]) that understanding the algebraic curve S associated to an equilibrium measure is enough to understand the asymptotic expansion of all expectation values to all orders. In the large N limit, the expectation value of a multi-resolvent, the joint probability of n-eigenvalues, can be expanded as

$$\langle(\mathrm{Tr}\frac{1}{x_1 - M})\dots(\mathrm{Tr}\frac{1}{x_n - M})\rangle \sim \sum_{g=0}^{\infty} N^{2-2g-n} W_{g,n}(x_1, ..., x_n).$$

Understanding the $W_{g,n}$ is what is necessary to compute all correlation functions of the matrix model. The $W_{g,n}$ are differential forms defined recursively on $2-2g-n$, satisfying a relation

$$W_{g,n+1} \sim W_{g-1,n+2} + \sum_{\substack{g_1+g_2=g \\ |I\cup J|=n}} W_{g_1,|I|+1}W_{g_2,|J|+1},$$

and depend only on the information of S.

4.2 Eynard-Orantin differentials

A natural question to ask is "what happens when we compute these $W_{g,n}$ for a *spectral curve*, an algebraic curve arising as the spectrum of an arbitrary matrix-valued function?" This defines a family of invariants $\{W_{g,n}\}$ of the curve called the *Eynard-Orantin differentials*.

Definition 4.2.1. *A **spectral curve** is a triple (S, x, y) where S is a compact Riemann surface, and $x, y : S \to \mathbb{P}^1$.*

We can view a spectral curve S as an $r : 1$ cover of $\mathbb{P}^1$ with covering map $x : S \to \mathbb{P}^1$. Viewing x as the local coordinate on S and y as the fiber coordinate of $T^*\mathbb{P}^1$, the spectral curve is defined by

$$\{(x, y) : P(x, y) = y^r + \sum_{i=1}^{r} f_i(x) y^{r-i} = 0\}. \tag{4.2.1}$$

Remark. If we have a rank r Higgs bundle $(\mathcal{E}, \phi)$ on $\mathbb{P}^1$, we can think of (4.2.1) as a local expression of (3.2.3)

In this section, we are interested in spectral curves of the form

$$P(x, y) = y^2 - f(x) = 0, \tag{4.2.2}$$

and we consider an affine coordinate z around p on S. The topological recursion lives on

$$\begin{tikzcd}
K_S \arrow[d] & & \\
S \arrow[rr] \arrow[dr, "x"'] & & T^*\mathbb{P}^1 \arrow[dl, "\pi"] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

where $\pi|_S = x$.

We will impose an additional condition on the ramification points of our spectral curve. While it is possible to be general, we will only be interested in this specific case.

Definition 4.2.2. *A **good spectral curve** over $\mathbb{P}^1$ is a spectral curve as above, with only simple ramification points and the additional condition that zeroes of dx and dy do not coincide.*

For the remainder of this section, we will be exclusively interested in good spectral curves, and will refer to them as spectral curves.

The ramification divisor R of a spectral curve consists of points where $f(x)$ has zeroes or poles. For the spectral curves that we are considering, these are points where zeroes of dx or poles of x are order 2 or greater. Around each ramification point $p \in R$, there is a local involution σ_p that fixes the ramification point and exchanges nearby points between the sheets (see Figure 4.1).

The Bergman kernel (Definition 2.6.2) plays an important role in the definition of the Eynard-Orantin invariants. As such, we need to choose a symplectic basis $\langle A_1, ..., A_{\tilde{g}}, B_1, ..., B_{\tilde{g}} \rangle$ for $H_1(X, \mathbb{Z})$.

Definition 4.2.3. *Let $p \in R$. The **recursion kernel** at p is a meromorphic section of $K_S \boxtimes K_S^*$ defined by*

$$K_p(z_0, z) = \frac{\int_{t=\alpha}^{z} B(t, z_0)}{(y(z) - y(\sigma_p(z)) dx(z)} \tag{4.2.3}$$

where α is an arbitrary base point, and B is the Bergman kernel.

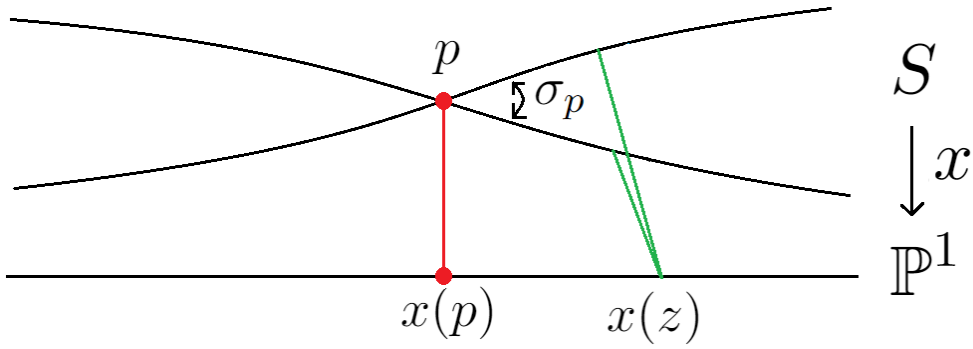

Figure 4.1: Local picture of a spectral curve S on $\mathbb{P}^1$.

Remark. The term $\frac{1}{dx(z)}$ is acting as a contraction operation with the vector field $(\frac{dx}{dz})^{-1}\frac{\partial}{\partial z}$. In this way the "division" acts by killing terms of the form $dx(z)$ in the numerator.

Definition 4.2.4. *The **Eynard-Orantin differentials** $W_{g,n}$ are meromorphic sections of the n-th exterior tensor product $K_S^{\boxtimes n}$, i.e. multi-differentials, defined as follows:*

The initial conditions of the recursion are given by:

- *$W_{0,1}$ is a meromorphic 1-form on S.*[1]
- *$W_{0,2}(z_1, z_2) = B(z_1, z_2)$.*

For all $g, n \in \mathbb{N}$ and $2g - 2 + n \geq 0$, define $W_{g,n}$ recursively by

$$W_{g,n+1}(z_0, \boldsymbol{z}) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z) \left[W_{g-1,n+2}(z, \sigma_p(z), \boldsymbol{z}) + \sum_{\substack{g_1+g_2=g \\ I \cup J = \boldsymbol{z}}}' W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \right] \tag{4.2.4}$$

where the prime signifies summation excluding the cases (g_1, I) or $(g_2, J) = (0, 0)$.
*The terms with $k = 2g + n - 1 \geq 2$ are called **stable differentials**.*

When performing computations in a later section of this thesis, it will be convenient to compress the recursion term (*i.e.*, the terms in the square brackets) into a single term for ease of bookkeeping. As such, we have the following notation:

$$\mathcal{R}^2 W_{g,n+1}(z, \sigma_p(z), \mathbf{z}) := W_{g-1,n+2}(z, \sigma_p(z), \mathbf{z}) + \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{z}}}' W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \tag{4.2.5}$$

$$= W_{g-1,n+2}(z, \sigma_p(z), \mathbf{z}) + \sum_{\text{stable}} W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \tag{4.2.6}$$

$$+ \sum_{i=1}^{n} W_{0,2}(z, z_i) W_{g,n}(\sigma_p(z), \mathbf{z} \backslash \{z_i\}) + W_{0,2}(\sigma_p(z), z_i) W_{g,n}(z, \mathbf{z} \backslash \{z_i\}).$$

To illustrate the general form of the Eynard-Orantin differentials, we compute the first few.

[1] *The choice of $W_{0,1}$ is related to the geometric problem that one is studying. In the original formulation [42], $W_{0,1}$ is taken to be 0. In many other examples, including the Airy spectral curve, it is taken to be ydx.*

$k = 1:$

$$W_{0,2}(z_1, z_2) = B(z_1, z_2) \tag{4.2.7}$$

$k = 2:$

$$W_{0,3}(z_0, z_1, z_2) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z)[W_{0,2}(z, z_1)W_{0,2}(\sigma_p(z), z_2) + W_{0,2}(\sigma_p(z), z_1)W_{0,2}(z, z_2)] \tag{4.2.8}$$

$$W_{1,1}(z_0) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z) W_{0,2}(\sigma_p(z), z) \tag{4.2.9}$$

$k = 3:$

$$\begin{aligned} W_{0,4}(z_0, z_1, z_2, z_3) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z) \times \\ [W_{0,3}(z, z_1, z_2)W_{0,2}(\sigma_p(z), z_3) + W_{0,3}(\sigma_p(z), z_1, z_2)W_{0,2}(z, z_3) \\ W_{0,3}(z, z_2, z_3)W_{0,2}(\sigma_p(z), z_1) + W_{0,3}(\sigma_p(z), z_2, z_3)W_{0,2}(z, z_1) \\ W_{0,3}(z, z_1, z_3)W_{0,2}(\sigma_p(z), z_2) + W_{0,3}(\sigma_p(z), z_1, z_3)W_{0,2}(z, z_2))] \end{aligned} \tag{4.2.10}$$

$$W_{1,2}(z_0, z_1) = \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z)[W_{0,3}(z, \sigma_p(z), z_1) + W_{1,1}(z)W_{0,2}(\sigma_p(z), z_1) + W_{1,1}(\sigma_p(z))W_{0,2}(z, z_1)] \tag{4.2.11}$$

The $W_{g,n}$'s satisfy three properties that will be used in future computations.

1. **Symmetry**

   For any $\sigma_{ij} \in S_2$ (a map that swaps i and j), the differentials satisfy:

$$W_{g,n}(\sigma_{ij}(\mathbf{z})) = W_{g,n}(\mathbf{z}) \tag{4.2.12}$$

2. **Location of poles**

   If a stable differential has a pole then it must be at a ramification point.

3. **Stable differentials are odd**

   The differentials $W_{g,n}$ satisfy

$$W_{g,n}(z_1, ..., z_n) + W_{g,n}(-z_1, ..., z_n) = 0 \tag{4.2.13}$$

   for $(g, n) \neq (0, 2)$. By symmetry, this property extends to all arguments.

   The differential $W_{0,2}$ satisfies a different property,

$$W_{0,2}(z_1, z_2) + W_{0,2}(-z_1, z_2) = \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}. \tag{4.2.14}$$

4.3 Airy spectral curve

To better understand the methods of doing the recursion, it is helpful to work through an example. The Airy spectral is a classic example of a spectral curve for topological recursion which is related to Witten-Kontsevich intersection

numbers.

The **Airy spectral curve** S is the spectral curve over $\mathbb{P}^1$ given by the local expression

$$y^2 - x = 0. \tag{4.3.1}$$

We would like to start by identifying the ramification points of this curve. We want to view this curve as living inside of the cotangent bundle of $\mathbb{P}^1$, recalling that $K_{\mathbb{P}^1} = \mathcal{O}(-2)$. Thinking in this way, we have that y^2 and x are sections of $\mathcal{O}(-4)$. Let U_0 and U_1 be the charts around 0 and ∞ on the base $\mathbb{P}^1$. Viewing (4.3.1) as being written in coordinate chart U_0, we expect that the defining equation for S will have a pole of order 5 at ∞ in U_1 (because -4 = 1 - 5). The spectral curve has two ramification points, one coming from the simple zero at $x = 0 \in \mathbb{P}^1$, and the other coming from a pole of order 5 at $x = \infty \in \mathbb{P}^1$. This means that S is a double cover of $\mathbb{P}^1$ with two ramification points, both with multiplicity 2. By the Riemann-Hurwitz formula, we have

$$\begin{aligned} g_S &= \frac{b}{2} - n(g_{\mathbb{P}^1} - 1) + 1 \\ &= \frac{2}{2} - 2(0 - 1) + 1 \\ &= 0, \end{aligned}$$

and so the genus of S is 0, meaning that this is a covering of $\mathbb{P}^1$ by $\mathbb{P}^1$.

We also want a defining equation for S in the U_1 chart. Let $u = \frac{1}{x}$ be the coordinate on U_1. Because we are viewing x as a section of $\mathcal{O}(-4)$, there is a section $s_1(u)$ on U_1 such that on the intersection $U_0 \cap U_1$, x and $s_1(u)$ are related by

$$x = x^{-4} s_1(u).$$

For this relation to make sense, we must have that $s_1(\frac{1}{x}) = x^5$ on the intersection, and thus on U_1 we have

$$s_1(u) = u^{-5}. \tag{4.3.2}$$

On U_1, the spectral curve is still a double cover, so we can find a local vertical coordinate w so that the spectral curve is defined by

$$w^2 - u^{-5} = 0. \tag{4.3.3}$$

Written in this coordinate chart, we can clearly see the pole of order 5 at $x = \infty$ ($u = 0$).

We now want to find local expressions for the components of topological recursion in terms of parametrizations on the spectral curve. Starting in the chart U_0, the spectral curve can be parametrized by

$$\begin{aligned} x(z) &= z^2, \\ y(z) &= z. \end{aligned}$$

In this parametrization, the ramification points are at $z = 0, \infty$. Around both of these ramification points we have a Galois involution given by

$$\sigma_p(z) = -z. \tag{4.3.4}$$

As we saw in Proposition 2.3.1, there are no holomorphic one-forms on $\mathbb{P}^1$. This means that in our coordinate chart, the Bergman kernel for the spectral curve $S = \mathbb{P}^1$ is given by

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{4.3.5}$$

With the information of $x(z)$, $y(z)$, and $B(z_1, z_2)$, the recursion kernel can be computed by

$$\begin{aligned} K_p(z_0, z) &= \frac{\int_{t=\alpha}^{z} B(t, z_0)}{(y(z) - y(\sigma_p(z)))dx(z)} \\ &= \frac{\int_{t=\alpha}^{z} \frac{dt dz_0}{(t-z_0)^2}}{(z-(-z))2zdz} \\ &= \frac{1}{4z^2 dz} \frac{-dz_0}{t - z_0}\Big|_{t=a}^{z} \\ &= -\frac{dz_0}{4z^2 dz}\Big[\frac{1}{z - z_0} - \frac{1}{\alpha - z_0}\Big]. \end{aligned} \tag{4.3.6}$$

We will choose our $W_{0,1}$ to be the canonical one-form ydx on $T^*\mathbb{P}^1$. In the local coordinate z, we can write $W_{0,1}(z)$ as

$$W_{0,1}(z) = y(z)dx(z) = zd(z^2) = 2z^2 dz. \tag{4.3.7}$$

To better understand the remaining $W_{g,n}$, we compute the $k = 2$ differentials of the recursion, $W_{0,3}$ and $W_{1,1}$.

$$\begin{aligned} W_{0,3}(z_0, z_1, z_2) &= \sum_{p \in R} \mathrm{Res}_{z=p} K_p(z_0, z)[W_{0,2}(z, z_1)W_{0,2}(-z, z_2) + W_{0,2}(-z, z_1)W_{0,2}(z, z_2)] \\ &= -dz_0 \mathrm{Res}_{z=0} \frac{1}{4z^2 dz}\Big(\frac{1}{z - z_0} - \frac{1}{\alpha - z_0}\Big)\Big[\frac{dzdz_1}{(z-z_1)^2}\frac{-dzdz_2}{(z+z_2)^2} + \frac{-dzdz_1}{(z+z_1)^2}\frac{dzdz_2}{(z-z_2)^2}\Big] \\ &= -dz_0 \lim_{z \to 0} \frac{\partial}{\partial z}\Big(z^2 \frac{1}{4z^2 dz}\Big(\frac{1}{z - z_0} - \frac{1}{\alpha - z_0}\Big)\Big[\frac{dzdz_1}{(z-z_1)^2}\frac{-dzdz_2}{(z+z_2)^2} + \frac{-dzdz_1}{(z+z_1)^2}\frac{dzdz_2}{(z-z_2)^2}\Big]\Big) \\ &= -dz_0 \lim_{z \to 0} \frac{\partial}{\partial z}\Big(\frac{1}{4dz}\Big(\frac{1}{z - z_0} - \frac{1}{\alpha - z_0}\Big)\Big[\frac{dzdz_1}{(z-z_1)^2}\frac{-dzdz_2}{(z+z_2)^2} + \frac{-dzdz_1}{(z+z_1)^2}\frac{dzdz_2}{(z-z_2)^2}\Big]\Big) \end{aligned}$$

We can see that only one term, coming from K_p, contributes a pole at $z = 0$. Differentiating the round bracketed terms results in

$$\begin{aligned} \frac{\partial}{\partial z}\Big(\frac{1}{z - z_0} - \frac{1}{\alpha - z_0}\Big)\Big|_{z=0} &= \frac{-1}{(z - z_0)^2}\Big|_{z=0} \\ &= \frac{-1}{z_0^2}, \end{aligned}$$

while the derivative of the second term vanishes when evaluated at $z = 0$,

$$\begin{aligned} &\frac{\partial}{\partial z}\left[\frac{1}{(z-z_1)^2}\frac{1}{(z+z_2)^2} + \frac{1}{(z+z_1)^2}\frac{1}{(z-z_2)^2}\right]\Bigg|_{z=0} = \\ &= \left[\frac{-2}{(z-z_1)^3(z+z_2)^2} + \frac{-2}{(z-z_1)^2(z+z_2)^3} + \frac{-2}{(z+z_1)^3(z-z_2)^2} + \frac{-2}{(z+z_1)^2(z-z_2)^3}\right]\Bigg|_{z=0} \\ &= \frac{-2}{(-z_1)^3 z_2^2} + \frac{-2}{(-z_1)^2 z_2^3} + \frac{-2}{z_1^3(-z_2)^2} + \frac{-2}{z_1^2(-z_2)^3} \\ &= 0. \end{aligned}$$

By the above computation of the derivatives, we see that only one term of the product rule derivative remains, and so we can proceed with computing $W_{0,3}$.

$$\begin{aligned}
W_{0,3}(z_0,z_1,z_2) &= -dz_0 \lim_{z\to 0} \frac{\partial}{\partial z}\Big(\frac{1}{4dz}\Big(\frac{1}{z-z_0}-\frac{1}{\alpha-z_0}\Big)\Big[\frac{dzdz_1}{(z-z_1)^2}\frac{-dzdz_2}{(z+z_2)^2}+\frac{-dzdz_1}{(z+z_1)^2}\frac{dzdz_2}{(z-z_2)^2}\Big]\Big)\\
&= \frac{dz_0}{4}\left[\frac{dz_1}{(z-z_1)^2}\frac{dz_2}{(z+z_2)^2}+\frac{dz_1}{(z+z_1)^2}\frac{dz_2}{(z-z_2)^2}\right]\Bigg|_{z=0}\frac{\partial}{\partial z}\left(\frac{1}{z-z_0}-\frac{1}{\alpha-z_0}\right)\Bigg|_{z=0}\\
&= \frac{dz_0}{4}\Big[\frac{dz_1}{(-z_1)^2}\frac{dz_2}{(z_2)^2}+\frac{dz_1}{(z_1)^2}\frac{dz_2}{(-z_2)^2}\Big]\Big(\frac{-1}{z_0^2}\Big)\\
&= \frac{-dz_0}{4z_0^2}\Big[\frac{2dz_1dz_2}{z_1^2z_2^2}\Big]\\
&= \frac{-dz_0dz_1dz_2}{2z_0^2z_1^2z_2^2}
\end{aligned} \tag{4.3.8}$$

It is clear that $W_{0,3}$ is symmetric and odd, and only has poles at $z_i = 0$, which is a ramification point. This serves as a good example of the properties of stable forms mentioned above

The $W_{1,1}$ differential also has a pole at $z = 0$ contributed only by the K_p term. By expanding the bracketed term into a Laurent series, we can easily pick out the coefficient of z^{-1}.

$$\begin{aligned}
W_{1,1}(z_0) &= \sum_{p\in R}\mathrm{Res}_{z=p}K_p(z_0,z)W_{0,2}(-z,z)\\
&= -dz_0\mathrm{Res}_{z=0}\frac{1}{4z^2dz}\Big[\frac{1}{z-z_0}-\frac{1}{\alpha-z_0}\Big]\frac{d(-z)dz}{((-z)-z)^2}\\
&= -dz_0\mathrm{Res}_{z=0}\frac{1}{4z^2dz}\Big[\frac{1}{z-z_0}-\frac{1}{\alpha-z_0}\Big]\frac{-dz^2}{4z^2}\\
&= dz_0\mathrm{Res}_{z=0}\frac{1}{16z^4}\Big[\frac{1}{z-z_0}-\frac{1}{\alpha-z_0}\Big]dz\\
&= \frac{dz_0}{16}\mathrm{Res}_{z=0}\frac{1}{z^4}\Big[-\frac{1}{z_0}-\frac{z}{z_0^2}-\frac{z^2}{z_0^3}-\frac{z^3}{z_0^4}-\cdots-\frac{1}{\alpha-z_0}\Big]dz\\
&= \frac{dz_0}{16}\mathrm{Res}_{z=0}\Big[-\frac{1}{z_0z^4}-\frac{1}{z_0^2z^3}-\frac{1}{z_0^3z^2}-\frac{1}{z_0^4z}-\cdots-\frac{1}{(\alpha-z_0)z^4}\Big]dz\\
&= \frac{-dz_0}{16z_0^4}
\end{aligned}$$

4.3.1 Airy and WKB

We will take a small digression to briefly review the Wentzel-Kramers-Brillouin (WKB) method and show how it relates to topological recursion for the Airy spectral curve. The WKB method is used for finding a solution to a linear differential equation whose highest order derivative is multiplied by a small parameter [23, 72, 96]. We wish to apply the WKB method to a second-order differential equation with potential $V(x)$ given by

$$\Big[\hbar^2\frac{d^2}{dx^2}-V(x)\Big]\psi(x)=0 \tag{4.3.9}$$

where $\hbar$ is a small parameter. Consider an ansatz of the form

$$\psi(x)=\exp\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^kS_k(x)\Big). \tag{4.3.10}$$

Computing both the first and second derivative of ψ yields:

$$\begin{aligned}\frac{d}{dx}\psi(x) &= \exp\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k(x)\Big)\cdot\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k'(x)\Big)\\ &= \psi(x)\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k'(x)\Big),\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dx^2}\psi(x) &= \exp\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k(x)\Big)\cdot\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k'(x)\Big)^2 + \exp\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k(x)\Big)\cdot\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k''(x)\Big)\\ &= \psi(x)\Big[\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k''(x)\Big) + \Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k'(x)\Big)^2\Big].\end{aligned}$$

We then plug the ansatz into the differential equation (4.3.9) and gather terms by powers of $\hbar$.

$$\begin{aligned}0 &= \Big[\hbar^2\frac{d^2}{dx^2} - V(x)\Big]\psi(x)\\ &= \Big(\hbar^2\Big[\Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k''(x)\Big) + \Big(\frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k S_k'(x)\Big)^2\Big] - V(x)\Big)\psi(x)\\ &= \Big(\sum_{k=0}^{\infty}\hbar^{k+1}S_k''(x) + \Big(\sum_{k=0}^{\infty}\hbar^k S_k'(x)\Big)^2 - V(x)\Big)\psi(x)\\ &= \Big(\sum_{k=0}^{\infty}\hbar^{k+1}S_k''(x) + \Big(\sum_{l=0}^{\infty}\hbar^l S_l'(x)\Big)\Big(\sum_{m=0}^{\infty}\hbar^m S_m'(x)\Big) - V(x)\Big)\psi(x)\\ &= \Big(\sum_{k=0}^{\infty}\hbar^{k+1}S_k''(x) + \Big(\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\hbar^{l+m} S_l'(x)S_m'(x)\Big) - V(x)\Big)\psi(x)\\ &= \Big(\sum_{k=1}^{\infty}\hbar^{k}S_{k-1}''(x) + \Big(\sum_{k=1}^{\infty}\sum_{m=0}^{k}\hbar^{k} S_{k-m}'(x)S_m'(x)\Big) - V(x)\Big)\psi(x)\\ &= \Big(\sum_{k=1}^{\infty}\hbar^{k}\Big[S_{k-1}''(x) + \sum_{m=0}^{k}S_{k-m}'(x)S_m'(x)\Big] + S_0'(x)^2 - V(x)\Big)\psi(x)\end{aligned}$$

Comparing terms in powers of $\hbar$ yields the following recursion relation on the coefficients

$$S_0'(x)^2 - V(x) = 0, \tag{4.3.11}$$

$$S_{k-1}''(x) + \sum_{m=0}^{k} S_{k-m}'(x)S_m'(x) = 0. \tag{4.3.12}$$

Computing the coefficients for $k = 0, 1, 2$:

$k = 0$

$$S_0'(x) = V(x)^{1/2} \tag{4.3.13}$$

$k = 1$

$$\begin{aligned}0 &= S_0''(x) + \sum_{m=0}^{1} S_m'(x)S_{1-m}'(x)\\ &= S_0''(x) + S_0'(x)S_1'(x) + S_1'(x)S_0'(x)\\ &= S_0''(x) + 2S_0'(x)S_1'(x)\\ &\Rightarrow S_1'(x) = \frac{-S_0''(x)}{2S_0'(x)}\end{aligned} \tag{4.3.14}$$

$k = 2$

$$
\begin{aligned}
0 &= S_1''(x) + \sum_{m=0}^{2} S_m'(x) S_{2-m}'(x) \\
&= S_1''(x) + S_0'(x)S_2'(x) + S_1'(x)^2 + S_2'(x)S_0'(x) \\
&= S_1''(x) + 2S_0'(x)S_2'(x) + S_1'(x)^2 \\
&\Rightarrow S_2'(x) = \frac{-(S_1''(x) + S_1'(x)^2)}{2S_0'(x)}
\end{aligned} \tag{4.3.15}
$$

We would like to use the relations above with the following differential equation:

$$
\left[\hbar^2 \frac{d^2}{dx^2} - x\right]\psi(x) = 0 \tag{4.3.16}
$$

Using equations (4.3.13), (4.3.14), and (4.3.15), we have that the first three WKB coefficients are

$$
\begin{aligned}
S_0'(x) &= \sqrt{V(x)} \\
&= \sqrt{x},
\end{aligned} \tag{4.3.17}
$$

$$
\begin{aligned}
S_1'(x) &= \frac{-S_0''(x)}{2S_0'(x)} \\
&= \frac{-\frac{1}{2}x^{-\frac{1}{2}}}{2\sqrt{x}} \\
&= \frac{-1}{4x},
\end{aligned} \tag{4.3.18}
$$

$$
\begin{aligned}
S_2'(x) &= \frac{-(S_1''(x) + S_1'(x)^2)}{2S_0'(x)} \\
&= \frac{\left(\frac{1}{4x^2} + (\frac{-1}{4x})^2\right)}{2\sqrt{x}} \\
&= \frac{\left(\frac{1}{4x^2} + \frac{1}{16x^2}\right)}{2\sqrt{x}} \\
&= \frac{-\frac{5}{16x^2}}{2\sqrt{x}} \\
&= \frac{-5}{32x^{\frac{5}{2}}}.
\end{aligned} \tag{4.3.19}
$$

We would like to compare these results to the $W_{g,n}$'s that we computed for the Airy spectral curve. Gather terms having the same $k = 2g + n - 1$ with some particular coefficients, and integrate each component n times to yield a function.

$$\begin{aligned}\int_{\infty}^{z} W_{0,1}(z) &= \int_{\infty}^{z} y(z)dx(z) \\ &= \int_{\infty}^{z} z(2zdz) \\ &= \int_{\infty}^{z} 2z^2 dz \\ &= \frac{2}{3}z^3 \\ &= \frac{2}{3}x^{\frac{3}{2}}\end{aligned} \tag{4.3.20}$$

$$\begin{aligned}\frac{-1}{2!}\int_{\infty}^{z}\int_{\infty}^{z} W_{0,2}(-z_1,z_2) &= \frac{1}{2}\int_{\infty}^{z}\int_{\infty}^{z} \frac{dz_1 dz_2}{(z_1+z_2)^2} \\ &= \frac{-1}{2}\int_{\infty}^{z} \frac{dz_2}{(z+z_2)} \\ &= \frac{-1}{2}\log(2z) \\ &= \frac{-1}{2}\log(2x^{\frac{1}{2}}) \\ &= \frac{-1}{4}\log(x)\end{aligned} \tag{4.3.21}$$

$$\begin{aligned}\frac{1}{3!}\int_{\infty}^{z}\int_{\infty}^{z}\int_{\infty}^{z} W_{0,3}(z_0,z_1,z_2) + \frac{1}{1!}\int_{\infty}^{z} W_{1,1}(z_0) &= \frac{1}{6}\int_{\infty}^{z}\int_{\infty}^{z}\int_{\infty}^{z} \frac{-dz_0 dz_1 dz_2}{2z_0^2 z_1^2 z_2^2} + \int_{\infty}^{z}\frac{-dz_0}{16z_0^4} \\ &= \frac{1}{12z}\int_{\infty}^{z}\int_{\infty}^{z}\frac{dz_1 dz_2}{z_1^2 z_2^2} + \frac{1}{48z^3} \\ &= \frac{-1}{12z^2}\int_{\infty}^{z}\frac{dz_2}{z_2^2} + \frac{1}{48z^3} \\ &= \frac{1}{12z^3} + \frac{1}{48z^3} \\ &= \frac{5}{48z^3} \\ &= \frac{5}{48x^{\frac{3}{2}}}\end{aligned} \tag{4.3.22}$$

Remark. In each of these terms we ignore the infinite constant, in the spirit of renormalization.

Comparing these integrals to the (suggestively labeled) S_k''s computed above, we see that the S_k' are the derivatives of the corresponding $W_{g,n}$ integrals. This suggests that there is a relationship between the Eynard-Orantin differentials and the WKB coefficients. The map,

$$x \to \widehat{x} = x, \tag{4.3.23}$$

$$y \to \widehat{y} = \hbar\frac{d}{dx}, \tag{4.3.24}$$

takes the Airy spectral curve to the differential operator $\hbar^2\frac{d^2}{dx^2} - x$ in (4.3.16), called the *Airy quantum curve*. This further suggests that we might be able to produce a WKB solution to a differential equation from the data of topological recursion which is the "quantization" of the initial spectral curve. Specifically, integrating a suitable linear combination of differentials for a specific $k = 2g + n - 1$ is proportional to the WKB coefficients S_k. One might ask if such a relationship always exists. We will show that the relationship is at least true for the Airy spectral curve, and address the general query in the next section.

4.3.2 Topological recursion for Airy spectral curve

In this section, we use the topological recursion to compute a WKB solution for the Airy quantum curve. The process involves roughly four steps:

- compute the $W_{g,n}$ using residues;
- integrate the $W_{g,n}$;
- specialization (set all variables $z_i = z$);
- produce a "wave-function".

