Multiresolution Analysis
on
Non-abelian Locally Compact Groups

A Thesis Submitted to the College of Graduate Studies and Research

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

In the Department of Mathematics

of the University of Saskatchewan

By

Qingde Yang

Saskatoon, Canada

Fall, 1999

0-612-43523-7

Canada

UNIVERSITY OF SASKATCHEWAN

College of Graduate Studies and Research

SUMMARY OF DISSERTATION

Submitted in partial fulfillment

of the requirements for the

DEGREE OF DOCTOR OF PHILOSOPHY

By

Qingde Yang

Department of Mathematics & Statistics
University of Saskatchewan

Fall 1999

Examining Committee:

Dr. G. Entwistle   X̶P̶e̶a̶n̶/̶A̶s̶s̶o̶c̶i̶a̶t̶e̶X̶D̶e̶a̶n̶/Dean's Designate, Chair

College of Graduate Studies and Research

Dr.M.Bickis   Chair, Department of Mathematics & Statistics

Dr.K.F.Taylor   Supervisor, Department of Mathematics Statistics

Dr.M.Keil   Department of Computer Science

Dr.M.Khoshkam   Department of Mathematics & Statistics

Dr.M.Marshall   Department of Mathematics & Statistics

Dr.G.Patrick   Department of Mathematics & Statistics

External Examiner:

Dr.L.Baggett Department of Mathematics

University of Colorado at Boulder

Campus Box 395

Boulder, CO 80309-0395, USA

ABSTRACT

Wavelets form a relatively new topic in mathematics. Multiresolution analysis(MRA) is an important mathematical tool because it provides a natural framework for understanding and constructing wavelets. In this thesis, we extend MRA to the setting of non-abelian locally compact groups. The main contributions of the thesis are the following:

- We create a new term, scalable, for a special class of groups. MRA can only be set up for the class of scalable groups. We approximately identify the class of scalable groups out of second countable, type I, unimodular locally compact groups.

- For a scalable group G, we formulate the definition of MRA for $L^2(G)$ by using the information exposed from the MRA of $L^2(\mathbf{R}^d)$. There are three things in MRA that mainly concerned us; that is, the density of the union, the triviality of the intersection of the nested sequence of closed subspaces and the existence of refinable functions. The intersection triviality property is derived from the other conditions of MRA. To get the union density property, we have to generalize the concept of the support of the Fourier transform. The new concepts, such as "strongly supported", left nonzero divisor in $L^2(G)$, and automorphism-absorbing subset of $\widehat{G}$, arise in this generalization. As to refinability, it depends very much on the individual function ϕ. We prove that refinable functions are present for general scalable groups as long as self-similar tiles are present.

- We provide a very interesting concrete example for our theory using Heisenberg groups. We prove the existence of scaling functions for the Heisenberg groups. These scaling functions are related to certain self-similar tilings of $\mathbf{H}^d$. The corresponding scaling functions are characteristic functions of appropriate sets. We generalize the construction of Strichartz's self-similar tiles to a more general case. We also obtain a theorem which says that there are $2^{(2d+2)}-1$ orthonormal wavelets for Heisenberg groups.

i

ACKNOWLEDGEMENTS

Contents

Chapter 1

Overview of the thesis

Wavelet analysis is one of the rapidly developing areas in the mathematical sciences. The main aim of the theory is to find nice ways to break down a given function into elementary building blocks. Historically, the Haar basis, constructed in 1910 long before the term "wavelet" was created, was the first orthonormal wavelet basis in $L^2(\mathbf{R})$. But it was only recently discovered that the construction works because of an underlying multiresolution analysis structure. In the early 80's, Strömberg [Strö] discovered the first continuous orthogonal wavelets. His wavelets had exponential decay and were in C^k (k arbitrary but finite). The next construction, independent of Strömberg, was of the Meyer wavelets [Me1]. The images of the Meyer wavelets under the Fourier transform were compactly supported and were in C^k (k arbitrary, may be ∞). Then Battle [Ba] and Lemarié [Le] used very different methods to construct their own orthonormal wavelet bases with exponentially decaying properties. With the notion of multiresolution analysis, introduced by Mallat [Mal2] and Meyer [Me2], a systematic framework for understanding these orthogonal expansions was developed, see [Mal1], [Mal2], [Mal3] and [Me2] for details. This framework gave a satisfactory explanation for all these constructions, and provided a tool for the construction of other bases. Thus, multiresolution analysis is an important mathematical tool to understand and construct a wavelet basis of $L^2(\mathbf{R}^d)$, i.e., a basis that consists of the scaled and integer translated versions of a finite number of functions. In recent years,

multiresolution analysis for the Euclidean group $\mathbf{R}^d$ has received extensive investigation; see [BD],[Dau],[JS],[Ma],[Mal2],[Me2] and [St] for example. Dahlke [Da] extended multiresolution analysis to abelian locally compact groups. Baggett, et al. [BC] considered the existence of wavelets in general Hilbert space based on the formulation of multiresolution analysis by using an abstract approach.

In this thesis, we plan on extending multiresolution analysis to the setting of non-abelian locally compact groups. Our motivation for the development of multiresolution analysis for non-abelian groups comes from Heisenberg groups. The function space $L^2(\mathbf{R}^{2d+1})$ can be identified with $L^2(\mathbf{H}^d)$, where $\mathbf{H}^d$ is the $2d + 1$-dimensional Heisenberg group. The group product of $\mathbf{H}^d$ leads to an alternative, but still natural "translation" on $L^2(\mathbf{R}^{2d+1})$. This thesis can be seen as an initial step towards wavelet theory of non-abelian groups. The main contributions of the thesis are the following:

(i) We create a new term, called scalable groups, for a special class of groups. Multiresolution analysis can only be set up for the class of scalable groups. We approximately identify the class of scalable groups out of second countable, type I, unimodular locally compact groups.

(ii) For a scalable group G, we formulate the definition of multiresolution analysis for $L^2(G)$ by using the information exposed from the multiresolution analysis of $L^2(\mathbf{R}^d)$. Generally speaking, there are two things in multiresolution analysis that mainly concerned us, that is, the density of the union and the triviality of the intersection of the nested sequence of closed subspaces. We set up the union density and intersection triviality theorems and other related things. We answer the basic question: under what conditions does a function ϕ generate a multiresolution analysis for $L^2(G)$.

(iii) A multiresolution analysis on Heisenberg groups is set up. We investigate the existence of scaling functions for the Heisenberg groups. These scaling functions are related to certain self-similar tilings of $\mathbf{H}^d$, that is, the corresponding scaling functions are characteristic functions of appropriate sets. We also obtain a theorem which says that there are $2^{(2d+2)}-1$ orthonormal wavelets for Heisenberg groups.

2

This thesis is organized into five chapters and one appendix.

In chapter 2, we describe the notions from wavelets and multiresolution analysis that are needed to understand the remainder of this thesis. We start from introducing the Haar basis of $L^2(\mathbf{R})$. The reason that we choose the Haar basis for illustration is its simplicity and there is a beautiful pattern of multiresolution analysis hidden in it. The properties obtained through analyzing the Haar basis for $L^2(\mathbf{R})$ leads us to formulate the definition of multiresolution analysis for the more general space $L^2(\mathbf{R}^d)$, where $\mathbf{R}^d$ is d-dimensional Euclidean space.

In chapter 3, we provide some background for the analysis to be presented in subsequent chapters. The key information for later use is the abstract Fourier transform on the class of second countable, type I, unimodular locally compact groups. We quickly go through the basic concepts and results centered around abstract Fourier analysis on second countable, type I, unimodular locally compact groups. This material is, for the most part, available from Folland's book [Fo1]. This chapter also serves to establish our main notations.

In chapter 4, we address two main topics. The first topic concerns how the definition of multiresolution analysis of $L^2(\mathbf{R}^d)$ can be generalized to $L^2(G)$, where G is a suitable second countable, type I, unimodular locally compact group. The second topic is to answer the basic question, that is, how can a function ϕ produce a multiresolution analysis of $L^2(G)$. We shall consider two basic issues: the union density and intersection triviality properties. Normally, the intersection triviality is less important because it is the consequence of the other conditions. We shall give a necessary and sufficient condition under which the nested sequence of subspaces generated by ϕ satisfies the union density. Also, we develop a number of conditions on a refinable function ϕ that are sufficient for the union of the associated nested sequence of subspaces to be dense in $L^2(G)$.

Let us describe the results in this thesis in more detail.

We start by first investigating the definition of multiresolution analysis of $L^2(\mathbf{R}^d)$ and finding some key points in the definition by properly interpreting it with a more general point of view. The definition of multiresolution analysis of $L^2(\mathbf{R}^d)$ is as follows:

Definition. *A multiresolution analysis of $L^2(\mathbf{R}^d)$ consists of a sequence of closed linear subspaces V_j, $j \in \mathbf{Z}$, of $L^2(\mathbf{R}^d)$ with the following properties:*

(i) $V_j \subset V_{j+1}$, $j \in \mathbf{Z}$;

(ii) $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$;

(iii) $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$;

(iv) $f \in V_j \iff U_D f \in V_{j+1}$. In other words, $V_j = U_D^j V_0$, $j \in \mathbf{Z}$, where D is the dilation matrix;

(v) V_0 is assumed to be shift-invariant, that is, if $f \in V_0$ then so is $T_k f$ for all k in $\mathbf{Z}^d$, where

$$T_x f(\cdot) := f(\cdot - x), \quad \forall f \in L^2(\mathbf{R}^d);$$

(vi) there is a function $\phi \in V_0$, called the scaling function, or the generator of the multiresolution analysis, such that the collection $\{T_k \phi \mid k \in \mathbf{Z}^d\}$ is a orthonormal basis of V_0.

In this definition, D is a matrix with integer entries, called a dilation matrix, it satisfies the following conditions:

- D leaves $\mathbf{Z}^d$ invariant. In other words, $D\mathbf{Z}^d \subset \mathbf{Z}^d$, where
$$DZ^d = \{\, y \mid y = Dx \text{ and } x \in \mathbf{Z}^d \,\}$$
- All the eigenvalues, λ_i, of D satisfy $|\lambda_i| > 1$

Such a D induces a unitary operator $U_D : f \longmapsto U_D f$ on $L^2(\mathbf{R}^d)$, defined by

$$U_D f(\cdot) = \delta_D^{1/2} f(D \cdot),$$

where $\delta_D = |\det(D)|$.

Notice that $\mathbf{Z}^d$ is a lattice subgroup of $\mathbf{R}^d$ and $\mathbf{R}^d/\mathbf{Z}^d \cong \mathbf{T}^d$, where $\mathbf{T}^d$ is the d-torus. The shift-invariance of V_0 can be interpreted as an invariant property with respect to the action of the discrete lattice subgroup $\mathbf{Z}^d$ of $\mathbf{R}^d$. The scaling matrix D can be viewed as the action of some group automorphism of $\mathbf{R}^d$, with the property $D\mathbf{Z}^d \subset \mathbf{Z}^d$ and $1 < [\mathbf{Z}^d : D\mathbf{Z}^d] < \infty$. Also, we have to observe one very special thing that the lattices $D^{-j}\mathbf{Z}^d$ in $\mathbf{R}^d$ form a nested sequence whose union $\bigcup_{j \in \mathbf{Z}} D^{-j}\mathbf{Z}^d$ is dense in $\mathbf{R}^d$. Actually, "approximation to $L^2(\mathbf{R}^d)$

4

by the nested sequence of closed subspaces $\{\ V_j\ \}_{j\in\mathbf{Z}}$ imitates and reflects the geometric approximation to $\mathbf{R}^d$ by the nested sequence of lattices $2^{-j}\mathbf{Z}^{d_n}$ (see p.69 [Me2] for details). Furthermore, we have to note the roles played by the dilation operator D and the translation operator T_x.

With this in mind, we extend the multiresolution analysis for $L^2(\mathbf{R}^d)$ to the multiresolution analysis for $L^2(G)$, where G is a second countable, type I, unimodular locally compact group. (a) First, we shall suppose that G contains a discrete subgroup Γ such that the quotient G/Γ is compact, where Γ is discrete means that the topology on Γ induced from G is the discrete topology. It is worthwhile noting that for connected nilpotent Lie groups, such a discrete lattice subgroup Γ of G often exists, see [Ra] for details. This Γ will play the same role in G as $\mathbf{Z}^d$ in $\mathbf{R}^d$. (b) Furthermore, we shall assume that there exists a dilative topological automorphism α (hence α^{-1} is contractive) of G onto G such that $\alpha\Gamma\subset\Gamma$ and $1 < [\Gamma : \alpha\Gamma] < \infty$, where α is a topological automorphism means that α is an automorphism and a homeomorphism and α^{-1} is contractive means that for any fixed compact subset K of G and for any neighborhood U of the identity e, there is a positive integer N, depending on K and U, such that

$$\alpha^{-j}K\subset U, \ \forall j > N.$$

A second countable, type I, unimodular locally compact group G with such a lattice Γ and compatible dilative automorphism α will be called a scalable group. The most important consequence of the above assumptions is that the union $\bigcup_{j\in\mathbf{Z}} \alpha^{-j}\Gamma$ is dense in G. This is Proposition 4.6 in this thesis:

Proposition 4.6 *Let G be a locally compact group. Suppose that G contains a discrete countable subgroup Γ such that the quotient G/Γ is compact and that there exists a dilative topological automorphism α of G onto G such that $\alpha\Gamma\subset\Gamma$ and $1 < [\Gamma : \alpha\Gamma] < \infty$. Then the union $\bigcup_{j\in\mathbf{Z}} \alpha^{-j}\Gamma$ is dense in G.*

We observe that the image under the Fourier transform of either a left shift-invariant subspace generated by ϕ or a left and right shift-invariant subspace generated by the same ϕ is both supported on the same subset $\text{supp}\mathcal{F}(\phi)$ in $\widehat{G}$. This fact suggests that we may

5

consider only one-sided translations for the definition of multiresolution analysis, either left translations or right translations. Let's fix left ones without loss of generality.

Now we can give a definition of multiresolution analysis for $L^2(G)$.

Definition 4.8 *Let G be a scalable group. We say that a sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(G)$ forms a multiresolution of $L^2(G)$ if the following conditions are satisfied:*

(i) $V_j\subset V_{j+1}$, $j\in\mathbf{Z}$;

(ii) $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(G)$;

(iii) $\bigcap_{j\in\mathbf{Z}}V_j = \{0\}$;

(iv) $f\in V_j \iff \sigma f\in V_{j+1}$. In other words, $V_j = \sigma^j V_0$, $j\in\mathbf{Z}$;

(v) V_0 is left shift-invariant, that is, if $f\in V_0$ then so is $L_\gamma f$ for all γ in Γ;

(vi) there is a function $\phi\in V_0$, called the scaling function, or generator of the multiresolution analysis, such that the collection $\{L_\gamma\phi \mid \gamma\in\Gamma\}$ is a orthonormal basis for V_0.

In the definition above, the dilation operator σ is defined by

$$\sigma f(x) := \delta_\alpha^{1/2} f(\alpha x), \quad \forall f\in L^2(G),\ x\in G,$$

where δ_α is a proper positive constant depending on α such that the operator σ becomes a unitary operator. And L_x is the left translation operator given by $L_x f(y) = f(x^{-1}y)$.

Since we only use left side translations in the definition of multiresolution analysis, to get the results similar to abelian cases, we have to put an extra condition on the scaling function ϕ. It is found that this extra requirement on ϕ is very natural from the explanation given in section 4 of chapter 4.

We say that ϕ is strongly supported on $\Omega\subseteq\widehat{G}$ if it satisfies the following condition: for any $F\in\mathcal{H}^2(\Omega)$,

$$\pi(x)\hat\phi(\pi)F(\pi) = 0 \text{ for all } x\in G \text{ and almost all } \pi\in\widehat{G} \text{ implies } F = 0,$$

where $\mathcal{H}^2(\Omega)$ stands for a Hilbert space that is introduced in chapter 3. Or equivalently, ϕ is strongly supported on Ω if, for any $F\in\mathcal{H}^2(\Omega)$, $< \hat{x\phi}|F > = 0$, for all $x\in G$ implies $F = 0$. This is equivalent to saying that the image under the Fourier transform of the left translation invariant subspace generated by ϕ is dense in $\mathcal{H}^2(\Omega)$. Similarly, we can define

6

the term "strongly supported" for a sequence of functions. We say that $\{\phi_i\}_{i\in I}$ is strongly supported on Ω if, for any $F\in\mathcal{H}^2(\Omega)$, $< \hat{x}\hat{\phi_i}|F >= 0$, for all $x\in G$ and $i\in I$ implies $F = 0$.

To help understand the idea, let's consider one particular case where $G = \mathbf{R}$ (hence $\widehat{G}$ can be identified with $\mathbf{R}$ as well) and supp$(\hat{\phi})=\mathbf{R}$. For convenience, assume that $\phi\in L^1(\mathbf{R})$. So $\hat{\phi}$ is a continuous function. Then ϕ is strongly supported on any non-null subset of $\mathbf{R}$. The fact that supp$(\hat{\phi})$ fills out $\mathbf{R}$ implies that the translation invariant closed subspace generated by ϕ must be the whole space $L^2(\mathbf{R})$. Or in other words, $\{e_t\hat{\phi} : t\in\mathbf{R}\}$ is a total set in $L^2(\mathbf{R})$, where $e_t(\gamma) = e^{2\pi i t\gamma}$ for $\gamma\in\mathbf{R}$, $t\in\mathbf{R}$.

Notice that the strong support concept is defined from the image of ϕ under the Fourier transform. Now if we interpret this fact from the other side without using the Fourier transform, then we find that: ϕ is strongly supported on the whole dual space $\widehat{G}$ if and only if, for any $f\in L^2(G)$, $\phi*f = 0$ implies $f = 0$. This can be easily checked by using the Fourier transform and the identity $(f*g)^\wedge(\pi) = \hat{f}(\pi)\hat{g}(\pi)$.

For a family $\{\phi_i\}_{i\in I}$ of functions in $L^2(G)$, we say that $\{\phi_i\}_{i\in I}$ is a left nonzero divisor in $L^2(G)$ if, for any $f\in L^2(G)$, $\phi_i*f=0$, for all $i\in I$, implies $f = 0$.

After introducing this term, we obtain the following theorems on the triviality of intersection and the density of the union.

Theorem 4.9(triviality of intersection). *Let ϕ be a refinable element of $L^2(G)$ and define $V_0 = \mathcal{V}(\phi)$ and $V_j = \sigma^j V_0$ for $j\in\mathbf{Z}$. Suppose that left shifts of ϕ, that is, $\{L_\gamma\phi \mid \gamma\in\Gamma\}$, constitutes a frame for V_0, then $\bigcap_{j\in\mathbf{Z}}V_j = \{0\}$.*

Theorem 4.11(density of union). *Let $V_0 = \mathcal{V}(\phi)$ be the left shift-invariant subspace generated by a refinable function $\phi\in L^2(G)$, and let V_j be the σ^j-dilate of V_0 for $j\in\mathbf{Z}$. Define $\phi_j(\cdot) := \sigma^j\phi(\cdot) = \delta_\alpha^{j/2}\phi(\alpha^j\cdot)$. Then the following are equivalent:*

(a) $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(G)$

(b) $\{\phi_j\}_{j\in\mathbf{Z}}$ is a left nonzero divisor in $L^2(G)$

(c) $\{\phi_j\}_{j\in\mathbf{Z}}$ is strongly supported on all of $\widehat{G}$

We also give some easily verified conditions on ϕ that imply $\{\phi_j\}_{j\in\mathbf{Z}}$ is a left nonzero divisor in $L^2(G)$ or is strongly supported on $\widehat{G}$. A measurable subset Ω of $\widehat{G}$ is called

α-absorbing if $\mu(\widehat{G}\backslash\bigcup_{j\in\mathbf{Z}}\alpha^j(\Omega)) = 0$.

We have a theorem as follows.

Theorem 4.19. *Let ϕ be a refinable function in $L^2(G)$. Let $\phi_j = \sigma^j\phi$. Similarly, let $V_0 = \mathcal{V}(\phi)$ and $V_j = \sigma^j V_0$. If ϕ satisfies either of the following conditions*

(i) ϕ is strongly supported on an α-absorbing subset Ω of $\widehat{G}$ and $\widehat{\phi}(\pi) = 0$, for almost all $\pi\in\widehat{G}\backslash\Omega$,

(ii) ϕ has compact support in G, $\phi{\geq}0$ and $\phi{\neq}0$,

then $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(G)$.

In addition, we obtain the following density of union result with respect to a special case where the scaling function ϕ is a characteristic function of some self-similar tile for the group G and discrete subgroup Γ. As for the notion of self-similar tiling, see details in chapter 4.

Theorem 4.20(density of union for a special case) *Let ϕ be a scaling function which is the characteristic function of some self-similar tile T. Let $\phi = \frac{1}{|T|^{1/2}}\chi_T$, where $|T|$ is the Haar measure of T. Finally, let $\{\ V_j\ \}_{j\in\mathbf{Z}}$ be as above with ϕ. Then ϕ generates a multiresolution analysis for $L^2(G)$.*

In chapter 5, we have some interesting topics. This chapter can serve as a concrete example of our theory. Our main contributions are: (i) For the Heisenberg group $\mathbf{H}^d$, we set up a multiresolution analysis on $\mathbf{H}^d$ by applying the theory developed in Chapter 4. (ii) We also investigate the existence of scaling functions for the Heisenberg groups. These scaling functions are related to certain self-similar tilings of $\mathbf{H}^d$. We extend the result obtained by Strichartz to a more general case. In addition, we obtain a theorem on Heisenberg groups which says that there exists $2^{(2d+2)}$-1 orthonormal wavelets for $L^2(\mathbf{H}^d)$. This means that we have an orthonormal basis for $L^2(\mathbf{H}^d)$ that consists of the dilated and translated versions of $2^{(2d+2)}$-1 functions.

Checking the definition of multiresolution analysis for $L^2(G)$ in section 3 of chapter 4, to build a multiresolution analysis on $\mathbf{H}^d$ we have to find those objects such as a lattice subgroup, left translation operators and dilation operator. In analogy with the role of $\mathbf{Z}^d$

in the multiresolution analysis of $L^2(\mathbf{R}^d)$, we choose the following lattice subgroup Γ of $\mathbf{H}^d$ which plays the same role in $\mathbf{H}^d$ as $\mathbf{Z}^d$ plays in $\mathbf{R}^d$:

$$\Gamma = \{ \ (l/2,\underline{m},\underline{n}) \ | \ l\in\mathbf{Z}, \ \underline{m}, \ \underline{n}\in\mathbf{Z}^d \ \}.$$

It is easy to check that Γ forms a subgroup. Similarly, for $h\in\mathbf{H}^d$, we define the left translation operators L_h from $L^2(\mathbf{H}^d)$ to $L^2(\mathbf{H}^d)$. We also have the following similar terms such as left shift-invariant subspace and so on. It is easy to check that the map α from $\mathbf{H}^d$ to $\mathbf{H}^d$ given by

$$\alpha(t,\underline{q},\underline{p}) := (2^2 t, 2\underline{q}, 2\underline{p})$$

is an topological automorphism of $\mathbf{H}^d$ and α^{-1} satisfies the contractive property. Also, $\bigcup_{j\in\mathbf{Z}}\alpha^{-j}\Gamma$ is dense in $\mathbf{H}^d$. With this α, we form the following unitary operator from $L^2(\mathbf{H}^d)$ to $L^2(\mathbf{H}^d)$

$$\sigma: \ L^2(\mathbf{H}^d)\longrightarrow L^2(\mathbf{H}^d),$$

which is given explicitly by

$$\sigma f(t,\underline{q},\underline{p}) := 2^{d+1}f(\alpha(t,\underline{q},\underline{p})) := 2^{d+1}f(2^2t, 2\underline{q}, 2\underline{p}).$$

We say that V is refinable if, for any $f\in V$, $\sigma^{-1}f$ is also in V. It is straightforward to verify that $L^2(\mathbf{H}^d)$ is left shift-invariant and refinable. For $\phi\in L^2(\mathbf{H}^d)$, we denote $\mathcal{V}(\phi)$ to be the smallest closed left shift-invariant subspace of $L^2(\mathbf{H}^d)$ containing ϕ. We say that ϕ is refinable if $\mathcal{V}(\phi)$ is.

Let $V_0 = \mathcal{V}(\phi)$ and $V_j=\sigma^j V_0$ for $j\in\mathbf{Z}$. If ϕ is refinable, then $\{V_j | j\in\mathbf{Z}\}$ forms a nested sequence of closed subspaces of $L^2(\mathbf{H}^d)$. Since $\bigcup_{j\in\mathbf{Z}}\alpha^{-j}\Gamma$ is dense in $\mathbf{H}^d$, $V = \overline{\bigcup_{j\in\mathbf{Z}}V_j}$ is a closed left translation invariant subspace. By theorem 4.11 in chapter 4, $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(\mathbf{H}^d)$ if and only if $\{\widehat{\phi_j}\}_{j\in\mathbf{Z}}$ is strongly supported on $\widehat{\mathbf{H}^d}$. If, in addition, the left shifts of ϕ, that is, $\{L_\gamma\phi \ | \ \gamma\in\Gamma \ \}$, constitutes a orthonormal basis for V_0, then by the intersection triviality theorem 4.9, we have $\bigcap_{j\in\mathbf{Z}}V_j = \{0\}$. Therefore, $\{V_j\}_{j\in\mathbf{Z}}$ forms a multiresolution analysis of $L^2(\mathbf{H}^d)$.

The following is a very simple sufficient condition for $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(\mathbf{H}^d)$.

Theorem 5.1. *Let ϕ be a function in $L^2(\mathbf{H}^d)$, let V_j be the σ^j-dilate of $\mathcal{V}(\phi)$. Assume that*

$\{V_j\}_{j\in\mathbf{Z}}$ *is nested. Then* $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(\mathbf{H}^d)$ *if there exists a neighborhood* E *of 0 in* $\mathbf{R}$ *such that* ϕ *is strongly supported on* E *and* $\hat{\phi}(\lambda) = 0$ *for a.e.* λ *not in* E.

As we know that the starting point of constructing a multiresolution analysis is the scaling function ϕ. With this scaling function ϕ, we form V_0 by applying the left shift operator and then generate V_j by the dilation operator. We investigate some special scaling functions which are related to certain self-similar tilings of $\mathbf{H}^d$, that is, the corresponding scaling functions are characteristic functions of appropriate sets. Let's first consider the standard tiling of Euclidean space $\mathbf{R}^d$ by unit cubes. Each tile is a translate of a single tile by an element of the lattice subgroup $\mathbf{Z}^d$. If we dilate a tile by a factor 2, then the enlarged tile consists of 2^d original tiles. The generalization of the above leads in many cases to interesting self-similar tilings with fractal boundaries. For example, for two dimensions, the fractal twin dragon tiles are constructed in Gröchenig and W.R.Madych [GM]. As for Heisenberg groups, which have both dilations and lattice groups, in order to construct the analogue of the cubic-like tiling, it seems that the tiles with fractal boundaries are the only choice. Such self-similar tilings are present for the Heisenberg groups from the work by Strichartz [Str].

Based on the above notations, we can state the following theorem, which is a generalized version of Strichartz's theorem. For further information, please see §5.4.

Theorem 5.3. *For a properly chosen finite subset* Γ_0, *there exists a unique self-similar stacked tiling* $\bigcup_{\gamma\in\Gamma}\gamma T$ *for* $\mathbf{H}^d$. *The function* $F(\underline{x})$ *is given explicitly by*

$$F(\underline{x}) = \sum_{n=1}^{\infty} \frac{1}{r_\alpha^n} S([D_\alpha^n(\underline{x})]_A \, mod \, (D_\alpha(\mathbf{Z}^{2d})), < D_\alpha^n(\underline{x}) >_A)$$

where a lattice point $\underline{k}$ *mod* $(D_\alpha(\mathbf{Z}^{2d}))$ *equals the representative of the coset which contains element* $\underline{k}$.

We show, in §5.4, that the existence of self-similar tilings for Heisenberg groups implies the existence of scaling functions. Thus, we can form a multiresolution analysis for $L^2(\mathbf{H}^d)$ by using these self-similar scaling functions. On the other hand, Baggatt and et al. [BC] studied the relationship between the existence of an orthonormal wavelet and the existence

10

of a multiresolution analysis for general Hilbert space based on the formulation of multiresolution analysis by using methods from noncommutative harmonic analysis. They obtained four theorems. Their first theorem Theorem 1 guaranteed the existence of an orthonormal wavelet once a multiresolution analysis was built on the space. One of the interesting connections between the existence of self-similar tilings for Heisenberg groups and the work by [BC] is the following theorem on the existence of orthonormal wavelets on Heisenberg groups.

Theorem 5.6 *There exists a wavelet set* { $\psi_1, \psi_2, \cdots, \psi_{2^{2(d+1)}-1}$ } *for the system* $(L^2(\mathbf{H}^d), \Gamma, \sigma)$.

