Inter-dimensional effects in quasirelativistic systems:Bosons

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Abstract

Recently, interest in low-dimensional systems with quasirelativistic dispersion relations has risen due to the discovery of many new types of phases of matter that are topological in nature. These phases present a variety of intiguing phenomena that are of great interest in the field of spintronics. We are motivated by this interest in the condensed matter physics community to derive a fully-analytical model for quasirelativistic inter-dimensional systems which allows for the calculation of the density of states, to be used for calculating the availability of charge carriers in these systems. This work is the first step in a process that will lead to a fully-analytical model for fermions in topological materials and their topological phase transition in various compounds.

In this thesis we examine an inter-dimensional system that consists of a three-dimensional bulk with a two-dimensional interface residing inside the bulk. We study how the propagation of bosons with a quasirelativistic dispersion relation is affected by the presence of the two-dimensional interface. In order to accomplish this, we model the system in terms of using a relativistic Hamiltonian for the bulk along with a relativistic term for the interface that only contains the kinetic energy and change-in-mass terms. The interface Hamiltonian is scaled by a length parameter that accounts for the interface thickness. The inclusion of the kinetic energy terms in the interface to model the propagation effects is consistent with previous work done in this area. We calculate the Green's function and corresponding density of states in the interface for the inter-dimensional quasirelativistic Hamiltonian. We find that the density of states in the interface approaches the free three-dimensional relativistic density of states in the limit that the length parameter of the interface approaches zero, as well as in the low energy limit when the change in the mass parameter due to the interface is small. We find that the density of states in the interface approaches the free two-dimensional relativistic density of states in the high-energy limit. Our results in the non-relativistic limit are in full agreement with the previous non-relativistic work from Ref. [1] in the parabolic band approximation with the constraint that the change in the mass parameter is small. We calculate the density of states using the band gap verified in Ref. [2] for PbTe as this material fits the criteria for our model.

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For my love, Joelle.

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Chapter 1

Introduction

1.1 Topological Insulators and Dirac Semimetals

Condensed matter physics has received a significant amount of interest over the last decade due to the discovery of a new type of phase transition in materials that are topological in nature. Practically, these new materials will help the spintronics industry by creating materials with intrinsic properties that allow for dissipationless conduction and superconductivity. Dissipation occurs in quantum spintronic devices due to the energy and information that flows through the system. Energy and information are both irreversible processes which lead to dissipation of heat into the surrounding environment. The first theoretical prediction to spark interest in this field was the intrinsic spin Hall effect [3, 4, 5]. References [3] and [5] theoretically predicted that an electric field could induce a large amount of dissipationless spin current at room temperature using hole-doped semiconductors such as Si, Ge and GaAs. In these materials, the effect is driven by spin-orbit coupling and leads to an intrinsic spin current in the direction perpendicular to the charge current. The quantum spin current predicted by [3],[5] is of interest because it allows for direct control of the spin without the use of an external magnetic field, and is dissipationless. Although the spin current predicted in Refs. [3, 5] is dissipationless, there is still a charge current that is not disspationless. Shortly after the prediction of the intrinsic dissipationless spin current present in semiconductors with strong spin-orbit coupling, Ref. [4] predicted the existence of a dissipationless spin-Hall effect without any dissipative charge current. It has been predicted [4] that band insulators of either "zero-band-gap semiconductors" [4] such as HgTe or narrow-band-gap semiconductors such as PbTe are materials suitable to realize the dissipationless spin-Hall

current. An important characteristic of these systems is that they adhere to time-reversal and inversion symmetry. Therefore from Kramer's theorem, which states that every energy eigenstate of a time-reversal-symmetric system of particles with half-integer spin must be doubly-degenerate, we must have a pair of degenerate energy states in the system.

Shortly after the prediction of the dissipationless spin-Hall current by [3, 5, 4] came the prediction of the quantum spin-Hall effect in [6, 7, 8]. The quantum spin-Hall effect and its associated quantum spin-Hall phase is a time-reversal-invariant electronic state. It is characterized by a Z_2 topological invariant that distinguishes the quantum spin-Hall phase from a convential insulator [8]. Like a conventional insulator, the quantum spin-Hall phase has a bulk energy gap. It differs from the conventional insulator by the presence of gapless edge states that facilitate the transport of spin and charge currents. The quantum spin-Hall effect predicted by Ref. [7] differs from the quantum Hall effect in that it does not require an external magnetic field to break time-reversal symmetry. The quantum spin-Hall effect exists in the absence of an external magentic field and has an intrinsic spin-Hall conductance in units of $2\frac{e}{4\pi}$ [7]. This spin-Hall conductance is created in a bilayer system in which spin-up electrons and spin-down electrons move in opposite directions on opposite sides of the bilayer. This system leads to a vanishing charge Hall conductance and a spin-Hall conductance that is quantized in units of $2\frac{e}{4\pi}$ [7]. Figure 1.1 offers a visualisation of the separation of spin currents.

Reference [6] expanded on the theoretical prediction of the quantum spin-Hall effect by predicting that it could be realized in HgTe/CdTe quantum wells. Both mercury telluride (HgTe) and cadmium telluride (CdTe) are semiconductors, which when arranged as CdTe/HgTe/CdTe creates a semiconductor quantum well. The number of helical edge states of the quantum spin-Hall state does not change under smooth variations or distortions of the Hamiltonian provided that the energy gap in the bulk does not collapse [6]. The topological quantity Z_2 is associated with this edge-state invariance. The mechanism with which the quantum spin-Hall effect is realized in the CdTe/HgTe/CdTe quantum well system is band inversion. As can be seen in Fig. 1.2, CdTe has a band structure in which the Γ_6 band which originates from the s-orbitals lies above the Γ_8 band which originates from the p-orbitals [6]. HgTe has an inverted band structure in that the Γ_6 band lies below the Γ_8 band [6].The

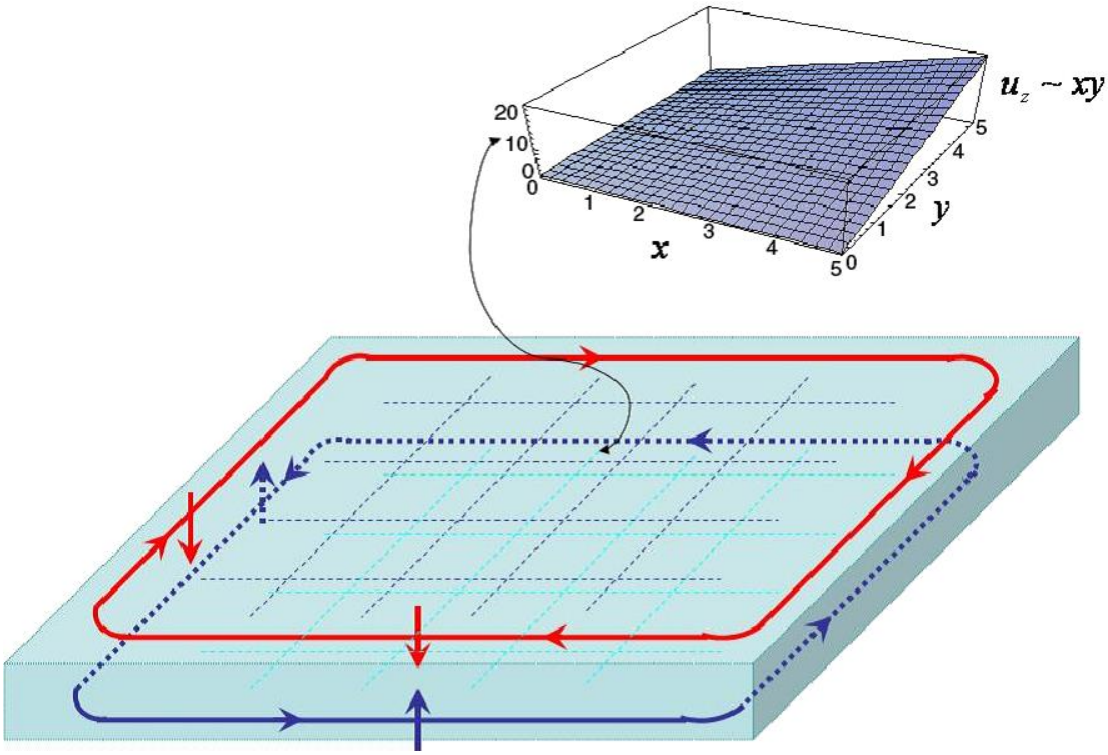

Figure 1.1: Spin-up and spin-down electrons have opposite chirality as they feel the opposite spin-orbit coupling force. Total charge conductance vanishes but the spin conductance is quantized. The inset shows the lattice displacement leading to the strain configuration. Figure from Ref. [7]

phase transition of the CdTe/HgTe/CdTe quantum well structure from the ordinary insulating phase to a quantum spin-Hall phase occurs when the thickness of the HgTe reaches a critical value $d = d_c$ [6]. At this critical thickness the $H1$ and $E1$ bands at the Γ point in the Brillouin zone invert, and the quantum well structure becomes a topological insulator with a single pair of helical edge states [6]. Helicity is defined as $h = \boldsymbol{S} \cdot \hat{\boldsymbol{p}}$ where $\boldsymbol{S}$ is the particles spin and $\hat{\boldsymbol{p}}$ is the direction of the particles momentum [9]. Near the Γ point, which is a symmetry point in the Brillouin zone with momentum $|\boldsymbol{k}_{\|}| = 0$, the energy states can be described using a $2+1$ dimensional Dirac equation. At the critical thickness when band inversion occurs, the mass term in the Dirac equation changes sign and leads to two $U(1)$-spin and Z_2 topological invariants. The theoretical prediction proposed in [6] has experimentally been realized [10]. The helical edge states in the quantum spin-Hall state are protected by time-reversal symmetry. Turning on a magnetic field destroys the time-reversal symmetry and thus opens a gap between the otherwise degerate helical edge states [10].

A second example of a model system for the quantum spin-Hall effect is graphene [11].

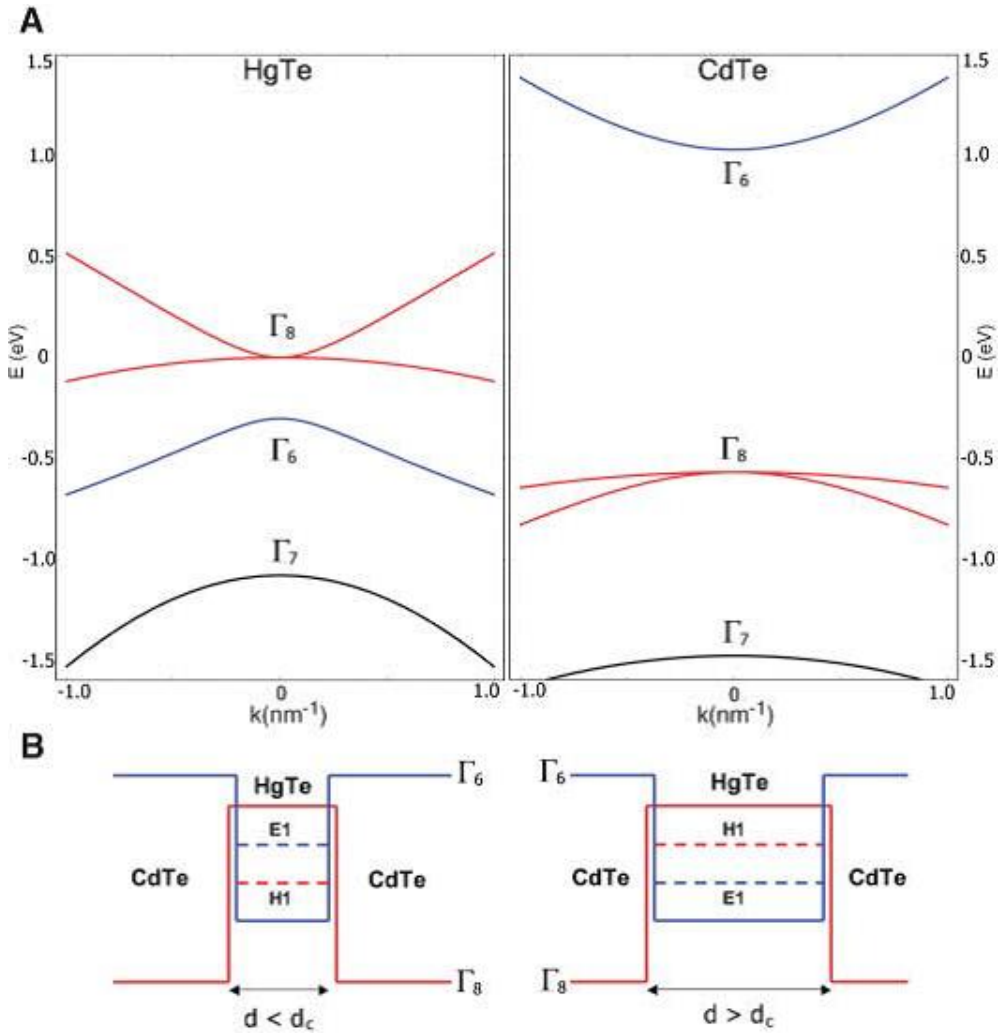

Figure 1.2: (***A***) Bulk energy band of HgTe and CdTe near the Γ point. (***B***) The CdTe-HgTe-CdTe quantum well in the normal regime E1 > H1 with $d < d_c$ and in the inverted regime H1 > E1 with $d > d_c$. Figure from Ref. [6]

The quantum spin-Hall effect in graphene is modelled as two copies of the quantum spin-Hall state [12, 11]. Therefore we will include a brief overview of the fundamental principles that describe the quantum spin-Hall state in graphene. Graphene is a model system for the quantum Hall effect when in the presence of a periodic magnetic field [13]. The two-dimensional model of a semimetal, which was also investigated in Ref. [14] as a two-dimensional graphite model, is used to model graphene that exhibits a quantized Hall conductance without an external magnetic field. It has been show in Ref. [13] that for critical values of the model parameters, the ground-state energy exhibits a transition from a normal semiconductor to a quantum Hall state. In this case the low-energy states can be modelled by a $(2 + 1)$-

dimensional field theory where at the critical points the Dirac fermions behave as if they are massless. The system modelled in Ref. [13] contains a degeneracy at isolated points that occur at the bottom of the valence band and the top of the conduction band. This degeneracy exists because of inversion symmetry and time-reversal invariance at these points. Under broken inversion symmetry, a gap in the band structure opens between the degenerate points in the valence and conduction bands and the system is a normal semiconductor. However if time-reversal symmetry is broken the system moves into a quantum Hall effect state [13]. The models of graphene discussed up to this point have neglected the effect of the spin-orbit interaction. This effect was included for graphene in Ref. [12] where it was shown that the presence of this effect in a low-temperature state yields a system that converts from a semimetallic state to a quantum spin-Hall insulator [12]. In the quantum spin-Hall state, the system contains a band gap in the bulk but has gapless edge states at the boundaries of the system that allow for charge and spin currents to propagate [12]. The quantum spin-Hall state in graphene is topologically distinct from a trivial insulator.

After the theoretical prediction of the two-dimensional topological insulator and its experimental discovery, it was realized that the two-dimensional topological insulator state was a special case of a three-dimensional topological insulator [11, 15, 16, 17]. The three-dimensional topological insulator is described by the existence of four Z_2 topological invariants [15, 16, 17, 11]. In the Brillouin zone for the surface of a three-dimensional crystal structure there are four time-reversal invariant points [11]. If there are surface states present at these points then from Kramer's theorem these points must be doubly degenerate [11]. The Kramer's degenerate points form Dirac points where the bottom of the valence band touches the top of the conduction bands [11]. Depending on how the four distinct Dirac points connect to each other, the system may be a trivial insulator or a three-dimensional topological insulator. If two time-reversal invariant momenta, occuring at the Dirac points, cross the Fermi energy an odd number of times, then the surface states are topologically protected and gapless with spin-filtered Dirac-like dispersion relations connecting the conduction band and the valence band. Therefore for an odd number of crossings of the Fermi energy, the edge states are examples of the quantum spin-Hall phase [11, 15]. If the edge states cross the Fermi energy an even number of times then the system is not topologically

protected and is in the trivial insulating phase [15]. The distinction between these two cases is determined by the four Z_2 topological invariants of the system [11]. There are two types of three-dimensional topological insulators. The first that is characterized by the Z_2 topological invariant $\nu_0 = 0$ is a weak topological insulator. The weak topological insulator is a system that is created by stacking layers of two-dimensional quantum spin-Hall insulators. However, unlike the strictly two-dimensional case of the quantum spin-Hall insulator, i.e., a single layer of the weak topological insulator, the helical edge states in the layered system are not protected by time-reversal symmetry [11]. The second type of three-dimensional topological insulator is termed a strong topological insulator and is associated with a Z_2 topological invariant $\nu_0 = 1$. This system is distinctly different from a two-dimenesional topological insulating state. The ν_0 value determines whether the Fermi surface circle encloses an even or odd number of Dirac points [11]. An even number of enclosed Dirac points is associated with a weak topological insulator, while an odd number of enclosed Dirac points is associated with a strong topological insulator. Reference [15] used a slab geometry to calculate the band structure of both weak and strong topological insulators in order to show the number of Dirac points present in each case. The edge states present in a strong topological insulator form a new type of topological metal [11, 15]. In this new metallic state, the edge states are not spin degenerate [11]. This can occur because the Dirac point partners, Dirac points of opposite spin, reside on opposite surfaces [15]. The strong topological insulator is robust against the weak disorder and thus cannot be localized. This is a consequence of the electron having the Berry phase of π upon circling along the Fermi surface. Experimentally, the three-dimensional topological insulator has been realized in the semiconducting alloy $Bi_{1-x}Sb_x$ using angle-resolved photoemission spectroscopy (ARPES) experiments [11, 18]. A second generation of three-dimensional topological insulators was discovered in the materials Bi_2Sb_3, Bi_2Te_3 and Sb_2Te_3 [11]. This discovery of these materials had been motivated by the search for $Bi_{1-x}Sb_x$ [11].

A different type of topological insulator termed a topological crystalline insulator is characterized by having crystal point symmetry and time-reversal symmetry which makes the edge states of the sample topologically protected [19]. These topological insulators are three-dimensional and are considered to be a counterpart to topological insulators without spin-

orbit coupling [19]. Topological crystalline insulators cannot be smoothly connected to a trivial insulator when time-reversal and crystal symmetries exist. In order for the crystal symmetry to be protected, the crystal structure must be symmetric upon rotational symmetry on the surfaces that contain the edge states. In Ref. [19], crystal structures that have fourfold C_4 and sixfold C_6 rotational symmetry as well as time-reversal symmetry were studied. It is shown that a new Z_2 topological invariant defines the topological nature of a system with time-reversal invariant band structures with C_4 or C_6 rotational symmetry [19]. The surface states traverse the band gap and are four-fold degenerate [19]. Similarly to the strong topological insulator, a Z_2 topological invariant of $\nu_0 = 1$ characterizes the topological crystalline insulator with gapless surface states [19]. Shortly after the theoretical prediction of the topological crystalline insulator came the first theoretical prediction of a material that could exist as a topological crystalline insulator [20]. The material studied in Ref. [20] was SnTe. SnTe contains an even number of Dirac cones on its high symmetry crystal surfaces which lead to metallic-like surface states which are topologically protected from disorder by crystal symmetry. Reference [20] compares the two insulators SnTe and PbTe, showing that when crystal symmetry is present there exists a mirror Chern number that is topologically-invariant and characterizes the topological crystalline insulator [20]. It has been shown in Ref. [20] that because SnTe has an inverted band structure at its fundamental band gaps which are located at four equivalent L points in the face-centered-cubic Brillouin zone, it is topologically non-trivial [20]. PbTe on the other hand has a band structure that can be smoothly connected to a trivial insulator [20]. This explains the band inversion in the material $Pb_{1-x}Sn_xTe$ where for increasing x, the conduction and valence bands become inverted at the L points in the band structure [20].

In 2011 Wan *et al*. discovered the theoretical existence of a new type of topological system in pyrochlore iridates such as $Y_2Ir_2O_7$ by means of first-principle calculations [21]. The new state of matter is a topological semimetal which can be viewed as a three-dimensional realization of graphene [21]. This topological semimetal has linearly dispersing excitations and shows the existence of Weyl fermions that obey a two-component Dirac equation [21]. The topologically-protected surface states are in the form of Fermi arcs, which can only exist in a three-dimensional magnetic system [21, 22]. Each point where the valence and

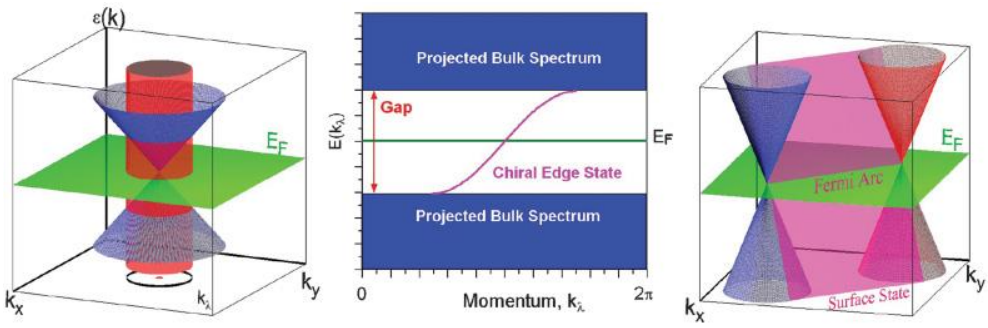

Figure 1.3: Illustration of surface states arising from bulk Weyl points. (a) The bulk states as a function of (k_x,k_y) (and arbitrary k_z) fill the inside of a cone. A cylinder whose base defines a one-dimensional circular Brillouin zone is also drawn. (b) The cylinder unrolled onto a plane gives the spectrum of the two-dimensional subsystem H(λ,k_z) with a boundary. On top of the bulk spectrum, a chiral state appears due to the nonzero Chern number. (c) Meaning of the surface states back in the three-dimensional system. The chiral state appears as a surface connecting the original Dirac cone to a second one, and the intersection between this plane and the Fermi level gives a Fermi arc connecting the Weyl points. Figure from Ref. [21]

conduction bands touch is described by the Weyl equation, which lends the term Weyl semimetal for these types of materials [21]. The Weyl fermions with chirality are gapless at the points where the valence and conduction bands meet. Unlike a topological insulator that requires time-reversal and inversion symmetry, no symmetry is required for the existence of the Weyl fermion nodes [21, 22]. In fact, the Weyl nodes do not exist in the presence of time-reversal and inversion symmetry. A Weyl node can be cancelled by an antinode with opposite chirality; therefore the existence of the Weyl nodes implies that they must be separated in momentum space [21, 22]. Therefore Weyl nodes come in pairs. There are Fermi arcs that connect these Weyl nodes of opposite chirality [21, 22]. It was shown in Ref. [21] that there would be 12 pairs of Weyl points in the entire Brillouin zone of $Y_2Ir_2O_7$. The surface states in this material are chiral edge states. In a three-dimensional system, the chiral edge state is a surface connecting one Dirac cone to another [21]. The intersection between this surface and the Fermi energy surface yields a Fermi arc that connects the two Weyl points of the Dirac cones [21]. Figure 1.3 displays schematically the Dirac cones and Fermi arc. It has been shown in [23] that the Weyl fermions characterizing the Weyl semimetal are generically present at the critical transition point between ordinary band insulators and topological insulators. Reference [22] considers a multilayer heterostructure consisting of

thin topological insulator films that are separated by normal insulators. They found that stable Weyl nodes connecting the conduction and valence bands existed in the system when the time-reversal symmetry was broken by magnetic impurities or an external magnetic field. These nodes lead to robust surface states, which can be considered chiral quantum Hall edge states [22].

In 2012, shortly after the theoretical prediction of Weyl semimetals, came the prediction of Dirac semimetals in compounds of the form A_3Bi where (A=Na, K, Rb) [24]. Weyl points as discussed in [21] and [22] had to be separated in momentum space in order to exist because of their opposite chiralities. In Ref. [24] it was shown that two overlapping Weyl points or a three-dimensional Dirac point existed by crystal symmetry in the compounds A_3Bi. Normally when two Weyl points overlap they cancel each other and a band gap opens between the conduction and valence bands. It is argued that in systems with crystal symmetry, the three-dimensional Dirac points can be protected and observed in the compounds A_3Bi, where low-energy states create a three-dimensional gas of massless Dirac fermions [24]. These compounds have Fermi arcs on their surfaces which make the system topologically non-trivial [24]. The compounds can be pushed into different topological phases by breaking explicit symmetries, and it has been shown that giant diamagnetism, linear quantum magnetoresistance, and the quantum spin-Hall effect are expected physical properties when the compounds are in a thin film configuration [24]. Following the theoretical prediction of Dirac semimetals in Ref. [24], their experimental verification was achieved for Cd_3As_2 [25, 26] and in Na_3Bi [27].

In 2010, A. Svane *et al.* calculated the electronic band structures of the compounds PbS, PbSe and PbTe [2]. It has been found [2] that a pressure-induced gap closure between the lowest point of the conduction band and the highest point of the valence band leads to the presence of three-dimensional Dirac points with linear dispersion relations [2]. An assumption made about these surface states is that they originate from massive Dirac bulk states. In 2013 this assumption was verified using the compound $Pb_{1-x}Sn_xTe$ [28] where the bulk states were indeed massive Dirac states in the bulk. In 2014, samples of $Pb_{1-x}Sn_xTe(111)$ were grown as thin films and it has been shown that by increasing the ratio of Sn/Pb the films undergo a topological phase transition between a trivial insulator and a topological

crystalline insulator [29]. They observed an even number of Dirac cones at distinct time-reversal invariant momenta, which is a requirement for the topological crystalline insulator [29]. At a critical value of x in $Pb_{1-x}Sn_xTe$ the band gap present in PbTe closes and re-opens as the amount of Sn is increased [29]. At this critical point the conduction and valence bands at the band gaps invert, which leads to a change in the topological invariant mirror Chern number characterizing the phase transition from trivial insulator to topological crystalline insulator [29]. The $Pb_{1-x}Sn_xTe(111)$ thin films contain non-trivial surface states with a Dirac-like dispersion for a low Sn/Pb ratio [29]. It was also found that by lowering the thickness of the film, a gap opens between coupled topological states on opposite surfaces [29].

In Sec. 1.3 we will explicitly show the connection between relativistic Bosons and the quasirelativistic system present in the topological crystalline insulator $Pb_{1-x}Sn_xTe$. The methodology that has been developed in Refs. [1, 30, 31, 32] allows for analogous development and analysis of the inter-dimensional system involving Bosons and Fermions. It is the purpose of this thesis to develop the formulation necessary to calculate the inter-dimensional quasirelativistic density of states for Bosons. Although not directly applicable to the systems of Dirac semimetals and topological crystalline insulators that involve Fermions, the work done in this thesis still has an applicable connection to these systems. Fermion fields which satisfy the Dirac equation

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - mc\right)\Psi(x) = 0,$$

are also solutions to the Klein-Gordon equation which is the equation of motion for relativistic scalar particles or Bosons. Therefore, the work done in this thesis and the results calculated will be related to the results found in the inter-dimensional quasirelativistic system for Fermions. In the following section, we will outline previous work done in the area of modelling inter-dimensional systems using the energy-dependent Green's function as this formulation will also be used in this thesis.

