OPTIMIZATION OF A SYSTEM USING STATISTICAL CONTROL THEORY

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by
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"OPTIMIZATION OF A SYSTEM USING STATISTICAL CONTROL THEORY"

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ABSTRACT

The earlier methods for finding the optimum settings of the variable control parameters of a speed governor of a hydro-electric turbine employed arbitrary performance indices and assumed deterministic system disturbances. In this thesis, the Generalized Mean Square Error, the GMSE, is used as a performance index. The GMSE is a function of the correlation functions of input and output and is ideally suited for situations where the disturbance is non-deterministic. A method is developed to describe the random input disturbance statistically and hence to evaluate the input autocorrelation. The other correlation functions required for the evaluation of the GMSE are then derived using the input autocorrelation function and the frequency response function of the system under study. The model used in this study is a particular example of a non-minimum phase, time invariant feedback control system with a transport delay subject to a random disturbance. It is an electric turbo-generator supplying an isolated load. The turbine is provided with the conventional mechanical type governor. The trends in the variations of the GMSE resulting from changes in the statistics of the input disturbance and in the control parameters are shown.

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PARTIAL LIST OF SYMBOLS

Symbol	
$C(t)$	the output response of a system
$E_N(T)$	the generalized mean square error
$H(\omega)$	the frequency response of a system
$P_\alpha(x_i)$	the probability of the variable α taking a value x_i
$P_{\beta/\alpha}(x_{\beta j}/x_{\alpha i};T)$	the conditional probability of variable β assuming a value $x_{\beta j}$ provided T seconds earlier $x_{\alpha i}$ is the value assumed by α
S	the Laplace operator
T_m	the mechanical starting time of turbine, generator and load
T_R	the reset time
T_s	servomotor time constant (seconds)
α, β	the random variables
δ	transient droop
δ', δ_i'	damping coefficients of a second order system
η	discrete random variable
σ	permanent droop
$\phi_{ii}(T)$	the input autocorrelation at time T
$\phi_{oo}(T)$	the output autocorrelation at time T

Symbol

$\phi_{io}(T)$	the input-output cross correlation
$\lambda_1, \lambda_2, \ldots, \lambda_6$	the constants*
λ, μ, μ_i, λ_i	the rates of departures
$\Phi_{ii}(\omega)$	input power spectral density
$\Phi_{oo}(\omega)$	output power spectral density
$\Phi_{io}(\omega)$	input output cross spectral density
a, b, c	the amplitudes of the discrete random process
$e(t, T)$	the error $\Delta P(t) - \Delta L(t)$
$r(t)$	the input to the system

* This reference is to Chapter 1 only.

1. INTRODUCTION

1.1 General

A considerable amount of work has been done by many researchers and practising engineers to find methods of establishing optimum settings for controllers in physical systems. One particular case of great economic importance is the optimization of the speed governor of a prime-mover driving an electric generator to minimize the variation in generated frequency and tie-line power interchanges. This problem is most difficult in cases where the prime-movers are hydraulic turbines with mechanical or electro-hydraulic governors, where the number of adjustable system parameters is large. This thesis is a report of studies carried out on the application of statistical control theory to problems of this particular type.

The majority of the earlier work in the area of optimization of governor parameters has been directed toward the consideration of systems with deterministic disturbances, for example step disturbances. In this thesis, a method is presented which will solve this specific problem of governor optimization where the disturbance is a more realistic one; namely, a pseudo-random function of time. This method in fact is a general one and may be applied to any linear control system with a real time transport delay. Non-minimum phase systems can also be handled by this method.

1.2 Review of the Previous Work

1.2.1 A general historical review

A detailed review of the work done in the past in connection with the choice of proper settings of variable parameters of governors and their stability limits can be found in the thesis by Harras[1]. The results of earlier work are summarized briefly in the following paragraphs.

The earliest analytical work was concerned with the determination of the operating stability boundaries for a generating unit. This was closely followed by the search within the stable region for the most suitable time response. In general, the Routh-Hurwitz Criterion has been employed for the calculation of the stability boundary. Empirical methods using the coefficients of the system characteristic equation were developed by Vyshnegradskii[2], Paynter[3] and others. These methods have been used to choose the proper operating points within the stable region. Hovey[5] and Bateman[4] have reported the successful application of these techniques in a particular situation to optimize the governor settings of a hydro-electric plant. Harras and Fleming[6] have shown that these basic techniques can be extended to handle systems of higher order than the one considered by Paynter.

The most recent research work has been directed towards the use of certain performance criteria related to the output responses as functions of time to assess the effectiveness of speed controller settings. Stein[7] has shown the value of this technique in power system situations. The work of Krasovskii[2] points the way toward the selection of suitable performance indices for general problems of this type.

The latest analytical work in this field has been concerned with the analysis of the location of the complex variable on the S-plane (root locus and sensitivity analysis) to determine how the particular control parameters affect the modes of system response. This approach has been exploited by Harras, Van Ness[8], Meloy[9] and others.

As a specific illustration of the methods of approach commonly used the basic principles underlying the Paynter-Hovey approach to the governor optimization are explained in section 1.2.2. This particular method is widely used by many power companies in North America. It illustrates the arbitrary nature of the criteria which have been used as the basis for controller adjustments.

1.2.2 The Paynter-Hovey Method

Paynter was the first person in the western world to publish the results on the detailed investigation of the choice of optimum parameters of a speed governor for a hydro-electric generator. The system he considered (shown in Figure 2.2) consisted of a hydro-electric turbine driving an inertial load controlled by a simplified two-mode, proportional-integral type, mechanical governor with the permanent droop $\sigma=0$. The Characteristic Equation of the system is given by

$$0.5\ T_R\ \delta\ T_m\ T_w\ S^3 + (T_R\ T_m\ \delta - T_R\ T_w)\ S^2 + (T_R - T_w)\ S + 1 = 0 \qquad 1.1$$

where, T_R and δ are adjustable governor parameters, T_w and T_m are fixed system parameters. To satisfy the Routh-Hurwitz criterion for stability,

$$0.5\ T_R\ \delta\ T_m\ T_w > 0 \qquad 1.2$$

$$\delta\ T_R\ T_m - T_R\ T_w > 0 \qquad 1.3$$

$$(\delta\ T_R\ T_m - T_R\ T_w)\ (T_R - T_w) > 0.5\ \delta\ T_R\ T_m\ T_m \qquad 1.4$$

It can be seen that equation* (1.2) is greater than 0 if and only if all variables δ, T_R, T_m and T_w have positive values. Since,

$$\delta\ T_R\ T_m - T_R\ T_w > 0,$$

then,

$$(T_R - T_w) > 0$$

Let

---

* Numbers within paremtheses denote the equation numbers.

$$\lambda_1 = (T_w / T_m \delta) \text{ and } \qquad \lambda_2 = T_w / T_R,$$

then, (1.4) is,

$$\left(\frac{1}{\lambda_1 \lambda_2} - \frac{1}{\lambda_2}\right) \left(\frac{1}{\lambda_2} - 1\right) > \frac{0.5}{\lambda_1 \lambda_2}$$

At the stability boundary,

$$(1 - \lambda_2)(1 - \lambda_1) = 0.5 \lambda_2$$

$$\lambda_1 = (1 - 1.5 \lambda_2) / (1 - \lambda_2) \qquad 1.5$$

Relation (1.5) gives a stability limit curve of the form shown in Figure 1.1. The parameters chosen from the region enclosed by this stability curve in the λ_1-λ_2 plane lead to stable operation of the system. Inside the region of stability Paynter found step responses for sets of λ_1 and λ_2 when the system was simulated on an analog computer. He chose a set of parameters for which "transient time was the shortest" as the optimum set of parameters.