An appendix to the thesis is devoted to approximate identification of the class of scalable groups. The second countable, type I, unimodular locally compact groups form a very large class of groups. It includes all connected nilpotent Lie groups and all connected semisimple Lie groups. Unfortunately, some intrinsic properties in defining multiresolution analysis prevent some groups from becoming the groups fit for multiresolution analysis. Let us restate these essential properties as follows:

- There exists a discrete subgroup Γ in G such that G/Γ is compact
- There exists a topological automorphism α of G such that $\alpha(\Gamma) \subseteq \Gamma$ and $\bigcup_{j \in \mathbb{Z}} \alpha^{-1}\Gamma$ is dense in G

$\left. \vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}} \right\}$ (*)

This key information leads us to the following definition:

Definition A.1. *Suppose G is a second countable, type I, unimodular locally compact group. If there is a discrete subgroup Γ in G such that G/Γ is compact and a topological automorphism of G such that the conditions above are satisfied, then we call G a scalable group*

Generally speaking, only those groups which are close to being abelian are scalable groups. We have the following results:

Theorem (see section 3 in the Appendix). *At most countably many solvable Lie groups are scalable groups. That is, at most countably many solvable Lie groups are suitable for building multiresolution analysis.*

Theorem (see section 4 in the Appendix). *A semisimple Lie group is not scalable. Thus, there is no multiresolution analysis in the sense of this thesis on $L^2(G)$ if G is a semisimple*

Lie group.

At the end of the thesis, a discussion of possible future research is included.

Chapter 2

Multiresolution Analysis on $\mathbf{R}^d$

1 Introduction

The goal for this chapter is to describe the theory of multiresolution analysis for $L^2(\mathbf{R})$ which will be abbreviated to "MRA". The idea of MRA was first introduced by Mallat [Ma] and [Me2] in the fall of 1986. It has become an important mathematical tool and provided a framework for the understanding and the construction of wavelet bases.

To introduce MRA, we have to start from wavelets. The name wavelets comes from the admissibility requirement that they should integrate to zero. This forces them "wave" up and down around the x-axis. Wavelets have generated tremendous interest in both theoretical and applied areas because they have powerful properties which make them superior to Fourier methods in some situations. Some of their properties are localization in time and frequency, and a basis generated by one single function through translations and dilations and so on.

For the convenience of the reader we provide, in section 2.2, the simple notations and definitions used in this thesis. More will be introduced as the thesis advances on.

Historically, the Haar basis, constructed in 1910 long before the term "wavelet" was created, was the first orthonormal wavelet basis in $L^2(\mathbf{R})$. In section 2.3, we introduce the Haar basis for $L^2(\mathbf{R})$. In the 1980's, people tried to construct several orthonormal wavelet

13

bases for $L^2(\mathbf{R})$ by using different techniques. See [Dau] §4.2 for details. This situation has changed with the arrival of the MRA formulated by Mallat and Meyer. The concept of MRA is hidden in the Haar basis. In section 2.4, we unveil this beautiful pattern by introducing the scaling function and scaling identity. This leads to the definition of multiresolution analysis for the more general space $L^2(\mathbf{R}^d)$.

2 General

Throughout this thesis, we use the standard notation $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ for the sets of natural, integer, real and complex numbers, respectively. As usual, $\mathbf{R}^d$ denotes the d-dimensional Euclidean space. On $\mathbf{R}^d$ we use the Lebesgue measure. Let $\int \cdots dx$ denote the Lebesgue integral and let $C_c(\mathbf{R}^d)$ denote the function space consisting of continuous compactly supported complex-valued functions on $\mathbf{R}^d$. It is easily seen that $C_c(\mathbf{R}^d)$ forms a linear space with respect to ordinary addition of functions and multiplication by constants. For $f \in C_c(\mathbf{R}^d)$, let

$$\|f\|_p = (\int |f(x)|^p dx)^{1/p}, \text{ for } 1 \le p < \infty.$$

We denote the completion of the normed linear space $(C_c(\mathbf{R}^d), \| \cdot \|_p)$, for $1 \le p < \infty$, by $L^p(\mathbf{R}^d)$. As usual, we treat the elements in $L^p(\mathbf{R}^d)$ as Lebesgue measurable functions.

The most important space for us is $L^2(\mathbf{R}^d)$ which becomes a Hilbert space when the inner product of two functions f and g in $L^2(\mathbf{R}^d)$ is defined as

$$< f, g > = \int_{\mathbf{R}^d} f(x)\overline{g(x)} dx.$$

If E is a subset of $\mathbf{R}^d$, let $L^2(E)$ denote the closed subspace of $L^2(\mathbf{R}^d)$ consisting of elements supported on E.

For $f \in L^1(\mathbf{R}^d)$, the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(\zeta) := \int_{\mathbf{R}^d} f(x)e^{2\pi i \zeta \cdot x} dx,$$

for all $\zeta \in \mathbf{R}^d$.

14

If $f \in L^1(\mathbf{R}^d) \bigcap L^2(\mathbf{R}^d)$, we have the following relation:

$$\|\hat{f}\|_2 = \|f\|_2.$$

This result asserts that the Fourier transform is a bounded linear operator defined on the dense subset $L^1(\mathbf{R}^d) \bigcap L^2(\mathbf{R}^d)$ of $L^2(\mathbf{R}^d)$. Therefore, there exists a unique bounded extension of this operator to all of $L^2(\mathbf{R}^d)$, we denote this bounded operator by $\mathcal{F}$. Actually, $\mathcal{F}$ is a unitary operator from $L^2(\mathbf{R}^d)$ onto $L^2(\mathbf{R}^d)$. $\mathcal{F}$ is called the Fourier transform on $L^2(\mathbf{R}^d)$. We shall also use the notation $\hat{f} = \mathcal{F}f$ whenever $f \in L^2(\mathbf{R}^d)$.

For each $p \in [1, \infty)$, let $l^p(\mathbf{Z}^d)$ be the Banach space of functions x on $\mathbf{Z}^d$ such that

$$\|x\|_p := (\sum_{n \in \mathbf{Z}^d} |x_n|^p)^{1/p} < \infty.$$

We also denote by $l_c(\mathbf{Z}^d)$ the linear space of all finitely supported sequences on $\mathbf{Z}^d$.

A shift-invariant subspace is another notion often used. Suppose V is a linear subspace of complex-valued functions in $L^p(\mathbf{R}^d)$, we say that V is shift-invariant if, for any $f \in V$ and $n \in \mathbf{Z}^d$, the shift $f(\cdot - n)$ of f is also in V. V is said to be refinable if, for any $f \in V$, its dyadic dilate $f(\cdot/2)$ is also in V. For $\phi \in L^2(\mathbf{R}^d)$, we define $\mathcal{V}(\phi)$ to be the smallest closed shift-invariant subspace of $L^p(\mathbf{R}^d)$ containing ϕ. The function ϕ is said to be refinable if $\mathcal{V}(\phi)$ is refinable. A refinable function ϕ is also called a scaling function.

We denote the set of all square summable functions from $\mathbf{Z}^d$ into the complex numbers by $l^2(\mathbf{Z}^d)$. $l^2(\mathbf{Z}^d)$ is a Hilbert space with the inner product for $\{x_n\}_{n \in \mathbf{Z}^d}$ and $\{y_n\}_{n \in \mathbf{Z}^d}$ defined by

$$< x, y >= \sum_{n \in \mathbf{Z}^d} x_n \overline{y_n}.$$

Suppose $\mathcal{H}$ is a separable Hilbert space. A countable subset $\{e_n\}$ of $\mathcal{H}$ is said to be a Riesz basis if

- Every element f of $\mathcal{H}$ can be written uniquely as $f = \sum_n a_n e_n$;
- There exists positive constants α and β such that
  $$\alpha\|f\|^2 \leq \sum_{n \in \mathbf{Z}} | < f, e_n > |^2 \leq \beta\|f\|^2.$$

We also need to know the concept of a frame which is a generalization of Riesz basis. A countable subset $\{e_n\}$ of $\mathcal{H}$ is said to be a frame if there exist two positive numbers α and β so that, for any f in $\mathcal{H}$,

$$\alpha\|f\|^2 \leq \sum_n |<f, e_n>|^2 \leq \beta\|f\|^2.$$

We call α and β the frame bounds. If the two frame bounds are equal, $\alpha=\beta$, then we call the frame a tight frame. Frames were introduced by Duffin and Schaeffer [DS] in 1952. Every $f \in \mathcal{H}$ can be written $f = \sum_n a_n e_n$. But we should know that this representation is not unique in general. For more information, see [Dau] or [DS]. An advantage of frames is that we do not require $\{e_n\}$ to be orthogonal nor the coefficients a_n to be unique. This often allows us the freedom to impose extra conditions we would like.

3　The Haar Basis for $L^2(\mathbf{R})$

In order to motivate the concept of MRA, let's first introduce the Haar basis of $L^2(\mathbf{R})$ which was invented by Haar in 1910 [Ha]. The reason that we choose the Haar basis for illustration is its simplicity and there is a beautiful pattern of MRA hidden in it. We first quickly prove that the Haar family constitutes an orthonormal basis for $L^2(\mathbf{R})$. Then using the approximation approach and introducing a scaling function and the scaling identity, we uncover the model of MRA.

The Haar function $\psi(x)$ is defined by

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

We will write

$$\psi_{j,n}(x) = 2^{j/2}\psi(2^j x - n) = \begin{cases} \sqrt{2^j}, & \frac{2n}{2^{j+1}} \leq x < \frac{2n+1}{2^{j+1}} \\ -\sqrt{2^j}, & \frac{2n+1}{2^{j+1}} \leq x < \frac{2n+2}{2^{j+1}} \\ 0, & \text{otherwise.} \end{cases}$$

16

Observe that $\text{supp}(\psi_{j,n}) = [2^{-j}n, 2^{-j}(n+1)]$. When j gets larger, $\text{supp}(\psi_{j,n})$ becomes narrower. Denote $\{\psi_{j,n}\}_{j,n\in\mathbf{Z}}$ by Ψ. We claim that

Theorem 2.1 *The set Ψ constitutes an orthonormal basis for $L^2(\mathbf{R})$.*

proof In order to prove this, we need to show two things:

(i) Ψ is orthonormal.

(ii) Ψ is complete. That is, given any $f\in L^2(\mathbf{R})$, f can be written as $\sum_{j,n\in\mathbf{Z}} a_{j,n}\psi_{j,n}$.

(i) is easy to check. In fact, two different elements with the same j never overlap, it means that $<\psi_{j,n_1},\psi_{j,n_2}> = \delta_{n_1,n_2}$, where δ_{n_1,n_2} equals 1 when $n_1 = n_2$ and 0 when $n_1 \neq n_2$. If two functions ψ_{j_1,n_1} and ψ_{j_2,n_2} with different first indexes, say, $j_1 < j_2$, have overlapping support, then $\text{supp}(\psi_{j_2,n_2})$ is completely contained in a subinterval of $\text{supp}(\psi_{j_1,n_1})$ where (ψ_{j_1,n_1}) is constant. It follows that

$$<\psi_{j_1,n_1},\psi_{j_2,n_2}> = \text{constant} \int_{\frac{n_2}{2^{j_2}}}^{\frac{n_2+1}{2^{j_2}}} \psi_{j_2,n_2}(x)dx$$

$$= \text{constant}[\sqrt{2^{j_2}}/2^{j_2+1} - \sqrt{2^{j_2}}/2^{j_2+1}] = 0.$$

The completeness of the span of the $\psi_{m,n}$ follows from the fact that the dyadic step functions are dense in $L^2(\mathbf{R})$, where a dyadic step function is a function of the form

$$f(x) = \sum_{n\in\mathbf{Z}} c_n \chi_{I_n}(x)$$

where χ_{I_n} is the function which is one on the interval I_n and zero elsewhere and I_n is an interval of the form $[\frac{n}{2^m}, \frac{n+1}{2^m}]$. For details, please see [Dau, Chapter 1].

In the following section, we will define a ladder of closed subspaces $\{V_j | j\in\mathbf{Z}\}$ of $L^2(\mathbf{R})$, and use V_j to approximate general functions in $L^2(\mathbf{R})$. By doing this, a beautiful hidden pattern, called MRA, will be exposed to us.

4   Multiresolution analysis hidden in the Haar Basis

In this section, we try to expose the concept MRA by analyzing the Haar basis. We begin with one of the simplest functions we can imagine, that is, the characteristic function of the

unit interval [0,1] which is one on the interval [0,1] and zero elsewhere. Let's denote this function by ϕ. As before, we write

$$\phi_{j,n}(x) = 2^{j/2}\phi(2^j x - n), \; j, n \in \mathbf{Z}.$$

Let V_0 be the collection of functions of the form

$$f = \sum_{n \in \mathbf{Z}} a_n \phi_{0,n}, \text{ where } \{a_n\} \in l^2(\mathbf{Z}).$$

One can see that V_0 is naturally isomorphic to $l^2(\mathbf{Z})$ by means of the mapping $f \longmapsto \{a_n\}$. Thus V_0 is a closed subspace of $L^2(\mathbf{R})$, and

$$\{\phi_{0,n}\}_{n \in \mathbf{Z}} \text{ is an orthonormal basis for } V_0. \tag{2.1}$$

It's obvious that V_0 is not all of $L^2(\mathbf{R})$, it is the subspace of piecewise constant functions with jump discontinuities at $\mathbf{Z}$. V_0 is invariant under integer transforms:

$$\text{if } f \in V_0, \text{ then } f(\cdot - n) \in V_0 \text{ for all } n \in \mathbf{Z}. \tag{2.2}$$

Next let V_1 denote the collection of functions of the form

$$f = \sum_{n \in \mathbf{Z}} a_n \phi_{1,n}, \text{ with } \{a_n\} \in l^2(\mathbf{Z}).$$

Again, V_1 is a closed subspace of $L^2(\mathbf{R})$ and $\{\phi_{1,n}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for V_1. It is the subspace of piecewise constant functions with jump discontinuities at $\frac{1}{2}\mathbf{Z}$.

The most important thing is that

$$\phi(x) = \phi_{0,0}(x) = \phi_{1,0}(x) + \phi_{1,1}(x). \tag{2.3}$$

(2.3) is often called a two-scale dilation equation, or simply the scaling identity. ϕ is called the scaling function. The meaning of (2.3) is that the dilates of ϕ are self-similar to ϕ. (2.3) implies that each $\phi_{0,n}$ can be written as a sum of $\phi_{1,2n}$ and $\phi_{1,2n+1}$. $\{\phi_{0,n}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for V_0 implies that $V_0 \subseteq V_1$. It is easy to see that $f(x) \in V_0$ if and only if $f(2x) \in V_1$. Or equivalently,

$$V_1 = \{f \in L^2(\mathbf{R}) | \; f(\frac{x}{2}) \in V_0\},$$

18

that is, V_1 is a scaled version of the V_0.

By continuing in this manner, we define V_2 to be the closed subspace of piecewise constant functions with jump discontinuities at $\frac{1}{2^2}\mathbf{Z}$, which is

$$V_2 = \{f{\in}L^2(\mathbf{R})|\ f(\tfrac{x}{2}){\in}V_1\}, \ \text{and} \ V_1{\subseteq}V_2.$$

Similarly, one can define subspaces $V_3{\subseteq}V_4{\subseteq}\ldots\ldots$ On the other hand one may define negatively indexed subspaces. For example, we let V_{-1} be the space consisting of the functions in $L^2(\mathbf{R})$ which are piecewise constant with jump discontinuities at $2\mathbf{Z}$, that is,

$$V_{-1} = \{f{\in}L^2(\mathbf{R})|\ f(2x){\in}V_0\},$$

which is contained in V_0. Again, one may continue in this way to construct

$$\cdots \subseteq V_{-3}{\subseteq}V_{-2}{\subseteq}V_{-1}.$$

What we have then is a sequence of closed subspaces of $L^2(\mathbf{R})$ which are scaled versions of the central space V_0 such that

$$\cdots{\subseteq}V_{-2}{\subseteq}V_{-1}{\subseteq}V_0{\subseteq}V_1{\subseteq}V_2{\subseteq}\cdots$$

where V_j consists of the piecewise constant L^2 functions with jump discontinuities at $2^{-j}\mathbf{Z}$. The functions $\{\phi_{j,n}\}_{n{\in}\mathbf{Z}}$ form an orthogonal basis for V_j. We can pass up and down among the spaces V_j by scaling:

$$f(x){\in}V_j \ \text{if and only if} \ f(2^{k-j}x){\in}V_k. \tag{2.4}$$

There are two properties satisfied by this sequence of subspaces which we will wish to generalize.

$$\text{The union of } V_j's \text{ is dense in } L^2(\mathbf{R}) : \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}). \tag{2.5}$$

$$\text{The intersection of } V_j's \text{ is trivial: } \bigcap_{j\in\mathbf{Z}} V_j = \{0\}. \tag{2.6}$$

19

We only sketch the proofs. One first notes that $C_c(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, so it suffices to show that we can approximate functions in $C_c(\mathbf{R})$ by dyadic step functions on L^2-norm. What one does then is very similar to taking a Riemann sum approximation over a dyadic partition of an interval containing the support of a function in $C_c(\mathbf{R})$. Too see (2.6), suppose $f \in \bigcap_{j \in \mathbf{Z}} V_j$, then f must be constant on intervals of length 2^j for all integers j. This is the same as saying that f is a constant function. But the only constant function in $L^2(\mathbf{R})$ is 0. In view of (2.1), it is natural to try to obtain one orthonormal basis for $L^2(\mathbf{R})$ by combining all the orthonormal bases $\{\phi_{j,n} | n \in \mathbf{Z}\}$ of V_j. But although $V_j \subseteq V_{j+1}$, the orthonormal basis for V_j is not contained in the orthonormal basis $\{\phi_{j+1,n} | n \in \mathbf{Z}\}$ for V_{j+1}.

To find an orthonormal basis for $L^2(\mathbf{R})$, we use the following way. For every $j \in \mathbf{Z}$, use W_j to denote the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$, where the symbol $\oplus$ stands for orthogonal direct sum. So we can decompose $L^2(\mathbf{R})$ into mutually orthogonal closed subspaces, $L^2(\mathbf{R}) = \oplus_{j \in \mathbf{Z}} W_j$ by (2.5) and (2.6).

The most important thing remaining unchanged is that, the spaces W_j, $j \in \mathbf{Z}$, still keep the scaling property from V_j:

$$f(x) \in W_j \Longleftrightarrow f(2^{k-j}x) \in W_k. \tag{2.7}$$

Our goal is reduced to finding an orthonormal basis for W_0. If we can find such an orthonormal basis for W_0, then by the scaling property (2.7), we can easily find an orthonormal basis for the whole space $L^2(\mathbf{R})$. Now we use the following little trick. Set

$$\psi(x) = \phi_{1,0}(x) - \phi_{1,1}(x).$$

Then $\{\psi(\cdot - n) | n \in \mathbf{Z}\}$ forms an orthonormal basis for W_0. If we write $\psi_{j,n} = 2^{j/2}\psi(2^j x - n)$, then $\{\psi_{j,n} | n \in \mathbf{Z}\}$ is a orthonormal basis for W_j. Therefore $\{\psi_{j,n} | j, n \in \mathbf{Z}\}$ forms an orthonormal basis for $L^2(\mathbf{R})$, where we put the factor $2^{j/2}$ in front to normalize $\psi(2^j x - n)$.

The properties revealed above through analyzing the Haar basis for $L^2(\mathbf{R})$ leads us to formulate the definition of MRA for the more general space $L^2(\mathbf{R}^d)$, where $\mathbf{R}^d$ is the d-dimension Euclidean space. To get a general definition of MRA for $L^2(\mathbf{R}^d)$, we follow one particularly interesting generalization given by K.Gröchenig and W.R.Madych [GM]. Their

dilation operator is a matrix D with integer entries such that

- D leaves $\mathbf{Z}^d$ invariant. In other words, $D\mathbf{Z}^d \subset \mathbf{Z}^d$, where
  $$D\mathbf{Z}^d = \{\ y \mid y = Dx \text{ and } x \in \mathbf{Z}^d\ \}$$ (2.8)
- All the eigenvalues, λ_i, of D satisfy $|\lambda_i| > 1$

We substitute the number 2 in (2.4) by a matrix D, called dilation matrix, The second condition in (2.8) implies that D is a strict dilation in all directions, or D^{-1} contracts in all directions. Such a D induces a unitary operator $U_D : f \longmapsto U_D f$ on $L^2(\mathbf{R}^d)$, defined by

$$U_D f(\cdot) = \delta_D^{-1/2} f(D^{-1}\cdot),$$

where $\delta_D = |\det(D)|$.

Definition 2.2 *A multiresolution analysis(MRA) of $L^2(\mathbf{R}^d)$ consists of a sequence of closed linear subspaces V_j, $j \in \mathbf{Z}$, of $L^2(\mathbf{R}^d)$ with the following properties:*

(i) $V_j \subset V_{j+1}$, $j \in \mathbf{Z}$;

(ii) $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$;

(iii) $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$;

(iv) $f \in V_j \iff U_D f \in V_{j+1}$. In other words, $V_j = U_D^j V_0$, $j \in \mathbf{Z}$, where D is the dilation matrix;

(v) V_0 is assumed to be shift-invariant, that is, if $f \in V_0$ then so is $T_k f$ for all k in $\mathbf{Z}^d$, where

$$T_x f(\cdot) := f(\cdot - x), \quad \forall f \in L^2(\mathbf{R}^d),$$

we refer to T_x ($x \in \mathbf{R}^d$) as translations;

(vi) There is a function $\phi \in V_0$, called the scaling function, or the generator of the MRA, such that the collection $\{T_k \phi \mid k \in \mathbf{Z}^d\ \}$ is a orthonormal basis of V_0.

Note . (1) Observe that (iv) in the definition 2.2 implies that $f \in V_j$ if and only if $f(D^{-j}\cdot)$ is in V_0. It follows that an MRA is essentially completely determined by the closed subspace V_0. But from (vi) and (v), V_0 is the closure of the linear span of the $\mathbf{Z}^d$ translates of the scaling function ϕ. In other words, the whole MRA may be regarded as being generated by the scaling function ϕ. Thus the starting point of the construction of MRA is the existence

21

of the scaling function ϕ. Therefore, it is especially important for us to give some conditions under which an initial function ϕ generates an MRA.

(2) If in (iv), the dilation matrix $D = 2I$, where I is the identity matrix, the MRA is often referred to as a dyadic MRA. This is the case to which most of the current work is devoted and is representative of the general case.

Chapter 3

Notations and Preliminaries

1 Introduction

Our goal is to extend MRA on $\mathbf{R}^d$ to MRA on G for more general non-abelian groups G. The purpose of this chapter is to provide some background for the analysis to be presented in subsequent chapters. The proofs of many theorems in this chapter are lengthy and technical and involve ideas beyond the scope of our requirement. Hence, to a large extent we shall content ourselves with providing definitions and statements of the theorems.

Although wavelet analysis has some good points not available in Fourier analysis, they can not entirely replace Fourier analysis. Indeed, Fourier analysis is used in constructing the wavelets and forming a theoretical basis for wavelets. To extend MRA to more general groups, we restrict our attention to the class of second countable, type I, unimodular locally compact groups. The reason for this restriction is that abstract Fourier analysis on such groups is available, (see [Fol] for details). We shall see that not all the groups in this class are suitable for building an MRA on them. Roughly speaking, only those groups which are close to being vector groups are suitable for building MRA.

In this chapter, the basic concepts and results centered around abstract Fourier analysis on second countable, type I, unimodular locally compact groups are presented. At the same time, we establish our notations. We quickly go through preliminaries such as

locally compact groups and Lie groups, group representations, Hilbert-Schmidt operators and trace-class operators, tensor products of Hilbert spaces, and direct integrals of Hilbert spaces. In the last section, we provide the abstract Fourier analysis for the class of second countable, type I, unimodular locally compact groups including the most important Plancherel Theorem. This material may, for the most part, be found in [Fo1], and will be directly applied in the sequel without additional explanation.

2 Locally compact groups and Lie groups

A topological group G is a group equipped with a topology with the following properties:

(i) The mapping $(x, y) \longmapsto xy$ of $G \times G$ into G is continuous,

(ii) The mapping $x \longmapsto x^{-1}$ of G into G is continuous.

Thus G has two structures defined on it, one algebraic and one topological, and they are connected by the above two properties. If the topology on G is locally compact and Hausdorff we call G a locally compact group.

A Lie group is a group which is an analytic manifold and the mappings in (i) and (ii) are analytic.

For two subgroups H_1, H_2 of a group G, $[H_1, H_2]$ denotes the subgroup generated by $\{aba^{-1}b^{-1} | a \in H_1, b \in H_2\}$. The derived series of G is the descending sequence of normal subgroups $D^k G$ defined inductively by setting $D^0 G = G$, $D^k G = [D^{k-1}G, D^{k-1}G]$. Similarly, the sequence $C^0 G = G$, $C^k G = [G, C^{k-1}G]$ is called the descending central series of G. A group is nilpotent if $C^k G = \{e\}$ for all large k. G is called solvable if $D^k G = \{e\}$ for all large k. Since $D^k G \subseteq C^k G$, a nilpotent group must be a solvable group.

Let G be a locally compact group. A positive measure λ on G is called a left Haar measure if: (i) λ is a nonzero Radon measure on G, (ii) $\lambda(xE) = \lambda(E)$ for any $x \in G$, and any Borel subset $E \subseteq G$. ρ is called right Haar measure if (ii) is replaced by: (ii)' $\rho(Ex) = \rho(E)$, $\forall x \in G$ and $\forall$ Borel subsets $E \subseteq G$. One of the fundamental results in harmonic analysis is that every locally compact group G has a left Haar measure which is unique up to multiplication by a constant. For the proof of this, see [Fo §2.2] for details. Let's

24

recall the definition the modular function. For $x \in G$, we define $\lambda_x(E) := \lambda(Ex)$. Then $\lambda_x(yE) = \lambda(yEx) = \lambda(y(Ex))$. Since λ is left Haar measure, $\lambda(y(Ex)) = \lambda(Ex) = \lambda_x(E)$. Thus λ_x is again a left Haar measure. By the uniqueness of left Haar measure (up to constant multiplications), there is a number $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$ and $\Delta(x)$ is independent of the original choice of λ. The function $\Delta : G \longrightarrow \mathbf{R}^+$ thus defined is called the modular function of G, where $\mathbf{R}^+$ is the multiplicative group of positive real numbers. Actually, Δ is a continuous homomorphism from G to $\mathbf{R}^+$. G is called unimodular if left Haar measure is also right Haar measure, or in other words, G is unimodular if $\Delta(x) = 1$ for any $x \in G$.

Unimodularity is a useful property that makes the situation simpler in a number of respects. Obviously, Abelian groups and discrete groups are unimodular, but many others are too. For instant, every connected semi-simple Lie group is unimodular and also every connected nilpotent Lie group is unimodular [Fo §2.4]. But we should point out that not every connected solvable Lie group which is not nilpotent is unimodular. To see this, consider the affine group of $\mathbf{R}$:

$$G = \{(b,a)|b \in \mathbf{R}, \ a \in \mathbf{R}^*\}, \quad \text{where } \mathbf{R}^* = \mathbf{R} \setminus \{0\}.$$

with the following group law:

$$(b_1, a_1)(b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2).$$

Then G is solvable but not nilpotent because

$$D^1 G = \{(r,1)|r \in \mathbf{R}\}, \ D^2 G = \{e\}$$

and $C^1 G = D^1 G$, $C^k G = C^1 G$ for any $k > 1$. It is easy to compute that $\frac{1}{|a|^2} da\,db$ is the left Haar measure of G and $\frac{1}{|a|} da\,db$ is the right Haar measure of G. Thus G is not unimodular.

Let G be a locally compact group with a fixed left Haar measure λ. We shall generally write dx for $d\lambda(x)$. Let $C_c(G)$ denote the function space consisting of continuous compactly supported complex-valued functions on G. For $f \in C_c(G)$, let $\|f\|_p = (\int_G |f(x)|^p dx)^{1/p}$ for

25

$1 \leq p < \infty$. Let $L^p(G)$ denote the completion of the normed linear space $(C_c(G), \| \cdot \|_p)$, for $1 \leq p < \infty$. We are most interested in $L^1(G)$ and $L^2(G)$.

If $f, g \in L^1(G)$, the convolution of f and g is the function defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy.$$

If $f \in L^1(G)$, the involution is defined by the relation

$$f^*(x) = \overline{f(x^{-1})}.$$

If f is a function on the topological group G and $y \in G$, we define the left and right translations of f through y by

$$L_y f(x) = f(y^{-1}x), \quad R_y f(x) = f(xy).$$

The reason for using y^{-1} in L_y and y in R_y is to make the maps $y \longmapsto L_y$ and $y \longmapsto R_y$ group homomorphisms:

$$L_{yz} = L_y L_z, \quad R_{yz} = R_y R_z.$$

3 Group representations

Let G be a locally compact group. A continuous unitary representation of G is a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a Hilbert space and π is a homomorphism from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators that is continuous with respect to the strong operator topology. More exactly, $\pi: G \longrightarrow \mathcal{U}(\mathcal{H}_\pi)$ satisfies $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and for which $x \longrightarrow \pi(x)\xi$ is continuous from G to $\mathcal{H}_\pi$ for any $\xi \in \mathcal{H}_\pi$. $\mathcal{H}_\pi$ is called the representation space of π.