1.2 Low-dimensional Systems in Quantum Mechanics

"Low-dimensional quantum mechanics as well as electrodynamics are commonly used to describe and analyze the properties of electrons, photons, and quasiparticles such as phonons in structures with attractive properties"[33]. Examples of low-dimensional systems in one and two dimensions include quantum wires, interfaces, thin films, and surfaces. The density of states of such low-dimensional systems allows for estimates to be made regarding the number of available carriers for both charge and heat. In order to calculate the density of states analytically, Refs. [31] and [30] proposed using Hamiltonians with two-dimensional and three-dimensional kinetic energy terms as a description of a two-dimensional system. Reference [31] proposed a dimensionally-hybrid Hamiltonian to describe a system of interacting particles in the presence of a thin, planar, homogenous layer that generates a potential $U(z)$, where z is the coordinate normal to the plane of the layer. The inter-dimensional Hamiltonian in this case is given by

$$\begin{aligned} H &= \frac{\hbar^2}{2\mu}\int d^2\boldsymbol{x}\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\nabla}\psi|_{z=0} \\ &+ \int d^2\boldsymbol{x}\int dz\left(\frac{\hbar^2}{2M}\left(\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\nabla}\psi+\partial_z\psi^\dagger\cdot\partial_z\psi\right)+\psi^\dagger U(z)\psi\right), \end{aligned} \tag{1.1}$$

where particles of mass M are at $z = 0$ and $\boldsymbol{x} = (x, y)$. An implicit assumption made about the form of Eq. (1.1) is that the same field ψ can describe particles in the bulk three-dimensional part of the system, as it can describe collective excitations of those particles in the layer. The parameter μ is given by

$$\mu = \lim_{L\to 0}\frac{m(L)}{L}, \tag{1.2}$$

where $m(L)$ is the mass of the excitation modes in the layer, L is the thickness of the layer, and μ has the dimension of mass per unit length. In terms of a parabolic band approximation for particle motion in the layer, the mass of the excitation modes is given by an effective mass $m(L) = m_*$. A second scenario presented in Ref. [31] which is of direct importance regarding the formulation of this thesis, involves an inter-dimensional Hamiltonian with particles of mass M in the bulk, and a layer with a localized impurity potential given by

$$V(\boldsymbol{x}, z) = u(\boldsymbol{x})\delta(z), \tag{1.3}$$

where $\boldsymbol{x}$ is the two-dimensional coordinate in the interface. In this case the inter-dimensional Hamiltonian is given by

$$\begin{aligned} H &= \frac{\hbar^2}{2M}\int d^2\boldsymbol{x}\int dz\left(\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\nabla}\psi+\partial_z\psi^\dagger\cdot\partial_z\psi\right) \\ &+ \int d^2\boldsymbol{x}\left(\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\nabla}\psi+\psi^\dagger u(\boldsymbol{x})\psi\right)\Bigg|_{z=0}. \end{aligned} \tag{1.4}$$

The stationary wave equation that corresponds to Eq. (1.4) is given by [31]

$$E\psi(\boldsymbol{x},z)=\delta(z)\left(-\frac{\hbar^2}{2\mu}\Delta+u(\boldsymbol{x})\right)\psi(\boldsymbol{x},0)-\frac{\hbar^2}{2M}\left(\Delta+\partial_z^2\right)\psi(\boldsymbol{x},z). \tag{1.5}$$

Here only the layer potential $u(\boldsymbol{x})$ is treated as a perturbation as opposed to using the entire two-dimensional contribution to Eq. (1.5) as the perturbation in the Born approximation of the field $\psi(\boldsymbol{x},z)$. The corresponding Green's function $G_K(\boldsymbol{x},z)$ which describes propagation of bulk plane waves with energy $E=\hbar^2K^2/2M=\hbar^2(\boldsymbol{K}_\parallel^2+K_\perp^2)/2M$ satisfies the equation,

$$\left(\Delta+\partial_z^2+K^2\right)G_K(\boldsymbol{x},z)+2\ell\delta(z)\Delta G_K(\boldsymbol{x},0)=-\delta(\boldsymbol{x})\delta(z), \tag{1.6}$$

where $\ell=M/2\mu$. Using a Fourier ansatz given by

$$G_K(\boldsymbol{x},z)=\frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}\int dk_\perp G_K(\boldsymbol{k},k_\perp)\exp[i(\boldsymbol{k}\cdot\boldsymbol{x}+k_\perp z)] \tag{1.7}$$

in Eq. (1.6) yields the following equation for the momentum space Green's function:

$$\left(k^2+k_\perp^2-K^2\right)G_K(\boldsymbol{k},k_\perp)+\frac{\ell}{\pi}k^2\int dk'_\perp G_K(\boldsymbol{k},k'_\perp)=1. \tag{1.8}$$

It is useful to reiterate that $K^2=\boldsymbol{K}_\parallel^2+K_\perp^2=2ME/\hbar^2$ is the energy of the particle given in terms of its wavenumber. $k_\perp$ is the momentum space component that is in the direction perpendicular to the plane of the layer. Of most direct importance to the formulation of this thesis is that the $k_\perp$ dependence of the propagator $G_K(\boldsymbol{k},k_\perp)$ is given by

$$G_K(\boldsymbol{k},k_\perp)=\frac{f(k)}{(k^2+k_\perp^2-K^2)}, \tag{1.9}$$

where $f(k)$ is given by

$$f(k)=\frac{\ell}{\hbar}k^2\int dk'_\perp\frac{G_K(\boldsymbol{k},k'_\perp)}{(k^2+{k'_\perp}^2-K^2)}. \tag{1.10}$$

There is no $k_\perp$ dependence on the value of $f(k)$ once the integral is solved using the residue theorem, where the poles of the integral in Eq. (1.10) are chosen to be $k_\perp = \pm(\sqrt{K^2 - k^2}+i\epsilon)$ in order to properly reflect that the scattered waves from the two-dimensional layer are outgoing given $k < K$. This method for solving the inter-dimensional Green's function will also be employed in this thesis.

Reference [32] analyzes the inter-dimensional behaviour between one and two dimensions, respectively. This system can be visualized as a flat two-dimensional layer in the (x, y) plane, with a quantum wire laying on the layer at $y = y_0$ and extending out in the x direction. The inter-dimensional system of a quantum wire on a two-dimensional surface is modelled using an inter-dimensional Hamiltonian given by

$$\begin{aligned} H &= \int dx \int dy \frac{\hbar^2}{2m}\left(\frac{\partial\psi^\dagger(x,y)}{\partial x}\frac{\partial\psi(x,y)}{\partial x} + \frac{\partial\psi^\dagger(x,y)}{\partial y}\frac{\partial\psi(x,y)}{\partial y}\right) \\ &+ \int dx \frac{\hbar^2}{2\mu}\frac{\partial\psi^\dagger(x,y_0)}{\partial x}\frac{\partial\psi(x,y_0)}{\partial x}, \end{aligned} \tag{1.11}$$

where $\mu = m_*/L_\perp$ is the mass per lateral extension of the substructure [32].

The retarded energy-dependent Green's function $\mathcal{G}(E) = (E - H + i\epsilon)^{-1}$ for this inter-dimensional Hamiltonian therefore satisfies the equation in (x, y) representation given by

$$\left(\frac{2m}{\hbar^2}E + \partial_x^2 + \partial_y^2 + \delta(y - y_0)\frac{m}{\mu}\partial_x^2\right)\langle x, y|G(E)|x', y'\rangle = -\delta(x - x')\delta(y - y'), \tag{1.12}$$

The Fourier transform of the matrix element of the Green's function is given by

$$\begin{aligned} \langle x, y|G(E)|x', y'\rangle &= \frac{1}{4\pi^2}\int dk_x \int dk'_x \int dk_y \int dk'_y \langle k_x, k_y|G(E)|k'_x, k'_y\rangle \\ &\times \exp[i(k_x x + k_y y - k'_x x' - k'_y y)]. \end{aligned} \tag{1.13}$$

Inserting this into Eq. (1.12) yields

$$\begin{aligned} \left(\frac{2m}{\hbar^2}E - k_x^2 - k_y^2\right)\langle k_x, k_y|G(E)|k'_x, k'_y\rangle &- \frac{m}{\mu}\frac{k_x^2}{2\pi}\int d\kappa \exp[i(\kappa - k_y)y_0] \\ &\times \langle k_x, \kappa|G(E)|k'_x, k'_y\rangle \\ &= -\delta(k_x - k'_x)\delta(k_y - k'_y). \end{aligned} \tag{1.14}$$

Making use of translation invariance of the energy-dependent Green's function in the x direction, $\langle k_x, k_y|G(E)|k'_x, k'_y\rangle = \langle k_y|G(E, k_x)|k'_y\rangle\delta(k_x - k'_x)$, the solution for the Green's

function matrix element $\langle k_y|G(E,k_x)|k'_y\rangle$ is found to be

$$\begin{aligned}\langle k_y|G(E,k_x)|k'_y\rangle &= \frac{1}{k_y^2+k_x^2-(2mE/\hbar^2)-i\epsilon} \\ &\times \left[\delta(k_y-k'_y)-\frac{k_x^2\ell}{\pi}\frac{\exp[i(k'_y-k_y)y_0]}{k_y'^2+k_x^2-(2mE/\hbar^2)-i\epsilon}\right. \\ &\times \left(\frac{\sqrt{\hbar^2k_x^2-2mE}\,\Theta(\hbar^2k_x^2-2mE)}{\sqrt{\hbar^2k_x^2-2mE}+\hbar k_x^2\ell}\right. \\ &+ \left.\left.\frac{\sqrt{2mE-\hbar^2k_x^2}\,\Theta(2mE-\hbar^2k_x^2)}{\sqrt{2mE-\hbar^2k_x^2}+i\hbar k_x^2\ell}\right)\right], \end{aligned} \tag{1.15}$$

where $\ell = m/2\mu = (mL_\perp)/(2m_*)$. Reference [32] uses a Green's function matrix element in which both transverse direction arguments are given in terms of the configuration space in order to derive the effects of a first order perturbation on a state $\psi_0(x,y)$ off an impurity potential $V(x,y)$. Using the Born approximation to first order, the state $\psi(x,y)$ is given by

$$\begin{aligned}\psi(x,y) &= \psi_0(x,y)-\frac{2m}{\hbar^2}\int dx'\int dy'\langle y|G(E,x-x')|y'\rangle V(x',y')\psi_0(x',y') \\ &= \psi_0(x,y)-\frac{m}{\pi\hbar^2}\int dx'\int dy'\int dk_x\langle y|G(E,k_x)|y'\rangle \\ &\times \exp[ik_x(x-x')]V(x',y')\psi_0(x',y'), \end{aligned} \tag{1.16}$$

where $\langle y|G(E,k_x)|y'\rangle$ is given by

$$\begin{aligned}\langle y|G(E,k_x)|y'\rangle &= \frac{\hbar\Theta(\hbar^2k_x^2-2mE)}{2\sqrt{\hbar^2k_x^2-2mE}}\left[\exp\left(-\sqrt{\hbar^2k_x^2-2mE}\frac{|y-y'|}{\hbar}\right)\right. \\ &- \left.\frac{\hbar k_x^2\ell}{\sqrt{\hbar^2k_x^2-2mE}+\hbar k_x^2\ell}\exp\left(-\sqrt{\hbar^2k_x^2-2mE}\frac{|y-y_0|+|y'-y_0|}{\hbar}\right)\right] \\ &+ \frac{\hbar\Theta(2mE-\hbar^2k_x^2)}{2\sqrt{2mE-\hbar^2k_x^2}}\left[i\exp\left(i\sqrt{2mE-\hbar^2k_x^2}\frac{|y-y'|}{\hbar}\right)\right. \\ &+ \frac{\hbar k_x^2\ell}{\sqrt{2mE-\hbar^2k_x^2}+i\hbar k_x^2\ell} \\ &\times \left.\exp\left(i\sqrt{2mE-\hbar^2k_x^2}\frac{|y-y_0|+|y'-y_0|}{\hbar}\right)\right]. \end{aligned} \tag{1.17}$$

Reference [32] shows interesting distance effects between the location of the wire on the surface and the location of the perturbation potential or impurity. It also shows that there is a dependence between the location of the wire and the y coordinate of the wavefunction.

In the evanescent case, where the wave exhibits exponential decay at the boundary of the nanowire, the impact of the wire on the impurity scattering is exponentially suppressed for two different cases. The first is when the location of the impurity is far from the location of the wire. The second is when the wave function is located far away from the wire. In the case where we do not see an exponential drop-off in the intensity of the wave upon meeting the nanowire, namely the non-evanescent case, the perturbation of the Green's function propagator due to the wire is a strongly-oscillating function of $\sqrt{2mE - \hbar^2 k_x^2}$ far from the wire. The impact of the wire is small when the wave packets are located far away from the nanowire.

It has also been shown in Ref. [32] that for a one-dimensional quantum wire located on a two-dimensional surface, the system can be modelled in terms of an inter-dimensional Hamiltonian with competing kinetic energy terms. Motion along the wire comes at a different kinetic energy cost due to the different effective mass for propagation along the wire. The effect of the wire on the Green's function is to introduce extra exponential terms. These exponential terms have distance effects between the particles and the location of the wire.

To conclude this section we will introduce work done in Ref. [34], to analytically derive the density of states of the inter-dimensional system containing a three-dimensional bulk with a two-dimensional quantum well. The Hamiltonian for this system, with quantum well thickness $2a$ and potential $V_0 = -\mathcal{W}/2a$, is

$$H = \frac{\boldsymbol{p}^2}{2m} - \mathcal{W}\delta(z - z_0). \tag{1.18}$$

As we have seen before, the energy-dependent Green's function which is translational invariant in the directions parallel to the quantum well satisfies the equation,

$$\left(\boldsymbol{\Delta} + \frac{2m}{\hbar^2}[E + \mathcal{W}\delta(z - z_0)]\right)\langle z|G(\boldsymbol{x}_\parallel, E)|z'\rangle = -\delta(\boldsymbol{x}_\parallel)\delta(z - z'), \tag{1.19}$$

where we have substituted

$$\begin{aligned}\langle \boldsymbol{x}_\parallel, z|G(E)|\boldsymbol{x'}_\parallel, z'\rangle &\equiv \langle z|G(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel, E)|z'\rangle \\ &\equiv -\frac{\hbar^2}{2m}\langle z|\mathcal{G}(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel, E)|z'\rangle.\end{aligned} \tag{1.20}$$

The equation for the energy-dependent Green's function in a mixed representation $(k_\perp, z')$

can be solved using the following Fourier transform of the Green's function matrix element,

$$\begin{aligned}\langle \boldsymbol{k}_\parallel, k_\perp|G(E)|\boldsymbol{k'}_\parallel, z'\rangle &= \frac{1}{\sqrt{2\pi}^5}\int d^2\boldsymbol{x}_\parallel \int d^2\boldsymbol{x'}_\parallel \int dz \langle \boldsymbol{x}_\parallel, z|G(E)|\boldsymbol{x'}_\parallel, z'\rangle \\ &\times \exp[i(\boldsymbol{k'}_\parallel \cdot \boldsymbol{x'}_\parallel - \boldsymbol{k}_\parallel \cdot \boldsymbol{x} - k_\perp z)] \\ &= \langle k_\perp|G(\boldsymbol{k}_\parallel, E)|z'\rangle \delta(\boldsymbol{k}_\parallel - \boldsymbol{k'}_\parallel), \end{aligned} \tag{1.21}$$

where

$$\begin{aligned}\langle k_\perp|G(\boldsymbol{k}_\parallel, E)|z'\rangle &= \frac{1}{\sqrt{2\pi}}\int d^2\boldsymbol{x}_\parallel \int dz \langle z|G(\boldsymbol{x}_\parallel, E)|z'\rangle \\ &\times \exp[-i(\boldsymbol{k}_\parallel \cdot \boldsymbol{x}_\parallel + k_\perp z)] \end{aligned} \tag{1.22}$$

and translational invariance in $\boldsymbol{x}_\parallel$ has been used. Substituting this Fourier transform into Eq. (1.19) and using $\kappa = m\mathcal{W}/\hbar^2$ yields

$$\begin{aligned}\frac{\exp[ik_\perp(z_0 - z')]}{\sqrt{2\pi}} &= \left(\boldsymbol{k}_\parallel^2 + k_\perp^2 - \frac{2mE}{\hbar^2}\right)\exp(ik_\perp z_0)\langle k_\perp|G(\boldsymbol{k}_\parallel, E)|z'\rangle \\ &- \frac{\kappa}{\pi}\int dq_\perp \exp(iq_\perp z_0)\langle q_\perp|G(\boldsymbol{k}_\parallel, E)|z'\rangle. \end{aligned} \tag{1.23}$$

The solution for the mixed representation of the Green's function matrix element is given by

$$\begin{aligned}\langle k_\perp|G(\boldsymbol{k}_\parallel, E)|z'\rangle &= \frac{1}{\sqrt{2\pi}}\frac{1}{k_\perp^2 + \boldsymbol{k}_\parallel^2 - (2mE/\hbar^2) - i\epsilon}\Bigg[\exp(-ik_\perp z') \\ &+ \frac{\hbar\kappa\Theta(\hbar^2\boldsymbol{k}_\parallel^2)}{\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE} - \hbar\kappa - i\epsilon}\exp\left(-ik_\perp z_0 - \sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE}\frac{|z' - z_0|}{\hbar}\right) \\ &+ \frac{i\hbar\kappa\Theta(2mE - \hbar^2\boldsymbol{k}_\parallel^2)}{\sqrt{2mE - \hbar^2\boldsymbol{k}_\parallel^2} - i\hbar\kappa} \\ &\times \exp\left(-ik_\perp z_0 + i\sqrt{2mE - \hbar^2\boldsymbol{k}_\parallel^2}\frac{|z' - z_0|}{\hbar}\right)\Bigg]. \end{aligned} \tag{1.24}$$

The quantum well located at $z = z_0$ breaks the translational invariance of the Green's function matrix element. Therefore we wish to Fourier-transform Eq. (1.24) with respect to $k_\perp$ in order to calculate the density of states for a given Green's function matrix element

$\langle z|G(\boldsymbol{k}_\parallel,E)|z'\rangle$. Doing this yields

$$\begin{aligned}\langle z|G(\boldsymbol{k}_\parallel,E)|z'\rangle &= \frac{\hbar\Theta(\hbar^2\boldsymbol{k}_\parallel^2-2mE)}{2\sqrt{\hbar^2\boldsymbol{k}_\parallel^2-2mE}}\left[\exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2-2mE}\frac{|z-z'|}{\hbar}\right)\right.\\ &+ \left.\frac{\hbar\kappa}{\sqrt{\hbar^2\boldsymbol{k}_\parallel^2-2mE}-\hbar\kappa-i\epsilon}\exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2-2mE}\frac{|z-z_0|+|z'-z_0|}{\hbar}\right)\right]\\ &+ i\frac{\hbar\Theta(2mE-\hbar^2\boldsymbol{k}_\parallel^2)}{2\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}}\left[\exp\left(i\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}\frac{|z-z'|}{\hbar}\right)\right.\\ &+ \left.\frac{i\hbar\kappa}{\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}-i\hbar\kappa}\exp\left(i\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}\frac{|z-z_0|+|z'-z_0|}{\hbar}\right)\right]. \quad (1.25)\end{aligned}$$

Using Refs. [30] and [35], the density of states can be calculated from the Green's function in the axially-symmetric mixed representation $\langle z|\mathcal{G}(E,\boldsymbol{k}_\parallel)|z\rangle$ using the equation,

$$\begin{aligned}\varrho(E,z) &= -\frac{g}{\pi}\Im\langle\boldsymbol{x}_\parallel,z|\mathcal{G}(E)|\boldsymbol{x}_\parallel,z\rangle\\ &= -\frac{g}{\pi(2\pi)^{d-1}}\Im\int d^{d-1}\boldsymbol{k}\langle z|\mathcal{G}(E,\boldsymbol{k}_\parallel)|z\rangle. \quad (1.26)\end{aligned}$$

Inserting the Green's function matrix element into this relation between the density of states and the imaginary part of the Green's function yields

$$\begin{aligned}\varrho(E,z) &= \frac{4m}{\pi\hbar^2}\Im\langle\boldsymbol{x}_\parallel,z|G(E)|\boldsymbol{x}_\parallel,z\rangle\\ &= \frac{m}{\pi^3\hbar^2}\Im\int d^2\boldsymbol{k}_\parallel\langle z|G(\boldsymbol{k}_\parallel,E)|z\rangle, \quad (1.27)\end{aligned}$$

where we have used $g=2$ for the number of helicity states for spin 1/2 particles. In order to analyze if there is any two-dimensional behaviour for the density of states, the density of states is calculated inside the interface at $z=z_0$. Substitution of Eq. (1.25) into Eq. (1.27) yields after evaulation of the integrals

$$\begin{aligned}\varrho(E,z_0) &= \Theta(2mE+\hbar^2\kappa^2)\kappa\frac{m}{\pi\hbar^2}\\ &+ \Theta(E)\frac{m}{\pi^2\hbar^3}\left[\sqrt{2mE}-\hbar\kappa\arctan\left(\frac{\sqrt{2mE}}{\hbar\kappa}\right)\right]\\ &= \kappa\varrho_{d=2}(E+(\hbar^2\kappa^2/2m))\\ &+ \varrho_{d=3}(E)\left[1-\frac{\hbar\kappa}{\sqrt{2mE}}\arctan\left(\frac{\sqrt{2mE}}{\hbar\kappa}\right)\right]. \quad (1.28)\end{aligned}$$

In the last line we have substituted the value of the d-dimensional density of states for both two and three dimensions. The constant κ represents the inverse penetration depth of particles into the quantum well. This system differs from the one cited in Ref. [30] in that the thickness of the quantum well does not appear in the Green's function or density of states. Instead, the Green's function is acceptable for any positive energy and any distance from the quantum well. The density of states inside the quantum well is clearly a superposition of the two-dimensional and three-dimensional density of states. The two-dimensional density of states term comes with a pre-factor which is the inverse penetration depth, giving the correct units for the density of states as states per energy and per volume. This two-dimensional density of states is given as a function of the binding energy the particle feels inside the quantum well, which is given by $\hbar^2\kappa^2/2m$. From the Heaviside step function in Eq. (1.28), it is clear that in order for the particle to escape the quantum well, it must have energy larger than the binding energy. The three-dimensional density of states is un-bounded for positive energy. In the limit $\kappa \to 0$, the density of states in the quantum well approaches the three-dimensional density of states for a free particle $\varrho_{d=3}(E)$ as it should.

1.3 Motivation for Research

The recent prediction [2] and verification [28] of the existence of massive Dirac hyperboloids in the bulk of the topological insulator $Pb_{1-x}Sn_xSe$ motivates an analytic model for calculating the density of states in low-dimensional substructures. Relativistic wave equations adhere to a dispersion relation given by $E^2 = p^2c^2 + m^2c^4$. In terms of the band structure of electrons, this dispersion relation can be realized through Dirac cones with a dispersion relation $E^2 = p^2c^2$. Dirac cones connect at three-dimensional Dirac points in Dirac semimetals [22]. These Dirac points are degenerate and protected by time-reversal and crystal mirror symmetries [22]. If these symmetries are broken, then degeneracies are lifted and a band gap opens. This results in the formation of massive Dirac hyperboloids bounded by the asymptotes $E = \pm pc$. The mass term of the massive Dirac hyperboloids can manifest itself in the relativistic dispersion relation by lifting this degeneracy near $p = 0$ which results in $E^2 = p^2c^2 + \Delta_g^2$ with $2\Delta_g$ representing the band gap created by destroying the symmetry protecting the Dirac

points. Expanding this dispersion relation for $cp \ll \Delta_g$ around $p = 0$ gives the relation between the band gap and the effective mass parameter of the massive Dirac hyperboloids. Using the identity $\sqrt{a^2 + x^2} = a + x^2/2a$ for $x \ll a$ yields for the energy E

$$E \approx \pm\left(\Delta_g + \frac{c^2p^2}{2\Delta_g}\right) \tag{1.29}$$

Therefore we confirm that $m = \Delta_g/c^2$. We wish to study Bosons in the presence of a two-dimensional interface that adhere to a dispersion relation $E^2 = p^2c^2 + \Delta_g^2$ in order to make the connection with Dirac semimetals. Even though the particles in the Dirac semimetals are fermions, beginning with Bosons is a necessary first step, and still applicable because Klein-Gordon fields are roots of Dirac fields. This model is only applicable in the energy and momentum range around these Dirac points, or on the surface of a topological insulator. In particular, we will use parameters for the band gap between massive Dirac hyperboloids for thin films of the topological insulator $Pb_{1-x}Sn_xTe$ in our results for the inter-dimensional density of states in the interface.

1.4 Layout of Thesis

The remainder of this thesis is composed as follows. Chapter 2 will outline the theoretical background needed to formulate the inter-dimensional Hamiltonian, as well as illustrate its relevance from current literature. Chapter 3 will solve for the inter-dimensional density of states. Chapter 4 will discuss the limiting cases of the inter-dimensional density of states in the interface in terms of energy ranges. I will also derive and analyze the non-relativistic limit of the inter-dimensional density of states in the interface as well as the inter-dimensional quasirelativistic Hamiltonian from which it was derived. Chapter 5 will summarize and discuss the findings of this thesis. Appendix A is a detailed derivation of the inter-dimensional Green's function used to calculate the inter-dimensional density of states.

Chapter 2

Theoretical Background

2.1 Density of states in non-relativistic systems with d-spatial dimensions

It is useful in the physics of quantum structures and solid state physics, to decompose the three-dimensional $\boldsymbol{k}$ space volume element $d^3\boldsymbol{k}$ into components that are parallel to the constant energy surface $E(\boldsymbol{k})$ and orthogonal to them. This can also be done in d dimensions, where the parallel components of the integration measure are written in terms of the hyper-area of a $(d-1)$-dimensional unit sphere. Given an isotropic dispersion relation $E(\boldsymbol{k}) = E(k)$, the measure $d^d\boldsymbol{k}$ yields

$$d^d\boldsymbol{k} = S_{d-1}k^{d-1}\frac{dE}{dE/dk} \tag{2.1}$$

where S_{d-1} is the $(d-1)$-dimensional hyper-area of a d-dimensional unit sphere, and is given by

$$S_{d-1} = \frac{2\sqrt{\pi}^d}{\Gamma(d/2)}.$$

Following the derivation found in Ref. [34], the density of states as a function of energy and number of helicity states g in d-spatial dimensions is

$$\varrho(E) = g\frac{V}{(2\pi)^d}\frac{2\sqrt{\pi}^d}{\Gamma(d/2)}\frac{k^{d-1}}{|dE/dk|} \tag{2.2}$$

Considering the free non-relativistic particle, the d-dimensional density of states can be solved using the following substitutions

$$E(k) = \frac{\hbar^2k^2}{2m}, \quad \frac{dE}{dk} = \frac{\hbar^2k}{m} \tag{2.3}$$

Substituting dE/dk in Eq. (2.2) yields for the d-dimensional density of states per volume and in the energy scale

$$\frac{\varrho_d(E)}{V} = g\Theta(E)\sqrt{\frac{m}{2\pi}}^d \frac{\sqrt{E}^{d-2}}{\Gamma(d/2)\hbar^d}. \tag{2.4}$$

In the remainder of this thesis we will simply denote the density of quantum states per volume in the energy scale as $\varrho(E)$.