Theorem 4.3.1. *(Eynard-Orantin, [42]) If $2g+n-1 \geq 1$, then*

$$\frac{W_{g,n+1}(z_0,\boldsymbol{z})}{dz_0} = \frac{1}{4z_0^2 dz_0^2}\Big(W_{g-1,n+2}(z_0,-z_0,\boldsymbol{z}) + \sum_{\substack{g_1+g_2=g\\ I\cup J=\boldsymbol{z}}}{}' W_{g_1,|I|+1}(z_0,I)W_{g_2,|J|+1}(-z_0,J)\Big)$$
$$+\sum_{i=1}^{n} d_{z_i}\left(\frac{1}{4z_i^2 dz_i}\left[\frac{1}{z_1-z_0}+\frac{1}{z_1+z_0}\right]W_{g,n}(-z_i,\boldsymbol{z}\backslash\{z_i\})\right) \tag{4.3.25}$$

Proof. The first step of computing this residue is to apply the Residue theorem. The residue at $z=0$ becomes a sum over residues at non-branch points.

$$\begin{aligned}
W_{g,n+1}(z_0,\mathbf{z}) &= -\mathrm{Res}_{z=0}\frac{dz_0}{4z^2dz}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]\mathcal{R}^{(2)}W_{g,n+1}(z,-z,\mathbf{z})\\
&= \sum_{\text{poles } Q\notin R}\mathrm{Res}_{z=Q}\frac{dz_0}{4z^2dz}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]\mathcal{R}^{(2)}W_{g,n+1}(z,-z,\mathbf{z})\\
&= \sum_{\text{poles } Q\notin R}\mathrm{Res}_{z=Q}\frac{dz_0}{4z^2dz}\Big[\underbrace{\frac{1}{z-z_0}}_{\text{pole at } z_0}+\frac{1}{\alpha-z_0}\Big]\Big(\underbrace{W_{g-1,n+2}(z,-z,\mathbf{z})}_{\text{pole at } 0}\\
&\quad+\sum_{\text{stable}}\underbrace{W_{g_1,|I|+1}(z,I)W_{g_2,|J|+1}(-z,J)}_{\text{pole at } 0}\\
&\quad+\sum_{i=1}^{n}\underbrace{W_{0,2}(z,z_i)W_{g,n}(-z,\mathbf{z}\backslash\{z_i\})}_{\text{pole at } z=z_i}+\underbrace{W_{0,2}(-z,z_i)W_{g,n}(z,\mathbf{z}\backslash\{z_i\})\Big)}_{\text{pole at } z=-z_i}
\end{aligned}$$

The terms in the square brackets has a single pole at $z=z_0$. Stable forms can only have poles at ramification points, so all stable forms in the above expressions can only have poles at 0. The remaining $W_{0,2}$ terms have poles at $z=\pm z_i$

$$\begin{aligned}
W_{0,2}(z,z_i) &= \frac{dzdz_i}{(z-z_i)^2},\\
W_{0,2}(-z,z_i) &= \frac{-dzdz_i}{(z+z_i)^2}.
\end{aligned}$$

At $z=z_0$, only the $\frac{1}{z-z_0}$ term contributes to the residue with $\mathrm{Res}_{z=z_0}\left(\frac{1}{z-z_0}\right)=1$

$$
\begin{aligned}
&\mathrm{Res}_{z=z_0}\frac{dz_0}{4z^2dz}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]\mathcal{R}^2W_{g,n+1}(z,-z,\mathbf{z})\\
&\quad=\lim_{z\to z_0}(z-z_0)\frac{dz_0}{4z^2dz}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]\mathcal{R}^2W_{g,n+1}(z,-z,\mathbf{z})\\
&\quad=\lim_{z\to z_0}\frac{dz_0}{4z^2dz}\Big(\,W_{g-1,n+2}(z,-z,\mathbf{z})+\sum_{g_1+g_2=g,\ I\cup J=\mathbf{z}}{}'\,W_{g_1,|I|+1}(z,I)W_{g_2,|J|+1}(-z,J)\Big)\\
&\quad=\frac{1}{4z_0^2dz_0}\Big(\,W_{g-1,n+2}(z_0,-z_0,\mathbf{z})+\sum_{g_1+g_2=g,I\cup J=\mathbf{z}}{}'\,W_{g_1,\ |I|+1}(z_0,I)W_{g_2,|J|+1}(-z_0,J)\Big)
\end{aligned}
$$

Remark. The last line of this computation involves a slight abuse of notation. The $\mathcal{R}^2W_{g,n+1}(z,-z,\mathbf{z})$ terms are $(n+2)$-differentials, and $K(z_0,z)$ is a one-differential, which come together in (4.2.4) to form an $(n+3)$-differential. Taking the residue and the $\frac{1}{dz}$ term together reduces the $(n+3)$-differential into an $(n+1)$-differential. The notation $\frac{1}{dz_0}$ is a shorthand way of denoting the removal of a one-differential and evaluation $z=z_0$.

At $z=\pm z_i$, only the $W_{0,2}(\pm z,z_i)$ term contributes to the residue.

$$
\begin{aligned}
&\mathrm{Res}_{z=\pm z_i}\frac{dz_0}{4z^2dz}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]W_{0,2}(\pm z,z_i)W_{g,n}(\mp z,\mathbf{z}\backslash\{z_i\})=\\
&\quad=\lim_{z\to\pm z_i}\frac{d}{dz}\Big((z_i\mp z)^2\frac{dz_0}{4z^2dz}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]\frac{\pm dzdz_i}{(z_i\mp z)^2}W_{g,n}(\mp z,\mathbf{z}\backslash\{z_i\})\Big)\\
&\quad=\lim_{z\to\pm z_i}\frac{d}{dz}\Big(\frac{\pm dz_0dz_i}{4z^2}\Big[\frac{1}{z-z_0}+\frac{1}{\alpha-z_0}\Big]W_{g,n}(\mp z,\mathbf{z}\backslash\{z_i\})\Big)\\
&\quad=d_{z_i}\Big(\frac{\mp1}{4z_i^2dz_i}\Big[\frac{1}{z_0\mp z_i}\Big]W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})\Big)dz_0
\end{aligned}
$$

Adding both of these residues together yields:

$$
d_{z_i}\Big(\frac{1}{4z_i^2dz_i}\Big[\frac{1}{z_i-z_0}+\frac{1}{z_i+z_0}\Big]W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})\Big)dz_0,
$$

where the z_0-z_i term comes from $z=z_i$ and the z_0-z_i term comes from $z=-z_i$.

The theorem then follows by summing over the poles and dividing by dz_0.

□

Corollary 4.3.2. *If $2g+n-1\geq 0$, then*[2]

$$
\begin{aligned}
&\frac{W_{g-1,n+2}(-z_0,z_0,\boldsymbol{z})}{dx(z_0)^2}-\sum_{\substack{g_1+g_2=g\\ I\cup J=\boldsymbol{z}}}\Big(\frac{W_{g_1,|I|+1}(-z_0,I)}{dx(z_0)}\Big)\Big(\frac{W_{g_2,|J|+1}(-z_0,J)}{dx(z_0)}\Big)\\
&+\sum_{i=1}^{n}\Big(\Big(\frac{dx(z_i)}{(x(z_0)-x(z_i))^2}\Big)\frac{W_{g,n}(-z_0,\boldsymbol{z}\backslash\{z_i\})}{dx(z_0)}-d_{z_i}\Big(\frac{1}{x(z_0)-x(z_i)}\frac{W_{g,n}(-z_i,\boldsymbol{z}\backslash\{z_i\})}{dx(z_i)}\Big)\Big)=0.
\end{aligned}
$$

If $(g,n)=(0,0)$, then

$$
\frac{W_{0,1}(-z_0)}{dx(z_0)}\frac{W_{0,1}(-z_0)}{dx(z_0)}-x(z_0)=0. \tag{4.3.26}
$$

[2] *Our version of this corollary differs from the analogous one in Eynard-Orantin [42]. We suspect that their calculation contains two sign errors, which we have rectified here. Correcting these is essential to establishing the validity of further results in the literature.*

Proof. We begin by considering the $2g+n-1 \geq 0$ case. In the final steps of the recursion, we want to set all of the z_i to be equal, but the $W_{0,2}(z_1, z_2)$ terms are problematic because

$$\lim_{z_1 \to z_2} W_{0,2}(z_1, z_2) = \infty.$$

However, if we negate one of the entries, it removes this problem,

$$\begin{aligned} W_{0,2}(-z, z) &= \frac{d(-z)dz}{(-z-z)^2} \\ &= \frac{-dzdz}{(-2z)^2} \\ &= \frac{-dz^2}{4z^2}. \end{aligned}$$

As such, we would like to replace all instances of $W_{0,2}(z_1, z_2)$ with $W_{0,2}(-z_1, z_2)$ using (4.2.14), which we verify is

$$\begin{aligned} W_{0,2}(z_1, z_2) + W_{0,2}(-z_1, z_2) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2} - \frac{dz_1 dz_2}{(z_1 + z_2)^2} \\ &= \frac{(z_1+z_2)^2 - (z_1 - z_2)^2}{(z_1 - z_2)^2 (z_1 + z_2)^2} dz_1 dz_2 \\ &= \frac{z_1^2 + 2z_1 z_2 + z_2^2 - z_1^2 + 2z_1 z_2 - z_2^2}{(z_1^2 - z_2^2)^2} dz_1 dz_2 \\ &= \frac{4z_1 z_2 dz_1 dz_2}{(z_1^2 - z_2^2)^2} \\ &= \frac{(2z_1 dz_1)(2z_2 dz_2)}{(z_1^2 - z_2^2)^2} \\ &= \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}. \end{aligned}$$

For $(g_1, |I|) = (0, 1)$ and $(g_2, |J|) = (g, n-1)$,

$$\begin{aligned} W_{g_1,|I|+1}(z_0, I) W_{g_2,|J|+1}(-z_0, J) &= W_{0,2}(z_0, z_i) W_{g,n}(-z_0, \mathbf{z} \backslash \{z_i\}) \\ &= \Big(- W_{0,2}(-z_0, z_i) + \frac{dx(z_0)dx(z_i)}{(x(z_0) - x(z_i))^2} \Big) W_{g,n}(-z_0, \mathbf{z} \backslash \{z_i\}) \\ &= -W_{0,2}(-z_0, z_i) W_{g,n}(-z_0, \mathbf{z} \backslash \{z_i\}) + \frac{dx(z_0)dx(z_i)}{(x(z_0) - x(z_i))^2} W_{g,n}(-z_0, \mathbf{z} \backslash \{z_i\}). \end{aligned}$$

Recalling that all stable forms $W_{g,n}$ are odd in all arguments, we have that the remaining terms, which are of the form $(g_1, |I|) \neq (0, 1)$, $(g_2, |J|) \neq (0, 1)$, satisfy

$$W_{g_1,|I|+1}(z_0, I) W_{g_2,|J|+1}(-z_0, J) = W_{g_1,|I|+1}(-z_0, I) W_{g_2,|J|+1}(-z_0, J).$$

Putting all of this together, we get:

$$\begin{aligned} \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{z}}}' W_{g_1,|I|+1}(z_0, I) W_{g_2,|J|+1}(-z_0, J) = - \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{z}}}' & W_{g_1,|I|+1}(-z_0, I) W_{g_2,|J|+1}(-z_0, J) \\ & + \sum_{i=1}^{n} \frac{dx(z_0)dx(z_i)}{(x(z_0) - x(z_i))^2} W_{g,n}(-z_0, \mathbf{z} \backslash \{z_i\}). \end{aligned} \tag{4.3.27}$$

Moreover, recall that

$$\begin{aligned} W_{0,1}(-z_0) &= y(-z_0) dx(-z_0) \\ &= -z_0 dx(-z_0) \\ &= -z_0 dx(z_0). \end{aligned}$$

Combining the above calculation with the fact that stable forms are odd, we can make the following observation

$$\begin{aligned}\frac{W_{g,n+1}(z_0,\mathbf{z})}{dz_0} &= \frac{-W_{g,n+1}(-z_0,\mathbf{z})}{dz_0}\\ &= \frac{-2z_0W_{g,n+1}(-z_0,\mathbf{z})}{2z_0dz_0}\\ &= -2z_0\frac{W_{g,n+1}(-z_0,\mathbf{z})}{dx(z_0)}\\ &= 2\frac{W_{0,1}(-z_0)}{dx(z_0)}\frac{W_{g,n+1}(-z_0,\mathbf{z})}{dx(z_0)}.\end{aligned} \tag{4.3.28}$$

Combining (4.3.27) and (4.3.28) we get a sum over all g and n,

$$\sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}\frac{W_{g_1,|I|+1}(-z_0,I)}{dx(z_0)}\frac{W_{g_2,|J|+1}(-z_0,J)}{dx(z_0)}. \tag{4.3.29}$$

The last term in theorem 4.3.1 can be simplified as

$$\begin{aligned}\frac{1}{4z_i^2dz_i}\Big[\frac{1}{z_i-z_0}+\frac{1}{z_i+z_0}\Big] &= \frac{1}{4z_i^2dz_i}\Big[\frac{z_i+z_0+z_i-z_0}{(z_i-z_0)(z_i+z_0)}\Big]\\ &= \frac{1}{4z_i^2dz_i}\Big[\frac{2z_i}{z_i^2-z_0^2}\Big]\\ &= \frac{1}{2z_idz_i}\Big[\frac{1}{z_i^2-z_0^2}\Big]\\ &= \frac{1}{dx(z_i)}\Big[\frac{1}{x(z_i)-x(z_0)}\Big]\\ &= \frac{-1}{dx(z_i)}\Big[\frac{1}{x(z_0)-x(z_i)}\Big].\end{aligned}$$

Starting from Theorem 4.3.1 and taking all of this together we have

$$\begin{aligned}&\frac{W_{g,n+1}(z_0,\mathbf{z})}{dz_0}\overbrace{=}^{\text{thm 4.3.1}}\\ &\frac{1}{4z_0^2dz_0^2}\Big(W_{g-1,n+2}(z_0,-z_0,\mathbf{z})+\sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}{}'W_{g_1,|I|+1}(z_0,I)W_{g_2,|J|+1}(-z_0,J)\Big)\\ &+\sum_{i=1}^n d_{z_i}\Big(\frac{1}{4z_i^2dz_i}\big[\frac{1}{z_1-z_0}+\frac{1}{z_1+z_0}\big]W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})\Big)\\ &=\frac{1}{4z_0^2dz_0^2}\Big(\sum_{i=1}^n\frac{dx(z_0)dx(z_i)}{(x(z_0)-x(z_i))^2}W_{g,n}(-z_0,\mathbf{z}\backslash\{z_i\})\Big)\\ &+\frac{1}{4z_0^2dz_0^2}\Big(W_{g-1,n+2}(z_0,-z_0,\mathbf{z})-\sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}{}'W_{g_1,|I|+1}(-z_0,I)W_{g_2,|J|+1}(-z_0,J)\Big)\\ &+\sum_{i=1}^n d_{z_i}\Big(\frac{-1}{dx(z_i)}\Big[\frac{1}{x(z_0)-x(z_i)}\Big]W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})\Big)\\ &=\sum_{i=1}^n\frac{dx(z_i)}{(x(z_0)-x(z_i))^2}\frac{W_{g,n}(-z_0,\mathbf{z}\backslash\{z_i\})}{dx(z_0)}\\ &+\frac{W_{g-1,n+2}(z_0,-z_0,\mathbf{z})}{dx(z_0)^2}-\sum_{\substack{g_1+g_2=g\\ I\cup J=\mathbf{z}}}{}'\frac{W_{g_1,|I|+1}(-z_0,I)}{dx(z_0)}\frac{W_{g_2,|J|+1}(-z_0,J)}{dx(z_0)}\\ &-\sum_{i=1}^n d_{z_i}\Big(\frac{1}{(x(z_0)-x(z_i))}\frac{W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})}{dx(z_i)}\Big).\end{aligned}$$

Gathering everything to one side and using (4.3.29) finishes this part of the proof.

For $(g,n)=(0,0)$ we can easily compute

$$\begin{aligned}\frac{W_{0,1}(-z_0)}{dx(z_0)}\frac{W_{0,1}(-z_0)}{dx(z_0)} &= \frac{y(-z_0)dx(-z_0)}{dx(z_0)}\frac{y(-z_0)dx(-z_0)}{dx(z_0)}\\ &= (-z_0)(-z_0)\\ &= z_0^2\\ &= x(z_0).\end{aligned}$$

□

Definition 4.3.3. *Define*

$$G_{g,n+1}(z_0,\boldsymbol{z}) \coloneqq \int_\infty^{z_1}\cdots\int_\infty^{z_n} W_{g,n+1}(-z_0,z_1,...,z_n) \tag{4.3.30}$$

where the integration is along a path with a base point at $z=\infty$ to the marked point z_i in each coordinate except for z_0.

In the second part of the recursion procedure, we integrate the $W_{g,n}$ and adapt our previous result to the $G_{g,n}$. As the differentials only have poles at $z=0$, the integrals converge and the $G_{g,n}$ are well-defined.

Lemma 4.3.4. *If $2g+n-1\geq 0$, then*

$$\begin{aligned}&\Big(\frac{\partial}{\partial x(z_{n+1})}\frac{G_{g-1,n+2}(z_0,\boldsymbol{z},z_{n+1})}{dx(z_0)}\Big)_{z_{n+1}=z_0}\\ &-\sum_{\substack{g_1+g_2=g\\|I|+|J|=z}}\Big(\frac{G_{g_1,|I|+1}(z_0,I)}{dx(z_0)}\Big)\Big(\frac{G_{g_2,|J|+1}(z_0,J)}{dx(z_0)}\Big)\\ &+\sum_{i=1}^{n}\frac{1}{x(z_0)-x(z_i)}\Big(\frac{G_{g,n}(z_0,\boldsymbol{z}\backslash\{z_i\})}{dx(z_0)}-\frac{G_{g,n}(z_i,\boldsymbol{z}\backslash\{z_i\})}{dx(z_i)}\Big)=0\end{aligned} \tag{4.3.31}$$

If $(g,n)=(0,0)$, then

$$\frac{G_{0,1}(z_0)}{dx(z_0)}\frac{G_{0,1}(z_0)}{dx(z_0)}-x(z_0)=0 \tag{4.3.32}$$

Proof. Each term in (4.3.31) is obtained from Corollary 4.3.2 by integrating over $z_1,\ldots,z_n$. The first term of this lemma is obtained from the first term of the corollary by

$$\begin{aligned}&\frac{\partial}{\partial x(z_{n+1})}\frac{G_{g-1,n+2}(z_0,\mathbf{z},z_{n+1})}{dx(z_0)}|_{z_{n+1}=z_0}\\ &=\frac{\partial}{\partial x(z_{n+1})}\Big(\int_\infty^{z_1}\cdots\int_\infty^{z_{n+1}}\frac{W_{g-1,n+2}(-z_0,\mathbf{z},z_{n+1})}{dx(z_0)}\Big)|_{z_{n+1}=z_0},\\ &=\int_\infty^{z_1}\cdots\int_\infty^{z_n}\frac{W_{g-1,n+2}(-z_0,\mathbf{z},z_{n+1})}{dx(z_0)dz_{n+1}}\frac{dz_{n+1}}{dx(z_{n+1})}|_{z_{n+1}=z_0}\ \text{(fundamental theorem of calculus)},\\ &=\int_\infty^{z_1}\cdots\int_\infty^{z_n}\frac{W_{g-1,n+2}(-z_0,\mathbf{z},z_0)}{dx(z_0)^2}\ \text{(evaluate } z_{n+1}=z_0),\\ &=\int_\infty^{z_1}\cdots\int_\infty^{z_n}\frac{W_{g-1,n+2}(-z_0,z_0,\mathbf{z})}{dx(z_0)^2}\ \text{(symmetry of } W_{g,n}).\end{aligned}$$

The second term is obtained from the $W_{g_1,|I|+1}(-z_0,I)W_{g_2,|J|+1}(-z_0,J)$ terms by applying the definition of $G_{g,n+1}$.

$$\begin{aligned}G_{g_1,|I|+1}(z_0,I)G_{g_2,|J|+1}(z_0,J) &= \Big(\int\cdots\int W_{g_1,|I|+1}(-z_0,I)\Big)\Big(\int\cdots\int W_{g_2,|J|+1}(-z_0,J)\Big)\\ &= \int\cdots\int W_{g_1,|I|+1}(-z_0,I)W_{g_2,|J|+1}(-z_0,J)\end{aligned}$$

The final term is obtained in part by integrating the last terms of the previous corollary.

$$\begin{aligned}&\int_\infty^{z_1}\cdots\int_\infty^{z_n}\frac{W_{g,n}(-z_0,\mathbf{z}\backslash\{z_i\})}{dx(z_0)}\frac{dx(z_i)}{(x(z_0)-x(z_i))^2}\\ &\quad= \int_\infty^{z_1}\cdots\int_\infty^{z_n}\int_\infty^{z_i}\frac{W_{g,n}(-z_0,\mathbf{z}\backslash\{z_i\})}{dx(z_0)}\frac{dx(z_i)}{(x(z_0)-x(z_i))^2}\\ &\quad= \underbrace{\int_\infty^{z_1}\cdots\int_\infty^{z_n}}_{\text{no } z_i \text{ integral}}\frac{W_{g,n}(-z_0,\mathbf{z}\backslash\{z_i\})}{dx(z_0)}\frac{1}{x(z_0)-x(z_i)}\\ &\quad= \frac{1}{x(z_0)-x(z_i)}\frac{G_{g,n}(z_0,\mathbf{z}\backslash\{z_i\})}{dx(z_0)}\end{aligned}$$

$$\begin{aligned}&\int_\infty^{z_1}\cdots\int_\infty^{z_n}d_{z_i}\Big(\frac{1}{x(z_0)-x(z_i)}\frac{W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})}{dx(z_i)}\Big)\\ &\quad= \int_\infty^{z_1}\cdots\int_\infty^{z_n}\int_\infty^{z_i}d_{z_i}\Big(\frac{1}{x(z_0)-x(z_i)}\frac{W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})}{dx(z_i)}\Big)\\ &\quad= \underbrace{\int_\infty^{z_1}\cdots\int_\infty^{z_n}}_{\text{no } z_i \text{ integral}}\Big(\frac{1}{x(z_0)-x(z_i)}\frac{W_{g,n}(-z_i,\mathbf{z}\backslash\{z_i\})}{dx(z_i)}\Big)\\ &\quad= \frac{1}{x(z_0)-x(z_i)}\frac{G_{g,n}(z_i,\mathbf{z}\backslash\{z_i\})}{dx(z_i)}\end{aligned}$$

For the $(g,n)=(0,0)$ case we have that $W_{0,1}(-z_0)=G_{0,1}(z_0)$, so (4.3.32) follows directly from the previous corollary.

□

Definition 4.3.5. *The **partial specialization** of $G_{g,n+1}(z_0,\boldsymbol{z})$ is*

$$\widehat{G}_{g,n+1}(z_0,z)=G_{g,n+1}(z_0,z,\dots,z), \tag{4.3.33}$$

*and the **full specialization** is*

$$\widehat{G}_{g,n+1}(z,z). \tag{4.3.34}$$

Proposition 4.3.6. *If $2g+n-1\geq 0$, then:*

$$\Big(\frac{\partial}{\partial x(z_{n+1})}\frac{G_{g-1,n+2}(z_0,\boldsymbol{z},z_{n+1})}{dx(z_0)}\Big)_{\substack{z_0=z\\ \vdots\\ z_{n+1}=z}} = \frac{1}{n+1}\Big(\frac{\partial}{\partial x(z_0)}\frac{\widehat{G}_{g-1,n+2}(z_0,z)}{dx(z_0)}\Big)_{z_0=z} \tag{4.3.35}$$

Proof. Let $z_i=z_i(z)$ for all z_i except z_0. Using the chain rule, we have

$$\begin{aligned}\frac{\partial}{\partial x(z)}\frac{G_{g-1,n+2}(z_0,\mathbf{z},z_{n+1})}{dx(z_0)} &= \sum_{i=1}^{n+1}\Big(\frac{\partial}{\partial z_i}\frac{G_{g-1,n+2}(z_0,\mathbf{z},z_{n+1})}{dx(z_0)}\Big)\frac{\partial z_i}{\partial x(z)}\\ &= \sum_{i=1}^{n+1}\Big(\frac{\partial}{\partial x(z_i)}\frac{G_{g-1,n+2}(z_0,\mathbf{z},z_{n+1})}{dx(z_0)}\Big)\frac{\partial x(z_i)}{\partial z_i}\frac{\partial z_i}{\partial x(z)}\\ &= \sum_{i=1}^{n+1}\Big(\frac{\partial}{\partial x(z_i)}\frac{G_{g-1,n+2}(z_0,\mathbf{z},z_{n+1})}{dx(z_0)}\Big)\frac{\partial x(z_i)}{\partial x(z)}.\end{aligned}$$

Letting $z_i = z$ for all z_i except z_0,

$$\frac{\partial}{\partial x(z)} \frac{G_{g-1,n+2}(z_0, z, \dots, z)}{dx(z_0)} = \sum_{i=1}^{n+1} \Big(\frac{\partial}{\partial x(z_i)} \frac{G_{g-1,n+2}(z_0, \mathbf{z}, z_{n+1})}{dx(z_0)} \Big) \Big|_{\substack{z_1=z \\ \vdots \\ z_{n+1}=z}} .$$

The $G_{g,n+1}$ inherit symmetry in the last n components from the $W_{g,n+1}$, meaning that the derivatives are symmetric after taking $z_1 = \cdots = z_n = z$. For all $i \neq 0$ the derivative terms are all equal,

$$\Big(\frac{\partial}{\partial x(z_i)} \frac{G_{g-1,n+2}(z_0, \mathbf{z}, z_{n+1})}{dx(z_0)} \Big) \Big|_{\substack{z_1=z \\ \vdots \\ z_{n+1}=z}} = \Big(\frac{\partial}{\partial x(z_{n+1})} \frac{G_{g-1,n+2}(z_0, \mathbf{z}, z_{n+1})}{dx(z_0)} \Big) \Big|_{\substack{z_1=z \\ \vdots \\ z_{n+1}=z}} ,$$

and so the sum becomes

$$\frac{\partial}{\partial x(z)} \frac{G_{g-1,n+2}(z_0, z, \dots, z)}{dx(z_0)} = (n+1) \Big(\frac{\partial}{\partial x(z_{n+1})} \frac{G_{g-1,n+2}(z_0, \mathbf{z}, z_{n+1})}{dx(z_0)} \Big) \Big|_{\substack{z_1=z \\ \vdots \\ z_{n+1}=z}} .$$

The proof is finished by setting $z_0 = z$ in the above expression. □

We now adapt lemma 4.3.4 to our partial specialization.

Lemma 4.3.7. *If $2g + n - 1 \geq 0$, then:*

$$-\frac{1}{n+1} \Big(\frac{\partial}{\partial x(z)} \frac{\widehat{G}_{g-1,n+2}(z_0, z)}{dx(z_0)} \Big)|_{z_0=z} \tag{4.3.36}$$

$$+ \sum_{g_1+g_2=g} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \frac{\widehat{G}_{g_1,m+1}(z,z)}{dx(z)} \frac{\widehat{G}_{g_2,n-m+1}(z,z)}{dx(z)} \tag{4.3.37}$$

$$- n \Big(\frac{\partial}{\partial x(z_0)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z_0)} \Big)|_{z_0=z} = 0 \tag{4.3.38}$$

If $(g, n) = (0, 0)$, then:

$$\frac{G_{0,1}(z)}{dx(z)} \frac{G_{0,1}(z)}{dx(z)} - x(z) = 0$$

Proof. We prove this lemma by applying partial specialization to Lemma 4.3.4.

The first term of follows immediately from proposition 4.3.6.

For the second term, we note that when all $z_i = z$, only the size of I and J matter. This means we need to count the number of ways to partition the variables $z_1, \dots, z_n$ into intervals of size $|I|$ and $|J|$, which is

$$\binom{n}{|I|} = \frac{n!}{(|I|)!(n-|I|)!}. \tag{4.3.39}$$

Denoting $|I| = m$, we have that

$$\sum_{\substack{g_1+g_2=g \\ |I|+|J|=\mathbf{z}}} \Big(\frac{G_{g_1,|I|+1}(z_0, I)}{dx(z_0)} \Big) \Big(\frac{G_{g_2,|J|+1}(z_0, J)}{dx(z_0)} \Big)$$
$$= \sum_{g_1+g_2=g} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \frac{\widehat{G}_{g_1,m+1}(z,z)}{dx(z)} \frac{\widehat{G}_{g_2,n-m+1}(z,z)}{dx(z)},$$

which is the second term in the lemma.

We begin the final term by taking a limit $z_i \to z_0$ of each individual summand.

$$\begin{aligned}
&\lim_{z_i \to z_0} \frac{1}{x(z_0) - x(z_i)} \Big(\frac{G_{g,n}(z_0, \mathbf{z} \backslash \{z_i\})}{dx(z_0)} - \frac{G_{g,n}(z_i, \mathbf{z} \backslash \{z_i\})}{dx(z_i)} \Big) \\
&= \lim_{z_i \to z_0} \frac{1}{x'(z_0)(z_i - z_0)} \Big(\frac{G_{g,n}(z_i, \mathbf{z} \backslash \{z_i\})}{dx(z_i)} - \frac{G_{g,n}(z_0, \mathbf{z} \backslash \{z_i\})}{dx(z_0)} \Big) \\
&= \frac{\partial}{\partial x(z_0)} \Big(\frac{G_{g,n}(z_0, \mathbf{z} \backslash \{z_i\})}{dx(z_0)} \Big).
\end{aligned}$$

Setting all $z_i = z$ and putting this term back into the sum yields

$$n \Big(\frac{\partial}{\partial x(z_0)} \frac{G_{g,n}(z_0, z, \ldots, z)}{dx(z_0)} \Big)|_{z_0 = z}. \tag{4.3.40}$$

This proves the first part of the lemma.

The $(g, n) = (0, 0)$ case follows immediately from Lemma 4.3.4 by setting $z_0 = z$. □

Definition 4.3.8. *Define*

$$\xi_1(z', z) := - \sum_{g,n=0}^{\infty} \frac{\hbar^{2g+n}}{n!} \frac{\widehat{G}_{g,n+1}(z', z)}{dx(z')} \tag{4.3.41}$$

Proposition 4.3.9.

$$\hbar \frac{d}{dx(z)} \xi_1(z, z) = - \sum_{2g+n-1 \geq 0} \Big[\frac{\hbar^{2g+n}}{(n+1)!} \Big(\frac{\partial}{\partial x(z)} \frac{\widehat{G}_{g-1,n+2}(z_0, z)}{dx(z_0)} \Big)|_{z_0 = z} + \frac{\hbar^{2g+n}}{(n-1)!} \Big(\frac{\partial}{\partial x(z_0)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z_0)} \Big)|_{z_0 = z} \Big]$$

Proof. We begin by writing the sum in a more convenient form

$$-\hbar \sum_{2g+n-1 \geq 0} \Big[\frac{\hbar^{2(g-1)+(n+1)}}{(n+1)!} \Big(\frac{\partial}{\partial x(z)} \frac{\widehat{G}_{g-1,n+2}(z_0, z)}{dx(z_0)} \Big)|_{z_0 = z} + \frac{\hbar^{2g+n-1}}{(n-1)!} \Big(\frac{\partial}{\partial x(z_0)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z_0)} \Big)|_{z_0 = z} \Big].$$

Reindexing the first term in the summation by $g - 1 \to g$ and $n + 2 \to n$, the sum becomes

$$-\hbar \sum_{2g+n-1 \geq 0} \frac{\hbar^{2g+n-1}}{(n-1)!} \left[\left(\frac{\partial}{\partial x(z)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z_0)} \Big)|_{z_0 = z} + \Big(\frac{\partial}{\partial x(z_0)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z_0)} \right) \Bigg|_{z_0 = z} \right].$$

To finish, observe that

$$\frac{d}{dx(z)} \frac{\widehat{G}_{g,n}(z, z)}{dx(z)} = \Big(\frac{\partial}{\partial x(z_0)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z_0)} \Big)|_{z_0 = z} + \Big(\frac{\partial}{\partial x(z)} \frac{\widehat{G}_{g,n}(z_0, z)}{dx(z)} \Big)|_{z_0 = z}. \tag{4.3.42}$$

We can substitute this into the square brackets in the above expression and conclude that this term is the derivative of ξ_1. □

Lemma 4.3.10.

$$\hbar \frac{d}{dx(z)} \xi_1(z, z) + \xi_1(z, z)^2 - x(z) = 0 \tag{4.3.43}$$

Proof. We begin by applying Proposition 4.3.9 to Lemma 4.3.7. We take the first and last term of Lemma 4.3.7, multiply by $\frac{\hbar^{2g+n}}{n!}$ and sum over g and n. We can then change the sum over $g, n = 0$ to a sum $2g - n + 1 \geq 0$, as terms where g or n are negative are 0.

$$-\sum_{g,n=0}^{\infty}\frac{\hbar^{2g+n}}{(n+1)!}\Big(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g-1,n+2}(z_0,z)}{dx(z_0)}\Big)|_{z_0=z}+\frac{\hbar^{2g+n}}{(n-1)!}\Big(\frac{\partial}{\partial x(z_0)}\frac{\widehat{G}_{g,n}(z_0,z)}{dx(z_0)}\Big)|_{z_0=z}$$
$$=-\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n}}{(n+1)!}\Big(\frac{\partial}{\partial x(z)}\frac{\widehat{G}_{g-1,n+2}(z_0,z)}{dx(z_0)}\Big)|_{z_0=z}+\frac{\hbar^{2g+n}}{(n-1)!}\Big(\frac{\partial}{\partial x(z_0)}\frac{\widehat{G}_{g,n}(z_0,z)}{dx(z_0)}\Big)|_{z_0=z}$$

This expression is exactly the right-hand side of Proposition 4.3.9, so it is equal to

$$\hbar\frac{d}{dx(z)}\xi_1(z,z).$$

Now, we expand the ξ_1^2 and try to relate it to Lemma 4.3.7.

$$\begin{aligned}\xi_1(z,z)^2&=\Big(-\sum_{g,n=0}^{\infty}\frac{\hbar^{2g+n}}{n!}\frac{\widehat{G}_{g,n+1}(z,z)}{dx(z)}\Big)^2\\&=\sum_{\substack{g_1,m=0\\g_2,k=0}}^{\infty}\frac{\hbar^{2(g_1+g_2)+(m+k)}}{m!k!}\frac{\widehat{G}_{g_1,m+1}(z,z)}{dx(z)}\frac{\widehat{G}_{g_2,k+1}(z,z)}{dx(z)}\\&=\sum_{g,n=0}^{\infty}\sum_{g_1+g_2=g}\sum_{m=0}^{n}\frac{\hbar^{2g+n}}{m!(n-m)!}\frac{\widehat{G}_{g_1,m+1}(z,z)}{dx(z)}\frac{\widehat{G}_{g_2,n-m+1}(z,z)}{dx(z)}\\&=\sum_{g,n=0}^{\infty}\frac{\hbar^{2g+n}}{n!}\sum_{g_1+g_2=g}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{\widehat{G}_{g_1,m+1}(z,z)}{dx(z)}\frac{\widehat{G}_{g_2,n-m+1}(z,z)}{dx(z)}\end{aligned}$$

The last term in the expression can be split into two pieces: $(g,n)=(0,0)$ and $2g+n-1\geq 0$.