Suppose $\mathcal{K}$ is a closed subspace of $\mathcal{H}_\pi$. $\mathcal{K}$ is called an invariant subspace for π if $\pi(x)\mathcal{K} \subseteq \mathcal{K}$ for all $x \in G$. If $\mathcal{K}$ is invariant and $\neq \{0\}$, the restriction of π to $\mathcal{K}$, $\pi^{\mathcal{K}}(x) := \pi(x)|_{\mathcal{K}}$, defines a representation of G on $\mathcal{K}$, called a subrepresentation of π. If π admits an invariant subspace of $\mathcal{H}_\pi$ that is nontrivial, then π is called reducible, otherwise π is called irreducible.

If $(\pi, \mathcal{H}_\pi)$ and $(\sigma, \mathcal{H}_\sigma)$ are two representations of G and $T \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\sigma)$, where $\mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\sigma)$ denotes all bounded linear operators from $\mathcal{H}_\pi$ to $\mathcal{H}_\sigma$, satisfies $T\pi(g) = \sigma(g)T$, $\forall g \in G$, then T

is said to be an intertwining operator for π and σ. If there exists a unitary map $U \colon \mathcal{H}_\pi \longrightarrow \mathcal{H}_\sigma$ which intertwines π and σ, then we say that π is equivalent to σ and write $\pi \sim \sigma$. Let $\text{IRR}(G) = \{$ irreducible representation of $G \}$ and $\widehat{G} = \text{IRR}(G)/ \sim$. For $\pi \in \text{IRR}(G)$, we still use π to denote its equivalent class. The dual space $\widehat{G}$ of G is the set of equivalence classes of irreducible unitary representations of G endowed with the Fell topology. See [Fo §7.1 and §7.2] for a discussion of this topology on $\widehat{G}$.

The space of all intertwining operators for π and σ is denoted by $\text{Hom}(\pi, \sigma)$. Irreducibility of π is related to the structure of $\text{Hom}(\pi, \pi)$ by a fundamental result:

Schur's Lemma: *A unitary representation π on $\mathcal{H}_\pi$ is irreducible if and only if $\text{Hom}(\pi, \pi)$ contains only scalar multiples of the identity.*

For the proof of this result, see [Fo §3.1].

A unitary representation π of G is called primary if the center of $\text{Hom}(\pi, \pi)$ is trivial, i.e., consists of scalar multiples of the identity operator I. By Schur's lemma, every irreducible representation is primary. More generally, if π is a direct sum of irreducible representations, π is primary if and only if all its irreducible subrepresentations are unitarily equivalent. The group G is said to be type I if every primary representation of G is a direct sum of copies of some irreducible representation.

4 Hilbert-Schmidt and trace-class operators

Let us start by recalling the definition of the Hilbert-Schmidt norm of an operator in a finite dimensional Hilbert space $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$ be any endomorphism in $\mathcal{H}$. Take any orthonormal basis $\{e_k\}_{k=1}^d$ of $\mathcal{H}$, where $d = \dim(\mathcal{H})$, and assume that T is replaced by the matrix (t_{kl}) in the basis $\{e_k\}_k$. Obviously

$$\|T\|_2 = (\sum_{k,l} |t_{kl}|^2)^{1/2} \tag{3.1}$$

defines a norm on $\mathcal{B}(\mathcal{H})$. It is called the Hilbert-Schmidt norm of T. If S is another endomorphism, represented by the matrix (s_{kl}) with respect to the same basis, a computation

shows that

$$\text{Tr}(S^*T) = \sum_{k,l} t_{k,l} \overline{s_{k,l}}.$$

This shows that the Hilbert-Schmidt norm (3.1) is derived from the following inner product

$$< T, S >= \sum_{k,l} t_{k,l} \overline{s_{k,l}} = \text{Tr}(S^*T) \qquad (3.2)$$

on $\mathcal{B}(\mathcal{H})$. We can show that $\|T\|_2$ and $< T, S >$ are independent of the choice of orthonormal basis $\{e_k\}_k$ of V.

Now we need some analogous results in arbitrary Hilbert space. Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be a continuous linear operator on $\mathcal{H}$. Let us take an orthonormal basis $\{e_k\}_k$ of $\mathcal{H}$. Then $\sum_k \|Te_k\|^2$ is independent of the choice of orthonormal basis $\{e_k\}_k$ of $\mathcal{H}$. In fact, suppose $\{e_l'\}_l$ is another orthonormal basis of $\mathcal{H}$, then with the Parseval's identity, we have

$$\sum_k \|Te_k\|^2 = \sum_{k,l} | < e_l', Te_k > |^2 = \sum_{k,l} | < T^*e_l', e_k > |^2 = \sum_l \|T^*e_l'\|^2 \qquad (3.3)$$

The last series is obviously independent from the choice of $\{e_k\}_k$. As a result of (3.3), we see that the following non-negative sequences

$$\{\|Te_k\|^2\}_k \qquad \{\|T^*e_l'\|^2\}_l \qquad \{| < Te_k, e_l' > |^2\}_{k,l}$$

are simultaneously summable or not, whenever they are summable, their sum is the same independent of $\{e_k\}_k$ and $\{e_l'\}_k$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Hilbert-Schmidt operator if for one, hence for any, orthonormal basis $\{e_k\}_k$ of $\mathcal{H}$, $\sum_k \|Te_k\|^2 < \infty$. By the preceding argument, this is well-defined. We use $\text{HS}(\mathcal{H})$ to denote the set of all Hilbert-Schmidt operators on $\mathcal{H}$. For $T \in \text{HS}(\mathcal{H})$, define norm of T as

$$\|T\|_2 = (\sum_k \|Te_k\|^2)^{1/2}. \qquad (3.4)$$

We have the following properties: If T is a Hilbert-Schmidt operator, so is T^*. If T and S are Hilbert-Schmidt operators, so is $\alpha T + \beta S$, for any constants α, β. So we see that

28

all the Hilbert-Schmidt operators on $\mathcal{H}$ form a linear space with respect to addition and scalar multiplication. Moreover, for any $T \in \mathrm{HS}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, TS and ST are both in $\mathrm{HS}(\mathcal{H})$. Thus $\mathrm{HS}(\mathcal{H})$ is also an ideal in $\mathcal{B}(\mathcal{H})$. We shall see very shortly that if T and S are in $\mathrm{HS}(\mathcal{H})$ and $\{e_k\}_k$ is any orthonormal basis of $\mathcal{H}$, then an inner product of these two operators can be defined by

$$< T, S >= \sum_k < Te_k, Se_k >= \mathrm{Tr}(S^*T). \tag{3.5}$$

This (3.5) corresponds to (3.2). With the inner product defined as in (3.5), $\mathrm{HS}(\mathcal{H})$ is a Hilbert space; the norm is given by

$$\|T\|_2 = \sqrt{\mathrm{Tr}(T^*T)}.$$

From the above, we know that the product of two bounded operators, of which at least one is in $\mathrm{HS}(\mathcal{H})$, is also in $\mathrm{HS}(\mathcal{H})$. Now we shall consider products of operators both of which are in $\mathrm{HS}(\mathcal{H})$.

Let T be the product of two operators in $\mathrm{HS}(\mathcal{H})$ and let $\{e_k\}_k$ be a given basis. Then the sequence $\{| < Te_k, e_k > |\}$ is summable. Consequently, $\{< Te_k, e_k >\}$ is also summable and its sum is independent of $\{e_k\}_k$. Indeed, since $K \in \mathrm{HS}(\mathcal{H})$ if and only if $K^* \in \mathrm{HS}(\mathcal{H})$, we may assume without loss of generality that $T = K^*L$ with both K and L in $\mathrm{HS}(\mathcal{H})$. Clearly,

$$| < Te_k, e_k > | = | < Le_k, Ke_k > | \leq \frac{1}{2}(\|Le_k\|^2 + \|Ke_k\|^2)$$

and therefore

$$\sum_k | < Te_k, e_k > | = \sum_k | < Le_k, Ke_k > | \leq \frac{1}{2}(\|L\|_2^2 + \|K\|_2^2).$$

By the polarization identity, we have

$$\mathcal{R}e < Le_k, Ke_k >= \frac{1}{4}(\|(L + K)e_k\|^2 - \|(L - K)e_k\|^2).$$

So we get

$$\mathcal{R}e \sum_k < Le_k, Ke_k >= \frac{1}{4}(\|(L + K)\|_2^2 - \|(L - K)\|_2^2).$$

29

The right hand side is clearly independent of $\{e_k\}_k$. Replacing L by iL, we see that

$$\mathcal{I}m < Le_k, Ke_k > = -\mathcal{R}e \sum_k < iLe_k, Ke_k >$$

is also independent of $\{e_k\}_k$. Therefore $\sum_k < Te_k, e_k >$ is independent of the choice of $\{e_k\}_k$.

We call a product of two operators in HS($\mathcal{H}$) a trace-class operator. By the preceding argument, if T is trace-class operator and $\{e_k\}_k$ is any orthonormal basis for $\mathcal{H}$, then

$$\mathrm{Tr}(T) := \sum_k < Te_k, e_k >$$

is well-defined and it is independent of $\{e_k\}_k$. We have the following properties for trace-class operators: every trace-class operator is a Hilbert-Schmidt operator and T is a Hilbert-Schmidt operator if and only if T^*T is a trace-class operator. The product of two Hilbert-Schmidt operators is a trace-class operator and any trace-class operator can be resolved into the product of two Hilbert-Schmidt operators. T is a trace-class operator if and only if T^* is a trace-class operator. If T and S are trace-class operators, so is $\alpha T + \beta S$, for any constants α and β. So with the obvious definition of addition and scalar multiplication, the set of all trace-class operators forms a linear space. Furthermore, if T is trace-class operator and $S \in \mathcal{B}(\mathcal{H})$, then TS and ST are both trace-class operators. Thus the set of all trace-class operators is a two-sided ideal in $\mathcal{B}(\mathcal{H})$.

For more about Hilbert-Schmidt operators and trace-class operators, see [Fo Appendix 2] and [Sc §2 and §3].

5 Tensor products of Hilbert spaces

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, we use $\mathcal{H}_2^*$ to denote the conjugate Hilbert space. That is, as a set, $\mathcal{H}_2^* = \mathcal{H}_2$, but $\mathcal{H}_2^*$ has the different algebraic structure and inner product:

$$(\xi, \eta) \longmapsto \xi + \eta : \ \mathcal{H}_2 \times \mathcal{H}_2 \longmapsto \mathcal{H}_2,$$

$$(\alpha, \xi) \longmapsto \overline{\alpha} \xi : \ \mathbb{C} \times \mathcal{H}_2 \longmapsto \mathcal{H}_2,$$

$$(\xi,\eta) \longmapsto\; < \xi, \eta >^-:\; \mathcal{H}_2 \times \mathcal{H}_2 \longmapsto \mathbb{C},$$

where

$$\alpha^-\xi = \overline{\alpha}\xi \;\text{ and }\; < \xi, \eta >^- = < \eta, \xi > .$$

Obviously, the conjugate Hilbert space of $\mathcal{H}_2^*$ is $\mathcal{H}_2$ and a bounded linear operator from $\mathcal{H}_2^*$ into $\mathcal{H}_1$ is just a bounded antilinear operator from $\mathcal{H}_2$ into $\mathcal{H}_1$.

We define the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ to be the set $\mathcal{H}_1 \otimes \mathcal{H}_2$ of all antilinear operators $T:\; \mathcal{H}_2 \longmapsto \mathcal{H}_1$ such that $\sum_k \|Te_k\|^2 < \infty$ for some, hence any, orthonormal basis $\{e_k\}$ for $\mathcal{H}_2$. If we set

$$\|T\|_2 = (\sum_k \|Te_k\|^2)^{1/2},$$

then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with the norm $\| \cdot \|_2$ and associated inner product

$$< T, S > = \sum_k < Te_k, Se_k >, \tag{3.6}$$

where $\{e_k\}$ is any orthonormal basis of $\mathcal{H}_2$.

Also, $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be constructed by an algebraic method. First, we treat both $\mathcal{H}_1$ and $\mathcal{H}_2$ as vector spaces. The tensor product of the vector spaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ is usually defined abstractly as a vector space $\mathcal{H}_0$ such that any bilinear map from the Cartesian product $\mathcal{H}_1 \times \mathcal{H}_2$ into another vector space factors uniquely through $\mathcal{H}_0$. More precisely, if

$$L:\; \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{K}$$

is a bilinear mapping from $\mathcal{H}_1 \times \mathcal{H}_2$ into another vector space $\mathcal{K}$, then L has a unique factorization $L = Tp$, with p a bilinear mapping from $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{H}_0$ and T a linear mapping from $\mathcal{H}_0$ into $\mathcal{K}$. $\mathcal{H}_0$ can be identified with the quotient of the linear space of all linear combinations of simple tensors: $\xi \otimes \eta$ for any $\xi \in \mathcal{H}_1$ and any $\eta \in \mathcal{H}_2$, by the subspace consisting of these finite sums that must vanish if p is to be bilinear. We can identify $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the completion of its everywhere dense subspace $\mathcal{H}_0$.

In particular, $\mathcal{H} \otimes \mathcal{H}^*$ is just the set of Hilbert Schmidt operators on $\mathcal{H}$ and it is a Hilbert space with respect to the inner product (3.6).

For more information, see [Fo §7.3] or [KR §2.6] for details.

31

6 Direct integrals of Hilbert spaces

In the following, we outline some notations and results of direct integrals. For further information on direct integrals, we refer the reader to [Fo §7.4].

A Borel structure on a set A is a σ-algebra of subsets of A, i.e., a family of subsets of A containing A itself and closed under complements and countable unions. A measurable space is a pair $(A, \mathcal{M})$ consisting of a set A and a Borel structure $\mathcal{M}$ on A. A family $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ of nonzero separable Hilbert spaces indexed by A will be called a field of Hilbert spaces over A. A map f on A such that $f(\alpha) \in \mathcal{H}_\alpha$ for each α will be called a vector field on A. We denote the inner product and norm on $\mathcal{H}_\alpha$ by $< \cdot, \cdot >_\alpha$ and $\| \cdot \|_\alpha$. A measurable field of Hilbert spaces over A is a field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ together with a countable set $\{e_j\}_1^\infty$ of vector fields with the following properties:

(i) the functions $\alpha \longmapsto < e_j(\alpha), e_k(\alpha) >_\alpha$ are measurable for all j, k.

(ii) the linear span of $\{e_j(\alpha)\}_1^\infty$ is dense in $\mathcal{H}_\alpha$ for each α.

Thus when we mention a measurable field of Hilbert spaces over A, there are two things involved: $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ and $\{e_j(\alpha)\}_1^\infty$. A constant field of Hilbert spaces over A is a measurable field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ over A such that $\mathcal{H}_\alpha = \mathcal{H}$ for all $\alpha \in A$, where $\mathcal{H}$ is a separable Hilbert space.

Given a measurable field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$, $\{e_j(\alpha)\}_1^\infty$ on A, a vector field f on A will be called measurable if $< f(\alpha), e_j(\alpha) >_\alpha$ is measurable function on A for each j.

Finally, we are ready to define direct integrals. Suppose $\{\mathcal{H}_\alpha\}_{\alpha \in A}$, $\{e_j(\alpha)\}_1^\infty$ is a measurable field of Hilbert spaces over A, and suppose μ is a measure on A. The direct integral of the spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ with respect to μ is denoted by

$$\int^\oplus \mathcal{H}_\alpha d\mu(\alpha).$$

This is the space of measurable vector fields f on A such that

$$\|f\|^2 = \int \|f(\alpha)\|_\alpha^2 d\mu(\alpha) < \infty.$$

32

Then it easily follows that $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$ is a Hilbert space with the inner product

$$< f, g > = \int_A < f(\alpha), g(\alpha) >_\alpha d\mu(\alpha).$$

In case of a constant field, that is, $\mathcal{H}_\alpha = \mathcal{H}$ for all $\alpha \in A$, $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha) = L^2(A, \mu, \mathcal{H})$, all the measurable functions $f : A \longmapsto \mathcal{H}$ defined on a measurable space (A, μ) with values in a Hilbert space $\mathcal{H}$ such that

$$\|f\|^2 = \int_A \|f(x)\|^2 d\mu(x) < \infty.$$

7 Abstract Fourier Analysis on second countable, type I, unimodular locally compact groups

This section contains a brief review of Fourier analysis on a second countable, type I, unimodular locally compact group, that will be needed in the next chapter, including the most important Plancherel Theorem. For details, we refer the reader to [Fo §7.5] and [Li]. Throughout this section, G will denote a second countable, type I, unimodular locally compact group.

There is a measure μ on $\widehat{G}$, called Plancherel measure, uniquely determined once the Haar measure on G is fixed. The family $\{ HS(\mathcal{H}_\pi) \}_{\pi \in \widehat{G}}$ of Hilbert spaces indexed by $\widehat{G}$ is a field of Hilbert spaces over $\widehat{G}$. The direct integral of the spaces $\{ HS(\mathcal{H}_\pi) \}_{\pi \in \widehat{G}}$ with respect to μ, denoted by $\int_{\widehat{G}}^{\oplus} HS(\mathcal{H}_\pi) d\mu(\pi)$, is the space of measurable vector fields F on $\widehat{G}$ such that

$$\|F\|^2 = \int_{\widehat{G}} \|F(\pi)\|_\pi^2 d\mu(\pi) < \infty.$$

For convenience, we denote $\int_{\widehat{G}}^{\oplus} HS(\mathcal{H}_\pi) d\mu(\pi)$ by $\mathcal{H}^2(\widehat{G})$. Given a μ-measurable subset $\Omega \subseteq \widehat{G}$, we write $\mathcal{H}^2(\Omega)$ instead of $\int_{\Omega}^{\oplus} HS(\mathcal{H}_\pi) d\mu(\pi)$. In other words,

$$\mathcal{H}^2(\Omega) = \{ F \in \mathcal{H}^2(\widehat{G}) \mid F(\pi) = 0 \text{ for } \mu - a.e. \ \pi \in \widehat{G} \backslash \Omega \}.$$

For F, H in $\mathcal{H}^2(\Omega)$, define

$$< F, H >_\Omega := \int_\Omega tr[H(\pi)^* F(\pi)] d\mu(\pi),$$

then $(\mathcal{H}^2(\Omega), < \cdot, \cdot >_\Omega)$ becomes a Hilbert space.

For $1 \le p < \infty$, we define

$$\mathcal{H}^p(\widehat{G}) := \{\ F\ :\ F \text{ is a measurable field of bounded operators on } \widehat{G} \text{ such that}$$

$$\|F(\pi)\|_p < \infty,\ \mu \text{ almost all } \pi \in \widehat{G} \text{ and } \int_{\widehat{G}} \|F(\pi)\|_p^p d\mu(\pi) < \infty\ \},$$

where for bounded operators T, $\|T\|_p = (tr|T|^p)^{1/p}$, $1 \le p < \infty$, $|T| = (T^*T)^{1/2}$. If we identify any two elements whenever they agree on $\widehat{G}$ except for a μ-null subset, then $\mathcal{H}^p(\widehat{G})$ becomes a Banach space with respect to the following norm

$$\|F\|_p = (\int_{\widehat{G}} \|F(\pi)\|_p^p d\mu(\pi))^{1/p}.$$

When $p = 2$, we recover the definition of the Hilbert space $\mathcal{H}^2(\widehat{G})$.

For a measurable subset Ω of $\widehat{G}$, we write

$$\mathcal{H}^p(\Omega) := \{\ F \in \mathcal{H}^p(\widehat{G})\ |\ F(\pi) = 0 \text{ for } \mu - a.e.\ \pi \in \widehat{G} \backslash \Omega\ \}.$$

For $p = \infty$, set

$$\mathcal{H}^\infty(\widehat{G}) := \{\ F\ :\ F \text{ is a measurable field of bounded operators on } \widehat{G} \text{ such that}$$

$$\|F\|_\infty = \text{esssup}_{\pi \in \widehat{G}} \|F(\pi)\|_\infty < \infty,\ \},$$

where $\|T\|_\infty =$ the operator norm of T. Again, $\mathcal{H}^\infty(\widehat{G})$ is a Banach space under the norm $\|\cdot\|_\infty$. Also, $\mathcal{H}^\infty(\widehat{G})$ is a Banach $*$-algebra under the product

$$(F \cdot H)(\pi) = F(\pi)H(\pi) \text{ and involution } F^*(\pi) = F(\pi)^*.$$

Note that, if G is non-abelian, then $\mathcal{H}^\infty(\widehat{G})$ is non-abelian.

Similarly, for a measurable subset Ω of $\widehat{G}$, we write

$$\mathcal{H}^\infty(\Omega) := \{\ F \in \mathcal{H}^\infty(\widehat{G})\ |\ F(\pi) = 0 \text{ for } \mu - a.e.\ \pi \in \widehat{G} \backslash \Omega\ \}.$$

If $f \in L^1(G)$, we define the Fourier transform of f to be the measurable field of operators over $\widehat{G}$ given by

$$\mathcal{F}f(\pi) = \hat{f}(\pi) = \int_G f(x)\pi(x)dx. \tag{3.7}$$

34

Let $\mathcal{J}^1 := L^1(G)\cap L^2(G)$ and $\mathcal{J}^2 :=$linear span of $\{f*g \mid f, g \in \mathcal{J}^1\}$. So the elements of $\mathcal{J}^2$ are finite linear combinations of convolutions of elements of $\mathcal{J}^1$. By the property of convolution in harmonic analysis, $\mathcal{J}^2 \subset C_0(G)$, where $C_0(G)$ is the space of all complex-valued continuous functions f on G such that for every $\epsilon > 0$, there exists a compact subset K of G (depending upon f and ϵ) such that $|f(x)| < \epsilon$ for all $x \in G \backslash K$. Thus, $\mathcal{J}^2$ is a vector space of well-behaved functions which can be shown to be dense in both $L^1(G)$ and $L^2(G)$. With the notations set as above, we have the following abstract Plancherel theorem. This theorem is due to Segal [Se2] and [Se3] and Mautner [Mau]; the proof may be found in Dixmier [Di §18.8].

Plancherel Theorem. *Suppose G is a second countable, type I, unimodular locally compact group. There is a measure μ on $\widehat{G}$, uniquely determined once the Haar measure on G is fixed, with the following properties. The Fourier transform $f \longrightarrow \hat{f}$ maps $\mathcal{J}^1$ into $\mathcal{H}^2(\widehat{G})$, and it extends to a unitary map from $L^2(G)$ onto $\mathcal{H}^2(\widehat{G})$. For $f, g \in \mathcal{J}^1$ one has the Parseval formula*

$$\int_G f(x)\overline{g(x)}dx = \int_{\widehat{G}} tr[\hat{g}(\pi)^*\hat{f}(\pi)]d\mu(\pi). \tag{3.8}$$

and for $h \in \mathcal{J}^2$ one has the Fourier inversion formula

$$h(x) = \mathcal{F}^{-1}\hat{h}(x) = \int_{\widehat{G}} tr[\pi(x)^*\hat{h}(\pi)]d\mu(\pi). \tag{3.9}$$

Remark. The Plancherel theorem states that the Fourier transform maps $\mathcal{J}^1$ into $\mathcal{H}^2(\widehat{G})$. This means that, when $f \in \mathcal{J}^1$, $\hat{f}(\pi)$ is Hilbert-Schmidt for μ-almost every π and its Hilbert-Schmidt norm is square-integrable on $\widehat{G}$. From section 4, we know that, if $f, g \in \mathcal{J}^1$, then $\widehat{f*g}(\pi) = \hat{f}(\pi)\hat{g}(\pi)$ is trace-class for μ-almost every π and its trace is integrable on $\widehat{G}$.

The inversion formula (3.9) still holds for a large class of more general functions. This was established by Lipsman in [Li]. For easy reference, we quote the Fourier Inversion Theorem from [Li] in the following.

Fourier Inversion Theorem: (i) *Let $F \in \mathcal{H}^1(\widehat{G})$ and suppose that*

$$f(x) = (\mathcal{F}^{-1}F)(x) = \int_{\widehat{G}} tr[\pi(x)^*F(\pi)]d\mu(\pi), \quad x \in G$$

35

is in $L^1(G)$. *Then* $\hat{f} = F$ μ-*a.e.*.

(ii) Let $f \in L^1(G)$ *be continuous. If* $F = \hat{f}$ *is in* $\mathcal{H}^1(\widehat{G})$, *then*

$$f(x) = \int_{\widehat{G}} tr[\pi(x)^* F(\pi)] d\mu(x), \ x \in G.$$

Finally, recall that the involution $f \longrightarrow f^*$ is defined by the relation

$$f^*(x) = \overline{f(x^{-1})} \ \text{for} f \in L^1(G).$$

Since $\mathcal{J}^1$ is dense in $L^2(G)$ and $\|f^*\|_2 = \|f\|_2$ for $f \in \mathcal{J}^1$, the involution on $\mathcal{J}^1$ extends to an involution on $L^2(G)$, which we will also denote by $*$. Simple computations show that the basic properties of the Fourier transform remain valid:

$$(af + bg)^\wedge(\pi) = a\hat{f}(\pi) + b\hat{g}(\pi) \tag{3.10}$$

$$(f*g)^\wedge(\pi) = \hat{f}(\pi)\hat{g}(\pi) \tag{3.11}$$

$$(L_x f)^\wedge(\pi) = \pi(x)\hat{f}(\pi) \tag{3.12}$$

$$(f^*)^\wedge(\pi) = \hat{f}(\pi)^* \tag{3.13}$$

where, $\hat{f}(\pi)^*$ is the adjoint operator of $\hat{f}(\pi)$.

To summarize, if G is a second countable, type I, unimodular locally compact group, then many of the tools of abelian Fourier analysis are still available. It must be remembered that, for $f \in L^1(G)$, $\hat{f}(\pi)$ is not a complex number, but an operator on the Hilbert space $\mathcal{H}_\pi$, for each $\pi \in \widehat{G}$.

Chapter 4

Extension of Multiresolution Analysis in $L^2(\mathbf{R}^d)$ to Multiresolution Analysis in $L^2(G)$

1 Introduction

This chapter is devoted to three topics. The first topic concerns the definition of MRA of $L^2(G)$, where G is a suitable second countable, type I, unimodular locally compact group. The second topic deals with some properties of an MRA generated by the scaling function ϕ. This includes two main things: intersection triviality and union density of the nested closed subspaces. The third topic is devoted to the conditions under which the given function ϕ becomes a father wavelet, that is, ϕ generates an MRA for $L^2(G)$.

In section 2, we summarize the developments on the question: under what conditions does an initial function ϕ generate an MRA of $L^2(\mathbf{R}^d)$. Also, we give a outline of the results by de Boor, Devore, and Ron.

In section 3, we first investigate the definition of MRA of $L^2(\mathbf{R}^d)$ and find some key points in defining the MRA of $L^2(\mathbf{R}^d)$ by properly interpreting it with a more general point of view. Then we formulate the definition for $L^2(G)$ by using the information exposed from

the MRA of $L^2(\mathbf{R}^d)$. Notice that we are dealing with nonabelian groups, so we have two types of translations available: left and right translations. We know that the image under the Fourier transform of either a left shift-invariant subspace generated by ϕ or a left and right shift-invariant subspace generated by the same ϕ is supported on the same subset supp$\mathcal{F}(\phi)$. This fact suggests that we may only pay attention to the one-sided translations without considering the other ones. But in order to do so, we introduce the concept of "strongly supported" for a function ϕ. This term arises in a very natural manner once one connects non-abelian cases to abelian cases.

As we mentioned after definition 2.2, the starting point of constructing an MRA is the scaling function ϕ. With this scaling function ϕ, we form V_0 by applying the shift operators and then generate V_j by the dilation operator. Generally speaking, there are three things in MRA that mainly concern us; that is, the refinability of ϕ, the density of the union and the triviality of the intersection of the nested sequence of closed subspaces. Since the intersection triviality property is a direct consequence of the other conditions of the definition of an MRA, we prove this property immediately after we give the definition. The importance of this theorem (intersection triviality) is that it is not necessary to assume the intersection triviality property in the definition.

Thus, essentially two basic questions remain: How can we check that union density property holds? And how do we find a refinable function with orthonormal shifts? Section 4 mainly contributes to the union density theorem. This theorem characterizes the union density property. Most of the work in this chapter is on the first question. The second question remains a focus of much of the basic research for $G = \mathbf{R}^d$ and we will be satisfied with the construction of explicit examples for the Heisenberg group. The significance of refinability of ϕ is to have a sequence of closed subspaces nested. The refinability depends very much on the individual function ϕ. We will discuss this issue for an example on the Heisenberg group in chapter 5.