2.2 Relation between the Green's function and the density of states in non-relativistic systems

Following the extensive work in Ref. [34], the d-dimensional Green's function and its relation to the density of states is introduced. In the case of non-relativistic particles, we are interested in the time-independent Hamiltonian for a free particle in d spatial dimensions.

$$H = \frac{\boldsymbol{p}^2}{2m} = \int d^d\boldsymbol{x}|\boldsymbol{x}\rangle \left(\frac{-\hbar^2}{2m}\boldsymbol{\Delta}\right) \langle\boldsymbol{x}|.$$

The energy-dependent Green's function $\mathcal{G}_d(E)$ satisfies the inversion equation for the Schrodinger operator

$$(E - H)\mathcal{G}_d(E) = 1.$$

In the $\boldsymbol{x}$ representation the condition reads

$$\left(E + \frac{\hbar^2}{2m}\boldsymbol{\Delta}\right) \langle\boldsymbol{x}|\mathcal{G}_d(E)|\boldsymbol{x'}\rangle = \delta(\boldsymbol{x} - \boldsymbol{x'})$$

The solution for the Green's function is

$$\mathcal{G}_d(E) = \frac{a}{E - H + i\epsilon} + \frac{1-a}{E - H - i\epsilon},$$

where the $a = 1$ solution corresponds to the *retarded Green's function*, and the $a = 0$ corresponds to the *advanced Green's function*. The particular case of interest to this thesis is the *retarded Green's function*. Re-scaling the Green's operator yields

$$G_d(E) = \frac{-\hbar^2}{2m}\mathcal{G}_d(E) = \frac{1}{\boldsymbol{k}^2 - 2mE/\hbar^2 - i\epsilon}.$$

Therefore the re-scaled retarded Green's function satisfies

$$\left(\Delta + \frac{2mE}{\hbar^2}\right)\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x'}\rangle = -\delta(\boldsymbol{x}-\boldsymbol{x'}).$$

The energy-dependent retarded Green's function in the $\boldsymbol{x}$ representation contains dimensional aspects of interest. This can be found by using the Fourier transform of this matrix element in d-spatial dimensions. Insertion of two identity operators in the form of d-dimensional momentum eigenstates yields

$$\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x'}\rangle = \int d^d\boldsymbol{k}\int d^d\boldsymbol{k'}\langle \boldsymbol{x}|\boldsymbol{k}\rangle\langle \boldsymbol{k}|G_d(E)|\boldsymbol{k'}\rangle\langle \boldsymbol{k'}|\boldsymbol{x'}\rangle.$$

The inner product of these bra-ket states evaluates to give

$$\langle \boldsymbol{x}|\boldsymbol{k}\rangle = \frac{1}{\sqrt{2\pi}^d}\exp(i\boldsymbol{k}\cdot\boldsymbol{x}).$$

Therefore the Fourier transform is

$$\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x'}\rangle = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}\int d^d\boldsymbol{k'}\langle \boldsymbol{k}|G_d(E)|\boldsymbol{k'}\rangle\exp[i\boldsymbol{k}\cdot\boldsymbol{x} - i\boldsymbol{k'}\cdot\boldsymbol{x'}].$$

The Green's function matrix element in the $\boldsymbol{k}$ representation can be evaluated from the expression for $G_d(E)$ in which the $\boldsymbol{k}$ term is an operator. Therefore the matrix element for different initial and final $\boldsymbol{k}$ states is

$$\langle \boldsymbol{k}|G_d(E)|\boldsymbol{k'}\rangle = \langle \boldsymbol{k}|\frac{1}{\boldsymbol{k}^2 - 2mE/\hbar^2 - i\epsilon}|\boldsymbol{k'}\rangle = \frac{\delta(\boldsymbol{k}-\boldsymbol{k'})}{\boldsymbol{k}^2 - 2mE/\hbar^2 - i\epsilon} \equiv G_d(\boldsymbol{k},E)\delta(\boldsymbol{k}-\boldsymbol{k'}).$$

Inserting this into the Fourier transform for $\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x'}\rangle$ and using $\boldsymbol{k}=\boldsymbol{k'}$ yields

$$\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x'}\rangle = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}G_d(\boldsymbol{k},E)\exp[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})] \equiv G_d(\boldsymbol{x}-\boldsymbol{x'},E).$$

In order to evaluate the d-dimensional Fourier transform, the integral will need to be evaluated. Extending $|\boldsymbol{k}| = k$ to the complex plane, the denominator of $G_d(\boldsymbol{k},E)$ can be re-written in terms of it's four poles, two for each of positive and negative energy E. This decomposition yields

$$\begin{aligned} G_d(\boldsymbol{k},E) &= \frac{\Theta(E)}{(k-\sqrt{2mE/\hbar^2}-i\epsilon)(k+\sqrt{2mE/\hbar^2}+i\epsilon)} \\ &+ \frac{\Theta(-E)}{(k-i\sqrt{-2mE/\hbar^2})(k+i\sqrt{-2mE/\hbar^2})}, \end{aligned} \tag{2.5}$$

where $\Theta(\pm E)$ is the Heaviside function for positive and negative energy respectively, the properties of which are defined in Ref. [36]. Inserting this decomposition into the Fourier transform of $G_d(\boldsymbol{x}, E)$ yields

$$G_d(\boldsymbol{x}, E) = \frac{1}{(2\pi)^d} \int d^d\boldsymbol{k} \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{x})}{\boldsymbol{k}^2 - (2mE/\hbar^2) - i\epsilon}, \tag{2.6}$$

where it is understood that the decomposition still exists for positive and negative energy values. We can evaluate the integral using d-dimensional polar coordinates and the d-dimensional integration measure decomposition in polar coordinates [34],

$$d^d\boldsymbol{x} = dr d\theta_1 ... d\theta_{d-1} r^{d-1} \sin^{d-2}\theta_1 \cdot \sin^{d-3}\theta_2 \cdot ... \cdot \sin\theta_{d-2}.$$

We can further simplify this by using the expression for the hyper-surface area of the $(d-1)$-dimensional unit sphere [34] given by

$$S_{d-1} = 2\pi \prod_{n=1}^{d-2} \int_0^\pi d\theta \sin^n\theta = \frac{2\sqrt{\pi}^d}{\Gamma(d/2)}.$$

We can use this hyper-area expression in place of the angular measures found in the decomposition of the d-dimensional integration measure in polar coordinates. Recognizing that the upper limit for powers of sine in the hyper-area expression is $d-2$ we can extract the $d\theta_1 \sin^{d-2}\theta_1$ component from $d^d\boldsymbol{x}$, and let the rest of the angular components be represented by S_{d-2}. The expression for S_{d-2} has an upper limit for powers of sine of $d-3$ which matches the remaining sine terms in $d^d\boldsymbol{x}$. Therefore the d-dimensional integration measure in $\boldsymbol{k}$-space can be written as

$$d^d\boldsymbol{k} = dk d\theta k^{d-1} \sin^{d-2}\theta S_{d-2}. \tag{2.7}$$

The expression for S_{d-2} can also be expressed without any angular dependence and can therefore be removed from the integral. Applying all of this to the expression for $G_d(\boldsymbol{x}, E)$ yields

$$G_d(\boldsymbol{x}, E) = \frac{S_{d-2}}{(2\pi)^d} \int_0^\infty dk \int_0^\pi d\theta \frac{k^{d-1}\exp(ikr\cos\theta)}{k^2 - (2mE/\hbar^2) - i\epsilon} \sin^{d-2}\theta.$$

The value of S_{d-2} is found by inserting $d-1$ into d in the second expression for S_{d-1}. In particular this yields

$$S_{d-2} = \frac{2\sqrt{\pi}^{d-1}}{\Gamma((d-1)/2)}.$$

We can perform the θ integral using the identity [37],

$$\int_0^\pi \exp(iz\cos x)\sin^{\nu-1}x dx = \sqrt{\pi}\left(\frac{2}{z}\right)^{(\nu-1)/2}\Gamma\left(\frac{\nu}{2}\right)J_{(\nu-1)/2}(z), \tag{2.8}$$

where $\Re(\nu) > 0; |argz| < \pi$. Applying this to the θ integral implies

$$x = \theta,$$

$$z = kr,$$

$$\nu - 1 = d - 2.$$

The constraint on ν means that $d > 1$. Thus, applying the integral identity to the θ integral yields

$$\begin{aligned} G_d(\boldsymbol{x}, E) &= \frac{2\sqrt{\pi}^{d-1}}{(2\pi)^d\Gamma((d-1)/2)}\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{r}\right)^{\frac{d-2}{2}} \\ &\quad\times\int_0^\infty dk\frac{k^{d-1}}{k^{(d-2)/2}}\frac{J_{(d-2)/2}(kr)}{k^2-(2mE/\hbar^2)-i\epsilon} \\ &= \frac{1}{(2\pi)^{d/2}\sqrt{r}^{d-2}}\int_0^\infty dk\frac{\sqrt{k}^d J_{(d-2)/2}(kr)}{k^2-(2mE/\hbar^2)-i\epsilon}. \end{aligned}$$

We can solve the $\boldsymbol{k}$ integral for both positive and negative energy. For $E > 0$ we use the integral identity [38],

$$\int_0^\infty \frac{x^{\nu+1}}{(x^2+z^2)^\rho}J_\nu(cx)dx = \frac{c^{\rho-1}z^{\nu-\rho+1}}{2^{\rho-1}\Gamma(\rho)}K_{\nu-\rho+1}(cz),$$

where $\Re(z) > 0; -1 < \Re(\nu) < 2\Re(\rho) - 1/2$. Applying this to the k integral for $E > 0$ implies

$$\nu = (d-2)/2,$$

$$x = k,$$

$$z = \sqrt{-2mE}/\hbar,$$

$$c = r,$$

$$\rho = 1.$$

The inequality between ν and ρ demands that $d > 0$. Applying the integral identity to the $\boldsymbol{k}$ integral for $E > 0$ yields

$$G_d(\boldsymbol{x}, E) = \frac{\Theta(E)}{(2\pi)^{d/2}} \left(\frac{\sqrt{-2mE}}{\hbar r} \right)^{\frac{d-2}{2}} K_{(d-2)/2} \left(\frac{r\sqrt{-2mE}}{\hbar} \right),$$

where $K_\alpha(x)$ is a modified Bessel function. For $E < 0$ we use the integral identity [38],

$$\int_0^\infty \frac{x^{\nu+1}}{x^2 - y^2} J_\nu(cx) = -\frac{\pi}{2} y^\nu Y_\nu(cy).$$

Applying this to the $\boldsymbol{k}$ integral for $E < 0$ implies

$$x = k,$$

$$\nu = (d-2)/2,$$

$$y = \sqrt{2mE}/\hbar,$$

$$c = r.$$

The inequality on ν demands that $d > 0$. Applying the integral identity to the $\boldsymbol{k}$ integral for $E < 0$ yields

$$G_d(\boldsymbol{x}, E) = -\frac{\pi\Theta(-E)}{(2\pi)^{d/2}} \left(\frac{\sqrt{2mE}}{\hbar r} \right)^{\frac{d-2}{2}} Y_{\frac{d-2}{2}} \left(\frac{r\sqrt{2mE}}{\hbar} \right).$$

Using $r = |\boldsymbol{x} - \boldsymbol{x'}| \to 0$ in the argument of the Bessel function of second kind yields the following limiting form for fixed ν and $z \to 0$. We have

$$Y_\nu(z) \approx -iH_\nu^{(1)}(z). \tag{2.9}$$

Applying this to our result for the energy-dependent Green's function yields

$$G_d(\boldsymbol{x}, E) = i\frac{\pi\Theta(-E)}{(2\pi)^{d/2}} \left(\frac{\sqrt{2mE}}{\hbar r} \right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}} \left(\frac{r\sqrt{2mE}}{\hbar} \right). \tag{2.10}$$

Thus we can combine our results for the energy dependent Green's function for both postive and negative energy which yields

$$\begin{aligned} G_d(\boldsymbol{x}, E) &= \frac{\Theta(-E)}{\sqrt{2\pi}^d} \left(\frac{\sqrt{-2mE}}{\hbar r} \right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}} \left(\sqrt{-2mE}\frac{r}{\hbar} \right) \\ &+ i\frac{\pi}{2}\frac{\Theta(E)}{\sqrt{2\pi}^d} \left(\frac{\sqrt{2mE}}{\hbar r} \right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}} \left(\sqrt{2mE}\frac{r}{\hbar} \right). \end{aligned} \tag{2.11}$$

Retreating back to the expression for the Green's operator, we can re-write it in terms of energy eigenvalues by having the Hamiltonian operator act on energy eigenstates:

$$H|E', \nu(E')\rangle = E'|E', \nu(E')\rangle.$$

The identity operator in terms of the energy eigenstates can be written as

$$I = \int dE' \sum d\nu(E')|E', \nu(E')\rangle\langle E', \nu(E')|,$$

where $\{\nu(E')\}$ are a set of degenerate indices specific to a given energy E'. Inserting this form of the identity operator into the Green's operator and using the Hamiltonian operator to extract the energy eigenvalue yields

$$\begin{aligned}\mathcal{G}_d(E)I &= \frac{1}{E - H + i\epsilon}\int dE' \sum d\nu(E')|E', \nu(E')\rangle\langle E', \nu(E')|. \\ &= \int dE' \sum d\nu(E')\frac{|E', \nu(E')\rangle\langle E', \nu(E')|}{E - E' + i\epsilon}\end{aligned}$$

Using the Sokhotsky-Plemelj relations,

$$\frac{1}{x \mp i\epsilon} = \mathcal{P}\frac{1}{x} \pm i\pi\delta(x)$$

allows for the connection between the re-scaled Green's function and the density of states. Applying the first of the Sokhotsky-Plemelj relations to the Green's function yields

$$\begin{aligned}\mathcal{G}_d(E) &= \mathcal{P}\int dE' \sum d\nu(E')\frac{|E', \nu(E')\rangle\langle E', \nu(E')|}{E - E'} \\ &- i\pi \int dE' \sum d\nu(E')\delta(E - E')|E', \nu(E')\rangle\langle E', \nu(E')|. \qquad (2.12)\end{aligned}$$

The density of states per volume for plane waves can be expressed as

$$\frac{dn}{V} = \frac{d^3\boldsymbol{k}}{(2\pi)^3} = d^3\boldsymbol{k}|\langle \boldsymbol{x}|\boldsymbol{k}\rangle|^2.$$

Analogously, the density of states per volume can be expressed in terms of quantum numbers α. The density of states per volume would then be

$$\frac{dn}{V}(\boldsymbol{x}) = d\alpha|\langle \boldsymbol{x}|\alpha\rangle|^2.$$

If the Hamiltonian H has discrete energy spectra E_n and continuous energy spectra for energies E where $E_{b1} \leq E \leq E_{b2}$, then the density of states can be written using $\alpha =$

$(E, \nu(E))$ as the quantum numbers. Here $\nu(E)$ represents a set of degenerate indices. This yields a definition of the density of states given in terms of the discrete energies E_n and the continuous energy range between E_{b1} and E_{b2}:

$$\begin{aligned} \varrho(\boldsymbol{x}, E) &= \sum_n \delta(E - E_n) \sum \int d\nu_n |\langle \boldsymbol{x} | E_n, \nu_n \rangle|^2 \\ &+ \sum_b \Theta(E - E_{b1}) \Theta(E_{b2} - E) \sum \int d\nu(E) |\langle \boldsymbol{x} | E, \nu(E) \rangle|^2. \end{aligned} \tag{2.13}$$

This equation can be written in an even simpler form as

$$\varrho(\boldsymbol{x}, E) = \sum \int dE' d\nu(E') \delta(E - E') |\langle \boldsymbol{x} | E', \nu(E') \rangle|^2,$$

where it is understood that the energy E' is within the spectra, discrete or continuous. Comparing this definition of the density of states with the equation for the Green's operator written in terms of energy eigenstates and eigenvalues suggests that we should sandwich the Green's operator between $\boldsymbol{x}$ representation eigenstates. Therefore the Green's function matrix element is

$$\begin{aligned} \langle \boldsymbol{x} | \mathcal{G}_d(E) | \boldsymbol{x} \rangle &= \mathcal{P} \int dE' \sum d\nu(E') \frac{\langle \boldsymbol{x} | E', \nu(E') \rangle \langle E', \nu(E') | \boldsymbol{x} \rangle}{E - E'} \\ -i\pi \int dE' \sum d\nu(E') \delta(E - E') \langle \boldsymbol{x} | E', \nu(E') \rangle \langle E', \nu(E') | \boldsymbol{x} \rangle \\ &= \mathcal{P} \int dE' \sum d\nu(E') \frac{|\langle \boldsymbol{x} | E', \nu(E') \rangle|^2}{E - E'} \\ -i\pi \int dE' \sum d\nu(E') \delta(E - E') |\langle \boldsymbol{x} | E', \nu(E') \rangle|^2. \end{aligned}$$

Therefore, comparing the Green's operator matrix element and the rescaled Green's function matrix element to the density of states yields the equation,

$$\varrho(\boldsymbol{x}, E) = -\frac{1}{\pi} \Im \langle \boldsymbol{x} | \mathcal{G}_d(E) | \boldsymbol{x} \rangle = \frac{2m}{\pi \hbar^2} \Im \langle \boldsymbol{x} | G_d(E) | \boldsymbol{x} \rangle.$$

This equation for the density of states reproduces the result for the non-relativistic density of states for a free particle per volume and per energy when the imaginary part of the Green's function written in terms of Hankel and Bessel functions is used. In particular,

$$\langle \boldsymbol{x} | G_d(E) | \boldsymbol{x'} \rangle = G_d(\boldsymbol{x} - \boldsymbol{x'}, E).$$

Defining $r = |\boldsymbol{x} - \boldsymbol{x'}| \to 0$ allows simplification of the modified Hankel function with entirely real argument found in Eq. (2.11). From Ref. [36] we are able to simplify the modified Hankel function using the property

$$\Re H^{(1)}_{\frac{d-2}{2}}\left(\sqrt{2mE}\frac{r}{\hbar}\right)|_{r\to 0} = J_{\frac{d-2}{2}}\left(\sqrt{2mE}\frac{r}{\hbar}\right)|_{r\to 0} \sim \frac{1}{\Gamma(d/2)}\left(\sqrt{\frac{mE}{2}}\frac{r}{\hbar}\right)^{\frac{d-2}{2}}.$$

In the limit $r \to 0$, $\boldsymbol{x'} \to \boldsymbol{x}$. Therefore applying the result from the Hankel-function property indeed yields the correct expression for the matrix element $\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle = G_d(\boldsymbol{x}, E)$. Using this property and simplifying further does indeed reproduce the result obtained in Sec. 2.1 for the non-relativistic density of states per volume and per energy in d-spatial dimensions.

2.3 Inter-dimensional effects for non-relativistic electrons

In this section, we overview the non-relativistic system in which electrons move in the presence of a two-dimensional interface with a different electron mass. This system has been analyzed in Ref. [1], so the purpose of this section is to highlight the important results that pertain explicitly to this thesis. As stated in Ref. [1], upon suppressing the motion of particles through lower dimensional structures, the system is modelled using low-dimensional quantum mechanics. One quantity of great interest that depends explicitly on d is the density of states. As was shown in Sec. 2.1, the density of states in d-spatial dimensions per volume and per energy in the standard parabolic band approximation for two helicity states is given by

$$\varrho_d(E) = 2\Theta(E)\sqrt{\frac{m}{2\pi}}^d \frac{\sqrt{E}^{d-2}}{\Gamma(d/2)\hbar^d}. \tag{2.14}$$

Using the relation between the Green's function and the density of states yields the formula

$$\varrho(E_n, \boldsymbol{x}) = \frac{4m}{\pi\hbar^2}\Im\langle \boldsymbol{x}|G(E)|\boldsymbol{x}\rangle. \tag{2.15}$$

This system is modelled as having a thin two-dimensional interface, inside a surrounding bulk. Of particular interest is that inside the interface, particles move with an effective mass m_*, while their mass in the bulk is simply m. The interface is located at $z = z_0$ so

that motion in the xy plane is parallel to the interface, while motion in the z direction is perpendicular to the face of the interface. The interface is assumed to have a thickness L. The relation between the components of the wave vector $\boldsymbol{k}$ and the de Broglie wavelength λ for periodic boundary conditions is given by

$$k_i = \frac{2\pi}{\lambda}\cos\theta_i.$$

The Hamiltonian for the inter-dimensional system can be modelled as a superposition of two-dimensional and three-dimensional kinetic energy terms provided that the wavenumber component orthogonal to the interface is small compared to the inverse width, $|k_\perp L| << 1$ [1]. Multiplying the wavevector component equation by the interface thickness L and isolating for the de Broglie wavelength yields the condition

$$\lambda = \frac{2\pi L}{k_\perp L}\cos\theta_\perp >> 2\pi L|\cos\theta_\perp|. \tag{2.16}$$

The component of the de Broglie wavelength orthogonal to the interface is large compared to the thickness of the interface L. Reference [32] argues that this means the component of the wavefunction orthogonal to the interface is essentially constant in the z direction. Therefore the z integral for the kinetic energy density in the interface only contributes a factor of L to the value of the kinetic energy for motion along the interface. Under this condition, the kinetic energy of the particles is expressed through the inter-dimensional second-quantized Hamiltonian given by [1]

$$H = \int d^2\boldsymbol{x}_\parallel \int dz \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{x}_\parallel, z)\cdot\boldsymbol{\nabla}\psi(\boldsymbol{x}_\parallel, z) + \int d^2\boldsymbol{x}_\parallel \frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_\parallel\psi^\dagger(\boldsymbol{x}_\parallel, z_0)\cdot\boldsymbol{\nabla}_\parallel\psi(\boldsymbol{x}_\parallel, z_0), \tag{2.17}$$

where $\mu = m_*/L$ and $\boldsymbol{\nabla} = (\boldsymbol{\nabla}_\parallel, \partial_z)$. The Green's functions for interfaces are commonly parameterized using an axially-symmetric mixed representation [1]. The energy-dependent Green's function matrix element in the axially-symmetric mixed representation is translationally-invariant in the directions parallel to the interface, while translational invariance is broken in the direction perpendicular to the interface. Therefore, in the axially-symmetric mixed representation the energy-dependent Green's function matrix element is

$$\langle \boldsymbol{k}_\parallel, z|G(E)|\boldsymbol{k'}_\parallel, z'\rangle = \langle z|G(E, \boldsymbol{k}_\parallel)|z'\rangle\delta(\boldsymbol{k}_\parallel - \boldsymbol{k'}_\parallel). \tag{2.18}$$

To allow for comparison later, it is useful to give the solution for the energy-dependent Green's function in the axially-symmetric mixed representation. In this situation, translational invariance is no longer broken in the direction perpendicular to the interface, because the interface does not exist. The Green's function matrix element can be written as

$$\langle \boldsymbol{k}_\parallel, z|G_0(E)|\boldsymbol{k'}_\parallel, z'\rangle = G_0(E;\boldsymbol{k}_\parallel, z-z')\delta(\boldsymbol{k}_\parallel - \boldsymbol{k'}_\parallel).$$

The energy-dependent Green's function satisfies [1]

$$\left(\partial_z^2 - \boldsymbol{k}_\parallel^2 + \frac{2mE}{\hbar^2}\right) G_0(E;\boldsymbol{k}_\parallel, z) = -\delta(z). \tag{2.19}$$

This equation yields a solution for the Green's function as given by

$$\begin{aligned} G_0(E;\boldsymbol{k}_\parallel, z) &= \frac{1}{2\pi}\int dk_\perp \frac{\exp(ik_\perp z)}{k_\perp^2 + \boldsymbol{k}_\parallel^2 - (2mE/\hbar^2) - i\epsilon} \\ &= \frac{\hbar\Theta(\hbar^2\boldsymbol{k}_\parallel^2 - 2mE)}{2\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE}} \exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE}\frac{|z|}{\hbar}\right) \\ &+ \frac{i\hbar\Theta(2mE - \hbar^2\boldsymbol{k}_\parallel^2)}{2\sqrt{2mE - \hbar^2\boldsymbol{k}_\parallel^2}} \exp\left(i\sqrt{2mE - \hbar^2\boldsymbol{k}_\parallel^2}\frac{|z|}{\hbar}\right). \end{aligned} \tag{2.20}$$

In the presence of an interface, the inter-dimensional Hamiltonian yields a Schrodinger-like equation [1]

$$E\psi(\boldsymbol{x}_\parallel, z) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{x}_\parallel, z) - \frac{\hbar^2}{2\mu}\delta(z-z_0)\boldsymbol{\nabla}_\parallel^2\psi(\boldsymbol{x}_\parallel, z)$$

Therefore the energy-dependent Green's function matrix element satisties the equation,

$$\left(\frac{2m}{\hbar^2}E + \Delta + \delta(z-z_0)\frac{m}{\mu}\boldsymbol{\nabla}_\parallel^2\right)\langle \boldsymbol{x}_\parallel, z|G(E)|\boldsymbol{x'}_\parallel, z'\rangle = -\delta(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel)\delta(z-z'). \tag{2.21}$$

The solution for the axially-symmetric mixed representation matrix element as found in Ref. [1] is given by

$$\begin{aligned} \langle z|G(E,\boldsymbol{k}_\parallel)|z'\rangle = \frac{\hbar\Theta(\hbar^2\boldsymbol{k}_\parallel^2 - 2mE)}{2\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE}}\Bigg[&\exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE}\frac{|z-z'|}{\hbar}\right) \\ &- \frac{\hbar\boldsymbol{k}_\parallel^2\ell}{\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE} + \hbar\boldsymbol{k}_\parallel^2\ell} \exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mE}\frac{|z-z_0| + |z'-z_0|}{\hbar}\right)\Bigg] \end{aligned}$$

$$+i\frac{\hbar\Theta(2mE-\hbar^2\boldsymbol{k}_\parallel^2)}{2\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}}\left[\exp\left(i\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}\frac{|z-z'|}{\hbar}\right)\right.$$

$$\left.-i\frac{\hbar\boldsymbol{k}_\parallel^2\ell}{\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}+i\hbar\boldsymbol{k}_\parallel^2\ell}\exp\left(i\sqrt{2mE-\hbar^2\boldsymbol{k}_\parallel^2}\frac{|z-z_0|+|z'-z_0|}{\hbar}\right)\right]. \tag{2.22}$$

The definition of ℓ is $\ell \equiv m/2\mu = Lm/2m_*$. Given the expression for the Green's function for a free particle $G_0(E;\boldsymbol{k}_\parallel,z)$, we notice that if we apply $\ell = 0$ to the inter-dimensional Green's function, we again arrive at the expression for the free Green's function. This is expected. Therefore, using the relation between the density of states and the Green's function matrix element, along with the solution for the inter-dimensional Green's function in the axially-symmetric mixed representation, yields the expression for the density of states as

$$\begin{aligned}
\varrho(E,z_0) &= \frac{m\Theta(E)\Theta(\hbar^2-8mE\ell^2)}{2\pi^2\hbar^2\ell\sqrt{\hbar^2-8mE\ell^2}} \\
&\times \left[2\hbar\arctan\left(\frac{\ell\sqrt{8mE}}{\hbar+\sqrt{\hbar^2-8mE\ell^2}}\right)-\frac{\pi}{2}\left(\hbar-\sqrt{\hbar^2-8mE\ell^2}\right)\right] \\
&+ \frac{m\Theta(8mE\ell^2-\hbar^2)}{2\pi^2\hbar^2\ell} \\
&\times \left[\frac{\hbar}{\sqrt{8mE\ell^2-\hbar^2}}ln\left(\frac{\ell\sqrt{8mE}-\sqrt{8mE\ell^2-\hbar^2}}{\hbar}\right)+\frac{\pi}{2}\right].
\end{aligned} \tag{2.23}$$

The density of states reduces to the two-dimensional density of states for large energies [1], if the energy states probe length scales smaller than ℓ. The three-dimensional limit is found in the small energy limit, when the energy states probe length scales larger than ℓ. Mathematically this can be stated as, given $8mE\ell^2 >> \hbar^2$,

$$\varrho(E,z_0) \to \Theta(E)\frac{m}{4\pi\hbar^2\ell} = \frac{1}{4\ell}\varrho_{(d=2)}(E). \tag{2.24}$$

Given the case $8mE\ell^2 << \hbar^2$,

$$\varrho(E,z_0) \to \Theta(E)\frac{\sqrt{2m^3}}{\pi^2\hbar^3}\sqrt{E} = \varrho_{(d=3)}(E). \tag{2.25}$$

2.4 Relativistic scalar Green's function in d-spatial dimensions

The derivation in Sec. 2.2 can be generalized to the case of relativistic scalar particles. The relativistic scalar Green's function for a free particle in the time domain for the Klein-Gordon Hamiltonian satisfies

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right) G_d(\boldsymbol{x}, t; \boldsymbol{x'}, t') = -\delta(\boldsymbol{x} - \boldsymbol{x'})\delta(t - t'). \tag{2.26}$$

The Green's function can be expressed in terms of plane wave states with $x = (t, \boldsymbol{x})$:

$$G_d(\boldsymbol{x}, t; \boldsymbol{x'}, t') \equiv \langle x|G_d|x'\rangle.$$

The Fourier transform of $G_d(\boldsymbol{x}, t; \boldsymbol{x'}, t')$ can be found by inserting the identity operator in the k representation where $k = (\omega, \boldsymbol{k})$. This yields

$$G_d(\boldsymbol{x}, t; \boldsymbol{x'}, t') = \int d\omega \int d\omega' \int d^d\boldsymbol{k} \int d^d\boldsymbol{k'} \langle t, \boldsymbol{x}|\omega, \boldsymbol{k}\rangle\langle\omega, \boldsymbol{k}|G_d|\omega', \boldsymbol{k'}\rangle\langle\omega', \boldsymbol{k'}|t', \boldsymbol{x'}\rangle. \tag{2.27}$$

The inner products are given by

$$\langle t, \boldsymbol{x}|\omega, \boldsymbol{k}\rangle = \frac{1}{\sqrt{2\pi}^{d+1}} \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)].$$

Expressing the inner products in this form, using the integral representations of the delta functions on the RHS, and evaluating the differential operators from Eq. (3.1) yields

$$\left(-\boldsymbol{k}^2 + \frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2}\right) \frac{1}{(2\pi)^{d+1}} \int d\omega \int d\omega' \int d^d\boldsymbol{k} \int d^d\boldsymbol{k'} \langle\omega, \boldsymbol{k}|G_d|\omega', \boldsymbol{k'}\rangle$$

$$\times \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k'}\cdot\boldsymbol{x'} - \omega t + \omega' t')] = \frac{-1}{(2\pi)^{d+1}} \int d\omega \int d^d\boldsymbol{k}\, \exp[i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x'})] \exp[-i\omega(t - t')] \tag{2.28}$$

Multiplying both sides of the equation by

$$\frac{1}{(2\pi)^{d+1}} \int d^d\boldsymbol{x} \int d^d\boldsymbol{x'} \int dt \int dt' \exp[i(\boldsymbol{\kappa'}\cdot\boldsymbol{x'} - \boldsymbol{\kappa}\cdot\boldsymbol{x} + \epsilon t - \epsilon' t')]$$

and using the integral representation of the delta functions yields

$$\left(-\boldsymbol{k}^2 + \frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2}\right) \int d\omega \int d\omega' \int d^d\boldsymbol{k} \int d^d\boldsymbol{k'} \langle\omega, \boldsymbol{k}|G_d|\omega', \boldsymbol{k'}\rangle$$

$$\times\delta(\boldsymbol{k}-\boldsymbol{\kappa})\delta(\boldsymbol{\kappa'}-\boldsymbol{k'})\delta(\epsilon-\omega)\delta(\omega'-\epsilon')$$

$$=-\int d^d\boldsymbol{k}\int d\omega\delta(\boldsymbol{k}-\boldsymbol{\kappa})\delta(\boldsymbol{\kappa'}-\boldsymbol{k})\delta(\epsilon-\omega)\delta(\omega-\epsilon'). \tag{2.29}$$

