New Exact Solutions in Einstein-Maxwell Theory

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By

Vineet Kumar

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Abstract

I construct several new exact solutions in Einstein-Maxwell theory with a focus on embedding Nutku helicoid-catenoid instantons in 5 and higher dimensions. Using the Nutku helicoid instanton, I also construct convoluted-like solutions in 6 and higher dimensional Einstein-Maxwell theory. These Nutku-embedded solutions may be interpreted as wormhole handles. I also construct similar solutions through an embedding of the Eguchi-Hanson instanton in higher dimensions. Additionally, I derive two new gravitational instantons using the general Nutku metric. Lastly, I construct a spacetime in which the electric and gravitational field strengths are shown to be determined by the same parameter, i.e. mass alone.

Acknowledgements

I thank my supervisor Dr Amir Masoud Ghezelbash for giving me the opportunity to do research with him and teaching me things I would probably never have learnt otherwise. I also thank Dr Michael Bradley for giving me the motivation and encouragement at a time when I needed them the most. I thank the thesis advisory committee for allowing me some flexibility in choosing courses and research topics.

I also would like to thank my friend Jason for enlightening discussions and encouragement and all other friends who had to bear general relativity gossip for two years. And lastly, I would like to thank my friend Lazăr for stimulating discussions and insight on increasing scientific output.

Preface

The official purpose of this thesis is to complete a degree requirement but I have also written it with the advancement of knowledge in mind. In doing so, I have taken certain liberties in the style of writing and the manner of presentation but wherever possible, I have attempted to be cogent enough with the ideas and arguments presented that an upper-year physics student should be able to follow them to an appreciable extent or at the very least, know where to look for further clarification. The thesis is replete with footnotes for various reasons, including a desire to not introduce jargon without explaining it and to put additional comments that I've found difficult to resist making. I have generally used geometrised units throughout this work but not always been consistent in dropping or keeping $G = 1$, $c = 1$ etc., in order to keep the argument at hand clearer. In this regard, I have followed the maxim of Einstein (who attributes it to Boltzmann)—"*matters of elegance are best left to the tailor and the cobbler*".[1] Although I appreciate mathematical rigour, I do not let it become a severely constraining factor. Thus, wherever an argument or deduced relation lacks mathematical rigour, I've tried to clarify it as such.

The first chapter is introductory where I have tried to give some historical account of gravitational thought and theories. By no means, it is sufficiently expansive or highly instructive but it serves the purpose of giving the reader a well-paced starting point through limited historiography. The sections on basic General Relativity are admittedly not didactic and are there only in case the reader may like to refer back to them if some idea in a later chapter demands it. The chapter on instantons and solitons is intentionally left quite basic because the typical use of instantons is not pertinent here. In any case, I have provided enough resources in the references for the reader to acquaint themselves further with the topics discussed. The third chapter comprises the bulk of the *official* research undertaken during the course of the master's programme. The fourth chapter is motivated

[1]Perhaps, the maxim is anachronistic in a time when cobblers and tailors are sadly rare.

by a desire to construct instantons which may have special properties as yet unclear to me; I may be wrong in my hunch about their special nature. Lastly, the fifth chapter is a result of work done near the end of the programme and as such, I did not have enough time to research it more deeply. In this chapter, I have attempted to ask the question if we truly understand electric charge or if (higher dimensional) General Relativity is not a good theory for even *classical* electromagnetism. I intend to continue the work on this chapter in a future publication.

I hope that some of the work done in this thesis will provide impetus for further scientific enquiry in some of the questions raised. Finally, I also hope nobody minds a bit of irreverence in the footnotes; scientific writing does not have to be dry.

I dedicate this work to nature and all those who appreciate it.

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List of Abbreviations

D	Dimensions
LHS	Left Hand Side
GR	General Relativity
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
RHS	Right Hand Side
SR	Special Relativity

Chapter 1

Gravity: An Introduction

The last thing one discovers in composing a work is what to put first.

Pensées
Blaise Pascal

1.1 History

Our knowledge about thoughts on gravitational phenomena goes back to at least as far as Aristotle who attributed the fall of things towards the ground to their specific "nature" and not to an external "cause". He erroneously believed that objects of different masses, when released together from the same height, hit the ground at different times—an idea that seemingly went unchallenged for two millennia till Galileo Galilei disproved it by dropping two spherical masses from a tower. The Indian mathematician Brahmagupta (598–c.670 CE) had theorised that the earth attracts things to itself and all things stand upright everywhere on earth [1], which was already known to be round from the works of Aryabhata (476–550 CE).

A mathematical theory of gravitation only came about in 1686 in Newton's *Principia* although the inverse square law of gravitation predates Newton's work.[1] In this work, Isaac Newton laid down the Laws of Motion and the Law of Universal Gravitation.

[1]There was a plagiarism dispute between Newton and Robert Hooke who had published works on the the attractive nature and inverse square law of gravitation that Newton never acknowledged in his works [2].

1.1.1 Newtonian Gravity

> Newton was not the first of the age of reason. He was the last of the magicians.
>
> John Maynard Keynes

As the famous anecdote goes, Newton was baffled by the fall of an apple and he concluded that there must be a reason why free things at a height only move in one direction, i.e., downward. He reasoned that there was an attractive force acted upon the apple by the earth (and vice-versa). And that force is given by:[2]

$$\mathbf{F} = -G\frac{Mm}{r^2}\mathbf{e}_r \tag{1.2}$$

where M and m are the masses of the earth and the apple, respectively, r, the distance between their centres of masses and G, a universal constant. The remarkable thing about this law is that it applies to all things. Every object applies an attractive gravitational force on every other object. Thus, he was able to apply this Law of Universal Gravitation to the motions of planets, the moon and tides and explain them.

In Newtonian gravitation, there is a notion of a gravitational "field" which is all-pervasive and allows instantaneous action of one mass on another without any explicitly stated mediating objects. Newton was deeply aware of this philosophical issue with his theory but he had no rigorous way to fill the gap. Newtonian physics also had "absolute" space and time which exist on their own as a rigid stage for physical phenomena to occur. His contemporary natural philosopher and mathematician Gottfried Leibniz had a different conception of space and time in which they only exist in relation to objects in them. The success of Newton's gravitational and mechanical laws overshadowed such philosophical concerns and for centuries, there was neither a strong contender, nor a strong reason to challenge his theories.

One of the key features of Newtonian physics is its inherently flat Euclidean space, i.e. assuming

[2]The potential formulation of Newtonian gravitation is written as:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\partial\Phi}{\partial r}\mathbf{e}_r, \qquad \nabla^2\Phi = 4\pi G\rho \tag{1.1}$$

space to have zero curvature everywhere. The German mathematician Carl Friedrich Gauss entertained the notion of curved space and reportedly might have even attempted measuring the curvature of space by transmitting light from three German cities. Then came Bernhard Riemann who not only did seminal work in non-Euclidean geometry but also proposed the idea that physical phenomena can be modelled through curvature in space.

1.1.2 Relativity and Newtonian Physics

It was not until 1905 when Einstein published his Theory of Special Relativity that there was a strong need to question Newtonian gravitation and mechanics or the predominance of absolute space and time. Einstein was deeply influenced by Riemann, Spinoza and Mach and like Leibniz, he did not believe in the absolute nature of space or time.

Special Relativity assumes the absence of gravitation because Einstein believed that it was fundamentally incompatible with Newtonian gravitational theory. One way to look at this incompatibility is considering an observer at rest, standing on a gravitating mass and another observer moving away from him at a constant speed but in deep space where there is effectively no gravity. Experiments done in one inertial frame cannot be replicated in another, thus not satisfying a postulate of special relativity. Obviously, the "instantaneous" propagation of gravitational force from one body to another in Newtonian gravity is among the other problems.

Einstein goes on to describe the special nature of inertial frames as problematic itself [3]:

> *Material particles sufficiently far removed from other material particles continue to move uniformly in a straight line or continue in a state of rest. We have also repeatedly emphasised that this fundamental law can only be valid for bodies of reference K which possess certain unique states of motion, and which are in uniform translational motion relative to each other. Relative to other reference-bodies K the law is not valid. Both in classical mechanics and in the special theory of relativity we therefore differentiate between reference-bodies K relative to which the recognised " laws of nature " can be said to hold, and reference-bodies K relative to which these laws do not hold.*
>
> *But no person whose mode of thought is logical can rest satisfied with this condition of things. He asks : " How does it come that certain reference-bodies (or their states of motion) are given priority over other reference-bodies (or their states of motion) ? What is the reason for this Preference? In order to show clearly what I mean by this question, I shall make use of a comparison.*
>
> *I am standing in front of a gas range. Standing alongside of each other on the range are two pans so much alike that one may be mistaken for the other. Both are half full*

of water. I notice that steam is being emitted continuously from the one pan, but not from the other. I am surprised at this, even if I have never seen either a gas range or a pan before. But if I now notice a luminous something of bluish colour under the first pan but not under the other, I cease to be astonished, even if I have never before seen a gas flame. For I can only say that this bluish something will cause the emission of the steam, or at least possibly it may do so. If, however, I notice the bluish something in neither case, and if I observe that the one continuously emits steam whilst the other does not, then I shall remain astonished and dissatisfied until I have discovered some circumstance to which I can attribute the different behaviour of the two pans.

Analogously, I seek in vain for a real something in classical mechanics (or in the special theory of relativity) to which I can attribute the different behaviour of bodies considered with respect to the reference systems K and K'.) Newton saw this objection and attempted to invalidate it, but without success. But E. Mach recognsed it most clearly of all, and because of this objection he claimed that mechanics must be placed on a new basis. It can only be got rid of by means of a physics which is conformable to the general principle of relativity, since the equations of such a theory hold for every body of reference, whatever may be its state of motion.

1.2 Relativity: from Special to General Theory

> Since the mathematicians have invaded the relativity theory, I do not understand it myself any more.
>
> Einstein

Einstein got to the heart of the idea behind general relativity through the "happiest thought" of his life ("*glückischste Gedanke meines Lebens*")—a *gedankenexperiment* in 1908. He realised that a freely falling man will never feel his own weight and that this implies that gravitational mass and inertial mass must be the same. So, in essence, the freely falling man experiences no gravitational "force" on his body. Then what is it that we experience as gravitational force in our everyday lives? The answer, according to general relativity, lies in the curvature of spacetime. But how is that an answer, one may ask. We will get to that as we progress.

1.2.1 Special Relativity

Einstein's special relativity was laid down as 4-dimensional geometry by his former professor, Hermann Minkowski in 1907. The key idea in Minkowski's work was that space and time together formed one geometric entity that he called spacetime. All physical phenomena we observe can be considered spacetime events, which are just points in this 4-dimensional geometry and the "distance" between these points are invariant under Lorentz transformations. The line element for this Lorentz-invariant distance is given by:[3]

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{1.3}$$

where t is a timelike coordinate and x, y and z are spacelike.[4] The coordinate system can be arbitrary but the "distance" between any two spacetime events is invariant. When the invariant distance between two events is less than 0, we say that they are timelike separated and as such, may be causally connected, i.e. a particle or signal moving at a speed lower than the speed of light, may be sent from one to another. When the distance is greater than 0, the events are spacelike separated and cannot be causally related. When the distance is 0, the events are null separated and can only be connected by a signal travelling at the speed of light [4].

Using special relativity, Einstein was able to show that Maxwell equations are invariant under Lorentz transformations and thus, an electric field can be seen as a magnetic field from another reference frame and *vice-versa*.[5] Among many other things, the theory establishes the relativity of not only length and time but also of simultaneity.

1.2.2 General Relativity

Special relativity did not account for gravitation and put inertial reference frames on a special pedestal. This prompted Einstein and others to come up with a theory of relativity that could account

[3]Or alternatively, $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.

[4]The coordinate t does not represent the time on some special universal clock but only serves the role of time as a coordinate that is distinct from space.

[5]If S is a reference frame with coordinates (t, x, y, z) and S' is a reference frame with coordinates (t', x', y', z'), moving with a velocity v in the x direction, as measured by an observer in the S frame, then the Lorentz transformation is given by: $t' = \gamma(t - vx/c^2)$, $x' = \gamma(-vt + x)$, $y' = y$, $z' = z$ where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

for gravity. The first major developments to this end were Hermann Minkowski's reformulation of Special Relativity and electrodynamics as a geometry of 4-dimensional "spacetime" [5]–[7] in 1907-1908 and Einstein's "equivalence principle" [8] around the same time. Early attempts towards a relativistic theory of gravitation tried to incorporate gravitation into Special Relativity but without success. The central challenge was believed to be writing Newton's Law of Gravitation in a Lorentz-invariant form. Einstein's 1911 paper [9] proposed the idea that the speed of light varies with the strength of the gravitational field and that both SR and Newton's Theory of Gravitation must give way to a more fundamental theory. Soon after this work was published, Max Abraham attempted to combine Einstein's new idea with Minkowski's 4-dimensional spacetime theory and proposed his own theory of gravitation [10]. Abraham's theory[6] was not accepted, not the least because it attempted to combine a variable-speed-of-light theory with a constant speed one [11]. In 1912 and 1913, Gunnar Nordström proposed two different scalar theories of gravitation. His 1913 work [12] was significant because it was a metric theory, unlike other competing theories of the time. Nordström's theory can be stated in terms of the Ricci scalar, the trace of the energy-momentum tensor and the Weyl tensor as:

$$\begin{aligned} \mathcal{R} &= 24\pi\mathcal{T} \\ C_{abcd} &= 0 \end{aligned} \tag{1.4}$$

Although Nordström's theory is not backed by experimental evidence, its novelty and importance rested on the fact that it related a purely geometric quantity, the Ricci scalar to a purely physical quantity, the trace of the energy-momentum tensor. The period 1912–1915 saw a flurry of scientific work searching for a consistent, relativistic theory of gravitation. In 1915, Albert Einstein[7] after several erroneous publications, finally gave the correct field equations of general relativity [13], where he used the Ricci curvature tensor $R_{\mu\nu}$—the elusive geometric quantity that Einstein and others had been looking for. The field equations of GR, also called the Einstein equations can be stated as:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.5}$$

[6]Einstein and Abraham were reading and criticising each other's works with tirades ranging from Einstein accusing Abraham of nostrification of Einstein's ideas and Abraham accusing Einstein of plagiarising Abraham's ideas. For an account of the drama, see Abaraham Pai's biography of Einstein in [11].

[7]David Hilbert who had exchanged ideas with Einstein, submitted his equivalent formulation of general relativity to a journal a few days before Einstein did but Hilbert's work was published later in 1916. Einstein did not like this "nostrification" of his theory by Hilbert and Hilbert never tried to take credit for the development of GR.

where $G_{\mu\nu}$ is the Einstein tensor[8], Λ the cosmological constant, $g_{\mu\nu}$, the spacetime metric and $T_{\mu\nu}$, the stress-energy tensor.[9] The Greek indices denote the coordinates. The LHS of these non-linear equations is a measure of curvature and depends only on the spacetime metric $g_{\mu\nu}$. The right hand side of the equations describes the distribution of matter and energy in spacetime. The equation is often summed up as "*matter tells spacetime how to curve and spacetime tells matter how to move*" [34].

1.3 GR: A ruthlessly short primer

> We use mathematics in physics so that we won't have to think.
>
> Bryce DeWitt

This section is a short refresher on the basic mathematical concepts used in GR and in particular, this thesis. By no means, it is complete or rigorous in its scope and it will only attempt to serve as a scaffolding to hold on to whenever the reader finds some general relativistic concept in the later chapters hard to grasp.

1.3.1 Vectors and manifolds

A manifold can be thought of as a collection of "points". The Euclidean plane, the 2-sphere and the set of all rigid rotations in $\mathbb{R}^3$ are examples of a manifold. More strictly speaking, we can define a manifold in the following non-rigorous manner.[10]

Manifold: An N-dimensional manifold $\mathcal{M}$ is a set of points in space which requires N independent real coordinates $(x^1, x^2, ..., x^N)$ to specify any point in it completely and is locally Euclidean.

One coordinate system may be transformed into another one by an $N \times N$ transformation matrix,

[8] $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$

[9] c and G have their usual meanings as the speed of light in vacuum and Newton's constant, respectively. In geometrised units commonly used in general relativity, $c = 1$ and $G = 1$.

[10] For a more rigorous definition of manifolds, see [16].

for example, to suit a particular problem or to avoid the degeneracy of a coordinate system.[11] Suppose that we wish to transform the coordinates of any point from $x^a \to x'^a$ where the new coordinate system is $x'^a = x'^a(x^1, x^2, .., x^N)$, where $a = 1, 2, ..., N$. If P and Q are neighbouring points with coordinates x^a and $x^a + dx^a$, respectively, then the distance between them in the new coordinate system x'^a is [14]:

$$dx'^a = \sum_{b=1}^{N} \frac{\partial x'^a}{\partial x^b} dx^b = \frac{\partial x'^a}{\partial x^b} dx^b \tag{1.6}$$

where we have used the Einstein summation convention to imply a summation sign over repeated indices (b only, in this case).

At any point P in a N-dimensional manifold, we can define a set of N basis vectors $\mathbf{e}_a$ on the tangent space T_P.[12] And any vector at P may be then written as a linear combination of these basis vectors. A local vector field $v(x)$ then is expressed as [14]:

$$\mathbf{v}(x) = v^a(x)\mathbf{e}_a(x) \tag{1.7}$$

where $v^a(x)$ are the contravariant components of $\mathbf{v}(x)$ in the given basis. One may also define a dual basis vector set $\mathbf{e}^a(x)$ using $\mathbf{e}^a(x) \cdot \mathbf{e}_b(x) = \delta^a_b$. Then $\mathbf{v}(x)$ can also be expressed in terms of its covariant components in the dual basis $\mathbf{e}^a(x)$ as [14]:

$$\mathbf{v}(x) = v_a(x)\mathbf{e}^a(x) \tag{1.8}$$

The scalar product of two vectors $\mathbf{v}$ and $\mathbf{w}$ may be expressed in different equivalent ways as:

$$\mathbf{v} \cdot \mathbf{w} = (v^a \mathbf{e}_a) \cdot (w^b \mathbf{e}_b) = g_{ab} v^a w^b = g^{ab} v_a w_b = v^a w_a = v_a w^a \tag{1.9}$$

The basis vectors and the dual basis vectors may themselves be expressed through the relations $\mathbf{e}_a = g_{ab}\mathbf{e}^b$ and $\mathbf{e}^a = g^{ab}\mathbf{e}_b$.

[11]Not every coordinate system can specify all points on a manifold uniquely. For example, no coordinate system can cover all points on a 2-sphere. In such a case, the coordinate system is "degenerate".

[12]The tangent space T_P is essentially the set of all local vectors at P [14].

1.3.2 Spacetime metric

The spacetime metric is paramount in general relativity. All physical measurements and causal structure of a spacetime can be extracted from its metric alone. The metric measures the length between two points (events) in spacetime. Recall that the distance between two points (x_1, y_1) and (x_2, y_2) on a plane may be given in Cartesian coordinates as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. This notion of length can be represented with an infinitesimal line element $ds^2 = dx^2 + dy^2$ where $x, y \in (-\infty, \infty)$. However, the coordinate choice is arbitrary and in polar coordinates, the same line element becomes $ds^2 = dr^2 + r^2 d\theta^2$, where $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$. The line element is not the only way to represent a metric. Another way to write the metric is through the metric tensor:

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (Cartesian), \qquad g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \qquad (polar) \tag{1.10}$$

The ijth element is simply the coefficient of $dx^i \otimes dx^j$. For 2-surfaces in $\mathbb{R}^3$, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the tangent vectors. More generally,

$$ds^2 = g_{ab}(x) dx^a dx^b \tag{1.11}$$

Since the distance between two points on a manifold is invariant under coordinate transformations, a metric $g_{ab}(x)$ can be transformed to another metric $g'_{cd}(x')$ as:

$$g'_{cd}(x') = g_{ab}(x(x')) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} \tag{1.12}$$

1.3.3 Raising and lowering of indices

Indices on any quantity may be lowered or raised using the metric by the following prescriptions:[13]

$$g_{ab} w^b = w_a \tag{1.13}$$

$$g^{ab} w_b = w^a \tag{1.14}$$

[13]Notice that there is an implied summation over the index b.

1.3.4 Covariant derivative

The covariant derivative of a vector field $v(x)$ is given in terms of its contravariant components v^a by:

$$\partial_b \mathbf{v} = (\partial_b v^a + \Gamma^a_{cb} v^c)\mathbf{e}_a \equiv (\nabla_b v^a)\mathbf{e}_a \tag{1.15}$$

The covariant derivative derivative is often indicated by a semicolon:

$$\nabla_b v^a = v^a_{;b} \tag{1.16}$$

In terms of the covariant components of $\mathbf{v}(x)$, the covariant derivative is:

$$v_{a;b} = \partial_b v_a - \Gamma^c_{ab} v_c \tag{1.17}$$

1.3.5 Riemann and Ricci curvature tensors

Let us not digress further and stick only to the bare minimum GR concepts we need to understand what lies ahead in this thesis. Since we have talked about gravity as curvature of spacetime, we need some notion of curvature in general relativity. There are several quantities that measure "curvature" of some type.

The most general one is the Riemann curvature tensor R^d_{abc} which is defined as:

$$R^d_{abc} \equiv \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ac} \tag{1.18}$$

where the affine connection or the Christoffel symbols (of the second kind) are given by:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{1.19}$$

The Ricci tensor can be computed from a contraction of the Riemann tensor in the following manner:

$$R_{ab} \equiv R^c_{acb} = \frac{\partial \Gamma^c_{ab}}{\partial x^c} - \frac{\partial \Gamma^c_{ac}}{\partial x^b} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \tag{1.20}$$

The trace of the Ricci tensor is called Ricci scalar and is given by:

$$\mathcal{R} \equiv R^a_a = g^{ab} R_{ab} \tag{1.21}$$

The Kretschmann scalar is a contraction of the Riemann tensor upon itself. Loosely speaking, it can be seen as the "square" of the Riemann tensor and its main utility lies in finding singularities.[14] The Kretschmann invariant is given by:

$$\mathcal{K} = R_{abcd} R^{abcd} \tag{1.22}$$

Now that we know how to calculate the Ricci tensor from the spacetime metric, we might as well state the Einstein tensor once more:

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R^c_c = R_{ab} - \frac{1}{2} g_{ab} \mathcal{R} \tag{1.23}$$

1.3.6 Einstein equations

> If all mathematics disappeared, physics would be set back exactly one week.
>
> RICHARD FEYNMAN

> Precisely the week in which God created the world.
>
> MARK KAC

The field equations of GR are given by the Einstein equations:[15]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.24}$$

[14] Any invariant quantity becoming singular at some point on a manifold indicates a "singularity" on the manifold at that point.