Hovey[5] used the same third order system as his model for simulation purposes. He used the criterion that the "speed transients following load changes must be almost critically damped". He followed the same

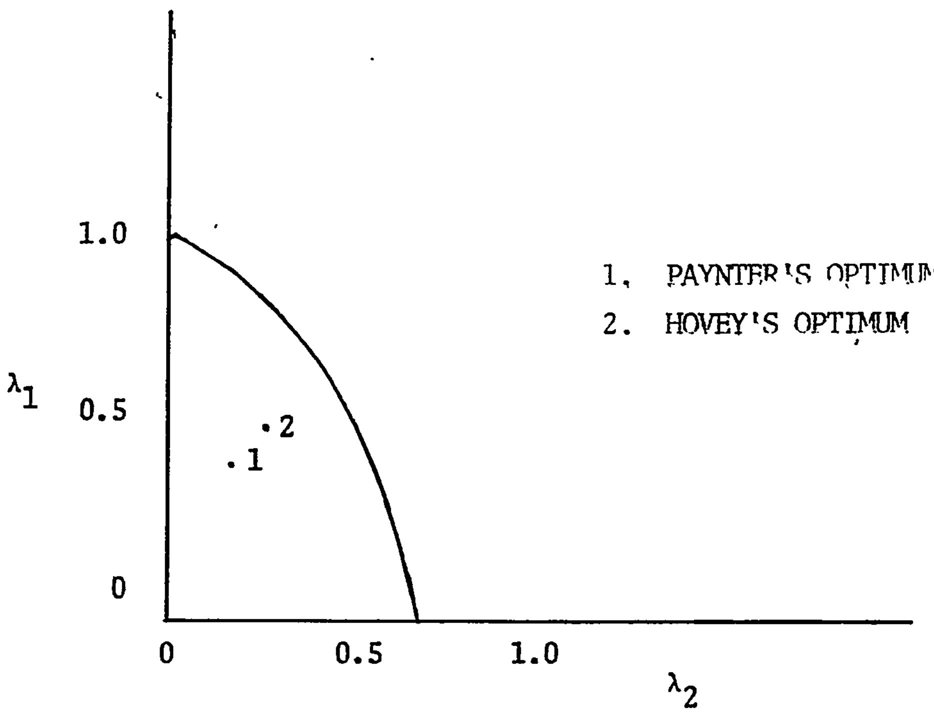


FIGURE-1.1 PAYNTER'S STABILITY BOUNDARY CURVE procedure as Paynter, but he chose those parameters that fulfilled his criterion as the optimum. The values chosen by Paynter and Hovey are nearly but not exactly the same. They are indicated on Figure 1.1. Harras also chose the third order system model and suggested a still different criterion for optimum governor settings - the so called "equal damping criterion" which forced the real values of the three system roots to be equal.

It can be now seen that the criterion for choosing optimum parameter settings on this basis is quite arbitrary and depends on experience and a subjective judgement of the quality of the response.

Harras also extended Paynter's method of finding stability limits to higher order systems. This gave rise to a number of parameters in addition to λ_1 and λ_2 in (1.5). For a representation of the stability limit curve in the λ_1-λ_2 plane, other parameters defined as λ_3 and λ_4 can be kept as constants. Therefore, instead of one curve of stability limit described by (1.5), there may be a number of these curves with different values for the parameters λ_3 and λ_4.

The parameters that have to be used in the case of a single machine feeding an isolated load when the servomotor time constant T_s and permanent droop σ are taken into account and are given by

$$\lambda_1 = \frac{T_w}{T_m \delta} \qquad \lambda_2 = \frac{T_w}{T_R}$$

$$\lambda_3 = 2 \delta T_w / T_s \qquad \lambda_4 = \sigma / \delta \qquad 1.6$$

Harras has shown that the region of stability varies with servomotor time constant T_s and permanent droop σ.

The stability limit curve drawn by Paynter using (1.5) is universal and does not vary from machine to machine. On the other hand, if the relation between λ_1 and λ_2 were drawn keeping λ_3 and λ_4 of (1.6) constant, then the results cannot be applied universally. Hence, the curve for λ_1 and λ_2 can be drawn for a particular σ or T_s so that the stability limit curve holds only for a particular machine. As noted above, Harras also used an arbitrary criterion for the optimization of governor parameters. The criterion was based on the location of the roots of characteristic equation on the complex S-plane.

So far most of the criteria or performance indices used to optimize the governor settings have been oriented towards choosing a set of parameters to give a specified time response to a single step disturbance. It has been found that the calculated optimum controller settings are often not entirely satisfactory in the field because of the fact that the system disturbances are actually pseudo-random in nature; the system seldom has a chance to settle down following a disturbance before another disturbance occurs. The actual transport delay or transportation lag was not taken into account in the earlier method and the performance index has been arbitrary.

1.3 Outline of Research Reported in This Thesis

In this thesis, a general method is evolved which can be used in the process of optimization of the controller parameters of a linear system with transport lag, subject to pseudo-random disturbances. The problem of choosing the proper settings for speed governors in power systems is a particular application of the general method developed in this thesis.

In this first chapter the pertinent background material has been reviewed briefly to illustrate that the criteria which are now used to establish governor settings are quite arbitrary and require a considerable measure of subjective judgement and experience if they are to be used successfully. Although these methods have served the industry well in establishing "workable" systems, they do not allow for the determination of whether or not the control is in any way optimum. This problem of controller optimization in the presence of a realistic pseudo-random disturbance is taken up in subsequent chapters of this thesis.

In the second chapter the analytical method proposed is outlined and the various mathematical concepts required to implement the method are discussed briefly. The particular mathematics required is drawn from the areas of complex variable theory, statistics, Markov processes and numerical techniques. A more detailed discussion of these pertinent mathematical subjects is given in an associated departmental report[12].

The key to the whole analytical procedure is the autocorrelation of the system disturbance function. The third chapter is devoted to a discussion of the autocorrelation calculation for the pseudo-random disturbance using the Poisson and Markov approaches. The Markov approach, which is more generally useful in these instances, is illustrated by way of a specific example.

The fourth chapter presents a discussion of the criterion proposed for the assessment of control quality. The choice of the Generalized Mean Square Error Criterion is justified and its calculation is illustrated by a specific numerical example.