Evaluating the trivial integrals with the help of the delta functions and making the substitutions $\epsilon\to\omega$, $\epsilon'\to\omega'$, $\boldsymbol{\kappa}\to\boldsymbol{k}$, $\boldsymbol{\kappa'}\to\boldsymbol{k'}$ after evaluation yields

$$\left(\boldsymbol{k}^2-\frac{\omega^2}{c^2}+\frac{m^2c^2}{\hbar^2}\right)\langle\omega,\boldsymbol{k}|G_d|\omega',\boldsymbol{k'}\rangle=\delta(\boldsymbol{k}-\boldsymbol{k'})\delta(\omega-\omega'). \tag{2.30}$$

Therefore, using the definition $G_d(\boldsymbol{k},\omega;\boldsymbol{k'},\omega')\equiv\langle\omega,\boldsymbol{k}|G_d|\omega',\boldsymbol{k'}\rangle$ yields

$$G_d(\boldsymbol{k},\omega;\boldsymbol{k'},\omega')=G_d(\boldsymbol{k},\omega)\delta(\boldsymbol{k}-\boldsymbol{k'})\delta(\omega-\omega'), \tag{2.31}$$

where

$$G_d(\boldsymbol{k},\omega)=\frac{1}{\left(\boldsymbol{k}^2-\frac{\omega^2}{c^2}+\frac{m^2c^2}{\hbar^2}-i\epsilon\right)}.$$

Promoting $k=(\omega/c,\boldsymbol{k})$ into the complex plane requires that a small imaginary component be added to the denominator of $G_d(\boldsymbol{k},\omega)$. This choice $-i\epsilon$, $\epsilon>0$, is such that in the non-relativistic limit

$$\omega\to\frac{mc^2+E}{\hbar}.$$

$G_d(\boldsymbol{k},\omega)$ reproduces the value in Sec. 2.2 for $G_d(\boldsymbol{k},E)$, provided that terms of order $\mathcal{O}(E^2)$ are neglected. It is important to remember that in the substitution to the non-relativistic limit, E is the total non-relativistic energy. In the relativistic case, this total non-relativistic energy E is the relativistic kinetic energy. For completeness, inserting the non-relativistic limit into $G_d(\boldsymbol{k},\omega)$ yields

$$G_d(\boldsymbol{k},\omega)\big|_{\omega\to\frac{mc^2+E}{\hbar}}=\frac{1}{\boldsymbol{k}^2-\frac{m^2c^4+2Emc^2+\mathcal{O}(E^2)}{c^2\hbar^2}+\frac{m^2c^2}{\hbar^2}-i\epsilon}$$

$$\approx\frac{1}{\boldsymbol{k}^2-\frac{2mE}{\hbar^2}-i\epsilon}=G_d(\boldsymbol{k},E), \tag{2.32}$$

which indeed agrees with our expression from Sec. 2.2. As was stated in Sec. 2.2, the resulting $G_d(\boldsymbol{k},E)$ is the retarded non-relativistic Green's function. In the relativistic case however, the relativistic Green's function $G_d(\boldsymbol{k},\omega)$ contains both retarded and advanced contributions in the time domain, namely for $G_d(\boldsymbol{k},t)$. The convention for the poles defines the relativistic

Green's function of Stuckelberg and Feynman [34]. The form of the relativistic Green's function $G_d(\boldsymbol{k},\omega)$ implies that the relativistic scalar Green's operator has the form

$$G_d = \frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - i\epsilon}, \tag{2.33}$$

where using the convention $\eta_{00} = -1$, $k = (k^0, \boldsymbol{k}) = (\omega/c, \boldsymbol{k})$, hence $k^2 = \boldsymbol{k}^2 - (k^0)^2$. Using plane wave states, the relativistic scalar Green's function in momentum space is

$$\langle k|G_d|k'\rangle = \frac{\delta(k-k')}{k^2 + \frac{m^2c^2}{\hbar^2} - i\epsilon} = G_d(\boldsymbol{k},\omega)\delta(k-k'), \tag{2.34}$$

where $G_d(\boldsymbol{k},\omega)$ in Eq. (2.34) has the same form as in Eq. (2.31). As in Sec. 2.2, we wish to solve for the Fourier transform of $G_d(\boldsymbol{k},\omega)$. Following the same strategy as in Sec. 2.2 using plane wave states, the function $G_d(\boldsymbol{x},t;\boldsymbol{x'},t')$ has the Fourier transform

$$\begin{aligned} G_d(\boldsymbol{x},t;\boldsymbol{x'},t') &= \langle \boldsymbol{x},t|G_d|\boldsymbol{x'},t'\rangle \\ &= \int d^d\boldsymbol{k}\int d^d\boldsymbol{k'}\int d\omega\int d\omega'\langle \boldsymbol{x},t|\boldsymbol{k},\omega\rangle\langle \boldsymbol{k},\omega|G_d|\boldsymbol{k'},\omega'\rangle\langle \boldsymbol{k'},\omega'|\boldsymbol{x'},t'\rangle \\ &= \frac{1}{(2\pi)^{d+1}}\int d^d\boldsymbol{k}\int d^d\boldsymbol{k'}\int d\omega\int d\omega' G_d(\boldsymbol{k},\omega)\delta(k-k') \\ &\times \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k'}\cdot\boldsymbol{x'} - \omega t + \omega' t')]. \end{aligned} \tag{2.35}$$

Using $\delta(k-k') = \delta(\boldsymbol{k}-\boldsymbol{k'})\delta(\omega-\omega')$ yields

$$\begin{aligned} G_d(\boldsymbol{x},t;\boldsymbol{x'},t') &= \frac{1}{(2\pi)^{d+1}}\int d^d\boldsymbol{k}\int d\omega G_d(\boldsymbol{k},\omega)\exp[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})]\exp[-i\omega(t-t')] \\ &\equiv G_d(\boldsymbol{x}-\boldsymbol{x'},t-t'). \end{aligned} \tag{2.36}$$

This implies that $G_d(\boldsymbol{x},t)$ has the Fourier transform,

$$G_d(\boldsymbol{x},t) = \frac{1}{(2\pi)^{d+1}}\int d^d\boldsymbol{k}\int d\omega G_d(\boldsymbol{k},\omega) exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]. \tag{2.37}$$

It will be shown in Sec. 3.1 that analogously to Sec. 2.2, in order to show the relation between the density of states and the relativistic scalar Green's function, we need to find the Green's function $G_d(\boldsymbol{x},\omega)$. This is given by

$$G_d(\boldsymbol{x},\omega) = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k} G_d(\boldsymbol{k},\omega) exp(i\boldsymbol{k}\cdot\boldsymbol{x}). \tag{2.38}$$

We can decompose the Green's function $G_d(\boldsymbol{k},\omega)$ into terms containing each of its four poles, two poles for each of positive and negative values of $\hbar|\omega| - mc^2$. Performing this decomposition yields

$$\begin{aligned} G_d(\boldsymbol{k},\omega) &= \frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - i\epsilon} \\ &= \frac{1}{\boldsymbol{k}^2 - \frac{\omega^2}{c^2} + \frac{m^2c^2}{\hbar^2} - i\epsilon} \\ &= \frac{\Theta(\hbar|\omega| - mc^2)}{\left(\boldsymbol{k} - \sqrt{\frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2}} - i\epsilon\right)\left(\boldsymbol{k} + \sqrt{\frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2}} + i\epsilon\right)} \\ &+ \frac{\Theta(mc^2 - \hbar|\omega|)}{\left(\boldsymbol{k} - i\sqrt{\frac{m^2c^2}{\hbar^2} - \frac{\omega^2}{c^2}}\right)\left(\boldsymbol{k} + i\sqrt{\frac{m^2c^2}{\hbar^2} - \frac{\omega^2}{c^2}}\right)}, \end{aligned} \tag{2.39}$$

where we have used $k^2 = \boldsymbol{k}^2 - (k^0)^2 = \boldsymbol{k}^2 - (\omega^2/c^2)$. Inserting the non-decomposed form of $G_d(\boldsymbol{k},\omega)$ into the Fourier transform of $G_d(\boldsymbol{x},\omega)$ yields

$$G_d(\boldsymbol{x},\omega) = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}\frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{x})}{\boldsymbol{k}^2 - \frac{\omega^2}{c^2} + \frac{m^2c^2}{\hbar^2} - i\epsilon}.$$

Following the same procedure from Sec. 2.2, we can rewrite the integral in terms of d-dimensional polar coordinates by rewrting the d-dimensional integration measure $d^d\boldsymbol{k}$ as

$$\begin{aligned} d^d\boldsymbol{x} &= dr d\theta_1 ... d\theta_{d-1} r^{d-1}\sin^{d-2}\theta_1 \cdot \sin^{d-3}\theta_2 \cdot ... \cdot \sin\theta_{d-2} \\ &= d^d\boldsymbol{k} = dk d\theta k^{d-1}\sin^{d-2}\theta S_{d-2}. \end{aligned}$$

Inserting this into the $\boldsymbol{k}$ integral yields

$$G_d(\boldsymbol{x},\omega) = \frac{S_{d-2}}{(2\pi)^d}\int_0^\infty dk\int_0^\pi d\theta\frac{k^{d-1}\exp(ikr\cos\theta)}{k^2 - \frac{\omega^2}{c^2} + \frac{m^2c^2}{\hbar^2} - i\epsilon}\sin^{d-2}\theta.$$

The θ integral is exactly the same as the one solved using Ref. [37] from Sec. 2.2. Therefore after solving the θ integral we have

$$G_d(\boldsymbol{x},\omega) = \frac{1}{(2\pi)^{d/2}\sqrt{r}^{d-2}}\int_0^\infty \frac{k^{d/2}}{k^2 - \frac{\omega^2}{c^2} + \frac{m^2c^2}{\hbar^2} - i\epsilon}J_{\frac{d-2}{2}}(kr),$$

where $d > 1$. The k integral can be solved for $\hbar|\omega| - mc^2 > 0$ using the same integral from Ref. [38] that was used for $E > 0$ from Sec. 2.2. The integral identity is

$$\int_0^\infty \frac{x^{\nu+1}}{x^2 - y^2}J_\nu(cx) = -\frac{\pi}{2}y^\nu Y_\nu(cy).$$

Applying this to the k integral implies that

$$x = k,$$

$$\nu = (d-2)/2,$$

$$y = \sqrt{\omega^2/c^2 - (m^2c^2)/\hbar^2},$$

$$c = r.$$

Thus, after evaluating the integral we have

$$G_d(\boldsymbol{x}, E) = -\frac{\pi}{2}\frac{\Theta(\hbar|\omega| - mc^2)}{(2\pi)^{d/2}}\left(\frac{\sqrt{\hbar^2\omega^2 - m^2c^4}}{\hbar cr}\right)^{\frac{d-2}{2}} Y_{\frac{d-2}{2}}\left(\sqrt{\hbar^2\omega^2 - m^2c^4}\frac{r}{\hbar c}\right). \quad (2.40)$$

Using $r = |\boldsymbol{x} - \boldsymbol{x'}| \to 0$ in the argument of the Bessel function of second kind yields the following limiting form for fixed ν and $z \to 0$ [36]:

$$Y_\nu(z) \approx -iH_\nu^{(1)}(z). \quad (2.41)$$

Applying this to our result for the energy-dependent relativistic scalar Green's function yields

$$G_d(\boldsymbol{x}, E) = i\frac{\pi}{2}\frac{\Theta(\hbar|\omega| - mc^2)}{(2\pi)^{d/2}}\left(\frac{\sqrt{\hbar^2\omega^2 - m^2c^4}}{\hbar cr}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}\left(\sqrt{\hbar^2\omega^2 - m^2c^4}\frac{r}{\hbar c}\right). \quad (2.42)$$

Similarly we can use the k integral solution from Ref. [38] for $E < 0$ for the case where $\hbar|\omega| - mc^2 < 0$. The integral identity is

$$\int_0^\infty \frac{x^{\nu+1}}{(x^2+z^2)^\rho} J_\nu(cx)dx = \frac{c^{\rho-1}z^{\nu-\rho+1}}{2^{\rho-1}\Gamma(\rho)} K_{\nu-\rho+1}(cz). \quad (2.43)$$

Applying this to the k integral implies that

$$x = k,$$

$$\nu = (d-2)/2,$$

$$z = \sqrt{(m^2c^2)/\hbar^2 - \omega^2/c^2},$$

$$\rho = 1,$$

$$c = r.$$

Therefore after evaluating the integral we have

$$G_d(\boldsymbol{x}, E) = \frac{\Theta(mc^2 - \hbar|\omega|)}{(2\pi)^{d/2}}\left(\frac{\sqrt{m^2c^4 - \hbar^2\omega^2}}{\hbar cr}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}\left(\sqrt{m^2c^4 - \hbar^2\omega^2}\frac{r}{\hbar c}\right). \quad (2.44)$$

2.5 Relativistic density of states in d-spatial dimensions

We can use the equation for the density of states per volume and per energy for d-spatial dimensions found in Sec. 2.1

$$\varrho(E) = g\frac{V}{(2\pi)^d}\frac{2\sqrt{\pi}^d}{\Gamma(d/2)}\frac{k^{d-1}}{|dE/dk|}. \tag{2.45}$$

In the relativistic case with free particles and anti-particles, the density of states is a sum of the particle density of states and anti-particle density of states. In order to calculate the density of states per volume and per energy in d-spatial dimensions for a relativistic scalar particle, we must use the relativistic dispersion relation,

$$E^2 = c^2p^2 + m^2c^4. \tag{2.46}$$

Using $p = \hbar k$ in the dispersion relation and isolating for E gives us an equation we can use to calculate the quantities k^{d-1} and dE/dk. Therefore using $E = \hbar c\sqrt{k^2 + (mc/\hbar)^2}$ yields

$$\frac{dE}{dk} = \frac{\hbar ck}{\sqrt{k^2 + (mc/\hbar)^2}}. \tag{2.47}$$

Applying this to the last term in the density of states equation yields

$$\frac{k^{d-1}}{|dE/dk|} = \frac{k^{d-2}\sqrt{k^2 + (mc/\hbar)^2}}{\hbar c}.$$

Isolating the dispersion equation for k yields

$$k = \frac{1}{\hbar c}\sqrt{E^2 - m^2c^4}.$$

Applying this to the terms k^{d-2} and k^2 inside the simplified last term from the density of states results in

$$\frac{k^{d-1}}{|dE/dk|} = \frac{|E|}{\hbar^2c^2}\left(\frac{1}{\hbar c}\right)^{d-2}\left(\sqrt{E^2 - m^2c^4}\right)^{d-2}. \tag{2.48}$$

Therefore the relativistic density of states per volume, per energy, and per helicity state g is given by

$$\hat{\varrho}(E) = \varrho(E) + \bar{\varrho}(\overline{E}) = \frac{2\Theta(E^2 - m^2c^4)}{(2\sqrt{\pi}\hbar c)^d\Gamma(d/2)}|E|\sqrt{E^2 - m^2c^4}^{d-2}, \tag{2.49}$$

where $\overline{E} = -E$.

2.6 Field theory background

In order to provide an introductory background into quantum field theory, we will follow some key points found in Ref. [34]. Relativistic wave equations adhere to the dispersion relation,

$$E^2 = c^2\boldsymbol{p}^2 + m^2c^4.$$

Klein-Gordon fields that adhere to this dispersion relation can modelled by the Lagrangian density,

$$\mathcal{L} = \hbar\dot{\phi}^\dagger \cdot \dot{\phi} - \hbar c^2 \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi - \frac{m^2c^4}{\hbar}\phi^\dagger \cdot \phi.$$

The fields are solutions to the equation of motion, namely the Klein-Gordon equation, for the Lagrangian density and are expressed by

$$\phi(\boldsymbol{x},t) = \frac{1}{\sqrt{2\pi}^3}\int \frac{d^3\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(a(\boldsymbol{k})\exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_{\boldsymbol{k}}t)] + b^\dagger(\boldsymbol{k})\exp[-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_{\boldsymbol{k}}t)]\right),$$

where the frequency is given by

$$\omega_{\boldsymbol{k}} = c\sqrt{\boldsymbol{k}^2 + \left(\frac{mc}{\hbar}\right)^2}.$$

The creation and annihilation operators obey the commutation relations,

$$\left[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})\right] = \left[b(\boldsymbol{k}), b^\dagger(\boldsymbol{k'})\right] = \delta\left(\boldsymbol{k} - \boldsymbol{k'}\right).$$

All other commutators are zero. From these relations come the equal-time commutation relations for the Klein-Gordon field,

$$\left[\phi(\boldsymbol{x},t), \dot{\phi}^\dagger(\boldsymbol{x'},t)\right] = \left[\phi^\dagger(\boldsymbol{x},t), \dot{\phi}(\boldsymbol{x'},t)\right] = i\delta(\boldsymbol{x} - \boldsymbol{x'}).$$

In order to derive the Hamiltonian density that corresponds to our Klein-Gordon Lagrangian, we must analyze how the action behaves under coordinate transformations. The action in d-dimensions containing N fields is given by

$$S = \int d^dx\mathcal{L}(\phi, \partial\phi).$$

Under the coordinate transformation

$$x'(x) = x - \epsilon(x)$$

the Klein-Gordon fields transform as

$$\phi'(x') = \phi(x) + \delta\phi(x).$$

If $\delta S \equiv 0$ for every integration volume under transformations $\epsilon, \delta\phi$, then there exists a local conservation law in terms of a conserved current:

$$\partial_\mu j^\mu = 0$$

where the conserved current is given by

$$j^\mu = \epsilon^\nu \left(\eta_\nu{}^\mu \mathcal{L} - \partial_\nu \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}.$$

For a constant coordinate transformation

$$\partial_\mu \epsilon^\nu = 0$$

the fields transform like scalars, $\delta\phi = 0$, and the conserved current is simplified to

$$j^\mu = \epsilon^\nu \left(\eta_\nu{}^\mu \mathcal{L} - \partial_\nu \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \epsilon^\nu \Theta_\nu{}^\mu,$$

where the conservation law is now written as

$$\partial_\mu \Theta_\nu{}^\mu = 0$$

with the conserved current given by

$$\Theta_\nu{}^\mu = \left(\eta_\nu{}^\mu \mathcal{L} - \partial_\nu \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right).$$

The Hamiltonian density that corresponds to the Klein-Gordon Lagrangian is found using the energy-momentum tensor. The Hamiltonian density is defined as

$$\mathcal{H} = -\Theta_0{}^0 = \hbar \dot{\phi}^\dagger \dot{\phi} + \hbar c^2 \boldsymbol{\nabla} \phi^\dagger \cdot \boldsymbol{\nabla} \phi + \frac{m^2 c^4}{\hbar} \phi^\dagger \phi.$$

Using the definition of the Hamiltonian operator, $H = \int d^3x \mathcal{H}$ we have

$$H = \frac{1}{\hbar} \int d^3x \left[\hbar^2 \dot{\phi}^\dagger \dot{\phi} + \hbar^2 c^2 \boldsymbol{\nabla} \phi^\dagger \cdot \boldsymbol{\nabla} \phi + m^2 c^4 \phi^\dagger \phi \right].$$

Chapter 3

Formulation of the Inter-dimensional Quasirelativitic System for Bosons

3.1 Relation between the Green's functions and the density of states in relativistic systems

In order to show the relation between the relativistic scalar Green's function matrix element $\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle$ and the relativistic density of states for scalar particles, we need to generalize the derivation in the non-relativistic case as discussed in Sec. 2.2 . From Sec. 2.4 we have the relativistic scalar Green's operator given by

$$G_d = \frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - i\epsilon}, \tag{3.1}$$

Where we are using the convention $\eta_{00} = -1$, hence $k^2 = \boldsymbol{k}^2 - (k^0)^2$. As was shown in Sec. 2.4, the momentum-space Green's function matrix element for the Green's operator Eq. (3.1) is given by

$$\langle k|G_d|k'\rangle = \langle k|\frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - i\epsilon}|k'\rangle = \frac{\delta(k-k')}{k^2 + \frac{m^2c^2}{\hbar^2} - i\epsilon}. \tag{3.2}$$

In order to make the connection between the Green's operator in Eq. (3.1) and the relativistic density of states, we need to find the energy-dependent Green's operator. This is accomplished by first writing the Green's operator in Eq. (3.1) in terms of energy $E = cp^0$, and the relativistic Hamiltonian for a free scalar particle $H = c\sqrt{\boldsymbol{p}^2 + m^2c^2}$, where it is understood that the energy is still an operator. Rewriting Eq. (3.1) in terms of the energy

and Hamiltonian operators yields

$$\begin{aligned} G_d &= \frac{1}{\frac{1}{\hbar^2 c^2}\left(c^2\boldsymbol{p}^2 - c^2(p^0)^2 + m^2c^4 - i\epsilon\right)} \\ &= \frac{-\hbar^2 c^2}{E^2 - H^2 + i\epsilon} \\ &= \frac{-\hbar^2 c^2}{2E}\left(\frac{1}{E - H + i\epsilon} + \frac{1}{E + H - i\epsilon}\right). \end{aligned} \tag{3.3}$$

The connection between the Green's operator in Eq. (3.3) and the relativistic scalar energy-dependent Green's function of interest is done through the matrix element in Eq. (3.2) . In particular, the energy-dependent Green's function, $G(E)$ with classical variable $E = c\hbar k^0$ is derived from Eq. (3.2) using $|k\rangle = |\boldsymbol{k}\rangle \otimes |k^0\rangle$. Then, we have

$$\langle k^0 | G_d | k'^0 \rangle = G_d(E)\delta(k^0 - k'^0). \tag{3.4}$$

The relativistic scalar energy-dependent Green's function is therefore given by

$$G_d(E) = \frac{-\hbar^2 c^2}{2E}\left(\frac{1}{E - H + i\epsilon} + \frac{1}{E + H - i\epsilon}\right)\Bigg|_{E = c\hbar k^0}. \tag{3.5}$$

From Ref. [34], the local density of states per volume and per energy of a particle with energy E is given by

$$\varrho(E, \boldsymbol{x}) = \sum \int dE' d\nu(E') \delta(E - E') |\langle \boldsymbol{x} | E'\nu(E') \rangle|^2. \tag{3.6}$$

Therefore, an anti-particle with energy $\overline{E} = -E$ has a density of states per volume and per energy given by

$$\bar{\varrho}(\overline{E}, \boldsymbol{x}) = \sum \int dE' d\nu(E') \delta(E + E') |\langle \boldsymbol{x} | E'\nu(E') \rangle|^2 \tag{3.7}$$

Therefore we are interested in the $\boldsymbol{x}$ representation matrix element of the energy-dependent Green's function, in order to relate the relativistic density of states with the energy-dependent Green's function. Using energy eigenstates $|E', \nu(E')\rangle$, where $\nu(E')$ is a set of degeneracy indices for each energy level E', the matrix element can be expressed after insertion of the

identity operator as

$$\begin{aligned}\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle &= \frac{-\hbar^2c^2}{2E}\langle \boldsymbol{x}|\left(\frac{1}{E-H+i\epsilon}+\frac{1}{E+H-i\epsilon}\right)\\ &\times \left(\sum\int dE'd\nu(E')|E',\nu(E')\rangle\langle E',\nu(E')|\right)|\boldsymbol{x}\rangle\Big|_{E=c\hbar k^0}\\ &= \frac{-\hbar^2c^2}{2E}\sum\int dE'd\nu(E')|\langle \boldsymbol{x}|E',\nu(E')\rangle|^2\\ &\times \left(\frac{1}{E-E'+i\epsilon}+\frac{1}{E+E'-i\epsilon}\right)\Big|_{E=c\hbar k^0}. \end{aligned} \tag{3.8}$$

Analogous to Sec. 2.2, we use the Sokhotsky-Plemelj relation, $\frac{1}{x-i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$, on the energy-dependent Green's function to separate its real and imaginary parts. Using the relation yields

$$G_d(E) = \frac{-\hbar^2c^2}{2E}\left[\mathcal{P}\frac{1}{E-H} - i\pi\delta(E-H) + \mathcal{P}\frac{1}{E+H} + i\pi\delta(E+H)\right]\Big|_{E=c\hbar k^0}.$$

As in Sec. 2.2, the final step to relate the relativistic density of states and the energy-dependent Green's function is to use the Sokhotsky-Plemelj relation in the first line of Eq. (3.8), and then keep only the imaginary part of this equation. We have

$$\Im\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle = \frac{\pi\hbar^2c^2}{2E}\sum\int\left[\delta(E-E')-\delta(E+E')\right]|\langle \boldsymbol{x}|E',\nu(E')\rangle|^2\Big|_{E=c\hbar k^0}. \tag{3.9}$$

Comparing Eq. (3.9) with Eqs. (3.6) and (3.7) therefore yields the relation between the relativistic density of states and the energy-dependent Green's function,

$$\Im\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle = \frac{\pi\hbar^2c^2}{2E}\left[\varrho(E)-\bar{\varrho}(\overline{E})\right]\Big|_{E=c\hbar k^0}. \tag{3.10}$$

From Sec. 2.4, we know that for a free relativistic scalar particle

$$\begin{aligned}G_d(\boldsymbol{x},\omega) &= \frac{\Theta(mc^2-\hbar|\omega|)}{\sqrt{2\pi}^d}\left(\frac{\sqrt{m^2c^4-\hbar^2\omega^2}}{\hbar cr}\right)^{\frac{d-2}{2}}K_{\frac{d-2}{2}}\left(\sqrt{m^2c^4-\hbar^2\omega^2}\frac{r}{\hbar c}\right)\\ &+ i\frac{\pi}{2}\frac{\Theta(\hbar|\omega|-mc^2)}{\sqrt{2\pi}^d}\left(\frac{\sqrt{\hbar^2\omega^2-m^2c^4}}{\hbar cr}\right)^{\frac{d-2}{2}}H^{(1)}_{\frac{d-2}{2}}\left(\sqrt{\hbar^2\omega^2-m^2c^4}\frac{r}{\hbar c}\right). \end{aligned} \tag{3.11}$$

Here, the functions $K_\alpha(z)$ and $H^{(1)}_\alpha$ are the modified Bessel functions and Hankel function of the first kind [36]. The argument of both the modified Bessel and Hankel functions is

real in this case. The modified Bessel fuction remains entirely real given its real argument. Defining $r = |\boldsymbol{x} - \boldsymbol{x'}|$, the modified Hankel function with real argument has the property,

$$\Re H_{\alpha}^{(1)}(z)\Big|_{z\to 0} = \frac{(z/2)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.12}$$

Therefore in the limit $r \to 0$, $\boldsymbol{x'} \to \boldsymbol{x}$ and the property of the modified Hankel function applies for the matrix element $\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle = G_d(\boldsymbol{x},\omega)$. Equation (3.10) tells us that just as in the non-relativistic case, the connection between the density of states and the relativistic scalar Green's function comes from the imaginary part of the Green's function. Therefore, taking only the imaginary part of the result for $G_d(\boldsymbol{x},\omega)$ and using the property of the modified Hankel function yields

$$\begin{aligned}
\Im\langle \boldsymbol{x}|G_d(E)|\boldsymbol{x}\rangle &= \Im G_d(\boldsymbol{x},\omega) \\
&= \frac{\pi}{2}\frac{\Theta(\hbar|\omega| - mc^2)}{\sqrt{2\pi}^d}\left(\frac{\sqrt{\hbar^2\omega^2 - m^2c^4}}{\hbar c r}\right)^{\frac{d-2}{2}} \\
&\times \frac{1}{\Gamma(d/2)}\left(\frac{r\sqrt{\hbar^2\omega^2 - m^2c^4}}{2\hbar c}\right)^{\frac{d-2}{2}}.
\end{aligned} \tag{3.13}$$

To check that this gives the correct equation for the density of states as in Sec. 2.4 we must calculate

$$\frac{2E}{\pi\hbar^2c^2}\Im G_d(\boldsymbol{x},\omega) = \left[\varrho(E) - \overline{\varrho}(\overline{E})\right]\Big|_{E=\hbar c k^0}.$$

Inserting the expression for $\Im G_d(\boldsymbol{x},\omega)$ and simplifying yields

$$\begin{aligned}
\left[\varrho(E) - \overline{\varrho}(\overline{E})\right]\Big|_{E=\hbar c k^0} &= \frac{2E}{\pi\hbar^2c^2}\frac{\pi}{2}\frac{\Theta(\hbar|\omega| - mc^2)}{\sqrt{2\pi}^d}\left(\frac{\sqrt{\hbar^2\omega^2 - m^2c^4}}{\hbar c r}\right)^{\frac{d-2}{2}} \\
&\times \frac{1}{\Gamma(d/2)}\left(\frac{r\sqrt{\hbar^2\omega^2 - m^2c^4}}{2\hbar c}\right)^{\frac{d-2}{2}} \\
&= \frac{2\Theta(\hbar|\omega| - mc^2)}{(2\sqrt{\pi}\hbar c)^d\Gamma(d/2)}E\sqrt{\hbar^2\omega^2 - m^2c^4}^{d-2}\Big|_{E=\hbar c k^0}.
\end{aligned} \tag{3.14}$$

Recognizing that $ck^0 = \omega$ yields

$$\left[\varrho(E) - \overline{\varrho}(\overline{E})\right]\Big|_{E=\hbar c k^0} = \frac{2\Theta(E^2 - m^2c^4)}{(2\sqrt{\pi}\hbar c)^d\Gamma(d/2)}|E|\sqrt{E^2 - m^2c^4}^{d-2}. \tag{3.15}$$

The Heaviside function has been written in terms of E^2, in order to replicate the result for the relativistic density of states calculated in Sec. 2.5.