[15] Some authors may use a minus sign on the RHS.

The cosmological constant term Λ was introduced by Einstein around 1916 because without it, the equations predicted an unstable universe that was either contracting or expanding [14]. In 1929, when Hubble, building on Vesto Slipher's work, announced that the universe is observed to be expanding, Einstein dropped the cosmological constant calling it "the biggest blunder" of his life. The Λ term now has a modern interpretation as a universal constant that determines the energy density of vacuum (ρ_{vac}) through [14][15]:

$$\rho_{vac} c^2 = \frac{\Lambda c^4}{8\pi G} \tag{1.25}$$

However, there is no theory that accurately predicts the vacuum energy density in a manner reconcilable with the small, non-zero observed value of the cosmological constant [14].

The RHS of Eq. (1.24) has the stress-energy tensor $T_{\mu\nu}$. For vacuum solutions, such as the Schwarzschild spacetime, $T_{\mu\nu} = 0$, which simply means that there is an absence of matter and energy. When matter or energy is present, it contributes to the curvature of spacetime in its vicinity. Just as there is a 3×3 stress tensor in classical mechanics, general relativity has a 4×4 stress-energy tensor in a $4D$ spacetime. Figure 1.3.1 explains the physical meaning of various components of the stress-energy tensor $T^{\mu\nu}$ (in 4 dimensions).

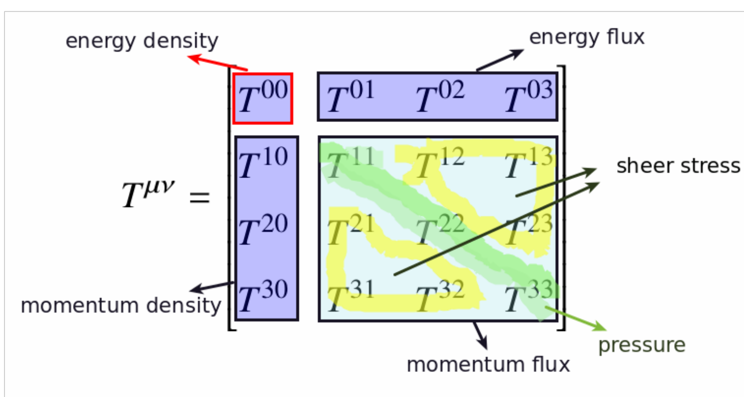


Figure 1.3.1: Physical meaning of the components of the stress-energy tensor. Adapted from Steane (2012) [4]

For example, in Cartesian coordinates, we may write the stress-energy tensor for an ideal fluid as [4]:

$$T^{\mu\nu} = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{1.26}$$

Any system with a diagonal stress-energy tensor $T^{\mu\nu}$ in some form can be called an ideal fluid [4]. If the pressure p is zero, then $T^{\mu\nu}$ above would represent "dust" [4].

1.3.7 Geodesics

Geodesics generalise the notion of a straight line in curved spacetime. One may think of a geodesic as the shortest path connecting two points. The geodesic equations are represented in the following form [15]:

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \tag{1.27}$$

where τ is an affine parameter such as proper time[16] The geodesic equation can also be written succinctly as:

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\alpha \tag{1.28}$$

1.4 Other Gravitational Theories

Einstein's General Relativity is not the only theory of gravitation. There have been several theories since that have attempted to go beyond GR in their scope. Few are backed by experimental evidence, however. Let's look at a few important gravitational theories, keeping in mind their impact on gravitational physics and the supporting experimental evidence (or lack thereof). For a brief history of scalar-tensor theories, see [17].

[16]The word "proper" is erroneous and misleading as it is a corruption of the French word *propre, meaning "self" or "own".*

1.4.1 Kaluza-Klein Theory

Kaluza-Klein theory is especially interesting because it attempted to combine gravitation with electromagnetism at a time when they were the only two known fundamental forces of nature.[17] Theodor Kaluza published[18] his work [19] in 1921, five years before Schrödinger published his eponymous wave equation [20]. Kaluza's theory proposed a fifth spatial dimension and incorporated the electromagnetic 4-vector potential A^μ and a scalar field ϕ into the spacetime metric. He further imposed the "cylinder condition", i.e., no part of the metric should depend on the 5th coordinate. This suppression of the 5th coordinate was criticised by Pauli, Einstein and others on the grounds that it seemed arbitrary and on the fact that we don't see this 5th dimension.

The theory proposes a 5-dimensional spacetime metric $\tilde{g}_{\alpha\beta}$, constructed from the 4-dimensional spacetime metric $g_{\mu\nu}$, in the following way:

$$\tilde{g}_{\alpha\beta} = \begin{bmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{bmatrix} \tag{1.29}$$

and applies the 5-dimensional Einstein equations of general relativity to this metric.

In 1926, Oskar Klein, inspired by the burgeoning quantum mechanical works of de Broglie, Schrödinger and Heisenberg, proposed [21] [22] that the 5th coordinate is periodic and closed (like a circle)[19] and that the discrete nature of electric charge can be seen as a consequence of the periodicity of the closed 5th coordinate. In [22], Klein had argued that the Schrödinger equation can be derived from a wave equation in 5-dimensional space, where the Planck constant h "does not appear originally, but is introduced in connexion with" the periodicity in the 5th dimension [21]. The full, self-consistent field equations of the Kaluza-Klein theory were given by Jordan and Müller in [29] and by Thiry in [23]. Einstein was deeply impressed [24] by Kaluza's theory and also published works on it but was reluctant to introduce a long-range massless scalar field (for example, in [25]) and turn GR into a scalar-tensor theory [26].

The beauty of the Kaluza-Klein theory lies in the fact that Lorentz force law arises naturally

[17]Nordström probably was the first to attempt a unification of gravity with electromagnetism in 1914, before Einstein developed GR in 1915 [18].

[18]Kaluza had sent his results to Einstein in 1919 [11].

[19]This means that the topology of the 5-dimensional Kaluza-Klein spacetime is $\mathbb{R}^4 \otimes S^1$.

out of it and the Maxwell equations and the Klein-Gordon equation (for the massless scalar field) are automatically satisfied [27].

Although Kaluza-Klein theory itself did not survive in its original form, it lives on vicariously through other "theories" such as the various string theories, that it inspired through its notion of compactification of spatial coordinates.

1.4.2 Jordan-Brans-Dicke Theory

> What's in a name? That which we call a rose by any other name would smell as sweet.
>
> *Romeo and Juliet*
> SHAKESPEARE

Until now, we have been treating Newton's constant G as a universal constant but there is no *a priori* reason to believe that this must be the case. Pascual Jordan [28]–[31] and later, Brans and Dicke [32] took the heretic approach towards the "universal gravitational constant" G. Starting from the equivalence principle, they constructed a scalar-tensor theory in which Newton's "constant" was not a universal constant of nature but has its value determined by a scalar field ϕ. In other words, the scalar field fixes the coupling strength of matter to gravitation [14].

Jordan-Brans-Dicke Theory is governed by the following equations:[20]

$$\Box\phi = 4\pi\lambda\mathcal{T}^{\mathcal{M}} \tag{1.30}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \frac{8\pi}{c^4\phi}(\mathcal{T}^{\mathcal{M}} + \mathcal{T}^{\phi}) \tag{1.31}$$

where $\mathcal{T}^{\mathcal{M}} = (T^M)^\mu_\mu$ and $\mathcal{T}^\phi = (T^\phi)^\mu_\mu$ are the respective traces of the energy-momentum tensors for matter and the scalar field $T^M_{\mu\nu}$ and $T^\phi_{\mu\nu}$. λ is a coupling constant[21] that fixes the scalar field ϕ which determines G.[22] The field equations relate the energy-momentum tensors of matter and

[20] Again, note that some authors use a negative sign on the RHS.

[21] The coupling constant is often given in terms of the Dicke coupling constant ω as $\lambda = \frac{2}{3+2\omega}$

[22] ϕ is a "long-range" scalar field that does not act directly upon matter but instead acts as an "indirectly coupling field", i.e., it merely is a "participant" in the field equations. By contrast, a scalar field directly coupled to matter is easier to detect and might, for example, produce slight deviations from geodesic motion or cause the mass of a body

the scalar field to the curvature of spacetime. In the limit $\lambda \rightarrow 0$, we are free to set $\phi = \frac{1}{G}$, thereby reducing Jordan-Brans-Dicke theory to Einstein's general relativity [14]. Jordan-Brans-Dicke theory allows G to vary with time and therefore has observational consequences on, for example, the motion of planets and the dates of solar eclipses [14].

1.5 Einstein-Maxwell Theory

Einstein-Maxwell theory concerns with the coupling of the Maxwell equations to the Einstein equations in a source-free region, i.e. a region of space with no current or charge or matter. Thus, only the electromagnetic field contributes to the stress-energy tensor. Because the coupled Einstein-Maxwell equations predate quantum theory, they are significant because they combined the two known forces of nature. However, it presupposes Maxwell equations [33] without starting from first principles.

The Maxwell field tensor in terms of the electromagnetic vector potential, in any reference frame, is given by [34]:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{1.32}$$

The Maxwell equations in curved spacetime[23] can be stated as [34]:

$$F^{\mu\nu}_{;\nu} = 4\pi J^{\mu} \tag{1.33}$$

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0 \tag{1.34}$$

and Lorentz force law is:

$$ma^{\mu} = F^{\mu\nu} Q u_{\nu} \tag{1.35}$$

where Q is the charge and u_{ν} is the covariant 4-velocity for the charged body. The stress-energy tensor for an electromagnetic field in a 4-dimensional spacetime is given by:

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\mu}F^{\mu}_{\beta} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right) \tag{1.36}$$

to depend on its position [34].

[23] The Christoffel symbols (affine connection terms) cancel out due to symmetries [34].

In Einstein-Maxwell theory, we take the current (or 4-current in 4-dimensional spacetime) J^μ to be zero. And in the source-free region, we only need:

$$F^{\mu\nu}_{;\nu} = 0 \tag{1.37}$$

Eq. (1.34) is automatically satisfied in the source-free case [34].

More generally, the trace-free[24] form of source-free Einstein-Maxwell equations in D dimensions in geometrised units are given by:

$$R_{\mu\nu} = F^{\lambda}_{\mu} F_{\nu\lambda} - \frac{1}{4 + 2(D-4)} g_{\mu\nu} F^2, \tag{1.38}$$

$$F^{\mu\nu}_{;\mu} = 0 \tag{1.39}$$

1.5.1 Reissner-Nordström spacetime

The Reissner-Nordström spacetime, discovered independently by Hans Reissner [35] and Gunnar Nordström [36] was the first solution in GR to have electric charge. The metric is spherically symmetric and is given by [34]:

$$ds^2 = -c^2 \left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{1.40}$$

where $r_s = \dfrac{2GM}{c^2}$ and $r_Q^2 = \dfrac{GQ^2}{4\pi\epsilon_0 c^4}$ along with the gauge field,

$$A_\mu = \left[\frac{Q}{r}, 0, 0, 0\right]. \tag{1.41}$$

The event horizon is located where g_{tt} vanishes, i.e.

$$1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2} = 0 \tag{1.42}$$

Therefore, $r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right)$. The outer horizon is the true event horizon and the inner

[24]Because $T_{\mu\nu}$ is trace-free, the Ricci scalar vanishes.

one, the "Cauchy" horizon. There is a curvature singularity at $r = 0$ as can be seen from its Kretschmann scalar:

$$\mathcal{K} = 4\,\frac{3\,r^2 {r_s}^2 - 6\,r Q^2 r_s + \frac{7}{2}\,Q^4}{r^8} \tag{1.43}$$

An interesting feature of this blackhole is that a neutral particle can reach the Cauchy horizon in finite proper time but never go beyond it. Instead, the neutral particle gets repelled away from the singularity [37]. The maximally extended solution also allows connecting one asymptotically flat region to another, thereby allowing wormholelike transport [37].

Chapter 2

Instantons

Hell, if I could explain it to the average person, it wouldn't have been worth the Nobel Prize.

Richard Feynman

2.1 Yang-Mills Theory

Yang-Mills theory is a gauge theory developed by Chen Ning-Yang and Robert Mills in 1954 [39]. To understand the notion of a gauge, recall that the four first-order Maxwell equations[1] can be written as two second-order differential equations [38]:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho \tag{2.1}$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \tag{2.2}$$

However, the set of $\mathbf{A}$ and V that can describe a particular electromagnetic field configuration is not unique. There can be another pair $(\mathbf{A}', V')$ that can describe the same electromagnetic fields.[2] This freedom in gauge transformations is a symmetry of the Maxwell equations, just as Lorentz invariance is. The Coulomb gauge and the Lorentz gauge are two examples of common gauges used. The theory of gauge symmetry was not known during Maxwell's time. It was not until Hermann Weyl's work in 1919 that a first theory of gauge invariance was published [40]. Weyl's work was

[1] The equations are written with SI units, as in the reference [38].

[2] More specifically, we may choose: $\mathbf{A}' = \mathbf{A} + \nabla\lambda, \qquad V' = V - \frac{\partial\lambda}{\partial t}$

inspired by the local nature of General Relativity in that it asked if the effects of gravitation can be described by a *connection*, then can electromagnetism[3] also be described in a similar manner? [41] His original work was not well-received, largely due to his physical interpretation of gauge invariance being incompatible with quantum theory [41]. His later work [42] and the works of Fock [43] and London [44] recognised that the gauge transformation in quantum theory was not a change of scale but in the wavefunction phase [41].

2.1.1 BPST Instanton

The theory of gauge invariance finally culminated in Yang and Mills proposing that local phase rotation invariance can be generalised to invariance under any continuous symmetry group [45]. The Yang-Mills equations are given by:

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \tag{2.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. $F_{\mu\nu}$ is the Yang-Mills field tensor and A_μ, the gauge potential. Belavin, Polyakov, Schwarz and Tyupkin showed [94] that the Yang-Mills equations are satisfied if we choose a self-dual field tensor [46]:

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu} \equiv \pm\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{2.4}$$

They found a Euclidean solution with finite action, self-dual $F_{\mu\nu}$ localised at $r = 0$ and with A_μ being asymptotically a pure gauge at infinity [46]. The authors call the instanton solution a "pseudoparticle" by which they mean long range fields A_μ with locally minimal finite Yang-Mills actions [94]. It was observed that the instantons are also related to quantum tunnelling between topologically distinct vacua [47]. Since Yang-Mills Theory allows quantisation of classical field theories, the BPST instanton solution sparked the question if such a solution can be found in General Relativity, thus paving a potential path to the quantisation of gravity [46].

[3] In 1919, gravity and electromagnetism were the only known forces.

2.2 Gravitational Instantons

In 1977, Stephen Hawking proposed that by analogy with Yang-Mills instantons, gravitational instantons are "solutions to classical Einstein equations which are non-singular on some section of complexified spacetime and in which the curvature dies away at large distances" [48]. In this work, he Euclideanised[4] both the Schwarzschild and the Taub-NUT spacetimes [49] [50] and proposed them as gravitational instantons. Gibbons and Hawking gave another definition of gravitational instantons as complete nonsingular positive definite solutions to the Einstein equations in vacuum, with or without a cosmological constant [105]. The Eguchi-Hanson metric [51] and the Atiyah-Hitchin metric [52] are also some of the early gravitational instantons found and have self-dual curvature.

2.2.1 Schwarzschild instanton

Recall that the Schwarzschild spacetime is given by the line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d_\theta^2 + sin^2(\theta) d_\phi^2) \tag{2.5}$$

where r, θ, ϕ are the spherical coordinates. If we perform a Wick rotation $t \longrightarrow i\tau$, we get:

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d_\theta^2 + sin^2(\theta) d_\phi^2) \tag{2.6}$$

The new coordinate τ is taken to be periodic with a period $8\pi M$ and $r \geq 2M$. Eq. (2.6) is the Schwarzschild instanton. Its Riemann curvature is not self-dual, [48] i.e.,[5]

$$R_{\mu\nu\lambda\rho} \neq \pm * R_{\mu\nu\lambda\rho} \equiv \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R^{\alpha\beta}_{\lambda\rho} \tag{2.7}$$

Hawking describes the Schwarzschild instanton as a "mass monopole of the ordinary 'electric'

[4]By Euclideanisation, one means changing the (Lorentzian) signature of the spacetime to Riemannian so that the timelike coordinate becomes spacelike. Essentially, this amounts to doing a Wick rotation on the timelike coordinate: $t \longrightarrow i\tau$.

[5]$\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol, a pseudo-tensor whose value is +1 for even permutations of the indices, −1 for odd and zero, otherwise.

type" and regards it as a "combination of two self-dual metrics which represent monopoles with 'electric' and equal or opposite imaginary 'magnetic' type mass", corresponding to two Taub-NUT metrics with electric mass M and magnetic mass (NUT parameter) $N = \pm iM$ [48].

2.2.2 Eguchi-Hanson instanton

Eguchi and Hanson used a different approach than Hawking to construct a gravitational instanton.[6] Their approach is closer to that taken by BPST in that they use the self-duality of the curvature form to construct a Euclidean solution to the Einstein vacuum equations.

They use the following ansatz:[7]

$$ds^2 = f^2(r)dr^2 + r^2(\sigma_x^2 + \sigma_y^2) + r^2 g^2(r)\sigma_z^2 \tag{2.8}$$

where $\sigma_x, \sigma_y, \sigma_z$ form a standard Cartan basis [51]:

$$\sigma_x = \frac{1}{2}(-\cos(\psi)d\theta - \sin(\theta)\sin(\psi)d\phi) \tag{2.9}$$

$$\sigma_y = \frac{1}{2}(\sin(\psi)d\theta - \sin(\theta)\cos(\psi)d\phi) \tag{2.10}$$

$$\sigma_z = \frac{1}{2}(-d\psi - \cos(\theta)d\phi) \tag{2.11}$$

θ, ϕ and ψ are Euler angles on S^3 with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$ [51]. They observe the curvature two-form R^a_b defined by $R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$ is self-dual then this yields:

$$fg = 1, \qquad f(2 - g^2) = g + rg' \tag{2.12}$$

Asymptotically flat solutions to the above equation are given by:

$$g(r) = \frac{1}{f(r)} = \sqrt{1 - \left(\frac{a}{r}\right)^4} \tag{2.13}$$

where a is a constant of integration [51]. The metric has an apparent singularity at $r = a$ which

[6] Eugenio Calabi independently published the same solution in 1979 in [53].

[7] Actually, this is one of the two solutions Eguchi-Hanson constructed and this ansatz is Type II and the resulting solution is generally referred to as *the* Eguchi-Hanson solution.

can be removed by a change of variables in the form $u^2 = r^2(1 - \left(\frac{a}{r}\right)^4)$. The metric is given by:

$$ds^2 = \frac{dr^2}{1 - \left(\frac{a}{r}\right)^4} + r^2 \left(1 - \left(\frac{a}{r}\right)^4\right) \sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2) \tag{2.14}$$

The metric describes the manifold $\mathbb{R} \otimes S^3$ [51]. The authors further observe that the curvature decays like $\frac{1}{r^6}$ for large r, unlike Taub-NUT and Schwarzschild instantons for which the curvature falls off like $\frac{1}{r^3}$. They thus conclude that their solution is like a gravitational "dipole" while Taub-NUT and Schwarzschild instantons are gravitational monopoles [51].

2.3 Instantons and Tunneling: An example

Example is the school of mankind, and
they will learn at no other.

Letters on a Regicide Peace
EDMUND BURKE

Instantons are not limited to Yang-Mills theory or General Relativity. Polyakov [54] observed that the double-well potential can be seen as an instanton problem. To see how that works, we will follow Huang's [55] textbook example later in this section but let's briefly discuss solitons first.

2.3.1 Solitons

Instantons are related to solitons which may be defined as "solutions to a classical field theory that describes a localised disturbance with finite energy" [55] or more generally, " a localised excitation which moves by preserving its shape" [57]. For the Sine-Gordon equation in 1+1 Minkowski spacetime, we have the Lagrangian,

$$\mathcal{L}(x,t) = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - W(1 - \cos\phi) \tag{2.15}$$

yielding the equation of motion:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + W \sin\phi = 0 \tag{2.16}$$

$\phi(x)$ is an angular field variable defined on a circle S^1 and the boundary condition $e^{i\phi} = 1$ identifies $x = \pm\infty$ [55]. Thus, $\phi(x)$ maps the circle to itself and has an associated topological invariant, the *winding number*. The lowest energy state, i.e. the classical vacuum is given by $\phi(x,t) = 0(mod\, 2\pi)$ but we may have a solution that approaches two equivalent classical vacua (say, $\phi = 0$ and $\phi = 2\pi$) for $x = \pm\infty$. Such a solution is called a topological soliton, i.e. a soliton stabilised by topology [55]. The following is a soliton solution to Eq. (2.16):

$$\phi(x,t) = 4\tan^{-1}(e^{\sqrt{W}(x-vt-x_0)}) \tag{2.17}$$

where x_0 is an arbitrary constant where the soliton appears as a localised "kink" and v is the speed of the travelling soliton [55]. A plot of the soliton solution is shown in Fig. 2.3.1. Notice that at any given time, ϕ approaches 2π for $x \to \infty$ and it approaches 0 for $x \to -\infty$.

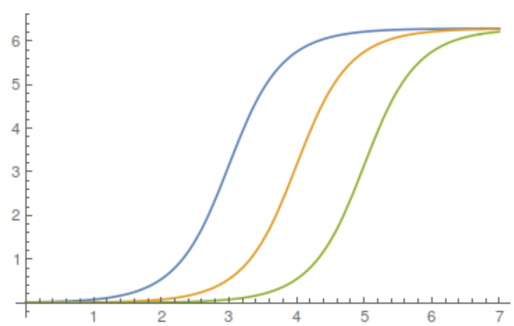


Figure 2.3.1: Plots of $\phi(x) = 4\tan^{-1}(e^{\sqrt{W}(x-vt-x_0)})$ with $v = 1$, $x_0 = 3$ for times $t = 0, 1, 2$ (solitary wave travelling from left to right)

The soliton is essentially a solitary wave, first described by the Scottish civil engineer and naval architect John Scott Russell[8] in 1845 as [56]:

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually*

[8]It's sinful to not quote J.S. Russell when talking about solitons.

diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Solitons appear in all sorts of non-linear natural phenomena. The pulse in blood circulation can be modelled as a Korteweg-de Vries (KdV) soliton [57]. Formation of blood vessels (angiogenesis) and their relationship to tumour formation have been studied as solitonic processes in [60]. Large amplitude internal hydrodynamic solitons are observed in the Andaman sea [58]. Solitons also appear in optics and plasmas [57]. The non-linear Schrödinger wave equation also admits soliton solutions. More recently, solitons have been studied in Bose-Einstein condensates where they appear to have negative mass, effectively getting accelerated through friction [59].