Chapter 5 presents the results of a number of studies carried out to illustrate the application of this technique to a particular class of problems. These results shed more light on the specific problem - the speed control of a hydro-electric generator subject to pseudo-random load disturbances; but what is more important is that they show that the proposed analytical technique can be used quite readily for the analysis of such systems.

In the sixth chapter, the conclusions drawn based on the overall research project are presented. These conclusions are grouped in catagories; those pertaining to the general nature of the problem, those pertaining to the solution technique proposed and those concerning the specific numerical example case which was studied. Some recommendations for further extensions of this work are also presented in Chapter 6.

2. MATHEMATICAL BACKGROUND

2.1 Introduction

In this chapter an outline of the method developed in this thesis is given. As mentioned earlier, this method is more general than those employed in the past and can be used in the optimization of controller parameters when the input to the system is not deterministic. The mathematical tools necessary for the development of this method are given briefly in this chapter; a more detailed treatment of the mathematical aspects dealt with in this chapter is given in a related report[12] by the author.

2.2 Outline of the Method

It has been suggested by Spooner and Rideout[10] that the Generalized Mean Square Error, otherwise known as the GMSE, can be used as the performance index to measure the effectiveness of system controller performance, when the input to the system is of random nature. The generalized mean square error is given by,

$$E_N(T) = 1 + \frac{\phi_{oo}(0)}{\phi_{ii}(0)} - \frac{2\,\phi_{io}(T)}{\phi_{ii}(0)} \qquad 2.1$$

where,

$E_N(T)$ is the GMSE in its normalized form,

$\phi_{ii}(0)$ and $\phi_{oo}(0)$ are the autocorrelation of the system input and output at T=0,

$\phi_{io}(T)$ is the input-output crosscorrelation at time T.

The derivation of the GMSE is given in section 2.3.1.

The basic steps in the procedure for the evaluation of the GMSE are as follows:

1. the input autocorrelation $\phi_{ii}(T)$ is derived from the statistics of the pseudo-random input to the system. For reasons mentioned in Chapter 3, the given continous pseudo-random input is approximated by a multi-level discrete random process. The Markov process is used for the evaluation of the input autocorrelation $\phi_{ii}(T)$ of the pseudo-random input;
2. the autocorrelation $\phi_{ii}(T)$ is transformed into its power spectral density function $\Phi_{ii}(w)$ by the use of the Fourier transform;
3. the spectral density function $\Phi_{oo}(w)$ and $\Phi_{io}(w)$ are calculated by using $\Phi_{ii}(w)$ and the system frequency response function H(w) in the relationships[11];

$$\Phi_{oo}(\omega) = |H(\omega)|^2 \Phi_{ii}(\omega) \qquad 2.2a$$

$$\Phi_{io}(\omega) = H(\omega) \quad \Phi_{ii}(\omega) \qquad 2.2b$$

where,

$\Phi_{oo}(\omega)$ and $\Phi_{io}(\omega)$ are the output spectral density and the input-output cross spectral density respectively,

$H(\omega)$ is the frequency response function of the system;

4. the inverse Fourier transforms of $\Phi_{oo}(\omega)$ and $\Phi_{io}(\omega)$ provide the correlation functions $\phi_{oo}(T)$ and $\phi_{io}(T)$ respectively.
5. the final evaluation of $E_N(T)$ is made using the appropriate correlation functions derived by steps 1 to 4 above in (2.1)

2.3 The Mathematical Background for the Evaluation of the GMSE

The five steps to be followed for the evaluation of the GMSE as outlined in section 2.2 involve a number of individual calculation steps. The particular mathematical procedures chosen to carry out these steps require some comment. In general the basis for the selection of the procedures to be used was that they should be calculable by digital computer techniques for the types of numerical problems expected.

This section briefly presents the procedures employed. To a large extent these are well known methods although they have been selected from a fairly wide range of applied mathematics and engineering disciplines. The derivation of the GMSE itself is presented and it is shown that the Fourier transformation and its inverse are of interest; this brings up the theory of complex variables. The essential basic aspects of Markov process are introduced since this method was found to be quite powerful for the calculation of the autocorrelation of the pseudo-random disturbance function. The QR-transformation and the Muller Algorithm used in root finding techniques associated with the numerical problems encountered are also introduced.

2.3.1 Derivation of the GMSE

Figure 2.1 is a schematic representation of a linear control system with negative feedback. The error e(t, T) is given by,

$$e(t, T) = r(t) - c(t + T) \qquad 2.3$$

where,

r(t) is the input to the system,

c (t + T) is the output of the given system which is predicted T seconds earlier.

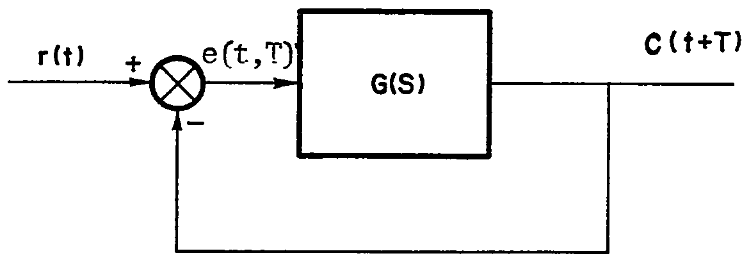


FIGURE 2.1

SCHEMATIC REPRESENTATION OF A LINEAR CONTROL SYSTEM WITH NEGATIVE FEEDBACK

The error $e(t, T)$ is an explicit function of the input and an implicit function of the variable parameters of the system. Hence, any criterion which uses a function of $e(t, T)$ as a performance index requires that the input be described explicitly.

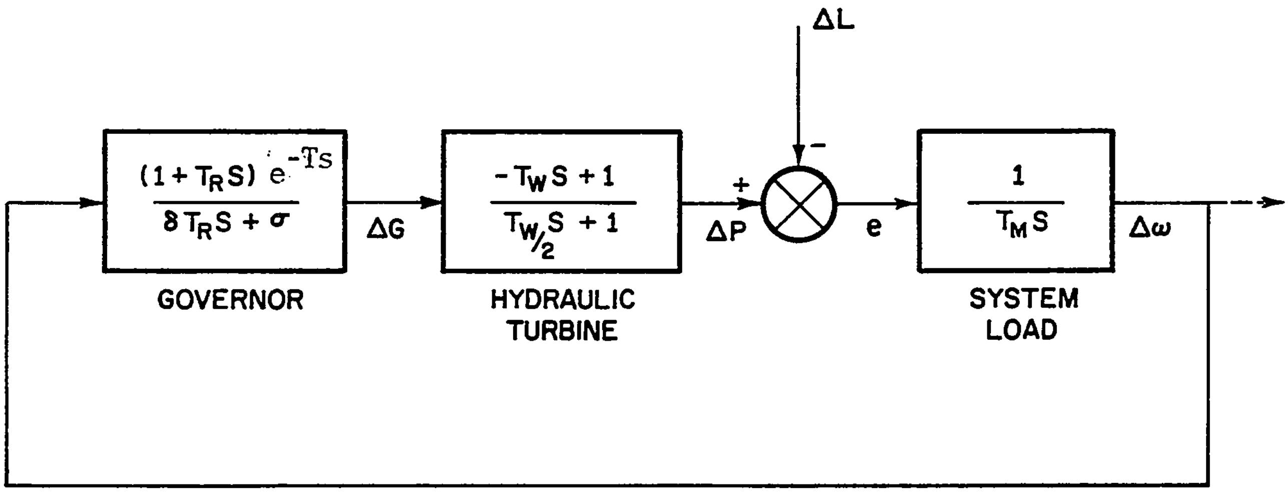
The block diagram of the particular case studied in this thesis is shown in Figure 2.2. The quantity that has to be optimized is the variation in frequency $\Delta\omega(t)$. It is shown in Appendix A that optimizing $\Delta\omega(t)$ is equivalent to optimizing $e(t, T)$. Since the function $\dot{e}(t,T)$ is a random function of time, a meaningful performance index could be the mean of the integral of some function of $e(t, T)$. The mean square error criterion was chosen as the performance index; the reasons for the choice are given in Chapter 4,