The $2g+n-1\geq 0$ term is the second term of Lemma 4.3.7 multiplied by $\frac{\hbar^{2g+n}}{n!}$ and summed over g and n, exactly as was done in the above computation. Using Lemma 4.3.7 we have that the sum of this term and $\hbar\frac{d}{dx(z)}\xi_1(z,z)$ equate to 0.

The $(g,n)=(0,0)$ is equal to $x(z)$, following the last statement of Lemma 4.3.7, and is thus killed by the remaining $-x(z)$, concluding the lemma. □

We can now define the wave-function. We expect this wave-function to be in the kernel of the quantum curve.

Definition 4.3.11. *The **perturbative wave-function** is defined by*

$$\psi(z)=\exp\Big[\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_{\infty}^{z}\cdots\int_{\infty}^{z}\Big(W_{g,n}(z_1,...,z_n)-\delta_{g,0}\delta_{n,0}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big)\Big].\tag{4.3.44}$$

Define

$$\psi_1(z',z):=\psi(z)\xi_1(z',z)\tag{4.3.45}$$

Proposition 4.3.12.

$$\hbar\frac{d}{dx(z)}\psi(z)=\psi_1(z,z)\tag{4.3.46}$$

Proof. Evaluating the derivative yields

$$\frac{d}{dx(z)}\psi(z)=\frac{\psi(z)}{x'(z)}\Big[\frac{d}{dz}\frac{1}{\hbar}\sum_{2g+n-1\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_{\infty}^{z}\cdots\int_{\infty}^{z}\Big(W_{g,n}(z_1,...,z_n)-\delta_{g,0}\delta_{n,0}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big)\Big].$$

Suppose that the integration bounds $z_i = z_i(z)$. Applying the chain rule and the fundamental theorem of calculus to the integral term, we get

$$\begin{aligned}
&\frac{1}{x'(z)}\frac{d}{dz}\Big(\int_{\infty}^{z_1(z)}\cdots\int_{\infty}^{z_n(z)} W_{g,n}(z_1,...,z_n) - \delta_{g,0}\delta_{n,0}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big)\\
&\qquad = \frac{1}{x'(z)}\sum_{i=1}^{n}\frac{1}{dz_i}\Big(\int_{\infty}^{z_1(z)}\cdots\widehat{\int_{\infty}^{z_i(z)}}\cdots\int_{\infty}^{z_n(z)} W_{g,n}(z_1,...,z_n) - \delta_{g,0}\delta_{n,0}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big).
\end{aligned}$$

Using the symmetry of the $W_{g,n}$, z_i can be moved into the first coordinate and then by picking up a minus sign, because $W_{g,n}$ is odd, it can be made into $-z_i$,

$$\begin{aligned}
&= -\frac{1}{x'(z)}\sum_{i=1}^{n}\frac{1}{dz_i}\Big(\int_{\infty}^{z_1(z)}\cdots\widehat{\int_{\infty}^{z_i(z)}}\cdots\int_{\infty}^{z_n(z)} W_{g,n}(-z_i,z_1,...,z_n)\Big)\\
&= -\frac{1}{x'(z)}\sum_{i=1}^{n}\frac{\widehat{G}_{g,n}(z_i,z)}{dz_i}.
\end{aligned}$$

Setting $z_i(z) = z$, we get from the above expression that the derivative of the integral is

$$-n\frac{\widehat{G}_{g,n}(z,z)}{dx(z)}. \tag{4.3.47}$$

Substituting this expression back into square brackets yields $\xi_1(z,z)$, and so the proof is complete.

□

Corollary 4.3.13.

$$\hbar^2\frac{d^2}{dx^2}\psi(z) = \psi_1(z,z)\xi_1(z,z) + \hbar\psi(z)\frac{d}{dx}\xi_1(z,z) \tag{4.3.48}$$

Proof. The proof follows from a simple computation employing the above proposition.

$$\begin{aligned}
\hbar^2\frac{d^2}{dx^2}\psi(z) &= \hbar\frac{d}{dx}\Big(\hbar\frac{d}{dx}\psi(z)\Big)\\
&= \hbar\frac{d}{dx}(\psi(z)\xi_1(z))\\
&= \hbar\Big(\frac{d}{dx}\psi(z)\Big)\xi_1(z,z) + \hbar\psi(z)\frac{d}{dx}\xi_1(z,z)\\
&= \hbar\psi_1(z,z)\xi_1(z,z) + \hbar\psi(z)\frac{d}{dx}\xi_1(z,z)
\end{aligned}$$

□

Theorem 4.3.14. *(Bouchard-Eynard, [19])*

$$\Big(\hbar^2\frac{d^2}{dx(z)^2} - x(z)\Big)\psi(z) = 0 \tag{4.3.49}$$

Proof. We begin by multiplying Lemma 4.3.10 by $\psi(z)$,

$$\begin{aligned}
0 &= \hbar\psi(z)\frac{d}{dx(z)}\xi_1(z,z) + \psi(z)\xi_1(z,z)^2 - \psi(z)x(z)\\
&= \hbar\psi(z)\frac{d}{dx(z)}\xi_1(z,z) + \psi_1(z,z)\xi_1(z,z) - \psi(z)x(z).
\end{aligned}$$

Applying Corollary 4.3.13 to the first two terms completes the proof,

$$0 = \hbar^2 \frac{d^2}{dx^2}\psi(z) - \psi(z)x(z). \tag{4.3.50}$$

□

As per our preliminary observations, Theorem 4.3.14 shows that from the information of topological recursion, we were able to produce a WKB solution to a "quantization" of the initial Airy spectral curve.

4.4 Topological recursion and WKB

The previous example of the Airy spectral suggests that there is a relationship between topological recursion and the WKB solution to a differential equation. In [19], they provide conditions on a spectral curve for which topological recursion recovers WKB solutions. This section will explore these conditions and the relationship between topological recursion and WKB.

Let S be a spectral curve defined by

$$\begin{aligned} P(x,y) &= p_0(x)y^r + p_1(x)y^{r-1} + \cdots + p_{r-1}(x)y + p_r(x) \\ &= \sum_{i=0}^{r} p_{r-1}(x)y^i = 0. \end{aligned} \tag{4.4.1}$$

For $m = 2, ..., r$ define

$$P_m(x,y) = \sum_{k=1}^{m-1} p_{m-1-k}(x)y^k. \tag{4.4.2}$$

When $r = 2$ we have only one term of this type

$$P_2(x,y) = p_0(x)y. \tag{4.4.3}$$

We can expand the $f_i(x)$ terms and consider $P(x,y)$ as a sum of monomials

$$P(x,y) = \sum_{(i,j)\in A} \alpha_{ij}x^i y^j = 0, \tag{4.4.4}$$

where $A \subset \mathbb{N}^2$ is the set of indices such that $\alpha_{ij} \neq 0$.

Definition 4.4.1. *The **Newton polygon** Δ_S of $P(x,y)$ is the convex hull of A.*

For a Newton polygon Δ_S, we can define two terms:

$$\alpha_m = \inf\{a|(a,m) \in \Delta\} \tag{4.4.5}$$

$$\beta_m = \sup\{a|(a,m) \in \Delta\}. \tag{4.4.6}$$

Using these terms, we can compute the number of integer points in Δ by

$$\text{integer points in } \Delta_S = \sum_{i=1}^{r-1}(\lceil\beta_i\rceil - \lfloor\alpha_i\rfloor - 1) \tag{4.4.7}$$

A useful theorem involving the interior integer points is Baker's formula:

Theorem 4.4.2 (Baker's Formula)**.** *If S is a spectral curve with Newton polygon Δ_S, then:*

$$g_S \leq \sum_{i=1}^{r-1} (\lceil \beta_i \rceil - \lfloor \alpha_i \rfloor - 1) \tag{4.4.8}$$

Definition 4.4.3. *A spectral curve is* ***admissible*** *if it satisfies:*

1. *Δ_S has no interior integer points*
2. *if $(0,0)$ is on S, then S is smooth at $(0,0)$*

Using Baker's formula, the first condition is equivalent to the spectral curve having genus zero.

Admissible spectral curves can be classified. They are given by $P(x,y) = 0$ that are either:

1. linear in x
2. have a Newton polygon given by the convex hull of $\{(0,0),(2,0),(0,2)\}$
3. such that they can be obtained from (1) or (2) by a transformation

$$(x,y) \mapsto (x^a y^b, x^c y^d), \tag{4.4.9}$$

with $ad - bc = 1$ together with a rescaling of x and y to get a irreducible polynomial.

Admissible spectral curves are the ones for which topological recursion recovers a WKB solution. This solution is the perturbative wave-function from Definition 4.3.11, but with more generic integration bounds.

Definition 4.4.4. *Let β be a simple pole of x. Define*

$$\psi(z;\beta) = \exp\Big[\frac{1}{\hbar} \sum_{2g+n-1\geq 0} \frac{\hbar^{2g+n-1}}{n!} \int_\beta^z \cdots \int_\beta^z \Big(W_{g,n}(z_1,...,z_n) - \delta_{g,0}\delta_{n,0} \frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2} \Big)\Big].$$

The differential equation that kills ψ is a quantization of $P(x,y) = 0$, called a *quantum curve*. A first naïve attempt to quantize $P(x,y) = 0$ would be to take the usual quantization map, taking coordinates (x,y) to differential operators $(\widehat{x},\widehat{y})$ defined by

$$x \to \widehat{x} = x, \tag{4.4.10}$$

$$y \to \widehat{y} = \hbar \frac{d}{dx}, \tag{4.4.11}$$

and consider the differential operator defined by $P(\widehat{x},\widehat{y})$ as the quantized spectral curve. The problem with the naïve attempt is that the operator variables are non-commutative; they satisfy the commutation relation $[\widehat{y},\widehat{x}] = \hbar$. This means that adding to $P(x,y)$ a term of the form $xy - yx$ leaves the spectral curve unchanged (x and y are commutative coordinate variables) but changes the quantized spectral curve to $P(\widehat{x},\widehat{y}) + \hbar$. This suggests that the definition of the quantized spectral curve needs to be more general.

Definition 4.4.5. *A* ***quantum curve*** *$\widehat{P}$ of a spectral curve S is a rank d linear differential operator in x such that after normal ordering (i.e. $\widehat{x}$ to the left of $\widehat{y}$), it has the form*

$$\widehat{P}(\widehat{x},\widehat{y};\hbar) = P(\widehat{x},\widehat{y}) + \sum_{n\geq 1} \hbar^n \tilde{P}_n(\widehat{x},\widehat{y}), \tag{4.4.12}$$

where the $\tilde{P}_n(\widehat{x},\widehat{y})$ are differential operators in x of rank at most $d-1$.
A quantum curve is ***simple*** *if there are only finitely many $\hbar$ terms.*

A theorem of [19] allows us to compute quantum curves, particularly the $\tilde{P}_n$ terms. To begin, we need to define two terms that are present in the theorem.

Definition 4.4.6. *Define for $k = 1, ..., r-1$ and $i = 1, ...r$*

$$C_k = \lim_{z_1 \to \beta} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor + 1}}, \tag{4.4.13}$$

$$D_i = \hbar \frac{x^{\lfloor \alpha_i \rfloor}}{x^{\lfloor \alpha_{i-1} \rfloor}} \frac{d}{dx}. \tag{4.4.14}$$

We can now state the main quantum curve theorem of this section.

Theorem 4.4.7. *(Bouchard-Eynard, [19]) If S is an admissible spectral curve, then $\psi(z;\beta)$ satisfies the following differential equation:*

$$\begin{aligned}\Big[D_1 D_2 \dots D_{r-1} \frac{p_0(x)}{x^{\lfloor \alpha_r \rfloor}} D_r + D_1 D_2 \dots D_{r-2} \frac{p_1(x)}{x^{\lfloor \alpha_{r-1} \rfloor}} D_{r-1} \\ + \dots + \frac{p_{r-1}(x)}{x^{\lfloor \alpha_1 \rfloor}} D_1 + \frac{p_r(x)}{x^{\lfloor \alpha_0 \rfloor}} - \hbar C_1 D_1 D_2 \dots D_{r-2} \frac{x^{\lfloor \alpha_{r-1} \rfloor}}{x^{\lfloor \alpha_{r-2} \rfloor}} \\ - \hbar C_2 D_1 D_2 \dots D_{r-3} \frac{x^{\lfloor \alpha_{r-2} \rfloor}}{x^{\lfloor \alpha_{r-3} \rfloor}} - \dots - \hbar C_{r-1} \frac{x^{\lfloor \alpha_1 \rfloor}}{x^{\lfloor \alpha_0 \rfloor}} \Big] \psi(z;\beta) = 0\end{aligned} \tag{4.4.15}$$

After normal ordering, this differential equation is equivalent to (4.4.12).

4.4.1 Example: Airy spectral curve

We saw in Section 4.3.2 that applying topological recursion to the Airy spectral curve $y^2 - x = 0$ recovered the WKB solution to the differential equation $\left(\hbar^2 \frac{d^2}{dx(z)^2} - x(z)\right)\psi(z) = 0$. We can check if Theorem 4.4.7 yields the same result for the Airy spectral curve.

We first verify that the Airy spectral curve,

$$P(x, y) = y^2 - x = 0, \tag{4.4.16}$$

with parametrization $x = z^2$ and $y = z$, is an admissible spectral curve. From this equation, we observe that

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= 0, \\ p_2(x) &= -x. \end{aligned}$$

The only non-zero α_{ij} are $\alpha_{0,2} = 1$ and $\alpha_{1,0} = -1$. The Newton polygon is the convex hull of $\{(1,0),(0,2)\}$, that is, the line segment joining $(1,0)$ and $(0,2)$. This set clearly has no interior integer points. The origin is a point on the curve; however $\nabla P(0,0) = (-1,0) \neq (0,0)$, so the curve is smooth at the origin. This means that the Airy spectral curve is an admissible spectral curve.

For the sake of matching the situation in the previous section, we will choose $\beta = \infty$. Theorem 4.4.7 says that $\phi(z)$ from (4.3.11) satisfies the equation

$$\Big[D_1 \frac{p_0(x)}{x^{\lfloor \alpha_2 \rfloor}} D_2 + \frac{p_1(x)}{x^{\lfloor \alpha_1 \rfloor}} D_1 + \frac{p_2(x)}{x^{\lfloor \alpha_0 \rfloor}} - \hbar C_1 \frac{x^{\lfloor \alpha_1 \rfloor}}{x^{\lfloor \alpha_0 \rfloor}} \Big] \psi(z) = 0. \tag{4.4.17}$$

To understand whether this is the same as (4.3.49), we need to compute the D_i's and C_1. From the Newton polygon, we have that $\lfloor \alpha_0 \rfloor = 1, \lfloor \alpha_1 \rfloor = \lfloor \alpha_2 \rfloor = 0$. Computing each D_i we get

$$D_1 = \hbar \frac{x^{\lfloor \alpha_1 \rfloor}}{x^{\lfloor \alpha_0 \rfloor}} \frac{d}{dx} = \hbar \frac{1}{x} \frac{d}{dx},$$
$$D_2 = \hbar \frac{x^{\lfloor \alpha_2 \rfloor}}{x^{\lfloor \alpha_1 \rfloor}} \frac{d}{dx} = \hbar \frac{d}{dx}.$$

To compute C_1, recall that we have $p_0(x) = 1, p_1(x) = 0$ and $p_2(x) = -x$. This means that $P_2(x, y) = p_0(x)y = y$, and

$$\begin{aligned} C_1 &= \lim_{z_1 \to \infty} \frac{P_2(x, y)}{x^{\lfloor \alpha_{2-1} \rfloor + 1}} \\ &= \lim_{z_1 \to \infty} \frac{y}{x} \\ &= \lim_{z_1 \to \infty} \frac{z_1}{z_1^2} = 0. \end{aligned} \tag{4.4.18}$$

Putting this all back into (4.4.17) yields

$$\begin{aligned} 0 &= \Big[\hbar \frac{1}{x} \frac{d}{dx} (\hbar \frac{d}{dx}) + 0 + \frac{-x}{x} - 0\Big] \psi(z) \\ &= \Big[\frac{1}{x} \hbar^2 \frac{d^2}{dx^2} - 1\Big] \psi(z). \end{aligned}$$

Rearranging terms in the bracket gives the expected differential equation

$$\Big[\hbar^2 \frac{d^2}{dx^2} - x\Big] \psi(z) = 0. \tag{4.4.19}$$

4.4.2 Another example

To highlight the problem with the "naïve" quantization and the need for the correction terms in Definition 4.4.5, we consider the following spectral curve,

$$P(x, y) = 4y^2 - x^2 + 4 = 0. \tag{4.4.20}$$

Again, we would like to apply Theorem 4.4.7 to this curve.

A parametrization for this curve is given by

$$(x, y) = (z + \frac{1}{z}, \frac{1}{2}(z - \frac{1}{z})).$$

With this parametrization, x has two poles at $z = 0, \infty$. The non-zero α_{ij} terms are $\alpha_{0,2} = 4$, $\alpha_{2,0} = -1$, and $\alpha_{0,0} = 4$. The Newton polygon is the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(0, 2)$. The interior of this shape contains no integer points. The point $(0, 0)$ is not a solution of $P(x, y) = 0$, so the smoothness condition does not apply to this curve. Hence, (4.4.20) defines an admissible spectral curve.

Applying Theorem 4.4.7, we get the following quantum curve

$$\Big[D_1 \frac{p_0(x)}{x^{\lfloor \alpha_2 \rfloor}} D_2 + \frac{p_1(x)}{x^{\lfloor \alpha_1 \rfloor}} D_1 + \frac{p_2(x)}{x^{\lfloor \alpha_0 \rfloor}} - \hbar C_1 \frac{x^{\lfloor \alpha_1 \rfloor}}{x^{\lfloor \alpha_0 \rfloor}}\Big] \psi(z; \beta) = 0. \tag{4.4.21}$$

We need to compute the D_i's and C_1 in order to understand this equation. From the Newton polygon, we have that $\lfloor \alpha_0 \rfloor = \lfloor \alpha_1 \rfloor = \lfloor \alpha_2 \rfloor = 0$, and so

$$D_1 = D_2 = \hbar \frac{d}{dx}. \tag{4.4.22}$$

In (4.4.20), we have $p_0(x) = 4$, $p_1(x) = 0$, and $p_2(x) = -x^2 + 4$. This means that $P_2(x, y) = 4y$. Computing C_1 we get

$$\begin{aligned} C_1 &= \lim_{z_1 \to \beta} \frac{P_2(x,y)}{x^{\lfloor \alpha_{2-1} \rfloor + 1}} \\ &= \lim_{z_1 \to \beta} \frac{4y}{x} \\ &= \lim_{z_1 \to \beta} \frac{4(\frac{1}{2}(z_1 - \frac{1}{z_1}))}{z_1 + \frac{1}{z_1}} \\ &= \lim_{z_1 \to \beta} \frac{2(z_1^2 - 1)}{z_1^2 + 1}. \end{aligned} \tag{4.4.23}$$

We have a choice of $\beta = 0, \infty$. Both will yield a different limit, but more importantly, neither choice of β will result in $C_1 = 0$ as we have $C_1(\beta = 0) = -2$ and $C_1(\beta = \infty) = 2$.

Gathering everything together we get,

$$\begin{aligned} 0 &= \left[\hbar \frac{d}{dx}(4(\hbar \frac{d}{dx})) + 0 + -x^2 + 4 - \hbar C_1\right] \psi(z;\beta) \\ &= \left[4\hbar^2 \frac{d^2}{dx^2} - x^2 + 4 - \hbar C_1\right] \psi(z;\beta). \end{aligned}$$

Choosing the pole $\beta = 0$,

$$\left[4\hbar^2 \frac{d^2}{dx^2} - x^2 + 4 + 2\hbar\right] \psi(z;0) = 0. \tag{4.4.24}$$

Choosing the pole $\beta = \infty$,

$$\left[4\hbar^2 \frac{d^2}{dx^2} - x^2 + 4 - 2\hbar\right] \psi(z;\infty) = 0. \tag{4.4.25}$$

Both choices for β present non-trivial quantizations of the spectral curve. The first three terms of both of the above expressions correspond to the "naïve" quantization, but an $\hbar$-correction term was necessary to quantize the curve in such a way that it is compatible with topological recursion.

5 Quantization of Hitchin spectral curves

In this chapter, we will discuss how topological recursion relates to Higgs bundles in the context of quantization of Hitchin spectral curves. Sections 5.1 to 5.4 largely follow Dumitrescu-Mulase [35] and their development of quantum curves for Higgs bundles and the relation to WKB analysis. Many technical details will be omitted in this document, and we refer the reader to [35] for further details. Once we have established a suitable notion for quantization, we will revisit the Airy spectral curve in the context of Higgs bundles. We will finish this chapter by highlighting some new interpretations of the quantum curve which incorporate the spectral correspondence and the $\mathbb{C}^*$-action.

5.1 Quantum curves for Higgs bundles

5.1.1 Rees $\mathcal{D}$-modules

Let X be a Riemann surface, and

$$D = \sum_{j=1}^{n} m_j p_j$$

where $m_j > 0$ be an effective divisor on X. We want to define a quantum curve for the Hitchin spectral curve of a meromorphic Higgs bundle with poles at D.

Definition 5.1.1. *The **compactified cotangent bundle** of X is a ruled surface defined by*

$$\overline{T^*X} \coloneqq \mathbb{P}(K \oplus \mathcal{O}_X) = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty}(K^{-n} \cdot I^0 \oplus K^{-n+1} \cdot I \oplus \cdots \oplus K^0 \cdot I^n)\right), \tag{5.1.1}$$

where I represents $1 \in \mathcal{O}_X$ being considered as a degree 1 element.

The divisor at infinity

$$X_\infty \coloneqq \mathbb{P}(K \oplus \{0\}) \tag{5.1.2}$$

is reduced in the ruled surface and supported on the subset $\mathbb{P}(K \oplus \mathcal{O}_X)\backslash T^*X$. The tautologocal section η on T^*X extends to a meromorphic 1-form on $\overline{T^*X}$ with simple poles along X_∞. The divisor of η in $\overline{T^*X}$ is given by

$$(\eta) = X_0 - X_\infty \tag{5.1.3}$$

where X_0 is the zero section in T^*X.

For a meromorphic Higgs bundle $(\mathcal{E}, \phi)$ with poles at D, the Higgs field is holomorphic on $X\backslash\operatorname{supp}(D)$. We can then define the divisor of zeros of the characteristic polynomial on $T^*(X\backslash\operatorname{supp}(D))$

$$S^0 = \{\det(\eta - \pi^*(\phi|_{X\backslash\operatorname{supp}(D)})) = 0\}. \tag{5.1.4}$$

The spectral curve S of $(\mathcal{E}, \phi)$ is the closure of S^0 with respect to the compactification

$$T^*(X\backslash\operatorname{supp}(D)) \subset \overline{T^*X}. \tag{5.1.5}$$

The sheaf $\mathcal{D}_X$ of differential operators on X is the subalgebra of $\mathbb{C}$-linear endomorphism algebra $End_{\mathbb{C}}(\mathcal{O}_{\mathcal{X}})$ generated by the anti-canonical sheaf K^{-1} and the structure sheaf $\mathcal{O}_X$, where K^{-1} acts on $\mathcal{O}_X$ as holomorphic vector fields, and $\mathcal{O}_X$ acts on itself by multiplication. In local coordinates, an element of $\mathcal{D}_X$ can be written

$$P(x) = \sum_{l=0}^{r} a_l(x) \left(\frac{d}{dx}\right)^{r-l},$$

where $a_l(x) \in \mathcal{O}_X$, for some $r \geq 0$. Fixing an r, we have a filtration of $\mathcal{D}_X$ by the order of differential operators

$$F_r\mathcal{D}_X = \left\{ \sum_{l=0}^{r} a_l(x) \left(\frac{d}{dx}\right)^{r-l} | a_l(x) \in \mathcal{O}_X \right\}.$$

We want to understand the relationship between the geometry of $\overline{T^*X}$ and $\mathcal{D}_X$. Let $gr_m\mathcal{D}_X = F_m\mathcal{D}_X \backslash F_{m-1}\mathcal{D}_X$. We first note that

$$\operatorname{Spec}\left(\bigoplus_{m=0}^{\infty} gr_m\mathcal{D}_X\right) = \operatorname{Spec}\left(\bigoplus_{m=0}^{\infty} K^{-m}\right) = T^*X. \tag{5.1.6}$$

By writing $I = 1 \in H^0(X, \mathcal{D}_X)$, we then have

$$\overline{T^*X} = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty} \left(gr_m\mathcal{D}_X \cdot I^0 \oplus gr_{m-1}\mathcal{D}_X \cdot I \oplus gr_{m-2}\mathcal{D}_X \cdot I^{\otimes 2} \oplus \cdots \oplus gr_0\mathcal{D}_X \cdot I^{\otimes m}\right)\right).$$

The *Rees* ring $\widetilde{\mathcal{D}}_X$ is defined by

$$\widetilde{\mathcal{D}}_X := \bigoplus_{r=0}^{\infty} \hbar^r F_r\mathcal{D}_X \subset \mathbb{C}[[\hbar]] \otimes_{\mathbb{C}} \mathcal{D}_X. \tag{5.1.7}$$

Similarly to (5.1.1), in local coordinates, an element of $\mathcal{D}_X$ can be written as

$$P(x, \hbar) = \sum_{l=0}^{r} a_l(x, \hbar) \left(\hbar\frac{d}{dx}\right)^{r-l}. \tag{5.1.8}$$

Definition 5.1.2. *The Rees construction*

$$\widetilde{\mathcal{M}} = \bigoplus_{r=0}^{\infty} \hbar^r F_r\mathcal{M} \tag{5.1.9}$$

associated with a filtered $\mathcal{D}_X$-module $(F_\bullet, \mathcal{M})$ is a ***Rees $\mathcal{D}$-module*** *if it satisfies the compatibility condition*

$$F_a\mathcal{D}_X \cdot F_b\mathcal{M} \subset F_{a+b}\mathcal{M}. \tag{5.1.10}$$

A left $\mathcal{D}_X$-module E on X is naturally an $\mathcal{O}_X$-module with a $\mathbb{C}$-linear integrable (*i.e.* flat) connection $\nabla : E \to K \otimes_{\mathcal{O}_X} E$. The construction is as follows:

$$\nabla : E \xrightarrow{\alpha} \mathcal{D}_C \otimes_{\mathcal{O}_X} E \xrightarrow{\nabla_{\mathcal{D}} \otimes id} (K \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} E \xrightarrow{\beta \otimes id} K \otimes_{\mathcal{O}_X} E \tag{5.1.11}$$

where

- α is the natural inclusion $v \mapsto 1 \otimes v \in \mathcal{D}_C \otimes_{\mathcal{O}_X} E$,
- $\nabla_{\mathcal{D}} : \mathcal{D}_X \to K \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is the connection defined by the $\mathbb{C}$-linear left-multiplication operation of K^{-1} on $\mathcal{D}_X$ satisfying

  $$\nabla_{\mathcal{D}}(f \cdot P) = f \cdot \nabla_{\mathcal{D}}(P) + df \cdot P \in K \otimes_{\mathcal{O}_X} \mathcal{D}_X \tag{5.1.12}$$

  for $f \in \mathcal{O}_X$ and $P \in \mathcal{D}_X$,
- β is the canonical right $\mathcal{D}_X$-module structure in K defined by the Lie derivative of vector fields.

In a local neighbourhood with coordinate x, we can write (5.1.12) in the following way. Let $P' = [\frac{d}{dx}, P] + P \cdot \frac{d}{dx}$, and define

$$\nabla_{\frac{d}{dx}}(P) := P \cdot \frac{d}{dx} + P'.$$

Equation (5.1.12) takes the form

$$\nabla_{\frac{d}{dx}}(f \cdot P) = f \cdot \nabla_{\frac{d}{dx}}(P) + \frac{df}{dx} \cdot P.$$

The connection ∇ is integrable because $d^2 = 0$. Note that there is no reason for E to be coherent as an $\mathcal{O}_X$-module.

Conversely, if an algebraic vector bundle $\mathcal{E}$ on X of rank r admits a holomorphic connection $\nabla : \mathcal{E} \to K \otimes_{\mathcal{O}_X} \mathcal{E}$, then $\mathcal{E}$ acquires the structure of a $\mathcal{D}_X$-module. This is because ∇ is automatically flat, and the covariant derivative ∇_Y for $Y \in K^{-1}$ satisfies

$$\nabla_Y(fv) = f\nabla_Y(v) + Y(f)v \tag{5.1.13}$$

for $f \in \mathcal{O}_X$ and $v \in \mathcal{E}$. A repeated application of this covariant derivative makes $\mathcal{E}$ a $\mathcal{D}_X$-module. The fact that every $\mathcal{D}_X$-module on a curve is principal implies that for every point $p \in X$, there is an open neighbourhood $p \in U \subset C$ and a linear differential operator P of order r on U, called a generator, such that $\mathcal{E}|_U \cong \mathcal{D}_U \backslash \mathcal{D}_U P$. Thus on an open curve U, a holomorphic connection in a vector bundle of rank r gives rise to a differential operator of order r. The converse is true if $\mathcal{D}_U \backslash \mathcal{D}_U P$ is $\mathcal{O}_U$-coherent.

Definition 5.1.3. *A **formal $\hbar$-connection** on a vector bundle $\mathcal{E} \to X$ is a $\mathbb{C}[[\hbar]]$-linear homomorphism*

$$\nabla^\hbar : \mathbb{C}[[\hbar]] \otimes \mathcal{E} \to \mathbb{C}[[\hbar]] \otimes K \otimes_{\mathcal{O}_X} \mathcal{E}$$

such that

$$\nabla^\hbar(f \cdot v) = f\nabla^\hbar(v) + \hbar df \otimes v, \tag{5.1.14}$$

where $f \in \mathcal{O}_X \otimes \mathbb{C}[[\hbar]]$ and $v \in \mathbb{C}[[\hbar]] \otimes \mathcal{E}$.

A priori, we are not assuming any form of holomorphic dependence of an $\hbar$-connection on $\hbar$. When considering the holomorphic dependence of a quantum curve with respect to the quantization parameter $\hbar$, we need to use a particular $\hbar$-deformation family of vector bundles.

Remark 5.1.4. The *classical limit* of a formal $\hbar$-connection is the evaluation $\hbar = 0$ of $\nabla^\hbar$, which is simply an $\mathcal{O}_X$-module homomorphism

$$\nabla^0 : \mathcal{E} \to K \otimes_{\mathcal{O}_X} \mathcal{E},$$

i.e., a holomorphic Higgs field on $\mathcal{E}$.

Remark 5.1.5. An $\mathcal{O}_X \otimes \mathbb{C}[[\hbar]]$-coherent $\widetilde{\mathcal{D}}_X$-module is equivalent to a vector bundle over X equipped with an $\hbar$-connection.

To motivate the definition of a quantum curve, we begin by studying the semi-classical limit of a differential operator. In analysis, the *semi-classical limit* of a differential operator $P(x, \hbar)$ of (5.1.8) is a function $S_0(x)$ defined by the equation

$$\lim_{\hbar \to 0} \left(e^{-\frac{1}{\hbar} S_0(x)} P(x, \hbar) e^{\frac{1}{\hbar} S_0(x)} \right) = \sum_{l=0}^{r} a_l(x, 0)(S_0'(x))^{r-l}, \tag{5.1.15}$$

where $S_0(x) \in \mathcal{O}_X(U)$. The equation

$$\lim_{\hbar \to 0} \left(e^{-\frac{1}{\hbar} S_0(x)} P(x, \hbar) e^{\frac{1}{\hbar} S_0(x)} \right) = 0 \tag{5.1.16}$$

determines the WKB asymptotic expansion

$$\psi(x, \hbar) = \exp \left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right) \tag{5.1.17}$$

of a solution $\psi(x, \hbar)$ to the differential equation

$$P(x, \hbar)\psi(x, \hbar) = 0$$

on U. Note that the WKB expansion for ψ is not meant to be a convergent series in $\hbar$.

Since $dS_0(x)$ is a local section of T^*X on U, $y = S_0'(x)$ gives a local trivialization of $T^*X|_U$, with $y \in T_x^*X$ a fiber coordinate. Combining Equations (5.1.15) and (5.1.16) in local coordinates, we get the equation

$$\sum_{l=0}^{r} a_l(x, 0) y^{r-l} = 0 \tag{5.1.18}$$

of a curve in $T^*X|_U$.