2.3.1.1 Tunnelling in a double-hill potential

't Hooft [65] coined the term "instantons" for static solitons because the vacuum state appears in an instant.[9] To see how instantons can be used to calculate the transition probability from one vacuum configuration to another, let's consider the simple but non-trivial example of a particle with mass m moving in a double-well potential $V(\phi)$, where ϕ is a coordinate [55]. The tunnelling amplitude for the $\phi_1 \to \phi_2$ at zero energy is given by:

$$T_{WKB} = \exp\left(-\int_{\phi_1}^{\phi_2} d\phi\sqrt{2mV(\phi)}\right) \tag{2.18}$$

Instead of using the WKB approximation, we can use the instanton approach. First, we perform a Wick rotation $t \to i\tau$. Then the transition amplitude is given by:

$$\langle\phi_2|T|\phi_1\rangle = \int \mathcal{D}\phi(\tau)e^{-S[\phi]} \tag{2.19}$$

where the Euclideanised action is:

$$S[\phi] = \int_{-\infty}^{\infty} d\tau \left[\frac{1}{2}\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^2 + V(\phi(\tau))\right] \tag{2.20}$$

[9]For instance, if we put $v = 0$ in Eq. (2.17), we get a static soliton.

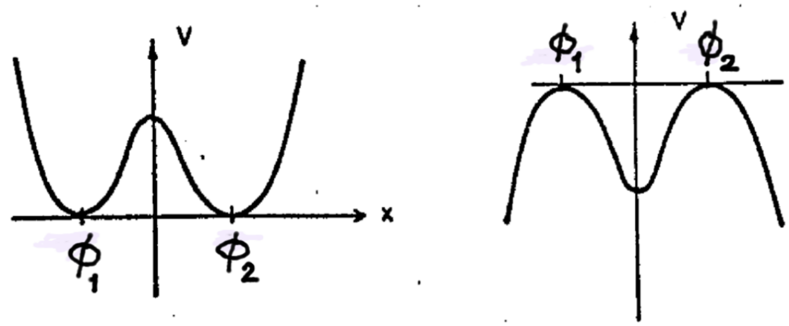


Figure 2.3.2: The double-well potential gets inverted and becomes a double-hill potential. Adapted from Coleman (1979) [61]

The problem now has become that of a double-hill potential, i.e. an inverted double-well potential (Figure 2.3.2) and the solution is classically allowed. The boundary conditions are given by:

$$\phi(\tau) \rightarrow \begin{cases} \phi_1 & (\tau \rightarrow \infty) \\ \phi_2 & (\tau \rightarrow -\infty) \end{cases} \tag{2.21}$$

The dominant path in the semi-classical limit is given the following equation of motion:

$$\frac{d^2\phi}{d\tau^2} - \frac{dV}{d\phi} = 0 \tag{2.22}$$

Since energy is a constant of motion and it is zero, we observe that:

$$\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 - V(\phi) = 0 \tag{2.23}$$

The action has a minimum at $S_0 = 2\int_{-\infty}^{\infty} d\tau V(\phi(\tau))$. Using Eq. (2.23), we can write $V = \sqrt{V}\sqrt{V} = \sqrt{V/2}(d\phi/d\tau)$. Thus,

$$S_0 = \int_{\phi_1}^{\phi_2} d\phi\sqrt{2mV(\phi)} \tag{2.24}$$

Using the saddle-point approximation, we get:

$$\langle\phi_2|T|\phi_1\rangle \approx e^{-S_0} = T_{WKB} \tag{2.25}$$

A more detailed treatment of such instanton applications is given by Coleman in [61].

Chapter 3

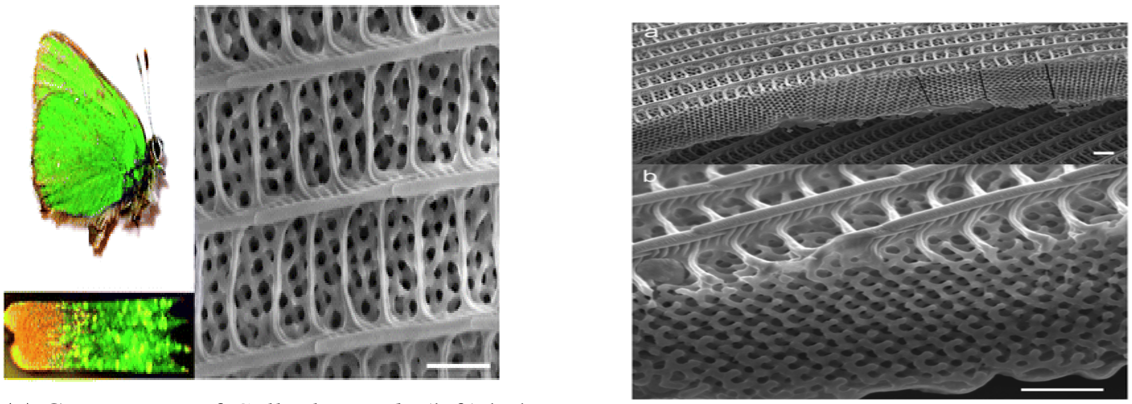
Some New Exact Solutions

> Doubtless we cannot see that other higher Spaceland now, because we have no eye in our stomachs.
>
> *Flatland: A Romance in Many Dimensions*
> Edwin Abbott

In this chapter, we will construct and explore several new, exact solutions in Einstein-Maxwell Theory of five and higher dimensions. The particular embedding ansatz used were inspired by M-theory and because of its similarity with Einstein-Maxwell theory, it motivates the question of what new insights we can gain from one theory into another. Ultimately, our goal is to understand the relationship between gravity and electromagnetism. The solutions on the embedding of Eguchi-Hanson and Nutku spaces were coauthored with A.M. Ghezelbash and published in [78] and [79], respectively. Since minimal surfaces play an important role in the Nutku-embedded solutions, we shall begin with an overview of the topic.

The mathematics of minimal surfaces is closely related to variational calculus developed by Euler and Lagrange. It was Lagrange who first asked the question on the existence of surfaces that minimize the area subject to some boundary constraints. The Belgian physicist Joseph Plateau studied minimal surfaces [72] through soap films on wire frames and the question of the existence of area-minimizing surfaces with given boundaries thus came to be known as Plateau's problem. Many mathematicians, including Weirstrass, Enneper and Riemann have made significant contributions to their investigation since.

Minimal surfaces have been observed elsewhere in nature and have found surprising applications in materials science. The wing scales of butterflies like *Parides sesostris, Teinopalpus imperialis, Cyanophrys remus, Mitoura gryneus, Callophrys dumetorum* and *Callophrys rubi* have triply periodic gryroid minimal surface structure which give rise to interesting optical phenomena [73]. Material scientists have long sought a photonic analogue of the transistor and triply periodic photonic crystals have been proposed as a viable candidate, motivating the study of biological photonic crystals. Figure 3.0.1 shows the microscopic structure of a butterfly wing.



(a) Green wings of *Callophrys rubi* (left) light micrograph of a single wing scale (right)

(b) Wing SEM micrograph

Figure 3.0.1: The microscopic structure of a butterfly wing scale with 1 μm scale bars. Source: Mille et al (2011) [74]

3.1 Minimal surface analysis

Let the function $u(x_1, ..., x_n)$, defined on an open bounded set Ω, represent the graph of a surface in $\mathbb{R}^{n+1}$. Then the area functional $\mathcal{A}(u)$ in terms of the gradient is:

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx_1 ... dx_n \tag{3.1}$$

If the function u minimises the area functional then it must satisfy the Euler-Lagrange equation:

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \tag{3.2}$$

The above equation is a quasi-linear partial differential equation and is the general minimal surface equation. For surfaces in $\mathbb{R}^3$, it can be written in the form of the "Lagrange equation":

$$(1 + |u_x|^2)u_{yy} - 2u_x u_y u_{xy} + (1 + |u_y|^2)u_{xx} = 0 \tag{3.3}$$

The plane is the trivial solution given by $u(x, y) = ax + by + c$. In 1742, Euler [82] found the first non-trivial minimal surface by forming the surface of revolution of the catenary; the surface is now called the catenoid [81]. The helicoid, Scherk surface, the gyroid, Enneper surface, Costa surface and several others have been found since [91]. Figure 3.1.1 shows some examples of minimal surfaces.

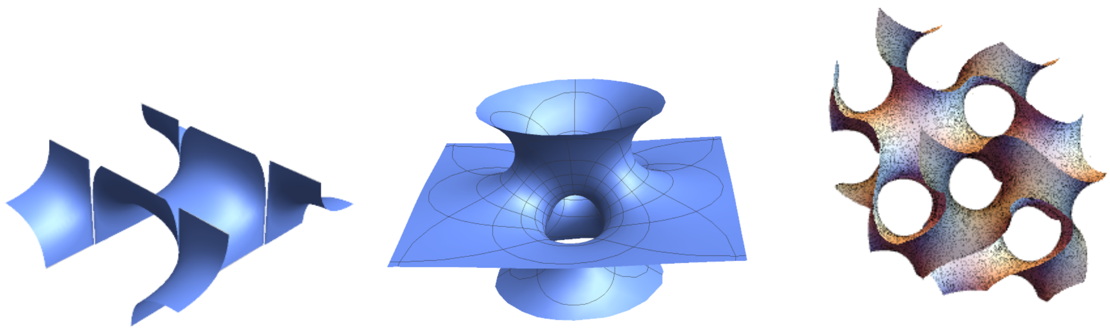


Figure 3.1.1: Scherk's 1st surface, Costa's surface, gyroid

3.1.1 Definition and examples

Let's state some equivalent definitions of minimal surfaces [81].

Definition 1:

A surface $M \subset \mathbb{R}^3$ is minimal if and only if it can locally be expressed as the graph of a solution to eq. (3.3).

Definition 2:

A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature vanishes identically.

Definition 3:

A surface $M \subset \mathbb{R}^3$ is minimal if and only if it is a critical point of the area functional for all compactly supported versions.

Definition 4:

A surface $M \subset \mathbb{R}^3$ is minimal if and only if every point $p \in M$ has a neighbourhood with least area relative to its boundary.

The following are some examples of minimal surfaces, written as solutions to Eq. (3.3):

$$catenoid: \quad u(x,y) = \frac{1}{a}\cosh^{-1}(a\sqrt{x^2+y^2}) \tag{3.4}$$

$$helicoid: \quad u(x,y) = a\tan^{-1}\left(\frac{y}{x}\right) \tag{3.5}$$

$$Scherk's\ 1st\ surface: \quad u(x,y) = \frac{1}{a}\ln\frac{\cos(ay)}{\cos(ax)} \tag{3.6}$$

3.1.2 The Helicoid

The helicoid was discovered by Jeun Baptiste Meusnier who proved its minimality in 1776 [92]. In $\mathbb{R}^3$, the helicoid and the plane are the only ruled 2-surfaces [81], i.e. surfaces generated by the rotation of a line. It can be defined as a surface on which every point has a helix passing through it. The *right circular* helicoid[1] can be parameterised as [83]:

$$\sigma(\theta,t) = \begin{pmatrix} t\cos\theta \\ t\sin\theta \\ c\theta \end{pmatrix}, \qquad t,\theta \in (-\infty,\infty) \tag{3.8}$$

where c is a constant that determines the pitch and chirality of the helices. In Figure 3.1.2, $c > 0$ but for $c < 0$, the helicoid will be "left-handed" or twisting in the opposite direction.

[1]Although the helicoid above is of most interest for this thesis, it is only a special case. The generalised helicoid is given by [83]:

$$\sigma(\theta,t) = (x(t)\cos\theta, x(t)\sin\theta, z(t)+c\theta), \qquad t,\theta \in (-\infty,\infty) \tag{3.7}$$

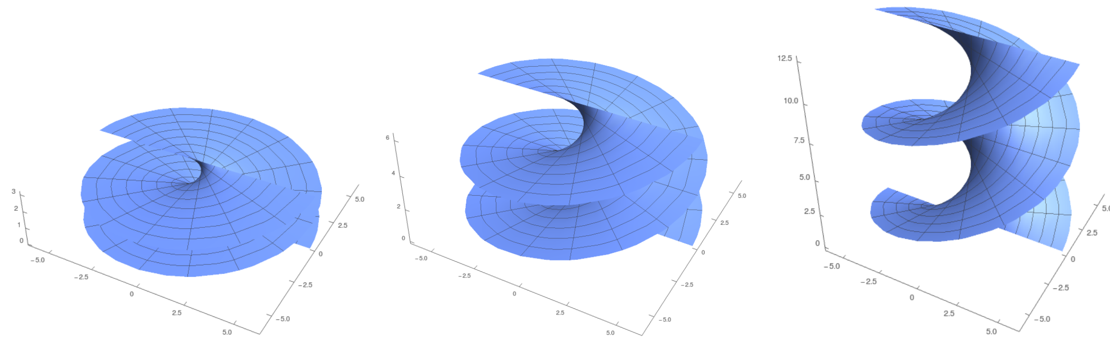


Figure 3.1.2: Left to right: Helicoids with $c = 0.5, 1, 2$

Let's replace t with r in eq. (3.1.2) and write the tangent vectors of the (right circular) helicoid as [84]:

$$\mathbf{e}_r = \begin{pmatrix} \cos z \\ \sin z \\ 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{e}_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ c \end{pmatrix} \tag{3.9}$$

Thus, the induced metric on the helicoid is:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{bmatrix} 1 & 0 \\ 0 & c^2 + r^2 \end{bmatrix} \tag{3.10}$$

$$\Longleftrightarrow ds^2 = dr^2 + (r^2 + c^2)d\theta^2 \tag{3.11}$$

The line element above shows that if $c = 0$, then it is just the metric on the flat plane. The second fundamental form b_{ij} in terms of the normal vector and the first partial derivatives of the tangent vectors is:

$$b_{ij} = \mathbf{e}_{i,j} \cdot \mathbf{n} = \mathbf{e}_{i,j} \cdot \frac{\mathbf{e}_r \times \mathbf{e}_\theta}{\sqrt{|g|}} = -\frac{c}{\sqrt{c^2 + r^2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.12}$$

The mean curvature is:

$$H = \frac{b_{ij}g^{ij}}{2} = \frac{b_{rr}g^{rr} + b_{\theta\theta}g^{\theta\theta}}{2} = 0 \tag{3.13}$$

which shows that the helicoid is a minimal surface.

Its Gaussian curvature is always negative:

$$K = \frac{|b|}{|g|} = -\frac{c^2}{(c^2 + r^2)^2} < 0 \tag{3.14}$$

Furthermore, its Riemann curvature[2], Kretschmann scalar, Ricci curvature and Ricci scalar, respectively are:

$$R_{r\theta r\theta} = -\frac{c^2}{c^2 + r^2} \tag{3.15}$$

$$\mathcal{K} = 4\,\frac{c^4}{\left(c^2 + r^2\right)^4} \tag{3.16}$$

$$R_{\mu\nu} = \begin{bmatrix} -\frac{c^2}{\left(c^2+r^2\right)^2} & 0 \\ 0 & -\frac{c^2}{c^2+r^2} \end{bmatrix} \tag{3.17}$$

$$\mathcal{R} = -2\,\frac{c^2}{\left(c^2 + r^2\right)^2} \tag{3.18}$$

Evidently, the helicoid is not Ricci flat but it satisfies the vacuum Einstein equations[3] $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}$ and has no curvature singularities.

3.1.3 The Catenoid

The catenoid is the only minimal surface of revolution in $\mathbb{R}^3$ other than the plane.[4] The helicoid and the catenoid are locally isometric[5] [93] and are conjugates[6] of each other [81]. A physical example of the catenoid is created by immersing two overlapping circular rings in a soap solution, taking them out and separating them slowly along the azimuthal axis. There is a critical distance, beyond which separating the rings will collapse the catenoid into two separate disc films inside the rings [101].

The catenoid is a surface of revolution generated by the catenary $y = c\cosh\left(\frac{x}{c}\right)$ revolved around

[2]In this case, we will write the only non-zero component of the Riemann tensor R_{abcd}.

[3]All 2-surfaces satisfy Einstein vacuum equations.

[4]A surface generated by rotating a curve about an axis is a surface of revolution.

[5]Local isometry implies that two points close to each other on one manifold can be mapped to another while keeping the distance preserved.

[6]Conjugate surfaces have the interesting property that a straight line in one surface can be mapped to a geodesic in the other and vice-versa.

the x-axis.[7] It can be parameterised as:

$$\sigma(u,v) = \begin{pmatrix} c\cosh\left(\frac{v}{c}\right)\cos u \\ c\cosh\left(\frac{v}{c}\right)\sin u \\ v \end{pmatrix}, \qquad u \in [0, 2\pi), \quad v \in (-\infty, \infty) \tag{3.19}$$

The constant c is the width of the "throat" of the catenoid. Figure 3.1.3 shows different slices for a catenoid for different ranges of parameters, with $c = 3$ in all of them. Figure 3.1.3 (a) has $v \in [0, 6]$, $u \in [0, 2\pi)$, (b) has $v \in [-6, 6]$, $u \in [0, \pi]$ and (c) has $v \in [-6, 6]$, $u \in [0, 2\pi)$.

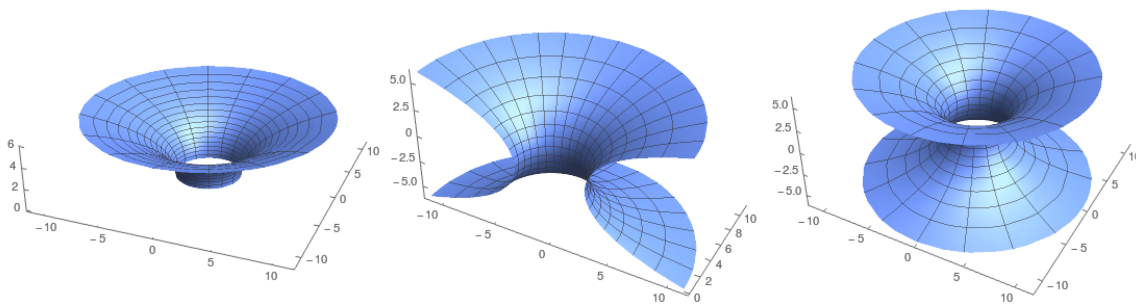


Figure 3.1.3: (a),(b),(c) Sections of a catenoid

The tangent vectors are given by:

$$\mathbf{e}_v = \begin{pmatrix} \sinh\left(\frac{v}{c}\right)\cos u \\ \sinh\left(\frac{v}{c}\right)\sin u \\ 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{e}_u = \begin{pmatrix} -c\cosh\left(\frac{v}{c}\right)\sin u \\ c\cosh\left(\frac{v}{c}\right)\cos u \\ 0 \end{pmatrix} \tag{3.20}$$

The induced metric on the catenoid is:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \cosh^2\left(\frac{v}{c}\right)\begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix} \tag{3.21}$$

$$\Longleftrightarrow ds^2 = \cosh^2\left(\frac{v}{c}\right)(dv^2 + c^2 du^2) \tag{3.22}$$

The Riemann curvature, Kretschmann scalar,[8] Ricci curvature and Ricci scalar, respectively

[7] A catenary is a transcendental curve that represents a heavy, flexible rope hanging from two ends. This problem arises in the calculus of variations.

[8] Notice that although the only non-zero component of the Riemann tensor is $R_{vuvu} = -1$, its "square", i.e. the

are:

$$R_{\mu\nu\mu\nu} = -1 \tag{3.23}$$

$$\mathcal{K} = \frac{4}{\left(c\cosh^2(\frac{v}{c})\right)^4} \tag{3.24}$$

$$R_{\mu\nu} = \begin{bmatrix} -\frac{1}{c^2\cosh^2(\frac{v}{c})} & 0 \\ 0 & -\frac{1}{\cosh^2(\frac{v}{c})} \end{bmatrix} \tag{3.25}$$

$$\mathcal{R} = -\frac{2}{c^2\cosh^4(\frac{v}{c})} \tag{3.26}$$

3.2 Minimal surfaces and instantons

In 1978, Comtet [96] showed that the multi-BPST-instanton solution of Witten [97] corresponds to minimal surfaces. In a similar vein, Yavuz Nutku [85] proved in 1996, that for every minimal surface in $\mathbb{R}^3$, there is a gravitational instanton with anti-self-dual curvature and gave some explicit examples. Despite the remarkable nature of his short article, it appears to be largely unnoticed and underresearched.

According to Nutku's work, a large class of gravitational instantons can be written with the line element:

$$ds^2 = \frac{1}{\mathcal{A}(t,x)}\{(f_t^2+\kappa)(dt^2+dy^2)+(1+f_x^2)(dx^2+dz^2)+2f_tf_x(dtdx+dydz)\}, \tag{3.27}$$

where $\mathcal{A}(t,x) = \sqrt{1+\kappa f_t^2+f_x^2}$, $f_t = \frac{\partial f(t,x)}{\partial t}$ and $f_x = \frac{\partial f(t,x)}{\partial x}$, if the function $f(t,x)$ satisfies the quasi-linear, elliptic-hyperbolic partial differential equation

$$(1+f_x^2)f_{tt} - 2f_tf_xf_{tx} + (\kappa+f_t^2)f_{xx} = 0. \tag{3.28}$$

Eq. (3.28) in the elliptic case ($\kappa = 1$) defines minimal surfaces in $\mathbb{R}^3$, as we can see from eq. (3.3). In the hyperbolic case ($\kappa = -1$), it becomes the Born-Infeld equation [85], which arises as a wavelike

Kretschmann invariant, $\mathcal{R} = R_{abcd}R^{abcd}$ is not a constant.

equation in Born-Infeld theory, a non-linear generalization of Maxwell electrodynamics. These two cases are also related through a Wick rotation ($t \to it$) [98]. Interestingly, the Born-Infeld equation is also related to the maximal surface equation in Lorentz-Minkowski space $\mathbb{L}^3$ by a Wick rotation $x \to ix$ [99].