In section 5, several sufficient conditions on ϕ are given that can be used to check that union density holds. We will answer the basic question: under what conditions does ϕ

generate an MRA of $L^2(G)$ provided ϕ satisfies the refinability property.

2 Outline of the results by de Boor, DeVore, and Ron

Remember that our final aim is to set up an MRA on more general groups. If we look back carefully into definition 2.2 in chapter 2, please keeping in mind that we are now still in the space $L^2(\mathbf{R}^d)$, we find that to build an MRA on $\mathbf{R}^d$, one can try to start the construction from an appropriate choice for the scaling function ϕ. Then generate a shift-invariant subspace $\mathcal{V}(\phi)$ by ϕ and take $\mathcal{V}(\phi)$ as V_0 (this is step (v) in the definition 2.2). Following that, one can use the dilation operator D to construct $V_j := D^j V_0$ by (iv). If, in addition, ϕ is refinable, that is, $D^{-1}V_0 \subset V_0$, then we get a sequence of closed nested subspaces $\{V_j\}_{j \in \mathbf{Z}}$ which satisfy (i),(iv),(v) and (vi) of the definition 2.2. We shall see later that (iii) is a direct consequence of (iv) and (vi). Thus, in order to make an MRA in $L^2(\mathbf{R}^d)$, we have to answer the following question:

$$\left.\begin{array}{l}\text{Accepting the refinable requirement for the starting } \phi,\\[4pt] \text{under what other conditions does an initial function } \phi\\[4pt] \text{generate an MRA of } L^2(\mathbf{R}^d) \text{ (i.e., satisfy the condition (ii))?}\end{array}\right\} \qquad (4.1)$$

We summarize the developments on the above question in the following.

First it is shown in [Mel], chapter 2, that $\mathcal{V}(\phi)$ generates an MRA of $L^2(\mathbf{R}^d)$ provided that:

i) ϕ is refinable;

ii) ϕ has stable shifts, that is, there exist two positive constants α and β such that

$$\alpha\|a\|_2 \leq \|\sum_{n \in \mathbf{Z}^d} \phi(\cdot - n)a(n)\|_2 \leq \beta\|a\|_2, \text{ for all } a \in l_2(\mathbf{Z}^d);$$

iii) ϕ satisfies the regularity conditions

$$|\phi(x)| \leq C_m(1 + \|x\|)^m \text{ for all } m \in \mathbf{N} \text{ and } x \in \mathbf{R}^d$$

where $\|x\|$ denotes the Euclidean norm of x, and C_m are positive constants.

Then in [JM], the regularity conditions iii) above were relaxed so that ϕ is only required to satisfy

$$\sum_{n\in\mathbf{Z}^d} |\phi(\cdot - n)| \in L^2([0,1]^d).$$

A few improvements to the above results were made in [Ma] and [St]. In [Ma], Madych verified that the function ϕ generates an MRA of $L^2(\mathbf{R}^d)$ if ϕ satisfies that

$$\lim_{j\to\infty} \frac{1}{|(D^*)^{-j}Q|} \int_{(D^*)^{-j}Q} |\phi(x)|^2 dx = 1 \tag{4.2}$$

for every cube Q of finite diameter in $\mathbf{R}^d$, where D is the dilation matrix used in the definition 2.2. Madych also gave some other conditions under which $\phi \in L^2(\mathbf{R}^d)$ generated an MRA. We should note that the condition (4.2) is related to the dilation matrix, and so are the other conditions given in [Ma]. Stöckler [St] found another criterion for the denseness condition in (ii) of Definition 2.2 by introducing two symbol functions.

But the ultimate solution to the question (4.1) concerning MRA was obtained by de Boor,DeVore, and Ron in [BD] by using a characterization of the closed translation invariant subspaces of $L^2(\mathbf{R}^d)$.

The definition of MRA put forward by de Boor, DeVore, and Ron in [BD] was a generalization of the definition formulated by Mallat [Ma] and Meyer [Me1]. It goes as follows. **Definition 4.1.** *Let $\phi \in L^2(\mathbf{R}^d)$. Let V_0 be the closed shift-invariant subspace $\mathcal{V}(\phi)$ of $L^2(\mathbf{R}^d)$. For $j \in \mathbf{Z}$, let V_j be the 2^j-dilate of V_0:*

$$V_j := \{ f(2^j \cdot) \mid f \in V_0 \}.$$

We say that $\{V_j\}$ forms an MRA of $L^2(\mathbf{R}^d)$ if the following three conditions are satisfied:

i) $V_j \subset V_{j+1}$, $j \in \mathbf{Z}$;

ii) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$; (Union density)

iii) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$. (Intersection triviality)

It is obvious that the condition i) is equivalent to saying that V_0 is refinable, or equally, saying that ϕ is refinable.

Under this definition, [BD] analyzed the conditions (ii) and (iii) above. They completely characterized the density property of (arbitrary) shift-invariant subspaces of $L^2(\mathbf{R}^d)$. First they proved that $\overline{\bigcup_{j\in\mathbf{Z}} V_j}$ is a closed translation invariant subspace of $L^2(\mathbf{R}^d)$ then they furnished a more general answer for this question.

Their results can be stated as follows.

Theorem 4.2(de Boor, Devore, and Ron). *Let $V_0 = \mathcal{V}(\phi)$ be the shift-invariant subspace generated by a function $\phi\in L^2(\mathbf{R}^d)$, and let V_j be the 2^j-dilate of V_0 for $j\in\mathbf{Z}$. If V_0 is refinable, then $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(\mathbf{R}^d)$ if and only if $\bigcup_{j\in\mathbf{Z}}supp(\mathcal{F}\phi_j) = \mathbf{R}^d$ modulo a null-set, where $\phi_j(\cdot) := D^j\phi(\cdot) = 2^{dj/2}\phi(2^j\cdot)$ and $supp(\mathcal{F}\phi_j) := \{\xi\in\mathbf{R}^d \mid \mathcal{F}\phi_j(\xi)\neq 0 \}$.*

Theorem 4.3(de Boor, Devore, and Ron). *Let $V_0 = \mathcal{V}(\phi)$ for some function $\phi\in L^2(\mathbf{R}^d)$. Then $\bigcap_{j\in\mathbf{Z}}V_j = \{0\}$.*

The following theorem gives a very simple sufficient condition for (ii) in the definition 4.1.

Theorem 4.4(de Boor, Devore, and Ron). *Let $\{V_j\}_{j\in\mathbf{Z}}$ be as in the theorem 4.2, then $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(\mathbf{R}^d)$ if $\mathcal{F}\phi$ is nonzero a.e. in some neighborhood of the origin.*

Remark. (i) Let $\widehat{\mathbf{R}^d}$ be the dual group of $\mathbf{R}^d$. It is a basic result in harmonic analysis that $\widehat{\mathbf{R}^d}$ can be identified with $\mathbf{R}^d$. So the condition "$\bigcup_{j\in\mathbf{Z}}supp(\mathcal{F}\phi_j) = \mathbf{R}^d$ modulo a null-set" in the theorem 4.3 can be replaced by "$\bigcup_{j\in\mathbf{Z}}supp(\mathcal{F}\phi_j) = \widehat{\mathbf{R}^d}$ modulo a null-set".

(ii) In theorem 4.2, de Boor, Devore, and Ron completely characterized the density property of (arbitrary) shift-invariant subspaces of $L^2(\mathbf{R}^d)$. First they proved that $\overline{\bigcup_{j\in\mathbf{Z}} V_j}$ is a closed translation invariant subspace of $L^2(\mathbf{R}^d)$ then they furnished a more general answer for this question based on the following well-known theorem on the characterization of the closed translation invariant subspaces of $L^2(\mathbf{R}^d)$. It's proof can be found in [Ru] Chapter 9.

Theorem 4.5. *A subspace $M\subseteq L^2(\mathbf{R}^d)$ is closed and translation invariant if and only if $\mathcal{F}(M) = L^2(\Omega)$ for some $\Omega\subseteq\mathbf{R}^d$, where $\mathcal{F}(M) := \{\mathcal{F}f\mid f\in M\}$.*

Our purpose in the next two sections is to formulate a definition of MRA for spaces of the form $L^2(G)$ and establish similar results to Theorem 4.2 and Theorem 4.3.

3   Extension of multiresolution analysis to $L^2(G)$ and intersection triviality

On the basis of the content of the above, now we can adapt the definition of MRA for $L^2(\mathbf{R}^d)$ to one for $L^2(G)$, where G is in a certain class of groups to be approximately identified in the appendix out of second countable, type I, unimodular locally compact groups. Since the intersection triviality property is a direct consequence of the other conditions of the definition of an MRA, we prove this property immediately after we give the difinition of an MRA.

First let us keep in mind definition 2.2 in chapter 2.

We begin by properly interpreting MRA of $L^2(\mathbf{R}^d)$. It is obvious that $\mathbf{Z}^d$ is a lattice subgroup of $\mathbf{R}^d$ and $\mathbf{R}^d/\mathbf{Z}^d \cong \mathbf{T}^d$, where $\mathbf{T}^d$ is the d-torus. The shift-invariance of V_0 in (v) of definition 2.2 can be interpreted as an invariance property with respect to the action of the discrete lattice subgroup $\mathbf{Z}^d$ of $\mathbf{R}^d$. The scaling matrix D can be viewed as the action of some group automorphism of $\mathbf{R}^d$, with the property $D\mathbf{Z}^d \subset \mathbf{Z}^d$ and $1 < [\mathbf{Z}^d : D\mathbf{Z}^d] < \infty$. Also, we have to observe one very special thing that the lattices $2^{-j}\mathbf{Z}^d$ in $\mathbf{R}^d$ form a nested sequence whose union $\bigcup_{j \in \mathbf{Z}} 2^{-j}\mathbf{Z}^d$ is dense in $\mathbf{R}^d$. According to Yves Meyer, "approximation to $L^2(\mathbf{R}^d)$ by the nested sequence of closed subspaces $\{ V_j \}_{j \in \mathbf{Z}}$ imitates and reflects the geometric approximation to $\mathbf{R}^d$ by the nested sequence of lattices $2^{-j}\mathbf{Z}^d$" (see p.69 [Me] for details). Furthermore, we have to note that the roles played by the dilation operator D and the translation operator T_x.

With this in mind, it is not difficult to conjecture the correct generalization of MRA to second countable, type I, unimodular locally compact groups. Indeed, suppose G is such a group. (a) First, we shall suppose that G contains a discrete subgroup Γ such that the quotient G/Γ is compact, where Γ is discrete means that the topology on Γ induced from

G is the discrete topology. It is worthwhile noting that for connected nilpotent Lie groups, such a discrete lattice subgroup Γ of G often exists, see [Ra] for details. This Γ will play the same role in G as $\mathbf{Z}^d$ in $\mathbf{R}^d$. (b) Furthermore, we shall assume that there exists a dilative topological automorphism α (hence α^{-1} is contractive) of G onto G such that $\alpha\Gamma\subset\Gamma$ and $1 < [\Gamma : \alpha\Gamma] < \infty$, where α is a topological automorphism means that α is an automorphism and a homeomorphism and α^{-1} is contractive means that for any fixed compact subset K of G and for any neighborhood U of the identity e, there is a positive integer N, depending on K and U, such that

$$\alpha^{-j}K\subset U, \ \forall j > N.$$

The most important consequence of the above assumptions is that the union $\bigcup_{j\in\mathbf{Z}} \alpha^{-j}\Gamma$ is dense in G:

Proposition 4.6. *Let G be a locally compact group. Suppose that G contains a discrete subgroup Γ such that the quotient G/Γ is compact and that there exists a dilative topological automorphism α of G onto G such that $\alpha\Gamma\subset\Gamma$ and $1 < [\Gamma : \alpha\Gamma] < \infty$. Then the union $\bigcup_{j\in\mathbf{Z}} \alpha^{-j}\Gamma$ is dense in G.*

proof. From the assumption (a), we can write $G = \bigcup_{\gamma\in\Gamma} \gamma^{-1}K$, where K is some compact subset of G. For any $x\in G$, given any neighborhood U of x, we have $x^{-1}U$ is a neighborhood of the identity e. By one of the basic properties of topological groups, there is a symmetric neighborhood V of e such that $V\subset x^{-1}U$. So we have $xV\subset U$. By the assumption (b), for V. there exists an integer $k\in\mathbf{Z}$ such that $\alpha^{-k}(K)\subset V$. So $G = \bigcup_{\gamma\in\Gamma} \gamma^{-1}K$ implies that $G = \alpha^{-k}(G) = \bigcup_{\gamma\in\Gamma} \alpha^{-k}(\gamma^{-1})\alpha^{-k}(K)\subset \bigcup_{\gamma\in\Gamma} \alpha^{-k}(\gamma^{-1})V = G$. Hence there exists some $\gamma\in\Gamma$ such that $x\in\alpha^{-k}(\gamma^{-1})V$. So $x = \alpha^{-k}(\gamma^{-1})v$ for some $v\in V$. But then $\alpha^{-k}(\gamma^{-1}) = xv^{-1}\in xV^{-1} = xV\subset U$. This means that intersection of $\bigcup_{j\in\mathbf{Z}} \alpha^{-j}\Gamma$ and U is not empty. Therefore, $\bigcup_{j\in\mathbf{Z}} \alpha^{-j}\Gamma$ is dense in G. This finishes the proof of proposition 4.6.

We note that this α will act as the dilation in G similar to D in $\mathbf{R}^d$. (c) Finally, we have to generalize two operators, that is, the dilation operator and translation operator. For the former, we can define the dilation operator σ as

$$\sigma f(x) := \delta_\alpha^{1/2} f(\alpha x), \ \forall f\in L^2(G), \ x\in G, \tag{4.3}$$

where δ_α is a proper positive constant depending on α such that the operator σ becomes a unitary operator.

As to the latter, we need to explain more. Remember that we are now dealing with non-abelian groups. So talking about the translation operator, there are two kinds of translations: left translation L_x and right translation R_x. But the following proposition suggests that we may only pay attention to the one sided translations without considering the other ones.

Proposition 4.7. *Given a function ϕ, let V^l be the closed subspace generated by the left shifts of ϕ, that is, $\{L_\gamma\phi|\gamma\in\Gamma\}$, V^r the closed subspace generated by the right shifts of ϕ. Then the images under the Fourier transform of V^l and V^r are both supported on the same subset $\mathrm{supp}\mathcal{F}(\phi)$.*

Proof. By the definition of V^l, we have

$$f(x) = \sum_{\gamma\in\Gamma} a(\gamma)L_\gamma\phi(x)$$

where $\{a(\gamma)\}_{\gamma\in\Gamma}\in l^2(\Gamma)$.

So we have

$$(\mathcal{F}f)(\pi) = \int_G \sum_{\gamma\in\Gamma} a(\gamma)L_\gamma\phi(x)\pi(x)dx$$

$$= \sum_{\gamma\in\Gamma} a(\gamma)\int_G \phi(\gamma^{-1}x)\pi(x)dx$$

The substitution $y = \gamma^{-1}\cdot x$ gives

$$(\mathcal{F}f)(\pi)$$

$$= \sum_{\gamma\in\Gamma} a(\gamma)\int_G \phi(y)\pi(\gamma\cdot y)dy$$

$$= \sum_{\gamma\in\Gamma} a(\gamma)\pi(\gamma)\int_G \phi(y)\pi(y)dy$$

$$= \sum_{\gamma\in\Gamma} a(\gamma)\pi(\gamma)(\mathcal{F}\phi)(\pi).$$

44

Similarly, for $f \in V^r$, we have

$$(\mathcal{F}f)(\pi) = \int_G \sum_{\gamma \in \Gamma} b(\gamma) R_\gamma \phi(x) \pi(x) dx$$

$$= \sum_{\gamma \in \Gamma} b(\gamma) \int_G \phi(x\gamma) \pi(x) dx$$

$$= \sum_{\gamma \in \Gamma} b(\gamma)(\mathcal{F}\phi)(\pi) \pi(\gamma^{-1}).$$

The proposition is proved.

After this preparation, we can give a definition of MRA for $L^2(G)$.

Definition 4.8. *We say that a sequence of closed subspaces $\{V_j\}_{j \in \mathbf{Z}}$ of $L^2(G)$ forms a multiresolution of $L^2(G)$ if the following conditions are satisfied:*

(i) $V_j \subset V_{j+1}$, $j \in \mathbf{Z}$;

(ii) $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(G)$;

(iii) $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$;

(iv) $f \in V_j \iff \sigma f \in V_{j+1}$. *In other words, $V_j = \sigma^j V_0$, $j \in \mathbf{Z}$, where σ is defined by (4.3);*

(v) V_0 *is left shift-invariant, that is, if $f \in V_0$ then so is $L_\gamma f$ for all γ in Γ;*

(vi) there is a function $\phi \in V_0$, called the scaling function, or generator of the MRA, such that the collection $\{L_\gamma \phi \mid \gamma \in \Gamma\}$ is an orthonormal basis for V_0.

Remarks. (1) For the scaling function, some people, e.g., Meyer [Me], impose regularity and decay conditions on ϕ. In our case, to make the argument simple and general, we generally require only that $\phi \in L^2(G)$.

(2) In analogy with $L^2(\mathbf{R}^d)$, we say that V is refinable if, for any $f \in V$, $\sigma^{-1} f$ is also in V. Thus the condition (i) is equivalent to saying that V_0 is refinable. Like the Euclidean case, for $\phi \in L^2(G)$, we denote $\mathcal{V}(\phi)$ to be the smallest closed left shift-invariant subspace of $L^2(G)$ containing ϕ. We say that ϕ is refinable if $\mathcal{V}(\phi)$ is. Thus, the basic question concerning MRA is whether the scaling function exists. We will see that for Heisenberg groups, such scaling functions do exist. We will enter into details in Chapter 5 for this special case. As for general groups, no existing argument is available. This would seem to be a reasonable next goal.

45

(3) Like in [Me], we call the function ϕ the father wavelet. The basic rule of father wavelet is to generate an MRA for the space $L^2(G)$. Thus, the father wavelet ϕ must be a refinable function in order to have a sequence of nested closed subspaces. As we know from the results by de Boor, Devore and Ron, the refinability of ϕ is not enough for ϕ to generate an MRA; we need other requirements for ϕ. We will consider this in detail in the next section.

Generally speaking, the way to construct an MRA is to start with a refinable function ϕ such that $\{L_\gamma \phi | \gamma \in \Gamma\}$ is an orthonormal set. Let V_0 be the closed linear span of $\{L_\gamma \phi | \gamma \in \Gamma\}$ and $V_j = \sigma^j V_0$, for $j \in \mathbf{Z}$. Then conditions (i),(iv),(v) and (vi) are automatically satisfied.

In closing this section, we prove the following intersection triviality theorem. This theorem shows that condition (iii) (intersection triviality) actually follows from (iv) and (vi).

Theorem 4.9(triviality of intersection). *Let ϕ be a refinable element of $L^2(G)$ and define $V_0 = \mathcal{V}(\phi)$ and $V_j = \sigma^j V_0$ for $j \in \mathbf{Z}$. Suppose that left shifts of ϕ, that is, $\{L_\gamma \phi \mid \gamma \in \Gamma\}$, constitutes a frame for V_0, then $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$.*

Note: The concept of a frame is a generalization of orthonormal basis. A countable subset $\{e_n\}$ of $\mathcal{H}$ is said to be a frame if there exist two positive numbers α and β so that, for any f in $\mathcal{H}$,

$$\alpha \|f\|^2 \leq \sum_n | < f, e_n > |^2 \leq \beta \|f\|^2.$$

We call α and β the frame bounds. If the two frame bounds are equal, $\alpha = \beta$, then we call the frame a tight frame. Frames were introduced by Duffin and Schaeffer [DS] in 1952.

Proof of Theorem 4.9. We know that $\{L_\gamma \phi \mid \gamma \in \Gamma\}$ constitutes a frame for V_0. This means that, there exist constants $A > 0$ and $B > 0$, such that for all $f \in V_0$,

$$A \|f\|_{L^2(G)}^2 \leq \sum_{\gamma \in \Gamma} | < f, L_\gamma(\phi) > |^2 \leq B \|f\|_{L^2(G)}^2. \tag{4.4}$$

Now we know that σ is an unitary operator from $L^2(G) \longrightarrow L^2(G)$ and $V_j = \sigma^j V_0$. Replacing f and $L_\gamma \phi$ by $\sigma^j f$ and $\sigma^j L_\gamma \phi$ in (4.4), respectively, we get that, for all $f \in V_j$,

$$A \|f\|_{L^2(G)}^2 \leq \sum_{\gamma \in \Gamma} | < f, \sigma^j L_\gamma(\phi) > |^2 \leq B \|f\|_{L^2(G)}^2. \tag{4.5}$$

46

Now let $f \in \bigcap_{j \in \mathbf{Z}} V_j$. Since $C_c(G)$ (all compactly supported and continuous functions on G) is dense in $L^2(G)$, for any $\epsilon > 0$, there exists a $f_1 \in C_c(G)$ such that $\|f - f_1\|_{L^2(G)} \leq \epsilon$. Recall that $\mathcal{P}_j$ is the orthogonal projection from $L^2(G)$ onto V_j. Then $\|f - \mathcal{P}_j f_1\|_{L^2(G)} = \|\mathcal{P}_j(f - f_1)\|_{L^2(G)} \leq \|f - f_1\|_{L^2(G)} \leq \epsilon$. Hence

$$\|f\|_{L^2(G)} \leq \epsilon + \|\mathcal{P}_j f_1\|_{L^2(G)}, \quad \text{for all } j \in \mathbf{Z}. \tag{4.6}$$

Now we estimate $\|\mathcal{P}_j f_1\|_{L^2(G)}$. By (4.5), we have

$$\|\mathcal{P}_j f_1\|_{L^2(G)} \leq A^{-\frac{1}{2}} [\sum_{\gamma \in \Gamma} | < \mathcal{P}_j f_1, \sigma^j L_\gamma \phi > |^2]^{\frac{1}{2}}$$

$$= A^{-\frac{1}{2}} [\sum_{\gamma \in \Gamma} | < f_1, \sigma^j L_\gamma \phi > |^2]^{\frac{1}{2}}$$

$$= A^{-\frac{1}{2}} [\sum_{\gamma \in \Gamma} | \int_G f_1(x) \delta_\alpha^{j/2} \overline{\phi(\gamma^{-1} \cdot \alpha^j(x))} dx|^2]^{\frac{1}{2}}$$

$$\leq A^{-\frac{1}{2}} [\sum_{\gamma \in \Gamma} (\int_G |f_1(x) \delta_\alpha^{j/2} \overline{\phi(\gamma^{-1} \cdot \alpha^j(x))}| dx)^2]^{\frac{1}{2}}$$

Choose a compact subset K of G so that $\text{supp}(f_1) \subseteq K$. Hence,

$$\|\mathcal{P}_j f_1\|_{L^2(G)}$$

$$\leq \delta_\alpha^{j/2} A^{-\frac{1}{2}} \|f_1\|_{L^\infty(G)} [\sum_{\gamma \in \Gamma} (\int_K |\phi(\gamma^{-1} \cdot \alpha^j(x))| dx)^2]^{\frac{1}{2}}$$

By the Cauchy-Schwartz inequality, we have

$$\|\mathcal{P}_j f_1\|_{L^2(G)}$$

$$\leq \delta_\alpha^{j/2} A^{-\frac{1}{2}} \|f_1\|_{L^\infty(G)} |K|^{1/2} [\sum_{\gamma \in \Gamma} \int_K |\phi(\gamma^{-1} \cdot \alpha^j(x))|^2 dx]^{\frac{1}{2}},$$

where $|K|$ denotes the measure of K. For $j < 0$, $|j|$ sufficiently large, we get

$$\|\mathcal{P}_j f_1\|_{L^2(G)}$$

$$\leq A^{-\frac{1}{2}} \|f_1\|_{L^\infty(G)} |K|^{1/2} [\int_{E_j} |\phi(x')|^2 dx']^{1/2},$$

47

where

$$E_j := \bigcup_{\gamma \in \Gamma} \gamma^{-1} \cdot \alpha^j(K).$$

Thus,

$$\|\mathcal{P}_j f_1\|_{L^2(G)}$$

$$\leq A^{-\frac{1}{2}} \|f_1\|_{L^\infty(G)} |K|^{1/2} [\int_G \chi_{E_j}(x) |\phi(x)|^2 dx]^{1/2} \tag{4.7}$$

where χ_E is the indicator function of E, i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ if x is not in E. By the contractive property of α^{-1}, for x not in Γ, we have $\chi_{E_j}(x) \to 0$ as $j \to -\infty$. It therefore follows from the Lebesgue dominated convergence theorem that (4.7) tends to 0 as $j \to -\infty$. So (4.6) says that $\|f\|_{L^2(G)} \leq \epsilon$. This yields $f = 0$. This concludes the proof.

The importance of theorem 4.9 is that it is not necessary to verify the intersection triviality property in the definition of an MRA. As for the density property, we consider this topic in the next section.

4   Union Density

This section is devoted to the theorem on the density of the union with respect to the definition 4.8. In the proof of proposition 4.7, we found that the image under the Fourier transform of either a left shift-invariant subspace generated by ϕ or a left and right shift-invariant subspace generated by the same ϕ is both supported on the same subset supp$\mathcal{F}(\phi)$. Now since we only use left side translations, to get the proof of the union density theorem for a left MRA, we have to put an extra condition on the scaling function ϕ. This condition involves a generalization of the concept of the support of the Fourier transform. It is found that this extra requirement on ϕ is very natural from the explanation given below. The result to be established will answer the basic question: under what conditions does a function ϕ produce an MRA of $L^2(G)$ provided that ϕ is a refinable function? Or equally, what are the requirements for a refinable function ϕ to be a father wavelet?

Let Ω be a non-null measurable subset of $\widehat{G}$. Recall that $\mathcal{H}^2(\Omega)$ is the subspace of $\mathcal{H}^2(\widehat{G})$ consisting of all $F \in \mathcal{H}^2(\widehat{G})$ such that $F = 0$ a.e. on $\widehat{G} \backslash \Omega$. Let $\{g_i\}_{i \in I}$ be a family of

elements of $L^2(G)$. We say that $\{g_i\}_{i\in I}$ is strongly supported on Ω if, for any $F\in\mathcal{H}^2(\Omega)$, $<\hat{x}\hat{g}_i|F>=0$, for all $x\in G$ and $i\in I$ implies $F=0$.

To help understand the concept of "strongly supported" consider the case of $G=\mathbf{R}$. Then $\hat{G}$ can be identified with $\mathbf{R}$ as well. For $\Omega\subseteq\mathbf{R}$ and a family $\{g_i\}_{i\in I}$ in $L^2(G)$, $\{g_i\}_{i\in I}$ is strongly supported on Ω if $\{e_t\hat{g}_i : t\in\mathbf{R}, i\in I\}$ is a total set in $L^2(\Omega)$, where $e_t(\gamma)=e^{2\pi it\gamma}$ for $\gamma\in\mathbf{R}$, $t\in\mathbf{R}$. More particularly, consider a single function $g\in L^1(\mathbf{R})\cap L^2(\mathbf{R})$ so $\hat{g}$ is a continuous function. Then g is strongly supported on any non-null subset of $\{\gamma\in\mathbf{R} : \hat{g}(\gamma)\neq0\}$ and is not strongly supported on any subset of $\{\gamma\in\mathbf{R} : \hat{g}(\gamma)=0\}$.

We need to introduce some other concepts for a family $\{g_i\}_{i\in I}$ of functions in $L^2(G)$. We say that $\{g_i\}_{i\in I}$ is a left nonzero divisor in $L^2(G)$ if, for any $f\in L^2(G)$, $g_i*f=0$, for all $i\in I$, implies $f=0$. Note that g, $f\in L^2(G)$ implies $g*f\in C_0(G)$, the space of continuous functions vanishing at infinity on G.

Let $V(\{g_i\}_{i\in I})$ denote the smallest closed left translation invariant subspace of $L^2(G)$ containing $\{g_i : i\in I\}$. That is,

$$V(\{g_i\}_{i\in I}) = \overline{<\{L_x g_i : x\in G, i\in I\}>}$$

$$= (\{L_x g_i : x\in G, i\in I\}^\perp)^\perp.$$

Let

$$\mathcal{F}V(\{g_i\}_{i\in I}) = \{\hat{g} : g\in V(\{g_i\}_{i\in I})\}$$

$$= \overline{<\{\widehat{L_x g_i} : x\in G, i\in I\}>}$$

$$= (\{\hat{x}\hat{g}_i : x\in G, i\in I\}^\perp)^\perp.$$

The following proposition brings these concepts together.