3.2 Inter-dimensional effects with quasirelativistic bosons

In order to create an inter-dimensional Hamiltonian, that models the presence of an interface, we include extra spatial derivative terms and a term that parameterizes the change in the bulk gap parameter m^2 due to motion in the interface at z_0. Therefore for an interdimensional Hamiltonian for quasirelativistic bosons we have

$$\begin{aligned} H &= \frac{1}{\hbar}\int d^2\boldsymbol{x}_\parallel \int dz \Big[\hbar^2 \dot{\phi}^\dagger(\boldsymbol{x}_\parallel, z, t)\dot{\phi}(\boldsymbol{x}_\parallel, z, t) + \hbar^2 c^2 \boldsymbol{\nabla}\phi^\dagger(\boldsymbol{x}_\parallel, z, t)\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}_\parallel, z, t) \\ &+ m^2c^4\phi^\dagger(\boldsymbol{x}_\parallel, z, t)\phi(\boldsymbol{x}_\parallel, z, t)\Big] \\ &+ \frac{\ell}{\hbar}\int d^2\boldsymbol{x}_\parallel \Big[\hbar^2 c^2 \boldsymbol{\nabla}_\parallel\phi^\dagger(\boldsymbol{x}_\parallel, z_0, t)\cdot\boldsymbol{\nabla}_\parallel\phi(\boldsymbol{x}_\parallel, z_0, t) \\ &+ \Delta m^2 c^4\phi^\dagger(\boldsymbol{x}_\parallel, z_0, t)\phi(\boldsymbol{x}_\parallel, z_0, t)\Big], \end{aligned} \tag{3.16}$$

where ℓ is a length-scale parameter that represents the thickness of the interface. The Δm^2 term represents the change in the bulk gap parameter m^2 due to particle motion in the interface which is located at z_0. The three-dimensional position vector $\boldsymbol{x}$ is broken into components parallel to the interface, $\boldsymbol{x}_\parallel$ and components perpendicular to the face of the interface z. Using this interdimensional Hamiltonian, we can use the twice-iterated Heisenberg equation to find an equation of motion for the Klein-Gordon fields. Only spatial gradient and mass terms have been added to the original Klein-Gordon Hamiltonian, which leaves the commutation relations for the Klein-Gordon fields unchanged. Therefore, solving the first iteration of the Heisenberg equation yields

$$\begin{aligned} i\hbar\frac{\partial\phi(\boldsymbol{x}_\parallel, z, t)}{\partial t} = \big[\phi(\boldsymbol{x}_\parallel, z, t), H\big] &= \frac{1}{\hbar}\int d^2\boldsymbol{x'}_\parallel \int dz' \left[\hbar^2\left[\phi(\boldsymbol{x}_\parallel, z, t), \dot{\phi}^\dagger(\boldsymbol{x'}_\parallel, z', t)\right]\right] + 0 \\ &= i\hbar\dot{\phi}(\boldsymbol{x}_\parallel, z, t), \end{aligned}$$

where the equal-time commutation relations have been used. Obviously the LHS and RHS of the equation are equivalent. Using the Heisenberg equation once more we have

$$-\hbar^2\frac{\partial\dot{\phi}(\boldsymbol{x}_\parallel, z, t)}{\partial t} = i\hbar\left[\dot{\phi}(\boldsymbol{x}_\parallel, z, t), H\right]$$

$$= i\int d^2\boldsymbol{x'}_\parallel \int dz' \Bigg(-\hbar^2c^2\left[\dot{\phi}(\boldsymbol{x}_\parallel, z, t), \phi^\dagger(\boldsymbol{x'}_\parallel, z', t)\right]\boldsymbol{\nabla'}^2\phi(\boldsymbol{x'}_\parallel, z', t)$$

$$+m^2c^4\left[\dot{\phi}(\boldsymbol{x}_\parallel,z,t),\phi^\dagger(\boldsymbol{x'}_\parallel,z',t)\right]\phi(\boldsymbol{x'}_\parallel,z',t)\Bigg)$$

$$+i\ell\int d^2\boldsymbol{x'}_\parallel\Bigg(-\hbar^2c^2\left[\dot{\phi}(\boldsymbol{x}_\parallel,z,t),\phi^\dagger(\boldsymbol{x'}_\parallel,z_0,t)\right]\boldsymbol{\nabla'}_\parallel^2\phi(\boldsymbol{x'}_\parallel,z_0,t)$$

$$+\Delta m^2c^4\left[\dot{\phi}(\boldsymbol{x}_\parallel,z,t),\phi^\dagger(\boldsymbol{x'}_\parallel,z_0,t)\right]\phi(\boldsymbol{x'}_\parallel,z_0,t)\Bigg).$$

In order to easily evaluate the commutators in the above expression, the surface term from $\boldsymbol{\nabla}\phi^\dagger(\boldsymbol{x}_\parallel,z,t)\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}_\parallel,z,t) = \boldsymbol{\nabla}\cdot\left(\phi(\boldsymbol{x}_\parallel,z,t)\boldsymbol{\nabla}\phi^\dagger(\boldsymbol{x}_\parallel,z,t)\right) - \phi^\dagger(\boldsymbol{x}_\parallel,z,t)\boldsymbol{\nabla}^2\phi(\boldsymbol{x}_\parallel,z,t)$ was integrated out using Gauss's law since the Klein-Gordon fields vanish at infinity. Therefore, after applying the commutator relations on the Klein-Gordon fields and evaluating the now trivial integrals over $\boldsymbol{x'}_\parallel$ and z, we have

$$-\hbar^2\partial_t^2\phi = \left(-\hbar^2c^2\boldsymbol{\nabla}^2 + m^2c^4 - \ell\delta(z-z_0)\hbar^2c^2\boldsymbol{\nabla}_\parallel^2 + \ell\delta(z-z_0)\Delta m^2c^4\right)\phi.$$

Multiplying this equation by $1/(\hbar^2c^2)$ and using the definition $\partial^2 = \boldsymbol{\nabla}^2 - (\partial_t/c)^2$ we arrive at the equation of motion for the Klein-Gordon fields in the presence of an interface:

$$\left[\partial^2 - \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2c^2}{\hbar}\right)\right]\phi = 0.$$

The corresponding inter-dimensional Green's function satisfies

$$\left(\partial^2 - \frac{m^2c^2}{\hbar^2}\right)\langle x|G|x'\rangle + \ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2c^2}{\hbar^2}\right)\langle x|G|x'\rangle = -\delta(x-x').$$

In order to relate the Green's function matrix element $\langle x|G|x'\rangle$ to the energy-dependent Green's function $\langle\boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle$, and hence the density of states, we observe that given

$$\langle k^0|G|k'^0\rangle = G(E)\delta(k^0-k'^0),$$

we can re-write the Green's function matrix element $\langle x|G|x'\rangle$ by inserting mixed representation unity operators. This results in

$$\langle x|G|x'\rangle = \int dk^0\int dk'^0\int d\boldsymbol{x}\int d\boldsymbol{x'}\langle x^0,\boldsymbol{x}|k^0,\boldsymbol{x}\rangle\langle k^0,\boldsymbol{x}|G|k'^0,\boldsymbol{x'}\rangle\langle k'^0,\boldsymbol{x'}|x'^0,\boldsymbol{x'}\rangle.$$

Using plane-waves states in a mixed representation yields the following for inner products of the form $\langle x^0,\boldsymbol{x}|k^0,\boldsymbol{x}\rangle$:

$$\langle x^0,\boldsymbol{x}|k^0,\boldsymbol{x}\rangle = \frac{1}{\sqrt{2\pi}}\exp(-ik^0x^0)\langle\boldsymbol{x}|\boldsymbol{x}\rangle$$

$$= \frac{1}{\sqrt{2\pi}} \exp(-ik^0 x^0)\delta(\boldsymbol{x} - \boldsymbol{x}).$$

Inserting this into the mixed representation for the Green's function matrix element $\langle x|G|x'\rangle$ and recognizing that through the delta functions $\delta(\boldsymbol{x}-\boldsymbol{x})$ and $\delta(\boldsymbol{x'}-\boldsymbol{x'})$ the $\boldsymbol{x}$ and $\boldsymbol{x'}$ integrals are now trivially equal to unitary,

$$\langle x|G|x'\rangle = \frac{1}{2\pi}\int dk^0 \int dk'^0 \langle k^0, \boldsymbol{x}|G|k'^0, \boldsymbol{x'}\rangle \exp[i(k'^0 x'^0 - k^0 x^0)]. \tag{3.17}$$

The mixed representation matrix element can then be written in terms of the energy-dependent Green's function $\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle$ using

$$\langle k^0, \boldsymbol{x}|G|k'^0, \boldsymbol{x'}\rangle = \langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle \delta(k^0 - k'^0)|_{E=\hbar c k^0}. \tag{3.18}$$

Inserting this into the inter-dimensional Green's function equation yields

$$\left(\partial^2 - \frac{m^2c^2}{\hbar^2}\right)\frac{1}{2\pi}\int dk^0 \langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle \exp[ik^0(x'^0 - x^0)]|_{E=\hbar c k^0}$$

$$+\ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2 c^2}{\hbar^2}\right)\frac{1}{2\pi}\int dk^0 \langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle \exp[ik^0(x'^0 - x^0)]|_{E=\hbar c k^0} = -\delta(x-x').$$

Using the defintion $\partial^2 = \boldsymbol{\nabla}^2 - \partial_0^2$ and evaulating the ∂_0^2 derivatives yields

$$\left(\boldsymbol{\nabla}^2 + (k^0)^2 - \frac{m^2c^2}{\hbar^2}\right)\frac{1}{2\pi}\int dk^0 \langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle \exp[ik^0(x'^0 - x^0)]|_{E=\hbar c k^0}$$

$$+\ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2 c^2}{\hbar^2}\right)\frac{1}{2\pi}\int dk^0 \langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle \exp[ik^0(x'^0 - x^0)]|_{E=\hbar c k^0} = -\delta(x-x').$$

Recognizing that the k^0 integral is simply the integral representation of $\delta(x'^0 - x^0)$ yields the inter-dimensional energy-dependent Green's function equation as

$$\left(\boldsymbol{\nabla}^2 + \frac{E^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2}\right)\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle + \ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2 c^2}{\hbar^2}\right)\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle = -\delta(\boldsymbol{x}-\boldsymbol{x'}). \tag{3.19}$$

The solution for the energy-dependent Green's function in the $\boldsymbol{x}$ representation is solved in appendix A in the form of the Hankel transform. It is given by

$$\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle = \frac{1}{2\pi}\int_0^\infty dk_\parallel J_0(k_\parallel|\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel|)\langle z|G(k^0, k_\parallel)|z'\rangle|_{E=\hbar c k^0}, \tag{3.20}$$

where

$$\langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle = i\frac{\Theta[(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2]}{2\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}}\left[\exp\left(i\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}|z-z'|\right)\right.$$

$$-i\frac{\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_{\parallel}^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}$$

$$\times\exp\left[+i\sqrt{(k^0)^2-\boldsymbol{k}_{\parallel}^2-(mc/\hbar)^2}(|z'-z_0|+|z-z_0|)\right]\Bigg]$$

$$+\frac{\Theta[\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2]}{2\sqrt{\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2}}\Bigg[\exp\left(-\sqrt{\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2}|z-z'|\right)$$

$$-\frac{\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}{2\sqrt{\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2}+\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}$$

$$\times\exp\left[-\sqrt{\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2}(|z'-z_0|+|z-z_0|)\right]\Bigg].$$

The density of states is given in terms of the relativistic scalar Green's function as in Sec. 3.1,

$$\frac{2E}{\pi\hbar^2c^2}\grave{I}m\langle\boldsymbol{x}|G(E)|\boldsymbol{x}\rangle=\varrho(E)-\overline{\varrho}(\overline{E}).$$

For the energy-dependent Green's function matrix element present in its relation with the density of states, we use $\boldsymbol{x}=\boldsymbol{x'}$. Thus in the mixed representation, the matrix element in the interface at $z=z'=z_0$ is given by

$$\begin{aligned}\langle z_0|G(k^0,\boldsymbol{k}_{\parallel})z_0\rangle &= i\frac{\Theta[(k^0)^2-\boldsymbol{k}_{\parallel}^2-(mc/\hbar)^2]}{2\sqrt{(k^0)^2-\boldsymbol{k}_{\parallel}^2-(mc/\hbar)^2}}\\ &\times\left[1-i\frac{\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_{\parallel}^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}\right]\\ &+\frac{\Theta[\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2]}{2\sqrt{\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2}}\\ &\times\left[1-\frac{\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}{2\sqrt{\boldsymbol{k}_{\parallel}^2-(k^0)^2+(mc/\hbar)^2}+\ell\left(\boldsymbol{k}_{\parallel}^2+(m_{\parallel}c/\hbar)^2\right)}\right].\end{aligned}$$

The density of states contains contributions only from the imaginary terms from the relativistic scalar Green's function. Therefore the second Heaviside function term can immediately be dropped because it contains no imaginary terms. Further simplification of the remaining

terms yields

$$\begin{aligned}
\langle z_0|G(k^0,\boldsymbol{k}_\parallel)z_0\rangle &= i\frac{\Theta[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2]}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}} \\
&\times \left[\frac{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)-i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}\right] \\
&= \frac{\Theta[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2]}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}} \\
&\times \left[\frac{2i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}\right] \\
&\times \frac{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}-i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}-i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}. \qquad (3.21)
\end{aligned}$$

Keeping only the imaginary terms, this yields

$$\begin{aligned}
\langle z_0|G(k^0,\boldsymbol{k}_\parallel)z_0\rangle &= \frac{\Theta[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2]}{\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}} \\
&\times \left[\frac{2i(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}{4((k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2)+\ell^2\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)^2}\right] \\
&= \Theta[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2] \\
&\times \left[\frac{2i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}}{4((k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2)+\ell^2\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)^2}\right].
\end{aligned}$$

Inserting this into the expression for the energy-dependent Green's function for $\boldsymbol{x}=\boldsymbol{x'}$ yields

$$\begin{aligned}
\Im\langle \boldsymbol{x}|G(E)|\boldsymbol{x}\rangle &= \Theta[(k^0)^2-(mc/\hbar)^2]\int_0^{\sqrt{(k^0)^2-(mc/\hbar)^2}} dk_\parallel k_\parallel \\
&\times \left[\frac{2\sqrt{(k^0)^2-k_\parallel^2-(mc/\hbar)^2}}{4((k^0)^2-k_\parallel^2-(mc/\hbar)^2)+\ell^2\left(k_\parallel^2+(m_\parallel c/\hbar)^2\right)^2}\right]. \qquad (3.22)
\end{aligned}$$

Inserting this into the density of states yields

$$\begin{aligned}
\varrho(E)-\overline{\varrho}(\overline{E}) &= \frac{E}{\pi^2\hbar^2c^2}\Theta[(k^0)^2-(mc/\hbar)^2]\int_0^{\sqrt{(k^0)^2-(mc/\hbar)^2}} dk_\parallel k_\parallel \\
&\times \left[\frac{2\sqrt{(k^0)^2-k_\parallel^2-(mc/\hbar)^2}}{4((k^0)^2-k_\parallel^2-(mc/\hbar)^2)+\ell^2\left(k_\parallel^2+(m_\parallel c/\hbar)^2\right)^2}\right]. \qquad (3.23)
\end{aligned}$$

Making the substitution $x = k_{\parallel}^2$, $\frac{dx}{2} = k_{\parallel} dk_{\parallel}$ in the integrand allows for numerical integration. Using $\hbar c k^0 = E$ into the result of the integral yields the density of states

$$\varrho(E) - \overline{\varrho}(\overline{E}) = \frac{E}{\pi^2 \hbar^2 c^2} \Theta[E^2 - m^2 c^4]$$

$$\times \frac{1}{4\ell} \left[2 \left(\arctan \left(\frac{\ell \sqrt{E^2 - m^2 c^4} + \sqrt{\ell^2 [E^2 - m^2 c^4 + \Delta m^2 c^4] - \hbar^2 c^2}}{\hbar c} \right) \right.\right.$$

$$\left.\left. + \arctan \left(\frac{\ell \sqrt{E^2 - m^2 c^4} - \sqrt{\ell^2 [E^2 - m^2 c^4 + \Delta m^2 c^4] - \hbar^2 c^2}}{\hbar c} \right) \right) \right]$$

$$+ \frac{E}{\pi^2 \hbar c} \Theta[E^2 - m^2 c^4] \frac{1}{4\ell \sqrt{\ell^2 [E^2 - m^2 c^4 + \Delta m^2 c^4] - \hbar^2 c^2}}$$

$$\times ln \left[\frac{2\ell(E^2 - m^2 c^4) + \ell \Delta m^2 c^4 - 2\sqrt{E^2 - m^2 c^4} \sqrt{\ell^2 [E^2 - m^2 c^4 + \Delta m^2 c^4] - \hbar^2 c^2}}{2\ell(E^2 - m^2 c^4) + \ell \Delta m^2 c^4 + 2\sqrt{E^2 - m^2 c^4} \sqrt{\ell^2 [E^2 - m^2 c^4 + \Delta m^2 c^4] - \hbar^2 c^2}} \right]$$

$$= \frac{E}{\pi^2 \hbar^2 c^2} \Theta[E^2 - m^2 c^4]$$

$$\times \frac{1}{2\ell} \left[\arctan \left(\frac{\ell \sqrt{E^2 - m^2 c^4} + \sqrt{g(E, \ell, m, \Delta m^2)}}{\hbar c} \right) \right.$$

$$\left. + \arctan \left(\frac{\ell \sqrt{E^2 - m^2 c^4} - \sqrt{g(E, \ell, m, \Delta m^2)}}{\hbar c} \right) \right]$$

$$+ \frac{E}{\pi^2 \hbar c} \Theta[E^2 - m^2 c^4] \frac{1}{4\ell \sqrt{g(E, \ell, m, \Delta m^2)}}$$

$$\times ln \left[\frac{2\ell(E^2 - m^2 c^4) + \ell \Delta m^2 c^4 - 2\sqrt{E^2 - m^2 c^4} \sqrt{g(E, \ell, m, \Delta m^2)}}{2\ell(E^2 - m^2 c^4) + \ell \Delta m^2 c^4 + 2\sqrt{E^2 - m^2 c^4} \sqrt{g(E, \ell, m, \Delta m^2)}} \right]$$

where $g(E, \ell, m, \Delta m^2) = \ell^2 [E^2 - m^2 c^4 + \Delta m^2 c^4] - \hbar^2 c^2$.

The density of states remains real for $g(E, \ell, m, \Delta m^2) < 0$. Therefore the density of states can be written only in terms of real functions using the following three identities. For $x \geq 0$, $y \geq 0$ and $\alpha\beta \neq -1$,

$$\frac{1}{i} ln \left(\frac{x + iy}{x - iy} \right) = 2 \arctan \left(\frac{y}{x} \right),$$

$$\arctan(x + iy) + \arctan(x - iy) = \arctan \left(\frac{x}{1 + y} \right) + \arctan \left(\frac{x}{1 - y} \right),$$

$$\arctan(\alpha) - \arctan(\beta) = \arctan\left(\frac{\alpha - \beta}{1 + \alpha\beta}\right).$$

The arctan terms can be written in terms of real functions using the identity:

$$\arctan\left(\frac{\ell\sqrt{E^2 - m^2c^4} + i\sqrt{-g(E, \ell, m, \Delta m^2)}}{\hbar c}\right)$$

$$+ \arctan\left(\frac{\ell\sqrt{E^2 - m^2c^4} - i\sqrt{-g(E, \ell, m, \Delta m^2)}}{\hbar c}\right)$$

$$= \arctan\left(\frac{\ell\sqrt{E^2 - m^2c^4}}{\hbar c + \sqrt{-g(E, \ell, m, \Delta m^2)}}\right) + \arctan\left(\frac{\ell\sqrt{E^2 - m^2c^4}}{\hbar c - \sqrt{-g(E, \ell, m, \Delta m^2)}}\right).$$

The logarithm term can be rewritten in terms of real functions using the first and third identities. Using the first yields

$$\frac{1}{i\sqrt{-g(E, \ell, m, \Delta m^2)}} ln\left[\frac{2\ell(E^2 - m^2c^4) + \ell\Delta m^2c^4 - 2i\sqrt{E^2 - m^2c^4}\sqrt{-g(E, \ell, m, \Delta m^2)}}{2\ell(E^2 - m^2c^4) + \ell\Delta m^2c^4 + 2i\sqrt{E^2 - m^2c^4}\sqrt{-g(E, \ell, m, \Delta m^2)}}\right]$$

$$= \frac{2}{\sqrt{-g(E, \ell, m, \Delta m^2)}} \arctan\left(\frac{-2\sqrt{E^2 - m^2c^4}\sqrt{-g(E, \ell, m, \Delta m^2)}}{2\ell(E^2 - m^2c^4) + \ell\Delta m^2c^4}\right).$$

The argument of the arctan function can be written in the form of the right hand side of the third identity. Therefore, $-2\sqrt{E^2 - m^2c^4}\sqrt{-g(E, \ell, m, \Delta m^2)} = \alpha - \beta$ and $2\ell(E^2 - m^2c^4) + \ell\Delta m^2c^4 = 1 + \alpha\beta$. The solutions for α and β are

$$\alpha = \frac{\ell\sqrt{E^2 - m^2c^4}}{\hbar c + \sqrt{-g(E, \ell, m, \Delta m^2)}},$$

$$\beta = \frac{\ell\sqrt{E^2 - m^2c^4}}{\hbar c - \sqrt{-g(E, \ell, m, \Delta m^2)}}.$$

Therefore inserting these values for α and β into the RHS yields

$$\frac{2}{\sqrt{-g(E, \ell, m, \Delta m^2)}} \arctan\left(\frac{-2\sqrt{E^2 - m^2c^4}\sqrt{-g(E, \ell, m, \Delta m^2)}}{2\ell(E^2 - m^2c^4) + \ell\Delta m^2c^4}\right)$$

$$= \frac{2}{\sqrt{-g(E, \ell, m, \Delta m^2)}}\left(\arctan\left(\frac{\ell\sqrt{E^2 - m^2c^4}}{\hbar c + \sqrt{-g(E, \ell, m, \Delta m^2)}}\right)\right.$$

$$\left. - \arctan\left(\frac{\ell\sqrt{E^2 - m^2c^4}}{\hbar c - \sqrt{-g(E, \ell, m, \Delta m^2)}}\right)\right).$$

Substituting this all into the density of states equation yields

$$\begin{aligned}\varrho(E,z_0)-\bar{\varrho}(\overline{E},z_0) &= \frac{E\Theta(E^2-m^2c^4)}{2(\pi\hbar c)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c+\sqrt{-g(E,\ell,m,\Delta m^2)}}\right)\right.\\ &+ \left.\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c-\sqrt{-g(E,\ell,m,\Delta m^2)}}\right)\right]\\ &+ \frac{E\Theta(E^2-m^2c^4)}{2\pi^2\hbar c\ell\sqrt{-g(E,\ell,m,\Delta m^2)}}\left[\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c+\sqrt{-g(E,\ell,m,\Delta m^2)}}\right)\right.\\ &- \left.\left.\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c-\sqrt{-g(E,\ell,m,\Delta m^2)}}\right)\right)\right]. \qquad (3.24)\end{aligned}$$

In order to preserve the continuity and smoothness of the Green's function which was used to derive the density of states, the arctan functions adhere to the inequality $0 \leq \arctan(x) < \pi$. If this were not the case, π would need to be added to every arctan function that has a negative argument.

As stated in the motivation section in Chapter 1, we will use parameters found in Refs. [2] and [29] for the bulk gap parameter Δ_g. As well Ref. [29] yields values for the momentum and energy ranges around the perturbed Dirac points that allow our model to be admissible. We find that the topological phase transition occurs when x is between 0.3 and 0.4. For $x \geq 0.3$ the momentum range is given by $\Delta k = 0.1\ \overset{\circ}{A}^{-1}$ and the energy range is given by $\Delta E = \pm 400$ meV, where $\pm$ refer to above and below the Fermi level. From [2] the gap parameter at 4.2 K is given by $\Delta_g = 95$ meV. This yields a velocity parameter given by $c \approx 2 \times 10^{-3} c_0 = 600$ km/s, where c_0 is the speed of light in the vacuum. These values are used in Chapter 4 in order to plot the density of states in the interface.

Chapter 4

Results and Discussion

4.1 Results for limiting cases of inter-dimensional density of states

In this section we will analyze specific limiting cases of the inter-dimensional density of states in the interface at $z = z_0$ derived in Chapter 3. We will show that the inter-dimensional density of states in the interface approaches two-dimensional behaviour in the high-energy limit, and that it approaches the three-dimensional limit in the low energy limit as well as the limit where $\ell \to 0$. Finally, we will derive the finite offset in the inter-dimensional density of states in the interface due to a negative gap shift $\Delta m^2 c^4 = -\Delta_g^2$ that is due to the generation of additional states that occur in the band gap.

We expect that the inter-dimensional density of states in the interface should behave like the three-dimensional relativistic density of states for a free particle in the limit that the size of the interface approaches zero. Applying this to the inter-dimensional density of states equation in the interface implies analyzing the limit that $\ell \to 0$. Specifically, in this limit

$-g(E,\ell,m,\Delta m^2)=\hbar^2c^2$. Therefore with this simplification the density of states reduces to

$$\begin{aligned}
[\varrho(E,z_0)-\overline{\varrho}(\overline{E},z_0)]_{\ell\to 0} &= \frac{E\Theta(E^2-m^2c^4)}{2(\pi\hbar c)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c+\hbar c}\right)\right.\\
&+ \left.\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c-\hbar c}\right)\right]\\
&+ \frac{E\Theta(E^2-m^2c^4)}{2(\pi\hbar c)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c+\hbar c}\right)\right.\\
&- \left.\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{\hbar c-\hbar c}\right)\right]\\
&= \frac{E\Theta(E^2-m^2c^4)}{(\pi\hbar c)^2\ell}\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{2\hbar c}\right). \qquad (4.1)
\end{aligned}$$

The argument of the arctan function is very small since $\ell\to 0$, therefore keeping only the linear order term of the Taylor series expansion yields for the density of states,

$$\begin{aligned}
[\varrho(E,z_0)-\overline{\varrho}(\overline{E},z_0)]_{\ell\to 0} &= \frac{\Theta(E^2-m^2c^4)E\sqrt{E^2-m^2c^4}}{2\pi^2(\hbar c)^3}\\
&= [\varrho(E)-\overline{\varrho}(\overline{E})]_{d=3}. \qquad (4.2)
\end{aligned}$$

This result is expected. The three-dimensional limit can also be derived for small energies. Given $|\Delta m^2c^4|\ll E^2-m^2c^4\ll\hbar^2c^2/\ell^2$ the function $-g(E,\ell,m,\Delta m^2)$ becomes $\hbar^2c^2$. Thus under these conditions, the density of states is

$$[\varrho(E,z_0)-\overline{\varrho}(\overline{E},z_0)]=\frac{E\Theta(E^2-m^2c^4)}{(\pi\hbar c)^2\ell}\arctan\left(\frac{\ell\sqrt{E^2-m^2c^4}}{2\hbar c}\right).$$

Given the inequality, the argument of the arctan function is again very small, and can be simplified by keeping only the first term in the Taylor series. Therefore

$$\begin{aligned}
[\varrho(E,z_0)-\overline{\varrho}(\overline{E},z_0)] &= \frac{\Theta(E^2-m^2c^4)E\sqrt{E^2-m^2c^4}}{2\pi^2(\hbar c)^3}\\
&= [\varrho(E)-\overline{\varrho}(\overline{E})]_{d=3}. \qquad (4.3)
\end{aligned}$$

Figure 4.1 shows the density of states in the interface with the three-dimensional density of states plotted for reference.