Definition 5.1.6. *Let $U \subset X$ be a local coordinate neigbourhood with coordinate x such that T^*X is trivial over U with fiber coordinate y. The **semi-classical limit** of a local section*

$$P(x, \hbar) = \sum_{l=0}^{r} a_l(x, \hbar) \left(\hbar \frac{d}{dx} \right)^{r-l}$$

of the Rees ring $\widetilde{\mathcal{D}}_X$ of the sheaf of differential operators $\mathcal{D}_X$ on U is the holomorphic function

$$\sum_{l=0}^{r} a_l(x, 0) y^{r-l} = 0$$

defined on $T^*X|_U$.

Definition 5.1.7. *Suppose a Rees $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ globally defined on X is written as*

$$\widetilde{\mathcal{M}}(U) = \widetilde{\mathcal{D}}_X(U) \backslash \widetilde{\mathcal{D}}_X(U) P_U \tag{5.1.19}$$

on every coordinate neighbourhood $U \subset X$ with a differential operator P_U of the form (5.1.8). Using this local expression for P_U, we construct a meromorphic function

$$p_U(x, y) = \sum_{l=0}^{r} a_l(x, 0) y^{r-l} \tag{5.1.20}$$

on $\overline{T^*X}|_U$, where y is the fiber coordinate of T^*X, which is trivialized on U. Define

$$S_U = \{p_U(x, y) = 0\}. \tag{5.1.21}$$

*If the S_U's glue together to form a spectral curve $S \subset \overline{T^*X}$, then we call S the **semi-classical limit** of the Rees $\widetilde{\mathcal{D}}_X$-module.*

Remark 5.1.8. For the local equation (5.1.20) to be consistent globally on X, the coefficients $a_l(x, 0)$ need to satisfy

$$a_l(x, 0) \in \Gamma(U, K^{\otimes l}). \tag{5.1.22}$$

Definition 5.1.9. *A **quantum curve** associated with the spectral curve $S \in T^*X$ of a holomorphic Higgs bundle on a Riemann surface X is a Rees $\widetilde{\mathcal{D}}_X$-module $\mathcal{E}$ whose semi-classical limit is S.*

A main interest of this thesis is to work with twisted objects, particularly on $\mathbb{P}^1$. We know from Proposition 2.3.1 that there are no non-trivial holomorphic one-forms on $\mathbb{P}^1$, which means there are also no non-trivial holomorphic connections either. We want to extend the work done above to define a quantum curve to include meromorphic connections so that we may work with twisted objects on $\mathbb{P}^1$.

Definition 5.1.10. *A $\mathbb{C}$-linear homomorphism*

$$\nabla : \mathcal{E} \to K(D) \otimes_{\mathcal{O}_X} \mathcal{E}$$

*is said to be a **meromorphic connection** with poles along an effective divisor D if*

$$\nabla(f \cdot v) = f\nabla(v) + df \otimes v$$

for every $f \in \mathcal{O}_C$ and $v \in \mathcal{E}$.

Define

$$\mathcal{O}_X(*D) := \lim_{\rightarrow} \mathcal{O}_X(mD),$$
$$\mathcal{E}(*D) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D).$$

A meromorphic connection ∇ extends to

$$\nabla : \mathcal{E}(*D) \to K(*D) \otimes_{\mathcal{O}_X} \mathcal{E}(*D).$$

Because ∇ is holomorphic on $X \backslash \mathrm{supp}(D)$, it induces a $\mathcal{D}_{X\backslash supp(D)}$-module structure in $\mathcal{E}|_{X\backslash \mathrm{supp}(D)}$. The $\mathcal{D}_X$-module direct image $\widetilde{\mathcal{E}} = j_*(\mathcal{E}|_{X\backslash \mathrm{supp}(D)})$ associated with the open inclusion map $j : X\backslash \mathrm{supp}(D) \to X$ is then naturally isomorphic to

$$\widetilde{\mathcal{E}} = j_*(\mathcal{E}|_{X\backslash \mathrm{supp}(D)}) \cong \mathcal{E}(*D) \tag{5.1.23}$$

as a $\mathcal{D}_X$-module.

Definition 5.1.11. *The above isomorphism is called the **meromorphic extension** of the $\mathcal{D}_{X\backslash supp(D)}$-module $\mathcal{E}|_{X\backslash supp(D)}$.*

Let x be a local coordinate of X around a pole $p_j \in \mathrm{supp}(D)$. If a generator $\widetilde{P}$ of $\widetilde{\mathcal{E}}$ near $x = 0$ has a local expression

$$\widetilde{P}\left(x, \frac{d}{dx}\right) = x^k \sum_{l=0}^{r} b_l(x) \left(x\frac{d}{dx}\right)^{r-l} \tag{5.1.24}$$

around p_j with locally defined holomorphic functions b_l, $b_0(0) \neq 0$, and an integer $k \in \mathbb{Z}$, then $\widetilde{P}$ has a *regular* singular point at p_j. Otherwise, p_j is an irregular singular point of $\widetilde{P}$.

Definition 5.1.12. *Let $(\mathcal{E}, \phi)$ be a meromorphic Higgs bundle on a Riemann surface X of any genus with poles along an effective divisor D, and $S \subset \overline{T^*X}$ its spectral curve. A **quantum curve** associated with S is the meromorphic extension of a Rees $\widetilde{\mathcal{D}}_X$-module E on $X\backslash supp(D)$ such that the complex topology closure of its semi-classical limit $S^0 \subset T^*X|_{X\backslash supp(D)}$ in the compactified cotangent bundle $\overline{T^*X}$ agrees with S.*

5.1.2 $SL(r,\mathbb{C})$-opers

We can construct a quantization of Hitchin spectral curves using a particular choice of isomorphism between a Hitchin section and the moduli of *opers*. The quantum deformation parameter $\hbar$ is a formal parameter in WKB analysis. Since we will be using the PDE recursion (5.3.5) for the analysis of quantum curves, it plays the role of a formal parameter for the asymptotic expansion. This point of view motivates the definition of quantum curves as Rees $\mathcal{D}$-modules in the previous section. The quantum curves appearing in the quantization of Hitchin spectral curves always depend *holomorphically* on $\hbar$. Therefore, a more geometric setup is needed to deal with this holomorphic dependence. The purpose of this subsection is to explain *holomorphic $\hbar$-connections as quantum curves*, and a possible geometric interpretation of $\hbar$. The key concept is *opers*, in the sense of Beilinson-Drinfeld [9], although we will focus solely on $SL(r,\mathbb{C})$-opers for $r \geq 2$.

For most of what follows, let X be a Riemann of genus $g \geq 2$.

Definition 5.1.13. *(cf. [61]) A* ***complex projective coordinate system*** *is a coordinate neighbourhood covering*

$$X = \bigcup_{\alpha} U_\alpha$$

with a local coordinate x_α of U_α such that for every $U_\alpha \cap U_\beta$, we have a Möbius coordinate transformation

$$x_\alpha = \frac{a_{\alpha\beta} x_\beta + b_{\alpha\beta}}{c_{\alpha\beta} x_\beta + d_{\alpha\beta}} \tag{5.1.25}$$

where

$$\begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \in SL(2,\mathbb{C}).$$

Remark. Later on we will be dealing with differential equations on X, so we will assume that each coordinate neighbourhood U_α is simply connected.

Fix a projective coordinate system on X. Taking the exterior derivative of (5.1.25), we have

$$\begin{aligned} dx_\alpha &= \frac{a_{\alpha\beta} d_{\alpha\beta} - b_{\alpha\beta} c_{\alpha\beta}}{(c_{\alpha\beta} x_{\alpha\beta} + d_{\alpha\beta})^2} dx_\beta \\ &= \frac{1}{(c_{\alpha\beta} x_{\alpha\beta} + d_{\alpha\beta})^2} dx_\beta. \end{aligned}$$

In these coordinates, the transition function for K is given by the cocycle

$$\{(c_{\alpha\beta} x_{\alpha\beta} + d_{\alpha\beta})^2\}$$

on $U_\alpha \cap U_\beta$. Also fix a *spin structure* on X, *i.e.* a holomorphic vector bundle $K^{\frac{1}{2}}$ such that $(K^{\frac{1}{2}})^{\otimes 2} \cong K$. Let $\{\xi_{\alpha\beta}\}$ denote the 1-cocycle corresponding to $K^{\frac{1}{2}}$. We have that

$$\xi_{\alpha\beta} = \pm(c_{\alpha\beta} x_\beta + d_{\alpha\beta}). \tag{5.1.26}$$

The choice of $\pm$ here is an element of $H^1(X, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$, indicating that there are 2^{2g} possible choices for the spin structure.

The use of a projective coordinate structure plays a significant role in the proofs of several important theorems in this subsection, specifically the main fact that $\partial_\beta^2 \xi_{\alpha\beta} = 0$. This simple property plays an essential role in our construction of global connections on X.

Remark 5.1.14. For the remainder of this subsection, we will be working with $SL(r,\mathbb{C})$ Higgs bundles and connections. We will use the notation $\mathcal{M}_X^0$ to denote the moduli space of $SL(r,\mathbb{C})$-Higgs bundles on X, and $\mathcal{M}_{deR}^0$ to denote the moduli space of $SL(r,\mathbb{C})$-connections on X, i.e. pairs $(\mathcal{E},\nabla)$ of an irreducible holomorphic $SL(r,\mathbb{C})$-connection $\nabla:\mathcal{E}\to\mathcal{E}\otimes K$ acting on a vector bundle $\mathcal{E}$.

For an $SL(r,\mathbb{C})$-Higgs bundle, the Hitchin base is

$$\mathcal{B}^0=\bigoplus_{l=2}^{r}H^0(X,K^l). \tag{5.1.27}$$

With the choice of spin structure $K^{\frac{1}{2}}$, we have a natural section $\kappa:\mathcal{B}^0\hookrightarrow\mathcal{M}_X^0$ defined by utilizing Konstant's *principal three-dimensional subgroup* (TDS) [71] as follows.

Let

$$\mathbf{q}=(q_2,q_3,\dots,q_r)\in\mathcal{B}$$

be an arbitrary point of the Hitchin base. Define

$$\begin{aligned}
X_- &\coloneqq [\sqrt{s_{i-1}}\delta_{i-1,j}]_{ij}=\begin{bmatrix}0&0&\cdots&0&0\\ \sqrt{s_1}&&&&0\\ 0&\sqrt{s_2}&&&0\\ \vdots&&\ddots&&\vdots\\ 0&0&\cdots&\sqrt{s_{r-1}}&0\end{bmatrix},\\
X_+&\coloneqq X_-^T,\\
H&\coloneqq[X_+,X_-],
\end{aligned} \tag{5.1.28}$$

where $s_i\coloneqq i(r-i)$. Note that the matrix H is diagonal with (i,i)-entry $H_{i,i}=s_i-s_{i-1}=r-2i+1$. The Lie algebra $\langle X_+,X_-,H\rangle\cong sl(2,\mathbb{C})$ is the Lie algebra of the principal TDS in $SL(r,\mathbb{C})$.

Define a Higgs bundle $(\mathcal{E}_0,\phi(\mathbf{q}))$ consisting of a vector bundle

$$\mathcal{E}_0\coloneqq\left(K^{\frac{1}{2}}\right)^{\otimes H}=\bigoplus_{i=1}^{r}\left(K^{\frac{1}{2}}\right)^{\otimes(r-2i+1)} \tag{5.1.29}$$

and a Higgs field

$$\phi(\mathbf{q})\coloneqq X_-+\sum_{l=2}^{r}q_lX_+^{l-1}. \tag{5.1.30}$$

Lemma 5.1.15. *$(\mathcal{E}_0,\phi(\boldsymbol{q}))$ is a stable $SL(r,\mathbb{C})$-Higgs bundle. The Hitchin section is defined by*

$$\kappa:\mathcal{B}\ni\boldsymbol{q}\mapsto(\mathcal{E}_0,\phi(\boldsymbol{q}))\in\mathcal{M}_X, \tag{5.1.31}$$

which gives a biholomorphic map between $\mathcal{B}$ and $\kappa(\mathcal{B})\subset\mathcal{M}_X$.

Remark 5.1.16. This is a generalization of the Hitchin section we discussed in Example 3.2.9. If we choose a rank 2 bundle $\mathcal{E}_0=K^{\frac{1}{2}}\oplus K^{-\frac{1}{2}}$, we have that

$$X_-=\begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

For a point $q \in \mathcal{B}^0 = H^0(X, K^2)$, and the Higgs field (5.1.31) is

$$\phi(q) = \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix},$$

which is exactly the Higgs bundle from Example 3.2.9.

Proof. First, note that $X_- : \mathcal{E}_0 \to \mathcal{E}_0 \otimes K$ is a globally defined $End_0(\mathcal{E}_0)$-valued one-form, since it is a collection of constant maps

$$\sqrt{s_i} : \left(K^{\frac{1}{2}}\right)^{\otimes(r-2i+1)} \xrightarrow{=} \left(K^{\frac{1}{2}}\right)^{\otimes(r-2(i+1)+1)} \otimes K. \tag{5.1.32}$$

Similarly, since X_+^{l-1} is an upper-diagonal matrix with non-zero entries along the $(l-1)$-th upper diagonal, we have

$$q_l : \left(K^{\frac{1}{2}}\right)^{\otimes(r-2i+1)} \to \left(K^{\frac{1}{2}}\right)^{\otimes(r-2i+1+2l)} = \left(K^{\frac{1}{2}}\right)^{\otimes(r-2(i-l+1)+1)} \otimes K.$$

This means that $\phi(\mathbf{q}) : \mathcal{E}_0 \to \mathcal{E}_0 \otimes K$ is globally defined as a Higgs field on $\mathcal{E}_0$. The pair $(\mathcal{E}_0, \phi(\mathbf{q}))$ is stable because no subbundle of $\mathcal{E}_0$ is invariant under $\phi(\mathbf{q})$, unless $\mathbf{q} = 0$. When $\mathbf{q} = 0$, the invariant subbundles all have positive degrees, since $g \geq 2$. □

To define $\hbar$-connections which depend holomorphically on $\hbar$, we need to construct a *one*-parameter holomorphic family of deformations of vector bundles

$$\begin{tikzcd} \mathcal{E}_\hbar \arrow[r] \arrow[d] & E \arrow[d] \\ X \times \{\hbar\} \arrow[r] & X \times H^1(X, K) \end{tikzcd}$$

and a $\mathbb{C}$-linear first-order differential operator

$$\hbar\nabla^\hbar : \mathcal{E}_\hbar \to \mathcal{E}_\hbar \otimes K$$

depending holomorphically on $\hbar \in H^1(X, K)$.

Definition 5.1.17. *A **filtered extension** of the vector bundle $\mathcal{E}_0$ parametrized by $\hbar \in H^1(X, K)$ is a one-parameter family of filtered holomorphic vector bundles $(\mathcal{F}_\hbar^\bullet, \mathcal{E}_\prime\hbar)$ on X with a trivialized determinant $\det(\mathcal{E}_\hbar) \cong \mathcal{O}_X$ satisfying the following conditions:*

- *$\mathcal{F}_\hbar^\bullet$ is a filtration of $\mathcal{E}_\hbar$*

$$0 = \mathcal{F}_\hbar^r \subset \mathcal{F}_\hbar^{r-1} \subset \cdots \subset \mathcal{F}_\hbar^0 = \mathcal{E}_\hbar. \tag{5.1.33}$$

- *The term $\mathcal{F}_\hbar^{r-1}$ is given by*

$$\mathcal{F}_\hbar^{r-1} = \left(K^{\frac{1}{2}}\right)^{\otimes(r-1)}. \tag{5.1.34}$$

- *For every $i = 1, 2, \ldots, r-1$, there is an $\mathcal{O}_X$-module isomorphism*

$$\mathcal{F}_\hbar^i / \mathcal{F}_\hbar^{i+1} \to (\mathcal{F}_\hbar^{i-1} / \mathcal{F}_\hbar^i) \otimes K. \tag{5.1.35}$$

Remark 5.1.18. Because we need to identify a deformation parameter $\hbar$ and extension class, we make the natural identification

$$Ext^1(E, F) = H^1(X, E^* \otimes F) \tag{5.1.36}$$

for every pair of vector bundles E and F. We also identify $\mathrm{Ext}(E,F)$ as the class of extensions

$$0 \to F \to V \to E \to 0$$

of E by a vector bundle V. These identifications are done by a choice of projective coordinate system on X as below.

Proposition 5.1.19. *For every choice of spin structure $K^{\frac{1}{2}}$ and a non-zero element $\hbar \in H^1(X,K)$, there is a unique non-trivial filtered extension $(\mathcal{F}^\bullet_\hbar, \mathcal{E}_{/\hbar})$ of $\mathcal{E}$.*

Proof. We will only consider the $r=2$ case here, and refer the reader to [35] for the full proof. We have that

$$\hbar \in H^1(X,K) = \mathrm{Ext}^1(E,F) \cong \mathbb{C},$$

and so there is a unique extension

$$0 \to K^{\frac{1}{2}} \to \mathcal{E}_\hbar \to K^{-\frac{1}{2}} \to 0 \tag{5.1.37}$$

corresponding to $\hbar$. We also have that

$$K^{\frac{1}{2}} \to \left(\mathcal{E}_\hbar / K^{\frac{1}{2}}\right) \otimes K,$$

which proves that it is a filtered extension. Note that as a rank 2 vector bundle, there is an isomorphism

$$\mathcal{E}_\hbar \cong \begin{cases} \mathcal{E}_1 & \hbar \neq 0 \\ \mathcal{E}_0 & \hbar = 0. \end{cases} \tag{5.1.38}$$

□

Definition 5.1.20. *A point $(\mathcal{E},\nabla) \in \mathcal{M}_{deR}$ is an $SL(r,\mathbb{C})$-**oper** if the following conditions are satisfied:*

Filtration: *there is a filtration $\mathcal{F}^\bullet$ by vector subbundles*

$$0 = \mathcal{F}^r \subset \mathcal{F}^{r-1} \subset \cdots \subset \mathcal{F}^0 = \mathcal{E}. \tag{5.1.39}$$

Griffiths transversality: *the connection respects the filtration:*

$$\nabla|_{\mathcal{F}^i} : \mathcal{F}^i \to \mathcal{F}^{i-1} \otimes K, \qquad i = 1, \ldots r. \tag{5.1.40}$$

Grading condition: *the connection induces $\mathcal{O}_X$-module isomorphisms*

$$\overline{\nabla} : \mathcal{F}^i / \mathcal{F}^{i+1} \xrightarrow{\sim} \left(\mathcal{F}^{i-1}/\mathcal{F}^i\right) \otimes K \qquad i = 1, \ldots, r-1. \tag{5.1.41}$$

The projective coordinate system on X serves two purposes. It will allow us to define differential operators globally on X, and to give a concrete $\hbar \in H^1(X,K)$-dependence in the filtered extensions. For example, the extension $\mathcal{E}_\hbar$ of $(\mathcal{E}_0, \phi(\mathbf{q}))$ (5.1.37) is given by a system of transition functions

$$\mathcal{E}_\hbar \longleftrightarrow \left\{ \begin{bmatrix} \xi_{\alpha\beta} & \hbar\sigma_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{bmatrix} \right\} \tag{5.1.42}$$

on each $U_\alpha \cap U_\beta$. The cocycle condition for the transition functions translates into a condition

$$\sigma_{\alpha\beta} = \xi_{\alpha\beta}\sigma_{\beta\gamma} + \sigma_{\alpha\beta}\xi_{\beta\gamma}^{-1}. \tag{5.1.43}$$

The application of the exterior derivative d to the cocycle condition $\xi_{\alpha\gamma} = \xi_{\alpha\beta}\xi_{\beta\gamma}$ yields

$$\frac{d\xi_{\alpha\gamma}}{dx_\gamma} dx_\gamma = \frac{d\xi_{\alpha\beta}}{dx_\beta} dx_\beta \xi_{\beta\gamma} + \xi_{\alpha\beta} \frac{d\xi_{\beta\gamma}}{dx_\gamma} dx_\gamma.$$

Noticing that

$$\xi_{\alpha\beta}^2 = \frac{dx_\beta}{dx_\alpha}, \tag{5.1.44}$$

we have

$$\sigma_{\alpha\beta} := \frac{d\xi_{\alpha\beta}}{dx_\beta} = \partial_\beta \xi_{\alpha\beta} \tag{5.1.45}$$

solves (5.1.43). Note that

$$\begin{bmatrix} \xi_{\alpha\beta} & \hbar\sigma_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{bmatrix} = \exp\left(\log \xi_{\alpha\beta} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \exp\left(\hbar\partial_\beta \log \xi_{\alpha\beta} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

In the multiplicative sense, the extension class is determined by $\partial_\beta \log \xi_{\alpha\beta}$.

Lemma 5.1.21. *The extension class $\sigma_{\alpha\beta}$ of* (5.1.45) *defined a non-trivial extension* (5.1.37)*.*

The class $\{\sigma_{\alpha\beta}\}$ of (5.1.45) gives a natural isomorphism $H^1(X, K) \cong \mathbb{C}$. We identify the deformation parameter $\hbar \in \mathbb{C}$ with the cohomology class $\{\hbar\sigma_{\alpha\beta}\} \in H^1(X, K) = \mathbb{C}$. Let $\mathbf{q} = (q_2, \ldots, q_r) \in \mathcal{B}^0$. We trivialize the line bundle $K^{\otimes l}$ with respect to the projective coordinate chart $X = \bigcup_\alpha U_\alpha$ and write each q_l as $\{(q_l)_\alpha\}$ that satisfy the transition relation

$$(q_l)_\alpha = (q_l)_\beta \xi_{\alpha\beta}^{2l}. \tag{5.1.46}$$

The transition function for the vector bundle $\mathcal{E}_0$ is given by

$$\xi_{\alpha\beta}^H = \exp(H \log \xi_{\alpha\beta}). \tag{5.1.47}$$

Because $X_- : \mathcal{E}_0 \to \mathcal{E}_0 \otimes K$ is a global Higgs field, its local expression $\{X_- dx_\alpha\}$ with respect to the projective coordinate system satisfies the transition relation

$$X_- dx_\alpha = \exp(H \log \xi_{\alpha\beta}) X_- dx_\beta \exp(-H \log \xi_{\alpha\beta}) \tag{5.1.48}$$

on every $U_\alpha \cap U_\beta$. The same relation holds for the Higgs field $\phi(\mathbf{q})$

$$\phi(\mathbf{q}) dx_\alpha = \exp(H \log \xi_{\alpha\beta}) \phi(\mathbf{q}) dx_\beta \exp(-H \log \xi_{\alpha\beta}). \tag{5.1.49}$$

Theorem 5.1.22 (Construction of $SL(r, \mathbb{C})$-opers)**.** *On each $U_{\alpha\beta}$, define a transition function by*

$$f_{\alpha\beta}^\hbar := \exp(H \log \xi_{\alpha\beta}) \exp(\hbar\partial_\beta \log \xi_{\alpha\beta} X_+). \tag{5.1.50}$$

where $\partial_\beta = \frac{d}{dx_\beta}$, and $\hbar\partial_\beta \log \xi_{\alpha\beta} \in H^1(X, K)$. Then:

- *the collection $\{f_{\alpha\beta}^\hbar\}$ satisfies the cocycle condition*

$$f_{\alpha\beta}^\hbar f_{\beta\gamma}^\hbar = f_{\alpha\gamma}^\hbar, \tag{5.1.51}$$

  and thus defines a holomophic bundle on X. It is, in fact, the filtered extension $\mathcal{E}_\hbar$ from Proposition (5.1.19)*.*

- *the locally defined differential operator*

$$\nabla_\alpha^\hbar(0) := d + \frac{1}{\hbar} X_- dx_\alpha \tag{5.1.52}$$

  for every $\hbar \neq 0$ forms a holomorphic connection on $\mathcal{E}_\hbar$, i.e.

$$\frac{1}{\hbar} X_- dx_\alpha = \frac{1}{\hbar} f_{\alpha\beta}^\hbar X_- dx_\beta \left(f_{\alpha\beta}^\hbar\right)^{-1} - df_{\alpha\beta}^\hbar \cdot \left(f_{\alpha\beta}^\hbar\right)^{-1}. \tag{5.1.53}$$

- *every point* $(\mathcal{E}_0, \phi(\boldsymbol{q})) \in \kappa(\mathcal{B}) \subset \mathcal{M}_X$ *of the Hitchin section gives rise to a one-parameter family of* $SL(r,\mathbb{C})$*-opers* $(\mathcal{E}_\hbar, \nabla^\hbar(\boldsymbol{q})) \in \mathcal{M}_{deR}$*. In other words, the locally defined differential operator*

$$\nabla^\hbar_\alpha(\boldsymbol{q}) \coloneqq d + \frac{1}{\hbar}\phi_\alpha(\boldsymbol{q})dx_\alpha \tag{5.1.54}$$

for every $\hbar \neq 0$ *determines a global holomorphic connection*

$$\nabla^\hbar_\alpha(\boldsymbol{q}) = f^\hbar_{\alpha\beta}\nabla^\hbar_\beta(\boldsymbol{q})\left(f^\hbar_{\alpha\beta}\right)^{-1} \tag{5.1.55}$$

on $\mathcal{E}_\hbar$ *satisfying Definition* (5.1.20)*.*

- *Deligne's* $\hbar$*-connection*

$$\left(\mathcal{E}_\hbar, \hbar\nabla^\hbar(\boldsymbol{q})\right) \tag{5.1.56}$$

interpolates the Higgs bundle and the oper, i.e. at $\hbar = 0$ *the Deligne connection gives the Higgs bundle* $(\mathcal{E}_0, \phi(\boldsymbol{q}))$*, and at* $\hbar = 1$ *it gives the* $SL(r,\mathbb{C})$*-oper* $(\mathcal{E}_1, \nabla^1(\boldsymbol{q}))$*.*

- *After a suitable gauge transformation depending on* $\hbar$*, the* $\hbar \to 0$ *limit of the oper* $\nabla^\hbar(\boldsymbol{q})$ *exists and is equal to* $\nabla^{\hbar=1}(0)$*.*

Remark 5.1.23. In the proof of the theorem (see [35]), the projective coordinate system is essential for the global connection (5.1.55) to make sense.

From the Theorem 5.1.22, we obtain the following theorem.

Theorem 5.1.24 (Biholomorphic quantization of Hitchin spectral curves)**.** *Let* X *be a compact Riemann surface of genus* $g \geq 2$ *with a chosen projective coordinate system subordinating its complex structure. For a fixed spin structure* $K^{\frac{1}{2}}$*, we have a Hitchin section* $\kappa(\mathcal{B}) \in \mathcal{M}_X$ *of* (5.1.31)*. Denote by* $Op \in \mathcal{M}_{deR}$ *the moduli space of* $SL(r,\mathbb{C})$*-opers with the condition that the second term of the filtration is given by* $\mathcal{F}^{r-1} = K^{\frac{r-1}{2}}$*. Then the map*

$$(\mathcal{E}_0, \phi(\boldsymbol{q})) \overset{\gamma}{\mapsto} (\mathcal{E}_\hbar, \nabla^\hbar(\boldsymbol{q})) \in Op \tag{5.1.57}$$

evaluated at $\hbar = 1$ *is a biholomorphic map with respect to the natural complex structures induced from the ambient spaces.*

The biholomorphic quantization (5.1.57) *is also* $\mathbb{C}^*$*-equivariant. The oper corresponding to the* $\mathbb{C}^*$*-action* $\lambda.(\mathcal{E}_0, \phi) = (\mathcal{E}_0, \lambda\phi) \in \kappa(\mathcal{B})$ *is* $d + \frac{\lambda}{\hbar}\phi(\boldsymbol{q})$*.*

A holomorphic connection on a compact Riemann surface X is automatically flat, so it defines a $\mathcal{D}$-module over X. We next show that for a fixed projective coordinate system on X, the $\hbar$-connection $\hbar\nabla^\hbar(\mathbf{q})$ defines a family of Rees $\mathcal{D}$-modules over X parametrized by $\mathcal{B}^0$, such that the semi-classical limit of the family agrees with the family of spectral curves over $\mathcal{B}^0$.

Fix a projective coordinate system on X. To calculate the semi-classical limit, let us trivialize $\mathcal{E}_\hbar$ over each simply connected neigbourhood U_α with coordinate x_α of the projective coordinate system. A flat section Ψ_α of $\mathcal{E}_\hbar$ over U_α

is a solution of

$$\hbar\nabla^{\hbar}_{\alpha}(\mathbf{q})\Psi_\alpha := (\hbar d + \phi_\alpha(\mathbf{q}))\begin{bmatrix}\psi_{r-1}\\ \psi_{r-2}\\ \vdots\\ \psi_1\\ \psi\end{bmatrix}_\alpha = 0, \tag{5.1.58}$$

with an appropriately unknown function ψ. Because $\Psi_\alpha = f^{\hbar}_{\alpha\beta}\Psi_\beta$, the function ψ on U_α satisfies the transition relation

$$(\psi)_\alpha = \xi^{-r+1}_{\alpha\beta}(\psi)_\beta,$$

meaning that ψ is a local section of $K^{-\frac{r-1}{2}}$. There are r linearly independent solutions of (5.1.58) because $q_2, \ldots, q_r$ are represented by holomorphic functions on U_α. The entries of X^l_+ are given by the formula

$$X^l_+ = \left[s^{(l)_{ij}}\right] \tag{5.1.59}$$

where

$$s^{(l)}_{ij} = \delta_{i+1,j}\sqrt{s_i s_{i+1}\cdots s_{i+l-1}}. \tag{5.1.60}$$

This means that (5.1.58) is equivalent to

$$\begin{aligned} 0 &= \sqrt{s_{r-k-1}}\psi_{k+1} + \hbar\psi'_k + \sqrt{s_{r-k}}q_2\psi_{k-1} \\ &\quad + \sqrt{s_{r-k}s_{r-k+1}}q_3\psi_{k-2} + \cdots + \sqrt{s_{r-k}s_{r-k+1}\cdots s_{r-1}}q_{k+1}\psi \\ &= \sqrt{s_{k+1}}\psi_{k+1} + \hbar\psi'_k + \sqrt{s_k}q_2\psi_{k-1} + \sqrt{s_k s_{k-1}}q_3\psi_{k-1} + \cdots + \sqrt{s_k s_{k-1}\cdots s_1}q_{k+1}\psi \end{aligned} \tag{5.1.61}$$

for $k = 0, 1, \ldots, r-1$, where $s_k = s_{r-k}$. Note that $\phi(\mathbf{q})$ given by Equation (5.1.30) takes the form

$$\phi(\mathbf{q}) = \begin{bmatrix} 0 & \sqrt{s_{r-1}}q_2 & \sqrt{s_{r-2}s_{r-1}}q_3 & \cdots & \cdots & \sqrt{s_2 s_3\cdots s_{r-1}}q_{r-1} & \sqrt{s_1 s_2\cdots s_{r-1}}q_r \\ \sqrt{s_{r-1}} & 0 & \sqrt{s_{r-2}}q_2 & \cdots & \cdots & \sqrt{s_2 s_3\cdots s_{r-2}}q_{r-2} & \sqrt{s_1 s_2\cdots s_{r-2}}q_{r-1} \\ & \sqrt{s_{r-2}} & 0 & \ddots & \cdots & \sqrt{s_2 s_3\cdots s_{r-3}}q_{r-3} & \sqrt{s_1 s_2\cdots s_{r-3}}q_{r-2} \\ & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \sqrt{s_3} & 0 & \sqrt{s_2}q_2 & \sqrt{s_1 s_2}q_3 \\ & & & & \sqrt{s_2} & 0 & \sqrt{s_q}q_2 \\ & & & & & \sqrt{s_1} & 0 \end{bmatrix}.$$

By solving (5.1.61) for $k = 0, 1, \ldots, r-2$ recursively, we obtain an expression of ψ_k as a linear combination of

$$\psi = \psi_0, \qquad \hbar\psi' = \hbar\frac{d}{dx_\alpha}\psi, \qquad \ldots \qquad h^k\psi^{(k)} = \hbar^k\frac{d^k}{dx^k_\alpha}\psi,$$

with coefficients in differential polynomials of $q_2, q_3, \ldots, q_k$. For example,

$$\begin{aligned} \psi_1 &= -\frac{1}{\sqrt{s_1}}\hbar\psi', \\ \psi_2 &= \frac{1}{\sqrt{s_1 s_2}}(\hbar^2\psi'' - s_1 q_2\psi), \\ \psi_3 &= \frac{1}{\sqrt{s_1 s_2 s_3}}\left(-\hbar^3\psi''' + \hbar(s_1+s_2)q_2\psi' + (\hbar s_1 q'_2 - s_1 s_2 q_3)\psi\right). \end{aligned}$$

Because ψ_1 is proportional to ψ', inductively, we can show that the linear combination expression of ψ_k by derivatives of ψ' does not contain the $(k-1)$-th order derivative of ψ. For $k = r-1$, Equation (5.1.61) is a differential equation

$$\hbar\psi'_{r-1} + \sqrt{s_1}q_2\psi_{r-2} + \sqrt{s_1 s_2}q_3\psi_{r-3} + \cdots + \sqrt{s_1 s_2 \ldots s_{r-1}}q_r\psi = 0, \tag{5.1.62}$$

which is an order r differential equation for $\psi \in K^{-\frac{r-1}{2}}$. Because we are using a fixed projective coordinate system, the connection $\nabla^\hbar(\mathbf{q})$ takes the same form on each coordinate neighbourhood U_α, so the shape of the differential equation (5.1.62) as an equation for $\psi \in K^{-\frac{r-1}{2}}$ is again, the same on every coordinate neighbourhood. The shape of this equation is what we usually refer to as the **quantum curve** of the spectral curve $\det(\eta + \phi(\mathbf{q})) = 0$.