Proposition 4.10. *Let $\{g_i\}_{i\in I}$ be a family of functions in $L^2(G)$. The following conditions are equivalent.*

(a) $V(\{g_i\}_{i\in I}) = L^2(G)$

(b) $\mathcal{F}V(\{g_i\}_{i\in I}) = \mathcal{H}^2(\hat{G})$

(c) $\{g_i\}_{i\in I}$ *is a left nonzero divisor in* $L^2(G)$

(d) $\{g_i\}_{i\in I}$ *is strongly supported on* $\widehat{G}$

Proof. (a) $\Longleftrightarrow$ (b) is clear since $\mathcal{F}$ is a unitary map onto $\mathcal{H}^2(\widehat{G})$.

(a) $\Longleftrightarrow$ (c). For g, $f \in L^2(G)$,

$$g*f(x) = \int_G g(y)f(y^{-1}x)dy$$

$$= \int_G g(y)\overline{f^*}(x^{-1}y)dy$$

$$= \int_G g(xy)\overline{f^*}(y)dy$$

$$= <L_{x^{-1}}g|f^*> .$$

Suppose $g_i*f = 0$, for all $i\in I$. Then $f^* \perp L_{x^{-1}}g_i$, for all $x\in G$, $i\in I$ which implies $f^* \perp V(\{g_i\}_{i\in I})$. If $V(\{g_i\}_{i\in I}) = L^2(G)$, then we must have $f^* = 0$ and, hence, $f = 0$. Thus, (a) implies (c). Conversely, suppose (c) holds and $f \perp V(\{g_i\}_{i\in I})$. Then $< L_{x^{-1}}g_i|f > = 0$, for all $x\in G$ and $i\in I$. Therefore, by the above computation again, $g_i*f^* = 0$, for all $i\in I$. By (c), this implies $f^*=0$; so $f = 0$. Thus $V(\{g_i\}_{i\in I}) = L^2(G)$.

(b) $\Longleftrightarrow$ (d) follows easily from the definition of strongly supported.

Recall that G has a fixed discrete subgroup Γ and a subspace X of $L^2(G)$ is called left shift invariant if $L_\gamma X \subseteq X$, for all $\gamma \in \Gamma$. For $\phi \in L^2(G)$, $\mathcal{V}(\phi)$ denotes the smallest left shift invariant closed subspace of $L^2(G)$ containing ϕ. So

$$\mathcal{V}(\phi) = \overline{<\{L_\gamma\phi : \gamma\in\Gamma\}>}$$

$$= (\{L_\gamma\phi : \gamma\in\Gamma\}^\perp)^\perp.$$

Define $V_j = \sigma^j \mathcal{V}(\phi)$, for all $j\in\mathbf{Z}$. Recall that ϕ is refinable if $V_0 \subseteq V_1$ (then $V_j \subseteq V_{j+1}$, for all $j\in\mathbf{Z}$). Define

$$\phi_j(x) = \delta_\alpha^{j/2}\phi(\alpha^j(x)), \quad \text{for any } x\in G, \ j\in\mathbf{Z}.$$

Theorem 4.11 (Density of the union). *Let* ϕ *be a refinable function in* $L^2(G)$ *and* V_j, $j\in\mathbf{Z}$ *defined as above. Then the following are equivalent:*

(a) $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = L^2(G)$

50

(b) $\{\phi_j\}_{j\in\mathbf{Z}}$ *is a left nonzero divisor in* $L^2(G)$

(c) $\{\phi_j\}_{j\in\mathbf{Z}}$ *is strongly supported on all of* $\hat{G}$.

Before proving Theorem 4.11, we present two preliminary propositions.

Proposition 4.12. *With* ϕ, ϕ_j, V_j, *for* $j\in\mathbf{Z}$ *as in Theorem 4.11, we have*

$$V_j = \overline{< \{L_\lambda \phi_j : \lambda\in\alpha^{-j}\Gamma\} >}$$ *and, hence,* V_j *is invariant under shifts from the discrete subgroup* $\alpha^{-j}\Gamma$.

Proof. The map $\sigma : L^2(G)\longrightarrow L^2(G)$ given by $\sigma g(x) = \delta_\alpha^{1/2}g(\alpha(x))$ for $x\in G$, $g\in L^2(G)$ is a unitary operator. For any $j\in\mathbf{Z}$, $\gamma\in\Gamma$ and $x\in G$,

$$\sigma^j L_\gamma\phi(x) = \delta_\alpha^{j/2} L_\gamma\phi(\alpha^j(x))$$

$$= \delta_\alpha^{j/2}\phi(\gamma^{-1}\alpha^j(x))$$

$$= \delta_\alpha^{j/2}\phi(\alpha^j(\alpha^{-j}(\gamma^{-1})x))$$

$$= L_{\alpha^{-j}(\gamma)}\phi_j(x).$$

Thus, the unitary σ^j maps V_0 onto V_j and the generating set $\{L_\gamma\phi : \gamma\in\Gamma\}$ onto $\{L_\lambda\phi_j : \lambda\in\alpha^{-j}\Gamma\}$. This proves the proposition.

Proposition 4.13. *Let* V_j, $j\in\mathbf{Z}$ *be the nested sequence of closed subspace of* $L^2(G)$ *as in Theorem 4.11. Then* $\overline{\bigcup_{j\in\mathbf{Z}}V_j}$ *is a left translation invariant subspace of* $L^2(G)$.

Proof. Let $X := \bigcup_{j\in\mathbf{Z}}V_j$. Then, for any j and $\lambda\in\alpha^{-j}\Gamma$, we have that $\lambda\in\alpha^{-k}\Gamma$ for all $k\geq j$. By Proposition 4.12, V_j is invariant under L_λ and V_k is invariant under L_λ, for all $k\geq j$. Since the union is nested ($V_n\subseteq V_{n+1}$), X is invariant under L_λ. Thus, X and $\overline{X}$ are invariant under L_λ for any $\lambda\in\bigcup_{j\in\mathbf{Z}}\alpha^{-j}\Gamma$.

For any $f\in\overline{X}$, $L_\lambda f\in\overline{X}$, for all $\lambda\in\bigcup_{j\in\mathbf{Z}}\alpha^{-j}\Gamma$. By continuity of the left regular represetation in the strong operator topology and denseness of $\bigcup_{j\in\mathbf{Z}}\alpha^{-j}\Gamma$, $L_x f\in\overline{X}$, for all $x\in G$. Thus, $\overline{X}$ is left translation invariant.

Proof of Theorem 4.11. Proposition 4.12 and 4.13 imply that $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = V(\{\phi_j\}_{j\in\mathbf{Z}})$. Thus, Theorem 4.11 follows directly from Proposition 4.10.

Continuing with the set up of Theorem 4.11, let $\mathcal{P}_j$ denote the orthogonal projection of $L^2(G)$ onto V_j. If $j\leq k$, then $\mathcal{P}_j\mathcal{P}_k = \mathcal{P}_k\mathcal{P}_j = \mathcal{P}_j$. We have the following corollary of 4.11.

Corollary 4.14. *If $\{\phi_j\}_{j\in\mathbb{Z}}$ is a left nonzero divisor in $L^2(G)$, then for all $f\in L^2(G)$, $\lim_{j\to\infty}\|\mathcal{P}_j f - f\| = 0$. That is, the $\mathcal{P}_j$ converge to the identity operator in the strong operator topology in $L^2(G)$.*

5 Sufficient conditions of the density property

In this section, we develop a number of conditions on a refinable function ϕ that are sufficient for the union of the associated nested sequence of subspaces, $\bigcup_{j\in\mathbb{Z}}V_j$, to be dense in $L^2(G)$. By Theorem 4.11, we have two conditions, (b) and (c) of 4.11, to work with. We are looking for easily verified conditions on ϕ that imply $\{\phi_j\}_{j\in\mathbb{Z}}$ is a left nonzero divisor in $L^2(G)$ or is strongly supported on $\widehat{G}$. We will develop these sufficient conditions one by one and then gather them into a single summary theorem.

We begin by seeing how the automorphism α moves sets in $\widehat{G}$. For any represetation π of G, let $\pi^\alpha = \pi\circ\alpha$. That is, $\pi^\alpha(x) = \pi(\alpha(x))$, for all $x\in G$. Likewise, $\pi^{\alpha^{-1}} = \pi\circ\alpha^{-1}$. It is easily checked that π^α and $\pi^{\alpha^{-1}}$ are (strongly continuous unitary) represetations of G and that the following properties hold:

(i) $(\pi^\alpha)^{\alpha^{-1}} = (\pi^{\alpha^{-1}})^\alpha = \pi$

(ii) $\pi_1^\alpha = \pi_2^\alpha$ if and only if $\pi_1\sim\pi_2$

(iii) π^α is irreducible if and only if π is irreducible.

Then (i), (ii) and (iii) imply that there is a well-defined bijection, also denoted $\pi\to\pi^\alpha$, of $\widehat{G}$ onto $\widehat{G}$. For Type I unimodular second countable groups such as G, the σ-algebra of the Plancherel measure μ can be taken to be the Borel subsets related to the Fell topology (see [Fo1], Chapter 7 for these concepts). What we need about this topology here is that a net $\{\pi_t\}$ in $\widehat{G}$ converges to $\pi\in\widehat{G}$ if and only if, for any $\xi\in\mathcal{H}_\pi$, there exist $\xi_t^i\in\mathcal{H}_{\pi_t}, i = 1,\cdots,n_t$, for each t such that

$$\sum_{i=1}^{n_t} <\pi_t(x)\xi_t^i|\xi_t^i> \longrightarrow <\pi(x)\xi|\xi>$$

uniformly on compact subsets of G. From this it follows that $\pi_t\to\pi$ if and only if $\pi_t^\alpha\to\pi^\alpha$. Thus, $\pi\to\pi^\alpha$ is a homeomorphism of $\widehat{G}$. All we need in this thesis is that this map is a

measurable bijection with inverse $\pi \rightarrow \pi^{\alpha^{-1}}$.

For a subset $E \subseteq \widehat{G}$, let $\alpha(E) = \{\pi^{\alpha^{-1}} : \pi \in E\}$. Define a new measure μ_α on $\widehat{G}$ by

$$\mu_\alpha(E) = \mu(\alpha(E)), \quad \text{for any measurable } E \subseteq \widehat{G}.$$

So for any integrable function F on $\widehat{G}$,

$$\int_{\widehat{G}} F(\pi^\alpha) d\mu(\pi) = \int_{\widehat{G}} F(\pi) d\mu_\alpha(\pi).$$

Proposition 4.15. *If μ is the fixed Plancherel measure on $\widehat{G}$, then*

$$\mu_\alpha = \delta_\alpha^{-1} \mu.$$

Proof. The Plancherel Theorem, listed in Section 3.7, says that, for $f, g \in L^2(G) \cap L^1(G)$,

$$\int_G f(x)\overline{g(x)} \, dx = \int_{\widehat{G}} tr[\hat{g}(\pi)^* \hat{f}(\pi)] \, d\mu(\pi)$$

and this uniquely determines the measure μ. In order to track the effect of α, define for $f \in L^2(G)$,

$$\tilde{\alpha}f(x) = f(\alpha^{-1}(x)), \quad \text{for all } x \in G.$$

Then, for $f \in L^2(G) \cap L^1(G)$,

$$\widehat{\tilde{\alpha}f}(\pi) = \int_G \tilde{\alpha}f(x)\pi(x) \, dx$$

$$= \int_G f(\alpha^{-1}(x))\pi(x) \, dx$$

$$= \delta_\alpha \int_G f(x)\pi(\alpha(x)) \, dx$$

$$= \delta_\alpha \int_G f(x)\pi^\alpha(x) \, dx$$

$$= \delta_\alpha \hat{f}(\pi^\alpha).$$

For $f, g \in L^2(G) \cap L^1(G)$, we then have

$$\int_G \tilde{\alpha}f(x)\overline{\tilde{\alpha}g}(x) \, dx = \int_{\widehat{G}} tr[\widehat{\tilde{\alpha}g}(\pi)^* \widehat{\tilde{\alpha}f}(\pi)] \, d\mu(\pi)$$

53

$$= \delta_\alpha^2 \int_{\widehat{G}} tr[\hat{g}(\pi^\alpha)^* \hat{f}(\pi^\alpha)] \, d\mu(\pi)$$

$$= \delta_\alpha^2 \int_{\widehat{G}} tr[\hat{g}(\pi)^* \hat{f}(\pi)] \, d\mu_\alpha(\pi).$$

On the other hand,

$$\int_G \bar{\alpha} f(x) \overline{\bar{\alpha} g}(x) \, dx = \int_G f(\alpha^{-1}(x)) \bar{g}(\alpha^{-1}(x)) \, dx$$

$$= \delta_\alpha \int_G f(x) \bar{g}(x) \, dx.$$

So

$$\int_{\widehat{G}} tr[\hat{g}(\pi)^* \hat{f}(\pi)] \, d\mu_\alpha(\pi) = \delta_\alpha^{-1} \int_G f(x) \bar{g}(x) \, dx$$

$$= \delta_\alpha^{-1} \int_{\widehat{G}} tr[\hat{g}(\pi)^* \hat{f}(\pi)] \, d\mu(\pi).$$

This implies that μ_α is the Plancherel measure associated with a new Haar measure on G (δ_α^{-1} times the original Haar measure) and that $\mu_\alpha = \delta_\alpha^{-1}\mu$.

From Proposition 4.15, we get the change of variable formula for integrable functions F on $\widehat{G}$,

$$\int_{\widehat{G}} F(\pi^\alpha) \, d\mu(\pi) = \delta_\alpha^{-1} \int_{\widehat{G}} F(\pi) \, d\mu(\pi).$$

Proposition 4.16. *Let $f \in L^2(G)$ and Ω be a measurable subset of $\widehat{G}$. Then f is strongly supported on Ω if and only if $\bar{\alpha} f$ is strongly supported on $\alpha(\Omega)$.*

Proof. For $F \in \mathcal{H}^2(\widehat{G})$, let $\widetilde{\alpha^{-1}} F(\pi) = F(\pi^{\alpha^{-1}})$, for all $\pi \in \widehat{G}$. Note that $F \in \mathcal{H}^2(\alpha(\Omega))$ if and only if $\widetilde{\alpha^{-1}} F \in \mathcal{H}^2(\Omega)$.

Now suppose f is strongly supported on Ω. Let $F \in \mathcal{H}^2(\alpha(\Omega))$ be such that

$$< \hat{x} \widehat{\bar{\alpha} f} | F > = 0, \text{ for all } x \in G.$$

That is,

$$0 = \int_{\widehat{G}} tr[F(\pi)^* \hat{x}(\pi) \widehat{\bar{\alpha} f}(\pi)] \, d\mu(\pi)$$

$$= \delta_\alpha \int_{\widehat{G}} tr[F(\pi)^* \pi(x) \hat{f}(\pi^\alpha)] \, d\mu(\pi)$$

54

$$= \int_{\widehat{G}} tr[F(\pi^{\alpha^{-1}})^* \pi^{\alpha^{-1}}(x) \hat{f}(\pi)] \, d\mu(\pi)$$

$$= \int_{\widehat{G}} tr[\widetilde{\alpha^{-1}} F(\pi)^* \widetilde{\alpha^{-1}}(x)(\pi) \hat{f}(\pi)] \, d\mu(\pi)$$

$$= < \widetilde{\alpha^{-1}}(x) \hat{f} | \widetilde{\alpha^{-1}} F >, \text{ for all } x \in G.$$

Since $\alpha^{-1}(x)$ runs through G as x runs through G and f is strongly supported on Ω, $\widetilde{\alpha^{-1}} F = 0$ which implies $F = 0$. Thus, $\tilde{\alpha} f$ is strongly supported on $\alpha(\Omega)$. The converse follows from the same argument applied to α^{-1}.

A measurable subset Ω of $\widehat{G}$ is called α-absorbing if $\mu(\widehat{G} \backslash \bigcup_{j \in \mathbf{Z}} \alpha^j(\Omega)) = 0$.

Proposition 4.17. *Let $\phi \in L^2(G)$ and suppose Ω is an α- absorbing subset of $\widehat{G}$ such that the following two conditions are satisfied:*

(i) ϕ is strongly supported on Ω.

(ii) $\hat{\phi}(\pi) = 0$, for a.e., $\pi \in \widehat{G} \backslash \Omega$.

Then $\{\phi_j\}_{j \in \mathbf{Z}}$ is strongly supported on $\widehat{G}$.

Proof. Because of (ii), $\phi \in \mathcal{H}^2(\Omega)$. Using the fact that $\phi_j = \delta^{j/2} \widetilde{\alpha^{-j}} \phi$, we see that ϕ_j is strongly supported on $\alpha^{-j}(\Omega)$ and $\phi_j \in \mathcal{H}^2(\alpha^j(\Omega))$, for any $j \in \mathbf{Z}$. Let $\mathcal{P}_j$ denote the projection of $\mathcal{H}^2(\widehat{G})$ onto $\mathcal{H}^2(\alpha^j(\Omega))$. Since Ω is α-absorbing, $\vee_{j \in \mathbf{Z}} \mathcal{P}_j$ is the identity operator on $\mathcal{H}^2(\widehat{G})$. Also $\mathcal{P}_j \hat{x} \hat{\phi}_j = \hat{x} \hat{\phi}_j$, for any $x \in G$ and $j \in \mathbf{Z}$.

Now suppose $F \in \mathcal{H}^2(\widehat{G})$ and $< \hat{x} \hat{\phi}_j | F >= 0$, for all $x \in G$, $j \in \mathbf{Z}$. Then, for each $j \in \mathbf{Z}$,

$$< \hat{x} \hat{\phi}_j | \mathcal{P}_j F >=< \mathcal{P}_j \hat{x} \hat{\phi}_j | F >$$

$$=< \hat{x} \hat{\phi}_j | F >$$

$$= 0,$$

for any $x \in G$, $j \in \mathbf{Z}$. Since ϕ_j is strongly supported on $\alpha^j(\Omega)$, $\mathcal{P}_j F = 0$, for each $j \in \mathbf{Z}$. Since $\vee_{j \in \mathbf{Z}} \mathcal{P}_j$ is the identity operator, this implies $F = 0$. Thus, $\{\phi_j\}_{j \in \mathbf{Z}}$ is strongly supported on $\widehat{G}$.

Proposition 4.17 gives conditions (i) and (ii) that, when combined with refinability of a function ϕ, is sufficient for ϕ to generate a multiresolution analysis. However, checking that (i) and (ii) hold requires very detailed knowledge of $\widehat{G}$ and $\mathcal{H}^2(\widehat{G})$. If G is abelian, we

have enough knowledge (in fact, (i) and (ii) can be combined to something like $\mathrm{supp}(\hat{\phi})=\Omega$ in the abelian case). Let us turn to properties that imply condition (b) of Theorem 4.11.

Recall that if $g \in L^1(G)$ and $f \in L^2(G)$, then $g*f \in L^2(G)$ and $\|g*f\|_2 \leq \|g\|_1 \|f\|_2$. A sequence $\{g_n : n = 1, 2, 3, \cdots\}$ in $L^1(G)$ is called a left approximate identity for $L^2(G)$ if

$$\lim_{n \to \infty} \|g_n*f - f\|_2 = 0, \text{ for all } f \in L^2(G).$$

Note that left approximate identities for $L^2(G)$ exist in great abundance as the following proposition shows.

Proposition 4.18. *Let* $g \in L^1(G)$ *such that the support of* g *is compact,* $g \geq 0$ *and* $\int_G g(x)\, dx = 1$. *Define* $g_n(x) = \delta_\alpha^n g(\alpha^n(x))$, *for* $x \in G$, $n \in \mathbf{N}$. *Then* $\{g_n : n = 1, n = 2, n = 3, \cdots\}$ *is a left approximate identity for* $L^2(G)$.

Proof. Since $\int_G g_n(x)\, dx = 1$, for any $f \in L^2(G)$,

$$g_n*f(y) - f(y) = \int_G g_n(x) f(x^{-1}y)\, dx - \int_G g_n(x) f(y)\, dx$$

$$= \int_G g_n(x)[f(x^{-1}y) - f(y)]\, dx$$

$$= \int_G g_n(x)[L_x f(y) - f(y)]\, dx.$$

There is a generalization of Minkowski's inequality that applies to integrals (see Dunford and Schwartz, VI.11.13). Applied here, it says

$$\|g_n*f - f\|_2 = \{\int_G |g_n*f(y) - f(y)|^2\, dy\}^{1/2}$$

$$= \{\int_G |\int_G [L_x f(y) - f(y)]g_n(x)\, dx|^2\, dy\}^{1/2}$$

$$\leq \int_G \{\int_G |L_x f(y) - f(y)|^2\, dy\}^{1/2} g_n(x)\, dx$$

$$= \int_G \|L_x f - f\|_2 g_n(x)\, dx.$$

Let K be a compact set so that $g(x) = 0$ for a.e., $x \in G \backslash K$. For any $\varepsilon > 0$, there is a neighbourhood U of e in G such that

$$\|L_x f - f\|_2 < \varepsilon, \text{ for all } x \in U.$$

56

This is simply the strong operator continuity of the left regular representation. Since α^{-1} is a contractive automorphism of G, there exists an $n_0 \in \mathbf{N}$ such that $\alpha^{-n}(K) \subseteq U$ for $n \geq n_0$. But the support of g_n is contained in $\alpha^{-n}(K)$ so $n \geq n_0$ and $g_n(x) \neq 0$ implies $\|L_x f - f\|_2 < \varepsilon$ for a.e., x. Therefore, $n \geq n_0$ implies

$$\|g_n * f - f\|_2 \leq \int_G \|L_x f - f\|_2 g_n(x) \, dx$$

$$\leq \varepsilon \int_G g_n(x) \, dx = \varepsilon.$$

Thus, $\{g_n : n = 1, n = 2, n = 3, \cdots\}$ is a left approximate identity for $L^2(G)$.

Now suppose $\phi \in L^1(G) \bigcap L^2(G)$ is a refinable function, we have $\phi_j(x) = \delta_\alpha^{j/2} \phi(\alpha^j(x))$, for $x \in G$ and all $j \in \mathbf{Z}$. Let's take the forward half of this sequence and renormalize it to have constant L^1-norm. Let

$$g_n = \left(\frac{\delta_\alpha^{n/2}}{\|\phi\|_1}\right)\phi_n, \text{ for } n = 1, n = 2, n = 3, \cdots.$$

If $\{g_n : n = 1, n = 2, n = 3, \cdots\}$ happens to form a left approximate identity for $L^2(G)$, then $\{\phi_j\}_{j \in \mathbf{Z}}$ is a left nonzero divisor in $L^2(G)$. To see this, suppose $f \in L^2(G)$ and $\phi_j * f = 0$, for all $j \in \mathbf{Z}$. Then

$$g_n * f = \left(\frac{\delta_\alpha^{n/2}}{\|\phi\|_1}\right)\phi_n * f = 0, \text{ for } n = 1, n = 2, n = 3, \cdots.$$

But $g_n * f \to f$ in $L^2(G)$, so this forces $f = 0$.

We have established two distint kinds of sufficient conditions for a refinable function ϕ to generate a nest of subspaces whose union is dense in $L^2(G)$ and we formally put these conditions in a theorem.

Theorem 4.19. *Let ϕ be a refinable function in $L^2(G)$. Let $\mathcal{V}(\phi) = \overline{< \{L_\gamma \phi : \gamma \in \Gamma\} >}$ and $V_j = \sigma^j \mathcal{V}(\phi)$, for $j \in \mathbf{Z}$. Let $\phi_j = \sigma^j \phi$ for $j \in \mathbf{Z}$. If ϕ satisfies either of the following conditions*

(i) ϕ is strongly supported on an α-absorbing subset Ω of $\widehat{G}$ and $\hat{\phi}(\pi) = 0$, for almost all $\pi \in \widehat{G} \backslash \Omega$,

or

57

(ii) ϕ has compact support in G, $\phi \geq 0$ and $\phi \neq 0$,

then $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(G)$.

Proof. (i) implies $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(G)$ by Proposition 4.17 and (ii) implies $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(G)$ by Proposition 4.18 and the discussion following its proof.

To illustrate how condition (ii) in Theorem 4.19 might be useful, we consider the possibility of the system (G, Γ, α) having a self-similar tiling.

For the Euclidean group $\mathbf{R}^d$ with lattice $\mathbf{Z}^d$ and α being dilation by 2, the standard tiling by unit cubes is self-similar in the following sense. If $T = [0,1)^d$, then $\mathbf{R}^d = \bigcup_{n \in \mathbf{Z}^d}(T + n)$, a disjoint union and

$$2T = \bigcup_{n \in F}(T + n),$$

where F is a certain finite subset of $\mathbf{Z}^d$.

In our general situation, a measurable subset T of G will be called a tile for (G, Γ) if $\overline{T}$ is compact and $G = \bigcup_{\gamma \in \Gamma} \gamma T$, a disjoint union . If T is tile for (G, Γ) and there is a finite subset F of Γ such that

$$\alpha(T) = \bigcup_{\gamma \in F} \gamma T,$$

then T will be called a self-similar tile for (G, Γ, α).

Theorem 4.20. *Suppose T is a self-similar tile for (G, Γ, α) and let $\phi = \frac{1}{|T|^{1/2}} \chi_T$, where $|T|$ is the Haar measure of T. Then ϕ generates a multiresolution analysis for $L^2(G)$.*

Proof. As usual $\mathcal{V}(\phi) = \overline{\{L_\gamma \phi : \gamma \in \Gamma\}}$ and $V_j = \sigma^j \mathcal{V}(\phi)$, for $j \in \mathbf{Z}$. We check that the $\{V_j\}_{j \in \mathbf{Z}}$ satisfy all the conditions of Definition 4.8.

Since

$$\sigma^{-1}\phi(x) = \delta_\alpha^{-1/2}\phi(\alpha^{-1}(x))$$

$$= \frac{1}{\delta_\alpha^{1/2}|T|^{1/2}}\chi_T(\alpha^{-1}(x))$$

$$= \frac{1}{\delta_\alpha^{1/2}|T|^{1/2}}\chi_{\alpha(T)}(x)$$

$$= \frac{1}{\delta_\alpha^{1/2}|T|^{1/2}}\sum_{\gamma \in F}\chi_{\gamma T}(x)$$

$$= \frac{1}{\delta_\alpha^{1/2}|T|^{1/2}}\sum\nolimits_{\gamma\in F}\chi_T(\gamma^{-1}x)$$

$$= \delta_\alpha^{-1/2}\sum\nolimits_{\gamma\in F}L_\gamma\phi(x),$$

we have, by applying the unitary operator σ,

$$\phi = \delta_\alpha^{-1/2}\sum\nolimits_{\gamma\in F}\sigma(L_\gamma\phi).$$

Thus, $V_0 \subseteq V_1$ and $V_j = \sigma^j V_0 \subseteq \sigma^j V_1 = V_{j+1}$, for all $j \in \mathbb{Z}$. This verifies condition (i), in other words ϕ is refinable, of 4.8.

Condition (iv) and (v) of 4.8 are immediate and condition (vi) holds because T is a tile for (G, Γ). Condition (iii), intersection triviality follows from (i),(iv),(v) and (vi) by Theorem 4.9. Finally, condition (ii), union density follows from Theorem 4.19 (ii).

Using some work of Strichartz [Str], we will see in the next chapter a concrete construction of a self-similar tile in the Heisenberg groups.

Chapter 5

Multiresolution Analysis on Heisenberg Groups

1 Introduction

In the present chapter, we shall consider the concrete example of building a left MRA on the Heisenberg group $\mathbf{H}^d$. This is a simply connected nilpotent Lie group whose irreducible representations are classified by the Stone-von Neumann Theorem. More precisely, up to unitary equivalence it has two kinds of irreducible representations: (i) infinite-dimensional irreducible representations which can be parameterized by one-parameter; (ii) one-dimensional irreducible representations. We shall see shortly that the one-dimensional representations have no contribution to the Plancherel formula and Fourier inversion transform because they form a set of representations that has zero Plancherel measure.

This chapter can serve as a concrete example of our theory. Our main contributions are: (i) For the Heisenberg group $\mathbf{H}^d$, we set up a multiresolution analysis on $\mathbf{H}^d$ by applying the theory developed in Chapter 4. (ii) We also investigate the existence of scaling functions for the Heisenberg groups. These scaling functions are related to certain self-similar tilings of $\mathbf{H}^d$. We obtain a theorem on Heisenberg groups which says that there exist orthonormal wavelets for the space $L^2(\mathbf{H}^d)$.

In §5.2 we summarize the Fourier analysis on $\mathbf{H}^d$ for the convenience of the reader. For further information, see [Ge] and [Ta]. In §5.3 we build an MRA on $\mathbf{H}^d$ by directly applying the theory built in Chapter 4. In §5.4, we investigate the existence of scaling functions for the Heisenberg groups. These scaling functions are related to certain self-similar tilings of $\mathbf{H}^d$, that is, the corresponding scaling functions are characteristic functions of appropriate sets. We generalize the construction of Strichartz's self-similar tiles for Heisenberg groups to a more general case. In §5.5, we consider the existence of orthonormal wavelets for Heisenberg groups $\mathbf{H}^d$. We show that the orthonormal wavelets do exist in space $L^2(\mathbf{H}^d)$.