In the opposite limit, namely, when $E^2-m^2c^4\gg(\hbar c/\ell)^2-\Delta m^2c^4$, the density of states can again be reduced. Using this condition yields $-g(E,\ell,m,\Delta m^2)/\ell^2\to-(E^2-m^2c^4)$. Thus the density of states becomes

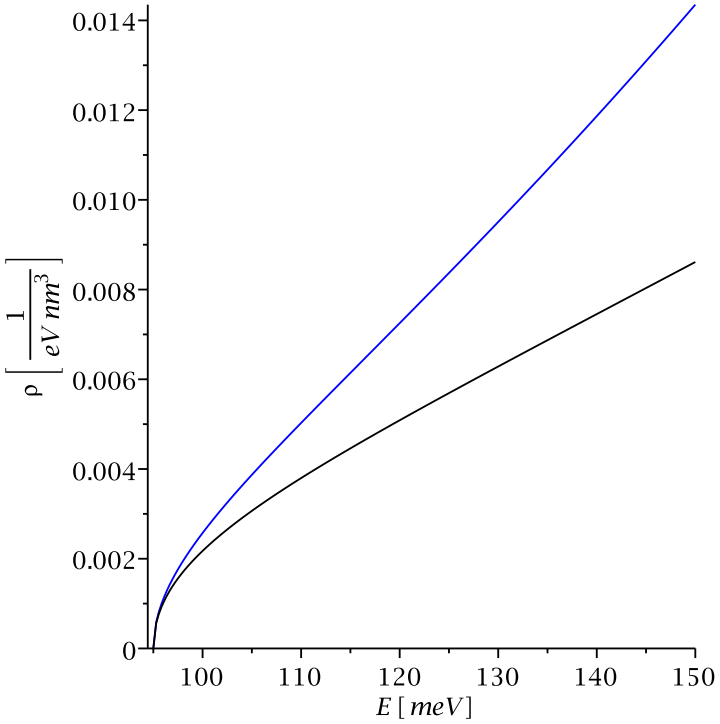

Figure 4.1: The upper line is the three-dimensional density of states. The lower line is the density of states in the interface for $\ell = 3$ nm, $\Delta_g = 95$ meV and $\Delta m^2c^4 = 0$.

$$
\begin{aligned}
[\varrho(E,z_0) - \overline{\varrho}(\overline{E},z_0)] &= \frac{E\Theta(E^2-m^2c^4)}{2(\pi\hbar c)^2\ell}\left[\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)+i\sqrt{E^2-m^2c^4}}\right)\right. \\
&+ \left.\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)-i\sqrt{E^2-m^2c^4}}\right)\right] \\
&+ \frac{E\Theta(E^2-m^2c^4)}{2\pi^2\hbar c\ell^2 i\sqrt{E^2-m^2c^4}}\left[\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)+i\sqrt{E^2-m^2c^4}}\right)\right. \\
&- \left.\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)-i\sqrt{E^2-m^2c^4}}\right)\right]. \qquad (4.4)
\end{aligned}
$$

Using the arctan sum and difference identities given by

$$\arctan(x) \pm \arctan(y) = \arctan\left(\frac{x \pm y}{1 \mp xy}\right)$$

allows for further simplification of the density of states. Applying the sum identity to the

first two arctan terms in Eq. (4.4) yields

$$\begin{aligned}\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)+i\sqrt{E^2-m^2c^4}}\right) &+ \arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)-i\sqrt{E^2-m^2c^4}}\right)\\ &= \arctan\left(\frac{2\ell\sqrt{E^2-m^2c^4}}{\hbar c}\right).\end{aligned}$$

From the condition $E^2-m^2c^4 \gg (\hbar c/\ell)^2-\Delta m^2c^4$, we can use $|\ell|\sqrt{E^2-m^2c^4} >> \hbar c$, to simplify the argument of the arctan function. The argument of the arctan function is very large and can be approximated by $\pi/2$. Thus the first term in the density of states reduces to

$$\begin{aligned}\frac{E\Theta(E^2-m^2c^4)}{4\pi(\hbar c)^2|\ell|} &= \frac{E\Theta(E^2-m^2c^4)}{2(\pi\hbar c)^2\ell}\left[\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)+i\sqrt{E^2-m^2c^4}}\right)\right.\\ &+ \left.\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)-i\sqrt{E^2-m^2c^4}}\right)\right]. \qquad (4.5)\end{aligned}$$

Using the arctan difference identity on the second two arctan terms yields

$$\begin{aligned}\arctan\left(\frac{-2i(E^2-m^2c^4)}{(\hbar c/\ell)^2+2(E^2-m^2c^4)}\right) &= \arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)+i\sqrt{E^2-m^2c^4}}\right)\\ &- \arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)-i\sqrt{E^2-m^2c^4}}\right). \qquad (4.6)\end{aligned}$$

In order to express the density of states in terms of entirely real functions, we will re-write the arctan function using the identity,

$$\frac{1}{i}\ln\left(\frac{x+iy}{x-iy}\right) = 2\arctan\left(\frac{y}{x}\right).$$

After applying the identity we have

$$\arctan\left(\frac{-2i(E^2-m^2c^4)}{(\hbar c/\ell)^2+2(E^2-m^2c^4)}\right) = \frac{1}{2i}\ln\left(\frac{(\hbar c/\ell)^2+4(E^2-m^2c^4)}{(\hbar c/\ell)^2}\right).$$

The numerator of the logarithm term reduces to $4(E^2-m^2c^4)$ under the given condition. Using the power rule for logarithms, $\ln(x^y) = y\cdot\ln(x)$, we have for the difference of arctan terms

$$\begin{aligned}-\frac{E\Theta(E^2-m^2c^4)}{\sqrt{E^2-m^2c^4}}\frac{\ln(2|\ell|\sqrt{E^2-m^2c^4}/\hbar c)}{2\pi^2\hbar c\ell^2} &= \frac{E\Theta(E^2-m^2c^4)}{2\pi^2\hbar c\ell^2 i\sqrt{E^2-m^2c^4}}\\ &\times \left[\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)+i\sqrt{E^2-m^2c^4}}\right)\right.\\ &- \left.\arctan\left(\frac{\sqrt{E^2-m^2c^4}}{(\hbar c/\ell)-i\sqrt{E^2-m^2c^4}}\right)\right]. \qquad (4.7)\end{aligned}$$

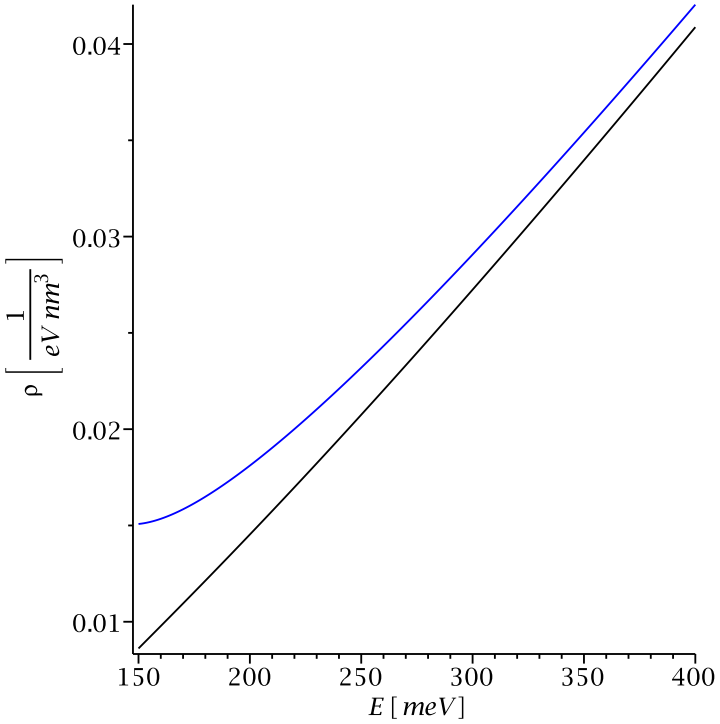

Figure 4.2: The upper line corresponds to the two-dimensional limit of the density of states in the interface plus the logarithmic correction. The lower curve is the density of states in the interface for $\ell = 3$ nm, $\Delta_g = 95$ meV and $\Delta m^2 c^4 = 0$.

Under the condition $E^2 - m^2c^4 \gg (\hbar c/\ell)^2 - \Delta m^2 c^4$, the density of states becomes

$$\begin{aligned}[\varrho(E, z_0) - \overline{\varrho}(\overline{E}, z_0)] &\simeq \frac{E\Theta(E^2 - m^2c^4)}{4\pi(\hbar c)^2|\ell|} - \frac{E\Theta(E^2 - m^2c^4)}{\sqrt{E^2 - m^2c^4}} \frac{\ln(2|\ell|\sqrt{E^2 - m^2c^4}/\hbar c)}{2\pi^2\hbar c\ell^2} \\ &= \frac{\varrho_{d=2}(E)}{2|\ell|} - \frac{E\Theta(E^2 - m^2c^4)}{\sqrt{E^2 - m^2c^4}} \frac{\ln(2|\ell|\sqrt{E^2 - m^2c^4}/\hbar c)}{2\pi^2\hbar c\ell^2}. \end{aligned} \tag{4.8}$$

Thus under the condition $E^2 - m^2c^4 \gg (\hbar c/\ell)^2 - \Delta m^2 c^4$, the density of states approaches the two-dimensional limit with an additional logarithmic correction factor. The curve in the upper blue line in Fig. 4.2 is the minimum in the logarithmic correction term, after which the line is dominated by the two-dimensional density of states term. This is shown in Fig. 4.2 along with the density of states in the interface.

A negative gap shift $\Delta m^2 c^4$ with $\Delta m^2 < 0$ in the interface induces a finite offset in the density of states about the energy gap Δ_g. The offset is calculated by using $E = \Delta_g = mc^2$

and $\Delta m^2 < 0$ in the quasi-relativistic density of states equation. In particular, this yields

$$\begin{aligned}
\varrho(\Delta_g, z_0)|_{\Delta m^2} &= \frac{\Delta_g}{2(\pi\hbar c)^2 \ell}\left[\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c + \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right. \\
&+ \left.\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c - \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right] \\
&+ \frac{\Delta_g}{2\pi^2\hbar c\ell\sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\left[\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c + \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right. \\
&- \left.\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c - \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right], \qquad (4.9)
\end{aligned}$$

where

$$g(\Delta_g, \ell, m, \Delta m^2) = \ell^2(\Delta_g^2 - m^2c^4 + \Delta m^2 c^4) - \hbar^2 c^2 = \ell^2 \Delta m^2 c^4 - \hbar^2 c^2.$$

The arctan functions in the density of states equation as defined within, follow the inequality $0 \leq \arctan(x) < \pi$. This means that if x is negative, we must add π to the value of $\arctan(-x)$ to properly ensure the continuity and smoothness properties of the Green's function matrix element. Clearly as $E^2 \to \Delta_g^2 = m^2c^4$, the arguments of all arctan functions in the density of states approach zero. Therefore, we must look at the two different arguments in the arctan functions to evaluate whether or not the arctan functions approach zero or π as their argument approaches zero. For the first arctan argument we have

$$\begin{aligned}
\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c + \sqrt{\hbar^2c^2 - \ell^2\Delta m^2 c^4}} &= \sqrt{\Delta_g^2 - m^2c^4}\left(\frac{\ell\hbar c - \ell\sqrt{\hbar^2c^2 - \ell^2\Delta m^2c^4}}{\ell^2\Delta m^2 c^4}\right) \\
&= \sqrt{\Delta_g^2 - m^2c^4}\left(\frac{-\ell\hbar c + \ell\sqrt{\hbar^2c^2 + \ell^2|\Delta m^2|c^4}}{\ell^2|\Delta m^2|c^4}\right) \\
&= \sqrt{\Delta_g^2 - m^2c^4}\left(\frac{-\ell\hbar c + \ell\hbar c\sqrt{1 + (\ell^2|\Delta m^2|c^2)/\hbar^2}}{\ell^2|\Delta m^2|c^4}\right). \qquad (4.10)
\end{aligned}$$

Therefore since $(\ell^2|\Delta m^2|c^2)/\hbar^2 > 0$ the term multiplying $\sqrt{\Delta_g^2 - m^2c^4}$ is a positive finite constant. To approximate the limiting value of the arctan function as its argument approaches zero from the positive side we use only the first term from its Taylor series expansion. We have

$$\arctan(\epsilon) = \epsilon \to 0^+$$

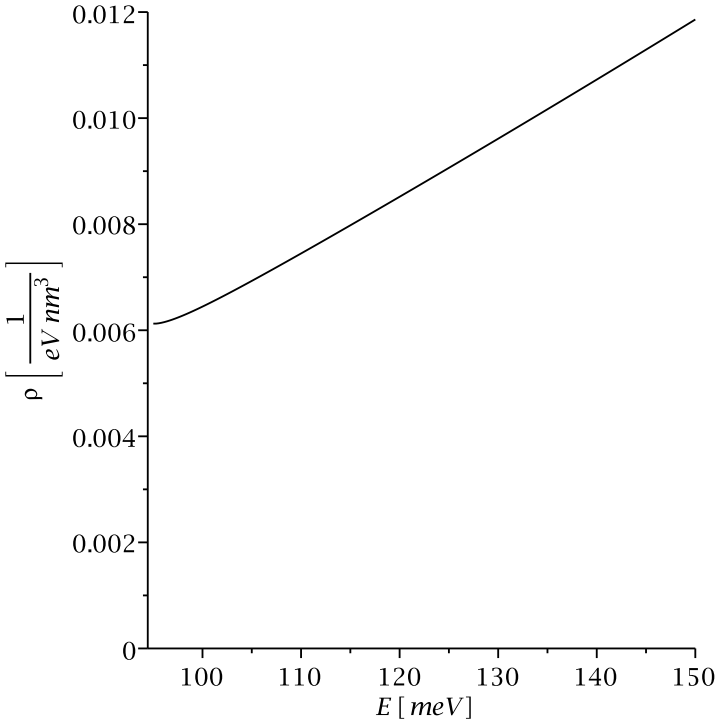

Figure 4.3: The density of states in the interface for $\ell = 3$ nm, $E > \Delta_g = 95$ meV, and a bulk gap shift parameter of $\Delta m^2 c^4 = -\Delta_g^2$.

for $\epsilon > 0$. We therefore have determined that this argument in the arctan function adheres to the equation,

$$\begin{aligned} \lim_{E^2 \to \Delta_g^2} \arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c + \sqrt{\hbar^2c^2 - \ell^2\Delta m^2 c^4}}\right) &= \sqrt{\Delta_g^2 - m^2c^4} \\ &\times \left(\frac{-\ell\hbar c + \ell\hbar c\sqrt{1 + (\ell^2|\Delta m^2|c^2)/\hbar^2}}{\ell^2|\Delta m^2|c^4}\right) \\ &\to 0 \end{aligned} \tag{4.11}$$

For the second arctan argument we have

$$\begin{aligned} \frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c - \sqrt{\hbar^2c^2 - \ell^2\Delta m^2c^4}} &= \sqrt{\Delta_g^2 - m^2c^4}\left(\frac{\ell\hbar c + \ell\sqrt{\hbar^2c^2 - \ell^2\Delta m^2c^4}}{\ell^2\Delta m^2c^4}\right) \\ &= \sqrt{\Delta_g^2 - m^2c^4}\left(\frac{-\ell\hbar c - \ell\sqrt{\hbar^2c^2 + \ell^2|\Delta m^2|c^4}}{\ell^2|\Delta m^2|c^4}\right) \\ &= \sqrt{\Delta_g^2 - m^2c^4}\left(\frac{-\ell\hbar c - \ell\hbar c\sqrt{1 + (\ell^2|\Delta m^2|c^2)/\hbar^2}}{\ell^2|\Delta m^2|c^4}\right). \end{aligned} \tag{4.12}$$

Since $(\ell^2|\Delta m^2|c^2)/\hbar^2 > 0$, the term multiplying $\sqrt{\Delta_g^2 - m^2c^4}$ is a negative finite constant. We therefore must add π to the value of the arctan function since it has a negative argument. Using the first term of the Taylor series to approximate the arctan function yields

$$\arctan(-\epsilon) = \pi - \epsilon \to \pi$$

for $\epsilon \to 0^+$. We have thus determined that this argument in the arctan function adheres to the equation,

$$\begin{aligned}
\lim_{E^2 \to \Delta_g^2} \arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c - \sqrt{\hbar^2c^2 - \ell^2\Delta m^2 c^4}}\right) &= \arctan\left(\sqrt{\Delta_g^2 - m^2c^4}\right. \\
&\times \left.\left(\frac{-\ell\hbar c - \ell\hbar c\sqrt{1 + (\ell^2|\Delta m^2|c^2)/\hbar^2}}{\ell^2|\Delta m^2|c^4}\right)\right) \\
&= \pi - \sqrt{\Delta_g^2 - m^2c^4} \\
&\times \left(\frac{\ell\hbar c + \ell\hbar c\sqrt{1 + (\ell^2|\Delta m^2|c^2)/\hbar^2}}{\ell^2|\Delta m^2|c^4}\right) \\
&\to \pi
\end{aligned} \tag{4.13}$$

Applying these results to the density of states equation yields

$$\begin{aligned}
\varrho(\Delta_g, z_0)|_{\Delta m^2} &= \frac{\Delta_g}{2(\pi\hbar c)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c + \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right. \\
&+ \left.\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c - \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right] \\
&+ \frac{\Delta_g}{2\pi^2\hbar c\ell\sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\left[\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c + \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right. \\
&- \left.\arctan\left(\frac{\ell\sqrt{\Delta_g^2 - m^2c^4}}{\hbar c - \sqrt{-g(\Delta_g, \ell, m, \Delta m^2)}}\right)\right] \\
&= \frac{\Delta_g}{2(\pi\hbar c)^2\ell}\left(0 + \pi\right) + \frac{\Delta_g}{2\pi^2\hbar c\ell\sqrt{\hbar^2c^2 - \ell^2\Delta m^2c^4}}\left(0 - \pi\right) \\
&= \frac{\Delta_g}{2\pi\hbar^2c^2\ell}\left(1 - \frac{\hbar c}{\sqrt{\hbar^2c^2 - \ell^2\Delta m^2c^4}}\right).
\end{aligned} \tag{4.14}$$

The gap offset for $\Delta m^2c^4 = -\Delta_g^2$ essentially closes the band gap and causes the conduction and valence bands to meet at $E = 0$. Therefore when the gap closes we see an increased number of permissible energy states given by the offset value for the density of states in the interface given by Eq. (4.14). The gap offset is plotted in Fig. 4.3.

4.2 Discussion of results in limiting cases of inter-dimensional density of states

The quasirelativistic density of states inside the interface at $z = z_0$ approaches the free three-dimensional relativistic density of states in the limit that the interface thickness ℓ approaches zero. This result is obviously expected as this limit is effectively removing the interface term from the quasirelativistic inter-dimensional Hamiltonian from which we started our calculation of the density of states. The free three-dimensional relativistic density of states is also found from the limit $|\Delta m^2 c^4| \ll E^2 - m^2c^4 \ll \hbar^2 c^2/\ell^2$ of the quasirelativistic density of states. This inequality tells us that for small energies given by $\sqrt{E^2 - m^2c^4}$, the quasirelativistic density of states approaches the three-dimensional limit provided that the change in the bulk gap parameter $|\Delta m^2|$ is small compared to the particles energy. This low-energy limit is analogous to the limit $8mE\ell^2 \ll \hbar^2$ used in Ref. [1]. The small-energy limit coincides with states that probed length scales that are larger than the parameter $\ell = Lm/2m_*$ where L is the interface thickness. This can be written in terms of the de Broglie wavelength in the direction perpendicular to the interface. In the non-relativistic case, we can substitute the de Broglie wavelength $\lambda = 2\pi\hbar/p$ into the inequality for three-dimensional behaviour, $8mE\ell^2 \ll \hbar^2$ [1]. Here E is the particle's kinetic energy and is given by $E = \hbar^2\boldsymbol{k}^2/2m = \boldsymbol{p}^2/2m$. Therefore the de Broglie wavelength in terms of the kinetic energy is

$$\begin{aligned} \lambda = \frac{2\pi\hbar}{|\boldsymbol{p}|} &= \frac{2\pi\hbar}{\sqrt{2mE}} \\ &= \frac{\sqrt{2}\pi\hbar}{\sqrt{mE}}. \end{aligned} \tag{4.15}$$

The inequality $8mE\ell^2 \ll \hbar^2$ can be written as

$$\begin{aligned} \sqrt{\ell^2} = \ell &\ll \sqrt{\frac{\hbar^2}{8mE}} \\ &= \frac{\hbar}{2\sqrt{2mE}} \\ &= \frac{\lambda}{4\pi} \end{aligned} \tag{4.16}$$

Thus for low energies that adhere to the inequality $8mE\ell^2 \ll \hbar^2$, the particle's de Broglie wavelength in the direction perpendicular to the interface is much larger compared to the interface thickness, $\ell \ll \lambda/4\pi$. Intuitively this suggests that the particle would see the bulk even though it is located in the interface, and in kind, have a free three-dimensional density of states. We can apply the same kind of analysis to the large-energy inequality, $8mE\ell^2 \gg \hbar^2$. This shows that for high energy, $\ell \gg \lambda/4\pi$. In this limit, the de Broglie wavelength in the direction perpendicular to the interface is much smaller than the thickness of the interface. It is thus expected that the density of states in the high-energy limit inside the interface resembles the two-dimensional density of states for a free particle.

We can use this argument in the quasirelativistic inter-dimensional system as well. The de Broglie wavelength equation holds in the relativistic case as long as we use the appropriate expression for momentum p. We know that relativistic particles follow the relativistic dispersion relation,

$$E^2 = p^2c^2 + m^2c^4.$$

Solving the equation for pc and inserting that into the definition of the de Broglie wavelength,

$$\lambda = \frac{hc}{pc},$$

results in

$$\lambda = \frac{2\pi\hbar c}{\sqrt{E^2 - m^2c^4}}. \tag{4.17}$$

Setting $|\Delta m^2| = 0$ in the low energy inequality $|\Delta m^2 c^4| \ll E^2 - m^2c^4 \ll \hbar^2c^2/\ell^2$ allows us to rearrange it in terms of the interface length parameter ℓ and the de Broglie wavelength. Isolating for ℓ and substituting our expression in the de Broglie wavelength yield

$$\begin{aligned} \ell = \sqrt{\ell^2} &\ll \sqrt{\frac{\hbar^2c^2}{E^2 - m^2c^4}} \\ &= \frac{\hbar c}{\sqrt{E^2 - m^2c^4}} \\ &= \frac{\lambda}{2\pi}. \end{aligned} \tag{4.18}$$

Thus in the low-energy limit, the de Broglie wavelength is much larger than the interface length parameter. In terms of the de Broglie wavelength in the direction perpendicular to the interface, this tells us that the wavelength extends past the edges of the interface into

the three-dimensional bulk. It then makes intuitive sense that as the boson's de Broglie wavelength extends into the bulk, the boson effectively sees more bulk than interface, and its density of states in the interface approaches the three-dimensional density of states.

We can apply this reasoning to the high-energy case as well. Once again we will use $|\Delta m^2| = 0$. The high-energy inequality is given by $E^2 - m^2c^4 \gg \hbar^2c^2/\ell^2$. Isolating for ℓ and inserting our expression for the de Broglie wavelength yield

$$\begin{aligned} \ell = \sqrt{\ell^2} \quad &>> \quad \sqrt{\frac{\hbar^2c^2}{E^2 - m^2c^4}} \\ &= \quad \frac{\hbar c}{\sqrt{E^2 - m^2c^4}} \\ &= \quad \frac{\lambda}{2\pi}. \end{aligned} \tag{4.19}$$

Therefore in the high-energy limit, the de Broglie wavelength is much smaller than the interface length parameter. Considering the de Broglie wavelength in the direction of the interface thickness, this tells us that the entire de Broglie wavelength of the boson resides inside the interface. It therefore makes intuitive sense that because the de Broglie wavelength is located entirely inside the interface, the density of states of the boson inside the interface would approach the two-dimensional density of states.

Both the high-energy and low-energy limits, as well as the $\ell \to 0$ limit, of the quasirelativistic inter-dimensional density of states behave exactly as in the non-relativistic case [1]. As well, we have confirmed that the relativistic interface length parameter ℓ is equal to 2ℓ in the non-relativistic case [1]. This can be seen by the respective high-energy and low-energy inequalities in quasirelativistc and non-relativistic systems. In the non-relativistic case the comparison between the interface length parameter ℓ_{nr} and the de Broglie wavelength λ was $\ell_{nr} \ll \lambda/4\pi$ in the low-energy limit and $\ell_{nr} \gg \lambda/4\pi$ in the high-energy limit. Therefore we can see that the only difference between the inequalities for high-energy and low-energy in the non-relativistic case is the 4π in the denominator of the term $\lambda/4\pi$. However if we use $\ell_{rel} = 2\ell_{nr}$ which we will also derive in Sec. 4.3, we see that the inequalities between the interface length parameter and the de Broglie wavelength are identical. As shown in Sec. 4.3, the condition $\ell_{rel} = 2\ell_{nr}$ is a result of the interface term in the non-relativistic limit of the of the quasirelativistic Hamiltonian necessarily being equal to the interface term from the inter-dimensional Hamiltonian [1].

4.3 Non-relativistic limit of the inter-dimensional quasi-relativistic system

In this section we will analyze the non-relativistic limit of the inter-dimensional relativistic Hamiltonian, the Green's function matrix element in $\boldsymbol{x}$ representation, $\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle$, and the density of states. To begin we will show that in the non-relativistic limit, $K = E - mc^2 \ll mc^2$, the energy-dependent Green's function matrix element from the relativistic inter-dimensional Hamiltonian reduces to the energy-dependent Green's function matrix element from the non-relativistic inter-dimensional system. The non-relativistic limit of the relativistic wave frequency

$$\omega_{\boldsymbol{k}} = c\sqrt{\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}}$$

is given by

$$\omega_{\boldsymbol{k}} \simeq \frac{mc^2}{\hbar} + \frac{\hbar\boldsymbol{k}^2}{2m}. \tag{4.20}$$

Denoting the non-relativistic kinetic energy K as

$$K = \frac{\hbar^2\boldsymbol{k}^2}{2m},$$

we have

$$\omega_{\boldsymbol{k}} \simeq \frac{mc^2}{\hbar} + \frac{K}{\hbar}.$$

Using the defintion $ck^0 = \omega_{\boldsymbol{k}}$ allows us to write k^0 as

$$k^0 = \frac{\omega_{\boldsymbol{k}}}{c} \simeq \frac{mc}{\hbar} + \frac{K}{\hbar c}. \tag{4.21}$$

Applying this to the expression for $\langle z|G(k^0, k_\parallel)|z'\rangle$ requires that we calculate the non-relativistic value for $(k^0)^2 - k_\parallel^2 - (mc/\hbar)^2$ and $k_\parallel^2 + (mc/\hbar)^2 - (k^0)^2$. For $(k^0)^2$ we have

$$(k^0)^2 = \frac{m^2c^2}{\hbar^2} + \frac{2mK}{\hbar^2} + \frac{K^2}{\hbar^2c^2}.$$

The $\mathcal{O}(K^2)$ term is very small compared to the other two terms and can be disregarded in the non-relativistic limit. Inserting this expression for $(k^0)^2$ into $(k^0)^2 - k_\parallel^2 - (mc/\hbar)^2$ and $k_\parallel^2 + (mc/\hbar)^2 - (k^0)^2$ yields

$$(k^0)^2 - k_\parallel^2 - (mc/\hbar)^2 = \frac{2mK}{\hbar^2} - k_\parallel^2$$

and

$$k_\parallel^2 + (mc/\hbar)^2 - (k^0)^2 = k_\parallel^2 - \frac{2mK}{\hbar^2}.$$

Applying these results to $\langle z|G(k^0, k_\parallel)|z'\rangle$ yields

$$\begin{aligned}
\langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle &= i\frac{\hbar\Theta[2mK - \hbar^2\boldsymbol{k}_\parallel^2]}{2\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2}}\left[\exp\left(i\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2}\frac{|z-z'|}{\hbar}\right)\right. \\
&- i\frac{\ell\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)}{(2/\hbar)\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2} + i\ell\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)} \\
&\times \left.\exp\left[+i\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2}\frac{(|z'-z_0| + |z-z_0|)}{\hbar}\right]\right] \\
&+ \frac{\hbar\Theta[\hbar^2\boldsymbol{k}_\parallel^2 - 2mK]}{2\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK}}\left[\exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK}\frac{|z-z'}{\hbar}|\right)\right. \\
&- \frac{\ell\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)}{(2/\hbar)\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK} + \ell\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)} \\
&\times \left.\exp\left[-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK}\frac{(|z'-z_0| + |z-z_0|)}{\hbar}\right]\right]. \qquad (4.22)
\end{aligned}$$

Multiplying the ℓ dependent terms by $1 = \hbar/\hbar$ yields

$$\begin{aligned}
\langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle &= i\frac{\hbar\Theta[2mK - \hbar^2\boldsymbol{k}_\parallel^2]}{2\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2}}\left[\exp\left(i\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2}\frac{|z-z'|}{\hbar}\right)\right. \\
&- i\frac{\ell\hbar\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)}{2\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2} + i\ell\hbar\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)} \\
&\times \left.\exp\left[+i\sqrt{2mK - \hbar^2\boldsymbol{k}_\parallel^2}\frac{(|z'-z_0| + |z-z_0|)}{\hbar}\right]\right] \\
&+ \frac{\hbar\Theta[\hbar^2\boldsymbol{k}_\parallel^2 - 2mK]}{2\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK}}\left[\exp\left(-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK}\frac{|z-z'}{\hbar}|\right)\right. \\
&- \frac{\ell\hbar\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)}{2\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK} + \ell\hbar\left(\boldsymbol{k}_\parallel^2 + \Delta m^2(c/\hbar)^2\right)} \\
&\times \left.\exp\left[-\sqrt{\hbar^2\boldsymbol{k}_\parallel^2 - 2mK}\frac{(|z'-z_0| + |z-z_0|)}{\hbar}\right]\right]. \qquad (4.23)
\end{aligned}$$

The non-relativistic kinetic energy K is the same as the non-relativistic energy E used in the energy-dependent Green's function discussed in Sec. 2.3. Comparing the non-relativistic limit

of $\langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle$ to the non-relativistic energy-dependent Green's function $\langle z|G(E, \boldsymbol{k}_\parallel)|z'\rangle$ from Sec. 2.3 shows that in order for the two expressions to agree, $\Delta m^2 = 0$ in the non-relativistic limit. Calculating $\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle$ in both the non-relativistic and relativistic cases is the same. Therefore the non-relativistic expression for $\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle$ in the relativistic case equals the non-relativistic value of $\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle$ provided that $\Delta m^2 = 0$ in the non-relativistic limit.