Theorem 5.1.25 (Quantization of holomorphic data)**.** *Let X be a compact Riemann surface of genus $g \geq 2$ with a chosen projective coordinate system subordinating its complex structure and spin structure $K^{\frac{1}{2}}$. Let $\mathcal{E}(\boldsymbol{q})$ denote the Rees $\mathcal{D}$-module $(\mathcal{E}_\hbar, \hbar\nabla^\hbar(\boldsymbol{q}))$ associated with the oper* (5.1.57)*. Then the semi-classical limit of $\mathcal{E}(\boldsymbol{q})$ is the spectral curve $\sigma^* S \subset T^*X$ of $-\phi(\boldsymbol{q})$ defined by the equation* $\det(\eta + \phi(\boldsymbol{q})) = 0$*, where σ is the involution of $\overline{T^*X}$ defined by fiberwise multiplication by -1.*

The equation (5.1.58) (and thus (5.1.62) as well) is equivalent to a single ordinary differential equation or order r

$$P_\alpha(x_\alpha, \hbar; \mathbf{q})\psi = 0, \tag{5.1.63}$$

where $P_\alpha(x_\alpha, \hbar; \mathbf{q})$ is the generator of the Rees $\mathcal{D}$-module $(\mathcal{E}_\hbar, \hbar\nabla^\hbar(\mathbf{q}))$. The shape of $P_\alpha(x_\alpha, \hbar; \mathbf{q})$ is non-trivial, and is not simply obtained from the naïve quantization $x \mapsto x$ and $y \mapsto \hbar\frac{d}{dx}$. In particular, the $P_\alpha(x_\alpha, \hbar; \mathbf{q})$ can contain various derivatives of the q_i's in the form of $\hbar^j q_i^{(j)}$ which do not appear in $\det(\eta + \phi(\mathbf{q})) = 0$. This coincides with Theorem 4.4.7 and what we saw for the case of quantum curves in classical topological recursion.

For every $\hbar \in H^1(X, K)$, the $\hbar$-connection $(\mathcal{E}_\hbar, \hbar\nabla^\hbar(\mathbf{q}))$ of (5.1.56) defines a global Rees $\mathcal{D}_X$-module structure in $\mathcal{E}_\hbar$. Thus, we have a universal family E_X of Rees $\mathcal{D}_X$-modules on X with a fixed spin strucutre and projective coordinate system:

$$\begin{tikzcd}
E_X \arrow[d] & (\mathcal{E}_\hbar, \hbar\nabla^\hbar(\mathbf{q})) \arrow[l, "\supset"'] \arrow[d] \\
X \times \mathcal{B} \times H^1(X, K) & X \times \{\mathbf{q}\} \times \{\hbar\} \arrow[l, "\supset"]
\end{tikzcd}$$

The universal family $\mathcal{S}_C$ of spectral curves is defined over $X \times \mathcal{B}$.

$$\begin{tikzcd}
\mathbb{P}(K \oplus \mathcal{O}_X) \arrow[dr] & \mathcal{S}_X \arrow[l, "\supset"'] \arrow[d] & (\det(\eta - \phi(\mathbf{q}))_0 \arrow[l, "\supset"'] \\
& X \times \mathcal{B} & X \times \{\mathbf{q}\} \arrow[l, "\supset"]
\end{tikzcd}$$

The semi-classical limit is thus a map of families.

$$\begin{tikzcd}
E_X \arrow[r] \arrow[d] & \mathcal{S}_X \arrow[d] \\
X \times \mathcal{B} \times H^1(X, K) \arrow[r] & X \times \mathcal{B}
\end{tikzcd}$$

The construction of the $SL(r,\mathbb{C})$-oper from a Hitchin section (5.1.54) using the projective coordinate system does not restrict to a holomorphic Higgs field $\phi(\mathbf{q})$. This means there is a generalization of Theorem 5.1.25 to the case of meromorphic Higgs bundles.

Let X be a Riemann surface of any genus with a chosen projective coordinate system subordinating its complex structure, and spin structure $K^{\frac{1}{2}}$. Let D be an effective divisor on C and

$$\mathbf{q} \in \mathcal{B}(D) := \bigoplus_{l=2}^{r} H^0(X, K(D)^{\otimes l}). \tag{5.1.64}$$

We can use equations (5.1.29) and (5.1.30) to define a meromorphic Higgs bundle $(\mathcal{E}_0, \phi(\mathbf{q}))$, and (5.1.54) to define a meromorphic oper $(\mathcal{E}_\hbar, \nabla^\hbar(\mathbf{q}))$. This meromorphic oper defines a meromorphic Rees $\mathcal{D}$-module

$$E(\mathbf{q}) = \left(\mathcal{E}_\hbar, \hbar\nabla^\hbar(\mathbf{q})\right).$$

Theorem 5.1.26 (Quantization of meromorphic data)**.** *The semi-classical limit of $E(\boldsymbol{q})$ is the spectral curve*

$$\{\det(\eta + \phi(\boldsymbol{q})) = 0\} \subset \overline{T^*X}. \tag{5.1.65}$$

Example 5.1.27. *Here we list some examples of characteristic polynomials and differential operators $P_\alpha(x_\alpha, \hbar; \boldsymbol{q})$ for $r = 2, 3, 4$ to highlight the non-trivial nature of the quantization. For the purposes of later sections, the $r = 2$ example will be of most importance to us.*

- $r = 2$

$$\det(y + \phi(\boldsymbol{q})) = y^2 - q_2 \tag{5.1.66}$$

$$P_\alpha(x_\alpha, \hbar; \boldsymbol{q}) = \left(\hbar\frac{d}{dx_\alpha}\right)^2 - q_2 \tag{5.1.67}$$

- $r = 3$

$$\det(y + \phi(\boldsymbol{q})) = y^3 - 4q_2 y + 4q_3$$
$$P_\alpha(x_\alpha, \hbar; \boldsymbol{q}) = \left(\hbar\frac{d}{dx_\alpha}\right)^3 - 4q_2\left(\hbar\frac{d}{dx_\alpha}\right) + 4q_3 - 2\hbar q_2'$$

- $r = 4$

$$\det(y + \phi(\boldsymbol{q})) = y^4 - 10q_2 + 24q_3 y - 35q_4 + 9q_2^2$$
$$P_\alpha(x_\alpha, \hbar; \boldsymbol{q}) = \left(\hbar\frac{d}{dx_\alpha}\right)^4 - 10q_2\left(\hbar\frac{d}{dx_\alpha}\right)^2 + (24q_3 - 10\hbar q_2')\left(\hbar\frac{d}{dx_\alpha}\right) - 36q_4 + 9q_2^2 + 3\hbar^2 q_2'' - 12\hbar q_3'.$$

5.2 Non-singular models of singular spectral curves

As described in the previous section, the quantization applies to all Hitchin spectral curves, regardless of any assumptions on smoothness. Topological recursion, however, requires a smooth spectral curve to be defined. One approach to defining topological recursion for a singular spectral curve, is to consider a non-singular model. In this section, we review the systematic construction of the non-singular models of singular $SL(2,\mathbb{C})$-Hitchin spectral curves from [34,35]. This process involves constructing a canonical blow-up space $Bl(\overline{T^*X})$ in which the non-singular model $\tilde{S}$ is realized

as a smooth divisor.

Let X be a Riemann surface of genus $g \geq 0$ with a fixed projective structure. Fix stable meromorphic $SL(2,\mathbb{C})$ Higgs bundle $(\mathcal{E}_0, \phi(q))$ with poles along D, an effective divisor on X, where

$$q = -\det(\phi(q)) \in H^0(X, K(D)^{\otimes 2}) \tag{5.2.1}$$

is a meromorphic quadratic differential with poles along D. The spectral curve of this Higgs bundle lives in the compactified cotangent bundle,

$$S = \{\eta^2 - \pi^*(q) = 0\} \subset \overline{T^*X}. \tag{5.2.2}$$

Recall that $\mathrm{Pic}(\overline{T^*X})$ is generated by the zero section $X_0 \subset T^*X$ and the fibres of the projection map $\pi : \overline{T^*X} \to X$. Because the spectral curve is a double cover of X, as a divisor it is expressed as

$$S = 2X_0 + \sum_{j=1}^{a} \pi^*(p_j) \in Pic(\overline{T^*X}), \tag{5.2.3}$$

where $\sum_{j=1}^{a} \pi^*(p_j) \in \mathrm{Pic}^a(X)$ is a divisor on X of degree a. As an element of the Néron-Severi group

$$\mathrm{NS}(\overline{T^*X}) = \mathrm{Pic}(\overline{T^*X})/\mathrm{Pic}^0(\overline{T^*X}),$$

it is given by

$$S = 2X_0 + aF \in \mathrm{NS}(\overline{T^*X}),$$

for a typical fiber class F. Because the intersection $F \cdot X_\infty = 1$, we have $a = S \cdot X_\infty$ in $\mathrm{NS}(\overline{T^*X})$. From the genus formula

$$p_a(S) = \frac{1}{2} S \cdot (S + K_{\overline{T^*X}}) + 1$$

and

$$K_{\overline{T^*X}} = -2X_0 + (4g-2)F \in NS(\overline{T^*X}), \tag{5.2.4}$$

the arithmetic genus of S is

$$p_a(S) = 4g - 3 + a, \tag{5.2.5}$$

where a is the number of intersections of S and X_∞.

Definition 5.2.1. *The **discriminant divisor** of the spectral curve is a divisor on X defined by*

$$\Delta := (q)_0 - (q)_\infty, \tag{5.2.6}$$

where

$$(q)_0 = \sum_{i=1}^{m} m_i r_i \tag{5.2.7}$$

$$(q)_\infty = \sum_{j=1}^{n} n_j p_j, \tag{5.2.8}$$

and $m_i, n_j > 0$ and $r_i, p_j \in X$. Because q is a meromorphic section of $K^{\otimes 2}$,

$$\deg \Delta = \sum_{i=1}^{m} m_i - \sum_{j=1}^{n} n_j = 4g - 4. \tag{5.2.9}$$

Theorem 5.2.2. *Define*

$$\delta = |\{i \mid m_i \equiv 1 \bmod 2\}| + |\{j \mid n_j \equiv 1 \bmod 2\}|. \tag{5.2.10}$$

The geometric genus of S is given by

$$p_g(S) = 2g - 1 + \frac{1}{2}\delta. \tag{5.2.11}$$

Note that (5.2.9) *implies* $\delta \equiv 0 \bmod 2$.

We need to construct the normalization $\nu : \tilde{S} \to S$ in a canonical way through a sequence of blow-ups of the ambient space $\overline{T^*X}$. This is done because we need to construct differential forms on $\tilde{S}$ that reflect the geometry of $S \hookrightarrow \overline{T^*X}$.

Definition 5.2.3. *The blow-up space $Bl(\overline{T^*X})$ is defined by blowing up $\overline{T^*X}$ in the following way:*

- *At each r_i of* (5.2.7)*, blow up $r_i \in S \cap X_0 \subset \overline{T^*X}$ a total of $\lfloor \frac{m_i}{2} \rfloor$ times.*
- *At each p_j of* (5.2.8)*, blow up at the intersection $S \cap \pi^{-1}(p_j) \subset X_\infty$ a total of $\lfloor \frac{n_j}{2} \rfloor$ times*

$$\begin{tikzcd}
\tilde{S} \arrow[rr, "\tilde{i}"] \arrow[dr, "\nu"] \arrow[dd, "\tilde{\pi}"'] & & Bl(\overline{T^*X}) \arrow[dr, "\nu"] & \\
& S \arrow[rr, "i"] \arrow[dl, "\pi"] & & \overline{T^*X} \\
X \arrow[urrr, "\pi"'] & & &
\end{tikzcd}$$

Theorem 5.2.4. *In the blow-up space $Bl(\overline{T^*X})$, we have the following:*

- *The proper transform $\tilde{S}$ of the spectral curve S by the birational morphism $\nu : Bl(\overline{T^*X}) \to \overline{T^*X}$ is a smooth curve with a holomorphic map $\tilde{\pi} = \pi \circ \nu : \tilde{S} \to X$.*
- *The Galois action $\sigma : S \to S$ lifts to an involution of $\tilde{S}$, and the morphism $\tilde{\pi} : \tilde{S} \to X$ is a Galois covering with Galois group $Gal(\tilde{S}/X) = \langle \tilde{\sigma} \rangle \cong \mathbb{Z}/2\mathbb{Z}$.*

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\nu"] \arrow[d, "\tilde{\sigma}"'] & S \arrow[d, "\sigma"] \arrow[dr, "\pi"] & \\
& & X \\
\tilde{S} \arrow[r, "\nu"] & S \arrow[ur, "\pi"'] &
\end{tikzcd}$$

5.3 Topological recursion and WKB analysis for Hitchin spectral curves

So far, we have constructed a quantization of a Hitchin spectral curve to a quantum curve through the construction of an $\hbar$-family of opers. As we saw in Section 4.4, ordinary topological recursion provides a method to produce a quantum curve, and a WKB expansion of the solution. We will see later in this section that these two ideas come together in the following way: topological recursion of the non-singular Hitchin spectral curve provides WKB analysis

of the quantum curve constructed through $\hbar$-families of opers for meromorphic $SL(2,\mathbb{C})$) Higgs bundles.

The topological recursion developed in *Chapter 4* is a local formulation. If we wish to use a Hitchin spectral curve as a component for the recursion, we start to pass into a more global picture. The tautological section η is a globally defined object, and the ramification divisor R is no longer just a chosen set of points, it is the eigenvalues of the Higgs field. We will only be concerned with topological recursion for degree 2 covers (such as those that arise from $SL(2,\mathbb{C})$-Higgs bundles. The definition below is suitable for higher degree covers that have only simple ramifications.

Let $\tilde{\pi} : \tilde{S} \to X$ be a degree 2 non-singular cover of a Riemann surface X (not necessarily arising from a Higgs bundle). Denote by R, the ramification divisor of π. The cover is a Galois covering with Galois group $\mathbb{Z}/2\mathbb{Z} = \langle\widetilde{\sigma}\rangle$, whose fixed point divisor is R. Choose a spin structure on S such that

$$\dim H^0(\tilde{S}, K_{\tilde{S}}^{\frac{1}{2}}) = 1, \tag{5.3.1}$$

and a symplectic basis $\langle A_1, ..., A_{\tilde{g}}, B_1, ..., B_{\tilde{g}}\rangle$ for $H_1(\tilde{S},\mathbb{Z})$.

Definition 5.3.1. *The* ***Eynard-Orantin differentials*** $W_{g,n}$ *are meromorphic sections of the n-th exterior tensor product $K^{\boxtimes n}$ defined as follows:*

- $W_{0,1}$ *is a meromorphic* 1*-form on $\tilde{S}$ prescribed according to the geometric setting.*
- $W_{0,2}$ *is given by*

$$W_{0,2}(z_1, z_2) = B(z_1, z_2). \tag{5.3.2}$$

For all $g, n \in \mathbb{N}$ and $2g - 2 + n \geq 0$, define $W_{g,n}$ recursively by

$$W_{g,n+1}(z_0, \boldsymbol{z}) = \sum_{p\in R} \mathrm{Res}_{z=p} \frac{\omega^{z-\widetilde{\sigma}(z)}(z_1)}{\Omega(z)} \left[W_{g-1,n+2}(z, \sigma_p(z), \boldsymbol{z}) + \sum_{\substack{g_1+g_2=g \\ I\cup J=\boldsymbol{z}}}' W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \right] \tag{5.3.3}$$

where ω is the normalized Cauchy kernel (Definition 2.6.3), $\Omega = W_{0,1} - \sigma^* W_{0,1}$, and the prime signifies summation excluding the cases (g_1, I) or $(g_2, J) = (0, 0)$.

Remark 5.3.2. As one might expect, this definition is remarkably similar to Definition 4.2.4. The major difference is that the components going into the definition, such as the spectral curve and the recursion kernel, have been defined in a coordinate-independent global manner. Nevertheless, it is important to remark that the recursion is inherently a local procedure, as it is built around the data of residues at the ramification points of the spectral curve. On local coordinate charts, where we can express S as the zero locus of a polynomial, this definition coincides with the topological recursion of *Chatper 4*.

Remark 5.3.3. When we have a non-singular Hitchin spectral curve S, we let $\tilde{S} = S$, $\tilde{\sigma} = \sigma$, the involution of $\overline{T^*X}$ defined by fiberwise multiplication by -1, and $W_{0,1} = \eta$.

There is another recursion related to the recursion for the $W_{g,n}$. This recursion on *free energies* is a recursion of differential equations rather than of multi-forms. In applications related to enumerative geometry, these free energies can be viewed as generating functions for numerous invariants, depending on choices of the spectral curve and $W_{0,1}$. We will not delve into this interpretation, but rather define the free energies and the related recursion as it plays an important role in Theorem 5.3.7.

Definition 5.3.4. *The **free energy** of type (g,n) is a function $F_{g,n}(z_1,\dots,z_n)$ defined on the universal covering $\mathcal{U}^n$ of $\widetilde{S}^n$ such that*

$$d_1\cdots d_n F_{g,n} = W_{g,n}. \tag{5.3.4}$$

Definition 5.3.5. *The **PDE topological recursion** is the following partial differential equation for all (g,n) subject to $2g-2+n\geq 2$:*

$$\begin{aligned} d_1F_{g,n}(\boldsymbol{z}) = &\sum_{j=2}^{n}\left[\frac{\omega^{z_j-\widetilde{\sigma}(z_j)}(z_1)}{\Omega(z_1)}d_1F_{g,n-1}(\boldsymbol{z}_{[\hat{j}]}) - \frac{\omega^{z_j-\widetilde{\sigma}(z_j)}(z_1)}{\Omega(z_j)}\cdot d_jF_{g,n-1}(\boldsymbol{z}_{[\hat{1}]})\right] \\ &+\frac{1}{\Omega(z_1)}d_{u_1}d_{u_2}\left[F_{g-1,n+1}(u_1,u_2,\boldsymbol{z}_{[\hat{1}]}) + {\sum_{\substack{g_1+g_2=g\\ I\cup J=\boldsymbol{z}_{[\hat{1}]}}}}' F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right]\Bigg|_{\substack{u_1=z_1\\ u_2=z_1}}, \end{aligned} \tag{5.3.5}$$

where the prime notation is the same as above, and $\boldsymbol{z}_{[\hat{j}]}$ denotes dropping the j-th component of the vector $\boldsymbol{z}$.

Remark 5.3.6. The PDE recursion is in fact a coordinate-free equation given in terms of exterior derivatives and contractions. The use of the $\mathbf{z}$ is to keep track of which factor of $\mathcal{U}^n$ the operations are taking place.

Let $(\mathcal{E}_0,\phi(\mathbf{q}))$ be a meromorphic $SL(2,\mathbb{C})$-Higgs bundle, with meromorphic quadratic differential $q\in H^0(X,K(D)^{\otimes 2})$ having poles along an effective divisor D on a curve X with arbitrary genus. Fix a spin structure $K^{\frac{1}{2}}$ and projective coordinate system on X. This gives rise to a Rees $\mathcal{D}_X$-module $E(q)=(\mathcal{E}_\hbar,\hbar\nabla^\hbar(q))$ which is generated by the single differential operator

$$P_\alpha(x_\alpha,\hbar) = \left(\hbar\frac{d}{dx_\alpha}\right)^2 - q_\alpha \tag{5.3.6}$$

on each projective coordinate neighbourhood U_α. By Theorem (5.1.26), E is the quantization of the (possibly singular) spectral curve

$$S=\{\eta^2-q=0\}\subset\overline{T^*X}. \tag{5.3.7}$$

Theorem 5.3.7 (WKB analysis for $SL(2,\mathbb{C})$-quantum curves)**.** *The PDE topological recursion* (5.3.5) *with an appropriate choice of initial data provides an all-order WKB analysis for the generator* (5.3.6) *of the Rees $\mathcal{D}_X$-module $E(q)$ on a small neighbourhood in X of each zero or pole of q of odd order, i.e. we can use the PDE topological recursion to construct a solution to*

$$P_\alpha(x_\alpha,\hbar)\psi_\alpha(x_\alpha,\hbar) = \left[\left(\hbar\frac{d}{dx_\alpha}\right)^2 - q_\alpha\right]\psi_\alpha(x_\alpha,\hbar) = 0 \tag{5.3.8}$$

of the form

$$\psi_\alpha(x_\alpha,\hbar) = \exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x_\alpha)\right). \tag{5.3.9}$$

5.4 Revisiting the Airy spectral curve

We would like to apply the theory that was developed in the previous sections to the Airy spectral curve. This will allow us to revisit an example that we treated in classical topological recursion in this new framework, and give us a launching point for the next section.

Consider now the meromorphic Higgs bundle $(\mathcal{E},\phi)$ on $\mathbb{P}^1$ given by

$$\mathcal{E} = K^{\frac{1}{2}}\oplus K^{-\frac{1}{2}} = \mathcal{O}(-1)\oplus\mathcal{O}(1) \tag{5.4.1}$$

with Higgs field $\phi : \mathcal{E} \to \mathcal{E} \otimes K(m)$

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \tag{5.4.2}$$

where $m \geq 0$. The components of ϕ are given by

$$\begin{aligned}
&\phi_{11} : \mathcal{O}(1) \to \mathcal{O}(1) \otimes K(m) \ \in H^0(\mathbb{P}^1, K(m)), \\
&\phi_{12} : \mathcal{O}(-1) \to \mathcal{O}(1) \otimes K(m) \ \in H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes K(m)), \\
&\phi_{21} : \mathcal{O}(1) \to \mathcal{O}(-1) \otimes K(m) \ \in H^0(\mathbb{P}^1, \mathcal{O}(-2) \otimes K(m)), \\
&\phi_{22} : \mathcal{O}(-1) \to \mathcal{O}(-1) \otimes K(m) \ \in H^0(\mathbb{P}^1, K(m)).
\end{aligned}$$

For now, let us consider the case $m = 4$. Recall that on $\mathbb{P}^1$, $K = \mathcal{O}(-2)$. With this choice of m, we have

$$K(4) = \mathcal{O}(-2) \otimes \mathcal{O}(4) = \mathcal{O}(2) = K^*.$$

This means we are working with a co-Higgs bundle on $\mathbb{P}^1$. A natural place to begin is the Hitchin section,

$$\phi = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} : \mathcal{E} \to \mathcal{E} \otimes K(4). \tag{5.4.3}$$

The map 1 is a section in

$$H^0(\mathbb{P}^1, \mathcal{O}(-2) \otimes K(4)) = H^0(\mathbb{P}^1, \mathcal{O}(-2) \otimes \mathcal{O}(2)) = H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C},$$

and α is a section in

$$H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes K(4)) = H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes \mathcal{O}(2)) = H^0(\mathbb{P}^1, \mathcal{O}(4)).$$

In our previous treatment of rank-2 Hitchin sections (Example 3.2.9), α was a section of K^2, a quadratic differential, however, in our current example α is a section of $\mathcal{O}(4) = K^{-2}$, a quadratic vector field.

We may give some geometric meaning to $\hbar$ by viewing it as a deformation parameter for extensions of holomorphic line bundles. This means that we consider

$$\hbar \in Ext^1(\mathcal{O}(1), \mathcal{O}(-1)) \cong H^1(\mathbb{P}^1, (\mathcal{O}(-1))^* \otimes \mathcal{O}(1)) \cong \mathbb{C}, \tag{5.4.4}$$

where the last equality follows from an application of Serre duality. In this way, we can view $\hbar$ both as a complex number, which we expect from usual quantum theory, as well as a local differential. The fact that $Ext^1(\mathcal{O}(1), \mathcal{O}(-1))$ is 1-dimensional is precisely why we study this example over other choices of parabolic Higgs bundles, as there is a unique parameter $\hbar$ rather than a number of independent parameters. The choice of a specific value of $\hbar$ defines an extension

$$0 \to \mathcal{O}(-1) \to \mathcal{E}_\hbar \to \mathcal{O}(1) \to 0, \tag{5.4.5}$$

where

$$\mathcal{E}_\hbar \cong \begin{cases} \mathcal{O}(-1) \oplus \mathcal{O}(1) & \hbar = 0 \\ \mathcal{O} \oplus \mathcal{O} & \hbar \neq 0. \end{cases} \tag{5.4.6}$$

Note that $\mathcal{E}_\hbar \cong \mathcal{E}_{\hbar'}$, whenever $\hbar, \hbar' \neq 0$, and so choices of $\hbar$ only affect the bundle projectively. This lets us identify $\hbar \neq 0$ with $\hbar = 1$, if and when it is convenient to us. When $\hbar \neq 0$, the bundle type changes to a globally, holomorphically trivial one, and now have the Higgs bundle $\mathcal{E}_\hbar = \mathcal{O} \oplus \mathcal{O}$ with Higgs field

$$\phi^\hbar = \begin{bmatrix} 0 & \alpha_\hbar \\ 1_\hbar & 0 \end{bmatrix}, \tag{5.4.7}$$

the components of which both live in $H^0(\mathbb{P}^1, \mathcal{O}(2))$. This new Higgs field $\phi^\hbar$ can be written in terms of the Higgs field on $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ and the data of the extension. The extension can be broken down as

$$\begin{tikzcd}
 & \mathcal{O} \arrow[dr, "\kappa_\hbar"] & \\
\mathcal{O}(-1) \arrow[ur] \arrow[rr, dashed, leftrightarrow, "1"] \arrow[dr, "\kappa_\hbar"'] \arrow[rr, dashed, "\alpha"'] & \oplus & \mathcal{O}(1) \\
 & \mathcal{O} \arrow[ur] &
\end{tikzcd}$$

where the dashed line indicates the action of the Higgs field ϕ. The map $\kappa_\hbar = \hbar z - a$, for some $a \in \mathbb{C}^*$, is a section in $H^0(\mathbb{P}^1, \mathcal{O}(1))$. This can be thought of as a meromorphic decomposition of $\hbar = 1$, namely $1 = \kappa_\hbar \kappa_\hbar^{-1}$. To make sense of the entries of $\phi^\hbar$ in terms of $\kappa_\hbar$, α, and 1, we must have that

$$\begin{aligned} 1_\hbar = \kappa_\hbar 1 \kappa_\hbar = \kappa_\hbar^2 \quad &\in H^0(\mathbb{P}^1, \mathcal{O}(1) \otimes \mathcal{O} \otimes \mathcal{O}(1)) = H^0(\mathbb{P}^1, \mathcal{O}(2)), \\ \alpha_\hbar = \kappa_\hbar^{-1} \alpha \kappa_\hbar^{-1} = \kappa_\hbar^{-2} \alpha \quad &\in H^0(\mathbb{P}^1, \mathcal{O}(-1) \otimes \mathcal{O}(4) \otimes \mathcal{O}(-1)) = H^0(\mathbb{P}^1, \mathcal{O}(2)). \end{aligned}$$

The new Higgs field is then a transformation of the old one given by

$$\phi^\hbar = \begin{bmatrix} 0 & \kappa_\hbar^{-2} \alpha \\ \kappa_\hbar^2 & 0 \end{bmatrix}. \tag{5.4.8}$$

The quantization (5.1.56) of the Higgs bundle $(\mathcal{E}, \phi)$ is the $\hbar$-Deligne connection $\hbar \nabla^\hbar$ on $\mathcal{E}_\hbar$, where

$$\nabla^\hbar = d + \frac{1}{\hbar} \phi : \mathcal{E}_\hbar \to \mathcal{E}_\hbar \otimes K(4). \tag{5.4.9}$$

We will demonstrate the connection to topological recursion of Hitchin spectral curves by reconstructing the Airy quantum curve from the Higgs bundle $(\mathcal{E}_0, \phi)$.

The spectral curve of our Higgs field (5.4.3) is

$$S = \{\det(\eta - \pi^* \phi) = \eta^2 - \alpha = 0\}. \tag{5.4.10}$$

The quadratic vector field $\alpha \in H^0(\mathbb{P}^1, \mathcal{O}(4))$ has 4 zeroes. Equivalently, α is determined by a polynomial of degree 4 in an affine chart x on $\mathbb{P}^1$. We can choose α so that it has a simple zero at $x = 0$ and a zero of order 3 at infinity (i.e. in the complement of the chart). In local coordinates, the tautological section η becomes y, α becomes x in the neighbourhood U_0 on $\mathbb{P}^1$, and S is given by the familiar local expression

$$y^2 - x = 0.$$

Notably, because $S \subset \mathrm{Tot}(K(4))$ instead of T^*X, the local expression on the U_1 chart will be different than our previous Airy example.

The quantization of $(\mathcal{E}, \phi(\alpha))$ is the meromorphic oper

$$\nabla^\hbar = d + \frac{1}{\hbar}\phi(\alpha). \tag{5.4.11}$$

On U_0, assume that there exists an analytic function in x, $\psi(x, \hbar)$, that satisfies the differential equation induced by the oper in the following way:

$$\hbar\nabla^\hbar(\alpha)\begin{bmatrix} -\hbar\frac{d\psi}{dx} \\ \psi \end{bmatrix} = 0, \tag{5.4.12}$$

for every $\hbar \neq 0$. It then follows that $\psi(x, \hbar)$ satisfies the Airy-Schrödinger equation

$$\Big(\hbar^2\frac{d^2}{dx^2} - x\Big)\psi(x, \hbar) = 0. \tag{5.4.13}$$

On the coordinate chart U_0, the quantum curve associated with S is

$$P(x, \hbar) := \left(\hbar\frac{d}{dx}\right)^2 - x. \tag{5.4.14}$$

Remark 5.4.1. To add some depth, we observe that we have directly transformed the Higgs field, which by Cayley-Hamilton must satisfy the polynomial equation given by its characteristic equation, into a differential operator modelled on that polynomial with distinguished solutions. From this point of view, there has been a swapping of roles: at first, the "classical" equation was the characteristic polynomial — i.e. the spectral curve — and its matrix-valued solution was the Higgs field ϕ, while the quantum equation is the non-trivial deformation of the Higgs field arising from the $\hbar$-deformation of the holomorphic structure on the bundle and its solution is a wave-function produced by the deformation of the spectral curve induced by topological recursion. In simpler terms, the classical equation becomes the quantum solution, and the classical solution becomes the quantum equation. What is fascinating is the underlying connection between deformations of holomorphic structures and quantum parameters. The Airy example suggests that $\hbar$ serves as a local coordinate on the moduli space of Higgs bundles[1]. The fact that there may be more than one linearly independent $\hbar$ parameter, depending on the size of the extension class of $\mathcal{E}_0$, is interesting to us.

5.5 New interpretation

While a lot of work has gone into studying quantization of Higgs bundles and quantum curves, no treatment of the topic has examined how the spectral correspondence and $\mathbb{C}^*$-action fit into the quantization. In this section, we look to introduce a new perspective on the quantization of Hitchin systems on $\mathbb{P}^1$ which includes these. In this section we will consider only $SL(2, \mathbb{C})$-co-Higgs bundles on $\mathbb{P}^1$. In this setting, a Higgs field has the form

$$\phi = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} : \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}(2), \tag{5.5.1}$$

[1] Or, more appropriately, on the moduli space of Deligne-Hitchin λ-connections, which is the moduli space of non-abelian Hodge theory [55].

where $\alpha \in H^0(\mathrm{End}E \otimes \mathcal{O}(4))$ is a quadratic vector field.

The heart of this interpretation is to view the quantization as a procedure on the spectral curve. This idea aligns with the quantum curve arising from topological recursion. Specifically, the data that goes into the recursion is the data of the spectral curve. The tautological section is an object that lives naturally on the spectral curve. Because we are working with co-Higgs bundle, η can be viewed naturally as a vector field. We want to take advantage of these two points of view of η and view, the tautological section as living two lives: a classical life as a $\pi^* K^*$-valued section, and a quantum life as the operator $\hbar \frac{d}{dx}$.