2 Fourier analysis on a Heisenberg group $\mathbf{H}^d$

The Heisenberg group $\mathbf{H}^d$ is a Lie group with underlying manifold $\mathbf{R}^{2d+1}$. We denote points in $\mathbf{H}^d$ by $(t_i, \underline{q_i}, \underline{p_i})$ with $t_i \in \mathbf{R}$, $\underline{q_i}, \underline{p_i} \in \mathbf{R}^d$, and define the group operation by

$$(t_1, \underline{q_1}, \underline{p_1})(t_2, \underline{q_2}, \underline{p_2}) = (t_1 + t_2 + \frac{1}{2}(\underline{p_1} \cdot \underline{q_2} - \underline{p_2} \cdot \underline{q_1}), \underline{q_1} + \underline{q_2}, \underline{p_1} + \underline{p_2}). \tag{5.1}$$

It is straightforward to verify that this is a group operation, with the origin $0 = (0, \underline{0}, \underline{0})$ as the identity element. Note that the inverse of $(t, \underline{q}, \underline{p})$ is given by $(-t, -\underline{q}, -\underline{p})$. It is also easy to show that Lebesgue measure on $\mathbf{R}^{2d+1} = \mathbf{H}^d$ is left invariant and right invariant under the group action defined by (5.1). Thus Lebesgue measure on $\mathbf{R}^{2d+1}$ gives the Haar measure on $\mathbf{H}^d$, and this group is unimodular.

We define the following map

$$\pi_1(t, \underline{q}, \underline{p})f(\underline{x}) = e^{i(t + \underline{q} \cdot \underline{x} + \underline{q} \cdot \underline{p}/2)}f(\underline{x} + \underline{p}), \quad \forall f \in L^2(\mathbf{R}^d).$$

Then it is proved that π_1 is a group homomorphism and a irreducible unitary representation of $\mathbf{H}^d$. For details, see [Ta, Theorem2.1, chapter 1].

Now we construct other irreducible representations of $\mathbf{H}^d$ on $L^2(\mathbf{R}^d)$ by using $\pi_1(t, \underline{q}, \underline{p})$. The first observation is that, for any $\lambda > 0$,

$$\delta_{\pm \lambda} : \mathbf{H}^d \longrightarrow \mathbf{H}^d$$

defined by

$$\delta_{\pm\lambda}(t,\underline{q},\underline{p}) = (\lambda t, \pm|\lambda|^{1/2}\underline{q}, |\lambda|^{1/2}\underline{p}),$$

or equivalently,

$$\delta_\lambda(t,\underline{q},\underline{p}) = (\lambda t, |\lambda|^{1/2}sign\lambda\underline{q}, |\lambda|^{1/2}\underline{p}),$$

is an automorphism of $\mathbf{H}^d$. It follows that, for each such $\lambda\neq 0$,

$$\pi_\lambda(h) := \pi_1(\delta_\lambda h), \quad \forall h\in\mathbf{H}^d$$

defines a representation of $\mathbf{H}^d$ on $L^2(\mathbf{R}^d)$. Since for each $\lambda\neq 0$, the set $\{\pi_\lambda(h) \mid h\in\mathbf{H}^d\}$ coincides with the set $\{\pi_1(h) \mid h\in\mathbf{H}^d\}$, it is clear that each representation π_λ is irreducible. Observe that

$$\pi_\lambda(t,\underline{q},\underline{p}) = e^{i(t\lambda+|\lambda|^{1/2}sign\lambda\underline{q}\cdot\underline{X}+|\lambda|^{1/2}\underline{p}\cdot\underline{D})}$$

is given explicitly on $L^2(\mathbf{R}^d)$ by

$$\pi_\lambda(t,\underline{q},\underline{p})f(\underline{x}) = e^{i(t\lambda+|\lambda|^{1/2}sign\lambda\underline{q}\cdot\underline{x}+\lambda\underline{q}\cdot\underline{p}/2)}f(\underline{x}+|\lambda|^{1/2}\underline{p}). \tag{5.2}$$

Except for the infinite-dimensional irreducible representations above, there are also the following one-dimensional irreducible representations of $\mathbf{H}^d$:

$$\pi_{(\underline{\xi},\underline{\eta})}(t,\underline{q},\underline{p}) = e^{i(\underline{\xi}\cdot\underline{q}+\underline{\eta}\cdot\underline{p})}. \tag{5.3}$$

It follows from the Stone-von Neumann theorem and Kirillov theory [see [CG] §2.2 for details] that the representations given by (5.2) and (5.3) exhaust the irreducible representations of $\mathbf{H}^d$. On the other side, no two different representations of $\mathbf{H}^d$ given by (5.2) and (5.3) are unitarily equivalent. In fact, suppose that there exists a unitary operator u such that

$$u\pi_{\lambda_1}(h)u^{-1} = \pi_{\lambda_2}(h), \quad \forall h\in\mathbf{H}^d, \tag{5.4}$$

for $\lambda_1\neq\lambda_2$. Let $h = (t,\underline{0},\underline{0})$, we see that (5.4) implies

$$ue^{i\lambda_1 t}u^{-1} = e^{i\lambda_2 t}, \quad \forall t\in\mathbf{R},$$

62

and since $e^{i\lambda t}$ is scalar, this implies $e^{i\lambda_1 t} = e^{i\lambda_2 t}$ for all $t \in \mathbf{R}$. Hence $\lambda_1 = \lambda_2$. Therefore, we can write $\widehat{\mathbf{H}}^d$ as $\{ \pi_\lambda \mid \lambda \in \mathbf{R} \backslash \{0\} \} \bigcup \{ \pi_{(\underline{\xi}, \underline{\eta})} \mid (\underline{\xi}, \underline{\eta}) \in \mathbf{R}^d \times \mathbf{R}^d \}$.

Let $L^p(\mathbf{H}^d)$ be the space consisting of all the measurable complex valued functions on $\mathbf{H}^d$ with $\int_{\mathbf{H}^d} |f(h)|^p dh < \infty$, where dh represents the Haar measure on $\mathbf{H}^d$. When $p = 2$, $L^2(\mathbf{H}^d)$ becomes a Hilbert space with inner product defined by $< f_1, f_2 > = \int_{\mathbf{H}^d} f_1(h)\overline{f_2(h)} dh$ for f_1, $f_2 \in L^2(\mathbf{H}^d)$. We associate to a function f in $L^2(\mathbf{H}^d) \bigcap L^1(\mathbf{H}^d)$ a "Fourier transform"

$$\mathcal{F}f(\pi_{(\underline{\xi}, \underline{\eta})}) = \iint\int_{\mathbf{H}^d} f(t, \underline{q}, \underline{p}) e^{i(\underline{\xi} \cdot \underline{q} + \underline{\eta} \cdot \underline{p})} dt d\underline{q} d\underline{p}, \quad \forall (\underline{\xi}, \underline{\eta}) \in \mathbf{R}^d \times \mathbf{R}^d$$

and

$$\mathcal{F}f(\pi_\lambda) = \int_{\mathbf{H}^d} f(h)\pi_\lambda(h) dh, \quad \forall \lambda \in \mathbf{R} \backslash \{0\}. \tag{5.5}$$

For short, we write $\mathcal{F}f(\lambda)$ instead of $\mathcal{F}f(\pi_\lambda)$. Then we have:

$$\int_{\mathbf{H}^d} |f(h)|^2 dh = c_d \int_{-\infty}^{+\infty} \|\mathcal{F}f(\lambda)\|_{HS}^2 |\lambda|^d d\lambda. \tag{5.6}$$

This is called the Plancherel theorem for the Heisenberg group. [Ta] gives a proof of this formula, see [Ta, Theorem 2.6, Chapter 1]. But the author did not say what c_d is there. The following computation shows that c_d must be $(2\pi)^{-(d+1)}$. We shall see shortly that the Euclidean Fourier transform plays a decisive role in this computation. Let's briefly go through the definition of the Euclidean Fourier transform. The Euclidean Fourier transform is defined by

$$\hat{f}(\underline{\xi}) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(\underline{x}) e^{i\underline{x} \cdot \underline{\xi}} d\underline{x}, \text{ for } f \in L^2(\mathbf{R}^d).$$

Its Fourier inversion formula is

$$\check{f}(\underline{x}) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(\underline{\xi}) e^{-i\underline{\xi} \cdot \underline{x}} d\underline{\xi}, \text{ for } f \in L^2(\mathbf{R}^d).$$

The Plancherel theorem for Euclidean space is

$$\int_{\mathbf{R}^d} |f(x)|^2 dx = \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 d\xi.$$

Let's denote the Euclidean Fourier transform of a function f on $\mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d$ with respect to its first, second, and third arguments by $\mathcal{F}_1 f$, $\mathcal{F}_2 f$, and $\mathcal{F}_3 f$ respectively. Thus, we have

$$\mathcal{F}_2 f(t, \underline{\xi}, \underline{p}) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(t, \underline{q}, \underline{p}) e^{i\underline{\xi} \cdot \underline{q}} d\underline{q}, \text{ etc.}$$

We know that $\mathcal{F}f(\lambda)$ is an operator on $L^2(\mathbf{R}^d)$ given by

$$[\mathcal{F}f(\lambda)u](\underline{x}) = \int_{\mathbf{R}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(t,\underline{q},\underline{p})e^{i(\lambda t + \sqrt{|\lambda|}\mathrm{sign}\lambda\underline{q}\cdot\underline{X} + \sqrt{|\lambda|}\underline{p}\cdot\underline{D})}u(\underline{x})dtd\underline{q}d\underline{p},$$

where $u(\underline{x})\in L^2(\mathbf{R}^d)$.

Hence, by (5.2), we have

$$[\mathcal{F}f(\lambda)u](\underline{x})$$

$$= \int_{\mathbf{R}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(t,\underline{q},\underline{p})e^{i(\lambda t + \sqrt{|\lambda|}\mathrm{sign}\lambda\underline{q}\cdot\underline{x} + \lambda\underline{q}\cdot\underline{p}/2)}u(\underline{x} + \sqrt{|\lambda|}\underline{p})dtd\underline{q}d\underline{p}$$

$$= (2\pi)^{1/2}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \mathcal{F}_1 f(\lambda,\underline{q},\underline{p})e^{i(\sqrt{|\lambda|}\mathrm{sign}\lambda\underline{q}\cdot\underline{x})}e^{i(\lambda\underline{q}\cdot\underline{p}/2)}u(\underline{x} + \sqrt{|\lambda|}\underline{p})d\underline{q}d\underline{p}$$

$$= (2\pi)^{1/2}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \mathcal{F}_1 f(\lambda,\underline{q},\frac{1}{\sqrt{|\lambda|}}(\underline{y} - \underline{x}))$$

$$e^{i(\sqrt{|\lambda|}\mathrm{sign}\lambda\underline{q}\cdot\underline{x})}e^{i(\frac{1}{2}\sqrt{|\lambda|}\mathrm{sign}\lambda\underline{q}\cdot(\underline{y}-\underline{x}))}|\lambda|^{-d/2}u(\underline{y})d\underline{q}d\underline{y}$$

$$= |\lambda|^{-d/2}(2\pi)^{1/2}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \mathcal{F}_1 f(\lambda,\underline{q},\frac{1}{\sqrt{|\lambda|}}(\underline{y} - \underline{x}))e^{i(\frac{1}{2}\sqrt{|\lambda|}\mathrm{sign}\lambda\underline{q}\cdot(\underline{y}+\underline{x}))}u(\underline{y})d\underline{q}d\underline{y}$$

$$= |\lambda|^{-d/2}(2\pi)^{(d+1)/2}\int_{\mathbf{R}^d} \mathcal{F}_1\mathcal{F}_2 f(\lambda,\frac{1}{2}\sqrt{|\lambda|}\mathrm{sign}\lambda(\underline{x} + \underline{y}),\frac{1}{\sqrt{|\lambda|}}(\underline{y} - \underline{x}))u(\underline{y})d\underline{y}.$$

Now, since the squared Hilbert-Schmidt norm of an operator

$$Au(\underline{x}) = \int_{\mathbf{R}^d} A(\underline{x},\underline{y})u(\underline{y})d\underline{y}, \quad \text{for } u\in L^2(\mathbf{R}^d)$$

is

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |A(\underline{x},\underline{y})|^2 d\underline{x}d\underline{y}.$$

Then the substitution

$$\begin{cases} \underline{u} = \frac{1}{2}\sqrt{|\lambda|}\mathrm{sign}\lambda(\underline{x} + \underline{y}) \\ \underline{v} = \frac{1}{\sqrt{|\lambda|}}(\underline{y} - \underline{x}) \end{cases}$$

together with the Euclidean Plancherel theorem gives

$$\|\mathcal{F}f(\lambda)\|^2_{HS}$$

$$= |\lambda|^{-d}(2\pi)^{(d+1)}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |\mathcal{F}_1\mathcal{F}_2 f(\lambda,\frac{1}{2}\sqrt{|\lambda|}\mathrm{sign}\lambda(\underline{x} + \underline{y}),\frac{1}{\sqrt{|\lambda|}}(\underline{y} - \underline{x}))|d\underline{x}d\underline{y}$$

$$= |\lambda|^{-d}(2\pi)^{(d+1)}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |\mathcal{F}_1\mathcal{F}_2 f(\lambda,\underline{u},\underline{v})|^2 d\underline{u}d\underline{v}$$

$$= |\lambda|^{-d}(2\pi)^{(d+1)} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |\mathcal{F}_1 f(\lambda, \underline{q}, \underline{p})|^2 d\underline{q}d\underline{p}.$$

Multiplying both sides by the factor $|\lambda|^d$ and applying the Euclidean Plancherel theorem again, we obtain the Plancherel theorem for the Heisenberg group:

$$\int_{-\infty}^{\infty} \|\mathcal{F}f(\lambda)\|_{HS}^2 |\lambda|^d d\lambda$$

$$= (2\pi)^{(d+1)} \int_{\mathbf{R}} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |\mathcal{F}_1 f(\lambda, \underline{q}, \underline{p})|^2 d\lambda d\underline{q}d\underline{p}$$

$$= (2\pi)^{(d+1)} \int_{\mathbf{H}^d} |f(h)|^2 dh.$$

Therefore, $c_d = (2\pi)^{-(d+1)}$.

(5.6) enables one to extend the Fourier transform to $L^2(\mathbf{H}^d)$. Note that polarization of (5.6) gives

$$\int_{\mathbf{H}^d} f_1(h)\overline{f_2(h)}dh = (2\pi)^{-(d+1)} \int_{-\infty}^{+\infty} tr[(\mathcal{F}f_2(\lambda))^* \mathcal{F}f_1(\lambda)]|\lambda|^d d\lambda. \tag{5.7}$$

Let $\{\ O_n\ |\ n{\in}\mathbf{N}\ \}$ be a neighborhood base at a element h in $\mathbf{H}^d$. For each n, let ψ_n be a function such that $\text{supp}(\psi_n)$ is compact and contained in O_n, $\psi{\geq}0$ and $\int \psi_n = 1$. Suppose f in (5.5) is the sequence $\{\ \psi_n\ |\ n{\in}\mathbf{N}\ \}$ and pass to the limit, we have

$$\mathcal{F}\delta_h(\pi_\lambda) = \pi_\lambda(h),$$

where δ_h is the point mass at h. Similarly, let f_2 in (5.7) be ψ_n and pass to the limit, we get the following Fourier inversion formula for the Heisenberg group:

$$f(h) = (2\pi)^{-(d+1)} \int_{-\infty}^{+\infty} tr[(\pi_\lambda(h))^* \mathcal{F}f(\lambda)]|\lambda|^d d\lambda.$$

where the integral converges absolutely and uniformly for $h{\in}\mathbf{H}^d$. See [Ge] for the proof.

If f_1 and f_2 are two functions in $L^1(\mathbf{H}^d)$, the convolution f_1*f_2 is defined by

$$f_1*f_2(g) = \int_{\mathbf{H}^d} f_1(h)f_2(h^{-1}g)dh.$$

It is easy to check that

$$\mathcal{F}(f_1*f_2)(\lambda) = \mathcal{F}f_1(\lambda)\mathcal{F}f_2(\lambda), \quad \forall\lambda{\in}\mathbf{R}\backslash\{0\}.$$

We remark that the representations (5.3) have no contribution to the Plancherel formula and Fourier inversion transform because this set of representations has zero Plancherel measure. In other words, Plancherel measure μ on $\hat{\mathbf{H}}^d$ is given by $d\mu(\pi_\lambda) = |\lambda|^d d\lambda$, $d\mu(\pi_{(\underline{\xi},\underline{\eta})}) = 0$. That is, $(2\pi)^{-(d+1)}|\lambda|^d$ on $\mathbf{R}\backslash\{0\}$ is the Plancherel measure on the set of equivalence classes of irreducible unitary representations of $\mathbf{H}^d$.

3 Multiresolution analysis on the Heisenberg group

In this section, we build an MRA on $\mathbf{H}^d$ by applying the theory developed in Chapter 4. From the definition of an MRA, it is clear that an MRA is determined by the scaling function ϕ. One fundamental question concerning an MRA on $\mathbf{H}^d$ is whether there exist scaling functions. We are going to see that the answer to this question is in the affirmative in the next section.

In analogy with the role of $\mathbf{Z}^d$ in the multiresolution analysis of $L^2(\mathbf{R}^d)$, we choose the following lattice subgroup Γ of $\mathbf{H}^d$ which plays the role in $\mathbf{H}^d$ that $\mathbf{Z}^d$ plays in $\mathbf{R}^d$:

$$\Gamma = \{ \ (l/2, \underline{m}, \underline{n}) \mid l \in \mathbf{Z}, \ \underline{m}, \ \underline{n} \in \mathbf{Z}^d \ \}.$$

It is easy to check that Γ forms a group under the group operation (5.1). Similarly, for $h \in \mathbf{H}^d$, we define the left translation operators L_h from $L^2(\mathbf{H}^d)$ to $L^2(\mathbf{H}^d)$. We also have the following similar terms such as left shift-invariant subspace and so on. It is easy to check that the map α from $\mathbf{H}^d$ to $\mathbf{H}^d$ given by

$$\alpha(t, \underline{q}, \underline{p}) := (2^2 t, 2\underline{q}, 2\underline{p})$$

is a topological automorphism of $\mathbf{H}^d$ and α^{-1} satisfies the contractive property. Also, $\bigcup_{j \in \mathbf{Z}} \alpha^{-j} \Gamma$ is dense in $\mathbf{H}^d$. With this α, we form the following unitary operator from $L^2(\mathbf{H}^d)$ to $L^2(\mathbf{H}^d)$

$$\sigma : \ L^2(\mathbf{H}^d) \longrightarrow L^2(\mathbf{H}^d),$$

which is given explicitly by

$$\sigma f(t, \underline{q}, \underline{p}) := 2^{d+1} f(\alpha(t, \underline{q}, \underline{p})) := 2^{d+1} f(2^2 t, 2\underline{q}, 2\underline{p}).$$

66

For $\phi \in L^2(\mathbf{H}^d)$, we denote $\mathcal{V}(\phi)$ to be the smallest closed left shift-invariant subspace of $L^2(\mathbf{H}^d)$ containing ϕ. We say that ϕ is refinable if $\mathcal{V}(\phi)$ is. We say that V is refinable if, for any $f \in V$, $\sigma^{-1} f$ is also in V. It is straightforward to verify that $L^2(\mathbf{H}^d)$ is left shift-invariant and refinable.

Let $V_0 = \mathcal{V}(\phi)$ and $V_j = \sigma^j V_0$ for $j \in \mathbf{Z}$. If ϕ is refinable, then $\{V_j | j \in \mathbf{Z}\}$ forms a nested sequence of closed subspaces of $L^2(\mathbf{H}^d)$. Since $\bigcup_{j \in \mathbf{Z}} \alpha^{-j} \Gamma$ is dense in $\mathbf{H}^d$, $V = \overline{\bigcup_{j \in \mathbf{Z}} V_j}$ is a closed left translation invariant subspace. By theorem 4.11 in chapter 4, $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{H}^d)$ if and only if $\{\phi_j\}_{j \in \mathbf{Z}}$ is strongly supported on $\mathbf{R}$. If in addition, the the left shifts of ϕ, that is, $\{L_\gamma \phi \mid \gamma \in \Gamma\}$, constitutes a orthonormal basis for V_0, then by the intersection triviality theorem 4.9, we have $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$. Therefore, $\{V_j\}_{j \in \mathbf{Z}}$ forms a multiresolution analysis of $L^2(\mathbf{H}^d)$.

Recall that $\phi_j(\cdot) := \sigma^j \phi(\cdot)$. We have the following statement: $\mathcal{F}\phi_j(\lambda) = \frac{1}{2^{j(d+1)}} \mathcal{F}\phi(2^{-2j}\lambda)$. It is easily checked.

The following is a very simple sufficient condition for $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{H}^d)$ to be true.

Theorem 5.1. *Let ϕ be a function in $L^2(\mathbf{H}^d)$, let V_j be the σ^j-dilate of $\mathcal{V}(\phi)$. Assume that $\{V_j\}_{j \in \mathbf{Z}}$ is nested. Then $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{H}^d)$ if there exists a neighborhood E of 0 in $\mathbf{R}$ such that the function ϕ is strongly supported on E and $\hat{\phi}(\lambda) = 0$ for a.e. λ not in E.*

Proof. Since $\phi_j = \sigma^j \phi(\cdot)$,

$$\mathcal{F}\phi_j(\lambda) = \frac{1}{2^{2j(d+1)}} \mathcal{F}\phi(\frac{\lambda}{2^{2j}}).$$

Suppose E is some neighborhood of the origin and ϕ is strongly supported on E. Then by the above computation, ϕ_j is strongly supported on $2^{-2j}E$. Hence we obtain that $\{\phi_j\}_{j \in \mathbf{Z}}$ is strongly supported on $\mathbf{R}^*$, since $\bigcup_{j \in \mathbf{Z}} 2^{-2j}E = \mathbf{R}^*$. By theorem 4.11, $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{H}^d)$ holds. This yields the result.

4 The existence of scaling functions for the Heisenberg groups

In the last section, we built an MRA on $\mathbf{H}^d$ under the assumption that scaling functions are present in $L^2(\mathbf{H}^d)$ by using the theory established in the previous chapter. This section is concerned with the existence of scaling functions for the Heisenberg groups $\mathbf{H}^d$. Looking

at the definition of an MRA, we see that the scaling function for an MRA is the starting point for constructing an MRA. With a scaling function ϕ, we first form $V_0 = \mathcal{V}(\phi)$ using left translations, then form $V_j = \sigma^j V_0$, $j \in \mathbf{Z}$ using the dilation operator. Thus, the first question we have to answer is whether there exists scaling functions for $\mathbf{H}^d$. In this section, we are going to show that the scaling functions for $\mathbf{H}^d$ do exist. These scaling functions are related to certain self-similar tilings of $\mathbf{H}^d$, that is, the corresponding scaling functions are characteristic functions of appropriate sets and the dilated tile consists of finitely many translates of the original tile.

We recall the notion of self-similar tiling. We have the standard tiling of Euclidean space $\mathbf{R}^d$ by unit cubes. Each tile is a translate of a single tile by an element of the lattice subgroup $\mathbf{Z}^d$. If we dilate a tile by a factor 2, then the enlarged tile consists of 2^d original tiles. The generalization of the above leads in many cases to interesting self-similar tilings with fractal boundaries. For example, for two dimensions, the fractal twin dragon tiles are constructed in Gröchenig and W.R.Madych [GM]. As for Heisenberg groups, which have both dilations and lattice groups, in order to construct the analogue of the cubic-like tiling, it seems that the tiles with fractal boundaries are the only choice. Such self-similar tilings are present for a class of nilpotent Lie groups from the work by Strichartz [Str]. In the following, we generalize the construction of Strichartz's self similar tiles for Heisenberg groups to a more general case and then prove that the existence of self-similar tilings for Heisenberg groups implies the existence of scaling functions.

We restrict ourself to constructing special self similar tiles named stacked self similar tiles. The fundamental idea to construct such self similar tiles for $\mathbf{H}^d$ is the following. Since $\mathbf{H}^d \cong \mathbf{R} \times \mathbf{R}^{2d}$, we first obtain self similar tiles A for the subspace $\mathbf{R}^{2d}$. Then over A, we build self similar tiles in the central direction $\mathbf{R}$. This means that we can decompose the process of constructing self similar tiles for $\mathbf{H}^d$ into two steps: first constructing in the direction $\mathbf{R}^{2d}$ then in the direction $\mathbf{R}$. From the work by [GM], we knew that there are an abundance of self similar tiles in $\mathbf{R}^{2d}$. Question is, given a self similar tile in $\mathbf{R}^{2d}$, can we construct a self similar tile for $\mathbf{H}^d$ in this way. In the following, we are going to see that, under certain

conditions, whenever we got a self similar tile for $\mathbf{R}^{2d}$, we can always build a self similar tile for $\mathbf{H}^d$. These special self similar tiles are called stacked self similar tiles because the whole space $\mathbf{H}^d$ is tiled by infinite many stacks of tiles on $\mathbf{R}^{2d}$.

Let $\Gamma = \{ (l/2, \underline{m}, \underline{n}) \mid l \in \mathbf{Z}, \underline{m}, \underline{n} \in \mathbf{Z}^d \}$. It is a lattice subgroup in $\mathbf{H}^d$. We say that a automorphism α of $\mathbf{H}^d$ is dilative for Γ if $\alpha(\Gamma) \subset \Gamma$, $[\Gamma : \alpha(\Gamma)] < \infty$ and α dilates in all directions. Bring back in mind that dilativeness of α (or contractiveness of α^{-1}) is a basic requirement in defining an MRA. From the book [Fo2] , we know that every automorphism α of $\mathbf{H}^d$ can be uniquely decomposed as a product of four factors $\alpha_1 \alpha_2 \alpha_3 \alpha_4$, with $\alpha_j \in G_j$, where G_j is defined as follows: G_1 denotes the symplectic group $sp(d, \mathbf{R})$; G_2 consists of inner automorphisms:

$$(c, \underline{a}, \underline{b})(t, \underline{q}, \underline{p})(c, \underline{a}, \underline{b})^{-1} = (c + \underline{a} \cdot \underline{p} - \underline{b} \cdot \underline{q}, \underline{q}, \underline{p});$$

G_3 consists of dilations $\delta[r]$ defined by

$$\delta[r](t, \underline{q}, \underline{p}) = (r^2 t, r\underline{q}, r\underline{p});$$

and G_4 consists of two elements, the identity and the automorphism i defined by

$$i(t, \underline{q}, \underline{p}) = (-t, \underline{p}, \underline{q}).$$

Since we confine ourselves to building stacked self similar tiles for $\mathbf{H}^d$ associated with Γ and α, such a α can be written as $\alpha_1 \alpha_3 \alpha_4$. That is,

$$\alpha(t, \underline{q}, \underline{p}) = (r_\alpha t, D_\alpha(\underline{q}, \underline{p})), \tag{5.8}$$

where r_α is some integer and D_α is a dilation operator from $\mathbf{R}^{2d}$ to $\mathbf{R}^{2d}$.

For simple notation reasons, let's use $(t, \underline{x})$ to denote the element $(t, \underline{q}, \underline{p})$ in the Heisenberg group, that is, $\underline{x} = (\underline{q}, \underline{p}) \in \mathbf{R}^{2d}$ and use $(l/2, \underline{a})$ to stand for $(l/2, \underline{m}, \underline{n}) \in \Gamma$. Then the group law becomes

$$(t, \underline{x})(t', \underline{x}') = (t + t' + S(\underline{x}, \underline{x}'), \underline{x} + \underline{x}')$$

where $S(\underline{x}, \underline{x}') = ((\underline{q}, \underline{p}), (\underline{q}', \underline{p}')) = 1/2(\underline{p} \cdot \underline{q}' - \underline{q} \cdot \underline{p}')$ is a skew-symmetric bilinear form from $\mathbf{R}^{2d} \times \mathbf{R}^{2d}$ to $\mathbf{R}$.

Suppose that there exists a tile A in $\mathbf{R}^{2d}$, that is, A is measurable and

$$A \bigcap (\underline{k} + A) = \emptyset \text{ for } \underline{k} \neq 0, \ \underline{k} \in \mathbf{Z}^{2d} \text{ and } \bigcup_{\underline{k} \in \mathbf{Z}^{2d}} (\underline{k} + A) = \mathbf{R}^{2d}.$$

Thus, the Lebegue measure of A must be 1, see lemma 1 in [GM]. Also suppose that A is a self similar tile corresponding to the dilation D_α, that is,

$$D_\alpha(A) = \bigcup_{i=1}^{s} (\underline{k}_i + A) \text{ disjoint union,}$$

where $\underline{k}_1, \ \underline{k}_2, \ \cdots, \ \underline{k}_s$ are lattice points that are representatives of distinct cosets in $\mathbf{Z}^{2d}/D_\alpha(\mathbf{Z}^{2d})$.