In (2.3) the error $e(t, T)$ is,

$$e(t, T) = r(t) - c(t + T)$$

where T is the prediction time and -T is the delay time in the system. The mean square error of $e(t, T)$ is,

$$\overline{e^2 (t, T)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2 (t, T)\, dt$$



τ — PURE TIME DELAY

T_W — WATER STARTING TIME (Secs.)

T_M — MECHANICAL STARTING TIME

T_R — WASHOUT TIME

δ — TEMPORARY DROOP

σ — PERMANENT DROOP

ΔG — CHANGE IN GATE OPENING

ΔP — CHANGE IN OUTPUT POWER

ΔL — CHANGE IN LOAD

$\Delta \omega$ — VARIATION IN FREQUENCY

δ, σ, τ and T_R ARE ADJUSTABLE CONTROLLER PARAMETERS

e — THE ERROR $e = (\Delta P - \Delta L)$

FIGURE 2.2-BLOCK DIAGRAM OF THE SYSTEM UNDER INVESTIGATION

$$\overline{e^2(t, T)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r^2(t)\,dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c^2(t + T)\,dt$$

$$- 2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r(t)\, c(t + T)\,dt$$

$$= \phi_{ii}(0) + \phi_{oo}(0) - 2\,\phi_{io}(T) \qquad 2.4$$

The normalization of (2.4) is done by multiplying through by $1/\phi_{ii}(0)$ to give the GMSE. Thus,

$$E_N(T) = 1 + \frac{\phi_{oo}(0)}{\phi_{ii}(0)} - \frac{2\,\phi_{io}(T)}{\phi_{ii}(0)} \qquad 2.1$$

It is to be emphasized that the keys to the evaluation of this function are the input autocorrelation function $\phi_{ii}(T)$ and its Fourier transform, the input spectral density $\Phi_{ii}(\omega)$; therefore $\phi_{ii}(T)$ must be transformable.

A treatise on the theory of complex variables and the evaluation of the Fourier transform and its inverse pertinent to this problem is given in a report associated with this thesis.[12]

In order that a function of time f(t) have a Fourier transform,

$$\int_{-\infty}^{\infty} |f(t)|\,dt$$

must be finite. It is shown in section 3.3 that the autocorrelation of the discrete pseudo-random functions are of the form

$$a + b\, e^{-k|T|}$$

The Fourier transform of the constant term is given by a $\delta(\omega)$, where $\delta(\omega)$ is the Fourier transform of a unit step function. The exponential term is readily transformable.

2.3.2 Power spectral density of a unit step function

The power spectral density by definition is,

$$\int_{\infty}^{\infty} e^{-j\omega T}\, dT = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} e^{-\varepsilon|T|}\, e^{-j\omega T}\, dT$$

$$= \lim_{\varepsilon \to 0} \left. \frac{e^{(\varepsilon - j\omega)T}}{\varepsilon - j\omega} \right|_{-\infty}^{0} - \left. \frac{e^{-(\varepsilon + j\omega)T}}{\varepsilon + j\omega} \right|_{0}^{\infty}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon - j\omega} + \frac{1}{\varepsilon + j\omega}$$

$$= \lim_{\varepsilon \to 0} \frac{2\varepsilon}{\varepsilon^2 + \omega^2}$$

For a finite ω the above expression is zero. But,

$$\lim_{\substack{\varepsilon \to 0 \\ \omega \to 0}} \frac{2\varepsilon}{\varepsilon^2 + \omega^2} \qquad 2.5$$

is indeterminate. Applying l'Hospital's rule (2.5) becomes,

$$\lim_{\varepsilon \to 0} \frac{2}{2\,\varepsilon} \quad \to \quad \infty$$

This can be represented by a dirac delta function $\delta(\omega)$. Hence the Fourier transform of a unit step function is

$$\Phi(\omega) = \delta(\omega) = \begin{cases} 0 \text{ for } \omega \neq 0 \\ \infty \text{ for } \omega = 0. \end{cases}$$

In this section and in the previous section it was mentioned that if the input autocorrelation $\phi_{ii}(T)$ is known and if it is Fourier transformable, then its Fourier transform $\Phi_{ii}(\omega)$ can be obtained. Using (2.2a) and (2.2b), the output spectral density $\Phi_{oo}(\omega)$ and the input-output cross-spectral density $\Phi_{io}(\omega)$ for the given H (ω) can be obtained. The inverse Fourier transform of $\Phi_{oo}(\omega)$ and $\Phi_{io}(\omega)$ give $\Phi_{oo}(T)$ and $\Phi_{io}(T)$ which when substituted appropriately in (2.1) give the GMSE. This shows the importance of finding an expression for $\Phi_{ii}(T)$. In the next section the evaluation of the input autocorrelation is discussed.

2.3.3 Evaluation of $\Phi_{ii}(T)$

It is impossible to find an analytic expression for the autocorrelation of a continuous pseudo-random function of time; however, if the continuous pseudo-random function is approximated by a discrete random function then an approximate autocorrelation can be readily obtained. Figure 2.3a shows a continuous pseudo-random disturbance function. Figure 2.3b shows the two-level approximation to the continuous disturbance function shown in Figure 2.3a. It is shown by Lee[11] that the input autocorrelation of a discrete random wave such as the one shown in Figure 2.3b is given by,

$$\phi_{ii}(T) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{\alpha i}\, x_{\beta j}\, P_{\alpha}(x_{\alpha i}) P_{\beta/\alpha}(x_{\beta j}/x_{\alpha i};\, T) \qquad 2.6$$