5.5.1 Spectral Correspondence

As outlined in Section 3.2.1, the spectral correspondence relates a line bundle $\mathcal{Q}$ on a spectral curve S, to a Higgs bundle $(\mathcal{E}, \phi)$ on X. In the classical picture, the tautological section η acts on the spectral line bundle by multiplication on $\mathcal{O}(\mathcal{Q})$,

$$\begin{aligned} \eta|_{S_a} :& \mathcal{Q} \to \mathcal{Q} \otimes \pi^* \mathcal{L} \\ & s \mapsto s \cdot y. \end{aligned}$$

Starting from $(\mathcal{E}, \phi)$, we produce its spectral curve

$$S = \{\eta^2 - \pi^* \alpha = 0\}.$$

As we have seen, this gives rise to a quantum curve

$$P(x, \hbar) = \left(\hbar \frac{d}{dx}\right)^2 - \pi^* \alpha \tag{5.5.2}$$

on each projective coordinate neighbourhood. In the quantum picture, we want to view $\eta = \hbar \frac{d}{dx}$. If we want to preserve the action of η on the spectral line bundle, then the sheaf of sections of the spectral line bundle $\mathcal{O}(\mathcal{Q})$ needs to manifest as an object on which $\eta = \hbar \frac{d}{dx}$ can act. It makes sense then to consider that in the quantum picture, η acts by differentiation on smooth sections of the spectral line bundle, $\mathcal{C}^\infty(\mathcal{Q})$ (or possibly square-integrable sections $L^2(\mathcal{Q})$ if we want to keep in line with usual quantum theories).

We also want to consider how the Higgs field fits into this picture. In the classical picture, the Higgs field is produced from η by pushing it forward under the direct image $\mathcal{E} = \pi_* \mathcal{Q}$. On the quantum side, this would involve pushing forward a differential operator to an object on the X. This produces a spectral network (*cf.* [52]), which is roughly a collection of trajectories on X that capture the behaviour of the spectral curve.

5.5.2 $\mathbb{C}^*$-action

The $\mathbb{C}^*$-action on the Higgs bundle

$$\lambda.(\mathcal{E}, \phi) = (\mathcal{E}, \lambda \cdot \phi)$$

for $\lambda \in \mathbb{C}^*$, lifts to an action on the spectral line bundle. Because the $\mathbb{C}^*$-action acts only on the Higgs field by multiplication by λ, it lifts to $\mathcal{Q}$ as

$$\begin{aligned} \mathbb{C}^* \times \mathcal{Q} &\to \mathcal{Q} \\ (\lambda, v) &\mapsto \lambda v, \end{aligned}$$

in other words, rescaling eigenvectors v of ϕ.

As was remarked in Section 3.2.2, fixed points of the $\mathbb{C}^*$-action have the form (3.2.6). In our setting, a fixed point is a Higgs field with $\alpha = 0$,

$$\phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{5.5.3}$$

The spectral curve is then given by

$$S = \{\eta^2 = 0\}, \tag{5.5.4}$$

meaning that it is ramified over all points. At a ramification point of the spectral curve, there are two copies of the spectral fibre coinciding. The multiplication action of the tautological section maps the first fibre to the second, and the second to zero (see Figure 5.1). This can be seen in the form of the Higgs field, as ϕ maps $(1, 0)$ to $(0, 1)$, and $(0, 1)$ to $(0, 0)$.

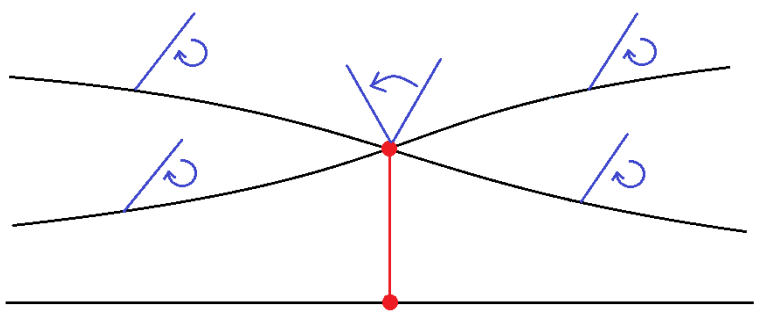

Figure 5.1: Action of the tautological section η on the spectral line bundle $\mathcal{Q}$.

Applying Theorem 5.3.7 to the spectral in the U_0 coordinate system, we get the quantum curve

$$P(x, \hbar) = \left(\hbar \frac{d}{dx}\right)^2. \tag{5.5.5}$$

We again want to capture the essence of the classical picture in the quantum setting. We will consider fixed-points as solutions ψ to the equation

$$\hbar^2 \frac{d^2}{dx^2} \psi = 0.$$

Observe that if we scale ψ by $\lambda \in \mathbb{C}^*$, then it is still a solution because this is a homogenous equation. Moreover, if we consider solutions to (5.5.2) for a generic α, they will not be invariant under this action.

In summary, we are seeking to replace the Higgs field-focused picture of quantization given by

	Classical	Quantum
action	multiplication ($\mathcal{O}_X$-linear)	differentiation ($\mathbb{C}$-linear)
operator	Higgs field ϕ	momentum $\hbar\frac{d}{dx}$
sheaf	$\mathcal{O}(\mathcal{E})$	$L^2(\mathcal{Q})$
spectrum	$\lambda^2 - \alpha = 0$	$\hbar^2\frac{d^2}{dx^2} - \alpha = 0$
$\mathbb{C}^*$-fixed point	$\phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\psi \in L^2(\mathcal{Q})$ with $\hbar^2\frac{d^2}{dx^2}\psi = 0$

with a tautological section-focused viewpoint given by

	Classical	Quantum
action	multiplication ($\mathcal{O}_S$-linear)	differentiation ($\mathbb{C}$-linear)
operator	tautological section η	momentum $\hbar\frac{d}{dx}$
sheaf	$\mathcal{O}(\mathcal{Q})$	$L^2(\mathcal{Q})$
spectrum	$\lambda^2 - \alpha = 0$	$\hbar^2\frac{d^2}{dx^2} - \alpha = 0$
$\mathbb{C}^*$-fixed point	$\alpha = 0$	$\psi \in L^2(\mathcal{Q})$ with $\hbar^2\frac{d^2}{dx^2}\psi = 0$

We also want to understand what happens when we start on the quantum side and let $\hbar \to 0$. We would expect that this returns us to the classic picture, however, that does not appear to be the case. In the limit $\hbar \to 0$, the tautological section on the quantum side also goes to 0. The direct image will be $(\mathcal{E}, 0)$, rather than the expected $(\mathcal{E}, \phi)$. We can give a rough comparison of this tautological quantization to the quantization of Higgs bundle via Deligne $\hbar$ connections (5.1.56) (see Figure 5.2). Observe that the tautological quantization maps all connections to the $(\mathcal{E}, 0)$, the quantization by Deligne $\hbar$-connections maps a connection $\hbar\nabla^\hbar = \hbar d + \phi$ to $(\mathcal{E}, \phi)$. While we have not remarked on non-abelian Hodge theory in this paper, we do note that it is also a method for producing Higgs bundles from flat connections to Higgs bundles.

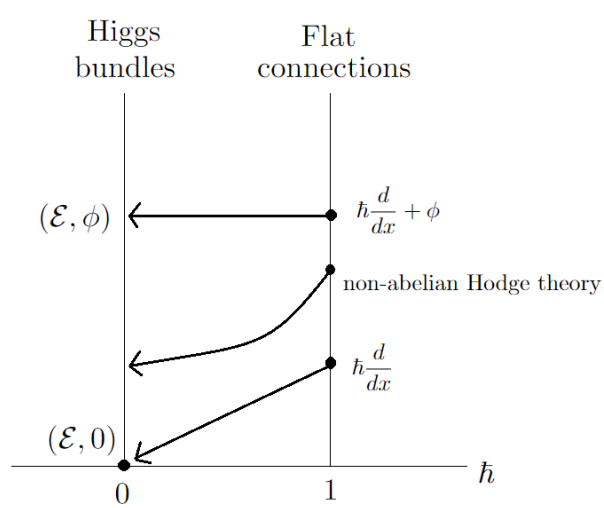

Figure 5.2: A depiction of the three procedures for producing a Higgs bundle from a flat connection.

6 Geometry of Hitchin spectral curves

6.1 Insights on an $\mathcal{L}$-twisted topological recursion

The Dumitrescu-Mulase picture outlined in *Chapter 5* fits into the larger scope of $\mathcal{L}$-twisted Higgs bundles, which are adaptable to all Riemann surfaces — in fact, all complex manifolds — without having to introduce punctures or special divisors. Notably, Dumitrescu-Mulase circumvent the meromorphic data when discussing topological recursion by defining the recursion on a non-singular model for the Hitchin spectral curve. To our knowledge, twisted Higgs bundles have not been explored in the context of topological recursion. Our interest in them is due to the potential they have for formulating a more invariant version of topological recursion and revealing intricacies that might otherwise not be clear in the already explored cases.

To this end, let $(\mathcal{E}, \phi)$ be an $\mathcal{L}$-twisted Higgs bundle on a Riemann surface X. The characteristic polynomial of ϕ gives rise to a spectral curve $S \subset Tot(\mathcal{L})$ with equation

$$\eta^{\otimes r} + \sum_{i=1}^{r} p_i \eta^{\otimes (r-i)} = 0. \tag{6.1.1}$$

We can proceed in a naïve way by reframing topological recursion for Hitchin spectral curves in the $\mathcal{L}$-twisted setting by replacing the appropriate objects with their $\mathcal{L}$-twisted versions. This $\mathcal{L}$-twisted recursion will take place in the diagram

$$\begin{tikzcd}
K_S \arrow[d] & \\
S \arrow[r] \arrow[dr, "\pi|_S"'] & Tot(\mathcal{L}) \arrow[d, "\pi"] \\
& X
\end{tikzcd}$$

with a Galois involution σ of S, and ramification divisor R of π.

As the Bergman kernel B depends only on the Riemann surface S, we choose $W_{0,2} = B$. In the $\mathcal{L}$-twisted setting, the recursion kernel at $p \in R$,

$$K_p = \frac{\omega^{z-\sigma(z)}}{(\eta - \sigma^* \eta)} \tag{6.1.2}$$

is a section of $(\pi^* \mathcal{L}|_S)^* \otimes K_S$. Using the recursion formula (5.3.3) gives rise to *$\mathcal{L}$-twisted Eynard-Orantin differentials*,

$$W_{g,n}^{\mathcal{L}} \in \Gamma(((\pi^* \mathcal{L}|_S)^* \otimes K_S)^{\boxtimes 2g+n-2} \otimes K_S^n). \tag{6.1.3}$$

A notable observation made apparent in the twisted case is the $(\pi^*\mathcal{L}|_S)^* \otimes K_S$ part of the differentials coming from the recursion kernel. From this observation, it is clear that the recursion kernel introduces the geometry of $\mathrm{Tot}(\mathcal{L})$ into the recursion. In the original formulation of topological recursion, the recursion kernel (4.2.3) introduces the geometry of $T^*\mathbb{P}^1$ through the canonical one-form ydx in the denominator, although it does not appear as obviously when looking at where the $W_{g,n}$'s live, as the embedding of S into K_X made $K_S = \pi^* K_X|_S$, and so the two terms cancel out.

With this naïve approach to framing an $\mathcal{L}$-twisted topological recursion, we have already revealed something about the $W_{g,n}$ that was not present in the regular case. A potentially interesting result of this generalization is a relationship between the $\mathcal{L}$-twisted Eynard-Ortantin differentials and the hyperkähler structure on the moduli space of Higgs bundles. While the moduli space of ordinary Higgs bundles in genus $g \geq 2$ possesses a canonical holomorphic symplectic structure (which derives from the hyperkähler structure), moduli spaces of Higgs bundles with twists have Poisson structures that come in a family, no element of which is necessarily canonical. The component of $W^{\mathcal{L}}_{g,n}$ in $\pi^*\mathcal{L}^{-1}|_S \otimes K_S$ defines a Poisson structure on a moduli space of $\pi^*\mathcal{L}^{-1}|_S$-twisted Higgs bundles defined on the spectral curve. This observation begs two important questions: in the ordinary topological recursion, the holomorphic symplectic form may be encoded in the Eynard-Orantin differentials (a partial result to this effect may already be present in the work of [6], who extract the so-called Donagi-Markman cubic from the differentials); and that Higgs bundles defined on the spectral curve itself may be relevant to topological recursion.

These observations serve as a reason to explore the naïve $\mathcal{L}$-twisted topological recursion. To validate the framework that is being developed, we could appeal to one of the three main areas to which topological recursion has been applied: enumerative geometry, quantum curves, or the geometry of the Hitchin moduli space. Any of these areas would prove meaningful and interesting for this framework, but as the main interest of this section lies in the relationship between Higgs bundles and topological recursion, we will choose the latter and study the relationship between $\mathcal{L}$-twisted Hitchin spectral curves and $\mathcal{L}$-twisted topological recursion.

6.2 $\mathcal{L} = K$ Hitchin moduli space

With our goal being to validate a framework for twisted topological recursion, we want to understand how (ordinary) topological recursion relates to the geometry of the (ordinary) Hitchin moduli space. In this section, we briefly go through the main results of [6], which we will seek to replicate in some capacity in the $\mathcal{L}$-twisted setting in subsequent sections.

In the $\mathcal{L} = K$ case, the moduli space of Higgs bundles $\mathcal{M}_X$ admits a complete hyperkähler metric [65]. This metric can be studied by studying a second related hyperkähler metric, called the *semi-flat metric*, defined over the regular locus of the Hitchin fibration $\mathcal{H} : \mathcal{M}_X^{reg} \to \mathcal{B}^{reg}$ (i.e. the locus containing non-singular fibres of $\mathcal{H}$). A theorem of Hitchin [66] says that under a set of mild assumptions, $\mathcal{B}^{reg}$ admits a special Kähler structure. The special Kähler metric on $\mathcal{B}^{reg}$ can be combined with a metric along the fibers to produce the semi-flat metric [24,50,66].

On $\mathcal{B}^{reg}$, we can define a local *conjugate coordinate system* $\{z_1, \ldots, z_{g_S}\}$ and $\{w_1, \ldots, w_{g_S}\}$, where S is the spectral

curve associated to a point in $\mathcal{B}^{reg}$, by

$$z_i = \int_{A_i} \theta, \tag{6.2.1}$$

$$w_j = \int_{B_j} \theta, \tag{6.2.2}$$

where θ is the canonical one-form on S, and $\langle A_1, ..., A_g, B_1, ..., B_g \rangle$ for $H_1(X, \mathbb{Z})$ is a symplectic basis. With respect to the *special coordinate system* $\{z_i\}$, the Kähler form ω can be written as

$$\omega = \frac{i}{2} Im(\tau_{ij}) dz^i \wedge d\bar{z}^j, \tag{6.2.3}$$

where τ_{ij} is the period matrix of the spectral curve. This means that the Kähler metric on $\mathcal{B}^{reg}$, and therefore the semi-flat metric on $\mathcal{M}_X$, can be written in terms of the period matrices τ_{ij} of spectral curves through (6.2.3).

We can apply the topological recursion for Hitchin spectral curves defined in *Chapter 5* to spectral curves associated to $\mathcal{B}^{reg}$. In this setting, the variational formula for the Eynard-Orantin invariants [42] provides a relationship between derivatives of the period matrix τ_{ij} about a point $b \in \mathcal{B}^{reg}$ and the $g = 0$ invariants $W_{0.n}$ of the spectral S_b associated to the point b.

Theorem 6.2.1 (Baraglia-Huang)**.**

$$\partial_{i_1} \partial_{i_2} \dots \partial_{i_{m-2}} \tau_{i_{m-1} i_m} = -\left(\frac{i}{2\pi}\right)^{m-1} \int_{p_{i_1} \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_{0,m}(p_1, ..., p_m) \tag{6.2.4}$$

This formula shows that the $g = 0$ invariants for the spectral curve S_b compute the Taylor series expansion of the period matrix about $b \in \mathcal{B}^{reg}$, and by extension of what was said above, the semi-flat metric on $\mathcal{M}_X^{reg}$. Notably, Theorem 6.2.1 states that the data of single spectral curve is enough to understand the geometry of nearby spectral curves.

In the $\mathcal{L} \neq K$ setting, the moduli space $\mathcal{M}_X^{\mathcal{L}}$ is no longer hyperkähler, so we cannot expect to make claims relating the $g = 0$ invariants to a hyperkähler metric on the moduli, however, we can (and will) seek to find an analogy for Theorem 6.2.1.

There are two concerns that impede our ability to immediately replicate this result. Firstly, in our twisted setting, the dimension of $\mathcal{M}_X^{\mathcal{L}}$ is not twice the dimension of the Hitchin base $\mathcal{B}$. This means that the base and fibres of the moduli space no longer have the same dimension. In fact, the dimension of the base is larger than that of the fibres, so we can no longer view $\mathcal{B}^{reg}$ as the deformation space of spectral curves. Secondly, in the K-case, we have a canonical coordinate system on T^*X given by the natural symplectic structure, which allows for many computations to be simplified, and relates the tautological section to the geometry of the total space. Without imposing a symplectic structure on $\mathrm{Tot}(\mathcal{L})$, and thus limiting our breadth of line bundles, we need to find a sufficiently natural choice of coordinate system for $\mathrm{Tot}(\mathcal{L})$. We seek to address these concerns in the next section by studying more in-depth the $\mathcal{L}$-twisted moduli space.

6.3 Deformation theory of the $\mathcal{L}$-twisted moduli space

With our attention focused on $\mathcal{L}$-twisted Higgs bundle, we want to better understand the moduli space $\mathcal{M}_X^{\mathcal{L}}$. In this section, we will intiate a thorough examination of the deformation theory $\mathcal{M}_X^{\mathcal{L}}$ by studying its hypercohomology

(*cf.* [59, pp. 438-447]).

On a Riemann surface X, an $\mathcal{L}$-twisted Higgs field satisfies

$$\phi \wedge \phi = 0 \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes \wedge^2 \mathcal{L}).$$

There is a natural complex of sheaves associated to a Higgs bundle $(\mathcal{E}, \phi)$ given by

$$\mathrm{End}(\mathcal{E}) \xrightarrow{-\wedge\phi} \mathrm{End}(\mathcal{E}) \otimes \mathcal{L} \xrightarrow{-\wedge\phi} \mathrm{End}(\mathcal{E}) \otimes \wedge^2 \mathcal{L} \xrightarrow{-\wedge\phi} \dots, \tag{6.3.1}$$

where $- \wedge \phi$ acts by the Lie bracket. Because $\phi \wedge \phi = 0$, we have that $- \wedge \phi$ is a differential, meaning we can use it to define cohomologies on $\mathrm{End}(\mathcal{E}) \otimes^i \mathcal{L}$. In this way, we have two cohomology theories, the cohomology of sheaves arising from the Čech complex, and the above theory arising from the Higgs field.

We consider the hypercohomology associated to a stable $\mathcal{L}$-twisted Higgs bundle $(\mathcal{E}, \phi)$ on a Riemann surface X given by the two double complexes

$$D = (\delta, \wedge\phi),$$
$$D' = (\wedge\phi, \delta),$$

where $\wedge\phi$ is the differential coming from the Higgs field, and δ is the Čech differential.

We start by computing the hypercohomology with the complex D. The hypercohomology fits into the short exact sequence

$$D: \ 0 \to E^{1,0} \to \mathbb{H}^1 \to E^{0,1} \to 0, \tag{6.3.2}$$

where

$$E^{1,0} = \frac{\ker H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) \xrightarrow{\wedge\phi} H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L}^{\otimes 2})}{\mathrm{im} H^0(\mathrm{End}\mathcal{E}) \xrightarrow{\wedge\phi} H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L})},$$
$$E^{0,1} = \ker H^1(\mathrm{End}\mathcal{E}) \xrightarrow{\wedge\phi} H^1(\mathrm{End}\mathcal{E} \otimes \mathcal{L}).$$

Here, $\mathbb{H}^1 = T_{(\mathcal{E},\phi)}\mathcal{M}_X^{\mathcal{L}}(r,d)$ is the tangent space to the moduli space of rank r, degree d, $\mathcal{L}$-twisted Higgs bundles. This is a short exact sequence around $\mathbb{H}^1$ because of vanishing due to stability and dimensionality (X is a curve and $rk\mathcal{L} = 1$).

To determine $\mathbb{H}^1$, we first need to better understand $E^{1,0}$ and $E^{0,1}$. We start by looking at $E^{0,1}$. Using Serre Duality, we compute the dimension of $H^1(\mathrm{End}\mathcal{E} \otimes \mathcal{L})$ as

$$h^1(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) = h^0(K \otimes \mathrm{End}\mathcal{E} \otimes \mathcal{L}^*) = 0, \tag{6.3.3}$$

which vanishes because $\deg \mathcal{L} > \deg K$. This means $E^{0,1} = H^1(\mathrm{End}\mathcal{E}) = \mathcal{U}_X(r,d)$, which is the moduli space of stable bundles on X. Its dimension is then

$$\dim E^{0,1} = \dim \mathcal{U}_X(r,d) = r^2(g-1) + 1. \tag{6.3.4}$$

Turning our attention to $E^{1,0}$, observe that

$$E^{1,0} = \frac{H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L})}{\mathrm{im}(\wedge\phi)}.$$

The map $\wedge\phi$ has kernel generated by 1 when acting on $H^0(\mathrm{End}\mathcal{E})$, as stability implies simplicity. Thus we have

$$\dim E^{1,0} = h^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) - h^0(\mathrm{End}\mathcal{E}) + 1.$$

To handle the first term, $h^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L})$, we apply Riemann-Roch and Serre Duality, recalling from (6.3.3) that $h^1(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) = 0$,

$$\begin{aligned} h^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) &= \deg(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) + \mathrm{rk}(\mathrm{End}\mathcal{E} \otimes \mathcal{L})(1-g) - h^1(\mathrm{End}\mathcal{E} \otimes \mathcal{L}) \\ &= r^2 \deg L + r^2(1-g). \end{aligned}$$

For the second term, $h^0(\mathrm{End}\mathcal{E})$, if we assume that $\mathcal{E}$ is stable as a vector bundle (which is generically true), then $h^0(\mathrm{End}\mathcal{E}) = 1$. We compute the $\dim E^{1,0}$ as

$$\begin{aligned} \dim E^{1,0} &= h^0(\mathrm{End}\mathcal{E} \otimes L) - h^0(\mathrm{End}\mathcal{E}) + 1 \\ &= r^2 \deg \mathcal{L} + r^2(1-g) - 1 + 1 \\ &= r^2 \deg \mathcal{L} + r^2(1-g). \end{aligned} \tag{6.3.5}$$

Combining (6.3.4) and (6.3.5), the dimension of $\mathbb{H}^1$ is

$$\dim \mathbb{H}^1 = r^2 \deg \mathcal{L} + r^2(1-g) + r^2(g-1) + 1 = r^2 \deg \mathcal{L} + 1.$$

Because $\mathbb{H}^1 = T_{(\mathcal{E},\phi)}\mathcal{M}_X^{\mathcal{L}}(r,d)$, we conclude that the dimension of $\mathcal{M}_X^{\mathcal{L}}(r,d)$ is

$$\dim \mathcal{M}_X^{\mathcal{L}}(r,d) = r^2 \deg \mathcal{L} + 1, \tag{6.3.6}$$

which agrees with the calculations from [81].

Denote by $\mathbb{L}$, the bundle over $\mathcal{U}_x(r,d)$ whose fibre at $\mathcal{E}$ is $H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L})$. By the fact that $(\mathcal{E},\phi)$ is stable for all $\phi \in H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L})$, and $\mathrm{Aut}(\mathcal{E})$ acts trivially on $H^0(\mathrm{End}\mathcal{E} \otimes \mathcal{L})$, we have an injection $\mathrm{Tot}\,(\mathbb{L}) \hookrightarrow \mathcal{M}_X^{\mathcal{L}}(r,d)$ that is open and dense (as their dimensions equal). In other words, $\mathcal{M}_X^{\mathcal{L}}(r,d)$ is the completion of $\mathbb{L}$ under stability, and

$$\begin{aligned} T_{(\mathcal{E},\phi)}\mathcal{M}_X^{\mathcal{L}}(r,d) &\cong T_{(\mathcal{E},\phi)}\mathbb{L} \\ &\cong \underbrace{H^0(\mathrm{End}\mathcal{E} \otimes L)}_{\mathbb{L}_{\mathcal{E}}} \times \underbrace{H^1(\mathrm{End}E)}_{T_{\mathcal{E}}\mathcal{U}_X(r,d)}, \end{aligned} \tag{6.3.7}$$

whenever $\mathcal{E}$ is a stable bundle. This means that $\mathbb{L}$ is the $\mathcal{L}$-twisted analogue of $T^*\mathcal{U}_X(r,d)$.

Remark 6.3.1. The dimension of $\mathcal{M}_X^{\mathcal{L}}(r,d)$ is actually independent of the stability of $\mathcal{E}$[1]. If we do not assume that $\mathcal{E}$ is stable, then

$$\begin{aligned} \dim \mathbb{H}^1 &= h^0(End\mathcal{E} \otimes \mathcal{L}) - h^0(End\mathcal{E}) + 1 + h^1(End\mathcal{E}) \\ &= r^2 \deg \mathcal{L} + r^2(1-g) + 1 - (h^0(End\mathcal{E}) - h^1(End\mathcal{E})) \\ &= r^2 \deg \mathcal{L} + 1 + r^2(1-g) - r^2(1-g) \\ &= r^2 \deg \mathcal{L} + 1. \end{aligned}$$

[1]The importance of stability in the previous calculations was to reveal the relationship to $\mathbb{L}$.

We now consider the hypercohomology of D'. We have the same type of short exact sequence as (6.3.2), with $E^{1,0}$ and $E^{0,1}$ now given by

$$E^{1,0} = H^1\left(\ker(\mathrm{End}\mathcal{E} \xrightarrow{\wedge\phi} \mathrm{End}\mathcal{E}\otimes\mathcal{L})\right),$$
$$E^{0,1} = H^0\left(\frac{\mathrm{End}\mathcal{E}\otimes\mathcal{L}}{\mathrm{im}(\wedge\phi)}\right).$$

Suppose that $(\mathcal{E},\phi)$ is not only stable but also regular, which means that $ker(\wedge\phi)$ is minimally generated, i.e. $ker(\wedge\phi)$ is a rank r subsheaf of $End\mathcal{E}$ — this is actually a generic property. Consider the compositions $\phi^{\otimes i} : \mathcal{E} \to \mathcal{E}\otimes\mathcal{L}^{\otimes i}$. Note that

$$\phi^i \in Hom(\mathcal{E},\mathcal{E}\otimes\mathcal{L}^i) = Hom(\mathcal{L}^{-i},\mathcal{E}^*\otimes\mathcal{E})$$

and so

$$\phi^i : \mathcal{L}^{-1} \to End(\mathcal{E}).$$

Notice that

$$\phi^i\wedge\phi = [\phi^i,\phi] = 0,$$

so we have $\phi^0,\phi^1,\ldots,\phi^{r-1} \in \ker(\wedge\phi)$, and in fact, they generate the sheaf $\ker(\wedge\phi)$, meaning that

$$\ker(\wedge\phi) = \mathcal{O}\oplus\mathcal{L}^{-1}\oplus\cdots\oplus\mathcal{L}^{-(r-1)}.$$

The sheaf $\frac{\mathrm{End}\mathcal{E}\otimes\mathcal{L}}{\mathrm{im}(\wedge\phi)}$ is the cokernel of $\wedge\phi$. Starting from the exact sequence

$$0\to\ker\wedge\phi \xrightarrow{\phi^i} \mathrm{End}\mathcal{E} \xrightarrow{\wedge\phi} \mathrm{End}\mathcal{E}\otimes\mathcal{L} \to \mathrm{coker}\wedge\phi \to 0,$$

we dualize the sequence to obtain

$$0\to(\mathrm{coker}\wedge\phi)^* \to \mathrm{End}\mathcal{E}\otimes\mathcal{L}^{-1} \xrightarrow{\phi^*\wedge} \mathrm{End}\mathcal{E} \to (\ker\wedge\phi)^* \to 0,$$

and tensor by $\mathcal{L}$ to return to the original sequence

$$0\to(\mathrm{coker}\wedge\phi)^*\otimes\mathcal{L} \to \mathrm{End}\mathcal{E} \to \mathrm{End}\mathcal{E}\otimes\mathcal{L} \to (\ker\wedge\phi)^*\otimes\mathcal{L} \to 0 \tag{6.3.8}$$

and so, we see that

$$\mathrm{coker}\wedge\phi = (\ker\wedge\phi)^* = \mathcal{L}\oplus\mathcal{L}^2\oplus\cdots\oplus\mathcal{L}^r.$$

This provides a different splitting of $\mathbb{H}^1$ as

$$\mathbb{H}^1 = H^1\left(\bigoplus_{i=0}^{r-1}\mathcal{L}^{-i}\right)\times H^0\left(\bigoplus_{i=1}^{r}\mathcal{L}^i\right). \tag{6.3.9}$$

This splitting is induced by the derivative of the Hitchin map,

$$\mathcal{H} : \mathcal{M}_X^{\mathcal{L}}(r,d) \to \mathcal{B},$$

where $\mathcal{B} = H^0\left(\bigoplus_{i=1}^r\mathcal{L}^i\right)$. Recall from Section 3.2.1, that $\mathcal{M}_X^{\mathcal{L}}$ is fibred by Jacobians of spectral curves. In other words, (6.3.9) tells us that

$$\begin{aligned} T_{(\mathcal{E},\phi)}\mathcal{M}_X^{\mathcal{L}}(r,d) &\cong T_{\mathcal{Q}}Jac(S)\times T_{\mathcal{H}(\mathcal{E},\phi)}\mathcal{B} \\ &= H^1\left(\bigoplus_{i=0}^{r-1}\mathcal{L}^{-i}\right)\times\mathcal{B}, \end{aligned}$$

where S is the spectral curve of $(\mathcal{E}, \phi)$, and $\mathcal{Q}$ is the spectral line bundle. In particular, the tangent space to a Hitchin fiber over the regular locus (call this $\mathcal{B}^{reg}$) is $\cong H^1\left(\bigoplus_{i=0}^{r-1} \mathcal{L}^{-i}\right)$, and the genus of the spectral curve is given by

$$g_S = \dim Jac(S) = \sum_{i=0}^{r-1} h^1(X, \mathcal{L}^i).$$

Serre duality on X pairs $T_M Jac(S) \cong H^1\left(\bigoplus_{i=0}^{r-1} \mathcal{L}^{-i}\right)$ with $H^0\left(\bigoplus_{i=0}^{r-1} \mathcal{L}^i \otimes K\right)$. We can map this space to a subvariety $\widetilde{\mathcal{B}} \subset \mathcal{B}$ by a choice of section $s \in H^0(X, K^* \otimes \mathcal{L})\backslash\{0\}$, which acts by multiplication. Recall that in the $\mathcal{L} = K$ setting, the Hitchin base has the same dimension as fibres. The vector space $H^0\left(\bigoplus_{i=0}^{r-1} \mathcal{L}^i \otimes K\right)$ is fulfilling this role in the $\mathcal{L}$-twisted picture.

Definition 6.3.2. *We call $\mathcal{B}_{eff} := H^0\left(\bigoplus_{i=0}^{r-1} \mathcal{L}^i \otimes K\right)$ the **effective Hitchin base**.*

By duality,

$$\dim \mathcal{B}_{eff} = \dim Jac(S) = g_s,$$

which means $h^{-1}(\mathcal{B}_{eff})$ is a moduli space of $\mathcal{L}$-twisted Higgs bundles in which the fibre and base are equidimensional.

In particular, when $\mathcal{L} = K$, $s \in H^0(K^* \otimes K)\backslash\{0\}$ is a nonzero multiple of the identity, and $\mathcal{B}_{eff} = \mathcal{B}$.

Example 6.3.3. *Consider the situation where $X = \mathbb{P}^1$, $r = 2$, $d = -1$, $\mathcal{L} = \mathcal{O}(2)$. The moduli space has dimension*

$$\dim \mathcal{M}_{\mathbb{P}^1}^{\mathcal{O}(2)}(2, -1) = 2^2 deg(\mathcal{O}(2)) + 1 = 2^2(2) + 1 = 9.$$

The Hitchin base is

$$\mathcal{B} = H^0(\mathcal{O}(2)) \oplus H^0(\mathcal{O}(4)) \cong \mathbb{C}^8,$$

and so the moduli space is fibred by 1-dimensional (elliptic) fibres.
The effective Hitchin base, on the other hand, is

$$\mathcal{B}_{eff} = H^0(\mathcal{O}(-2)) \oplus H^0(\mathcal{O}(2) \otimes \mathcal{O}(-2)) \cong \mathbb{C},$$

which has the same dimension as the fibre, and which is also that of the moduli space of elliptic curves.