Now, we are going to construct a self similar tile for $\mathbf{H}^d \cong \mathbf{R} \times \mathbf{R}^{2d}$ associated with α based on the given self similar tile A related to D_α. Before we do this, first we introduce some useful notations. Since the measure of A is 1 and the disjoint union $\bigcup_{k \in \mathbf{Z}^{2d}} (k + A)$ fill out the whole space $\mathbf{R}^{2d}$, we could arrange a one to one correspondence between the lattice points in $\mathbf{Z}^{2d}$ and the tiles. Or simply speaking, without loss of generality, we can assume that each tile only contains one lattice point. For $\underline{x} \in \mathbf{R}^{2d}$, we use $[\underline{x}]_A$ to denote the lattice point that corresponds to the tile which contains $\underline{x}$. Let $< \underline{x} >_A = \underline{x} - [\underline{x}]_A \in A$.

Let F be a bounded measurable real-valued function defined first on A and then extended periodically to the whole space $\mathbf{R}^{2d}$. Thus, we have $F(\underline{x}) = F(< \underline{x} >_A)$. We are going to produce a tile, denoted by T, for the Heisenberg group $\mathbf{H}^d$ as follows:

$$T = \{ \ (t, \underline{x}) \in \mathbf{H}^d \mid \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2 \ \},$$

where F is to be determined later. To get a geometric picture for T, we can view $F(\underline{x})$ as a piece of surface over A. So we can think of T as a solid over A bounded between two surfaces $F(\underline{x})$ and $F(\underline{x}) + 1/2$. Thus the volume of T is equal to $1/2$. So in some sense, we can think of the "thickness" (in the direction of t-axis) of tile T as $1/2$.

For an element $\gamma = (l/2, \underline{a}) \in \Gamma$, the image of T under the left translation by γ is given by

$$\gamma T = \{ \ (t, \underline{x}) \in \mathbf{H}^d \mid \underline{x} - \underline{a} \in A, 0 \leq t - l/2 - S(\underline{a}, \underline{x} - \underline{a}) - F(\underline{x}) < 1/2 \ \}.$$

To show that $\bigcup_{\gamma\in\Gamma}\gamma T$ is a tiling of $\mathbf{H}^d$, we need to check two things. (a) $\bigcup_{\gamma\in\Gamma}\gamma T$ is a disjoint union. (b) $\bigcup_{\gamma\in\Gamma}\gamma T$ fills out the whole space $\mathbf{H}^d$. For (a), if $\underline{a}\neq\underline{a}'$, then $(l/2,\underline{a})T\bigcap(l/2,\underline{a}')T = \emptyset$ since the image $(l/2,\underline{a})T$ of T is in a stack of tiles lying over the tile $(0,\underline{a})T$. If l and l' are different integers, then $(l/2,\underline{a})T$ and $(l'/2,\underline{a})T$ are two different tiles in one stack located at tile $(0,\underline{a})T$, but $(l/2,\underline{a})T\bigcap(l'/2,\underline{a})T = \emptyset$ since the thickness for each tile is $1/2$. As for (b), for any $(t,\underline{x})\in\mathbf{H}^d$, there exists a unique element $\underline{a}\in\mathbf{Z}^{2d}$ such that $\underline{x}-\underline{a}\in A$. And also there exist a unique element $l\in\mathbf{Z}$ with the property

$$0 \le t - l/2 - S(\underline{a},\underline{x}-\underline{a}) - F(\underline{x}-\underline{a}) < 1/2.$$

A stacked tiling is called self-similar if there exists a finite subset Γ_0 of Γ such that

$$\alpha T = \bigcup_{\gamma\in\Gamma_0}\gamma T$$

or equivalently

$$T = \bigcup_{\gamma\in\Gamma_0}\alpha^{-1}(\gamma T).$$

Here, Γ_0 is a finite subset of lattice points.

Now, we can start constructing a self-similar stacked tiling related to the given tile A in $\mathbf{R}^{2d}$. From the explanation above, we know that the key point is to determine the surface described by the equation $t = F(\underline{x})$ on A. We start by choosing

$$\Gamma_0 = \{\ (c,\underline{k_i})\ |\ i = 1,2,\cdots,s,\ \text{and}\ c = 0,1/2,1,3/2,\cdots,(r_\alpha-1)/2\ \}.$$

Then we have

Lemma 5.2. $\bigcup_{\gamma\in\Gamma}\gamma T$ *is a self-similar stacked tiling for* $\mathbf{H}^d$ *with the above choice of the finite set* Γ_0 *if and only if the function* $F(\underline{x})$ *on* A *satisfies*

$$F(\underline{x}) = \frac{1}{r_\alpha}F(<D_\alpha(\underline{x})>_A) + \frac{1}{r_\alpha}S([D_\alpha(\underline{x})]_A, <D_\alpha(\underline{x})>_A).$$

Proof. By the choice of Γ_0, we have

$$\bigcup_{\gamma\in\Gamma_0}\gamma T\ \text{(disjoint finite union)}$$

71

$$= \{ (t, \underline{x}) \mid \underline{x} \in D_\alpha(A) = \bigcup_{i=1}^s (\underline{k_i} + A) \text{ and } 0 \le t - S([\underline{x}]_A, < \underline{x} >_A) - F(< \underline{x} >_A) < \frac{r_\alpha}{2} \}$$

$$= \{ (t, \underline{x}) \mid \underline{x} \in D_\alpha(A) = \bigcup_{i=1}^s (\underline{k_i} + A) \text{ and } 0 \le \frac{1}{r_\alpha} t - \frac{1}{r_\alpha} S([\underline{x}]_A, < \underline{x} >_A) - \frac{1}{r_\alpha} F(< \underline{x} >_A) < \frac{1}{2} \}.$$

Geometrically speaking, there are s stacks of tiles in $\bigcup_{\gamma \in \Gamma_0} \gamma T$. For each stack there are r_α tiles with "thickness" for each tile $1/2$, so the "thickness" for each stack is $r_\alpha \times 1/2$. On the other side,

$$\alpha T = \alpha \{ (t, \underline{x}) \in \mathbf{H}^d \mid \underline{x} \in A, 0 \le t - F(\underline{x}) < 1/2 \}$$

$$= \{ (r_\alpha t, D_\alpha(\underline{x})) \in \mathbf{H}^d \mid \underline{x} \in A, 0 \le t - F(\underline{x}) < 1/2 \}$$

$$= \{ (t, \underline{x}) \in \mathbf{H}^d \mid D_\alpha^{-1}(\underline{x}) \in A, 0 \le \frac{t}{r_\alpha} - F(D_\alpha^{-1}(\underline{x})) < 1/2 \}$$

$$= \{ (t, \underline{x}) \in \mathbf{H}^d \mid \underline{x} \in D_\alpha(A) = \bigcup_{i=1}^s (\underline{k_i} + A) \text{ and } 0 \le \frac{t}{r_\alpha} - F(D_\alpha^{-1}(\underline{x})) < 1/2 \}.$$

These two sets are equal if and only if

$$F(D_\alpha^{-1}(\underline{x})) = \frac{1}{r_\alpha} F(< \underline{x} >_A) + \frac{1}{r_\alpha} S([\underline{x}]_A, < \underline{x} >_A).$$

Or equivalently

$$F(\underline{x}) = \frac{1}{r_\alpha} F(< D_\alpha(\underline{x}) >_A) + \frac{1}{r_\alpha} S([D_\alpha(\underline{x})]_A, < D_\alpha(\underline{x}) >_A).$$

This lemma yields the technical theorem below.

Theorem 5.3. *For the choice of* Γ_0 *given above, there exists a unique self-similar stacked tiling* $\bigcup_{\gamma \in \Gamma} \gamma T$ *for* $\mathbf{H}^d$. *The function* $F_0(\underline{x})$ *is given explicitly by*

$$F_0(\underline{x}) = \sum_{n=1}^\infty \frac{1}{r_\alpha^n} S([D_\alpha^n(\underline{x})]_A \bmod (D_\alpha(\mathbf{Z}^{2d})), < D_\alpha^n(\underline{x}) >_A)$$

where a lattice point $\underline{k} \bmod (D_\alpha(\mathbf{Z}^{2d}))$ *equals the representative of the coset which contains element* $\underline{k}$.

Proof. Define a mapping M from $L^\infty(A)$ to $L^\infty(A)$ by

$$Mf(\underline{x}) = \frac{1}{r_\alpha} f(< D_\alpha(\underline{x}) >_A) + \frac{1}{r_\alpha} S([D_\alpha(\underline{x})]_A \bmod (D_\alpha(\mathbf{Z}^{2d})), < D_\alpha(\underline{x}) >_A),$$

72

where $L^\infty(A)$ is a Banach space with the supermum norm. Given f, $g \in L^\infty(A)$, we have

$$\|Mf - Mg\|_{L^\infty(A)} = \|\frac{1}{r_\alpha}f(<D_\alpha(\underline{x})>_A) - \frac{1}{r_\alpha}g(<D_\alpha(\underline{x})>_A)\|_{L^\infty(A)}$$

$$\leq \frac{1}{r_\alpha}\|f - g\|_{L^\infty(A)}.$$

So M is a contractive mapping. The completeness of $L^\infty(A)$ guarantees the existence of the fixed point of M and the contractiveness of M guarantees the uniqueness of such fixed point. The fixed point, denoted by $F_0(\underline{x})$, is given by $\lim_{n \to \infty} M^n f$ for any $f \in L^\infty(A)$. Especially, we have

$$F_0 = \lim_{n \to \infty} M^n 0.$$

Thus,

$$F_0(\underline{x}) = \sum_{n=1}^{\infty} \frac{1}{r_\alpha^n} S([D_\alpha^n(\underline{x})]_A \bmod (D_\alpha(\mathbf{Z}^{2d})), <D_\alpha^n(\underline{x})>_A).$$

This finishes the proof of theorem 5.3.

Examples 1: Our first example produces the Strichartz like self similar tile for $\mathbf{H}^d$, see [Str].

Taking α from $\mathbf{H}^d$ to $\mathbf{H}^d$ defined by

$$\alpha(t, \underline{q}, \underline{p}) := (2^2 t, 2\underline{q}, 2\underline{p}). \tag{5.9}$$

We know that α is an automorphism and it dilates strictly in all directions and $\alpha(\Gamma) \subset \Gamma$ and $[\Gamma : \alpha(\Gamma)] < \infty$, where $\Gamma = \{ (l/2, \underline{m}, \underline{n}) \mid l \in \mathbf{Z}, \underline{m}, \underline{n} \in \mathbf{Z}^d \}$. The given α comes from the group G_3. Comparing (5.8) with (5.9), we see that

$$\alpha(t, \underline{q}, \underline{p}) = (r_\alpha t, D_\alpha(\underline{q}, \underline{p})) = (4t, 2(\underline{q}, \underline{p})).$$

Thus, $r_\alpha = 4$ and D_α is the dilation operator which dilates by a factor 2. Let $A = \{ \underline{x} \in \mathbf{R}^{2d} \mid 0 \leq x_j < 1, j = 1, 2, \cdots, 2d \}$ denote the "half open and half closed standard tile in the Euclidean space $\mathbf{R}^{2d}$, where x_j denotes the jth component of $\underline{x}$. Then it is obvious that $\bigcup_{\underline{a} \in \mathbf{Z}^{2d}}(A + \underline{a})$ (disjoint union) fills out the whole space $\mathbf{R}^{2d}$. Clearly, A is a self similar tile. If we choose

$$\Gamma_0 = \{ (b, \underline{a}) \mid a_j = 0 \text{ or } 1, 1 \leq j \leq 2d, b = 0, 1/2, 1 \text{ or } 3/2 \}.$$

Then by theorem 5.3, we have the result:

$$T = \{\ (t, \underline{x}) \in \mathbf{H}^d \mid \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2\ \}$$

is a self similar tile for $\mathbf{H}^d$ associated with α defined in (5.9) and the lattice group Γ, where

$$F(\underline{x}) = \sum_{n=1}^{\infty} \frac{1}{4^n} S([D_\alpha^n(\underline{x})]_A \bmod (D_\alpha^n(\mathbf{Z}^{2d})), < D_\alpha^n(\underline{x}) >_A)$$

$$F(\underline{x}) = \sum_{n=1}^{\infty} \frac{1}{4^n} S([2^n \underline{x}] \bmod 2, < 2^n \underline{x} >)$$

here, $[2^n \underline{x}] \bmod 2$ means $([2^n x_1] \bmod 2, [2^n x_2] \bmod 2, \cdots, [2^n x_{2d}] \bmod 2)$.

Examples 2: Another example is not the Strichartz like self similar tile for $\mathbf{H}^d$. This example is related to the automorphism α given by

$$\alpha(t, \underline{q}, \underline{p}) := (6t, 2\underline{q}, 3\underline{p}). \tag{5.10}$$

This α can be decomposed as $\alpha = \alpha_1 \alpha_3$, where

$$\alpha_1(t, \underline{q}, \underline{p}) := (t, \sqrt{\frac{2}{3}}\underline{q}, \sqrt{\frac{3}{2}}\underline{p})$$

and

$$\alpha_3(t, \underline{q}, \underline{p}) := ((\sqrt{6})^2 t, \sqrt{6}\underline{q}, \sqrt{6}\underline{p}) = (6t, \sqrt{6}\underline{q}, \sqrt{6}\underline{p}).$$

We can rewrite (5.10) as

$$\alpha(t, \underline{q}, \underline{p}) = (6t, D_\alpha(\underline{q}, \underline{p})),$$

where D_α is a dilation from $\mathbf{R}^{2d}$ to $\mathbf{R}^{2d}$ defined by $D_\alpha(\underline{q}, \underline{p}) := (2\underline{q}, 3\underline{p})$. Comparing with (5.8), we have $r_\alpha = 6$. Still using the same Γ as in the previous example, we choose Γ_0 as the following

$$\Gamma_0 = \{(c, \underline{a}) \mid a_j = 0 \text{ or } 1 \text{ for } 1 \leq j \leq d, a_j = 0, 1 \text{ or } 2 \text{ for } d < j \leq 2d \text{ and } c = 0, 1/2, 1, \cdots, \frac{5}{2}\}.$$

We still use the set A in the example 1 as a tile for $\mathbf{R}^{2d}$ related to the lattice group $\mathbf{Z}^{2d}$. Thus the dilated tile by D_α consists of 6 original tiles. With this self similar tile in $\mathbf{R}^{2d}$, by theorem 5.3 we obtain another self-similar tile in $\mathbf{H}^d$:

$$T = \{\ (t, \underline{x}) \in \mathbf{H}^d \mid \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2\ \}$$

74

with $F(\underline{x})$ constructed by

$$F(\underline{x}) = \sum_{n=1}^{\infty} \frac{1}{6^n} S([D_\alpha{}^n \underline{x}]_A \bmod (D_\alpha(\mathbf{Z}^{2d})), < D_\alpha{}^n \underline{x} >_A),$$

where $[D_\alpha^n \underline{x}]_A \bmod (D_\alpha(\mathbf{Z}^{2d}))$ means $[2^n x_1] \bmod 2, [2^n x_2] \bmod 2, \cdots, [2^n x_d] \bmod 2$ and $[3^n x_{d+1}] \bmod 3, [3^n x_{d+2}] \bmod 3, \cdots, [3^n x_{2d}] \bmod 3$.

Generally speaking, whenever an automorphism α from $\mathbf{H}^{2d}$ to $\mathbf{H}^{2d}$ can be decomposed as

$$\alpha(t, \underline{q}, \underline{p}) = (r_\alpha t, D_\alpha(\underline{q}, \underline{p}))$$

and there exists a self similar tile A in $\mathbf{R}^{2d}$ associated with D_α, then with this A, we can always construct a self similar tile in $\mathbf{H}^d$ associated with α.

The theorem 5.3 guarantees the existence of self-similar tiling for the system $(\mathbf{H}^d, \Gamma, \alpha)$. Suppose T is such a self-similar tile for $(\mathbf{H}^d, \Gamma, \alpha)$. Let

$$\phi = \frac{1}{|T|^{1/2}} \chi_T,$$

where χ_T is the characteristic function of T. By Theorem 4.20, the characteristic function of any self-similar tile for $(\mathbf{H}^d, \Gamma, \alpha)$ guarantees a multiresoluion analysis for $L^2(\mathbf{H}^d)$. Thus, we have proven the following theorem.

Theorem 5.4. *There exists a function $\phi \in L^2(\mathbf{H}^d)$ such that, if $V_0 := \mathcal{V}(\phi) =$ the smallest closed left Γ-shift-invariant subspace of $L^2(\mathbf{H}^d)$ containing ϕ and $V_j := \sigma^j(V_0)$, then $\{V_j\}_{j \in \mathbf{Z}}$ forms a multiresolution analysis for $L^2(\mathbf{H}^d)$.*

5 The existence of orthonormal wavelets for Heisenberg groups

This section is devoted to a study of the existence of orthonormal wavelets for Heisenberg groups $\mathbf{H}^d$.

In the last section, we showed that self-similar tilings are present for the Heisenberg groups from the work by Strichartz [Str]. Such self-similar tilings provide us with scaling

functions for Heisenberg groups if we consider the characteristic functions of the tilings. Thus, we can form a multiresolution analysis for $L^2(\mathbf{H}^d)$ by using these self-similar scaling functions. On the other hand, Baggett and et al. [BC] studied the relationship between the existence of an orthonormal wavelet and the existence of an MRA for general Hilbert space based on the formulation of MRA by using methods from noncommutative harmonic analysis. They obtained four theorems. Their first theorem Theorem 1 guaranteed the existence of an orthonormal wavelet once an MRA was built on the space. One of the interesting connections between the last section and the work by [BC] is that an MRA for $L^2(\mathbf{H}^d)$ in the sense of [BC] can be set up by taking only left translations on $\mathbf{H}^d$ and the unitary operator σ defined in the last section. Then theorem 1 [BC] concludes that there exists an orthonormal wavelet for $L^2(\mathbf{H}^d)$.

In the following, we first outline the definition of an MRA and theorem 1 in [BC]. Then we build an MRA for the space $L^2(\mathbf{H}^d)$. Finally the theorem 1 in [BC] proves that an orthonormal wavelet exists for $L^2(\mathbf{H}^d)$.

Let H be a separable Hilbert space. Let Π be a group of unitary operators on H and σ another unitary operator on H for which $\sigma^{-1}\pi\sigma$ is an element of Π for every $\pi\in\Pi$. It is easy to check that $\sigma^{-1}\Pi\sigma$ is a subgroup of Π. Assume that $n = [\Pi, \sigma^{-1}\Pi\sigma] < \infty$. We call the pair (Π, σ) an affine structure on H. A (Π, σ)-wavelet relative to this affine structure is a finite set $\{ \psi_1, \cdots, \psi_n \}$ of vectors in H such that the collection $\{ \sigma^j(\pi(\psi_i)) \mid -\infty < j < \infty, \pi\in\Pi, 1\leq i\leq n \}$ forms an orthonormal basis of H.

The definition of multiresolution analysis of H is as follows:

Definition 5.5. *Let (Π, σ) be an affine structure on H. An MRA of H consists of a sequence of closed linear subspaces V_j, $j\in\mathbf{Z}$, of H with the following properties:*

(i) $V_j\subset V_{j+1}$, for all $j\in\mathbf{Z}$;

(ii) $\overline{\bigcup_{j\in\mathbf{Z}}V_j} = H$;

(iii) $\bigcap_{j\in\mathbf{Z}}V_j = \{0\}$;

(iv) V_0 is invariant under each $\pi\in\Pi$;

(v) $V_j = \sigma^j V_0$;

(vi) there is a scaling function $\phi \in V_0$ such that the collection $\{ \pi\phi \mid \pi \in \Pi \}$ is a orthonormal basis of V_0.

Remarks. Comparing between this definition and the definition 4.8, we refer to the elements π of Π as translations, refer to the operator σ as a dilation.

Theorem 5.6(Baggett and et al.). *Let (Π, σ) be an affine structure of H and n the finite index of $\Pi_1 := \sigma^{-1}\Pi\sigma$ in Π. Suppose $\{ V_j \}_{j \in \mathbf{Z}}$ is an MRA of H in the sense of definition 5.5. Then n must be greater than 1 and there exists a (Π, σ)-wavelet $\psi_1, \psi_2, \cdots, \psi_{n-1}$ for H.*

Next, we build an MRA for $L^2(\mathbf{H}^d)$. Let's start by recalling some notations used in previous sections. The dilation operator σ is defined by

$$\sigma f(t, \underline{q}, \underline{p}) := 2^{d+1} f(\alpha(t, \underline{q}, \underline{p}))$$

for any $f \in L^2(\mathbf{H}^d)$, where α is the topological automorphism of $\mathbf{H}^d$ given by

$$\alpha(t, \underline{q}, \underline{p}) := (2^2 t, 2\underline{q}, 2\underline{p}).$$

The set

$$\Gamma = \{ (l/2, \underline{m}, \underline{n}) \mid l \in \mathbf{Z}, \underline{m}, \underline{n} \in \mathbf{Z}^d \}$$

forms a group under the group law (5.1).

We define the group of unitary operators on $L^2(\mathbf{H}^d)$ as

$$\Pi := \{ L_\lambda \mid \lambda \in \Gamma \}.$$

Let $\Pi_1 := \sigma^{-1}\Pi\sigma$. Then Π_1 is a subgroup of Π. The identity $\sigma^{-1}L_\lambda\sigma = L_{\alpha(\lambda)}$ is proved by the following computation:

$$\sigma^{-1}L_\lambda\sigma f(x) = 2^{-(d+1)}(L_\lambda\sigma f)(\alpha^{-1}(x))$$

$$= 2^{-(d+1)}(\sigma f)(\lambda^{-1}\cdot\alpha^{-1}(x))$$

$$= 2^{-(d+1)}2^{(d+1)}f(\alpha(\lambda^{-1})\cdot(x))$$

$$f((\alpha(\lambda))^{-1}\cdot x)$$

$$= (L_{\alpha(\lambda)}f)(x),$$

for any $f \in L^2(\mathbf{H}^d)$.

Thus $\sigma^{-1}\Pi\sigma = \{\ L_{\alpha(\lambda)} \mid \lambda \in \Gamma\ \}$ is a proper subgroup of Π and the index of the subgroup Π_1 in Π is equal to $2^{d+d+2} = 2^{2(d+1)}$. So (Π, σ) is an affine structure on $L^2(\mathbf{H}^d)$.

Now, let V_0 be a closed linear subspace of $L^2(\mathbf{H}^d)$ defined by

$$V_0 = \overline{\{\ L_\lambda\phi \mid \lambda \in \Gamma\ \}},$$

where ϕ is the $\phi(x) = |T|^{-1/2}\chi_T(x)$ with T as a self-similar tile of $\mathbf{H}^d$. In other words, $\{\ L_\lambda\phi \mid \lambda \in \Gamma\ \}$ is an orthonormal basis for V_0. Let $V_j := \sigma^j V_0$. Then $\{V_j\}_{j \in \mathbf{Z}}$ forms an MRA for $L^2(\mathbf{H}^d)$ with ϕ as a scaling function, by Theorem 5.4.

Therefore, by theorem 5.6, we finally have

Theorem 5.7 *There exists a (Π, σ)-wavelet $\psi_1, \psi_2, \cdots, \psi_{2^{2(d+1)}-1}$ for the Hilbert space $L^2(\mathbf{H}^d)$.*

Appendix A

Groups Suitable for Building Multiresolution Analysis on Them

1 Introduction

In this appendix, we are going to describe some groups on which an MRA can be constructed. We know that the class of second countable, type I, unimodular locally compact groups contains all connected semisimple Lie groups and also all connected nilpotent Lie groups. However, the conditions of having a lattice subgroup Γ and a compatible dilation α impose strong restrictions. We shall find out that connected semisimple Lie groups are not suitable for building MRA on them. For connected nilpotent Lie groups, only countable many are suitable for building MRA on them. Roughly speaking, only those groups which are close to being vector groups admit the properties for building MRA on them. For the purpose of this initial study, we restrict our attention to connected groups.

We would like to make it clear that our theory established in Chapter 4 is applicable to all second countable, type I, unimodular locally compact groups. Unfortunately, some intrinsic properties in defining MRA prevent some groups from becoming the groups fit for

MRA. Let us state these essential properties as follows:

- There exists a discrete subgroup Γ in G such that G/Γ is compact,
- There exists a topological automorphism α of G such that
  $\alpha(\Gamma)\subseteq\Gamma$ and $\bigcup_{j\in Z}\alpha^{-j}\Gamma$ is dense in G.

$$(A.1)$$

Recall that a uniform subgroup of G is a subgroup Γ of G such that:

(i) Γ is discrete,

(ii) G/Γ is compact.

And a lattice subgroup of G is a subgroup Γ of G such that:

(i) Γ is discrete,

(ii) G/Γ has a finite G-invariant measure on it.

But for nilpotent Lie groups and solvable Lie groups, G/Γ is compact if and only if G/Γ carries a G-invariant finite measure. For details, see [Ra, Theorem 2.1 and Theorem 3.1]. Thus, for nilpotent Lie groups and solvable Lie groups, the first condition in (A.1) can be simplified as "there exists a lattice subgroup Γ in G.

We use this key information to give the following definition

Definition A.1 *Suppose G is a connected second countable, type I, unimodular locally compact group. If there is a subgroup Γ in G and a topological automorphism of G such that the conditions in (A.1) above are satisfied, then we call G a scalable group*

In this appendix, we shall roughly determine how large this class of scalable groups is.

The requirement of existence of uniform subgroup naturally leads us to restrict our attention to the class of solvable and semisimple Lie groups because a detailed account of theory of lattice subgroups of solvable Lie groups and semisimple Lie groups has been laid out. It turns out that a lattice subgroup often exists for solvable Lie groups and semisimple Lie groups. See [Ra] for details. But then, one will naturally ask whether solvable Lie groups or semisimple Lie groups satisfy the second condition in (A.1).

This appendix is organized in the following way. Section 2 contains definitions and some basic facts about linear algebraic groups, most of which are from [Bo]. Based on the knowledge of section 2, in section 3, we rule out the possibility for those solvable Lie groups

which are not nilpotent Lie groups to become scalable groups. Then quoting a well-known theorem from [Ra], we conclude that, for connected solvable Lie groups, at most countably many nilpotent Lie groups are scalable groups. In section 4, we exclude all connected semisimple Lie groups from the scalable group class.

2   Terminologies and Basic facts about linear algebraic groups

The purpose of this section is to establish the language, conventions and some basic results of linear algebraic groups. We shall not give any proofs for the results. We refer the reader to the general references [Bo],[Hu] or any other standard textbook on algebraic groups.

Let **K** be an algebraic closed field of arbitrary characteristic.

Definition A.2 *The set* $\mathbf{K}^d := \mathbf{K} \times \mathbf{K} \times \cdots \times \mathbf{K}$ *will be called affine d−space over* **K** *and denoted by* $\mathbf{A}^d$. *An algebraic variety in* $\mathbf{A}^d$ *or an embedded affine algebraic variety is a subset in* $\mathbf{A}^d$ *defined by a system of equations:*

$$f(X_1, X_2, \cdots, X_d) = 0 \quad (f \in S),$$

where S *is a finite collection of polynomials.*

Let $\mathbf{K}[X] := \mathbf{K}[X_1, X_2, \cdots, X_d]$ be the polynomial ring in d-indeterminates. Then the ideal in $\mathbf{K}[X]$ generated by a set of polynomials $\{f_\alpha(X)\}$ has precisely the same common zeros as $\{f_\alpha(X)\}$. Moreover, since $\mathbf{K}[X]$ is noetherian, each ideal in $\mathbf{K}[X]$ has a finite set of generators, so every ideal corresponds to an algebraic variety.

There are two operators: (1) to each ideal I in $\mathbf{K}[X]$ we assign the set $\mathcal{V}(I)$ of its common zeros in $\mathbf{A}^d$. (2) to each subset $S \subset \mathbf{A}^d$ assign the collection $\mathcal{J}(S)$ of all polynomials vanishing on S.

The following idea of topologizing affine d-space $\mathbf{A}^d$ turns out to be very fruitful.

Definition A.3 *A subset of* $\mathbf{A}^d$ *is called Zariski closed if it is an algebraic variety. The topology on* $\mathbf{A}^d$ *defined in this way is called Zariski topology.*

Naturally, it has to be checked that the axioms for a topology are satisfied: (1) $\mathbf{A}^d$ and the empty set are closed as the respective zero sets of the ideals $\{0\}$ and $\mathbf{K}[X]$. (2) If I, J

81

are two ideals, then clearly

$$\mathcal{V}(I) \bigcup \mathcal{V}(J) \subseteq \mathcal{V}(I \bigcap J).$$

To establish the reverse inclusion, suppose x is a zero of $I \bigcap J$, but not of I or J. Suppose $f \in I$, $g \in J$ with $f(x) \neq 0$ and $g(x) \neq 0$. Since $fg \in I \bigcap J$, we must have $f(x)g(x) = 0$, which is contradiction to the assumption. This shows that finite union of closed sets is closed. (3) Let I_α be an arbitrary collection of ideals, so $\sum_\alpha I_\alpha$ is the ideal generated by this collection. Then it is clear that $\bigcap_\alpha \mathcal{V}(I_\alpha) = \mathcal{V}(\sum_\alpha I_\alpha)$, i.e., arbitrary intersections of closed sets are closed.