where,

$\phi_{ii}(T)$ is the input autocorrelation at time T,

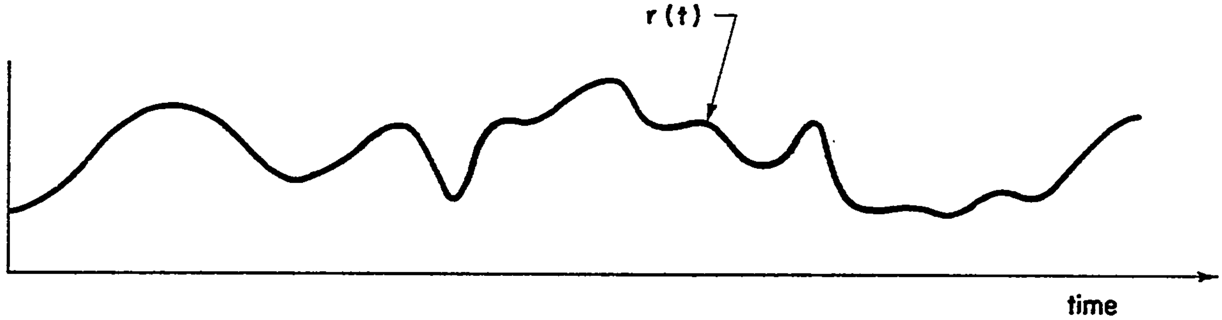
α and β are the variables T seconds apart,

$x_{\alpha i}$ and $x_{\beta j}$ are the amplitudes of the variables α and β assume,

$P_{\alpha}(x_{\alpha i})$ is the probability of the variable α assuming an amplitude x_i and

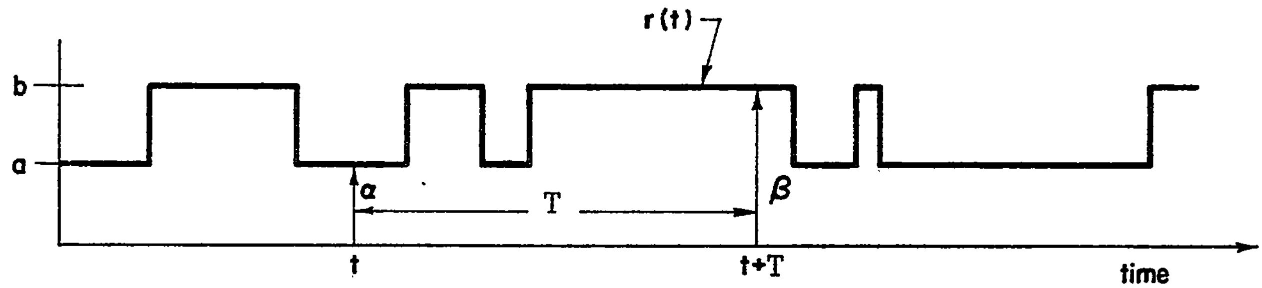
$P_{\beta/\alpha}(x_{\beta j}/x_{\alpha i};\, T)$ is the conditional probability of finding amplitude $x_{\beta j}$ provided T seconds earlier the amplitude was $x_{\alpha i}$.

Hence, the essential step is to estimate the conditional probabilities required in (2.6). Due to reasons given in Chapter 3, the stationary Markov process was used to evaluate these conditional probabilities.



(a)

FIGURE 2.3a-A CONTINUOUS PSEUDO-RANDOM DISTURBANCE FUNCTION



(b)

FIGURE 2.3b - TWO-LEVEL QUANTIZED APPROXIMATION OF THE FUNCTION IN FIGURE 2.3a

In the next section the basic principles of the stationary Markov process are briefly explained.

2.3.4 Stationary Markov process

Let the process described by P have two states denoted by A and B. At any given time t it can be found either in state A or in state B. The so-called state space diagram of such a process is given in Figure 2.4.

Let the process be in state A at time t and the next T seconds be divided into n small intervals of time so that

$$T = n \Delta t \qquad 2.7$$

It is assumed during the small interval Δt, it can go to B with a probability of 1/2 or remain in the same state with probability 1/2. Figure 2.5 shows the so-called "tree-diagram" of this process on a time axis.

In this case it is assumed that at time t the process P is in state A. The total probability of the process being in state A at time $t+\Delta t$,