To see $\mathcal{B}_{eff}$ in $\mathcal{B}$, we need to choose an $s \in H^0(\mathcal{O}(-2)^* \otimes \mathcal{O}(2))\backslash 0 = H^0(\mathcal{O}(4))\backslash 0$. Given a Higgs bundle (E, ϕ), there is a canonical choice; in this case: $s = \det \phi$. With this choice of s, $\mathcal{B}_{eff}$ is embedded as the line

$$\widetilde{\mathcal{B}} = \{(0, c \det \phi) \mid c \in \mathbb{C}\} \subset \mathcal{B}.$$

The effective Hitchin base $\mathcal{B}_{eff}$ serves as a moduli space of deformations spectral curves and can be related to the period data as follows.

Let $\pi : Tot(\mathcal{L}) \to X$ be the natural projection. The derivative of $\pi|_S$ is

$$d\pi : TS \to \pi^* TX \in Hom(\pi^* K, K_S),$$

where $K = K_X$. Note that $\pi^* s : \pi^* K \to \pi^* \mathcal{L}$, so if η is the tautological section in $H^0(S, \pi^* \mathcal{L})$, then we have that $(\pi^* s)^{-1}\eta$ is a (meromorphic) section of $\pi^* K$. We can define a meromorphic 1-form on S by

$$\Theta = d\pi((\pi^* s)^{-1}\eta) \in H^0(S, K_S). \tag{6.3.10}$$

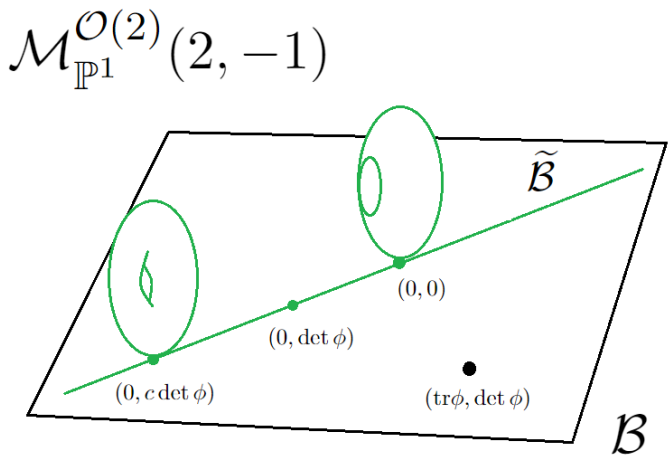

Figure 6.1: $\mathcal{B}_{eff}$ mapped into $\mathcal{B}$ by $s = \det\phi$.

Definition 6.3.4. *The meromorphic* 1*-form* Θ *is called the* ***twisted canonical*** 1*-form.*

If $\{a_1, b_1, \ldots, a_{g_S}, b_{g_S}\}$ is a basis for $H_1(S,\mathbb{Z})$, then $\{Re\int_{a_i(b_i)}\Theta, Im\int_{a_i(b_i)}\Theta\}$ form local (singular) coordinates on B_{eff}. We may choose a basis $\omega_1, \ldots, \omega_{g_s}$ of $H^0(S, K_S)$ so that $\int_{a_i}\omega_j = \delta_{ij}$, and $\tau = \left[\int_{b_i}\omega_j\right]$ is the $g_S \times g_S$ period matrix. Define $\lambda_i = Re\int_{a_i}\Theta + iIm\int_{a_i}\Theta$.

Definition 6.3.5. *We refer to the collection of derivatives* $\frac{\partial\tau_{jk}}{\partial\lambda_i}$ *as the* ***twisted Donagi-Markman cubic****.*

The spectral curve S itself is a divisor in $Tot(\mathcal{L})$, and the normal bundle of a divisor is given by the line bundle associated to the divisor, which in this case is $\pi^*\mathcal{L}^r$ (since S is the zero locus of a degree-r polynomial in $\eta \in H^0(S, \pi^*\mathcal{L})$). By adjunction,

$$K_S = K_{Tot(\mathcal{L})|_S} \otimes N_S = K_{Tot(\mathcal{L})}|_S \otimes \pi^*\mathcal{L}^r.$$

We remark that in the $\mathcal{L} = K$ setting, $K_{Tot(K)} \cong \mathcal{O}_{Tot(K)} \Rightarrow K_S = \pi^*K^r$.

Note (cf. [8]) that

$$\pi_*\mathcal{O}_S \cong \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-(r-1)}.$$

Tensoring with $\mathcal{L}^r$, we have

$$\pi_*\mathcal{O}_S \otimes \mathcal{L}^r \cong \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^r.$$

Looking at cohomology

$$H^0(X, \pi_*\mathcal{O}_S \otimes \mathcal{L}^r) \cong \bigoplus_{i=1}^{r} H^0(X, \mathcal{L}^i)$$

$$H^0(S, \mathcal{O}_S \otimes \pi^*\mathcal{L}^r) \cong \mathcal{B}$$

$$H^0(S, \pi^*\mathcal{L}^r) \cong \mathcal{B},$$

by properties of the direct image functor from Section 2.5.

In other words, deformations of $(\mathcal{E}, \phi)$ in the normal direction to S correspond to deformations along the full Hitchin base, while first-order deformations of S correspond to deformations along the effective Hitchin base.

6.4 Geometry of b-manifolds

We take a brief digression to discuss the language of b-manifolds. This will prove to be a useful tool when working with Higgs bundles in the twisted setting. These ideas were pioneered in [75] in the context of differential operators on manifolds with boundary. The category of b-manifolds was further developed in [60], [77] in a more general setting in order to study so-called log-symplectic structures. Our interest in b-geometry is the basic language of b-manifolds, and their b-tangent and b-cotangent bundles, so we will focus on the concepts of b-geometry most relevant to our needs, following [60].

Definition 6.4.1. *A* ***b-manifold*** *is a pair (M, Z), where M is an oriented manifold and Z is an oriented co-dimension one submanifold of M. A* ***b-map*** *is a smooth map $f : (M_1, Z_1) \to (M_2, Z_2)$ such that $f^{-1}(Z_2) = Z_1$, and f is transverse to Z_2, i.e.*

$$T_{f(p)M_2} = Im(d_p f) + T_{f(p)} Z_2,$$

for all $p \in Z_1$.

Remark 6.4.2. When the setting is clear, we will write M in place of (M, Z) for convenience.

Definition 6.4.3. *Let (M, Z) be a b-manifold. A b-vector field on M is a vector that is tangent to Z for all $p \in Z$. Denote the set of b-vector fields by ${}^b\mathfrak{X}(M)$.*

An "ordinary" vector field $X \in \mathfrak{X}(M)$ is a b-vector field on (M, Z) *iff* for all $p \in Z$, there is a neighbourhood $(U, x_1, \ldots, x_n)$ where $Z \cap U$ is defined by $x_1 = 0$ and

$$X|_U = f_1 x_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \cdots + f_n \frac{\partial}{\partial x_n}$$

for a unique collection of smooth functions $f_1, \ldots, f_n \in C^\infty(U)$. This makes ${}^b\mathfrak{X}(M)$ a locally free $C^\infty(M)$-module with local bases given by

$$\begin{aligned} &\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\} && \text{away from } Z, \\ &\left\{x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right\} && \text{near to } Z. \end{aligned}$$

An application of the Serre-Swan theorem (c.f. [94] Proposition 7.6.5) tells us that the set of b-vectors are sections of a vector bundle on M.

Definition 6.4.4. *Let (M, Z) be a b-manifold. The* ***b-tangent bundle*** *of M, denoted ${}^bTM$, is the vector bundle whose sections are ${}^b\mathfrak{X}(M)$.*

Starting with a b-vector field v on M, if we take the restriction of v to Z, we get a vector field $v|_Z$ which is tangent to Z for all $p \in Z$. Hence, $v|_Z$ defines a tangent vector field on Z. Using this restriction, we have a morphism of $C^\infty(Z)$-modules $\Gamma({}^bTM|_Z) \to \Gamma(TZ)$, which is induced by a vector bundle isomorphism

$${}^bTM|_Z \to TZ.$$

The kernel of this isomorphism is a line bundle with a canonical non-trivial section w, call the **normal b-vector field** of M. In local coordinates, the vector field w can be written in a coordinate independent way as

$$w = x_1 \frac{\partial}{\partial x_1},$$

where $\{x_1 = 0\}$ locally defines Z.

At points $p \in M \backslash Z$, the b-tangent space at p coincides with the usual tangent space, and at points $p \in Z$, there is a surjective map

$${}^bT_pM \to T_pZ$$

whose kernel is spanned by w_p, the normal b-vector field at p. So we can write

$${}^bT_pM = \begin{cases} T_pM & p \in M \backslash Z \\ T_pZ \oplus span\{x_1 \frac{\partial}{\partial x_1}|_p\} & p \in Z \end{cases}.$$

Remark 6.4.5. As an ordinary vector field, $x_1 \frac{\partial}{\partial x_1}$ vanishes along Z; however, it is non-vanishing when viewed as a b-vector field. Around Z, we can think of $x_1 \frac{\partial}{\partial x_1}$ as a formal object from the view point of b-geometry.

Definition 6.4.6. *Let (M, Z) be a b-manifold. The b-**cotangent bundle** is the vector bundle ${}^bT^*M$ dual to ${}^bTM$.*

At points $p \in M \backslash Z$, we have that ${}^bT_p^*M = ({}^bT_pM)^* = (T_pM)^* = T_p^*M$ coincides with the usual cotangent space. At points $p \in Z$, the dual of the map for b-tangent spaces above gives us an embedding

$$T_p^* \to^b T_p^*M$$

whose image is $\{l \in ({}^bT_pM)^* | l(w_p) = 0\}$.

Let $\{x_1, \ldots, x_n\}$ be coordinates around p such that Z is locally defined by $\{x_1 = 0\}$. Consider the one-form $\frac{dx_1}{x_1}$. At points away from Z, $\frac{dx_1}{x_1}$ is a well-defined one-form. The pairing of μ with any b-vector field extends smoothly over Z because

$$\langle f_1 x_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \cdots + f_n \frac{\partial}{\partial x_n}, \frac{dx_1}{x_1} \rangle = f_1.$$

This means $\frac{dx_1}{x_1}$ can be extended over Z as a section of ${}^bT^*M$. We will denote this section as $\frac{dx_1}{x_1}$, viewed as a formal object around Z in a similar manner as with the b-tangent spaces around Z. Moreover, as $\frac{dx_1}{x_1}(w_p) = 1$ for $p \in Z$, we know that $\frac{dx_1}{x_1} \notin \{l \in ({}^bT_pM)^* | l(w_p) = 0\}$, and so we can write

$${}^bT_p^*M = \begin{cases} T_p^*M & p \in M \backslash Z \\ T_p^*Z \oplus span\{\frac{dx_1}{x_1}|_p\} & p \in Z \end{cases}.$$

Definition 6.4.7. *Denote by ${}^b\Omega^k(M)$ the space of b-**de Rham k-forms**, i.e. sections of $\bigwedge^k \left({}^bT^*M\right)$.*

We can see that ${}^b\Omega^k(M)$ sits inside the usual space of k-forms in the following way. Let $\mu \in \Omega(M)$ be a k-form. We interpret it as a section of $\bigwedge^k \left({}^bT^*M\right)$ by

- at $p \in M \backslash Z$

$$\mu_p \in \bigwedge^k (T_p^*M) = \bigwedge^k \left({}^bT_p^*M\right),$$

- at $p \in Z$

$$\mu_p = (i^*\mu)_p \in \bigwedge^k \left(T_p^* Z\right) \subset \bigwedge^k \left({}^bT_p^* M\right),$$

where $i : Z \to M$ is the inclusion map. For a fixed defining function f, i.e. a b-map $f : (M, Z) \to (\mathbb{R}, 0)$, we can write a b-de Rham k-form $\omega \in^b \Omega^k(M)$ as

$$\omega = \alpha \wedge \frac{df}{f} + \beta \tag{6.4.1}$$

for some $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$. While α and β themselves are not unique, their values at every $p \in Z$ are unique. This decomposition of b-de Rham k-forms (6.4.1) allows us to extend the exterior derivative to ${}^b\Omega(M)$ by defining its action on $\omega \in {}^b\Omega^k(M)$ by

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta. \tag{6.4.2}$$

This operation is well-defined, and extends smoothly over M as a section of $\bigwedge^{k+1}\left({}^bT^*\right)$. Moreover, because the usual exterior derivative satisfies $d^2 = 0$, it is clear by (6.4.2) that the extended exterior derivative also satisfies $d^2 = 0$, so we can form the complex of b-forms, the *b-de Rham complex*

$$0 \to {}^b\Omega^0(M) \xrightarrow{d} {}^b\Omega^1(M) \xrightarrow{d} {}^b\Omega^2(M) \xrightarrow{d} \cdots \to 0. \tag{6.4.3}$$

We can define the notion of symplectic in the b-category, and introduce a particular example that will be useful to us.

Definition 6.4.8. *Let (M, Z) be a $2n$-dimensional b-manifold. A b-**symplectic form** on M is a b-form $\omega \in {}^b\Omega^2(M)$ that is closed and non-degenerate, i.e. $d\omega = 0$ and for all $p \in M$, $\omega|_p$ is of maximal rank as an element of $\Lambda({}^bT_p^*M)$.*

Remark 6.4.9. Given a symplectic manifold M, the cotangent bundle TM carries a natural symplectic structure. In the b-setting we can also define a natural b-symplectic structure on the b-cotangent bundle of a b-manifold.

Let (M, Z) be a b-manifold. Let $\{x_1, \ldots, x_n\}$ be local coordinates on M such that Z is defined by $\{x_1 = 0\}$, $\{y_1, \ldots, y_n\}$ fiber coordinates on ${}^bT^*M$. The canonical one-form is given by

$$\theta = y_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} y_i dx_i, \tag{6.4.4}$$

and the corresponding b-symplectic form on ${}^bT_p^*M$ is given by

$$\omega = d\theta = dy_1 \wedge \frac{dx_1}{x_1} + \sum_{i=2}^{n} dy_i \wedge dx_i.$$

6.5 Variations of spectral curves

Let $(\mathcal{E}, \phi)$ be an $\mathcal{L}$-twisted Higgs bundle on a Riemann surface X such that $\deg(\mathcal{L}) > \deg(K)$, with spectral curve S. From our work studying the deformation theory of $\mathcal{M}_X^{\mathcal{L}}$ in Section 6.3, associated to $(\mathcal{E}, \phi)$, we can choose a section $s \in H^0(X, K \otimes \mathcal{L})\backslash\{0\}$ which maps $\mathcal{B}_{eff}$ to $\widetilde{\mathcal{B}} \subset \mathcal{B}$, and from it produce the twisted canonical one-form Θ on K_S.

The chosen section $s \in H^0(X, K^* \otimes \mathcal{L})\backslash\{0\}$ defines a divisor in $Z \subset X$, namely the divisor of zeroes of s. We want to restrict for now to the case where s has distinct zeroes. This gives us a relationship $\mathcal{L} \cong K(Z)$. We can view X together with the divisor Z as a b-manifold (X, Z). Because S lives naturally inside of $K(Z)$, it inherits a natural

divisor given by $S \cap \pi^*(Z)$ and thus also carries the structure of a b-manifold which we will denote as $(S, \pi^*(Z))$ (see Figure 6.2). The projection map π restricted to S can be viewed as a map of b-manifolds with derivative $d\pi : TS(\pi^*(Z)) \to \pi^*(TX(Z)) \ \in Hom(\pi^*(K(Z)), K_S(\pi^*(Z)))$, where K_S denotes the canonical bundle of S. In this formulation, we have that the twisted canonical one-form is given by $\Theta := d\pi(\theta)$, where θ is the canonical one-form for the log-symplectic structure on $K(Z)$, and Θ is, in particular, the canonical one-form for the log-symplectic structure on $K_S(\pi^*(Z))$.

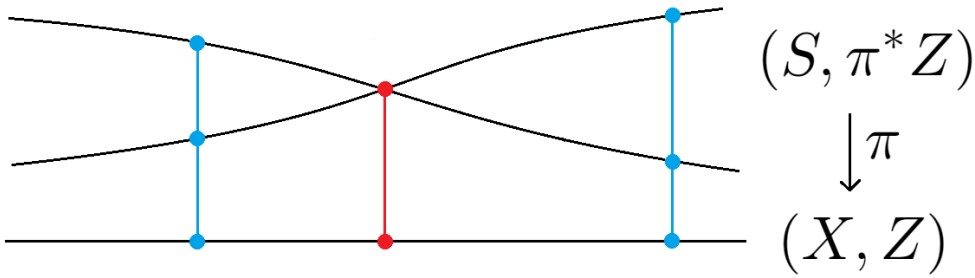

Figure 6.2: Spectral curve S inside of $K(Z)$.

Let $Z^{reg}_{eff} = \{(x, b) \in T^*X(Z) \times \tilde{\mathcal{B}}^{reg} : p_b(x) = 0\}$ be the universal moduli space of spectral curves, where $p_b(x)$ is the degree r polynomial with coefficients $b_1, \ldots, b_r$. The space Z^{reg}_{eff} is equipped with two maps: $q : Z^{reg}_{eff} \to \tilde{\mathcal{B}}^{reg}$ the projection onto the second component and $j : Z^{reg}_{eff} \to T^*X(Z)$ which maps a spectral curve $\{p_b(x) = 0\}$ to its image in $T^*X(Z)$.

$$\begin{tikzcd}
 & Z^{reg}_{eff} \arrow[dl, "j"'] \arrow[dr, "q"] & \\
S \subset T^*X(Z) \arrow[d, "\pi"'] & & \tilde{\mathcal{B}}^{reg} \supset U \ni b \\
X & &
\end{tikzcd}$$

For all $b \in \tilde{\mathcal{B}}^{reg}$ we can choose an open neighbourhood U of b such that in local coordinates $Z^{reg}_{eff}|_U \cong U \times S$, where S is the spectral curve viewed as a topological surface with a family of b-complex structures $I(t)$ that vary in $U \subset \tilde{\mathcal{B}}^{reg}$. The composition map $\tilde{\pi} = \pi \circ j : Z^{reg}_{eff} \to X$ corresponds to a family of maps $\pi_t : S \to X$ that is holomorphic with respect to the complex structure $I(t)$. Let $\partial \in T_bU$ be a tangent vector. If we differentiate the condition that π_t is holomorphic with respect to $I(t)$, we get

$$\pi_*(\kappa(\partial)) = \overline{\partial} Y, \tag{6.5.1}$$

where $\kappa(\partial) = -\frac{i}{2}\partial I$ is the Kodaira-Spencer class of the deformation I, and $Y = \tilde{\pi}_*(\partial) \in H^0(S, \pi^*TX(Z))$, where we have lifted ∂ to a vector field on S by assigning to each point on S the vector ∂.

Let $D = \sum_a (v(\pi, a) - 1)[a]$ be the divisor of ramification points on S. Define $W = \frac{Y}{\tilde{\pi}_*} \in H^0(S, TS(Z))(D)$ a b-vector field on S with poles along D, where we are viewing $\frac{1}{\pi_*}$ as a section of $TS(Z) \otimes \pi^*(T^*X)(Z)$ that is dual to π^*. From (6.5.1), we have

$$\kappa(\partial) = \bar{\partial} W. \tag{6.5.2}$$

The b-vector field W depends on both the choice of ∂ and the choice of local differentiable trivialization U. When we wish to show the dependence on ∂ we will write $W(\partial)$.

Fix a $(1,0)$-vector field ∂ on $\mathcal{B}^{reg}_{eff}$. Define the vector field $\delta \in H^0(Z^{reg}_{eff}, TZ^{reg}_{eff})(D)$ as the unique lift of ∂ to $Z^{reg}_{eff}\backslash D$ (i.e. $q_*(\delta) = \partial$) such that $\tilde{\pi}_*(\delta) = 0$ (i.e. $\tilde{\pi}$ is constant along integral curves of δ). In a local differentiable trivialization $Z^{reg}_{eff}|_U = U \times S$, we can write

$$\delta = \partial - W(\partial). \tag{6.5.3}$$

We are interested in differentiating objects on Z^{reg}_{eff} by δ. Let $V = \chi(\partial)$ be the normal vector field corresponding to ∂ .

Proposition 6.5.1. *The variation of θ with respect to δ is independent of the trivialization of Z^{reg}_{eff}, and is given by*

$$\delta\theta = i_{\widehat{V}} d\theta|_S. \tag{6.5.4}$$

Proof. Choose a local differentiable trivialization $Z^{reg}_{eff}|_U = U \times S$. Define two lifts of V to $T(T^*X(Z))$ by $\widetilde{V} = j_*(\partial) \in H^0(S, T(T^*X(Z))$ and $\widehat{V} = j_*(\delta) \in H^0(S, T(T^*X(Z)))(D)$. We will start by applying π_* to both b-vector fields. The vector field δ satisfies $\tilde{\pi}_*(\delta) = 0$, and so we have that $\pi_*\widehat{V} = \pi_*(j_*(\delta)) = \tilde{\pi}_*\delta = 0$. The map $j_* : T(U \times S)(Z) \to T(S)(Z)$ restricted to $TS(Z)$ acts as the identity map, so $\widehat{V} = j_*(\delta) = j_*(\partial - W) = j_*(\partial) - j_*(W) = \widetilde{V} - W$. Applying $\bar{\partial}$ to $\pi_*(\widetilde{V}) = \pi_*(W)$ yields $\pi_*(\bar{\partial}\widetilde{V}) = \pi_*(\bar{\partial}W) = \pi_*\kappa$. The map $\pi_* : TS(Z) \to \pi^*TX(Z)$ is generically an isomorphism, meaning that $\kappa = \bar{\partial}W = \bar{\partial}V$. From this, we can say that $\widehat{V}$ is a meromorphic $b-$vector field, i.e. $\bar{\partial}\widehat{V} = 0$ with extra poles at the ramification points of π. In the local differentiable trivialization Z^{reg}_{eff}, flowing along the vector field ∂ produces a family of spectral curves. Pushing forward ∂ to $T^*X(Z)$, this family of spectral curves is obtained by flowing along $\widetilde{V} = j_*(\partial)$. Varying θ, which is an object S, by the ∂ is then given by $\partial\theta = \mathcal{L}_{\widetilde{V}}\theta|_S$. We can understand the variation of δ on θ by

$$\delta\theta = \partial\theta - \mathcal{L}_W\theta = \mathcal{L}_{\widetilde{V}-W}\theta|_S = \mathcal{L}_{\widehat{V}}\theta|_S = \iota_{\widehat{V}}d\theta|_S + d(\iota_{\widehat{V}}\theta)|_S. \tag{6.5.5}$$

We can claim that $\iota_{\widehat{V}}\theta$ is a holomorphic one-form. There are two potential areas of concern: the b-geometry, and the poles of $\widehat{V}$. The interior product of a b-vector field with θ eliminates terms of the form $\frac{dx}{x}$, i.e. eliminating poles that would occur along Z. The canonical one-form θ vanishes at the ramification points with order at least that of π_* which eliminates poles coming from $\widehat{V}$. This means that $\iota_{\widehat{V}}\theta$ is indeed a holomorphic one-form, and so in particular it is constant on S, and thus $d(\iota_{\widehat{V}}\theta)|_S = 0$.

We interpret the term $\iota_{\widehat{V}}d\theta|_S$ as being a one-form on S, i.e. taking in only tangent vectors to S. In this way, any S-tangential component of $\widehat{V}$ will vanish when a tangent vector in S is inserted into the resultant one-form, and so only the S-normal component of $\widehat{V}$ will contribute to a nonzero term. Because $\widehat{V}$ is a lift of V to $T(T^*X(Z))$, this normal component is V, and so we have $\iota_{\widehat{V}}d\theta|_S = \iota_V d\theta|_S$.

□

Let $\xi \in H^0(T^*X(Z), T(T^*X(Z)))$ be the the vector field generating the $\mathbb{C}^*$-action on $T^*X(Z)$. In local coordinates (x, y), were x is the local coordinate on X and y is the y is the fiber coordinate (i.e. thinking of (x, y) as $(x, y\frac{dx}{x})$), we have that $\xi = y\frac{\partial}{\partial y}$.

Lemma 6.5.2. *If $V, \widehat{V}$ are as in the above proposition, then*

$$\widehat{V} = \frac{\alpha}{\theta}\xi, \tag{6.5.6}$$

where $\alpha = \iota_V d\theta|_S$.

Proof. Recall from the proof above that $\widehat{V}$ is a lift of V that satisfies $\pi_*(\widehat{V}) = 0$. Because $\ker(\pi_*) \cap TS(Z)$ is generically zero, we have that $\widehat{V}$ is characterized by these two conditions. Let $V^* = \frac{\alpha}{\theta}\xi$. By uniqueness, it is sufficient to show that V^* satisfies two properties: V^* is a lift of V to $T(T^*X(Z))(D)$, and $\pi_*(V^*) = 0$.

We have that $\pi*(V^*) = 0$ for free because ξ is a vector field on T^*X in the cotangent fiber direction, i.e. $\xi \in \ker(\pi_*)$.

To check that $V^* \in H^0(S, T(T^*X(Z))(D))$ we need to check the bundle in which each component of V^* lives. For terms in the "numerator", we have $\xi \in H^0(T^*X(Z), T(T^*X(Z)))$, $\alpha = \iota_V d\theta|_S \in H^0(S, T^*(T^*S(Z)))$. In the "demoninator", we are viewing $\frac{1}{\theta}$ as the dual vector field to θ, i.e. satisfying $\theta(\frac{1}{\theta}) = 1$. In a local frame with coordinates (x, y) on $T^*X(Z)$ where we have $\theta = y\frac{dx}{x}$, we must have that $\frac{1}{\theta} = \frac{1}{y}x\frac{\partial}{\partial x}$. This local insight tells us that while $\frac{1}{\theta}$ lives in the dual bundle to θ, it has poles at zeros of θ, which occur along the ramification divisor D. In this way we have $\frac{1}{\theta} \in H^0(T^*X(Z), T(T^*X(Z)))(D)$. Putting it all together (and restricting to $S \in T^*X(Z)$ where necessary), we see that $V^* \in H^0(S, T(T^*X(Z)))(D)$.

We need now to check that V^* is a lift of V. We can do this by showing that

$$\iota_{V^*} d\theta|_S = \iota_V d\theta|_S.$$

First observe that in a local frame of $T^*X(Z)$ with coordinates (x, y), we have that

$$\theta(\xi) = y\frac{dx}{x}(y\frac{\partial}{\partial y}) = 0,$$

and

$$\begin{aligned}\iota_\xi d\theta &= (dy \wedge \frac{dx}{x})(y\frac{\partial}{\partial y}) \\ &= y\frac{dx}{x} \\ &= \theta.\end{aligned}$$

Computing $\iota_{V^*} d\theta|_S$ we have

$$\begin{aligned}\iota_{V^*} d\theta|_S &= \frac{\alpha}{\theta|_S}\iota_\xi d\theta|_S \\ &= \frac{\alpha}{\theta|_S}\theta|_S \\ &= \alpha = \iota_V d\theta|_S,\end{aligned}$$

which shows that V^* is a lift of V.

We have thus shown that V^* satisfies the same properties as $\widehat{V}$, and so by uniqueness we have $\widehat{V} = V^* = \frac{\alpha}{\theta}\xi$.

□

We want to use the vector field δ to differentiate the Bergman kernel, and the twisted analogues of the Eynard-Orantin differentials. This will produce twisted analogues of certain variational formulas [6, 42]. We first need to adapt these objects to the b-geometric framework.

Choose a symplectic basis $\langle A_1, ..., A_g, B_1, ..., B_g\rangle$ for $H_1(S, \mathbb{Z})$ such that none of the cycles intersect with Z. The spectral curve comes equipped with a Bergman kernel $B(z_1, z_2)$ as in Definition (2.6.2). When we make the association $L \cong K(Z)$, and S is realized as the b-manifold $(S, \pi^* Z)$, we can impose the b-strucutre onto B. The local expression changes near to images of the divisor Z in $S \times S$, being written as a bilinear b-form, i.e. on $(S, \pi^* Z)$ we view $B \in K_S(Z) \otimes K_S(Z)$. As an example, near a point $(p, q) \in Z \times U$, for some neighbourhood U away from Z, we have

$$B(z_1, z_2) = \frac{\frac{dz_1}{z_1} dz_2}{(z_1 - z_2)^2} + O(1)\frac{dz_1}{z_1} dz_2.$$

Such a local expression makes plain the fact that B is not a symmetric differential. While this causes no inherent problem for defining B, it will become a problem when we try to define the Eynard-Orantin differentials in this setting. Specifically, the symmetry of the $W_{g,n}$'s come from the symmetry of B. We will remedy this considering a symmetrized version of B in this setting.

Definition 6.5.3. *Define* $\widehat{B}(z_1, z_2) := \frac{1}{2}\left(B(z_1, z_2) + B(z_2, z_1)\right)$.

Notably, away from Z, we can choose a coordinate system U such that $\widehat{B}|_U = B|_U$.

Let $v_1, \ldots, v_g$ be a basis of holomorphic differentials, normalized by

$$\int_{A_j} v_i = \delta_{ij}. \tag{6.5.7}$$

We can impose the b-structure onto the v_i, which notably does not affect their normalization over the A_i-cycles. From the properties of the ordinary Bergman kernel on S, we have that the symmertrized Bergman kernel on $(S, \pi^* Z)$ is related to the differentials by

$$\int_{z_1 \in B_j} \widehat{B}(z_1, z_2) = 2\pi i v_j(z_2), \tag{6.5.8}$$

and the period matrix τ by

$$\int_{z_1 \in B_j} \int_{z_2 \in B_k} \widehat{B}(z_1, z_2) = \tau_{ij}. \tag{6.5.9}$$

We now want to differentiate $\widehat{B}$ with respect to the vector field δ, we obtain a twisted version of the Rauch Variational Formula.

Theorem 6.5.4 (Twisted-Rauch Variational Formula)**.** *If $\partial \in \tilde{B}^{reg}$, $p, q \in S \backslash D$ are distinct points, then*

$$\delta \widehat{B}(p, q) = -\sum_{a \in D} Res_{u \to a} \frac{\delta\theta \widehat{B}(u, p)\widehat{B}(u, q)}{dx(u)dy(u)} \tag{6.5.10}$$

Proof. Choose a local differentiable trivialization $Z^{reg}_{eff}|_U = U \times S$ so that $\delta = \partial - W$. By Theorem 6.5.1, we know that δ is independent of the choice of trivialization, we can choose the trivialization to be beneficial to us as we see fit. In particular we will choose the trivialization so that W vanishes in a neighbourhood of p and q. Applying δ to $\widehat{B}$ in the trivialization we have

$$\delta \widehat{B}(p, q) = \partial \widehat{B}(p, q) - \mathcal{L}_{W(p)} \widehat{B}(p, q) - \mathcal{L}_{W(q)} \widehat{B}(p, q) = \partial \widehat{B}(p, q)$$

In this trivialization, and choosing $\kappa = \overline{\partial} W = 0$, the Bergman kernel satisfies the variational formula [45, pg. 57]

$$\partial \widehat{B}(p,q) = \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} W(u) \widehat{B}(u,p) \widehat{B}(u,q), \tag{6.5.11}$$

where the sum is taken over all poles of $W(u)\widehat{B}(u,p)\widehat{B}(u,q)$, specifically, a is a ramification point of π (poles of W), $a = p$ (pole of $\widehat{B}(u,p)$), or $a = q$ (pole of $\widehat{B}(u,q)$), and γ_a is a contour around a. Because W vanishes in a neighbourhood of p and q, there will be no residue contribution, and we can therefore consider the sum to be over ramification points of π. Choosing the contours γ_a to be sufficiently small, we can ensure that the interiors do not contain any zeros of s, and thus need not be concerned with residues coming from Z.

In a coordinate chart around a ramification point of π, we can write $W = -(\delta\theta)\zeta + W'$, where $\zeta = \frac{1}{dydx}$ and W' is smooth. Using this decomposition for W, we have

$$\begin{aligned}
\delta \widehat{B}(p,q) &= \partial \widehat{B}(p,q) \\
&= \frac{1}{2\pi i} \sum_a \int_{u \in \gamma_a} (-(\delta\Theta)\zeta + W') \widehat{B}(u,p) \widehat{B}(u,q) \\
&= -\sum_a Res_{u \to a} (\delta\Theta)(u) \zeta(u) \widehat{B}(u,p) \widehat{B}(u,q) \\
&= -\sum_a Res_{u \to a} \frac{(\delta\Theta)(u) \widehat{B}(u,p) \widehat{B}(u,q)}{dx(u)dy(u)}
\end{aligned}$$

□

6.6 Twisted topological recursion

We next want to define a twisted version of the Eynard-Orantin differentials. Continuing with the naïve approach, we will define the necessary objects as living in $K(Z)$ instead of K.