3.3 The Nutku helicoid solution

Recall from Section 3.1.1 that the helicoid represented by $f(t,x) = a\tan^{-1}(\frac{x}{t})$ is a solution to equation (3.28). Substitution into Eq. (3.28) gives:

$$\mathcal{A}(t,x) = \sqrt{\frac{a^2+t^2+x^2}{t^2+x^2}}, \quad f_t = -\frac{ax}{t^2+x^2} \tag{3.29}$$

$$f_x = \frac{at}{t^2+x^2}, \quad f_{xx} = -2\,\frac{atx}{\left(t^2+x^2\right)^2} \tag{3.30}$$

$$f_{tt} = 2\,\frac{atx}{\left(t^2+x^2\right)^2}, \quad f_{tx} = \frac{\left(-t^2+x^2\right)a}{\left(t^2+x^2\right)^2} \tag{3.31}$$

Now the instanton metric from Eq. (3.27) becomes:

$$ds^2 = \frac{(a^2x^2+(t^2+x^2)^2)(dt^2+dy^2) - (2a^2tx)(dtdx+dydz) + (a^2t^2+(t^2+x^2)^2)(dx^2+dz^2)}{\sqrt{\frac{a^2+t^2+x^2}{t^2+x^2}}\left(t^2+x^2\right)^2} \tag{3.32}$$

The Kretschmann scalar,

$$\mathcal{K} = 24\,\frac{a^4\left(a^4+3\,a^2t^2+3\,a^2x^2+3\,t^4+6\,t^2x^2+3\,x^4\right)}{\left(a^2+t^2+x^2\right)^3\left(t^2+x^2\right)^3} \tag{3.33}$$

is singular at $t = x = 0$.

3.3.1 Wick rotation: 4-D spacetime

What happens when we do a Wick rotation $t \to \pm it$ to the metric in Eq. (3.32) Intriguingly, the metric changes its signature to Lorentzian and becomes a spacetime! The spacetime metric is:

$$ds^2 = \frac{(a^2x^2 + (x^2 - t^2)^2)(dy^2 - dt^2) - (2a^2tx)(\mp dydz - dtdx) + ((x^2 - t^2)^2 - a^2t^2)(dx^2 + dz^2)}{\sqrt{\frac{a^2 - t^2 + x^2}{x^2 - t^2}}\left(x^2 - t^2\right)^2} \tag{3.34}$$

The $dtdx$ in the metric above indicates a coupling between the timelike coordinate t and the spacelike coordinate x. The Kretschmann scalar for this spacetime also reflects the Wick rotation.

$$\mathcal{K} = 24\,\frac{a^4\left(a^4 - 3\,a^2t^2 + 3\,a^2x^2 + 3\,t^4 - 6\,t^2x^2 + 3\,x^4\right)}{\left(a^2 - t^2 + x^2\right)^3\left(x^2 - t^2\right)^3} \tag{3.35}$$

$$= 24\,\frac{a^4\left(a^2 - t^2 + x^2\right)^2 + 2\,\left(x^2 - t^2\right)^2 - a^2\left(x^2 - t^2\right)}{\left(a^2 - t^2 + x^2\right)^3\left(x^2 - t^2\right)^3} \tag{3.36}$$

However, $\mathcal{K}$ is no longer only singular at one point on the manifold but for all such points with $x^2 - t^2 + a^2 = 0$ or $x = \pm t$. Figure 3.3.1 shows the singular points as a graph in the xy-plane. It bears a striking resemblance to the invariant Minkowski hyperbolae in special relativity. Figure 3.3.2 shows the plots for $\mathcal{K}$ as a function of the coordinates t and x.

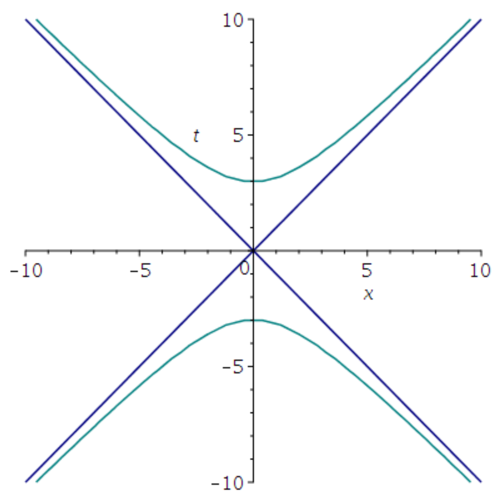


Figure 3.3.1: Singular points $x = \pm t$ and $x^2 - t^2 + a^2 = 0$ with $a = 3$

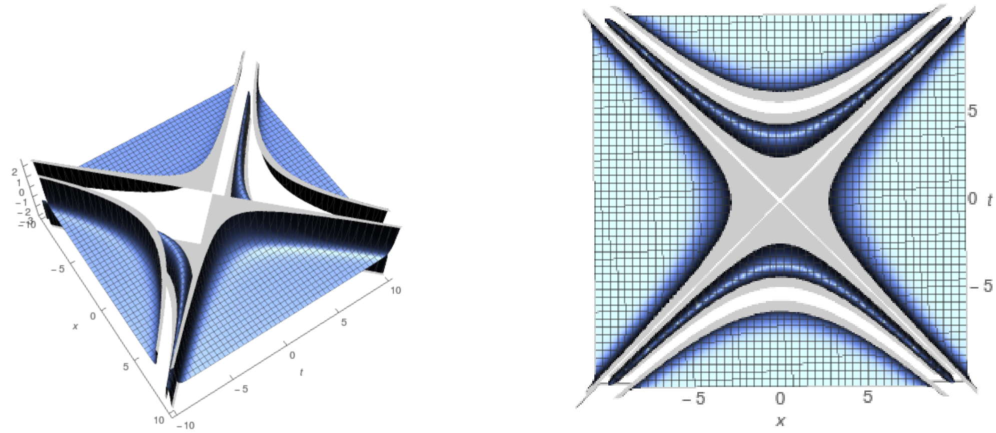


Figure 3.3.2: Plots of $\mathcal{K}(x,t)$ for the 4-D helicoid spacetime

The event horizons are present where g_{tt} vanishes, i.e.

$$\frac{a^2x^2 + \left(-t^2 + x^2\right)^2}{\left(-t^2 + x^2\right)^{3/22}} \frac{1}{\sqrt{a^2 - t^2 + x^2}} = 0 \tag{3.37}$$

The only $x, t, a \in \mathbb{R}$ which can satisfy the above equation are $x = t = 0$. Recall that $x = t = 0$ was a curvature singularity point for the positive signature instanton. This spacetime is unusual because event horizons generally cover curvature singularities[9] but in this spacetime, it appears that the curvature singularities are covering up an event horizon, if it can even be called one. Of course, I have tacitly implied $x \in (-\infty, \infty)$ but the coordinates can be chosen and transformed arbitrarily.

3.3.2 Coordinate transform of the instanton

Let's go back to the Euclidean signature Nutku instanton. After coordinate transformations $x = r\cos\theta$ and $t = r\sin\theta$, we obtain the original form of the Nutku helicoid instanton [86]:

$$ds^2 = \frac{dr^2 + \left(a^2 + r^2\right) d\theta^2 + \left(1 + \frac{a^2(\sin(\theta))^2}{r^2}\right) dy^2 - \frac{a^2 \sin(2\,\theta) dy dz}{r^2} + \left(1 + \frac{a^2(\cos(\theta))^2}{r^2}\right) dz^2}{\sqrt{1 + \frac{a^2}{r^2}}}, \tag{3.38}$$

Nutku and others take $0 < r < \infty$, $0 \leq \theta \leq 2\pi$ and y and z as the periodic coordinates on the 2-torus [86]. It helps avoid the singularity at $r = 0$ but restricts the extent of the manifold. In

[9]It's been conjectured that naked singularities cannot exist in nature to avoid offending prudish physicists. The conjecture is called the Cosmic Censorship Hypothesis.

[100], the authors studied the Dirac equation in the background of the Nutku metric (3.38) and the resulting singularity structure, concluding that the helicoid system is not stable while the catenoid is.

The metric (3.38) is asymptotically Euclidean and would correspond to a catenoidal solution if a^2 is replaced with $-a^2$. From here on, we will use $\epsilon = \pm 1$ to differentiate between the helicoid ($\epsilon = 1$) and catenoid ($\epsilon = -1$) cases, respectively.

$$ds^2_{Nutku} = \frac{dr^2 + \left(\epsilon a^2 + r^2\right) d\theta^2 + \left(1 + \epsilon \frac{a^2(\sin(\theta))^2}{r^2}\right) dy^2 - \epsilon \frac{a^2 \sin(2\,\theta) dy dz}{r^2} + \left(1 + \epsilon \frac{a^2(\cos(\theta))^2}{r^2}\right) dz^2}{\sqrt{1 + \epsilon \frac{a^2}{r^2}}}. \tag{3.39}$$

The Kretschmann invariant of the helicoid or catenoid instanton is given by

$$\mathcal{K} = \frac{72 a^4}{r^4 (r^2 + \epsilon a^2)^2} + \frac{24 a^8}{r^6 (r^2 + \epsilon a^2)^3}. \tag{3.40}$$

The helicoid has only a curvature singularity at $r = 0$, while for the catenoid, there is another singularity at $r = a$.

3.4 The Ansatz

To find new solutions, we shall use the following ansatz [76] and solve the Einstein-Maxwell equations:

$$ds^2 = -\frac{dt^2}{H(x^i)^2} + H(x^i)^{\frac{2}{D-3}} (d\chi^2 + \chi^2 d\Omega_{D-6} + ds_n{}^2) \tag{3.41}$$

where $H(x^i)$ is a gauge function depending on any number of coordinates x^i of the D-dimensional spacetime.[10] For example, it could be $H(r)$ or $H(r, x)$. We also demand that the only non-zero component of A_μ be:

$$A_t = \sqrt{\frac{D-2}{D-3}} \frac{1}{H(x^i)}. \tag{3.42}$$

The exponents -2 and $\frac{2}{D-3}$ of $H(x^i)$ in the metric were chosen such that any resulting differential

[10] For this ansatz to work, I conjecture that ds_n^2 must be Ricci-flat. All examples known to me of the successful embeddings through this ansatz are of Ricci-flat spaces.

equations for $H(x^i)$ are linear and homogeneous. However, I do not provide any mathematical proof for this claim but instead encourage the curious mind to verify this for any given embedding using this ansatz and for the more mathematically inclined, prove or disprove the claim. The linear nature of any resulting differential equation would solving them easier and also allow linear superposition of solutions.

The Majumdar-Papapetrou solution [87][88] can be seen as an application of the ansatz in 4 dimensions and the Kastor-Treschen solution [89] in 5 dimensions.

3.4.1 Embedding in 5-D

We will begin with a 5-dimensional ansatz $ds^2 = -\frac{dt^2}{H(r)^2} + H(r)(ds_{Nutku}{}^2)$ and solve the Einstein-Maxwell equations. Remarkably, these result in:

$$\mathcal{G}_{\mu\nu} = \begin{bmatrix} f_{11}(r)ODE & 0 & 0 & 0 & 0 \\ 0 & f_{22}(r)ODE & 0 & 0 & 0 \\ 0 & 0 & f_{33}(r)ODE & 0 & 0 \\ 0 & 0 & 0 & f_{44}(r)ODE & f_{45}(r)ODE \\ 0 & 0 & 0 & f_{45}(r)ODE & f_{55}(r)ODE \end{bmatrix} \tag{3.43}$$

$$F^{\mu\nu}_{;\mu} = \begin{bmatrix} g(r)ODE & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.44}$$

where, in order to satisfy the Einstein-Maxwell equations, we only need to solve one differential equation by setting the expression ODE to be zero:

$$\epsilon a^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} H(r) \right) + \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} H(r) \right) r^2 + \left(\frac{\mathrm{d}}{\mathrm{d}r} H(r) \right) r = 0 \tag{3.45}$$

$f(r) = \ln\left(r + \sqrt{r^2 + \epsilon a^2}\right)$ and $g(r) = c_1$ are solutions to the above differential equation, where c_1 is a constant.[11] For the helicoidal case, we can check that they are linearly independent on the

[11] $f(-r) = \ln\left(-r + \sqrt{r^2 \pm \epsilon a^2}\right)$ is also a solution for both cases but we ignore it on the grounds of unconditional linear independence.

interval $-\infty < r < \infty$ by taking the Wronskian:

$$W(f,g)(r) = f(r)g'(r) - g(r)f'(r) = -\frac{c_1}{\sqrt{r^2+\epsilon a^2}} \neq 0 \tag{3.46}$$

3.5 Helicoid-catenoid solutions in 5-D Einstein-Maxwell theory

We consider the source-free Einstein-Maxwell equations with geometrised units in D-dimensions that are given by

$$R_{\mu\nu} = F_{\mu}^{\lambda} F_{\nu\lambda} - \frac{1}{4+2(D-4)} g_{\mu\nu} F^2, \tag{3.47}$$

$$F^{\mu\nu}_{;\mu} = 0, \tag{3.48}$$

where the electromagnetic field tensor is given by

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \tag{3.49}$$

in terms of the electromagnetic potential A^{μ}. We will take its only non-zero component as

$$A_t = \sqrt{\frac{D-2}{D-3}} \frac{1}{H(r)}. \tag{3.50}$$

In 5 dimensions, we consider the following 5-D ansatz by adding a time coordinate to the Nutku helicoid or catenoid instanton

$$ds^2 = -\frac{dt^2}{(H(r))^2} + \frac{H(r)}{\sqrt{1+\frac{a^2}{r^2}}} (ds^2_{Nutku}). \tag{3.51}$$

We find that all Einstein and Maxwell equations are satisfied if the metric function $H(r)$ satisfies the differential equation

$$\epsilon \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} H(r) \right) a^2 + \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} H(r) \right) r^2 + \left(\frac{\mathrm{d}}{\mathrm{d}r} H(r) \right) r = 0. \tag{3.52}$$

The solutions to (3.52) are given by

$$H(r) = c_1 + C_2 \ln\left(\frac{r + \sqrt{\epsilon a^2 + r^2}}{a}\right), \tag{3.53}$$

where c_1 and C_2 are two constants.

The logarithmic solutions above correspond to $\sinh^{-1}(\frac{r}{a})$ and $\cosh^{-1}(\frac{r}{a})$ functions. Interestingly, these functions appear in the study of collapsing catenoidal soap films [101] and in the embedding of wormhole handles [90]. Notice that a submanifold of the $5-D$ helicoidal spacetime is conformally equivalent to a wormhole handle $(dr^2 + \left(a^2 + r^2\right) d\theta^2)$ [90][102] but for an interpretation as as a wormhole, the coordinate $r \in (-\infty, \infty)$.

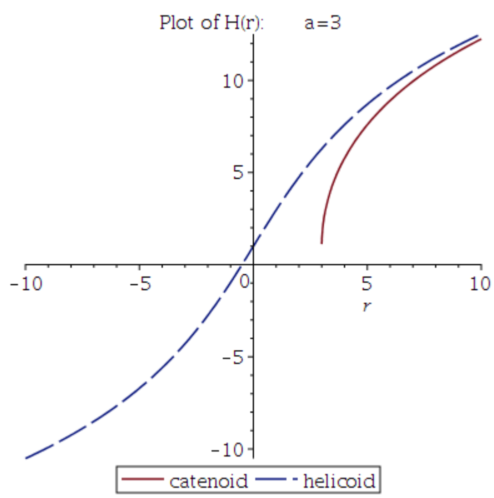


Figure 3.5.1: $H(r)$, for the helicoid and catenoid solutions, with $a = 3, c_1 = 1, c_2 = 6$.

Inspired by the soap film solutions [101], we set $c_1 = 1$ and $C_2 = c_2\, a$,[12] so that in the limit $a \longrightarrow 0$, the metric (3.51) becomes Minkowski spacetime. Thus, we have

$$H(r) = 1 + c_2\, a \sinh^{-1}(\frac{r}{a}), \tag{3.54}$$

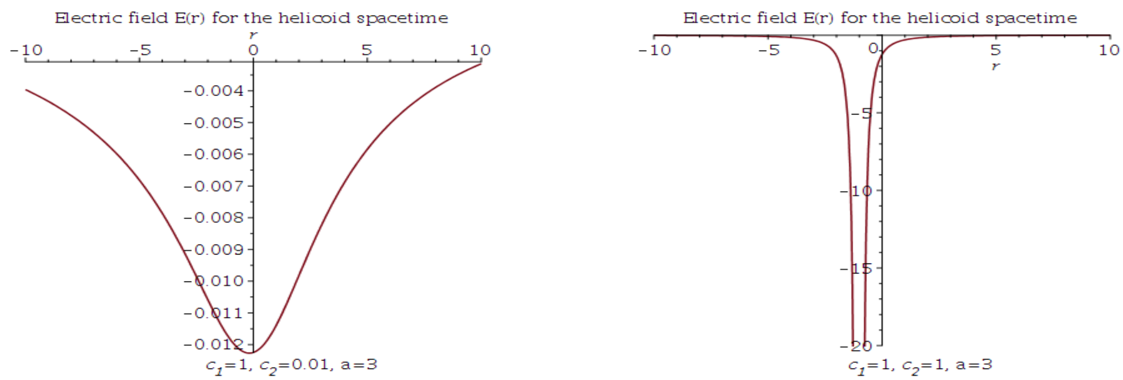
$$H(r) = 1 + c_2\, a \cosh^{-1}(\frac{r}{a}), \tag{3.55}$$

for the helicoid and the catenoid solutions, respectively. Figure 3.5.1 shows the typical behaviour

[12]In fact, we may even get rid of the constant parameter c_2 by setting it to be $c_2 = 2$ in order to match the height function for a catenoidal soap film. It is useful to keep c_2 for now to illustrate what it may do to the electric field.

of the metric functions $H(r)$ for helicoid and catenoid solutions. It is important to note here that in case of the helicoid, the function $H(r)$ can become negative and hence change the metric signature from Lorentzian to Riemannian for some value of the radial coordinate r. Although such signature changes are of interest in quantum cosmology [103], and may be dealt with through a Wick rotation, we wish to keep the metric well-behaved and thus require $a \geq 0$ and $c_2 \geq 0$.

Figure 3.5.2 shows how the electric field may or may not diverge in the helicoid spacetime, depending on the choice of constants. The region $r < 0$ is included in the plots only to illustrate this behaviour. The electric field for the catenoid spacetime ($a < r < \infty$) is shown in Figure 3.5.3.



(a) $a = 3, c_1 = 1, c_2 = 0.01$ **(b)** $a = 3, c_1 = 1, c_2 = 1$

Figure 3.5.2: The r-component of the electric field as a function of r, for the helicoid solution

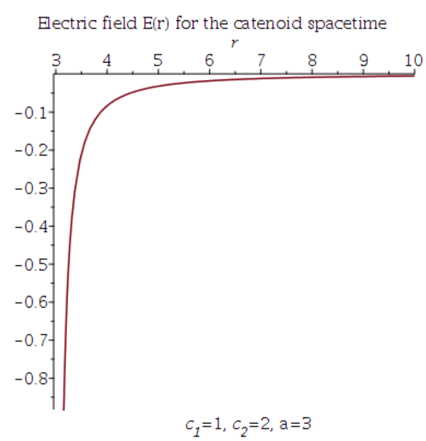


Figure 3.5.3: $E_r(r)$ for the catenoid solution with $a = 3, c_1 = 1, c_2 = 2$.

If we replace $H(r)$ with $H(r,t)$ in the metric ansatz (3.51), we can get a dynamic solution to the

Einstein-Maxwell equations with a cosmological constant Λ, which is given by

$$H(r,t) = 1 \pm \sqrt{\Lambda}t + c_2\, a \ln\left(\frac{r + \sqrt{\epsilon a^2 + r^2}}{a}\right). \qquad (3.56)$$

Again, if we are to avoid the metric signature change while keeping a and c_2 positive, we should only consider the $+\sqrt{\Lambda}t$ solution. Figure 3.5.4 illustrates the electric field as a function of r and t for the helicoid and catenoid spacetimes with a cosmological constant.

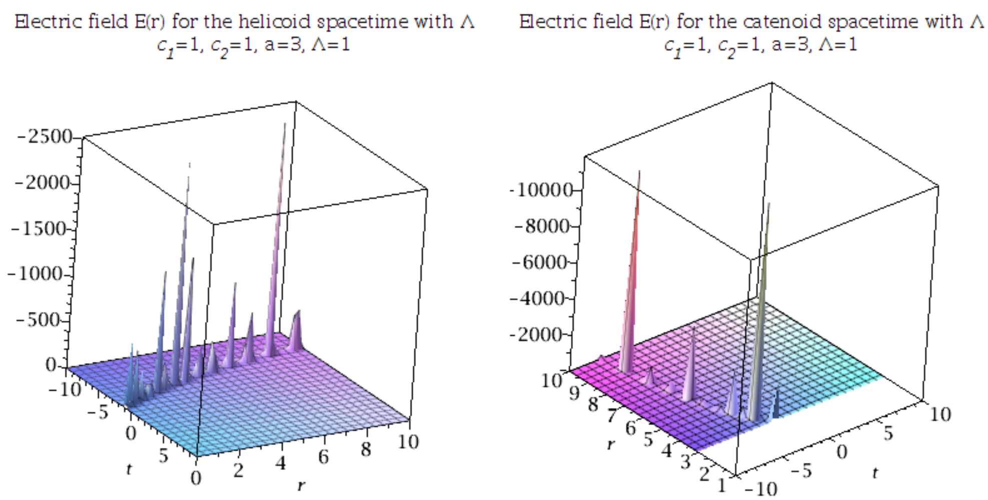


Figure 3.5.4: $E_r(r,t)$ with $a = 3, c_1 = 1, c_2 = 1, \Lambda = 1$, for the helicoid solution (left) and the catenoid solution (right).