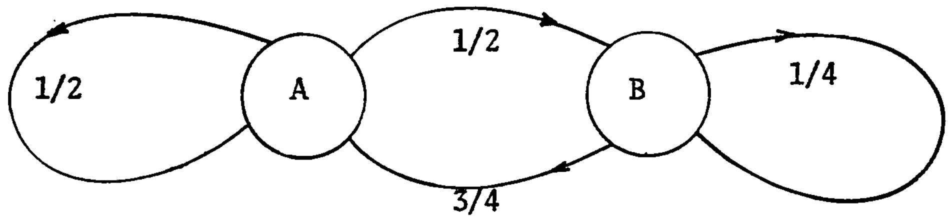


FIGURE 2.4 - SPACE DIAGRAM FOR A PARTICULAR CASE

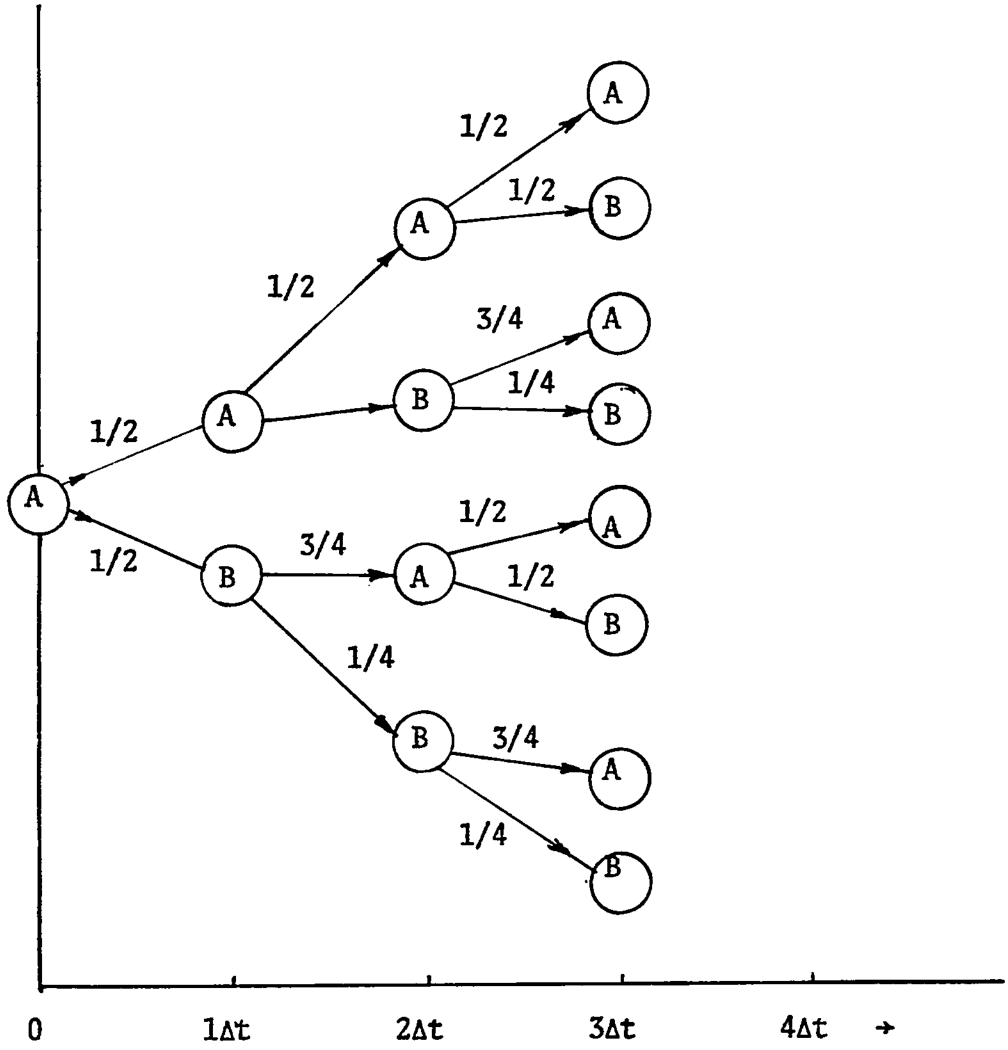


FIGURE 2.5 - TREE-DIAGRAM FOR A PARTICULAR CASE

$t+2\Delta t$, and $t+3\Delta t$ are 1/2, 5/8 and 19/32 respectively. It is assumed that during the time Δt only one transition occurs.

Instead of the constant probabilities assumed in Figure 2.4, Let it now be assumed that the process P has constant rates of departure. Let λ and μ be the constant rates of departures from levels A and B respectively. The test for the validity of constant rate of departure assumption and the estimation of the rates of departures for a particular random process are given in Chapter 3. For the constant rates of departures λ and μ, the revised state space diagram and the tree-diagram are given in Figures 2.6 and 2.7.

If it is assumed that at time t the probability of the process being in state A is 1, then $P_A(t+\Delta t)$ becomes the conditional probability. An elegant expression for the conditional probability can be obtained by applying a limiting process. This procedure avoids the cumbersome work involved in finding the conditional probabilities by tree-diagram.

For a multi-level quantization, the conditional probabilities are derived by using Cramer's rule. A detailed discussion of the derivation of the conditional probabilities is given in Chapter 3. The eigenvalues of the denominator matrix in (3.23) are obtained by using the QR-transformation. In the next sections the basic principles of the QR-transformation are explained briefly.

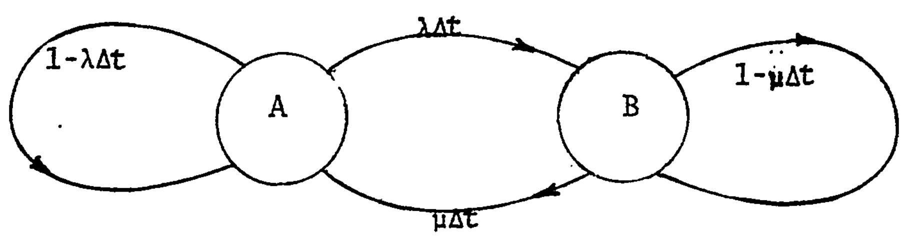


FIGURE 2.6 - SPACE DIAGRAM FOR A GENERAL CASE

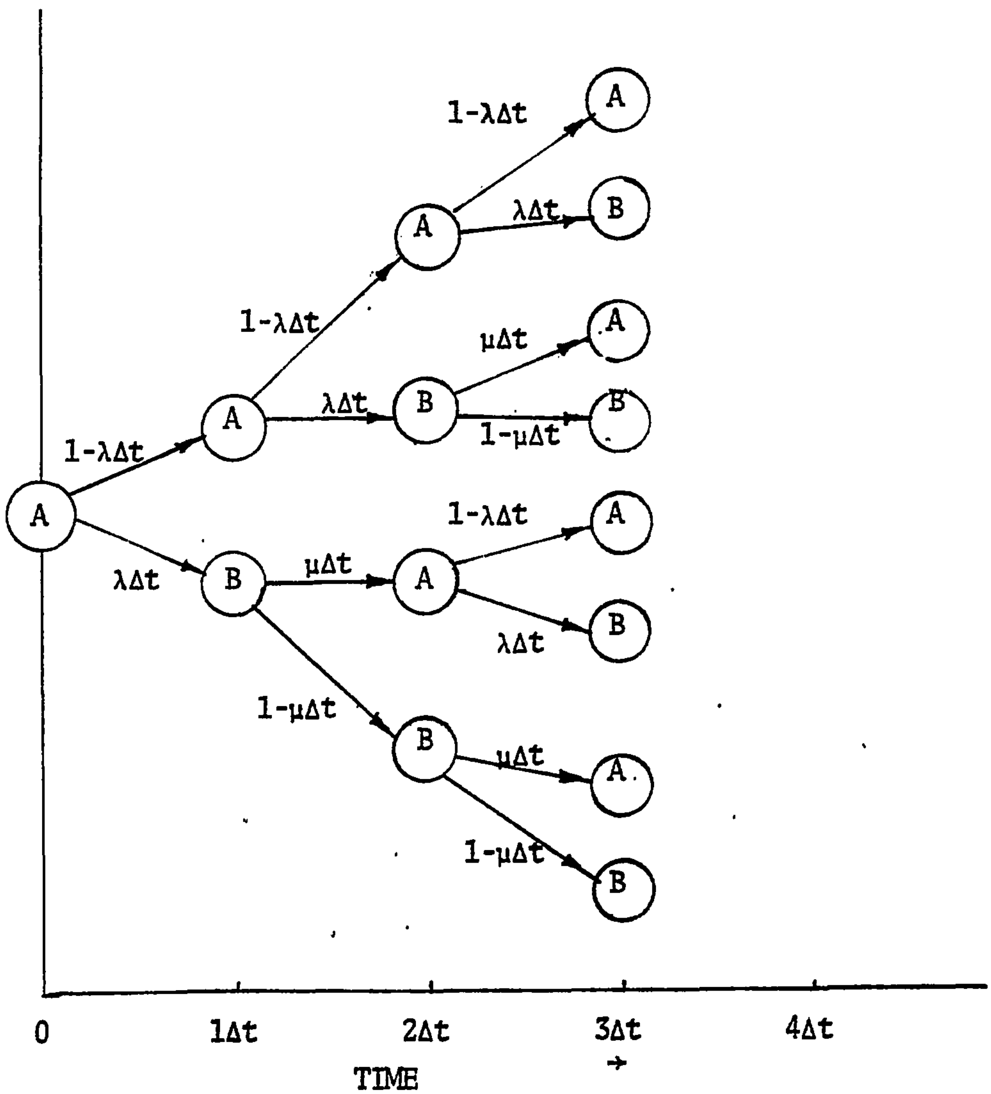


FIGURE 2.7 - TREE-DIAGRAM FOR A GENERAL CASE

2.3.5 QR-transformation

Let A be a matrix and let X be its modal matrix. Then,

$$X A X^{-1} = C$$

$$X A X^{-1} - \lambda I = C - \lambda I$$

$$X (A - \lambda I) X^{-1} = C - \lambda I$$

Taking the determinant on both sides,

$$\text{Det } |A - \lambda I| = \text{Det } |C - \lambda I| \qquad 2.8$$

(2.8) shows that the eigenvalues do not change under similarity transformation. It is well known that if A is symmetric then C is a diagonal matrix, called the canonical form. On the other hand, if the matrix A is asymmetric, it can be proved that the matrix A can be reduced to the triangular canonical form by an elementary similarity transformation or by elementary unitary transformations. The diagonal members are then the eigenvalues of the given matrix.