We can apply the same non-relativistic approximations to the results for the inter-dimensional relativistic density of states calculated in Sec. 3.2. In particular this will involve using the non-relativistic approximation

$$E = c\hbar k^0 \simeq m^2c^4 + 2mc^2K, \tag{4.24}$$

where once again we have neglected the $\mathcal{O}(K^2)$ term. Applying this to the density of states calculated in Sec. 3.2 involves finding the non-relativistic limit of the expression $E^2 - m^2c^4$. In the non-relativistic limit this becomes

$$E^2 - m^2c^4 = c^2\hbar^2(k^0)^2 - m^2c^4 \simeq 2mc^2K. \tag{4.25}$$

Thus the function $-g(E, \ell, m, \Delta m^2)$ becomes

$$-g(E, \ell, m, \Delta m^2) = \hbar^2c^2 - \ell^2(E^2 - m^2c^4 + \Delta m^2c^4) \simeq \hbar^2c^2 - \ell^2(2mc^2K + \Delta m^2c^4). \tag{4.26}$$

As well, in the non-relativistic limit, the relativistic energy reduces to $E \simeq mc^2$. Applying these approximations to our result for the density of states in Sec. 3.2 yields

$$\begin{aligned}
\varrho(E, z_0) - \overline{\varrho}(\overline{E}, z_0) \simeq \varrho(K, z_0) \quad &\simeq \quad \frac{m\Theta(K)}{2(\pi\hbar)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2c^2)}}\right)\right. \\
&+ \quad \left.\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2c^2)}}\right)\right] \\
&+ \quad \frac{m\Theta(K)}{2\pi^2\hbar\ell\sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2c^2)}} \\
&\times \quad \left[\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2c^2)}}\right)\right. \\
&- \quad \left.\left.\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2c^2)}}\right)\right)\right].
\end{aligned} \tag{4.27}$$

Analyzing the same limiting cases for the relativistic density of states but now in the non-relativistic limit amounts to analyzing the cases $\ell \to 0$, $|\Delta m^2 c^4| \ll 2mc^2 K \ll \hbar^2 c^2/\ell^2$, and $2mc^2 K \gg (\hbar c/\ell)^2 - \Delta m^2 c^4$. First we analyze the case $\ell \to 0$ in the non-relativistic limit. In this case the expression $\sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}$ reduces to $\hbar$. Applying this to the non-relativistic limit for the density of states yields for the density of states,

$$\begin{aligned} \varrho(K, z_0)|_{\ell\to 0} &\simeq \frac{m\Theta(K)}{2(\pi\hbar)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{2mK}}{2\hbar}\right) + \arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - \hbar}\right)\right] \\ &+ \frac{m\Theta(K)}{2\pi^2\hbar^2\ell}\left[\arctan\left(\frac{\ell\sqrt{2mK}}{2\hbar}\right) - \arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - \hbar}\right)\right)\right]. \end{aligned} \tag{4.28}$$

The arctan terms with vanishing denominator cancel each other, and the remaining terms add to give

$$\varrho(K, z_0)|_{\ell\to 0} \simeq \frac{m\Theta(K)}{(\pi\hbar)^2\ell}\arctan\left(\frac{\ell\sqrt{2mK}}{2\hbar}\right).$$

The argument of the arctan function approaches zero as $\ell \to 0$, therefore we can approximate the arctan function using the Taylor series expansion and keeping only the first term. This yields for the density of states

$$\varrho(K, z_0)|_{\ell\to 0} \simeq \Theta(K)\frac{\sqrt{m^3 K}}{\sqrt{2}\pi^2\hbar^3} = \frac{1}{2}\varrho_{d=3}(K). \tag{4.29}$$

The factor 1/2 exists because in the non-relativistic case discussed in Sec. 2.4 we explicitly included an extra factor of 2 for the number of helicity states available to the electron. Therefore multiplying our result by 2 shows that in the limit $\ell \to 0$ the relativistic density of states equals the three-dimensional non-relativistic density of states in the non-relativistic limit.

The next case to consider is the inequality $|\Delta m^2 c^4| \ll 2mc^2 K \ll \hbar^2 c^2/\ell^2$. In this case the expression $\sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}$ reduces to $\hbar$. Applying this to the non-relativistic limit of the relativistic density of states yields again for the density of states,

$$\varrho(K, z_0) \simeq \frac{m\Theta(K)}{(\pi\hbar)^2\ell}\arctan\left(\frac{\ell\sqrt{2mK}}{2\hbar}\right).$$

From the inequality we know that the argument of the arctan function is very small, so we once again keep only the first term in the Taylor series expansion. Remembering the extra

factor of 2 missing from the density of states when comparing with the result from Sec. 2.4 shows that when $|\Delta m^2c^4| \ll 2mc^2K \ll \hbar^2c^2/\ell^2$, the relativistic density of states reduces to the three-dimensional non-relativistic density of states in the non-relativistic limit.

The last case to consider is the inequality $2mc^2K \gg (\hbar c/\ell)^2 - \Delta m^2c^4$. In this case, the expression $\sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2c^2)}$ reduces to $i\ell\sqrt{2mK}$. Applying this to the non-relativistic limit of the relativistic density of states yields for the density of states

$$\begin{aligned}\varrho(K, z_0) &\simeq \frac{m\Theta(K)}{2(\pi\hbar)^2\ell}\left[\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + i\ell\sqrt{2mK}}\right)\right. \\ &+ \left.\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - i\ell\sqrt{2mK}}\right)\right] \\ &+ \frac{m\Theta(K)}{2\pi^2\hbar\ell^2 i\sqrt{2mK}}\left[\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + i\ell\sqrt{2mK}}\right)\right. \\ &- \left.\left.\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - i\ell\sqrt{2mK}}\right)\right)\right]. \end{aligned} \tag{4.30}$$

Using the arctan sum and difference identities given in Sec. 3.2 we can simplify the result for the density of states. Using the sum identity on the first two arctan functions yields

$$\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + i\ell\sqrt{2mK}}\right) + \arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - i\ell\sqrt{2mK}}\right) = \arctan\left(\frac{\ell\sqrt{8mK}}{\hbar}\right).$$

Thus given the inequality $2mc^2K \gg (\hbar c/\ell)^2 - \Delta m^2c^4$, the argument of the arctan function is very large. The arctan function therefore can be approximated as $\pi/2$ just as in Sec. 3.2.

The arctan difference identity can be used on the second two arctan functions. Using the identity yields

$$\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + i\ell\sqrt{2mK}}\right) - \arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - i\ell\sqrt{2mK}}\right) = \arctan\left(\frac{-4i\ell^2mK}{\hbar^2 + 4\ell^2mK}\right).$$

In order to write this result in terms of a real function we again use the identity,

$$\frac{1}{2i}\ln\left(\frac{x + iy}{x - iy}\right) = arctan\left(\frac{y}{x}\right).$$

Using the identity yields

$$\arctan\left(\frac{-4i\ell^2 mK}{\hbar^2+4\ell^2 mK}\right)=\frac{1}{2i}\ln\left(\frac{\hbar^2+8\ell^2 mK}{\hbar^2}\right)\simeq\frac{1}{2i}\ln\left(\frac{8\ell^2 mK}{\hbar^2}\right),$$

where we have imposed the inequality $2mc^2K\gg(\hbar c/\ell)^2-\Delta m^2c^4$. Using the power rule for logarithms yields

$$\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar+i\ell\sqrt{2mK}}\right)-\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar-i\ell\sqrt{2mK}}\right)\simeq\frac{1}{i}\ln\left(\frac{|\ell|\sqrt{8mK}}{\hbar}\right).$$

Inserting our reduced expressions for the arctan terms present in the density of states yields

$$\begin{aligned}\varrho(K,z_0) &\simeq \frac{m\Theta(K)}{4\pi\hbar^2|\ell|}-\frac{\sqrt{m}\Theta(K)}{\pi^2\hbar\ell^2\sqrt{8K}}\ln\left(\frac{|\ell|\sqrt{8mK}}{\hbar}\right)\\ &= \frac{\varrho_{d=2}}{2|\ell|}-\frac{\sqrt{m}\Theta(K)}{\pi^2\hbar\ell^2\sqrt{8K}}\ln\left(\frac{|\ell|\sqrt{8mK}}{\hbar}\right),\end{aligned}\tag{4.31}$$

where the first term has had the extra factor of two inserted that is present in the non-relativistic case from Sec. 2.3.

From Ref. [34] and Sec. 2.6 the solution to the free Klein-Gordon equation is given by

$$\begin{aligned}\phi(\boldsymbol{x},t) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3\boldsymbol{k}}{\sqrt{2\omega_{\boldsymbol{k}}}}\Big(a(\boldsymbol{k})\exp[i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)]\\ &+ b^\dagger(\boldsymbol{k})\exp[-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)]\Big).\end{aligned}\tag{4.32}$$

In the non-relativistic limit, namely $\hbar|\boldsymbol{k}|\ll mc$, the relativistic frequency can be approximated using the Taylor expansion for $\sqrt{1+x}$, which is given by

$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\ldots\quad,$$

where the series converges for $x\leq|1|$. For $x\ll1$ the series can be truncated after the linear order term. Applying this approximation to the relativistic frequency yields

$$\begin{aligned}\omega_{\boldsymbol{k}}=\sqrt{c^2\boldsymbol{k}^2+\frac{m^2c^4}{\hbar^2}} &= \frac{mc^2}{\hbar}\sqrt{\frac{\hbar^2\boldsymbol{k}^2}{m^2c^2}+1}\\ &\approx \frac{mc^2}{\hbar}\left(1+\frac{\hbar^2\boldsymbol{k}^2}{2m^2c^2}\right)\\ &= \frac{mc^2}{\hbar}+\frac{\hbar\boldsymbol{k}^2}{2m}.\end{aligned}\tag{4.33}$$

The non-relativistic limit of the complex scalar field solution to the free Klein-Gordon equation must reduce to a solution proportional to the free Schrödinger field. Applying the non-relativistic approximation to the relativistic frequency in the solution to the free Klein-Gordon equation yields

$$\begin{aligned}\phi(\boldsymbol{x},t) &\approx \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\boldsymbol{k}\sqrt{\frac{\hbar}{2mc^2}}\Bigg(a(\boldsymbol{k})\exp\left[i\left(\boldsymbol{k}\cdot\boldsymbol{x}-\left(\frac{mc^2}{\hbar}+\frac{\hbar\boldsymbol{k}^2}{2m}\right)t\right)\right]\\ &+ b^\dagger(\boldsymbol{k})\exp\left[-i\left(\boldsymbol{k}\cdot\boldsymbol{x}-\left(\frac{mc^2}{\hbar}+\frac{\hbar\boldsymbol{k}^2}{2m}\right)t\right)\right]\Bigg). \qquad (4.34)\end{aligned}$$

In order to have the standard time evolution of the free Schrödinger field, $\psi\sim\exp(-i\hbar\boldsymbol{k}^2t/2m)$, we extract the rest mass term $\exp(-imc^2t/\hbar)$ from each of the two exponential terms [34]. The first exponential term indeed has the correct time evolution after extraction of the rest mass term, the second exponential term however has an even faster oscillating term $\exp(2imc^2t/\hbar)$ after extraction of the rest mass term. It is therefore necessary to compare the amplitude $|\langle b^\dagger(\boldsymbol{k})\rangle|$ with the amplitude $|\langle a(\boldsymbol{k})\rangle|$. Therefore extraction of the rest mass term implies that in the non-relativistic limit, we must have $|\langle b^\dagger(\boldsymbol{k})\rangle|\ll|\langle a(\boldsymbol{k})\rangle|$. Applying all of this to the solution to the free Klein-Gordon equation yields

$$\phi(\boldsymbol{x},t)\simeq\sqrt{\frac{\hbar}{2mc^2}}\psi(\boldsymbol{x},t)\exp\left(-i\frac{mc^2}{\hbar}t\right), \qquad (4.35)$$

where the free Schrödinger field $\psi(\boldsymbol{x},t)$ is given by

$$\psi(\boldsymbol{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\boldsymbol{k}a(\boldsymbol{k})\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x}-\frac{i\hbar\boldsymbol{k}^2t}{2m}\right). \qquad (4.36)$$

Applying this approximation to the quasi-relativistic inter-dimensional Hamiltonian requires that we calculate the first-order time derivatives of the wavefunction $\phi(\boldsymbol{x},t)$. Therefore for the time derivative we have

$$\dot{\phi}(\boldsymbol{x},t)=\frac{\partial}{\partial t}\phi(\boldsymbol{x},t)\simeq\sqrt{\frac{\hbar}{2mc^2}}\left(\frac{-imc^2}{\hbar}\psi(\boldsymbol{x},t)+\frac{\partial\psi(\boldsymbol{x},t)}{\partial t}\right)\exp\left(-i\frac{mc^2}{\hbar}t\right).$$

The $\partial\psi(\boldsymbol{x},t)/\partial t$ term can be re-written using the Schrödinger equation:

$$\frac{\partial}{\partial t}\phi(\boldsymbol{x},t)\simeq\sqrt{\frac{\hbar}{2mc^2}}\left(\frac{-imc^2}{\hbar}\psi(\boldsymbol{x},t)+\frac{i\hbar}{2m}\boldsymbol{\Delta}\psi(\boldsymbol{x},t)\right)\exp\left(-i\frac{mc^2}{\hbar}t\right).$$

Given the $\boldsymbol{x}$ dependence of the free Schrödinger field, and the non-relativistic inequality $\hbar^2|\boldsymbol{k}|^2 \ll mc^2$, we can keep only the first of the two free Schrödinger field terms in the non-relativistic expression for $\dot{\phi}(\boldsymbol{x},t)$. We then have

$$\frac{\partial}{\partial t}\phi(\boldsymbol{x},t) \simeq \sqrt{\frac{\hbar}{2mc^2}}\left(\frac{-imc^2}{\hbar}\psi(\boldsymbol{x},t)\right)\exp\left(-i\frac{mc^2}{\hbar}t\right). \tag{4.37}$$

Inserting our expressions for $\phi(\boldsymbol{x},t)$ and $\partial\phi(\boldsymbol{x},t)/\partial t$ as well as their Hermitian conjugates into the quasirelativistic inter-dimensional Hamiltonian

$$\begin{aligned} H &= \frac{1}{\hbar}\int d^2\boldsymbol{x}_\parallel \int dz\Big[\hbar^2\dot{\phi}^\dagger(\boldsymbol{x}_\parallel,z,t)\dot{\phi}(\boldsymbol{x}_\parallel,z,t) + \hbar^2c^2\boldsymbol{\nabla}\phi^\dagger(\boldsymbol{x}_\parallel,z,t)\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x}_\parallel,z,t) \\ &+ m^2c^4\phi^\dagger(\boldsymbol{x}_\parallel,z,t)\phi(\boldsymbol{x}_\parallel,z,t)\Big] \\ &+ \frac{\ell}{\hbar}\int d^2\boldsymbol{x}_\parallel\Big[\hbar^2c^2\boldsymbol{\nabla}_\parallel\phi^\dagger(\boldsymbol{x}_\parallel,z_0,t)\cdot\boldsymbol{\nabla}_\parallel\phi(\boldsymbol{x}_\parallel,z_0,t) \\ &+ \Delta m^2c^4\phi^\dagger(\boldsymbol{x}_\parallel,z_0,t)\phi(\boldsymbol{x}_\parallel,z_0,t)\Big] \end{aligned} \tag{4.38}$$

yields

$$H \simeq \int d^2\boldsymbol{x}_\parallel \int dz\left[\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{x}_\parallel,z)\cdot\boldsymbol{\nabla}\psi(\boldsymbol{x}_\parallel,z) + mc^2\psi^\dagger(\boldsymbol{x}_\parallel,z)\cdot\psi(\boldsymbol{x}_\parallel,z)\right]$$

$$+\ell\int d^2\boldsymbol{x}_\parallel\left[\frac{\hbar^2}{2m}\boldsymbol{\nabla}_\parallel\psi^\dagger(\boldsymbol{x}_\parallel,z_0)\cdot\boldsymbol{\nabla}_\parallel\psi(\boldsymbol{x}_\parallel,z_0) + \frac{\Delta m^2c^2}{2m}\psi^\dagger(\boldsymbol{x}_\parallel,z_0)\cdot\psi(\boldsymbol{x}_\parallel,z_0)\right],$$

where we have reduced to Schrödinger picture operators. We notice that the Δm^2 term does not reduce to the parabolic band approximation with effective mass $m_* = m \pm \sqrt{|\Delta m^2|}$ for motion inside the interface. The rest-mass energy term is present because we began with a relativistic theory. It can be removed by removing the phase factor term $\exp(-imc^2t/\hbar)$ from the wavefunction $\psi(\boldsymbol{x},t)$. In order to recover the two-dimensional Schrödinger Hamiltonian for the interface term in the non-relativistic limit, we require that the term involving Δm^2 is small compared to the kinetic energy interface terms.

In Ref. [1], ℓ is defined as $\ell \equiv Lm/2m_*$. Comparing our non-relativistic approximation for the quasirelativistic inter-dimensional Hamiltonian with the non-relativistic inter-dimensional Hamiltonian from Ref. [1]

$$H = \int d^2\boldsymbol{x}_\parallel \int dz\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{x}_\parallel,z)\cdot\boldsymbol{\nabla}\psi(\boldsymbol{x}_\parallel,z) + \int d^2\boldsymbol{x}_\parallel\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_\parallel\psi^\dagger(\boldsymbol{x}_\parallel,z_0)\cdot\boldsymbol{\nabla}_\parallel\psi(\boldsymbol{x}_\parallel,z_0),$$

shows that for the two Hamiltonians to be equivalent, ℓ in the non-relativistic approximation Hamiltonian must be 2ℓ in the non-relativistic Hamiltonian from Ref. [1]. To see this let ℓ_{rel} be the ℓ term from the non-relativistic approximation Hamiltonian and let ℓ_{nr} be the ℓ term defined in Ref. [1]. Therefore for the kinetic energy term in the interface part of the non-relativistic approximation Hamiltonian we have

$$\ell_{rel}\frac{\hbar^2}{2m} = \frac{\hbar^2}{2\mu} \equiv \frac{\hbar^2 L}{2m_*} = \ell_{nr}\frac{\hbar^2}{m}.$$

Therefore in order for the two terms to be equal, we must have that $2\ell_{nr} = \ell_{rel}$.

4.4 Discussion on results from non-relativistic limit of inter-dimensional quasirelativistic system

The non-relativistic approximation to the quasirelativistic inter-dimensional Hamiltonian found in Sec. 4.3 is given by

$$\begin{aligned} H &\simeq \int d^2\boldsymbol{x}_\parallel \int dz \left[\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{x}_\parallel, z)\cdot\boldsymbol{\nabla}\psi(\boldsymbol{x}_\parallel, z) + mc^2\psi^\dagger(\boldsymbol{x}_\parallel, z)\cdot\psi(\boldsymbol{x}_\parallel, z)\right] \\ &+ \ell\int d^2\boldsymbol{x}_\parallel \left[\frac{\hbar^2}{2m}\boldsymbol{\nabla}_\parallel\psi^\dagger(\boldsymbol{x}_\parallel, z_0)\cdot\boldsymbol{\nabla}_\parallel\psi(\boldsymbol{x}_\parallel, z_0) + \frac{\Delta m^2 c^2}{2m}\psi^\dagger(\boldsymbol{x}_\parallel, z_0)\cdot\psi(\boldsymbol{x}_\parallel, z_0)\right] \end{aligned} \quad (4.39)$$

It contains rest mass terms in the bulk and interface that are present only because we began with a relativistic theory. The interface term containing the change in the bulk gap parameter Δm^2, does not reduce to a parabolic band approximation term for motion in the interface with the effective mass m_*, which is what we would expect given the non-relativistic inter-dimensional Hamiltonian given in Ref. [1]. If the non-relativistic approximation interface term did reduce to a parabolic band approximation term with the effective mass m_* for motion in the interface, we would expect the rest mass energy term in the interface would be given by

$$\ell\int d^2\boldsymbol{x}_\parallel m_* c^2\psi^\dagger(\boldsymbol{x}_\parallel, z_0)\cdot\psi(\boldsymbol{x}_\parallel, z_0).$$

The rest mass energy term present in the interface term in the non-relativistic approximation of the quasirelativistic Hamiltonian would only reduce to the expected rest mass energy term for a parabolic band approximation if the change in the bulk gap parameter Δm^2 were equal

to m_*^2. From our definition $m_* = m \pm \sqrt{|\Delta m^2|}$, this is clearly not the case. Therefore, in order for us to use a two-dimensional Schrödinger Hamiltonian for the interface in the non-relativistic approximation of the quasirelativistic Hamiltonian, we require that the change in bulk gap parameter Δm^2 term be small compared to the interface term containing the gradients of the Schrödinger picture operators. Under this constraint, the quasirelativistic inter-dimensional Hamiltonian reduces to the non-relativistic inter-dimensional Hamiltonian given in Ref. [1].

In terms of the kinetic energy $K = E - mc^2 \ll mc^2$, the Δm^2 term effectively interferes with a parabolic band approximation for both the bulk and interface in the form $2mK + \Delta m^2 c^2$. The term $2mK + \Delta m^2 c^2$ is present in all expressions for the density of states after substitution of the appropriate non-relativistic approximation for (k^0) in the expression for the quasirelativistic inter-dimensional density of states,

$$\begin{aligned}
\varrho(K, z_0) &\simeq \frac{m\Theta(K)}{2(\pi\hbar)^2\ell}\Bigg[\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}}\right) \\
&+ \arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}}\right)\Bigg] \\
&+ \frac{m\Theta(K)}{2\pi^2\hbar\ell\sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}} \\
&\times \Bigg[\arctan\left(\frac{\ell\sqrt{2mK}}{\hbar + \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}}\right) \\
&- \arctan\left(\frac{\ell\sqrt{2mK}}{\hbar - \sqrt{\hbar^2 - \ell^2(2mK + \Delta m^2 c^2)}}\right)\Bigg]. \qquad (4.40)
\end{aligned}$$

The $\Delta m^2 c^2$ term contains the relativistic speed c, which we would like to not be present in the non-relativistic limit of the quasirelativistic density of states. This means that if we are to have simultaneous applicability of the parabolic band approximation in the bulk and interface, the following inequality must hold:

$$\frac{|\Delta m^2| c^2}{m} \ll K \ll mc^2. \qquad (4.41)$$

This energy domain exists provided that the change in the energy gap between the Dirac cones in the interface $|\Delta m^2|$ is much smaller than the bulk gap parameter m. Therefore we require $|\Delta m^2| \ll m^2$ for simultaneous parabolic band approximation to be valid in both the bulk and interface.

Chapter 5

Conclusion

5.1 Conclusion

We have analyzed the inter-dimensional quasirelativistic system in which bosons propagate in a three-dimensional system with a two-dimensional interface. We have derived the respective Green's function and the density of states for this system using an inter-dimensional relativistic Hamiltonian using the assumption that the presence of the two-dimensional interface only affects the kinetic and rest mass energy terms. This assumption is consistent with the work done in previous inter-dimensional nonrelativistic systems [34, 1, 31, 30, 32]. In Sec. 3.2 we derived the fully analytic result for the inter-dimensional quasirelativistic density of states inside the interface $\varrho(E, z_0)$. In Chapter 4 we derived and analyzed the density of states inside the interface for both large and small energies. We have found that the density of states in the interface approaches the free three-dimensional relativistic density of states in the low-energy limit, provided that the change in the bulk gap parameter Δm^2 satisfies the inequality $\ell^2|\Delta m^2| \ll \hbar^2/c^2$. We have shown that in the low-energy limit, the de Broglie wavelength in the direction of the interface thickness satisfies $\lambda/2\pi \gg |\ell|$. In Chapter 4 we argued that this inequality suggests that the three-dimensional behaviour of the density of states in the interface in the low-energy limit is a result of the de Broglie wavelength extending into the bulk region outside of the interface. We have also found in Chapter 4 that as the scaling parameter for the interface thickness ℓ approaches zero, the density of states in the interface approaches the free three-dimensional relativistic density of states provided that the change in bulk gap parameter is again small. In the high-energy limit, we have found that the density of states in the interface approaches the free two-dimensional relativistic

density of states plus a logarithmic correction term, even when the change in the bulk gap parameter is not small. Similar to the low-energy case, we have shown in Chapter 4 that in the high-energy limit the de Broglie wavelength in the direction of interface thickness satisfies the inequality $\lambda/2\pi \ll |\ell|$. We argued that the two-dimensional behaviour of the density of states in the interface is a result of the de Broglie wavelength residing entirely inside the interface.

We have derived the non-relativistic limits for the inter-dimensional quasirelativistic Hamiltonian and for the inter-dimensional quasirelativistic density of states in the interface. In the non-relativistic limit for the density of states in the interface, we have found that for large kinetic energies and small change in the bulk gap parameter, the density of states in the interface approaches the free two-dimensional non-relativistic density of states with a logarithmic correction term. Similarly, we have found that for small kinetic energies the relativistic density of states in the interface approaches the free three-dimensional non-relativistic density of states. The non-relativistic limit of the relativistic density of states in the interface also approaches the free three-dimensional non-relativistic density of states in the limit that the interface scaling parameter ℓ approaches zero. The non-relativistic limit of the quasirelativistic inter-dimensional Hamiltonian reduces to the inter-dimensional non-relativistic Hamiltonian from Ref. [1] only when the change in the bulk gap parameter is small compared to the kinetic energy terms in the interface part of the Hamiltonian. This is because in the non-relativistic limit, the Δm^2 term does not reduce to the parabolic band approximation term with effective mass $m_* = m_*(m, \Delta m^2)$. The bulk term in the non-relativistic limit is in agreement with the parabolic band approximation from Ref. [1]. The non-relativistic limit of the inter-dimensional quasirelativistic Hamiltonian also yields the result that the interface thickness scaling parameter (ℓ_{rel}) in the quasirelativistic case corresponds to twice the interface scaling parameter ($2\ell_{nr}$) in the non-relativistic inter-dimensional Hamiltonian. This correspondence manifests itself in the de Broglie wavelength versus the interface scaling parameter comparisons in the relativistic case as compared to the non-relativistic case. As well, we see this result in the non-relativistic limit of the relativistic inter-dimensional density of states for both small and large kinetic energies. We have shown that the presence of the change in bulk gap parameter Δm^2 lingers in the non-relativistic

limit of the inter-dimensional quasirelativistic density of states in the interface in the form $2mc^2K + \Delta m^2c^2$.