3.6 Convoluted solutions in six dimensions

Now we may embed the Nutku helicoid instanton in 6 dimensions and drop the catenoid case to avoid any clutter that may ensue from keeping it. The ansatz (Eq. 3.41) for 6 dimensions gives us:

$$ds^2 = -H(r,x)^{-2}dt^2 + H(r,x)^{2/3}(dx^2 + ds^2_{Nutku}), \qquad (3.57)$$

where ds^2_{Nutku} is the 4-dimensional Nutku space given by Eq. (3.39). The Maxwell gauge field is again given by Eq. (3.42):

$$A_t(r,x) = \frac{2\sqrt{3}}{3H(r,x)}. \qquad (3.58)$$

Both the Einstein field equations and the Maxwell equations, in order to be satisfied, require that the metric function $H(r, x)$ satisfy the partial differential equation

$$r^2\sqrt{\frac{a^2+r^2}{r^2}}\frac{\partial^2}{\partial x^2}H(r,x)+(r^2+a^2)\left(\frac{\partial^2}{\partial r^2}H(r,x)\right)+\left(\frac{\partial}{\partial r}H(r,x)\right)r=0. \tag{3.59}$$

Because the PDE (3.59) above is linear, we can use separation of variables as $H(r, x) = 1 + R(r)X(x)$. Thus, we have two ordinary differential equations for $R(x)$ and $X(x)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r)+\varepsilon c^2R(r)\left(\sqrt{1+\frac{a^2}{r^2}}\right)^{-1}+\frac{\left(\frac{\mathrm{d}}{\mathrm{d}r}R(r)\right)r}{a^2+r^2}=0, \tag{3.60}$$

$$\frac{d^2}{dx^2}X(x)-\varepsilon c^2X(x)=0, \tag{3.61}$$

where $\varepsilon = \pm 1$ represents the two possible cases for the separation constant. Let's write Eq. (3.60) after transforming the independent variable according to:

$$z(r)=\sqrt{1+\frac{a^2}{r^2}} \tag{3.62}$$

For the $\varepsilon = +1$ case[13], we get:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}y(z)-\frac{\left(-2z^5+4z^3-2z\right)\frac{\mathrm{d}}{\mathrm{d}z}y(z)}{(z-1)^3(z+1)^3}+\frac{a^2c^2zy(z)}{(z-1)^3(z+1)^3}=0 \tag{3.63}$$

The above equation can be seen as a Double Confluent Heun Equation whose general form is:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}y(z)-\frac{\left(\alpha z^4-2z^5+4z^3-\alpha-2z\right)}{(z-1)^3(z+1)^3}\frac{\mathrm{d}}{\mathrm{d}z}y(z)=\frac{\left(-z^2\beta+(-\gamma-2\alpha)z-\delta\right)}{(z-1)^3(z+1)^3}y(z), \tag{3.64}$$

with the boundary conditions $y(0) = 1$, $\frac{dy}{dz}|_{z=0} = 0$.

A solution to Eq. (3.64) is given as the special "Heun-D" function $\mathcal{H}_{\mathcal{D}}(\alpha, \beta, \gamma, \delta, z)$. Comparing it to our Eq. (3.63), we see that $\alpha = \beta = \delta = 0$ and $\gamma = a^2c^2$.

[13] $\varepsilon = -1$ leads to divergent solutions. So, we won't analyse that case.

Thus the solution to (3.60) can be expressed as[14]:

$$R(r) = A_1 \mathcal{H}_{\mathcal{D}} \left(0, 0, a^2c^2, 0, \sqrt{1 + \frac{a^2}{r^2}} \right) \tag{3.65}$$

where A_1 is a constant. $\mathcal{H}_{\mathcal{D}}(\alpha, \beta, \gamma, \delta, z)$ is convergent in the disk $|z| < 1$ but $z(r) = \sqrt{1 + \frac{a^2}{r^2}} > 1$. For an analytic continuation in the domain $|z| > 1$, we use the identity:

$$\mathcal{H}_{\mathcal{D}} (\alpha, \beta, \gamma, \delta, z) = \mathcal{H}_{\mathcal{D}} \left(-\alpha, -\delta, -\gamma, -\beta, \frac{1}{z} \right) \tag{3.66}$$

$$\Longrightarrow \mathcal{H}_{\mathcal{D}} \left(0, 0, a^2c^2, 0, \sqrt{1 + \frac{a^2}{r^2}} \right) = \mathcal{H}_{\mathcal{D}} \left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{r^2 + a^2}} \right) \tag{3.67}$$

Hence, the solution to Eq. (3.60) is:

$$R(r) = A_1 \mathcal{H}_{\mathcal{D}} \left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{r^2 + a^2}} \right) \tag{3.68}$$

Figure 3.6.1 shows the typical behaviour of the Heun-D function for different values of a with fixed c and for different values of c with fixed a.

---

[14]Again, a second solution can be given as $A_2 \mathcal{H}_{\mathcal{D}} \left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{r^2+a^2}} \right) \int^{\sqrt{\frac{r^2}{a^2+r^2}}} \frac{1}{(f^2-1)\left(\mathcal{H}_{\mathcal{D}}(0,0,-a^2c^2,0,f)\right)^2} df$ but it is divergent and therefore, we set $A_2 = 0$.

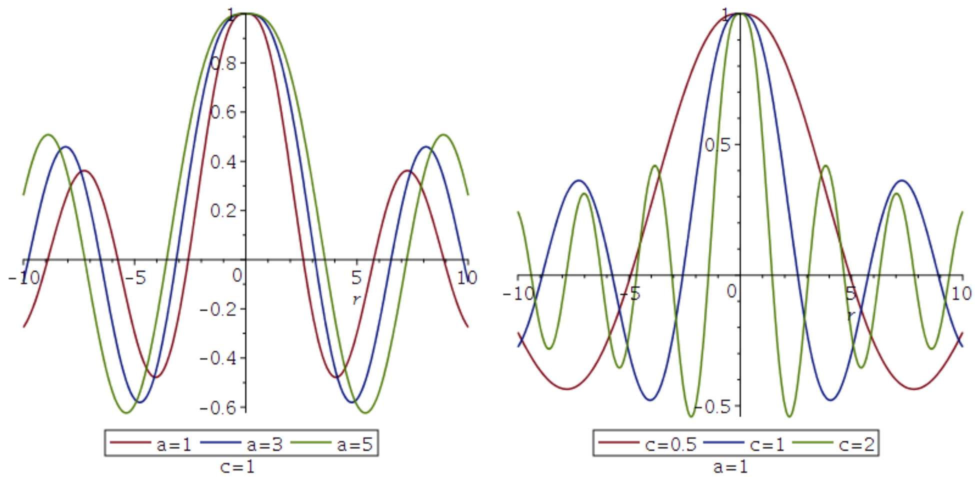


Figure 3.6.1: $\mathcal{H}_{\mathfrak{D}}\left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{a^2+r^2}}\right)$ for constant c (left) and constant a (right)

The solutions to (3.61) with $\varepsilon = +1$ for $X(x)$ are given by

$$X(x) = B_1\, \mathrm{e}^{cx} + B_2\, \mathrm{e}^{-cx}, \tag{3.69}$$

where B_1 and B_2 are two constants. To construct the most general convoluted solution to the partial differential equation (3.59), we superimpose all the solutions (3.68) and (3.69) while integrating over the separation constant c in the domain $c \in (0, \infty)$.

$$H(r, x) = 1 + \int_0^\infty \mathcal{H}_{\mathfrak{D}}\left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{r^2 + a^2}}\right)(f(c)e^{cx} + g(c)e^{-cx})\, dc, \tag{3.70}$$

where $f(c)$ and $g(c)$ are two arbitrary functions of c. To fix the arbitrary functions $f(c)$ and $g(c)$, we note that in the limit of $a \to 0$, the Nutku space describes a four-dimensional space $D^2 \otimes T^2$ with the line element[15]

$$ds^2_{a=0} = dr^2 + r^2 d\theta^2 + dy^2 + dz^2. \tag{3.71}$$

Quite interestingly, in this limit, we find an exact analytical solution to six-dimensional Einstein-

[15] D^2 is the two-dimensional disc and T^2, the 2-torus.

Maxwell theory with the line element

$$dS^2 = -H_0(r,x)^{-2}dt^2 + H_0(r,x)^{2/3}(dx^2 + ds^2_{a=0}), \tag{3.72}$$

and the Maxwell gauge field

$$\mathcal{A}_t(r,x) = \frac{2\sqrt{3}}{3H_0(r,x)}, \tag{3.73}$$

where the exact analytic metric function $H_0(r,x)$ is

$$H_0(r,x) = 1 + \frac{h_0}{\sqrt{r^2 + x^2}}, \tag{3.74}$$

and h_0 is a constant. We can now fix the functions $f(c)$ and $g(c)$ by requiring that the metric (3.57) and the gauge field (3.58), must approach the exact analytical metric (3.72) and the gauge field (3.73), respectively, in the limit $a \to 0$. These requirements imply that the convoluted metric function $H(r,x)$ in equation (3.70), must be equal to the exact analytic metric function $H_0(r,x)$ in Eq. (3.74), in the limit of $a \to 0$. The integrand of the convoluted metric function $H(r,x)$ in equation (3.70) contains the Heun-D function which is the solution to the differential equation (3.60) with $\varepsilon = +1$. This equation in the limit of $a \to 0$ reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r) + c^2 R(r) + \frac{\frac{\mathrm{d}}{\mathrm{d}r}R(r)}{r} = 0, \tag{3.75}$$

for which the solutions are given by the Bessel functions $J_0(cr)$ and $Y_0(cr)$. The Bessel function $Y_0(cr)$ is divergent and not oscillatingly decaying like the Heun-D function. However, the Bessel function $J_0(cr)$ appears similar to the Heun-D function.

So, we find an integral equation for the functions $f(c)$ and $g(c)$ which is

$$\int_0^\infty J_0(cr)(f(c)e^{cx} + g(c)e^{-cx})\, dc = \frac{h_0}{\sqrt{r^2 + x^2}}. \tag{3.76}$$

Integral equations are much harder to solve than differential equations and since there is no general method to solve the above integral equation, we are left with intuition and trial and error. Interestingly, the unique solutions to this integral equation for the functions $f(c)$ and $g(c)$ are given by the constant functions:

$$f(c) = \frac{h_0}{2}\xi_1, \;\; g(c) = \frac{h_0}{2}\xi_2, \tag{3.77}$$

where ξ_1 and ξ_2 are constants and $\xi_1 + \xi_2 = 1$. Furnished with all the necessary ingredients, we can now express the most general solution for the convoluted metric function $H(r, x)$ as

$$H(r, x) = 1 + \frac{h_0}{2}\int_0^\infty \mathcal{H}_\mathcal{D}\left(0, 0, a^2c^2, 0, \sqrt{\frac{r^2}{a^2 + r^2}}\right)(\xi e^{cx} + (1 - \xi)e^{-cx})\, dc, \tag{3.78}$$

where h_0 and ξ are two constants.

3.7 Convoluted solutions in higher dimensions

We can generalise the above results to dimensions higher than six. The metric Eq. (3.41) for D dimensions with Nutku instanton embedding is:

$$ds^2 = -H(r, x)^{-2}dt^2 + H(r, x)^{\frac{2}{D-3}}(dx^2 + x^2 d\Omega^2_{D-6} + ds^2_{Nutku}), \tag{3.79}$$

where $d\Omega^2_{D-6}$ is the metric on a $(D-6)$-dimensional unit sphere. Again, the the gauge field is given by Eq. (3.42), with its only non-zero component as:

$$A_t = \sqrt{\frac{D-2}{D-3}}\frac{1}{H(r, x)}. \tag{3.80}$$

We find that all the D-dimensional Einstein and Maxwell equations are satisfied if $H(r, x)$ satisfies the partial differential equation[16]:

$$r\sqrt{a^2 + r^2}\left(\frac{\partial^2}{\partial x^2}H(r, x) + \frac{D-6}{x}\frac{\partial}{\partial x}H(r, x)\right) + (r^2 + a^2)\left(\frac{\partial^2}{\partial r^2}H(r, x)\right) + \left(\frac{\partial}{\partial r}H(r, x)\right)r = 0. \tag{3.81}$$

As in the six-dimensional case, we separate the coordinates in $H(r, x)$ by $H(r, x) = 1 + R(r)X(x)$. The partial differential equation (3.81) then separates into two ordinary differential equations for $R(r)$ and $X(x)$. The differential equation for the radial function $R(r)$ is given by (3.60) where the

[16]It must be emphasized that I do not provide any proof that this is guaranteed to work for any dimension $D > 6$ but I have verified it up to $D = 11$ which allows us to deduce the general PDE in Eq. (3.81).

solutions are given in terms of Heun-D functions and the differential equation for $X(x)$ is

$$\frac{d^2}{dx^2}X(x) - \varepsilon c^2 X(x) + \frac{D-6}{x}\frac{dX(x)}{dx} = 0. \tag{3.82}$$

The solutions to (3.82) with $\varepsilon = 1$ for $D > 6$ are given by

$$X(x) = x^{\frac{7-D}{2}} I_{\frac{D-7}{2}}(cx) + x^{\frac{7-D}{2}} K_{\frac{D-7}{2}}(cx), \tag{3.83}$$

in terms of modified Bessel functions. As a result, we can write the most convoluted solution for the metric function in D dimensions $H_D(r,x)$ as

$$H_D(r,x) = 1 + \int_0^\infty \mathcal{H}_{\mathcal{D}}\left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{a^2+r^2}}\right)\left(f_D(c) I_{\frac{D-7}{2}}(cx) + g_D(c) K_{\frac{D-7}{2}}(cx)\right) x^{\frac{7-D}{2}} dc, \tag{3.84}$$

where $f_D(c)$ and $g_D(c)$ are two arbitrary functions in terms of separation constant c. We need to fix these two arbitrary functions to find a closed form for the metric function $H(r,x)$ in D-dimensions. As in the case of six-dimensional theory, we consider the limit of $a \to 0$, where the Nutku space reduces to $D^2 \otimes T^2$, with the line element (3.71). In this limit, we find an exact analytical solutions to D-dimensional Einstein-Maxwell theory with the line element

$$d\mathcal{S}^2 = -H_{D0}(r,x)^{-2}dt^2 + H_{D0}(r,x)^{\frac{2}{D-3}}(dx^2 + x^2 d\Omega^2_{D-6} ds^2_{a=0}), \tag{3.85}$$

and the Maxwell gauge field

$$\mathcal{A}_t(r,x) = \sqrt{\frac{D-2}{D-3}}\frac{1}{H_{D0}(r,x)}, \tag{3.86}$$

where the exact analytic metric function $H_{D0}(r,x)$ is

$$H_{D0}(r,x) = 1 + \frac{h_{D0}}{(r^2+x^2)^{\frac{D-5}{2}}}, \tag{3.87}$$

where h_{D0} is a constant. To fix the functions $f_D(c)$ and $g_D(c)$, we demand that the metric (3.79) and the gauge field (3.80), must reduce to the exact analytical metric (3.85) and the gauge field (3.86), respectively, in the limit of $a \to 0$. In other words, these requirements imply that the convoluted metric function $H_D(r,x)$ in equation (3.84), must be equal to the exact analytic metric function

$H_{D0}(r, x)$ in equation (3.87), in the limit of $a \to 0$. As in the case of six-dimensional solutions, the integrand of the convoluted metric function $H_D(r, x)$ in equation (3.84) contains the Heun-D function which is the solution to the differential equation (3.60) with $\varepsilon = +1$. This equation in the limit of $a \to 0$ reduces to equation (3.75) for which the solutions are given by the Bessel functions $J_0(cr)$ and $Y_0(cr)$. As a result, we find an integral equation for the functions $f_D(c)$ and $g_D(c)$ which is

$$\int_0^\infty J_0(cr)(f_D(c)I_{\frac{D-7}{2}}(cx) + g_D(c)I_{\frac{D-7}{2}}(cx))x^{\frac{7-D}{2}}\, dc = \frac{h_{D0}}{(r^2 + x^2)^{\frac{D-5}{2}}}. \tag{3.88}$$

After several trial and error attempts, we find that the unique solutions to this integral equation for the functions $f_D(c)$ and $g_D(c)$ are given by

$$f_D(c) = 0\ , g_D(c) = \alpha_D h_{D0} c^{\frac{D-5}{2}}, \tag{3.89}$$

where α_D's are given by $\alpha_7 = 1$, $\alpha_8 = \sqrt{\frac{2}{\pi}}$, $\alpha_9 = \frac{1}{2}$, $\alpha_{10} = \frac{1}{3}\sqrt{\frac{2}{\pi}}$, $\cdots$. So, we can write the most general solution for the convoluted metric function $H_D(r, x)$ which is given by

$$H_D(r, x) = 1 + \alpha_D h_{D0} \int_0^\infty \mathcal{H}_{\mathcal{D}}\left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{a^2 + r^2}}\right) K_{\frac{D-7}{2}}(cx)\, x^{\frac{7-D}{2}}\, c^{\frac{D-5}{2}}\, dc. \tag{3.90}$$

As in the five-dimensional case, we notice that asymptotically, a two-dimensional submanifold of the solutions (3.57) and (3.79) corresponds to half of an Ellislike wormhole handle [102].

3.8 The cosmological convoluted solutions

We consider the Einstein-Maxwell theory with a cosmological constant in six dimensions, where the metric function H also depends explicitly on the timelike coordinate t:

$$ds^2 = -H(t, r, x)^{-2}dt^2 + H(t, r, x)^{2/3}(dx^2 + ds^2_{Nutku}). \tag{3.91}$$

We also consider the Maxwell gauge field as

$$A_t(t, r, x) = \frac{2\sqrt{3}}{3H(t, r, x)}. \tag{3.92}$$

The Einstein equations and Maxwell equations in presence of cosmological constant lead to a second order partial differential equation for $H(t, r, x)$, which is given by

$$r\left(\left(\frac{5}{3}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{3\Lambda}{2}\right)H^{5/3} + \left(\frac{\partial^2 H}{\partial t^2}\right)H^{8/3} - \frac{\partial^2 H}{\partial x^2}\right)\sqrt{a^2+r^2} - \left(a^2+r^2\right)\frac{\partial^2 H}{\partial r^2} - r\frac{\partial H}{\partial r} = 0. \tag{3.93}$$

The form of partial differential equation (3.93) leads us to separate the coordinates as

$$H(t, r, x) = 1 + T(t) + R(r)X(x). \tag{3.94}$$

We find three ordinary uncoupled differential equations for $R(r)$, $X(x)$ and $T(t)$ functions. The partial differential equations for $R(r)$ and $X(x)$ are given by (3.60) and (3.61) and the solutions to the differential equation for $T(t)$ are

$$T(t) = \alpha t + \beta, \tag{3.95}$$

where $\alpha = 3\sqrt{\frac{\Lambda}{10}}$ and β is an arbitrary constant. Requiring in the limit of $\Lambda \to 0$, the solution (3.94) for the metric function $H(t, r, x)$ approaches to the metric function $H(r, x)$ in asymptotically flat spacetime, yields $\beta = 0$. Moreover, we find the following exact analytical non-stationary solutions to Einstein-Maxwell theory with a cosmological constant

$$ds^2 = -H_0(t, r, x)^{-2}dt^2 + H_0(t, r, x)^{2/3}(dx^2 + ds^2_{a=0}), \tag{3.96}$$

and the time-dependent Maxwell gauge field

$$A_t(t, r, x) = \frac{2\sqrt{3}}{3H_0(t, r, x)}, \tag{3.97}$$

where the exact analytic metric function $H_0(t, r, x)$ is given by

$$H_0(t, r, x) = 1 + 3\sqrt{\frac{\Lambda}{10}}t + \frac{h_0}{\sqrt{r^2 + x^2}}, \tag{3.98}$$

and h_0 is a constant. We note that $ds^2_{a=0}$ in (3.96) is given by (3.71). We then can find the most general convoluted cosmological non-stationary solutions in six-dimensional Einstein-Maxwell theory with a cosmological constant where the metric function $H(t, r, x)$ approaches $H_0(t, r, x)$ in the limit of $a \longrightarrow 0$. The metric function $H(t, r, x)$ is equal to

$$H(t, r, x) = 1 + 3\sqrt{\frac{\Lambda}{10}}t + \frac{h_0}{2}\int_0^{\infty} \mathcal{H}_{\mathcal{D}}\left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{a^2 + r^2}}\right)(\xi e^{cx} + (1 - \xi)e^{-cx})\, dc, \quad (3.99)$$

where h_0 and ξ are two constants. The result (3.99) for the metric function $H(t, r, x)$ in six-dimensions can simply be generalized to solutions in D-dimensions, where we get

$$\begin{aligned} H_D(t, r, x) = 1 + (D - 3)\sqrt{\frac{2\Lambda}{(D - 1)(D - 2)}}\, t \\ + \alpha_D h_{D0} \int_0^{\infty} \mathcal{H}_{\mathcal{D}}\left(0, 0, -a^2c^2, 0, \sqrt{\frac{r^2}{a^2 + r^2}}\right) K_{\frac{D-7}{2}}(cx)\, x^{\frac{7-D}{2}}\, c^{\frac{D-5}{2}}\, dc. \end{aligned} \quad (3.100)$$

3.9 Wormholes

The Nutku embedded solutions in 5 dimensions and above can be interpreted as conformal wormholes but only if allow $-\infty < r < \infty$. To see why that is, let's first understand some basic ideas behind wormholes. Harris [90] gives a concise introduction to them, which I will follow here.

Consider a two-sided plane sheet[17] where the top and bottom are not connected to each other, i.e., no continuous path exists that connects one point on one side to the other. We may introduce polar coordinates (ρ, ϕ) on each side with the origin on one side being directly above or below the origin on the other side. To connect the two sides, we cut out a circular hole of radius a, centred at the origin. Now the two spaces are connected as it is possible to connect any point on one side to another through a continuous path. We can now use a single coordinate system to chart both sides by introducing the coordinate R through $\rho = \sqrt{R^2 + a^2}$, where $-\infty < R < \infty$.[18] Let's identify the positive values of R as belonging to points in the top space and the negative values to those in the bottom space. $R = 0$, of course, would correspond to the edge of the hole.

[17]Or think of two thin plane sheets lying on top of each other.

[18]We must take only the positive square root. Another possible coordinate transform can be given by $\rho = a \cosh R$.