Definition 6.6.1. *Let $p \in R$. The **$\mathcal{L}$-twisted recursion kernel** (associated to s) at p is a meromorphic section of $K_S(Z) \boxtimes K_S(Z)^*$ defined by*

$$K_p(z_0, z) = \frac{\int_{t=\alpha}^{z} \widehat{B}(t, z_0)}{(y(z) - y(\sigma_p(z)) \frac{dx(z)}{x(z)}} \tag{6.6.1}$$

where α is an arbitrary base point.

Definition 6.6.2. *The **$\mathcal{L}$-twisted Eynard-Orantin differentials** (associated to s) $W_{g,n}$ are meromorphic sections of the n-th exterior tensor product $K_S(Z)^{\boxtimes n}$, i.e. multi-b-differentials, defined as follows:*

The initial conditions of the recursion are given by:

$$W_{0,1}(z) = y(z) \frac{dx(z)}{x(z)} \tag{6.6.2}$$

$$W_{0,2}(z_1, z_2) = \widehat{B}(z_1, z_2). \tag{6.6.3}$$

For all $g, n \in \mathbb{N}$ and $2g - 2 + n \geq 0$, define $W_{g,n}$ recursively by

$$W_{g,n+1}(z_0, \boldsymbol{z}) = \sum_{p\in R} \mathrm{Res}_{z=p} K_p(z_0, z) \left[W_{g-1,n+2}(z, \sigma_p(z), \boldsymbol{z}) + \sum_{\substack{g_1+g_2=g \\ I\cup J=\boldsymbol{z}}}' W_{g_1,|I|+1}(z, I) W_{g_2,|J|+1}(\sigma_p(z), J) \right] \tag{6.6.4}$$

where the prime signifies summation excluding the cases (g_1, I) *or* $(g_2, J) = (0, 0)$.

Remark 6.6.3. In the ordinary setting, the Eynard-Orantin differentials satisfy a suite of useful properties, most notably that they are symmetric differentials. It turns out that the twisted Eynard-Orantin differentials satisfy the same suite of properties. This is largely because the structure of the recursion revolves around local data, *i.e.* the residues. The choices made to keep zeroes of $s \in H^0(X, K^* \otimes L)$ away from ramification points and the symplectic basis of cycles means that the local computations in the recursion do not see the b-structure, although the differentials themselves are b-objects. This observation means that there is, in principle, a family of collections of twisted Eynard-Orantin differentials, parametrized by the choice of s. That said, they will all live in different spaces, as the divisor Z depends on s. For each collection of twisted Eynard-Orantin differentials in this family, the proofs of these properties in [42, Appendix A] hold because they depend only on the recursion structure and local computations around the residues.

Let $(\lambda_1, ..., \lambda_{g_S})$ be the local singular coordinates on B_{eff}, $\partial_i := \frac{\partial}{\partial \lambda_i}$, and $\delta_i := \delta(\partial_i)$.

Theorem 6.6.4 (Variational Formula for twisted-E-O invariants)**.** *For* $g + k > 1$,

$$\delta_i W_{g,k}(p_1, ..., p_k) = -\frac{1}{2\pi i} \int_{p\in b_i} W_{g,k+1}(p, p_1, ..., p_k), \tag{6.6.5}$$

where the cycle b_i *is chosen so that it contains no ramification points.*

Proof. This theorem is essentially the same as Theorem 5.1 of [42], but in the twisted setting. The original proof only relies on the Rauch variational formula and the diagrammatic representation of $W_{g,n}$. We proved an analogous form the Rauch variational formula in Theorem 6.5.4. The diagrammatic representation relies on only the properties of the differentials and the recursion formula, both of which are unchanged in the twisted setting.

□

To better understand how this variational formula relates to the topology of the twisted setting, let us apply the variational formula to $W_{0,2}(p_1.p_2) = \widehat{B}(p_1, p_2)$,

$$\delta_i \widehat{B}(p_1, p_2) = \delta_i W_{0,2}(p_1, p_2) = -\frac{1}{2\pi i} \int_{p\in b_i} W_{0,3}(p, p_1, p_2). \tag{6.6.6}$$

Integrating the left hand side twice then yields

$$\begin{aligned} \int_{p_1\in b_j} \int_{p_2\in b_k} \delta_i \widehat{B}(p_1, p_2) &= \partial_i \int_{p_1\in b_j} \int_{p_2\in b_k} \widehat{B}(p_1, p_2) \\ &= 2\pi i \partial_i \tau_{jk} \\ &= 2\pi i c_{ijk}, \end{aligned}$$

written more cleanly

$$c_{ijk} = \frac{1}{2\pi i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} \delta_i \widehat{B}(p_1, p_2).$$

Utilizing (6.6.6), we have a relationship between the twisted Donagi-Markman cubic and $W_{0,3}$ given by

$$c_{ijk} = -\left(\frac{1}{2\pi i}\right)^2 \int_{p_1 \in b_j} \int_{p_2 \in b_k} \int_{p \in b_i} W_{0,3}(p, p_1, p_2). \tag{6.6.7}$$

We can continue the computation on the right-hand side by adapting a lemma from [42, Appendix A]. The proof of this lemma again only depends on the properties of twisted Eynard-Orantin differentials.

Lemma 6.6.5.

$$W_{0,3}(p, p_1.p_2) = \sum_a Res_{q \to a} \frac{B(p,q)B(p_1,q)B(p_2,q)}{dx(q)dy(q)} \tag{6.6.8}$$

where the sum is taken over ramification points a of the spectral curve.

From this lemma we obtain

$$\begin{aligned} c_{ijk} &= -\left(\frac{1}{2\pi i}\right)^2 \sum_a \int_{p_1 \in b_j} \int_{p_2 \in b_k} \int_{p \in b_i} Res_{q \to a} \frac{B(p,q)B(p_1,q)B(p_2,q)}{dx(q)dy(q)} \\ &= -2\pi i \sum_a Res_{q \to a} \frac{v_i(q)v_j(q)v_k(q)}{dx(q)dy(q)}. \end{aligned}$$

This proves a local analogue of the residue formula for the Donagi-Markman cubic as presented in Baralgia-Huang.

Lemma 6.6.6 (Local residue formula for the twisted Donagi-Markman cubic)**.**

$$c_{ijk} = 2\pi i \sum_a Res_{q \to a} \frac{v_i(q)v_j(q)v_k(q)}{dx(q)dy(q)}$$

We want to return our attention back to (6.6.7). We can continue this process of applying the variational formula to $W_{0,2}$ multiple times with different choices of δ_i and obtain:

Theorem 6.6.7.

$$\partial_{i_1}\partial_{i_2}\ldots\partial_{i_{m-2}}\tau_{i_{m-1}i_m} = -\left(\frac{i}{2\pi}\right)^{m-1} \int_{p_{i_1} \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_{0,m}(p_1, ..., p_m). \tag{6.6.9}$$

We can interpret this theorem in the following way. The Higgs bundle $(\mathcal{E}, \phi)$ produces a spectral curve S. This spectral curve lives over a point $b \in \widetilde{\mathcal{B}}^{reg}$, the image of the deformation space of spectral curves inside of the Hitchin base induced by the chosen section $s \in H^0(X, K^* \otimes \mathcal{L})$. The $g = 0$ twisted Eynard-Orantin variants compute the Taylor series expansion of the period matrix at the point b. This means that with the information of a single spectral curve S, we can use the twisted Eynard-Orantin invariants to understand local deformations of S through their period matrices.

It is worth noting that while the coordinates λ_i depend on s, this result would hold for any s that does not have zeros along the symplectic basis of cycles. This aligns with the observation in Remark 6.6.3, as the variational formula only depends on local data away from the zeroes of s. While Theorem (6.6.7) will hold for any choice of s compatible with the symplectic basis of cycles, we have to be careful about the interpretation between various choices of s. The period matrix is an inherent object on S, depending only on the choice of symplectic basis, regardless of where or how we view S as residing. It makes sense, then, that choices of s should not change the period matrix — this is true, as we

write the period matrix in terms of $W_{0,2} = \widehat{B}$ by (6.5.9), for any compatible s. The information of the Taylor series, however, gives information about how the period matrix changes with respect to coordinates on $\widetilde{\mathcal{B}}^{reg}$, which does, *a priori*, depend on s (it has also not been addressed in the case of the twisted invariants). This dependence on s is a subtly that has not been addressed. In principle, deformations are controlled by the effective Hitchin base $\mathcal{B}_{eff}$, which is independent of s, rather than $\widetilde{\mathcal{B}}$. We would like to say that the full interpretation of Theorem 6.6.7 is independent of s, however, without further understanding the dependence of $\widetilde{B}$ on s, we leave this statement as a conjecture for now.

7 Next steps

7.1 Further perspectives on $\mathcal{L}$-twisted Hitchin geometry

The comments at the end of Section 6.6 suggest that we need to better understand the relationship between the various choices of $s \in H^0(X, K^* \otimes \mathcal{L})$ and the geometry of the spectral curve. To study this dependence, we consider the vector bundle over $H^0(X, K^* \otimes \mathcal{L})$ with fibres $\mathcal{B}$ (see Figure 7.1). Because $H^0(X, K^* \otimes \mathcal{L})$ is a vector space, this is just the trivial bundle $\mathcal{B} \times H^0(X, K^* \otimes \mathcal{L})$.

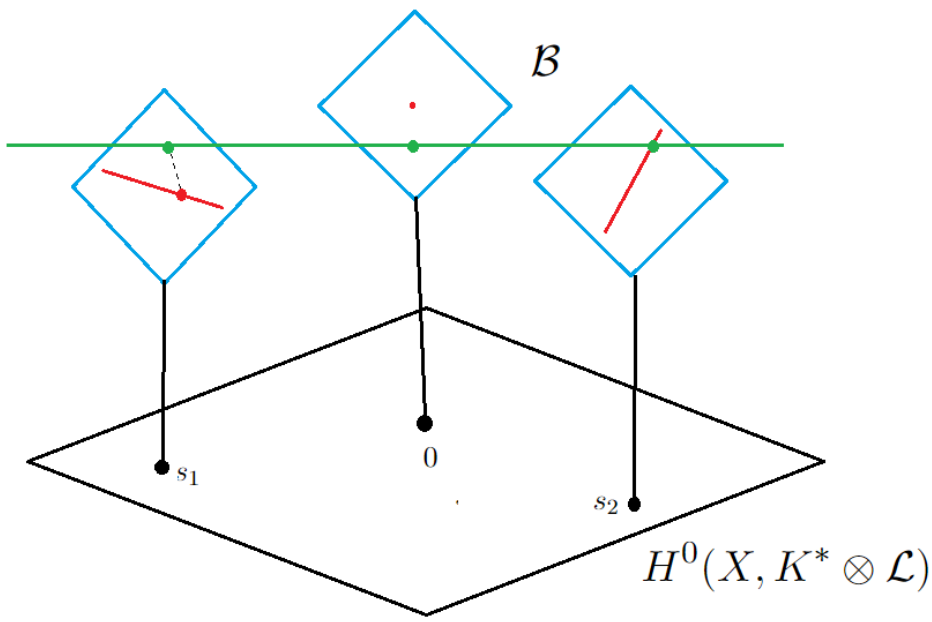

Figure 7.1: Vector bundle over $H^0(X, K^* \otimes L)$ whose fibres are $\mathcal{B}$ (in blue). Each fibre has a distinguished subspace $\widetilde{\mathcal{B}}_s$ (in red). For a chosen Higgs bundle, there is a constant section given by the characteristic coefficients (in green).

Let $(\mathcal{E}, \phi)$ be an $\mathcal{L}$-twisted Higgs bundle with spectral curve S. Choose a symplectic basis $\langle A_1, ..., A_g, B_1, ..., B_g \rangle$ for $H_1(S, \mathbb{Z})$. There is a natural constant section on this vector bundle associated to $(\mathcal{E}, \phi)$, which assigns to each point in the base the characteristic coefficients of the Higgs field in the fibre. In each fibre, there is also an identified subspace given by $\widetilde{\mathcal{B}}_s$, the image of $\mathcal{B}_{eff}$ under multiplication by s inside of $\mathcal{B}$. Recall from Section 6.3 that $\mathcal{B}_{eff}$, and by extension $\widetilde{\mathcal{B}}_s$, is the space of deformations of the spectral curve. Away from the point $s \equiv 0$ where $\dim \widetilde{\mathcal{B}}_s = 0$, the dimension of each $\widetilde{\mathcal{B}}_s$ is equal, although they do not define the same subspaces. Notably, the constant section defined by $(\mathcal{E}, \phi)$ need not intersect the identified subspace in the fibre. There is a subset of points in the base whose zeroes intersect the symplectic cycles. We will call this set the *incompatible locus*. Away from the incompatible locus (*i.e.*,

along points in the base whose zeroes do not intersect the symplectic cycles) we can, in a holomorphic way, identify S with a point on $\widetilde{\mathcal{B}}_s$, which describes S with "zero deformations". Like this, we have a natural section on the base minus the incompatible locus coming from S.

To determine the dependence on s, we need to understand three subsets of $H^0(X, K^* \otimes \mathcal{L})$: the incompatible locus, the set of sections with non-distinct zeroes, and the set of sections whose zeroes overlap with ramification points of S. Removing these subsets from $H^0(X, K^* \otimes \mathcal{L})$ will yield a space of sections that are suitable for the constructions in *Chapter* 6. The topology of this space will determine which sections are "equivalent", in the sense that we can move from one to another by moving zeroes of s without crossing any of the noted subsets above. We can then look at a slightly weakened version of the conjecture made at the end of the previous chapter: the interpretation of Theorem 6.6.7 holds on equivalence classes of sections.

There are also two directions that we can proceed to further investigate this vector bundle:

1. How do sections that pass through the identified subspaces in the fibres correspond to spectral curves? For example, can a section defined over only part of $H^0(X, K^* \otimes \mathcal{L})$ define a incompatible locus, and thus produce a set of cycles on S? Can it make that set of cycles a symplectic basis?
2. We can pull this bundle back onto itself. The tautological section on this pullback bundle contains the information of all spectral curves. Can we use this to construct a *universal spectral curve*, which contains all the important data of spectral curves? Suppose we have an $\mathcal{L}$-twisted Higgs bundle $(\mathcal{E}, \phi)$. Does this universal spectral curve see the properties of the spectral curve that are invariant under choices of s? (Or compatible choices of s?)

Remark 7.1.1. It is important to bring attention to an additional choice that was made at the beginning of Section 6.5, specifically, only choosing s to have distinct zeroes. This choice was made to ensure that we could work in the context of b-geometry, and that our Higgs fields only have simple poles. If we dropped this condition, we would be entering the setting of b^k-geometry [91] and wild Higgs bundles [14, 15, 49]. Assuming that we could carry out similar calculations in this setting, it would open up a larger class of sections in $H^0(X, K^* \otimes \mathcal{L})$, which correspond to situations where we allow zeroes to cross as we wary s. We will not investigate these ideas here, but wish to bring attention to them as a further direction of generalization.

7.2 Tautological recursion

Even though the tautological section appears in topological recursion, it is effectively a recursion built out of only the data of a spectral curve S. In fact, our arguments in Section 6.6 suggest that there are scenarios where the ambient space does not play a role in the interpretation of the Eynard-Orantin differentials. When we start with an $\mathcal{L}$-twisted Higgs bundle, the tautological section on $\mathrm{Tot}(\mathcal{L})$ is a natural object that comes with it. We can ask if it is possible to build a meaningful recursion out of the tautological section without appealing to the geometry of the spectral curve. We now have the opportunity to fashion a fully invariant, *tautological recursion*, on a rank 2, $\mathcal{L}$-twisted Higgs bundle by recognizing that:

- the choice of a quadratic section α of a holomorphic line bundle $\mathcal{L}$ on a Riemann surface X generates a canonical $\mathcal{L}$-twisted Higgs bundle in the Hitchin section of the corresponding moduli space;
- the tautological section of the pullback of $\mathcal{L}$ can be used as the seed $W_{0,1}$;
- the pullback of the preceding pullback generates a second tautological section whose square is a fully holomorphic replacement for the Bergman kernel as $W_{0,2}$.

This initiates a geometric process by which Higgs bundles of Hitchin section-type are produced successively over spaces of higher dimension. These spaces correspond to products of α's spectral curve with itself, reconstructing the product in Eynard-Orantin recursion. The limit of this process is a rank 2 Higgs bundle on an infinite-dimensional manifold.

To start the process, let $(\mathcal{E}_1, \phi_1)$ be the $\mathcal{L}$-twisted Higgs bundle on X given by $\mathcal{E}_1 = \mathcal{O} \oplus \mathcal{L}$ with Higgs field

$$\phi_1 = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} : \mathcal{E}_1 \to \mathcal{E}_1 \otimes \mathcal{L}. \tag{7.2.1}$$

The characteristic equation of the Higgs bundle defines a spectral curve S is given by

$$S = \{\eta_1^2 - \pi_1^*(\alpha) = 0\}, \tag{7.2.2}$$

where $\pi_1 : \mathrm{Tot}(\mathcal{L}) \to X$, and $\eta_1 \in \Gamma(\pi_1^*\mathcal{L})$ is the tautological section. On the spectral curve S, we can define another twisted Higgs bundle by $\mathcal{E}_2 = \mathcal{O}_S \oplus \pi_1^*\mathcal{L}|_S$ with Higgs field

$$\phi_2 = \begin{bmatrix} 0 & \eta_1^2 + \pi_1^*\alpha \\ 1_{\pi_1^*\mathcal{L}|_S} & 0 \end{bmatrix} : \mathcal{E}_2 \to \mathcal{E}_2 \otimes \pi_1^*\mathcal{L}. \tag{7.2.3}$$

Again, this will produce a spectral curve defined by the equation

$$\eta_2^2 + \pi_2^*\big(\eta_1^2 + \pi_1^*(\alpha)\big) = 0, \tag{7.2.4}$$

where $\pi_2 : \mathrm{Tot}(\pi_1^*\mathcal{L}) \to X$, and $\eta_2 \in \Gamma(\pi_2^*\pi_1^*\mathcal{L})$ is the tautological section, and η_1 is being viewed as a coordinate on $S \subset \mathrm{Tot}(\mathcal{L})$. We can continue this process, each time defining a new rank 2 twisted Higgs bundle *branched precisely over the spectral cover of the previous Higgs bundle.* At the kth step of this process, we will have the Higgs bundle $\mathcal{E}_k = \mathcal{O}_{S^{k-1}} \oplus (\pi^*)^{k-1}\mathcal{L}|_{S^{k-1}}$ with Higgs field

$$\phi_k = \begin{bmatrix} 0 & \eta_{k-1}^2 + \pi^*\big(\eta_{k-2}^2 + \pi^*(\cdots + \pi^*\alpha)\big) \\ 1_{(\pi^*)^{k-1}\mathcal{L}} & 0 \end{bmatrix} : \mathcal{E}_k \to \mathcal{E}_k \otimes \pi_{k-1}^* \dots \pi_1^*\mathcal{L}, \tag{7.2.5}$$

where S^k is the k-fold Cartesian product of S with itself.

We may regard the formal limit $(\mathcal{E}_\infty, \phi_\infty)$ as a rank 2 Higgs bundle of Hitchin section-type on the formal variety S^∞, which is an infinite-dimensional product of algebraic curves. Its spectrum is a double cover of S^∞ branched over the series in infinitely many variables corresponding to the upper-right corner of ϕ_∞. If we turn on an $\hbar$-deformation of the holomorphic structure on $\mathcal{E}_\infty$ — it is worth noting that there are now infinitely many such deformations that are linearly independent — then ϕ_∞ deforms into a Hitchin oper, $\nabla_\hbar^\infty$ on S^∞, which induces an Airy-Schrödinger equation whose potential is a series in infinitely many variables. We expect that the various $\hbar$ directions give rise to

a hierarchy of equations, in the same spirit as the KdV hierarchy, that move through non-trivial bundle types and terminate with the oper on the trivial bundle over S^∞.

We want to understand the relationship between this limiting Higgs bundle, our proposed tautological recursion, and solutions to Schrödinger equations in infinitely many variables. This may be related to other scenarios involving infinitely many variables, such as Toda flows in infinitely many variables, cf. [27].[1]

7.3 Quantum Airy structures

The stable Eynard-Orantin differentials, $W_{g,n}$ with $2g+n-1>1$, are symmetric, meaning that they live in some symmetric power $Sym^n(V)$ of an infinite vector space V of meromorphic forms. The initial conditions of topological recursion, $W_{0,1}$ and $W_{0,2}$, are special among the differentials. The form $W_{0,1}$ only appears in defining the recursion kernel K and is given canonically by the tautological section η, and the differential $W_{0,2}$ is not a true symmetric differential. Quantum Airy structures, formulated by Kontsevich-Soibleman [70], and studied in [4, 17, 25, 40], are generalization of Eynard-Orantin topological recursion where the initial data of the recursion procedure is replaced by four tensors, (A, B, C, D), defined on various tensor powers of a vector space V, where A is roughly $W_{0,3}$, D is $W_{1,1}$, and B and C can be computed recursively, such that all elements of the recursion naturally live in some $Sym^n(V)$. The quantum Airy approach encodes the same information as Eynard-Orantin topological recursion, in particular, the data of a spectral curve can be used to produce a corresponding quantum Airy structure. From this new perspective, topological recursion does not rely on a spectral curve, but rather the vector space V, which can be taken as finite or infinite-dimensional. Topological recursion now acts as a case of deformation quantization, giving a wave function of a deformation quantization module corresponding to a quadratic Lagrangian manifold (possibly singular) in $V \oplus V^*$.

The motivation for the definition of a quantum Airy structure begins with a classical Airy structure. A classical Airy structure is given by only three tensors (A, B, C). It is shown in [4, 70] to be equivalent to the Lagrangian variety mentioned above.

Let V be an n-dimensional vector space over $\mathbb{C}$ with ordered basis $y_1, \ldots, y_n$. On the dual space V^* denote by $x_1, \ldots, x_n$ the dual coordinates. The vector space $W := T^*V^* = V \oplus V^*$ has a symplectic structure whose corresponding Poisson bracket is given by

$$\{y_i, x_j\} = \delta_{ij}$$
$$\{x_i, x_j\} = 0$$
$$\{y_i . y_j\} = 0$$

in coordinates on W. Denote $Sym_{\leq 2}W := Sym^0 W \oplus Sym^1(W) \oplus Sym^2(W)$ the Lie algebra of polynomial function on W^* of degree ≤ 2 with the Lie algebra structure induced by the Poisson bracket.

[1] The invocation of Toda is not random as there are relevant connections to certain Schrödinger problems. An intermediary is the Riccati equation for the Stieltjes function of the orthogonal polynomials [95], which plays a fundamental role in the Bouchard-Eynard reconstruction of WKB solutions.

Definition 7.3.1. *A **classical Airy structure** on V is a collection of polynomials $H_i \in Sym_{\leq 2}W$ of the form*

$$H_i = y_i - \frac{1}{2}\sum_{ab} A^i_{ab} x_a x_b - \sum_{ab} B^i_{ab} x_a y_b - \frac{1}{2}\sum_{ab} C^i_{ab} y_a y_b \tag{7.3.1}$$

such that $\oplus_{1\leq i\leq n}\mathbb{C}\cdot H_i$ is Lie subalgebra of $Sym^2(W)$, i.e.

$$\{H_i, H_j\} = \sum_a g^a_{ij} H_a \tag{7.3.2}$$

In a coordinate-free form, the coefficients can be defined by three tensors:

1. $A \in \mathrm{Hom}(V^{\otimes 3}, \mathbb{C})$ defined by $A(e_i \otimes e_j \otimes e_k) = A^i_{jk}$,
2. $B \in \mathrm{Hom}(V^{\otimes 2}, V)$ defined by $A(e_i \otimes e_j) = B^i_{ja} e_a$,
3. $C \in \mathrm{Hom}(V, V^{\otimes 2})$ defined by $C(e_i) = C^i_{ab} e_a \otimes e_b$,

subject to constraints imposed by (7.3.2).

Let V be a vector space over $\mathbb{C}$ with basis $(e_i)_{i\in I}$ and dual basis $(x_i)_{i\in I}$. Consider the Weyl algebra

$$\mathcal{W}^\hbar_V = \mathbb{C}[\hbar]\langle (x_i, \partial_i)_{i\in I}\rangle / \langle [\partial_i, x_i] = \hbar\rangle, \tag{7.3.3}$$

where $\partial_i = \frac{\partial}{\partial x_i}$.

Definition 7.3.2. *A **quantum Airy structure** on V is a sequence $L = (L_i)_{i\in I}$ of elements of $\mathcal{W}^\hbar_V$ of the form*

$$L_i = \hbar\partial_i - \frac{1}{2}\sum_{ab} A^i_{ab} x_a x_b - \hbar\sum_{ab} B^i_{ab} x_a \partial_b - \frac{\hbar^2}{2}\sum_{ab} 2C^i_{ab}\partial_a\partial_b - \hbar D^i \tag{7.3.4}$$

where $\hbar$ is a formal parameter, $A^i_{jk}, B^i_{jk}, C^i_{jk}$ and D^i are scalars, such that the collection of L_i spans a Lie subalgebra of $\mathcal{W}^\hbar_V$, i.e.

$$[L_i, L_j] = \hbar\sum_a f^a_{ij} L_a. \tag{7.3.5}$$

In analogy to classical Airy structures, the coefficients can be defined in a coordinate-free form by four tensors:

1. $A \in \mathrm{Hom}(V^{\otimes 3}, \mathbb{C})$ defined by $A(e_i \otimes e_j \otimes e_k) = A^i_{jk}$,
2. $B \in \mathrm{Hom}(V^{\otimes 2}, V)$ defined by $A(e_i \otimes e_j) = B^i_{ja} e_a$,
3. $C \in \mathrm{Hom}(V, V^{\otimes 2})$ defined by $C(e_i) = C^i_{ab} e_a \otimes e_b$,
4. $D \in \mathrm{Hom}(V, \mathbb{C})$ defined by $(e_i) = D^i$.

The Lie algebra condition (7.3.5) imposes strong constraints on (A, B, C, D), which happen to be necessary and sufficient conditions for a sequence L of the form (7.3.4) to be a quantum Airy structure. Contained in these constraints are a set of constraints on (A, B, C), which coincide with the constraints imposed on a classical Airy structure by (7.3.2). This means that a quantum Airy structure is a classical Airy structure together with D and some additional constraints.

The differentials L_i act on V as differential operators. The Lie algebra condition is a sufficient condition for the existence of a function Z on V which is a common solution to $L_i Z = 0$. Such a Z is uniquely given by the formal series

$$Z = \exp\Big(\sum_{\substack{g\geq 0\\ n\geq 1}} \frac{\hbar^{g-1}}{n!} \sum_{i_1,\dots,i_n\in I} F_{g,n}(i_1,\dots,i_n)x_{i_1}\dots x_{i_n}\Big) \tag{7.3.6}$$

where the $F_{g.n}$ are scalars that are invariant under permutation of the i_k such that $F_{0,1}(i) = F_{0,2} = 0$ for all i, j. The coefficients are given in terms of (A, B, C, D) by

$$F_{0,3}(i,j,k) = A^i_{jk}, \tag{7.3.7}$$

$$F_{1,1}(i) = D^i, \tag{7.3.8}$$

and for $2g - 2 + n \geq 2$, they are defined recursively by

$$\begin{aligned} F_{g,n}(i_1,...,i_n) &= \sum_{m=2}^{n} B^{i_1}_{i_m a} F_{g,n-1}(a, i_2, ..., \hat{i_m}, ..., i_n) \\ &+ \frac{1}{2} C^{i_1}_{ab}\Big(F_{g-1,n+2}(a,b,i_2,...,i_n) + \sum_{\substack{g_1+g_2=g\\ J_1\cup J_2=\{k_1,\dots,k_n\}}} F_{g_1,|J_1|+1}(a,J_1)F_{g_2,|J_2|+1}(b,J_2)\Big). \end{aligned} \tag{7.3.9}$$

It is shown in [38], that the $W_{g,n}$ can be decomposed over a family of one-forms with some coefficients $\omega_{g,n}$. The connection between topological recursion and quantum Airy structures is that the $F_{g,n}$ produced by a quantum Airy structure as above are the same as the coefficients $\omega_{g,n}$ arising from the Eynard-Orantin differentials.

Quantum Airy structures have already shown themselves to be a generalization of topological recursion. We would like to investigate how (or if) $\mathcal{L}$-twisted topological recursion fits into the current framework of quantum Airy structures. There are also two other natural directions to investigate in regards to quantum Airy structures. The first is asking how this picture generalizes when the initial data is a vector bundle instead of a vector space. The second is to allow the L_i to be differential operators of degree ≥ 2. The latter has been studied in [17] in the context of vector spaces, but we wish to simultaneously extend the set up to vector bundles and to a $\mathbb{Z}$-grading, thereby transforming a truncation of a Fock space into a complete $\mathbb{Z}$-graded Fock sheaf.

7.4 Topological materials

Perhaps the most ambitious direction concerns applications of topological recursion and Higgs bundles to condensed matter physics. Higgs bundles have enjoyed numerous applications to classical integrable systems [31,65,67] as well as high energy physics via string theory and mirror symmetry [5,62]. Recent work has related the theory of topological materials, specifically the band structure, to Higgs bundles [68].

The theory of topological materials is studied via electric band structure (*cf.* for instance [51,97]). Electrons are arranged in a real d-dimensional lattice, to which is associated a reciprocal lattice in momentum space. The system is described by a Bloch Hamiltonian $H(k)$, a periodic Hermitian operator in momentum space defined on the *Brillouin zone*, the primitive cell of the momentum lattice, acting on a Hilbert space $\mathcal{H}_k$. The evolution of the eigen-energies

$E_n(k)$ of the Bloch Hamiltonian defines the band structure (see Figure 7.2). The existence of a gap between energy bands dictates whether the material is an insulator or a metal. Electron states of a gap system cannot be excited by a small perturbation of the system, such as the introduction of an electric field, and thus no current can be created. Classically, these are classified by the symmetries of the system. However, it has become evident that the topological information of the band structure leads to a more robust breadth of materials: topological insulators, semi-metals, topological superconductors, etc.

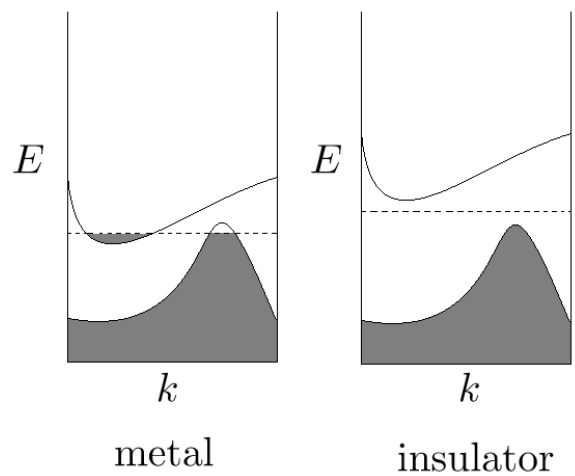

Figure 7.2: Band structure of metal and insulator.

The mathematics of such a quantum material can be roughly broken down as follows: The periodicity of $H(k)$ gives the Brillouin zone (BZ) the topology of a d-dimensional torus T^d. For each $k \in BZ$, the Hamiltonian is acting on a Hilbert space $\mathcal{H}_k$. The collection of spaces $\mathcal{H}_k$ forms a finite rank "Hilbert bundle" $\mathcal{H}$ on the base space BZ. In this setting, the Hamiltonian $H(k)$ acts as a Higgs field for the bundle, and together, they form what we might term a "Hilbert-Higgs" bundle over BZ. The energy functions $E_n(k)$ are the spectral data of $H(k)$, that is to say, the spectral curve and the corresponding eigenvectors form the spectral line bundle. The question of topological information of the band structure is now asking for topological properties of the spectral line bundle, such as its degree.

Advancements in the theory of topological materials, both experimental [69] and theoretical [68, 73, 74, 79] have led to the notion of hyperbolic band theory. This is a generalization of band theory, where the classic Euclidean Brouillon zone is replaced with representations of a hyperbolic crystal lattice. In [68], it is shown that there are two Higgs bundles associated to hyperbolic band theory: one arising as the moduli space of crystal lattices, the other as parametrizing the complex crystal momenta. The simple case where these structures are applied to 2D Euclidean crystals admits a meromorphic Higgs bundle. We can apply the framework from *Chapter 5* to these meromorphic Higgs fields to produce a quantum curve. A natural question is how such a quantum curve relates to the original Hamiltonian. If they coincide, then this procedure could potentially provide a means of reconstructing a physical lattice model for a particular Schrödinger equation from the information of a (complex) Brillouin zone in momentum space. If they do not coincide, then what is the physical meaning of the quantum curve for the given material?

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