Rutishauser's[14] LR-transformation and Francis'[15] QR-transformation achieve this reduction to triangular canonical form by an iterative process. More details on the QR-transform are also given in Reference 12.

For a given random sample the probabilities $P_\alpha(x_i)$ can be estimated. With the derived conditional probabilities, the expression for $\phi_{ii}(T)$ in (2.6) can be derived. The Fourier transform of $\phi_{ii}(T)$ is then found. In the next section the method of evaluation of $\phi_{oo}(0)$ and $\phi_{io}(T)$ are briefly explained.

2.3.6 Evaluation of $\phi_{oo}(0)$ and $\phi_{io}(T)$

Once the input power spectral density is derived, the output power spectral density $\Phi_{oo}(\omega)$ and the input-output cross-spectral density $\phi_{io}(\omega)$ can be obtained using the relations (2.2a) and (2.2b). For a particular set of system parameter values, the frequency response function $H(\omega)$ can be found and the spectral density functions $\Phi_{oo}(\omega)$ and $\Phi_{io}(\omega)$ can be evaluated. The inverse Fourier transforms defined in (2.9a) and (2.9b) give the required correlation functions.

$$\phi_{oo}(T) = \int_{-\infty}^{\infty} \Phi_{oo}(\omega) \, e^{j\omega T} \, d\omega \qquad 2.9a$$

$$\Phi_{io}(T) = \int_{-\infty}^{\infty} \Phi_{io}(\omega) \, e^{j\omega T} \, d\omega \qquad 2.9b$$

The detailed evaluation of the inverse transform and a discussion of their uniqueness are given in Reference 12. Essentially (2.9a) and (2.9b) are evaluated by substituting $j\omega$=s and evaluating the indefinite integrals with the Bromovitch on the imaginary axis of the S-plane. This is the inverse double-sided Laplace transform; the Fourier transform is a special case of the double-sided Laplace transform. The evaluation of the inverse of $\Phi_{oo}(s)$ and $\Phi_{io}(s)$ require partial fraction expansions. In order to carry out these expansions the Muller Algorithm and Newton-Raphson methods are used.

2.3.7 Muller Algorithm[12,16]

The Muller Algorithm is used to find the approximate roots or zeros of the characteristic equations of H(S) and $\Phi_{ii}(S)$. The approximate roots are used as initial values in the Newton-Raphson method[17]. The Muller Algorithm has almost quadratic convergence. These values of the roots are found in the ascending order of the absolute value and the polynomial is deflated after each root is found. This reduces the error of deflation. In the Muller Algorithm, the previous root found is used as the initial value for finding the next root. This has definite advantage over the Newton-Raphson method[17]. It was found that the time of convergence is reduced if the initial assumed values are closer to the actual values. It was also found that the number of iterations is not crucial. In case of a realizable system H(S) such as the one mentioned here, the shifting of the initial values can be of some value with regard to saving of computation time. The principle of the Muller Algorithm is to take three points and then approximate the given polynomial by a second order polynomial. The given polynomial and the unique second order polynomial assume the same values at least at three given points. A more detailed discussion on this method is given in Reference 12.

Once the values of $\phi_{ii}(T)$, $\phi_{oo}(T)$ and $\phi_{io}(T)$ are found, then $\phi_{oo}(0)$ and $\phi_{ii}(0)$ are found by merely substituting T=0 in their respective expressions. In order to evaluate $E_N(T)$ the values of $\phi_{ii}(0)$, $\phi_{oo}(0)$ and $\phi_{io}(T)$ are substituted in (2.1).

3. DESCRIPTION OF THE INPUT AND DERIVATION OF THE INPUT AUTOCORRELATION

3.1 Introduction

In Chapter 2, the generalized mean square error (2.1), hereafter referred to as the GMSE, was derived. It can be seen from (2.2a) and (2.2b) that if the input autocorrelation $\phi_{ii}(T)$ is known and if it is Fourier transformable, then $\phi_{oo}(\omega)$ and $\phi_{io}(\omega)$ can be obtained. The inverse Fourier transforms of these functions are $\phi_{oo}(T)$ and $\phi_{io}(T)$ respectively. This shows the need for describing a pseudo-random load input disturbance in a form which makes possible the derivation of the input autocorrelation $\phi_{ii}(T)$.

In this chapter, justification for the particular statistical description of the input is given. The theory for the derivation of the input autocorrelation is developed. A practical verification for the theory is also given. The derivation of the input autocorrelation for the case of multi-level quantization of the input is illustrated.

3.2 Description of the Input

It has been mentioned that the autocorrelation $\phi_{ii}(T)$ must be found in order to evaluate (2.1). The variations of the input, in this particular case the load input to the system in Figure 2.2, are continuous and pseudo-random. It is difficult to find an exact expression for the input autocorrelation of this continuous pseudo-random input directly; therefore, it was approximated by a discrete random process for which the autocorrelation could be found easily.

3.3 Autocorrelation of $\phi_{ii}(T)$ of the Discrete Random Process

The input autocorrelation of a discrete random signal can be calculated by using the Poisson process provided the random variable conforms with certain requirements. Due to the close relation between the Markov process and the Poisson process, the Markov process can also be used in particular cases. It is more flexible than the Poisson process for higher order quantization of the signal. In section 3.3.1 and 3.3.2, these two methods are described.

3.3.1 Poisson approach

It was mentioned in section 3.2 that the given continuous pseudo-random input could be approximated by a discrete random process whose input autocorrelation $\phi_{ii}(T)$ can be evaluated. As an example, let the input shown in Figure 2.3a be approximated by a discrete random process with two levels a and b. Now the autocorrelation $\phi_{ii}(T)$ of the two-level approximation to the given input can be derived easily. This is the simplest case. The two-level approximation to the given input in Figure 2.3b, can be considered to be equivalent to a step input of a magnitude (a+b)/2 with a pulse width modulated wave of amplitude (b-a)/2 superimposed on it.

Let the random variable, which in this case is the change of level points or number of zero crossings, be denoted by η. The random variable η can be assumed a priori to have a Poisson distribution, because it can be assumed that it has the following properties[18]:

1. the distribution is time invariant;
2. the probability of having one zero crossing in a small time Δt is approximately equal to $\lambda \Delta t$, where λ is a positive constant; and
3. the probability of having more than one zero crossing in this interval is zero.