The polar line element $ds^2 = d\rho^2 + \rho^2 d\phi^2$ with $a < \rho < \infty$ and $0 \le \phi \le 2\pi$ is replaced with:

$$ds^2 = \frac{R^2 dR^2}{R^2 + a^2} + (R^2 + a^2)d\phi^2 \tag{3.101}$$

The scalar curvature for the manifold is singular at $R = 0$. To avoid the singularity at the edge of the hole, we bring a small change to the metric:

$$ds^2 = \frac{(R^2 + b^2)dR^2}{R^2 + a^2} + (R^2 + a^2)d\phi^2 \tag{3.102}$$

Now the manifold described by this metric is regular everywhere for non-zero b. Suppose we want to visualise the new manifold by embedding it in 3-dimensional space. To do that, we will use cylindrical coordinates with the metric for the 3-space as:

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \tag{3.103}$$

where the new coordinate $-\infty < z < \infty$. If the equation of the surface is given by $z = z(R)$, then

$$ds^2 = \left(\left(\frac{d\rho}{dR}\right)^2 + \left(\frac{dz}{dR}\right)^2\right) dR^2 + \rho^2 \phi^2 \tag{3.104}$$

After comparing Eq. (3.102) and (3.104), we can conclude:

$$\frac{dz}{dR} = \pm \frac{b}{R^2 + a^2} \tag{3.105}$$

$$\Rightarrow z(R) = \pm b \ln\left(\frac{R + \sqrt{R^2 + a^2}}{a}\right) \tag{3.106}$$

$$\Rightarrow z(\rho) = \pm b \ln\left(\frac{\rho + \sqrt{\rho^2 - a^2}}{a}\right) \tag{3.107}$$

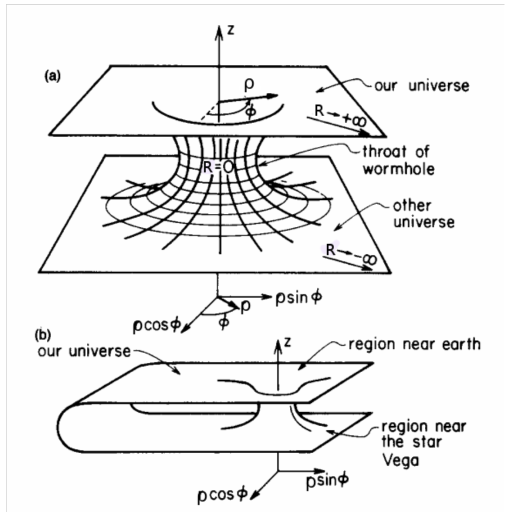


Figure 3.9.1: Embedding diagrams for the equatorial slice ($\theta = \frac{\pi}{2}$) for (a) a wormhole connecting two different universes (b) a wormhole connecting two regions of the same universe.
Source: Morris, Thorne (1987) [66]

A sketch of wormhole embedding diagrams is shown in Figure 3.9.1. The parameter a is the radius of the wormhole throat and b determines its "thickness" [90]. A common temptation in wormhole research is to demand $a = b$ to get a one-parameter wormhole with the metric:

$$ds^2 = dR^2 + (R^2 + a^2)d\phi^2 \tag{3.108}$$

Compare the above metric with the Nutku helicoid metric in Eq. (3.38). If we identify r with R and θ with ϕ and, also open our minds to considering $-\infty < r < \infty$, then the Nutku helicoid metric has a submanifold conformally equivalent to the wormhole ("handle") we just constructed. The main reason for choosing $0 < r < \infty$ for Nutku and its embeddings was avoiding the curvature singularity at $r = 0$. If we are fine with the singularity, then there should be no shame or hesitation in calling the Nutku helicoid metric or any of its higher dimensional embeddings a conformal wormhole because not all wormholes are traversable.[19]

Ludwig Flamm's 1916 work [62] was probably the first on spacetime bridges although there is no explicit mention of any such thing in it. Later, Hermann Weyl published a work in 1921 describing matter through multiply connected space that he called "*Schlauch*" (tube). The credit

[19]Of course, I don't recommend the Nutku wormholes for travelling.

for starting wormhole[20] physics, however, is generally given to Einstein and Rosen's 1935 paper [63] on the "particle problem"[21] in GR, wherein the authors do a coordinate transform on the radial coordinate to join two Schwarzschild spacetimes at the event horizon. The Einstein-Rosen bridge, however, was unstable and had a singularity and as such, it was not traversable. The motivation behind their work was simply to propose such bridges as models for particles.

In 1973, Bronnikov [64] and Homer Ellis [102] independently published exact wormhole solutions. However, these wormholes require exotic matter to be stable. To see what is meant by "exotic matter", consider the following spherically symmetric wormhole metric:

$$ds^2 = -dt^2 + dR^2 + (R^2 + a^2)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.109}$$

Einstein's equations are given by $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ which may be rewritten in its diagonalised form as [90]:

$$G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu} \tag{3.110}$$

To satisfy the Einstein equations, either one starts with calculating the Einstein tensor $G_{\mu\nu}$ and then finds a suitable $T_{\mu\nu}$ or the other way around. For the metric in question, we start with:

$$G^{\mu}_{\nu} = \frac{a^2}{(R^2 + a^2)^2} \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{3.111}$$

What kind of stress-energy tensor will be able satisfy the Einstein equations with the above G^{μ}_{ν}? First, let's consider a resting fluid with scalar pressure, P and mass-energy density ϱ . The energy-momentum tensor is given by:

$$T^{\mu}_{\nu} = (\varrho + P)u^{\mu}_{\nu} - \delta^{\mu}_{\nu} P \tag{3.112}$$

[20]Misner and Wheeler first coined the term "wormhole" in [106], in the domain of physics. Biological wormholes, however, are just tunnels that bugs bore into wood.

[21]Einstein and Rosen attempted to answer if GR can account for atomic phenomena.

Since the fluid is it at rest, its 4-velocity, $u^\mu = (-1, 0, 0, 0)$. Thus,

$$T^\mu_\nu = \begin{bmatrix} \varrho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \tag{3.113}$$

Alas, the signs of the diagonal elements are inconsistent with G^μ_ν. Therefore, we must look for something else. Let us consider an electric field that only depends on a radial variable. Its energy-momentum tensor then is:

$$T^\mu_\nu = \frac{\pi}{4}(F_{\nu\lambda}F^{\lambda\nu} + \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta}) = \frac{q^2}{8\pi(R^2 + a^2)^2} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{3.114}$$

where q is an integration constant. Again, the signs of the diagonal elements are inconsistent with those G^μ_ν. Finally, let's consider a massless scalar field with the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \tag{3.115}$$

Assuming that Φ only depends on the radial coordinate, the energy-momentum tensor then is:

$$T^\mu_\nu = \frac{Q^2}{8\pi(R^2 + a^2)^2} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.116}$$

Notice that the signs are the opposite of G^μ_ν components in (3.110). Therefore, a massless scalar field can satisfy the Einstein equations for the metric given by Eq. (3.109) but with a polarity opposite to the conventional ones [90].

It is important to note here that the above discussion applies to stable, traversable wormholes[22] and that we have imposed certain conditions to simplify the analysis, such as $a = b$ and spherical symmetry. None of the above analysis implies, by any means, that topological handles cannot exist in nature, for any scale of time and space, without exotic matter.

Wormholes have also been studied extensively in Euclidean quantum gravity [68] and they may have some surprising implications, if they exist. In [69], Coleman has shown that if wormholes exist, they can make the cosmological constant vanish. Building upon Coleman's work, the authors of [70] argue that a small number of "warm" universes, i.e. universes with matter and energy, can remain in contact with cold (empty) universes, effectively making the cosmological constant in the warm universe vanish. Inspired by Coleman's work, Preskill showed that wormholes can drive Newton's constant G to its minimum, thereby determining, in principle, all other constants of Nature [71].

3.10 Eguchi-Hanson space and convoluted solutions

The convoluted solutions presented above were inspired by similar supergravity solutions in [75] embedding the Taub-NUT metric. This work was adapted to Einstein-Maxwell theory in [76] and [77], whose author and I applied the idea to the Eguchi-Hanson space in [78]. The following is a summary of the work.

The ansatz remains the same as before. The metric is:

$$ds_D^2 = -H(r,x)^{-2}dt^2 + H(r,x)^{2/(D-3)}(dx^2 + x^2 d\Omega_{D-6} + ds_{EH}^2), \tag{3.117}$$

where $d\Omega_{D-6}$ is the metric on the unit sphere S^{D-6} and ds_{EH}^2 is the Eguchi-Hanson metric (Eq. 2.8). The ansatz, as given by Eq.(3.42) is:

$$A_t = \sqrt{\frac{D-2}{D-3}}H^{-1}(r,x). \tag{3.118}$$

The PDE for $H(r,x)$ is separable in D dimensions, if we set $H(r,x) = \alpha + \beta R(r)X(x)$ where α

[22]For a highly instructive and accessible article on traversable wormholes, see [66]. For their use in time travel and the resulting "causality" violation, see [67].

and β are two constants. Without loss of generality, we set $\alpha = 1$. After separation of variables, the ODE for $R(r)$ is given by:

$$\frac{d^2}{dr^2}R(r) + \frac{\left(a^4 + 3\,r^4\right)}{r\left(r^4 - a^4\right)}\frac{d}{dr}R(r) + \frac{R(r)\,r^4c^2}{r^4 - a^4} = 0, \tag{3.119}$$

where a is the Eguchi-Hanson parameter of the metric in Eq. (2.8) and c^2 is the separation constant. Similarly, the ODE for $X(x)$, after the separation of variables is:

$$x\frac{d^2}{dx^2}X(x) + (D-6)\frac{d}{dx}X(x) - c^2xX(x) = 0. \tag{3.120}$$

The solution for $R(r)$ is given in terms of the Heun-C function[23] by:

$$R(r) = r_1\,\mathcal{H}_C\left(0, 0, 0, -\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2 - r^2}{2a^2}\right) \tag{3.121}$$

where r_1 is a constant and $\mathcal{H}_C(\alpha, \beta, \gamma, \delta, \rho, z)$ is a solution to the Confluent Heun Differential Equation:

$$\frac{d^2\,y(z)}{dz^2} + \frac{(\alpha z^2 + (\beta + \gamma + 2 - \alpha)z - \beta - 1}{z(z-1)}\frac{d\,y(z)}{dz} + \frac{\{[(\beta + \gamma + 2)\alpha + 2\delta]z - (\beta + 1)\alpha + (\gamma + 1)\beta + 2\rho + \gamma\}}{2z(z-1)}y(z) =$$

with the boundary conditions $y(0) = 1$ and $\frac{dy}{dz}|_{z=0} = \frac{(1+\gamma-\alpha)\beta+\gamma+2\rho-\alpha}{2(\beta+1)}$. Figure 3.10.1 shows some plots for the $\mathcal{H}_C$ function.

[23]The Heun functions have appeared in a few works on curved spacetimes, including one [80] on the Eguchi-Hanson space.

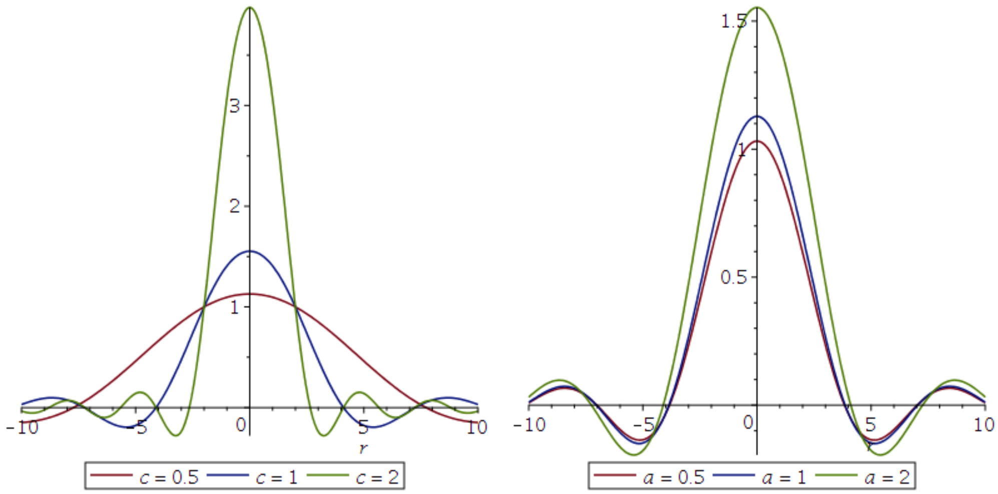


Figure 3.10.1: Plots of $\mathcal{H}_C\left(0, 0, 0, -\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right)$ for $a = 2$ (left) and $c = 1$ (right)

The solution for $X(x)$ is given by:

$$X(x) = x^{\frac{7-D}{2}}(x_1\, I_{\frac{D-7}{2}}(cx) + x_2\, K_{\frac{D-7}{2}}(cx)), \tag{3.122}$$

where x_1 and x_2 are constants. Since any solution with an arbitrary separation constant c is a solution to a linear PDE, a linear combination must also be a solution. Therefore, an integral—which is just an infinite sum—over the separation constant must also be a solution. Therefore, the general convolution-like solution is given by:

$$\begin{aligned} H(r, x) &= 1 + \int_0^\infty dc\, x^{\frac{7-D}{2}}\, H_C\left(0, 0, 0, -\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right)\{h_1(c)\, I_{\frac{D-7}{2}}(cx) \\ &+ h_2(c) K_{\frac{D-7}{2}}(cx)\}, \end{aligned} \tag{3.123}$$

where $h_1(c)$ and $h_2(c)$ are measure functions of the separation constant c.

To fix the functions $h_1(c)$ and $h_2(c)$, in Eq. (3.123), we exploit the fact that in the limit $a \to 0$ (or equivalently, $r \to \infty$), the 4-dimensional Eguchi-Hanson space (2.8) is asymptotically $\mathbb{R}^4$ with the line element $dr^2 + r^2 d\Omega_3^2$.

Suppose we embed $\mathbb{R}^4$ in the usual ansatz for the D-dimensional metric given by:

$$ds_0^2 = -\frac{1}{H_0^2(r,x)}dt^2 + H_0(r,x)^{2/(D-3)}(dx^2 + x^2 d\Omega_{D-6} + dr^2 + r^2 d\Omega_3^2), \tag{3.124}$$

and the gauge field,

$$A_t = \sqrt{\frac{D-2}{D-3}} H_0^{-1}(r,x), \tag{3.125}$$

Then, the metric function $H_0(r,x)$ in D-dimensions is given by

$$H_0(r,x) = 1 + \frac{\chi}{\left(r^2+x^2\right)^{\frac{D-3}{2}}}, \tag{3.126}$$

where χ is a constant. To find and fix the functions $h_1(c)$ and $h_2(c)$ in (3.123), we notice that, in the limit of $a \to 0$, the metric function (3.123) for the spacetime (3.117), must approach the metric function (3.126) for the spacetime (3.124), since in the limit of $a \to 0$, the Eguchi-Hanson-embedded spacetime in Eq. (3.117) becomes the spacetime given by Eq. (3.124). Thus, we have an integral equation involving $h_1(c)$ and $h_2(c)$

$$\int_0^\infty dc\, x^{\frac{7-D}{2}} \lim_{a\to 0} H_C\left(0,0,0,-\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right)\{h_1(c)\, I_{\frac{D-7}{2}}(cx) + h_2(c) K_{\frac{D-7}{2}}(cx)\} = \frac{\chi}{\left(r^2+x^2\right)^{\frac{D-3}{2}}}. \tag{3.127}$$

We also note that equation (3.127) guarantees the gauge field (3.42) also approaches the gauge field (3.125), in the limit $a \to 0$ (or $r \to \infty$). We solve Eq. (3.119) in the limit $a \to 0$ and compare the solution to Eq. (3.68) and observe that

$$\lim_{a\to 0} H_C\left(0,0,0,-\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right) = \frac{2}{cr} J_1(cr), \tag{3.128}$$

where J_1 is a Bessel function of the first kind.

Using trial and error on the exponents of c, we find that the following integral equation is true:

$$\int_0^\infty dc\, c^{\frac{D-3}{2}} \frac{J_1(cr)}{r} x^{-\frac{D-7}{2}} K_{\frac{D-7}{2}}(cx) = \frac{\xi_D}{\left(r^2+x^2\right)^{\frac{D-3}{2}}}, \tag{3.129}$$

where the constant ξ_D depends on dimension D. For even dimensions, we notice a general formula:[24]

$$\xi_{6+2n} = \sqrt{\frac{\pi}{2}}(2n+1)!!, \tag{3.130}$$

and for odd dimensions

$$\xi_{7+2n} = (2n+2)!!, \tag{3.131}$$

where $n = 0, 1, \cdots$. So, we find that the measure functions $h_1(c)$ and $h_2(c)$ in D-dimensions, are given by

$$h_1(c) = 0,\ h_2(c) = \frac{\chi}{2\xi_D} c^{\frac{D-1}{2}}, \tag{3.132}$$

and so the most general solution for the metric function $H(r,x)$ in the metric (3.117) and the gauge field (3.42), is given by:

$$H(r,x) = 1 + \frac{\chi}{2\xi_D}\int_0^\infty dc\, x^{\frac{7-D}{2}}\, c^{\frac{D-1}{2}} H_C\left(0,0,0,-\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right) K_{\frac{D-7}{2}}(cx) \tag{3.133}$$

We can analytically continue the separation constant c in Eq. (3.119) and (3.120) to ic and employ a similar procedure as we did above and find another independent exact solution given by:

$$\tilde{H}(r,x) = 1 + \frac{\chi\pi\epsilon_D}{4\xi_D}\int_0^\infty dc\, c^{\frac{D-1}{2}}\, x^{\frac{7-D}{2}}\, H_C\left(0,0,0,\frac{a^2c^2}{2}, -\frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right) J_{\frac{D-7}{2}}(cx), \tag{3.134}$$

where ϵ_D is equal to 1 for even dimensions and -1 for odd dimensions.

A class of cosmological solutions in D dimensions are given by:

$$H(t,r,x) = 1 + \rho_D t + \frac{\chi}{2\xi_D}\int_0^\infty dc\, x^{\frac{7-D}{2}}\, c^{\frac{D-1}{2}} H_C\left(0,0,0,-\frac{a^2c^2}{2}, \frac{a^2c^2}{4}, \frac{a^2-r^2}{2a^2}\right) K_{\frac{D-7}{2}}(cx). \tag{3.135}$$

where $\rho_D = \frac{D-3}{\ell}$ in which the D-dimensional length scale ℓ is related to the D-dimensional cosmological constant Λ by $\ell = \sqrt{\frac{(D-1)(D-2)}{2\Lambda}}$.

[24] I am not going to a pull a Fermat and claim in a footnote that I have a proof without stating it. In reality, these formulae are deduced through different numbers in different dimensions and I do not offer a proof for them. Thus, they should be considered conjectures for the general case of arbitrarily high number of dimensions.

The other class, corresponding to an imaginary separation constant is given by:

$$\tilde{H}(t,r,x) = 1 + \rho_D t + \frac{\chi\pi\epsilon_D}{4\xi_D}\int_0^\infty dc\, c^{\frac{D-1}{2}}\, x^{\frac{7-D}{2}}\, H_C\left(0,0,0,\frac{a^2c^2}{2},-\frac{a^2c^2}{4},\frac{a^2-r^2}{2a^2}\right) J_{\frac{D-7}{2}}(cx). \tag{3.136}$$

3.11 Concluding remarks

Inspired by the existence of helicoid-catenoid instantons in four dimensions, we constructed exact solutions to the five and higher dimensional Einstein-Maxwell theory with and without a cosmological constant. The solutions in five dimensions are given by the line element (3.51), gauge field (3.50) and the metric functions (3.53). We discussed the physical properties of the solution. We also found exact convoluted-like solutions to six and higher dimensional Einstein-Maxwell theory in which the metric functions are convoluted integrals of two special function with some measure functions. We fix the measure functions for all the solutions by considering the solutions in some appropriate limits and comparing them with some other exact solutions in D-dimensions that are given by (3.57) and (3.85) with the metric functions (3.78) and (3.87), respectively. We used a special separation of variables to construct the solutions to Einstein-Maxwell theory with positive cosmological constant. In this case, the metric function depends on time and two spatial directions. The solutions are given by the metric (3.91) and gauge field (3.92) where the cosmological metric functions in six and higher than six dimensions are given by (3.99) and (3.100), respectively. As noted in [86], the issue of singularities remains unresolved and needs further analysis to get a complete spacetime.

Finally, we found a physical interpretation for the Nutku-embedded solutions as wormholes and studied some basic ideas and applications concerning wormholes. We also studied Eguchi-Hanson-embedded solutions in Einstein-Maxwell theory with and without a cosmological constant and constructed similar convolution-like general solutions.

Chapter 4

Some New Gravitational Instantons

> The important thing is not to stop questioning. Curiosity has its own reason for existence. One cannot help but be in awe when he contemplates the mysteries of eternity, of life, of the marvelous structure of reality. It is enough if one tries merely to comprehend a little of this mystery each day.
>
> *Old Man's Advice to Youth: 'Never Lose a Holy Curiosity.' LIFE (2 May 1955)*
> Einstein

In this chapter, I construct some new gravitational instantons based on the general Nutku instanton metric. The instantons of special interest would be those that satisfy both the elliptic and hyperbolic cases of Eq. (3.28). The motivation for finding such a solution is simply curiosity. Elliptic and hyperbolic equations generally represent different types of systems or phenomena. Elliptic equations generally model equilibria, such as the Laplace equation in electrostatics; hyperbolic equations generally model systems evolving with time, such as waves. One may say that the "effects" of any disturbance in the former get "instantaneously" transmitted across the whole system but in the latter, they "propagate" as a wave would. So, it would be interesting to find out solutions which satisfy both cases of a partial differential equation.

4.1 The elliptic-hyperbolic quasi-linear equation

Recall that the quasi-linear partial differential equation[1]

$$(1 + f_x^2)f_{tt} - 2f_t f_x f_{tx} + (\kappa + f_t^2)f_{xx} = 0 \tag{4.1}$$

is elliptic for $\kappa = 1$ and represents the minimal surface equation in $\mathbb{R}^3$. It is hyperbolic for $\kappa = -1$, when it reduces to the Born-Infeld equation.

To demand that a solution satisfy both cases, we must write:

$$(1 + f_x^2)f_{tt} - 2f_t f_x f_{tx} + (1 + f_t^2)f_{xx} = (1 + f_x^2)f_{tt} - 2f_t f_x f_{tx} + (-1 + f_t^2)f_{xx} \tag{4.2}$$

$$\implies f_{xx} = -f_{xx} \tag{4.3}$$

$$\implies f_{xx} = 0 \tag{4.4}$$

Therefore,

$$f(x,t) = F(t)x + g(t) \tag{4.5}$$

If we substitute $f(x,t) = F(t)x + g(t)$ into Eq. (4.1), we get:

$$\left(1 + F^2(t)\right)\left(\frac{\mathrm{d}^2F(t)}{\mathrm{d}t^2}x + \frac{\mathrm{d}^2g(t)}{\mathrm{d}t^2}\right) - 2\,F(t)\left(\frac{\mathrm{d}F(t)}{\mathrm{d}t}x + \frac{\mathrm{d}g(t)}{\mathrm{d}t}\right)\frac{\mathrm{d}F(t)}{\mathrm{d}t} = 0 \tag{4.6}$$

4.1.1 An instanton

I cannot think of a straightforward way to solve this. One way to find a solution is to assume $g(t) = \textit{constant}$ and get:

$$\left(1 + F^2(t)\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}F(t) - 2\,F(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}F(t)\right)^2 = 0 \tag{4.7}$$

$$\implies F(t) = \tan(\omega t + \delta) \tag{4.8}$$

[1]A PDE is quasilinear if the coefficient of the highest order term only involves the dependent variable, lower order derivatives and the independent variable.

where ω and δ are constants. Let's put this value of $F(t)$ back into Eq. (4.6) to get a differential equation for g(t):

$$-2\ \sin(\omega\, t+\delta)\left(\frac{\mathrm{d}}{\mathrm{d}t}g\,(t)\right)\omega+\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\,(t)\right)\cos(\omega\, t+\delta)=0 \tag{4.9}$$

which yields:

$$g(t)=a\tan(\omega t+\delta)+b \tag{4.10}$$

where a and b are constants. b is not going to play any role ahead. So, let's just set it to zero. Thus, one solution to Eq. (4.1) is:

$$\begin{aligned} f_1(x,t) &= x(\tan(\omega t+\delta))+a\tan(\omega t+\delta)\\ &= (x+a)\tan(\omega t+\delta) \end{aligned} \tag{4.11}$$

Figure 4.1.1 shows a plot of the function $f_1(x,t)=(x+a)\tan(\omega t+\delta)$ for $a=-2$, $\omega=0.5$, $\delta=\frac{\pi}{2}$.