We have used a bulk gap parameter $\Delta_g = 95$ meV in order to present our results for the inter-dimensional quasirelativistic density of states. This value corresponds to the experimental value for the bulk band gap in [2] for PbTe which was verified theoretically in Ref. [2]. We have shown that for a change in the bulk gap parameter $\Delta m^2c^4 = -\Delta_g^2$, the band gap in the interface closes as we see the presence of gap states. This result is derived in Chapter 4. The process of gap closure and then band inversion leads to the topological insulator for increasing x in the compound $Pb_{1-x}Sn_xTe$ with topologically-protected gapless surface states. While our analytic results do not explain this, it is our hope that the work done in this thesis will serve as a stepping stone to working with relativistic fermions in an inter-dimensional Hamiltonian so as to better replicate real systems of interest in condensed matter physics. Our results show that in the high-energy limit, the quasirelativistic inter-dimensional density of states always approaches the free two-dimensional relativistic density of states even without the presence of an attractive potential, which is a very interesting result.

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Appendix A

Derivation of energy dependent Green's function in $\boldsymbol{x}$ representation in the form of a Hankel trasform

A.1 Derivation of the Green's function

The full Fourier transform of the Green's function matrix element is

$$\langle x|G|x'\rangle = \langle \boldsymbol{x}_\parallel, z, t|G|\boldsymbol{x'}_\parallel, z', t'\rangle = \frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel \int dk^0 \int dk'^0 \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle$$

$$\times \exp\left[i\left(\boldsymbol{k}_\parallel\cdot\boldsymbol{x}_\parallel - \boldsymbol{k'}_\parallel\cdot\boldsymbol{x'}_\parallel - k^0x^0 + k'^0x'^0\right)\right].$$

Substitution of the Fourier transformed Green's function in a mixed representation into the equation that the Green's function matrix must satisfy yields

$$\left[\partial^2 - \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2c^2}{\hbar^2}\right)\right]$$

$$\times\frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel \int dk^0 \int dk'^0 \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle$$

$$\times exp\left[i\left(\boldsymbol{k}_\parallel\cdot\boldsymbol{x}_\parallel - \boldsymbol{k'}_\parallel\cdot\boldsymbol{x'}_\parallel - k^0x^0 + k'^0x'^0\right)\right] = -\delta(x-x').$$

The delta function $\delta(x-x')$ can be written in the integral representation as

$$\begin{aligned}\delta(x-x') = \delta(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel)\delta(z-z')\delta(x'^0 - x^0) &= \frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int dk^0 \exp\left[i\boldsymbol{k}_\parallel\cdot\left(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel\right)\right] \\ &\times \exp\left[-ik^0\left(x^0 - x'^0\right)\right]\delta(z-z'). \qquad \text{(A.1)}\end{aligned}$$

Substituting this in the inter-dimensional Green's function equation we have

$$\begin{aligned}\left[\partial^2 - \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(\boldsymbol{\nabla}_\parallel^2 - \frac{\Delta m^2c^2}{\hbar^2}\right)\right] &\times \frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel \int dk^0 \int dk'^0 \\ &\times \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle \\ &\times \exp\left[i\left(\boldsymbol{k}_\parallel\cdot\boldsymbol{x}_\parallel - \boldsymbol{k'}_\parallel\cdot\boldsymbol{x'}_\parallel - k^0x^0 + k'^0x'^0\right)\right] \\ &= -\frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int dk^0 \exp\left[i\boldsymbol{k}_\parallel\cdot\left(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel\right)\right] \\ &\times \exp\left[-ik^0\left(x^0 - x'^0\right)\right]\delta(z-z'). \qquad \text{(A.2)}\end{aligned}$$

Expanding $\partial^2 = -\partial_0^2 + \boldsymbol{\nabla}_\parallel^2 + \partial_z^2$ and evaluating the derivatives yield

$$\frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel \int dk^0 \int dk'^0$$

$$\times \left[-\left(-ik^0\right)^2 + \left(i\boldsymbol{k}_\parallel\right)^2 + \partial_z^2 - \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(\left(i\boldsymbol{k}_\parallel\right)^2 - \frac{\Delta m^2 c^2}{\hbar^2}\right)\right]$$

$$\times \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle \exp\left[i\left(\boldsymbol{k}_\parallel\cdot\boldsymbol{x}_\parallel - \boldsymbol{k'}_\parallel\cdot\boldsymbol{x'}_\parallel - k^0x^0 + k'^0x'^0\right)\right]$$

$$= -\frac{1}{(2\pi)^3}\int d^2\boldsymbol{k}_\parallel \int dk^0 \exp\left[i\boldsymbol{k}_\parallel\cdot\left(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel\right)\right]\exp\left[-ik^0\left(x^0 - x'^0\right)\right]\delta(z-z').$$

Multiplying both sides of the equation by

$$\frac{1}{(2\pi)^3}\int d^2\boldsymbol{x}_\parallel \int d^2\boldsymbol{x'}_\parallel \int dx^0 \int dx'^0 \exp\left[i\left(\boldsymbol{\kappa'}_\parallel\cdot\boldsymbol{x'}_\parallel - \boldsymbol{\kappa}_\parallel\cdot\boldsymbol{x}_\parallel + \epsilon x^0 - \epsilon' x'^0\right)\right]$$

yields

$$\begin{aligned}
\frac{1}{(2\pi)^6}\int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel \int dk^0 \int dk'^0 \quad &\times \quad \left[(k^0)^2 - \boldsymbol{k}_\parallel^2 + \partial_z^2 - \frac{m^2c^2}{\hbar^2}\right. \\
&+ \quad \left.\ell\delta(z-z_0)\left(-\boldsymbol{k}_\parallel^2 - \frac{\Delta m^2c^2}{\hbar^2}\right)\right] \\
&\times \quad \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle \\
&\times \quad \exp\left[i\boldsymbol{x}_\parallel\cdot\left(\boldsymbol{k}_\parallel - \boldsymbol{\kappa}_\parallel\right)\right]\exp\left[i\boldsymbol{x'}_\parallel\cdot\left(\boldsymbol{\kappa'}_\parallel - \boldsymbol{k'}_\parallel\right)\right] \\
&\times \quad \exp\left[ix^0\left(\epsilon - k^0\right)\right]\exp\left[ix'^0\left(k'^0 - \epsilon'\right)\right] \\
&= \quad -\frac{1}{(2\pi)^6}\int d^2\boldsymbol{k}_\parallel \int dk^0 \int d^2\boldsymbol{x}_\parallel \\
&\times \quad \int d^2\boldsymbol{x'}_\parallel \int dt \int dt' \\
&\times \quad \exp\left[i\boldsymbol{x}_\parallel\cdot\left(\boldsymbol{k}_\parallel - \boldsymbol{\kappa}_\parallel\right)\right] \\
&\times \quad \exp\left[i\boldsymbol{x'}_\parallel\cdot\left(\boldsymbol{\kappa'}_\parallel - \boldsymbol{k}_\parallel\right)\right]\exp\left[ix^0\left(\epsilon - k^0\right)\right] \\
&\times \quad \exp\left[ix'^0\left(k^0 - \epsilon'\right)\right]\delta(z-z'). \qquad \text{(A.3)}
\end{aligned}$$

The position-space integrals can be eliminated by making use of the integral representation of the delta functions. Thus, the equation further simplifies to

$$\begin{aligned}
\int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel \int dk^0 \int dk'^0 \quad &\times \quad \left[(k^0)^2 - \boldsymbol{k}_\parallel^2 + \partial_z^2 - \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(-\boldsymbol{k}_\parallel^2 - \frac{\Delta m^2c^2}{\hbar^2}\right)\right] \\
&\times \quad \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle \\
&\times \quad \delta^2(\boldsymbol{k}_\parallel - \boldsymbol{\kappa}_\parallel)\delta^2(\boldsymbol{\kappa'}_\parallel - \boldsymbol{k'}_\parallel\delta(\epsilon - k^0)\delta(k'^0 - \epsilon') \\
&= \quad -\int d^2\boldsymbol{k}_\parallel \int dk^0\delta^2(\boldsymbol{k}_\parallel - \boldsymbol{\kappa}_\parallel)\delta^2(\boldsymbol{\kappa'}_\parallel - \boldsymbol{k}_\parallel)\delta(\epsilon - k^0) \\
&\times \quad \delta(k^0 - \epsilon')\delta(z-z'). \qquad \text{(A.4)}
\end{aligned}$$

Simplifying the remaining integrals by use of the delta functions and making the variable substitutions $\boldsymbol{\kappa}_\parallel \to \boldsymbol{k}_\parallel$, $\boldsymbol{\kappa'}_\parallel \to \boldsymbol{k'}_\parallel$, $\epsilon \to k^0$ and $\epsilon' \to k'^0$ yields

$$\begin{aligned}\left[\boldsymbol{k}_\parallel^2 - (k^0)^2 - \partial_z^2 + \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(\boldsymbol{k}_\parallel^2 + \frac{m_\parallel^2 c^2}{\hbar^2}\right)\right] &\times \langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle \\ &= \delta(\boldsymbol{k}_\parallel - \boldsymbol{k'}_\parallel)\delta(k^0 - k'^0) \\ &\times \delta(z-z'). \qquad \text{(A.5)}\end{aligned}$$

The matrix element of the inter-dimensional Green's function is translationally invariant in both the $\boldsymbol{k}_\parallel$ and k^0 plane. Therefore

$$\langle k^0, \boldsymbol{k}_\parallel, z|G|k'^0, \boldsymbol{k'}_\parallel, z'\rangle = \langle z|G(\boldsymbol{k}_\parallel, k^0)|z'\rangle\delta(\boldsymbol{k}_\parallel - \boldsymbol{k'}_\parallel)\delta(k^0 - k'^0)$$

and thus

$$\left[\boldsymbol{k}_\parallel^2 - (k^0)^2 - \partial_z^2 + \frac{m^2c^2}{\hbar^2} + \ell\delta(z-z_0)\left(\boldsymbol{k}_\parallel^2 + \frac{m_\parallel^2 c^2}{\hbar^2}\right)\right]\langle z|G(\boldsymbol{k}_\parallel, k^0)|z'\rangle = \delta(z-z').$$

Fourier transform of the Green's function matrix element with respect to z yields

$$\langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle = \int dk_\perp \langle z|k_\perp\rangle\langle k_\perp|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\int dk_\perp exp(ik_\perp z)\langle k_\perp|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle.$$

Once again using the integral representation of the delta functions, the Fourier transform of the inter-dimensional Green's function equation yields

$$\frac{1}{\sqrt{2\pi}}\int dk_\perp \left[\boldsymbol{k}_\parallel^2 - (k^0)^2 - \partial_z^2 + \frac{m^2c^2}{\hbar^2} + \frac{\ell}{2\pi}\int d\kappa_\perp \exp\left[i\kappa_\perp(z-z_0)\right]\left(\boldsymbol{k}_\parallel^2 + \frac{m_\parallel^2 c^2}{\hbar^2}\right)\right]$$

$$\times \exp(ik_\perp z)\langle k_\perp|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle = \frac{1}{(2\pi)}\int dk_\perp \exp\left[ik_\perp(z-z')\right]$$

Evaluating the z derivatives yields

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int dk_\perp \left[\boldsymbol{k}_\parallel^2 - (ik_\perp)^2 - (k^0)^2 + \frac{m^2c^2}{\hbar^2}\right] &\times \exp\left(ik_\perp z\right)\langle k_\perp|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle \\ &+ \frac{\ell}{2\pi}\int d\kappa_\perp \int dk_\perp \exp\left[i\kappa_\perp(z-z_0)\right]\exp\left(ik_\perp z\right) \\ &\times \left(\boldsymbol{k}_\parallel^2 + \frac{\Delta m^2 c^2}{\hbar^2}\right)\langle k_\perp|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle \\ &= \frac{1}{(2\pi)}\int dk_\perp \exp\left[ik_\perp(z-z')\right]. \qquad \text{(A.6)}\end{aligned}$$

Mulitplying both sides of the equation by $\int dz \exp\left(-ik'_\perp z\right)$ yields

$$\frac{1}{\sqrt{2\pi}}\int dk_\perp \int dz \left[\boldsymbol{k}_\parallel^2 + k_\perp^2 - (k^0)^2 + \frac{m^2c^2}{\hbar^2}\right] \exp\left[iz\left(k_\perp - k'_\perp\right)\right] \langle k_\perp | G(k^0, \boldsymbol{k}_\parallel) | z' \rangle$$

$$+\frac{\ell}{\sqrt{2\pi}^3}\int d\kappa_\perp \int dk_\perp \int dz \exp(-i\kappa_\perp z_0) \exp\left[iz\left(\kappa_\perp + k_\perp - k'_\perp\right)\right]$$

$$\times \left(\boldsymbol{k}_\parallel^2 + \frac{\Delta m^2 c^2}{\hbar^2}\right) \langle k_\perp | G(k^0, \boldsymbol{k}_\parallel) | z' \rangle$$

$$= \frac{1}{2\pi}\int dk_\perp dz \exp(-ik_\perp z') \exp\left[iz\left(k_\perp - k'_\perp\right)\right].$$

The z integrals can be written as delta functions using their integral representation:

$$\begin{aligned}
\sqrt{2\pi}\int dk_\perp \left[\boldsymbol{k}_\parallel^2 + k_\perp^2 - (k^0)^2 + \frac{m^2c^2}{\hbar^2}\right] \quad &\times \quad \delta(k_\perp - k'_\perp)\langle k_\perp | G(k^0, \boldsymbol{k}_\parallel) | z' \rangle \\
&\times \quad \frac{\ell}{\sqrt{2\pi}}\int d\kappa_\perp \int dk_\perp \exp(-i\kappa z_0)\delta(\kappa_\perp + k_\perp - k'_\perp) \\
&\times \quad \left(\boldsymbol{k}_\parallel^2 + \frac{\Delta m^2 c^2}{\hbar^2}\right) \langle k_\perp | G(k^0, \boldsymbol{k}_\parallel) | z' \rangle \\
&= \quad \int dk_\perp \exp(-ik_\perp z')\delta(k_\perp - k'_\perp). \qquad \text{(A.7)}
\end{aligned}$$

Using the delta functions to eliminate the $k_\perp$ integrals and renaming variables yields

$$\left[\boldsymbol{k}_\parallel^2 + k_\perp^2 - (k^0)^2 + \frac{m^2c^2}{\hbar^2}\right] \langle k_\perp | G(\boldsymbol{k}_\parallel, k^0) | z' \rangle$$

$$+\frac{\ell}{2\pi}\int d\kappa_\perp \exp\left[iz_0(\kappa_\perp - k_\perp)\right] \left[\boldsymbol{k}_\parallel^2 + \frac{m_\parallel^2 c^2}{\hbar^2}\right] \langle \kappa_\perp | G(\boldsymbol{k}_\parallel, k^0) | z' \rangle = \frac{1}{\sqrt{2\pi}} \exp\left(-ik_\perp z'\right).$$

Multiplying both sides of the equation by $\exp(ik_\perp z_0)$ and rearranging the equation yield

$$\left[\boldsymbol{k}_\parallel^2 + k_\perp^2 - (k^0)^2 + \frac{m^2c^2}{\hbar^2}\right] \exp(ik_\perp z_0)\langle k_\perp | G(\boldsymbol{k}_\parallel, k^0) | z' \rangle$$

$$+\frac{\ell}{2\pi}\int d\kappa_\perp \exp(i\kappa_\perp z_0) \left(\boldsymbol{k}_\parallel^2 + \frac{\Delta m^2 c^2}{\hbar^2}\right) \langle \kappa_\perp | G(\boldsymbol{k}_\parallel, k^0) | z' \rangle = \frac{1}{\sqrt{2\pi}} \exp\left[ik_\perp\left(z_0 - z'\right)\right].$$

The $\kappa_\perp$ dependence of the second term in the equation will be integrated out, which means the second term can be represented by a function that depends only on k^0, $\boldsymbol{k}_\parallel$ and z'. Therefore we have

$$-\frac{\ell}{2\pi}\int d\kappa_\perp \exp(i\kappa_\perp z_0) \left(\boldsymbol{k}_\parallel^2 + \frac{\Delta m^2 c^2}{\hbar^2}\right) \langle \kappa_\perp | G(\boldsymbol{k}_\parallel, k^0) | z' \rangle = f(k^0, \boldsymbol{k}_\parallel, z').$$

This implies that $\langle k_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle$ has the form

$$\exp\left(ik_\perp z_0\right)\langle k_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle=\frac{1}{k^2+(mc/\hbar)^2-i\epsilon}\left[\frac{\exp[ik_\perp(z_0-z')]}{\sqrt{2\pi}}+f(k^0,\boldsymbol{k}_\parallel,z')\right],$$

where $k^2=-(k^0)^2+\boldsymbol{k}_\parallel+k_\perp^2$. Noticing that the entire $\kappa_\perp$ dependence of $f(k^0,\boldsymbol{k}_\parallel,z')$ comes from $\exp(i\kappa_\perp z_0)\langle\kappa_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle$, and that upon the substitution $\kappa_\perp\to k_\perp$, a condition can be stated for the function $f(k^0,\boldsymbol{k}_\parallel,z')$. Rewriting the above equation in terms of $f(k^0,\boldsymbol{k}_\parallel,z')$ yields

$$\left[\boldsymbol{k}_\parallel^2+k_\perp^2-(k^0)^2+\frac{m^2c^2}{\hbar^2}\right]\left[\frac{1}{\sqrt{2\pi}}\exp\left[ik_\perp\left(z_0-z'\right)\right]+f(k^0,\boldsymbol{k}_\parallel,z')\right]\frac{1}{k^2+(mc/\hbar)^2-i\epsilon}$$

$$+\frac{\ell}{2\pi}\int d\kappa_\perp\frac{\boldsymbol{k}_\parallel^2+(\Delta m^2c^2/\hbar^2)}{k^2+(mc/\hbar)^2-i\epsilon}\left[\frac{1}{\sqrt{2\pi}}\exp\left[i\kappa_\perp\left(z_0-z'\right)\right]+f(k^0,\boldsymbol{k}_\parallel,z')\right]=$$

$$\frac{1}{\sqrt{2\pi}}\exp\left[ik_\perp\left(z_0-z'\right)\right].$$

The function $f(k^0,\boldsymbol{k}_\parallel,z')$ has to satisfy the condition

$$f(k^0,\boldsymbol{k}_\parallel,z')+\frac{\ell}{2\pi}\int dk_\perp\frac{\boldsymbol{k}_\parallel^2+(\Delta m^2c^2/\hbar^2)}{k^2+(mc/\hbar)^2-i\epsilon}\left(\frac{\exp[ik_\perp(z_0-z')]}{\sqrt{2\pi}}+f(k^0,\boldsymbol{k}_\parallel,z')\right)=0.$$

The pole shift $-i\epsilon$ in the denominator is such that given $\ell=0$, $\exp\left(ik_\perp z_0\right)\langle k_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle$ reproduces the expression for the free retarded Green's function. The denominator of the integrand $\frac{\exp(ik_\perp z)g(k_\perp)}{k_\perp^2+\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2-i\epsilon}$ where $z=z_0-z'$ can be expanded in order to recognize the four simple poles:

$$\begin{aligned}k_\perp^2+\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2-i\epsilon &= \left(k_\perp-\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}-i\epsilon\right)\\ &\times \left(k_\perp+\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\epsilon\right).\end{aligned}\quad\text{(A.8)}$$

For $(k^0)^2>\boldsymbol{k}_\parallel^2+\frac{m^2c^2}{\hbar^2}$, the denominator of the integrand is not analytic at:

$$k_\perp=\pm\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}\pm i\epsilon.$$

For $(k^0)^2<\boldsymbol{k}_\parallel^2+\frac{m^2c^2}{\hbar^2}$, the denominator of the integrand is not analytic at:

$$k_\perp=\pm i\sqrt{\boldsymbol{k}_\parallel^2+(mc/\hbar)^2-(k^0)^2}\pm i\epsilon.$$

The integral is evaulated for $\pm z$ using the residue theorem. For $z>0$ the integration contour is closed by an infinite counter-clockwise semi-circle in the upper half plane. For $z<0$ the integration contour is chosen to be an infinite clockwise semicircle in the lower half plane. Considering the case $z>0$, the contour encloses the residues $i\sqrt{\boldsymbol{k}_\parallel^2+(mc/\hbar)^2-(k^0)^2}+i\epsilon$

and $\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2} + i\epsilon$. The integration contour is decomposed into two parts. The first part is the semi-circle of infinite radius. In this case the integral is evaluated by promoting $k_\perp$ to be a complex variable. Thus $k_\perp = Re^{i\theta}$, $dk_\perp = iRe^{i\theta}d\theta$, $\theta = 0 \to \pi$,$R \to \infty$. Using these definitions in the integrand yields

$$\int \frac{dk_\perp}{2\pi} \frac{exp(ik_\perp z)g(k_\perp)}{k_\perp^2 + \boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2 - i\epsilon} = \int \frac{d\theta iRe^{i\theta}e^{iRe^{i\theta}z}}{(2\pi)\left((Re^{i\theta})^2 + \boldsymbol{k}_\parallel^2 + (mc/\hbar)^2 - (k^0)^2\right)}.$$

Evaluation of the integral using the residue theorm yields

$$\begin{aligned}
\int \frac{dk_\perp}{2\pi} \frac{\exp(ik_\perp z)g(k_\perp)}{k_\perp^2 + \boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2 - i\epsilon} &= i\Theta(z)\Theta\left[(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2\right] \\
&\frac{exp\left[i\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}z\right] g\left(\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}\right)}{2\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}} \\
&+i\Theta(-z)\Theta\left[(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2\right] \\
&\frac{\exp\left[-i\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}z\right] g\left(-\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}\right)}{2\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}} \\
&+\Theta(z)\Theta\left[\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2\right] \\
&\frac{\exp\left[-\sqrt{\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2}z\right] g\left(i\sqrt{\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2}\right)}{2\sqrt{\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2}} \\
&+\Theta(-z)\Theta\left[\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2\right] \\
&\frac{\exp\left[\sqrt{\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2}z\right] g\left(-i\sqrt{\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2}\right)}{2\sqrt{\boldsymbol{k}_\parallel^2 - (k^0)^2 + (mc/\hbar)^2}}.
\end{aligned}$$

Recognizing that we can simplify this expression using $|z|$ in place of $\pm z$ in the exponential terms allows for simplification of the integral result. Applying the simplification and using it in the condition on $f(k^0, \boldsymbol{k}_\parallel, z')$ yields

$$\begin{aligned}
&f\left(k^0, \boldsymbol{k}_\parallel, z'\right) + \frac{\ell}{2}\left(\boldsymbol{k}_\parallel^2 + \frac{m_\parallel^2 c^2}{\hbar^2}\right) \\
&\times\left[\frac{i\Theta\left[(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2\right]}{\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}}\right. \\
&\left.\times\left(\frac{\exp\left[i\sqrt{(k^0)^2 - \boldsymbol{k}_\parallel^2 - (mc/\hbar)^2}|z' - z_0|\right]}{\sqrt{2\pi}} + f\left(k^0, \boldsymbol{k}_\parallel, z'\right)\right)\right.
\end{aligned}$$

$$+\frac{\Theta\left[\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2\right]}{\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}}$$

$$\times\left(\frac{\exp\left[-\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}|z'-z_0|\right]}{\sqrt{2\pi}}+f(k^0,\boldsymbol{k}_\parallel,z')\right)\Bigg]=0.$$

Solving for $f(k^0,\boldsymbol{k}_\parallel,z')$ yields

$$f(k^0,\boldsymbol{k}_\parallel.z')=-i\frac{\Theta\left[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2\right]\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$

$$\times\frac{\exp\left[i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}|z'-z_0|\right]}{\sqrt{2\pi}}$$

$$-\frac{\Theta\left[\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2\right]\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}+\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$

$$\times\frac{\exp\left[-\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}|z'-z_0|\right]}{\sqrt{2\pi}}.$$

Substituting this into our expression for $\langle k_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle$ yields

$$\langle k_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{k_\perp^2+\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}}\Bigg[\exp\left(-ik_\perp z'\right)$$

$$-i\frac{\Theta\left[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2\right]\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$

$$\times\exp\left[-ik_\perp z_0+i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}|z'-z_0|\right]$$

$$-\frac{\Theta\left[\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2\right]\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}+\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$

$$\times\exp\left[-ik_\perp z_0-\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}|z'-z_0|\right]\Bigg].$$

Fourier transform with respect to $k_\perp$ yields

$$\langle z|G(k^0,\boldsymbol{k}_\parallel)|z'\rangle=\int dk_\perp\langle z|k_\perp\rangle\langle k_\perp|G(k^0,\boldsymbol{k}_\parallel)|z'\rangle$$

$$=\frac{1}{\sqrt{2\pi}}\int dk_\perp\exp(ik_\perp z)\langle k_\perp|G(k^0,\boldsymbol{k}_\parallel)|z'\rangle$$

$$\frac{1}{\sqrt{2\pi}}\int dk_\perp \exp(ik_\perp z)\langle k_\perp|G(\boldsymbol{k}_\parallel,k^0)|z'\rangle = \frac{1}{2\pi}\int dk_\perp \exp(ik_\perp z)$$
$$\times\frac{1}{\sqrt{k_\perp^2+\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}}\Bigg[\exp\left(-ik_\perp z'\right)$$
$$-i\frac{\Theta\left[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2\right]\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$
$$\times\exp\left[-ik_\perp z_0+i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}|z'-z_0|\right]$$
$$-\frac{\Theta\left[\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2\right]\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}+\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$
$$\times\exp\left[-ik_\perp z_0-\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}|z'-z_0|\right]\Bigg].$$

Once again the residue theorem is used to solve the $k_\perp$ integral. The form of this integral is identical to the previous one solved with the residue theorem. Therefore making use of that result yields

$$\langle z|G(k^0,\boldsymbol{k}_\parallel)|z'\rangle = i\frac{\Theta[(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2]}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}}\Bigg[\exp\left(i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}|z-z'|\right)$$
$$-i\frac{\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}+i\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$
$$\times\exp\left[+i\sqrt{(k^0)^2-\boldsymbol{k}_\parallel^2-(mc/\hbar)^2}(|z'-z_0|+|z-z_0|)\right]\Bigg]$$
$$+\frac{\Theta[\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2]}{2\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}}\Bigg[\exp\left(-\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}|z-z'|\right)$$
$$-\frac{\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}{2\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}+\ell\left(\boldsymbol{k}_\parallel^2+(m_\parallel c/\hbar)^2\right)}$$
$$\times\exp\left[-\sqrt{\boldsymbol{k}_\parallel^2-(k^0)^2+(mc/\hbar)^2}(|z'-z_0|+|z-z_0|)\right]\Bigg].$$

The energy-dependent Green's function in $\boldsymbol{x}$ space is

$$\langle\boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle = \frac{1}{(2\pi)^2}\int d^2\boldsymbol{k}_\parallel\int d^2\boldsymbol{k'}_\parallel\langle\boldsymbol{x}_\parallel,z|\boldsymbol{k}_\parallel,z\rangle\langle\boldsymbol{k}_\parallel,z|G(k^0)|\boldsymbol{k'}_\parallel,z'\rangle\langle\boldsymbol{k'}_\parallel,z'|\boldsymbol{x'}_\parallel,z'\rangle$$

$$= \frac{1}{(2\pi)^2} \int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel exp\left[i\left(\boldsymbol{k}_\parallel \cdot \boldsymbol{x}_\parallel - \boldsymbol{k'}_\parallel \cdot \boldsymbol{x'}_\parallel\right)\right] \langle \boldsymbol{k}_\parallel, z|G(k^0)|\boldsymbol{k'}_\parallel, z'\rangle.$$

Translational invariance of the energy-dependent Green's function in the direction parallel to the interface yields

$$\langle \boldsymbol{x}|G(E)|\boldsymbol{x'}\rangle = \frac{1}{(2\pi)^2} \int d^2\boldsymbol{k}_\parallel \int d^2\boldsymbol{k'}_\parallel exp\left[i\left(\boldsymbol{k}_\parallel \cdot \boldsymbol{x}_\parallel - \boldsymbol{k'}_\parallel \cdot \boldsymbol{x'}_\parallel\right)\right] \langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle \delta^2(\boldsymbol{k}_\parallel - \boldsymbol{k'}_\parallel)$$

$$= \frac{1}{(2\pi)^2} \int d^2\boldsymbol{k}_\parallel exp\left[i\boldsymbol{k}_\parallel \cdot \left(\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel\right)\right] \langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle$$

$$= \frac{1}{2\pi} \int dk_\parallel k_\parallel J_0(k_\parallel|\boldsymbol{x}_\parallel - \boldsymbol{x'}_\parallel|)\langle z|G(k^0, \boldsymbol{k}_\parallel)|z'\rangle.$$