To have an idea how Zariski topology $\mathbf{A}^d$ looks, let us illustrate a couple of facts. Points are closed, since $x = (x_1, x_2, \cdots, x_d)$ is the only common zero of polynomials $X_1 - x_1, X_2 - x_2, \cdots, X_d - x_d$. From the measure theory point of view, all nonempty open sets in $\mathbf{A}^d$ are very "large", since it is the complement of a very small "curve". For example, $GL_d(\mathbf{K})$, the group of all invertible $n \times n$ matrices over $\mathbf{K}$, is an open set in $\mathbf{A}^{n^2}$ defined by the nonvanishing of $\det(X_{i,j})$. Also notice that the Zariski topology on $\mathbf{A}^{n+m}$ does not coincide with the direct product topology on $\mathbf{A}^n \times \mathbf{A}^m$. For example, the set in $\mathbf{A}^2$ determined by the equation $X_1 = X_2$ is closed but it is not closed in $\mathbf{A}^1 \times \mathbf{A}^1$.

Definition A.4 *For an ideal I, the radical $\sqrt{I}$ of I is $\{f \in \mathbf{K}[X] | f^r \in I$ for some $0 \leq r\}$. $\sqrt{I}$ is easily seen to be an ideal, it includes I. An ideal is called a radical ideal in $\mathbf{K}[X]$ if it is equal to its radical.*

Theorem A.5 (Hilbert's Nullstellensatz)*If I is any ideal in $\mathbf{K}[X]$, then $\sqrt{I} = J(V(I))$.*

The Nullstellensatz theorem implies that the operators $\mathcal{V}$ and $\mathcal{J}$ set up 1-1 correspondence between the collection of all radicals in $\mathbf{K}[X]$ and the collection of all algebraic varieties in $\mathbf{A}^d$. We mentioned that every ideal in $\mathbf{K}[X]$ corresponds to a algebraic variety in $\mathbf{A}^d$. But note that this correspondence is not 1-1. For example, the ideals generated by X and X^2 are distinct, but have the same zero set $\{0\}$ in $\mathbf{A}^1$.

Definition A.6 *A morphism of an algebraic variety $M \subset \mathbf{A}^n$ into an algebraic variety $N \subset \mathbf{A}^m$ is any polynomial map $\phi: M \longrightarrow N$, i.e., a map that can be determined by polynomials. More precisely, it means that there are polynomials $\phi_1, \phi_2, \cdots, \phi_m \in \mathbf{K}[X_1, X_2, \cdots, X_n]$*

such that the map ϕ transforms a point $x \in M$ into the point of the variety N with coordinates

$$\phi(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_m(x)).$$

Given a morphism ϕ: $M \longrightarrow N$. We can induce a homomorphism of algebras of functions defined by the formula

$$(\phi^* g)(x) = g(\phi(x)),$$

where g is a function on N, $x \in M$. The definition of the morphism clearly implies that if g is a polynomial on N, then $\phi^* g$ is a polynomial on M. So we get a homomorphism of algebras ϕ^*: $\mathbf{K}[N] \longrightarrow \mathbf{K}[M]$.

Theorem A.7 *(1) Morphisms of algebraic varieties are continuous in Zariski topology.*
(2) For any algebra homomorphism ϕ : $\mathbf{K}[N] \longrightarrow \mathbf{K}[M]$ there exists a unique morphism f : $M \longrightarrow N$ such that $f^* = \phi$.

Thus, to define a morphism of embedded affine algebraic varieties is the same as to define a homomorphism of the algebras of polynomials on these varieties. Clearly, the product gf of morphisms f : $M \longrightarrow N$ and g : $N \longrightarrow P$ is a morphism and $(gf)^* = f^* g^*$. A morphism f : $M \longrightarrow N$ is called an isomorphism if there exists an inverse morphism f^{-1} : $N \longrightarrow M$, i.e., if f is bijective and the inverse map is also a polynomial map. This is equivalent to the fact that f^* is an isomorphism of algebras.

The class of isomorphic embedded affine algebraic varieties is called, in an abstract sense, an affine algebraic variety, or, in short, affine variety. And its representatives will called embeddings of this variety into the affine space, practically, an affine variety is identified with one of its embeddings.

Definition A.8 *An algebraic group is a group endowed with the structure of an affine algebraic variety so that the maps:*

$$\mu : G \times G \longrightarrow G, \ (x, y) \longmapsto xy$$

$$l : G \longrightarrow G, \ x \longmapsto x^{-1}$$

are morphisms of algebraic varieties.

Remarks. (1) The definition of an algebraic group is similar to that of a Lie group, except

that differentiable manifolds are replaced by a algebraic varieties and differentiable maps by morphisms of algebraic varieties. (2) A morphism of algebraic groups is a morphism of varieties which is also a homomorphism of groups. (3) The most important example of an algebraic group is the general linear group, i.e., $GL_d(\mathbf{K})$, all $d \times d$ invertible matrices with entries in $\mathbf{K}$, this is a group under multiplication. $GL_d(\mathbf{K})$ may be embedded in $\mathbf{A}^{d^2}$. Thus $GL_d(\mathbf{K})$ can be identified with the open subset defined by the nonvanishing of the polynomial det. Hence $GL_d(\mathbf{K})$ is an affine variety and has its algebra of polynomial functions generated by the d^2 coordinate functions $X_{i,j}$ along with $1/\det(X_{i,j})$. The formulas for matrix multiplication and inversion make it clear that $GL_d(\mathbf{K})$ is an algebraic group.

An algebraic subgroup of a general linear group is called an algebraic linear group. We have the following:

Theorem A.9 *Any algebraic group is isomorphic to an algebraic linear group.*

Thus, with this theorem, by an algebraic group we will always mean a subgroup $G \subseteq GL_d(\mathbf{K})$

Definition A.10 *An element A of an algebraic group G is called unipotent if $(A - I)^m = 0$ for some m, where I is the identity matrix.*

3 At most countably many solvable Lie groups are suitable for building multiresolution analysis

In this section, we are going to conclude that at most countably many solvable Lie groups are suitable for building multiresolution analysis on them. We reach this conclusion by first showing that connected non-nilpotent solvable Lie groups fail to satisfy the second condition of $(A.1)$ and then citing a well-known theorem from [Ra]. Its proof requires some knowledge of the basic theory of solvable algebraic groups. In the following, we first collect the theorems we need on algebraic groups. For detailed information, see [Bo] and [BM].

Lemma A.11 [Bo,Theorem 10.6, pp.137-138] *Let G be a connected, solvable algebraic group. Then*

(i) The set U of all unipotent elements of G is a normal subgroup, which is

called the unipotent radical of G;

(ii) G/U is abelian.

Let G^0 denote the connected component of the identity e in an algebraic group G. Then we have

Lemma A.12[Bo,proposition,p.46] *Let G be an algebraic group. Then G^0 is a normal subgroup of G and also of finite index in G.*

Let G be a connected, simply connected solvable Lie group. Let $\mathcal{G}$ be the Lie algebra of G and let Ad as usual denote the adjoint representation of G on $\mathcal{G}$. The image Ad(G) is contained in the full automorphism group Aut$(\mathcal{G})$ of $\mathcal{G}$, and the latter is an algebraic group. Therefore, we can consider the Zariski closure of Ad(G) in the algebraic group Aut$(\mathcal{G})$=Aut(G) since G is simply connected. Let Ad$(G)^*$ be the Zariski closure of Ad(G). The group Ad$(G)^*$ is connected as an algebraic group and solvable since G is solvable. The set of unipotent elements in Ad$(G)^*$ forms a normal subgroup U^* by lemma A.11 above, this is called the unipotent radical, and Ad$(G)^*/U^*$ is abelian by lemma A.11, which we denote by T^*.

We often use G^* to denote the Zariski topology closure of G. Sometimes, it is called "algebraic hull" of G. Now let π: $G^* \longmapsto T^*$ be the natural homomorphism and let p: $G \longmapsto T^*$ be the map one gets by composing Ad and π. At this place, we can state a fundamental theorem.

Lemma A.13[BM,Theorem 2.1, pp.576-577] *Let G be a connected, simply connected solvable Lie group and Γ be a closed cocompact subgroup, that is, G/Γ is compact. Then $p(\Gamma)$ is topologically discrete in T^*.*

Now, after this preparation, we have the following theorem:

Theorem A.14 *Let G be a connected, solvable Lie group and Γ be a lattice subgroup of G such that G/Γ is compact. Suppose the second condition of $(A.1)$ is satisfied, that is, there is a continuous automorphism α of G such that $\alpha(\Gamma) \subseteq \Gamma$ and $\bigcup_{j \in \mathbb{Z}} \alpha^j(\Gamma)$ is dense in G. Then G is nilpotent.*

Proof of the Theorem A.14 Replacing G by its universal cover, we may assume that G

is simply connected. Thus we can identify $\text{Aut}(G)$ with $\text{Aut}(\mathcal{G})$ since G is simply connected. Let $< \alpha >$ be the subgroup of $\text{Aut}(G)=\text{Aut}(\mathcal{G})$ generated by α. Let G^* be the identity component of the Zariski closure $(\text{Aut}(G) < \alpha >)^*$ of $\text{Aut}(G) < \alpha >$ in the algebraic group $\text{Aut}(\mathcal{G})=\text{Aut}(G)$. Let U^* be the set of all unipotent elements in G^*, and let $T^* = G^*/U^*$. Then by the lemma above, T^* is abelian.

Now since the identity component of an algebraic group always has finite index , so we have $\alpha^n \in G^*$ for some $n \in \mathbb{Z}^+$. In fact, all the left cosets $\{ \alpha^n G^* \mid n \in \mathbb{N} \}$ can not be distinct from each other. So since G^* has finite index, there must exist two distinct positive integers n_1 and n_2, say $n_2 > n_1$, such that

$$\alpha^{n_2} G^* = \alpha^{n_1} G^*, \text{ then } \alpha^{n_2-n_1} G^* = G^*,$$

that is, $\alpha^{n_2-n_1} \in G^*$.

Now let $\pi\colon G^* \longmapsto T^*$ be the natural homomorphism. Since $\text{Aut}(G)^*$ is connected, the image $\text{Aut}(G)$ is contained in G^*. Thus, $p\colon G \longmapsto T^*$, defined by the composition of Ad and π, is well-defined. Since $\alpha^n \in G^*$, $\pi(\alpha^n) \in T^*$. Because T^* is abelian by the lemma A.11, this means that $\pi(\alpha^n)$ normalizes $p(\Gamma)$. Therefore, we have $p(\alpha^n(\Gamma)) = p(\Gamma)$. Because $\cdots \subseteq \alpha^2(\Gamma) \subseteq \alpha(\Gamma) \subseteq \Gamma$, this implies that $p(\alpha^j(\Gamma)) = p(\Gamma)$, for all $j \in \mathbb{Z}$. Therefore, $p(\bigcup_{j \in \mathbb{Z}} \alpha^j(\Gamma)) = p(\Gamma)$. Now using the lemma A.13, we conclude that $p(\Gamma)$ is discrete, so $p(\bigcup_{j \in \mathbb{Z}} \alpha^j(\Gamma)) = p(\Gamma)$ is discrete. But then, since we are assuming that $\bigcup_{j \in \mathbb{Z}} \alpha^j(\Gamma)$ is dense in G, we deduce that $p(G)$ is discrete.

On the other hand, because G is connected, we know that $p(G)$ is connected. Hence $p(G)$ must be the trivial group, so image $\text{Ad}(G)$ is contained in U^*, that is, $\text{Ad}(G)$ is unipotent. Therefore, G is nilpotent. This establishes the theorem.

Theorem A.14 says that in the class of connected solvable Lie groups only nilpotent Lie groups are possible for constructing MRA on them. Unfortunately, not all nilpotent Lie groups have lattices. In the simply connected case, only countably many do. For a more concrete explanation of why not all nilpotent Lie groups have lattices, see Theorem 2.12. P34 and Remark 2.14, P38 of [Ra]. For the convenience of the reader, we quote the following theorem from [Ra] P32.

Theorem A.15 *Let G be a simply connected nilpotent Lie group and $\mathcal{G}$ be its Lie algebra. Then G admits a lattice if and only if $\mathcal{G}$ admits a basis with respect to which the constants of structure are rational.*

Remark. The term of structural constants comes from Lie algebra. Suppose $\mathcal{G}$ is a Lie algebra. One simple way to describe the multiplication in $\mathcal{G}$ is by choosing a basis $X_1, X_2, \cdots, X_d$ and expressing the products $[X_i, X_j]$, $(i, j = 1, \cdots, d)$ as linear combinations of these basic elements:

$$[X_i, X_j] = \sum_{k=1}^{d} \lambda_{ijk} X_k \quad (i, j = 1, \cdots, d).$$

The coefficients λ_{ijk} $(i, j, k = 1, \cdots, d)$ are called the structural constants of $\mathcal{G}$ with respect to the given basis.

4 Exclusion of semisimple Lie groups

In this section we are going to show that semisimple Lie groups are not scalable. As we did for nonnilpotent connected solvable Lie groups, we prove this result by illustrating that they fail to satisfy the second condition of $(A.1)$.

Theorem A.16 *For a connected semisimple Lie group, suppose α is an automorphism of G and Γ is a lattice subgroup of G, then the image of Γ under α can not be properly inside Γ.*

Theorem A.16 demonstrates that there does not exist any topological automorphism of G such that the second condition of $(A.1)$ is satisfied. Hence, semisimple Lie groups are not scalable groups. The proof of theorem A.16 will be finished by considering the following lemmas.

We use $\text{Aut}(G)$ to denote all automorphisms of G. It is obviously a group under composition operation. Similarly, the inner automorphism of G denoted by $\text{Int}(G)$ constitutes a subgroup of $\text{Aut}(G)$. Actually, $\text{Int}(G)$ is a normal subgroup of $\text{Aut}(G)$. Indeed, for any $\tau_x \in \text{Int}(G)$ $(\tau_x : y \mapsto xyx^{-1})$ and any $\alpha \in \text{Aut}(G)$, we have

$$(\alpha \tau_x \alpha^{-1})(y) = (\tau_x \alpha^{-1})(\alpha(y)) = \alpha^{-1}(x\alpha(y)x^{-1})$$

87

$$= \alpha^{-1}(x) y \alpha^{-1}(x^{-1}) = \alpha^{-1}(x) y \alpha(x) = \tau_{\alpha^{-1}(x)}(y).$$

So $\alpha \tau_x \alpha^{-1} \in \text{Int}(G)$. Therefore, the space of cosets $\text{Aut}(G)/\text{Int}(G)$ has group structure.

Lemma A.17([Gi]) *Aut(G)/Int(G) is not only discrete but finite.*

Lemma A.18 *For a connected semisimple Lie group, all automorphisms are volume preserving.*

Proof of Lemma A.18. We know that $\text{Int}(G)$ is identical with the component denoted by $\text{Aut}_0(G)$ of the group $\text{Aut}(G)$ which is connected to the identity:

$$\text{Aut}_0(G) = \text{Int}(G).$$

Furthermore, by the Lemma A.17, the factor group $\text{Aut}(G)/\text{Int}(G) = \text{Aut}(G)/\text{Aut}_0(G)$ is not only discrete but finite.

Now let's prove that all automorphisms of semisimple group G are volume preserving. To show this, we first establish a mapping Θ from $\text{Aut}(G)$ to the multiplicative group R^+ of positive real numbers. We note that, for G, the left Haar measure on G is uniquely determined up to constant multiplication. Let's denote this Haar measure by λ. For any $\alpha \in \text{Aut}(G)$, we define $\lambda_\alpha(E) = \lambda(\alpha(E))$ for any Borel subset $E \subseteq G$, then λ_α is again a left Haar measure . By the uniqueness theorem, there is number $\Theta(\alpha) > 0$ such that $\lambda_\alpha = \Theta(\alpha)\lambda$. We claim that

$$\Theta: \text{Aut}(G) \longrightarrow R^+$$

$$\alpha \longmapsto \Theta(\alpha)$$

is a homomorphism. In fact, for any α, $\beta \in \text{Aut}(G)$ and $E \subseteq G$,

$$\lambda_{\alpha\beta}(E) = \Theta(\alpha\beta)\lambda(E) = \lambda(\alpha\beta(E))$$

$$= \lambda(\alpha(\beta(E))) = \Theta(\alpha)\lambda(\beta(E)) = \Theta(\alpha)\Theta(\beta)\lambda(E).$$

For any $\tau_x \in \text{Int}(G)$, one can see that τ_x is volume preserving because, for any Borel subset $E \subseteq G$, we have that $\lambda(\tau_x(E)) = \lambda(xEx^{-1}) = \lambda(E)$ by the unimodularity of G. Since

$\text{Int}(G)$ is a normal subgroup of $\text{Aut}(G)$ and Θ is a homomorphism of $\text{Aut}(G)$ into $\mathbf{R}^+$, we can define

$$\overline{\Theta} : \text{Aut}(G)/\text{Int}(G) \longrightarrow \mathbf{R}^+$$

by $\overline{\Theta}(\alpha\text{Int}(G)) := \Theta(\alpha)$. This is well-defined. In fact, suppose α_1 and α_2 are two different representatives, that is, $\alpha_1\text{Int}(G) = \alpha_2\text{Int}(G)$, so $\alpha_1\alpha_2^{-1} \in \text{Int}(G)$ and hence $\Theta(\alpha_1\alpha_2^{-1}) = \Theta(\alpha_1)\Theta(\alpha_2^{-1}) = 1$. Therefore $\overline{\Theta}(\alpha_1\text{Int}(G)) = \overline{\Theta}(\alpha_2\text{Int}(G))$. $\overline{\Theta}$ is also a homomorphism:

$$\overline{\Theta}(\alpha_1\text{Int}(G)\alpha_2\text{Int}(G)) = \overline{\Theta}(\alpha_1\alpha_2\text{Int}(G)) = \Theta(\alpha_1)\Theta(\alpha_2)$$

$$= \overline{\Theta}(\alpha_1\text{Int}(G))\overline{\Theta}(\alpha_2\text{Int}(G))$$

It follow that the image of $\overline{\Theta}$ is a group of finite numbers from the fact that the quotient group $\text{Aut}(G)/\text{Int}(G)$ is finite. But the finite subgroup of $\mathbf{R}^+$ is only trivial one. Hence we proved that for any $\alpha \in \text{Aut}(G)$, α is volume preserving.

Proof of the Theorem. Now we can prove the theorem by using Lemma A.18. Assume that $\alpha(\Gamma)$ is a subgroup of Γ. We are going to show that $\alpha(\Gamma) = \Gamma$. Let F be a fundamental domain of Γ. Then F has finite volume since G/Γ is compact. We claim that this volume is the same for all fundamental domains. Indeed, suppose F_1 and F_2 are two fundamental domains of Γ, then $G = \bigcup_{\gamma\in\Gamma} \gamma F_1$, $G = \bigcup_{\gamma\in\Gamma} \gamma F_2$. Let's denote the Haar measure on G by $d\lambda$. Then for any nonnegative measurable function f on G, we have

$$\int_G f(x)d\lambda(x) = \int_{\sum_{\gamma\in\Gamma} \gamma F_1} f(x)d\lambda(x)$$

$$= \sum_{\gamma\in\Gamma} \int_{\gamma F_1} f(x)d\lambda(x)$$

$$= \sum_{\gamma\in\Gamma} \int_{\gamma F_1} f(\gamma x)d\lambda(\gamma x)$$

$$= \sum_{\gamma\in\Gamma} \int_{\gamma F_1} f(\gamma x)d\lambda(x).$$

Similarly,

$$\int_G f(x)d\lambda(x) = \sum_{\gamma\in\Gamma} \int_{\gamma F_2} f(\gamma x)d\lambda(x).$$

Thus

$$\sum_{\gamma \in \Gamma} \int_{\gamma F_1} f(\gamma x) d\lambda(x) = \sum_{\gamma \in \Gamma} \int_{\gamma F_2} f(\gamma x) d\lambda(x). \qquad (A.2)$$

Let $f = \chi_{F_1}$, where χ_E is the characteristic function of E, then the left side of (A.2) is $\lambda(F_1)$. it equals

$$\sum_{\gamma \in \Gamma} \int_{\gamma F_2} \chi_{F_1}(\gamma x) d\lambda(x) = \sum_{\gamma \in \Gamma} \int_{\gamma F_2} \chi_{\gamma^{-1} F_1}(x) d\lambda(x)$$

$$= \sum_{\gamma \in \Gamma} \int_G \chi_{F_2 \bigcap \gamma^{-1} F_1}(x) d\lambda(x)$$

$$= \sum_{\gamma \in \Gamma} (F_2 \bigcap \gamma^{-1} F_1) = \lambda(F_2).$$

So since α is an automorphism of G, $\alpha(F)$ will be a fundamental domain for $\alpha(\Gamma)$. Since α is volume preserving by the lemma above, $\alpha(F)$ has the same volume as a fundamental domain for the large group Γ. This is impossible unless $\alpha(\Gamma) = \Gamma$. Therefore we establish that $\alpha(\Gamma)$ can not be set properly inside Γ.

90

Conclusion

Our purpose was to build multiresolution analysis in $L^2(G)$, where G is a non-abelian locally compact group. We proved that within connected solvable Lie groups, at most countably many nilpotent Lie groups are scalable groups, that is, only countably many nilpotent Lie groups are candidates for constructing multiresolution analysis. The most importantly, Heisenberg groups are interesting concrete scalable examples. We also showed that all connected semisimple Lie groups are not suitable for setting up multiresolution analysis. This gives us an impression that only those groups which are close to being abelian are scalable groups. The first question remaining to be done is that how close are the scalable groups from abelian groups, or is there any scalable groups between nilpotent and semisimple. How to identify the class of scalable groups?

We gave the necessary and sufficient conditions on scaling function for non-abelian groups such that the scaling function generates a multiresolution analysis. That is, for certain non-abelian groups, for example Heisenberg groups, we can build multiresolution analysis on them. This is the first step towards constructing wavelets for non-abelian groups. The second question is how to construct concrete wavelets for the given scalable groups by analyzing the properties of the scaling function and the scaling identity like the case for the space $L^2(\mathbf{R}^d)$.

Since the dual spaces of Heisenberg groups have a simple one-dimensional structure, we may probably take this advantage to create the continuous wavelet transforms for the Heisenberg groups. That is, we represent elements by using translations and dilations of one fixed function with the translation and dilation parameters varying continuously. Also, it would be especially important to establish a theory on discrete version of the continuous wavelet transforms.

Bibliography

[Ba]   G.Battle, *A block spin construction of ondelettes. Part I: Lemarié functions,* Comm.Math.Phys., 110(1987), pp.601-615.

[BC]   L.Baggett, A.Carey, W.Moran and P.Ohring, *General existence theorems for orthonormal wavelets, an abstract approach.* Publ.RIMS,Kyoto Univ. 31 (1995), pp.95-111.

[BD]   C.de Boor, R.Devore, and A.Ron, *On the construction of multivariate (pre)wavelets,* Constr. Approx. 9(1993), pp.123-166.

[BM]   J.Brezin and C.Moore *Flows on homogeneous spaces: a new look.* American Journal of Mathematics, 103(1981), pp.571-613.

[Bo]   A.Borel, *Linear algebraic groups,* Second enlarged edition, Springer-Verlag, New York, 1991.

[CG]   L.Corwin and F.P.Greenleaf *Representations of nilpotent Lie groups and their applications. Part 1: Basic theory and examples,* Cambridge studies in advanced mathematics 18, Cambridge university press, 1990.

[Da]   S.Dahlke, *Multiresolution analysis and wavelets on locally compact Abelian groups,* in "Wavelets, Images, and Surface Fitting"(P.J.Laurent, A.Le Méhauté, and L.L.Schumaker, Eds.), pp.141-156, A.K.Peters, Wellesley, Massachusetts, 1994.

[Dau]   I.Daubechies, *Ten lectures on wavelets*, CBMF conference series in applied mathematics, Vol.61, SIAM, Philadelphia,1992.

[Di]   J.Dixmier, *C* Algebras*, North Holland, Amsterdam, 1981.

[DS]   R.J.Duffin and A.C.Schaeffer, *A class of nonharmonic Fourier series*, Trans.Amer.Math.Soc., 72(1952), pp.341-366.

[Fo1]   G.B. Folland, *A course in abstract harmonic analysis*, Studies in advanced mathematics, Boca Raton, 1995.

[Fo2]   G.B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton New Jersey, 1989.

[Ge]   D.Geller, *Fourier analysis on the Heisenberg group.I. Schwartz space.* J.Funct.Anal. 36(1980), pp.205-254.

[Gi]   R.Gilmore, *Lie groups, Lie algebras, and some of their applications.* Wiley-Interscience, New York, 1974.

[GM]   K.Gröchenig and W.R.Madych, *Multiresolution analysis, Haar bases, and self-similar tilings of* R^n, IEEE Transactions on information theory, 38(1992), 556-568.

[Ha]   A.Haar, *Zur theorie der orthogonalen funktionen-systeme*, Math.Ann., 69, pp.331-371.

[Hu]   J.E.Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975.

[JM]   R.Q.Jia and C.A.Micchelli, *Using the refinement equation for the construction of pre-wavelets II: Powers of two*, in "Curves and surfaces (P.J.Laurent, A.Le Méhauté, and L.L.Schumaker, eds.), Academic Press, New York, 1991, pp.209-246.

[JS] R.Q.Jia and Z.Shen, *Multiresolution and wavelets*, Proc.Edinburgh Math.Soc. 37(1994), pp.271-300.

[KR] R.V.Kadison and J.R.Ringrose, *Fundamentals of the theory of operator algebras*, vol.1, *Elementary theory*, Pure and applied Mathematics,100,Academic Press, New York-London,1983.

[Le] P.G.Lemarié, *Une nouvelle base d'ondelettes de $L^2(\mathbf{R}^n)$*, J.de Math.Pures et Appl., 67(1988), pp.227-236.

[Li] R.L.Lipsman, *Non-abelian Fourier analysis*, Bull.Sc.Math.,2e Série, 98(1974), pp.209-233.

[Ma] W.R.Madych, *Some elementary properties of multiresolution analyses of $L^2(\mathbf{R}^d)$*, in "Wavelets– A tutorial in theory and applications", Academic press, New York, 1992, pp.259-294.

[Mal1] S.Mallat, *Multifrequency channed decompositions of images and wavelet models*, IEEE Trans.Acoust.Speech Signal Process., 37(12)(1989), pp.2091-2110.

[Mal2] S.Mallat, *Multiresolution approximation and wavelets*, Trans.Amer.Math.Soc., 315(1989), pp.69-88.

[Mal3] S.Mallat, *A theory for multiresolution signal decomposition: The wavelet representation*, IEEE Trans.Patt.Anal.Mach.Intell., 11(7)(1989), pp.674-693.

[Mau] F.I.Mautner, *Unitary representations of locally compact groups*, II, Ann.of Math. 52(1950), pp.528-556.

[Me1] Y.Meyer, *Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs*, Séminaire Bourbaki, 1985-1986, no.662.

[Me2] Y.Meyer, *Wavelets and operators*,Combridge, Combridge University Press, 1992.

[Ra]   M.S.Raghunathan, *Discrete subgroups of Lie groups*, Springer Verlag, Berlin, 1972.

[Sc]   R.Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, 1960.

[Se1]   I.E.Segal, *The two-sided regular representation of a unimodular locally compact group*, Ann.of Math. 51(1950), pp.293-298.

[Se2]   I.E.Segal, *An extension of Plancherel's formula to separable unimodular groups*, Ann.of Math. 52(1950), pp.272-292.

[Se3]   I.E.Segal, *Decompositions of operator algebras*, I,Mem.Amer.Math.Soc. 9(1951), pp.1-67.

[St]   J.Stöckler, *Multivariate wavelets*, in "Wavelets– A tutorial in theory and applications", Academic press, New York, 1992, pp.325-355.

[Str]   R.S.Strichartz, *Self-similarity on nilpotent Lie groups*, in Contemporary Mathematics, 140(1992), pp.123-157.

[Strö]   J.O.Strömberg, *A modified Franklin system and higher order spline systems on R^n as unconditional bases for Hardy spaces*, In Beckner et al., editor, Conference on Harmonic analysis in honor of Antoni Zygmund, volume II, Univ. of Chicago Press,1981, pp.475-494.

[Ta]   M.E.Taylor, *Noncommutative harmonic analysis*. Mathematics Surveys and Monographs, No.22, American Mathematical Society, Providence, Rhode Island, 1986.