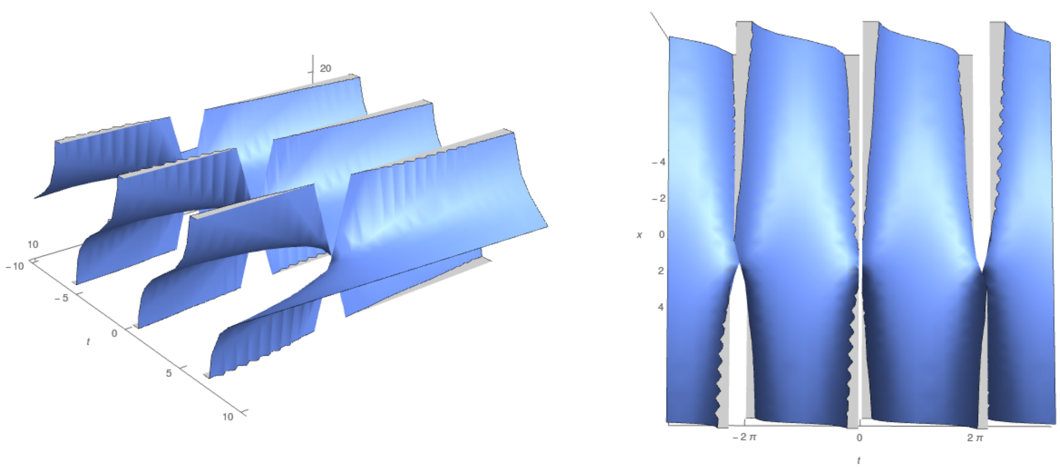


Figure 4.1.1: Plots of $f_1(x,t)=(x-2)\tan(0.5t+\pi/2)$

Now that we have a solution $f_1(x,t)=(x+a)\tan(\omega t+\delta)$ to Eq. (3.28), we can construct a gravitational instanton from it using Eq. (3.27) with $\kappa=1$. The line element after setting $\delta=0$ is given by:

$$ds^2_{one} = \frac{\left(\frac{cos^4(\omega t) + \omega^2(x+a)^2}{cos^2(\omega t)}\right)(dt^2 + dy^2) + 2\omega(x+a)\tan(\omega t)(dtdx + dydz) + dx^2 + dz^2}{\sqrt{cos^2(\omega t) + \omega^2(x+a)^2}} \tag{4.12}$$

4.1.2 Another instanton

This time we will not restrict ourselves to real solutions but instead seek solutions of the form:

$$f(x,t) = \pm i(F(t)x + g(t)) \tag{4.13}$$

Let's plug this back into Eq. (4.1) and get:

$$\begin{aligned} x&\left((F(t))^2 \frac{d^2}{dt^2}F(t) - 2F(t)\left(\frac{d}{dt}F(t)\right)^2 - \frac{d^2}{dt^2}F(t)\right) \\ &+ (F(t))^2 \frac{d^2}{dt^2}g(t) - 2F(t)\left(\frac{d}{dt}F(t)\right)\frac{d}{dt}g(t) - \frac{d^2}{dt^2}g(t) = 0 \end{aligned} \tag{4.14}$$

Now we will set $g(t) = k$, a constant and proceed with:

$$\left(\left(F^2(t)\right)\frac{d^2}{dt^2}F(t) - 2F(t)\left(\frac{d}{dt}F(t)\right)^2 - \frac{d^2}{dt^2}F(t)\right) = 0 \tag{4.15}$$

$$\implies F(t) = \pm\frac{ae^{2\lambda t}+1}{ae^{2\lambda t}-1} \tag{4.16}$$

Since k is not going to play any role in the instanton, we might as well set it to zero. Therefore, the following is a solution to Eq. (4.1):

$$f_2(x,t) = i\left(\pm x\frac{ae^{2\lambda t}+1}{ae^{2\lambda t}-1}\right) \tag{4.17}$$

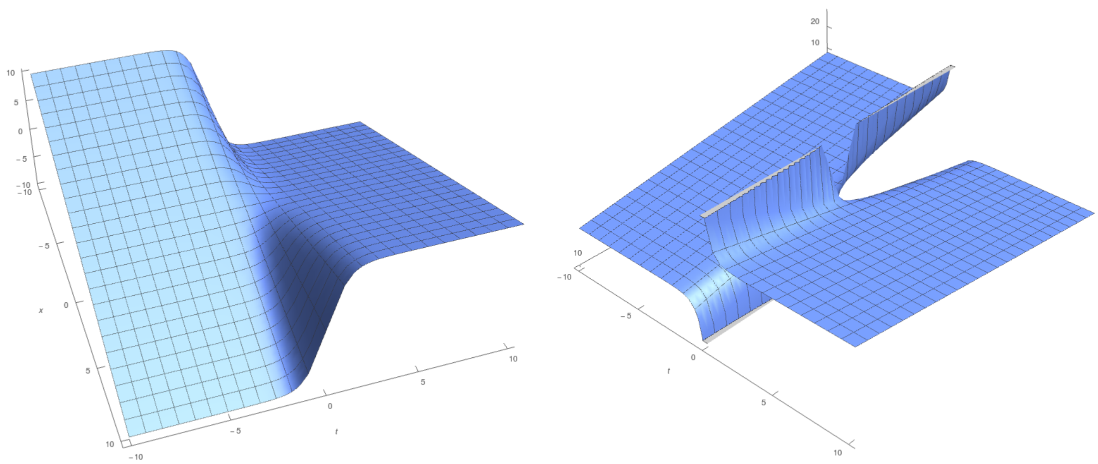


Figure 4.1.2: Plots of $\Im\mathfrak{m}(f_2(x,t)) = x\frac{ae^{2\lambda t}+1}{ae^{2\lambda t}-1}$ with $a=-1,\ \lambda=1$ (left) and $a=1,\ \lambda=1$ (right)

Figure 4.1.2 shows a plot of the real part of this function. Now we can put the above function (Eq. 4.17)[2] into Eq. (3.27) with $\kappa = 1$, to get a gravitational instanton with the line element:

$$ds^2_{two} = \frac{\dfrac{p(x,t)}{2(ae^{2\lambda t}-1)^2}(dt^2+dy^2) + q(x,t)(dtdx+dydz) - 2ae^{\lambda t}(dx^2+dz^2)}{\sqrt{-a(4a\,e^{2\,\lambda\,t}\lambda^2x^2 + a^2e^{4\,\lambda\,t} - 2a\,e^{2\,\lambda\,t}+1)}} \tag{4.18}$$

where $p(x,t)$ and $q(x,t)$ are given by:

$$p(x,t) = a^4e^{7\,\lambda\,t} - 16a^2\,e^{3\,\lambda\,t}\lambda^2x^2 - 4a^3\,e^{5\,\lambda\,t} + 6a^2\,e^{3\,\lambda\,t} - 4a\,e^{\lambda\,t} + e^{-\lambda\,t} \tag{4.19}$$

$$q(x,t) = 2\lambda ae^{\lambda t}\frac{ae^{2\lambda t}+1}{ae^{2\lambda t}-1}x \tag{4.20}$$

The coordinates y and z are Killing coordinates.

4.2 Some New Einstein-Maxwell Solutions

Now that we have two new gravitational instantons, we can embed them to find new solutions in Einstein-Maxwell Theory. The purpose is simply to show that these instantons can be embedded in the usual fashion with the ansatz we have been using all along. The choice of the gauge function

[2]The ± sign does not make a difference.

coordinate variables is arbitrary and I have chosen $H(x)$ instead of $H(x,\tau)$, for example, only for ease of calculation.[3]

We will use the metric ansatz for 5 dimensions but first we take the instanton given by Eq. (4.12) and replace the coordinate t with τ and embed it as:

$$ds^2 = -\frac{dt^2}{H(x)} + H(x)ds^2_{one} \tag{4.21}$$

We take the usual gauge function given by (Eq. 3.42):

$$A_t = \sqrt{\frac{3}{2}}\frac{1}{H(x)} \tag{4.22}$$

The Einstein-Maxwell equations are satisfied if the following ODE is satisfied:

$$\left(\cos^4(\omega\,\tau) + \omega^2 x^2\right)\left(\omega^2 x^2 + \cos^2(\omega\,\tau)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}H(x) - 2\,\cos^2(\omega\,\tau)\left(\frac{\mathrm{d}}{\mathrm{d}x}H(x)\right)\omega^2 x = 0 \tag{4.23}$$

The following is a solution represented as an integral:

$$H(x) = \int\left(\frac{\omega^2 x^2 + n^2}{\omega^2 x^2 + n}\right)^{\frac{1}{1-n}}\,\mathrm{d}x \tag{4.24}$$

where $n = cos^2(\omega t)$. A somewhat trivial but simple solution to Eq. (4.23) is:

$$H(x) = \frac{x}{x_Q} \tag{4.25}$$

where x_Q is a constant.

[3] Indeed, we could also do a suitable coordinate transform on the instanton before embedding it.

Chapter 5

The Riddle of Charge

> Study hard what interests you the most in the most undisciplined, irreverent and original manner possible.
>
> — Richard Feynman

Finally, it is time to address and dare I say, even question the notion of electric charge.[1] In this chapter, I construct a new exact solution to Einstein-Maxwell theory by embedding the Kerr instanton [108] in 5-dimensional Einstein-Maxwell theory and explicitly show that the same parameter ("mass"[2]) that determines the gravitational field strength also determines the electric field strength, thereby eliminating any need for the notion of an electric "charge". Furthermore, I also show how two things with different masses can appear to have the same charge, as is the case with elementary particles.

5.1 History

> I transmit but I do not create; I am sincerely fond of the ancients.
>
> *Analects 7:1*
> Confucius

[1] I understand that this chapter may sound *non-sequitur* or even brazen but the purpose of this thesis, as stated earlier, is the advancement of knowledge of natural phenomena and as such, I am prepared to receive any criticism or risks.

[2] The quotation marks are not preposterous; Rainich-Misner-Wheeler geometrodynamics even offers "*mass without mass*".

The idea of electric charge as an illusion goes back to at least as far as John Wheeler [3] who referred to the chargelike property of wormholes as "charge without charge". In their classic treatise on "*geometrodynamics*", John Wheeler and Charles Misner [106] do a thorough exploration of Rainich's work [107] on the geometrisation of Einstein and Maxwell equations and illustrate how electric field lines may emerge out of "holes" or handles in multiply connected space but appear to a macroscopic observer as emerging from a point source in simply connected space and thus making him erroneously apply Gauss's law on a perceived "point charge".

5.2 Charge from Mass

Now let's get straight to the point and face the exact solution in question. The metric for the Kerr instanton [108] is given by the line element [109]:

$$ds^2_{KerrInst} = (r^2 - a^2\cos^2\theta)\left(\frac{dr^2}{\Delta(r)} + d\theta^2\right) + \frac{1}{r^2 - a^2\cos^2\theta}(\Delta(r)(d\tau + a\sin^2\theta\, d\phi)^2 + \sin^2\theta((r^2 - a^2)d\phi - a\, d\tau)^2) \tag{5.1}$$

where $\Delta(r) = r^2 - r_s r - a^2$, $a = \dfrac{\mathcal{J}}{Mc}$ and $r_s = \dfrac{2GM}{c^2}$.

For $a = 0$, i.e., $\mathcal{J} = 0$, the Kerr instanton becomes the Schwarzschild instanton [109]. The coordinates r, θ, ϕ are the spherical coordinates and the coordinate τ is obtained from the Kerr spacetime by a Wick rotation of the timelike coordinate via $t \rightarrow i\tau$. The now spacelike coordinate τ must be periodic [108].

Then one proceeds with the usual embedding ansatz for five dimensions:[4]

$$ds^2 = -c^2\frac{dt^2}{H^2(r)} + H(r)\, ds^2_{KerrInst} \tag{5.2}$$

The only non-zero component of the gauge field is given by the usual gauge ansatz, i.e. Eq. (3.42) as:[5]

$$A_t = \sqrt{\frac{3}{2}}\frac{1}{H(r)} \tag{5.3}$$

[3] I hope I am forgiven for shamelessly ripping off the liberally translated but apt Confucian aphorism from Misner and Wheeler's ancient treatise "*Classical Physics as Geometry*".

[4] I have included c^2 in the metric to aid the analysis ahead although it is not necessary.

[5] At this point, it should be noted that the Kaluza "cylinder condition" is tacitly assumed here, i.e., that the metric and the gauge function are independent of the coordinate τ.

Einstein and Maxwell equations, in order to be satisfied, demand that the metric function $H(r)$ satisfy the following ODE:

$$(-c^2r^2 + 2GMr + a^2c^2)\frac{d^2H(r)}{dr^2} + (2GM - 2c^2r)\frac{dH(r)}{dr} = 0 \tag{5.4}$$

A nice, healthy solution to the above equation is:

$$H_+(r) = c_1 + C_2 \tanh^{-1}\left(\frac{GM - c^2r}{\sqrt{a^2c^4 + G^2M^2}}\right) \tag{5.5}$$

where c_1 and C_2 are constants. Figure 5.2.1 shows a plot of the function $H(r)$.

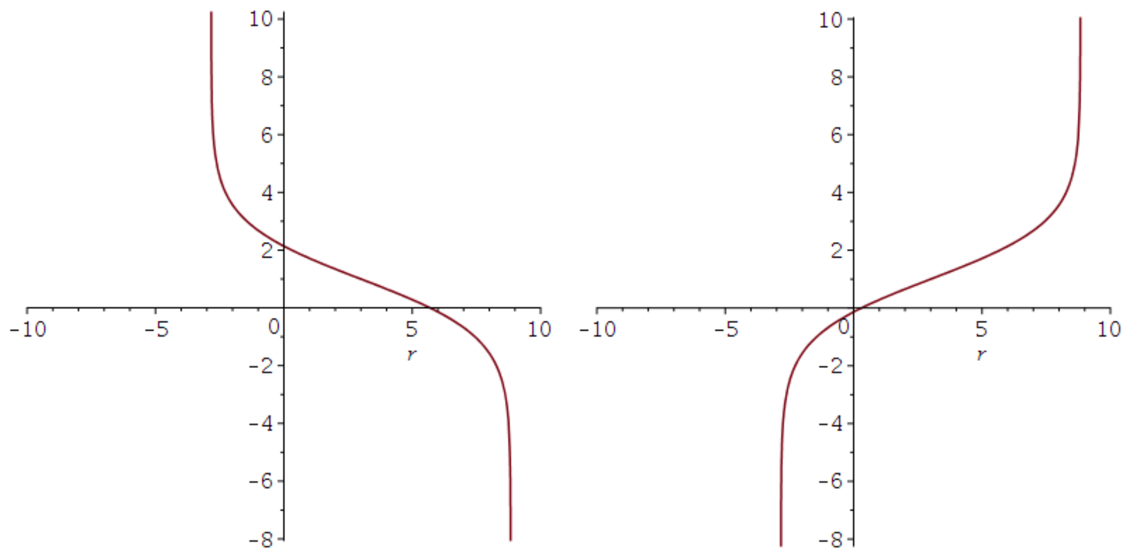


Figure 5.2.1: Plots of $H_+(r)$ with $c = 1$, $c_1 = 1$, $G = 1$, $M = 3$, $a = 5$, $C_2 = 2$ (left) and $C_2 = -2$ (right)

Looks like the solution $H_+(r)$ suffers from a pathology as it is not defined for $r > \frac{GM - \sqrt{a^2c^4+G^2M^2}}{c^2}$.

Evidently, the pathology is not easily cured by a sign change but we should not give up and instead convert $\tanh^{-1}$ to its logarithmic form using:

$$\tanh^{-1}(x) = \frac{1}{2}\ln\frac{1+x}{1-x} \tag{5.6}$$

Therefore,

$$H_{ext}(r) = c_1 + \frac{C_2}{2}\ln\left(\frac{1+\dfrac{GM-c^2r}{\sqrt{a^2c^4+G^2M^2}}}{1-\dfrac{GM-c^2r}{\sqrt{a^2c^4+G^2M^2}}}\right) \tag{5.7}$$

$$= c_1 + \frac{C_2}{2}\ln\left(\frac{\sqrt{a^2c^4+G^2M^2}+GM-c^2r}{\sqrt{a^2c^4+G^2M^2}-GM+c^2r}\right) \tag{5.8}$$

$$= c_1 + \frac{C_2}{2}\ln(f(r)) \tag{5.9}$$

is a solution to Eq. (5.4), with $f(r)$ representing the argument of the logarithmic function. Clearly, $H_{ext}(r)$ is not valid for $f(r) < 0$. Thus, we might as well write another solution $H_{int}(r) = c_1 + \frac{C_2}{2}\ln(-f(r))$. Figure 5.2.2 shows plots of the two functions for some arbitrary values of the constants.

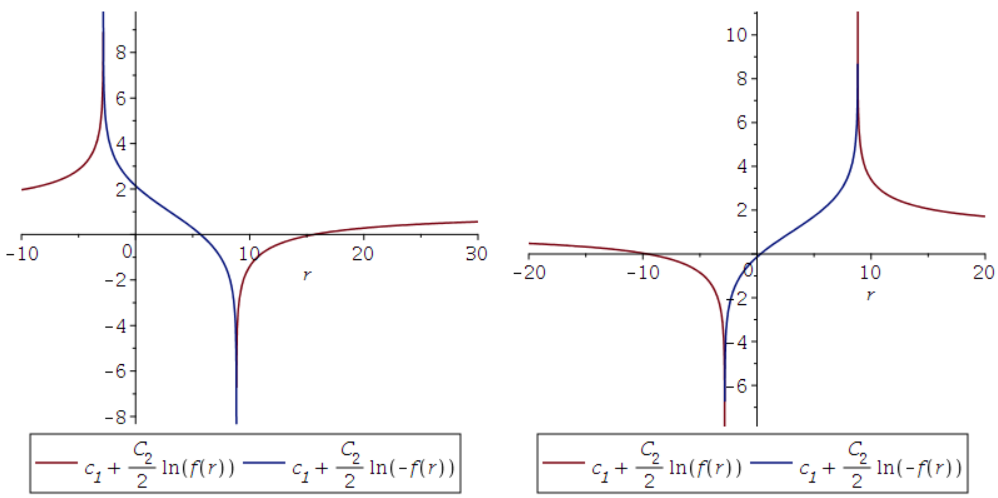


Figure 5.2.2: Plots of $H_{ext}(r)$ and $H_{int}(r)$ with $c = 1,\ c_1 = 1,\ G = 1,\ M = 3,\ a = 5,\ C_2 = 2$ (left) and $C_2 = -2$ (right)

Ultimately, the most wholesome solution, that comprises of both $H_{ext}(r)$ and $H_{int}(r)$, can be written as:[6]

$$
\begin{aligned}
H(r) &= c_1 + \frac{c_2}{4}\ln(|f(r)|) \\
&= c_1 + \frac{c_2}{4}\ln\left(\left|\frac{\sqrt{a^2c^4+G^2M^2}+GM-c^2r}{\sqrt{a^2c^4+G^2M^2}-GM+c^2r}\right|\right)
\end{aligned}
\tag{5.10}
$$

where $c_2 = 2\,C_2$.

5.3 Event horizons

The event horizons are located where g_{tt} vanishes:

$$
g_{tt} = -\frac{c^2}{\left(c_1 + \frac{c_2}{4}\ln\left(|\frac{\sqrt{a^2c^4+G^2M^2}+GM-c^2r}{\sqrt{a^2c^4+G^2M^2}-GM+c^2r}|\right)\right)^2} = 0 \tag{5.11}
$$

This implies that the logarithm function needs to approach infinity. The only possible location for g_{tt} to vanish for $r > 0$ then is:

$$
\Longrightarrow r_{eh} = \frac{\sqrt{a^2c^4+G^2M^2}+GM}{c^2} \tag{5.12}
$$

which, of course, implies that in the limit $a \to 0$, r_{eh} becomes the Schwarzschild radius $\frac{2GM}{c^2}$.

5.4 Weak field limit: Linearised gravity

To find the strength of the gravitational field at infinity, we expand g_{tt} as series and take the linear term in $\frac{1}{r}$:

$$
g_{tt} = -\frac{c^2}{c_1^2} - c_2\frac{\sqrt{a^2c^4+G^2M^2}}{c_1^3 r} + O\left(\frac{1}{r^2}\right) \tag{5.13}
$$

[6] If you find the absolute value distasteful but are fond of even integers, then writing $H(r) = c_1 + \frac{1}{4}\ln\left(\left(\frac{\sqrt{a^2c^4+G^2M^2}+GM-c^2r}{\sqrt{a^2c^4+G^2M^2}-GM+c^2r}\right)^{c_2}\right)$ and demanding that c_2 be an even integer, should assuage your immediate concern but it may raise a new question—“why integers?!”

In the weak field limit for a spherically symmetric, non-rotating, uncharged, gravitating body, we expect:

$$g_{tt} = -c^2 + \frac{2GM}{r} \tag{5.14}$$

Clearly, $c_1 = \pm 1$. And c_2 can be fixed[7] by noting that for $a = 0$ (i.e., the embedded Schwarzschild instanton where there is no coupling between τ and r coordinates), we should have $c_2 = \mp 2$ in order to model the attractive gravitational field of a non-rotating, spherically symmetric body.

Thus, the metric function $H(r)$ is given by:[8]

$$H(r) = c_1 + \frac{1}{4} \ln \left(\left(\frac{\sqrt{a^2c^4 + G^2M^2} + GM - c^2 r}{\sqrt{a^2c^4 + G^2M^2} - GM + c^2 r} \right)^{c_2} \right) \tag{5.15}$$

where $c_1 = \pm 1$ and $c_2 = \mp 2$.

5.5 Electric field

Recall that the r-component (the only component, in this case) of the electric field is given by:

$$E_r(r) = F_{tr} = -\sqrt{\frac{3}{2}} \frac{1}{H^2(r)} \frac{dH(r)}{dr} \tag{5.16}$$

$$= -\frac{c_2}{2} \sqrt{\frac{3}{2}} \frac{1}{H^2(r)} \frac{\sqrt{a^2c^4 + G^2M^2}}{(c^2r^2 - a^2c^2 - 2GMr)} \tag{5.17}$$

---

[7]The fixing of c_2 may not be as simple as that of c_1 and here, I have opted for the method motivated by a desire to model Newtonian gravity in the weak field limit.

[8]Now that $c_2 = \mp 2$ is an even integer, the universe does not have to compute an unnecessary *if-else* conditional arising from the absolute value.

which can be expanded as series at infinity as:

$$E_r(r) = -\frac{\sqrt{3}}{2\sqrt{2}}\frac{c_2}{c^2c_1^2}\frac{\sqrt{a^2c^4+G^2M^2}}{r^2} \tag{5.18}$$

$$= -c_2\frac{\sqrt{3}}{2\sqrt{2}}\frac{1}{c^2}\frac{\sqrt{a^2c^4+G^2M^2}}{r^2} \tag{5.19}$$

$$= -c_2\frac{\sqrt{3}}{2\sqrt{2}}\frac{1}{c^2}\frac{\sqrt{\frac{\mathcal{J}^2}{M^2}c^2+G^2M^2}}{r^2} \tag{5.20}$$

$$= -\frac{c_2}{c^2}\frac{\sqrt{3}}{2\sqrt{2}}\frac{\sqrt{\frac{\mathcal{J}^2}{M^2}c^2+G^2M^2}}{r^2} \tag{5.21}$$

It's obvious from the above expression that the mass parameter m can also determine the strength of the electric field, regardless of whether $\mathcal{J} = 0$ or not. The plot of $\sqrt{\frac{\mathcal{J}^2}{M^2}c^2 + G^2M^2}$ in Figure 5.5.1 shows how two different masses can appear to have the same electric field strength far away from the origin.

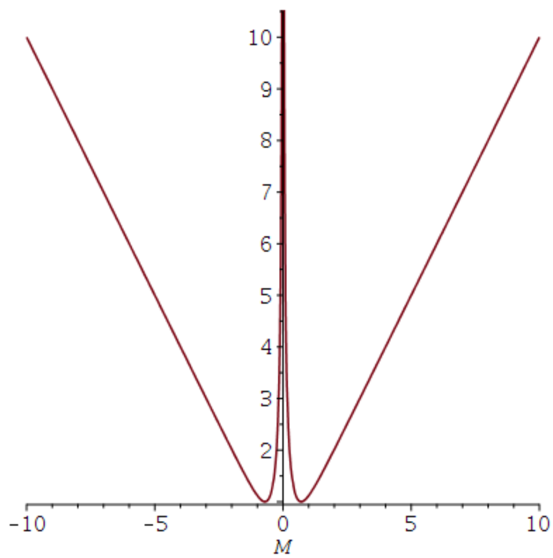


Figure 5.5.1: A plot of $\sqrt{\frac{\mathcal{J}^2}{M^2}c^2 + G^2M^2}$ with $\mathcal{J} = \frac{1}{2}$, $G = 1$ and $c = 1$

It's also obvious from Eq. (5.17) that c_2 being negative or positive makes an important difference. Evidently, from the same equation, $c_2 < 0$ makes the solution appear to have "positive" charge.

However, Eq. (5.13) is not so simple. It is clear that $c_1 = 1$ and $c_1 = -1$ both work but the particular choice of sign can impose constraints on the sign of c_2 if we are to model the familiar (attractive) Newtonian gravity and vice-versa. Therefore, they must have opposite signs. Else, consider the case when we permanently fix $c_1 = +1$ and never allow $c_1 = -1$. Then Eq. (5.13) demands that $c_2 = -2$ for attractive gravity. However, by Eq. (5.17), this would imply that the solution will appear positively charged to a far-off observer, thereby either limiting the scope of the solution or making the negatively "charged" solution have a repulsive gravitational field far away from the origin in case of the negatively charged solution.

So, we arrive at these questions: Can the mass M of an object determine the strength of an electric field? If M is not mass then what is this property (M) of the object that determines both its electric and gravitational field strengths?

5.6 Some remarks

Although I cannot resolve the questions I have raised above, I have found some relevant literature on the matter. For instance, Kilmister and Stephenson have raised objection to Rainich theory in [110] purely on the grounds that the "interdependence of gravitational and magnetic fields" seems to "conflict with experience (in macroscopic theories)". In [111], Klotz and Lynch show an approximate solution to Rainich theory in which they show that the electric field depends on a mass parameter. I am not persuaded that showing a yet unseen or cryptic relationship between mass and charge is proof that the underlying theory is wrong. In [112], the same authors again argue that a spherically symmetric charged body solution in Rainich's Already Unified Theory they constructed will have a smaller size than a neutral one and that this is a "contradiction" since they believe that "electrical repulsion *should*[9] tend to counteract gravitational attraction to make a charged sphere bigger than an uncharged one". However, this argument too is unconvincing because it is solely rested upon personal intuition based on everyday experience with charge and gravity and nothing more than that. To show that some object is impossible to exist in nature, one must show a resulting violation of some fundamental law and not personal sentiments.

The Schwarzschild-Kerr instanton embbeded solution shown above is in 5 dimensions and no currently accepted physical theory has dimensions other than 4. Therefore, one may reject the

[9]Emphasis mine. I don't think scientists should dictate what nature should do.

5-dimensional solutions on those grounds alone but that is a cop-out, considering theorists are still researching potentially viable string theories in higher dimensions. Another point to note here is that the parameter M does not necessarily have to be interpreted as mass. It can be called by any other name but it is still going to dictate the strength of both the electric and magnetic fields. This *liaison* between mass and electric field should not be brushed under the carpet without rigorously dismantling either Einstein-Maxwell theory altogether (or at the very least, higher dimensional Einstein-Maxwell theory) or experimentally disproving the theory.

5.7 Thesis summary

We have demonstrated that solutions from M-theory [75] can be translated to Einstein-Maxwell theory by embedding the Eguchi-Hanson space and the Nutku instanton in higher dimensions. The Nutku solution demonstrates that helicoidal, catenoidal and other minimal surface geometries are possible in Einstein-Maxwell theory, thereby opening new avenues for investigating electromagnetism in curved spacetimes, beyond the usual spherical solutions. We also found that the Nutku-embedded solutions may be interpreted as wormholes and that the electric field need not be singular at the origin even if there is a curvature singularity there. Finally, this thesis demonstrates an interesting relationship between mass and electric charge in Einstein-Maxwell theory when one embeds Schwarzschild or Kerr instanton in higher dimensions by showing that without an explicit charge term, mass alone can determine the electric field strengths in some spacetimes.

Bibliography

[1] C. Pickover *Archimedes to Hawking: Laws of Science and the Great Minds Behind Them* (Oxford University Press 2008)

[2] R.D. Purrington *The First Professional Scientist: Robert Hooke and the Royal Society of London* (Springer 2009)

[3] A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie : (Gemeinverständlich)* (Vieweg 1916); R. W. Lawson (transl.), A. Einstein, *Relativity: the Special and General Theory* (Pi Press 2005)

[4] A.M. Steane, *Relativity Made Relatively Easy* (Oxford University Press 2012)

[5] H. Minkowski, "Das Relativitätsprinzip." *Ann. Phys.* **47**, 927 (1915)

[6] H. Minkowski, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern." *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klaße*, **53** (1908)

[7] H. Minkowski, "Raum und Zeit." *Physik. Z.* **10**, 104 (1909)

[8] A. Einstein, "Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen." *Jahrbuch der Radioaktivität und Elektronik* **4**, 411 (1907)

[9] A. Einstein, "Über den Einfluß der Schwercraft auf die Ausbreitung des Lichtes." *Ann. Phys.* **35** (1911)

[10] M. Abraham, "Das Elementargesetz der Gravitation." *Physik. Z.* **13**, 4 (1912); "Das Gravitationsfeld." *Physik. Z.* **13**, 793 (1912); "Zur Theorie der Gravitation." *Physik. Z.* **13**, 1 (1912)

[11] A. Pais, *Subtle is the Lord: The Science and Life of Albert Einstein* (Oxford University Press 1982)

[12] G. Nordström, "Zür Theories der Gravitation vom Standpunkt des Relativitätsprinzips." *Ann. Phys.* **42**, 533 (1913)

[13] A. Einstein, "Die Feldgleichungen der Gravitation." *Sitzungsber. Preuss. Akad. Wiss. Berlin*, **844** (1915)

[14] M.P. Hobson, G.P. Efstathiou, A.N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press 2006)

[15] J.B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison-Wesley 2003)

[16] R.M. Wald, *General Relativity* (University of Chicago Press 1984)

[17] H. Goenner, "Some remarks on the genesis of scalar-tensor theories." *Gen. Relativ. Gravit.* **44** (8), 2077 (2012)

[18] T. Applequist, A. Chodos, P.G.O. Freund (eds.), *Modern Kaluza-Klein Theories* (Addison-Wesley 1987)

[19] Th. Kaluza, "Zum Unitätsproblem in der Physik." *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **966** (1921)

[20] E. Schrödinger, "An undulatory theory of the mechanics of atoms and molecules." *Phys. Rev.* **28** (6), 1049 (1926)

[21] O. Klein, "The atomicity of electricity as a quantum theory law." *Nature* **118**, 516 (1926)

[22] O. Klein, "Quantum Theory And Five-Dimensional Theory Of Relativity." *Z. Phys.* **37**, 895 (1926)

[23] M.Y. Thiry, "Geometrie—Les équations de la théorie unitaire de Kaluza." *Compt. Rend. Acad. Sci. Paris* **226**, 216 (1948)

[24] A.J. Kox, J. Eisenstaedt *The Universe of General Relativity* (Springer Science & Business Media 2006)

[25] A. Einstein, P. Bergmann, "On A Generalization Of Kaluza's Theory Of Electricity." *Annals of Mathematics* **39**, 683 (1938)

[26] E. Witten, "A Note On Einstein, Bergmann, and the Fifth Dimension." *arXiv*:1401.8048 (2014)

[27] J.A. Harvey, S. Kachru, E. Silverstein (eds.), *Strings, Branes and Gravity*, M.J. Duff, "TASI Lectures on Branes, Black Holes and Anti-De Sitter Space" (World Scientific 2001)

[28] P. Jordan, "Relativistische Gravitationstheorie mit variabler Gravitationskonstante." *Die Naturwissenschaften* **11**, 250 (1946)

[29] P. Jordan, C. Müller, "Field equations with a variable 'constant' of graviation." *Z. Naturfosch.* **2a**, 1 (1947)

[30] P. Jordan, *Schwerkraft und Weltall. Grundlage der theoretischen Kosmologie* (Vieweg 1952)

[31] P. Jordan, *Schwerkraft und Weltall* (Vieweg 1955)

[32] C.H. Brans, R.H. Dicke, "Mach's Principle and a Relativistic Theory of Gravitation." *Phys. Rev.* **124** (3), 925 (1961)

[33] A.H. Klotz, *Macrophysics and Geometry: From Einstein's Unified Field Theory to Cosmology* (CUP Archive 1982)

[34] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (W.H. Freeman 1973)

[35] H. Reissner, "Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie." it Ann. Phys. **50**, 106 (1916)

[36] G. Nordström, "On the Energy of the Gravitational Field in Einstein's Theory." *Verhandl. Koninkl. Ned. Akad. Wetenschap. Afdel. Natuurk. Amsterdam* **26**, 1201 (1918)

[37] R. D'Inverno, *Introducing Einstein's Relativity* (Clarendon Press Oxford 1992)

[38] D.J. Griffiths, *Introduction to Electrodynamics* (Pearson 2008)

[39] C.N. Yang, R. Mills, "Conservation of isotopic spin and isotopic gauge invariance." *Phys. Rev.* **96**, 191 (1954)

[40] H. Weyl, "A new extension of relativity theory." *Ann. Physik* **59**, 101 (1919)

[41] K. Moriyasu, *An Elementary Primer for Gauge Theory* (World Scientific 1983)

[42] H. Weyl, "Elektron und Gravitation. I." *Z. Phys.* **56**(5), 330 (1929)

[43] V. Fock, "Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt." *Z. Phys.* **39**(2), 226 (1926)

[44] F. London, "Quantenmechanische Deutung der Theorie von Weyl." *Z. Phys.* **42**(5), 375 (1927)

[45] M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory* (Westview 1995)

[46] T. Eguchi, A.J. Hanson, "Self-dual solutions to Euclidean gravity." *Ann. Phys.* **120**(1), 82 (1979)

[47] T. Eguchi, P.G.O. Freund "Quantum Gravity and World Topology." *Phys. Rev. Lett.* **37**, 1251 (1976)

[48] S.W. Hawking, "Gravitational Instantons." *Phys. Lett.* **60A** (1977)

[49] A.H. Taub, "Empty Space-Times Admitting a Three Parameter Group of Motions." *Ann. Math.* **53**, 472 (1951)

[50] E. Newman, L. Tamburino, T. Unti, "Empty-space generalization of the Schwarzschild metric." *J. Math. Phys.* **53**, 472 (1951)

[51] T. Eguchi, A.J. Hanson, "Asymptotically flat self-dual solutions to Euclidean gravity." *Phys. Lett.* **B74**, 249 (1978)

[52] M.F. Atiyah, A.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles* (Princeton University Press 1988)

[53] E. Calabi, ""Métriques kählériennes et fibrés holomorphes." *Ann. Sci. Ecole Norm. Sup. (4)* **12**(2), 269 (1979)

[54] A.M. Polyakov, "Quark confinement and topology of gauge theories." *Nucl. Phys. B 120*(3), 429 (1977)

[55] K. Huang, *Quantum Field Theory: From Operators to Path Integrals* (John Wiley & Sons 2010)

[56] J.S. Russell, "Report on Waves." *Proceedings of the British Association for the Advancement of Science* (London 1845)

[57] T. Dauxois, M. Peyrard, *Physics of Solitons* (Cambridge University Press 2006)

[58] A.R. Osborne, T.L. Burch, "Internal Solitons in the Andaman Sea." *Science* (1980)

[59] L.M. Aycock, H.M. Hurst, D.K. Efimkin, D. Genkina, H.I. Lu, V.M. Galitski, I.B. Spielman, "Brownian motion of solitons in a Bose-Einstein condensate." *Proc. Natl. Acad. Sci. USA* **114**(10), 2503 (2017)

[60] L.L. Bonilla, M. Carretero, F. Terragni, B. Birnir, "Soliton driven angiogenesis." *Nature Scientic Reports* **6**, 31296 (2016)

[61] S.R. Coleman, *"The Uses of Instantons"* Subnucl. Ser., vol. 15, 805, (1979)

[62] L. Flamm, "Beiträge zur Einsteinschen Gravitationstheorie." *Phys. Z.* **XVII**, 448 (1916)

[63] A. Einstein, N. Rosen, "The particle problem in the general theory of relativity." *Phys. Rev.* **48**, 73 (1935)

[64] K.A. Bronnikov, "Scalar-tensor theory and scalar charge." *Acta. Phys. Pol. B* **4**, 251 (1973)

[65] G. 'tHooft, "Symmetry breaking through Bell-Jackiw anomalies." *Phys. Rev. Lett.* **37**(1), 8 (1976)

[66] M.S. Morris, K.S. Thorne, "Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity." *Am. J. Phys.* **56**(5), 395 (1988)

[67] M.S. Morris, K.S. Thorne, U. Yurtsever, "Wormholes, time machines, and the weak energy condition." *Phys. Rev. Lett.* **61**(13), 1446 (1988)

[68] G.W. Gibbons, S.W. Hawking (eds.), *Euclidean Quantum Gravity* (World Scientific 1993)

[69] S. Coleman, "Why there is nothing rather than something: a theory of the cosmological constant." *Nuc. Phys. B* **310**, 643 (1988)

[70] I. Klebanov, L. Susskind, T. Banks, "Wormholes and the cosmological constant." *Nuc. Phys. B.* **317**, 665 (1989)

[71] J. Preskill, "Wormholes in spacetime and the constants of nature." *Nuc. Phys. B* **323**, 141 (1989)

[72] J.A.F. Plateau, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires* Vol. **2** (Gauthier-Villars 1873)

[73] K. Michielson, D.G. Stavenga, "Gyroid cuticular structures in butterfly wing scales: Biological photonic crystals." *J. R. Soc. Interface* **5**(18), 85 (2008)

[74] C. Mille, E.C. Tyrode, R.W. Corkery, "Inorganic chiral 3-D photonic crystals with bicontinuous gyroid structure replicated from butterfly wing scales." *Chem. Commun.* **47**, 9873 (2011)

[75] S. A. Cherkis , A. Hashimoto, "Supergravity solution of intersecting branes and AdS/CFT with flavor." *JHEP* **0211**, 036 (2002)

[76] A.M. Ghezelbash, "Exact convoluted solutions in higher-dimensional Einstein-Maxwell theory." *Phys. Rev. D* **90**, 084047 (2014)

[77] A.M. Ghezelbash, "Cosmological solutions in five-dimensional Einstein-Maxwell-dilaton theory." *Phys. Rev. D* **91**, 084003 (2015)

[78] A.M. Ghezelbash, V. Kumar, "Exact Solutions to Einstein-Maxwell theory on Eguchi-Hanson Space." *Int. J. Mod. Phys. A* **32**, 1750098 (2016)

[79] A.M. Ghezelbash, V. Kumar, "Exact helicoidal and catenoidal solutions in Einstein-Maxwell theory." *Phys. Rev. D* **95**, 124045 (2017)

[80] A. Malmendier, "The eigenvalue equation on the Eguchi-Hanson space." *J. Math. Phys.* **44**, 4308 (2003)

[81] W. Meeks, J. Perez, *A Survey on Classical Minimal Surface Theory* (American Mathematical Soc. 2012)

[82] L. Euler, *Methodus inveniendi lineas curvas maximi minimive propietate gaudeates sive solutio problematis isoperimetrici latissimo sensu accepti* (1744)

[83] K. Tapp, *Differential Geometry of Curves and Surfaces*, (Springer 2016)

[84] M. Deserno, *Lecture Notes on Differential Geometry: with special emphasis on surfaces in* $\mathbb{R}^3$ (2004)

[85] Y. Nutku, "Gravitational instantons and minimal surfaces." *Phys. Rev. Lett.* **77** (23), 4702 (1996)

[86] A. N. Aliev, M. Hortaçsu, J. Kalaycı, Y. Nutku, "Gravitational instantons from minimal surfaces." *Class. Quant Grav* **16**, 631 (1999)

[87] S.D. Majumdar, "A class of exact solutions of Einstein's field equations." *Phys. Rev.* **72**(5), 391 (1947)

[88] A. Papapetrou, "A static solution of the equations of the gravitational field for an arbitrary charge-distribution." *Proc. R. Irish Acad.*, **A51**, 191 (1947)

[89] D. Kastor, J. H. Traschen, "Cosmological Multi-Blackhole Solutions." *Phys. Rev. D* **47**, 5370 (1993) [arXiv:hep-th/9212035]

[90] E.G. Harris, "Wormhole connecting two Reissner-Nordstrom universes." *Am. J. Phys.* **61** (12) (1993)

[91] A. T. Fomenko, A. A. Tuzhilin, *Elements of the Geometry and Topology of Minimal Surfaces in Three-dimensional Space* (American Mathematical Soc. 2005)

[92] S. Krivoshapko, V.N. Ivanov, *Encyclopedia of Analytical Surfaces* (Springer 2015)

[93] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry* (Springer Science Business Media 2004)

[94] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, "Pseudoparticle solutions of the Yang-Mills equations." *Phys.Lett.* **B59** (1975)

[95] J. B. Hartle and S.W. Hawking, "Path-integral derivation of black-hole radiance." *Phys. Rev.* **D13**, 2188 (1976)

[96] A. Comtet, "Instantons and minimal surfaces." *Phys. Rev.* **D18**, 3890 (1978)

[97] E. Witten, "Some Exact Multipseudoparticle Solutions of Classical Yang-Mills Theory." *Phys. Rev. Lett.* **38**, 121 (1977)

[98] R. Dey, "The Weierstrass-Enneper representation using hodographic coordinates on a minimal surface." *Proc. Indian Acad. Sci. (Math.Sci.)* **113**, 2 (2003)

[99] R. Dey and R. K. Singh, "Born-Infeld solitons, maximal surfaces, and Ramanujan's identities." *Arch. Math.* (2017)

[100] T. Birkandan and M. Hortacsu, *J. Phys.* **A40**, 1105 (2007); *J. Math. Phys.* **48**, 092301 (2007); *J. Math. Phys.* **49**, 054101 (2008); *EAS Publ. Ser.* **30**, 265 (2008)

[101] M. Ito and T. Sato, *Eur. J. Phys.* **31**, 357 (2010); L. Durand, *Am. J. Phys.* **49**, 334 (1981)

[102] H.G. Ellis, "Ether flow through a drainhole: A particle model in general relativity." *J. Math. Phys.* **14**, 104 (1973)

[103] G. Ellis, A. Sumeruk, D. Coule and C. Hellaby, "Change of signature in classical relativity." *Class. Quant. Grav.* **9**, 1535 (1992)

[104] F. Leblond, R. C. Myers, D. Marolf, "Tall tales from de Sitter space I: Renormalization group flows." *JHEP* **06** 052 (2002)

[105] G. W. Gibbons, S. W. Hawking, "Gravitational Multi-instantons." *Phys. Lett.* **78B**, 430 (1978)

[106] C.W. Misner, J.A. Wheeeler, “Classical physics as geometry.” *Ann. Phys.* **2**, 525 (1957)

[107] G.Y. Rainich, “Electrodynamics in the general relativity theory.” *Trans. Amer. Math. Soc.* **27**, 106 (1925)

[108] G.W. Gibbons, S.W. Hawking, “Action integrals and partition functions in quantum gravity.” *Phys. Rev. D* **15**, 2752 (1977)

[109] D.J. Gross, M.J. Perry, L.G. Yaffe, “Instability of flat space at finite temperature.” *Phys. Rev. D* 25, 330 (1982)

[110] C.W. Kilmister, G. Stephenson, “An axiomatic criticism of unified field theories—II.” *Nuovo Cim.* **11** (Suppl. 1), 118 (1954)

[111] A.H. Klotz, J. Lynch, “Is Rainich’s theory tenable?.” *Lett. Nuovo Cim.* **4** (6), 248 (1970)

[112] A.H. Klotz, J. Lynch, “A spherically symmetric, charged body in Rainich’s Already Unified Field Theory.” *Mon. Not. R. Astron. Soc.* **150**, 149 (1970)