The input autocorrelation of the discrete random input is given by(2.6).

Hence in this case,

$$\phi_{ii}(T) = a.a.P_\alpha(a).P_{\beta/\alpha}(a/a;\ T) + a.b.P_\alpha(a)\ P_{\beta/\alpha}(b/a;\ T)$$
$$+ b.a.P_\alpha(b).P_{\beta/\alpha}(a/b;\ T) + b.b.P_\alpha(b)\ P_{\beta/\alpha}(b/b;\ T) \quad 3.1$$

$P_\alpha(a)$ and $P_\alpha(b)$ are the estimations of the probabilities of the variable α taking the value a and b respectively. The conditional probabilities $P_{\beta/\alpha}(a/a;\ T)$ and $P_{\beta/\alpha}(b/b;\ T)$ are the total probabilities of the even numbers of zero corssings η, in the time interval T. Hence,

$$\phi_{ii}(T) = a.a.P_\alpha(a) \sum_{n=0}^{\infty} P_\eta(2n;\ T) + a.b.P_\alpha(a) \sum_{n=0}^{\infty} P_\eta((2n+1);\ T)$$
$$+ b.a.P_\alpha(b) \sum_{n=0}^{\infty} P_\eta((2n+1);\ T) + b^2.\ P_\alpha(b)$$
$$. \sum_{n=0}^{\infty} P_\eta(2n;\ T) \quad 3.2$$

$P_\eta(n;\ T)$ is the probability of finding n zero crossings in an interval of time T and it is

$$P_\eta(n;\ T) = e^{-kT} \frac{(kT)^n}{n!}$$

where k is the average number of zero crossings per unit time. Using this relation, (3.2) can be rewritten as

$$\phi_{ii}(T) = a^2 P_\alpha(a) \left(1 + \frac{(kT)^2}{2!} + \frac{(kT)^4}{4!} + \frac{(kT)^6}{6!} + \dots + \frac{(kT)^{2n}}{2n!}\right) e^{-kT}$$

$$+ ab\, P_\alpha(a) \left(kT + \frac{(kT)^3}{3!} + \frac{(kT)^5}{5!} + \frac{(kT)^7}{7!} + \dots + \frac{(kT)^{2n+1}}{(2n+1)!}\right) e^{-kT}$$

$$+ b^2 P_\alpha(b) \left(1 + \frac{(kT)^2}{2!} + \frac{(kT)^4}{4!} + \frac{(kT)^6}{6!} + \dots + \frac{(kT)^{2n}}{2n!}\right) e^{-kT}$$

$$+ ba\, P_\alpha(b) \left(kT + \frac{(kT)^3}{3!} + \frac{(kT)^5}{5!} + \frac{(kT)^7}{7!} + \dots + \frac{(kT)^{2n+1}}{(2n+1)!}\right) e^{-kT}$$

$$= a^2 P_\alpha(a) \left(\frac{e^{kT} + e^{-kT}}{2}\right) e^{-kT} + a\, b\, P_\alpha(a)\, e^{-kT}$$

$$\cdot \left(\frac{e^{kT} - e^{-kT}}{2}\right) + b\, a\, P_\alpha(b) \left(\frac{e^{kT} - e^{-kT}}{2}\right) e^{-kT}$$

$$+ b^2 P_\alpha(b)\, e^{-kT} \left(\frac{e^{kT} + e^{-kT}}{2}\right)$$

$$= \frac{a^2}{2} P_\alpha(a) \left(1 + e^{-2kT}\right) + \frac{ab}{2} P_\alpha(a) \left(1 - e^{-2kT}\right)$$

$$+ \frac{ba}{2} P_\alpha(b) \left(1 - e^{-2kT}\right) + \frac{b^2}{2} P_\alpha(b) \left(1 + e^{-2kT}\right) \qquad 3.3$$

The value of $\phi_{ii}(T)$ in (3.3) is irrespective of the sign of T. Hence (3.3) is also given by (3.4).

$$\phi_{ii}(T) = \frac{a^2 P_\alpha(a) + b^2 P_\alpha(b)}{2} (1 + e^{-2k|T|})$$

$$+ \frac{a\,b}{2} (1 - e^{-2k|T|}) \qquad 3.4$$

If the given input were to be quantized into three levels, then it would be difficult to find conditional probabilities by the method described above. Both the Poisson and the stationary Markov process are memoryless. It is shown in section 3.3.2 that using the Markov process leads to the same autocorrelation as in (3.4). The Markov process is more flexible than the Poisson process for higher levels of quantization.

3.3.2 The Markov approach

To overcome the problem encountered for more than two levels of quantization, the Markov process was chosen apriori. The basic principles of Markov process are given in the subsection 2.3.4.

In Figure 2.3b, let the level a be called state "a" and level b be called state "b". The derivation of the conditional probabilities for this signal by the Markov process requires certain conditions. These conditions are similar to the ones used in conjunction with the Poisson process. They are[19] embodied in the statements 1 and 2 below.

1. The process is in state a at time t. It can leave this state in the next small interval Δt with a conditional probability $K_a \Delta t$, where K_a is a constant. Alternatively, the process could be in state b at time t and it could leave this state in the next small interval Δt with a conditional probability $K_b \Delta t$, where K_b is a constant.
2. Both the events in 1 cannot happen in that small interval Δt. This is equivalent to saying that the probability of more than one change of level occuring in a small interval Δt is equal to zero.

Condition 1 also means that the probability distribution function of the random variable, the period of residence in each level, is a negative exponential function. From these conditions, it appeared that there was some connection between the assumption of the Poisson process for variable η, and the Markov process. It is shown hereunder that both methods lead to the same result for the two-level approximation. The Markov process was applied to the three-level aproximation case also and found to be valid.

Figure 3.1 shows the state space diagram for this two-level approximation of the pseudo-random input shown in Figure 2.4a. λ is the rate of departure from state a to state b; μ is the rate of departure from state b to a. From the assumptions mentioned earlier, it may be stated that,

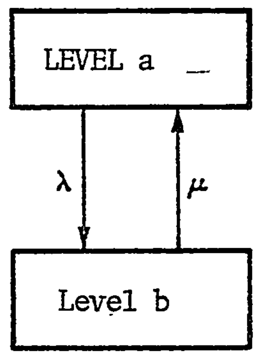


FIGURE 3.1 - STATE SPACE DIAGRAM FOR THE TWO-LEVEL QUANTIZED VARIABLE

$$P_a(t+\Delta t) = P_a(t)\,(1-\lambda\Delta t) + P_b(t)\,\mu\Delta t$$

$$P_b(t+\Delta t) = P_b(t)\,(1-\mu\Delta t) + P_a(t)\,\lambda\Delta t$$

where,

$P_a(t+\Delta t)$ is the probability of finding the variable in state a at time $(t+\Delta t)$,

$P_b(t+\Delta t)$ is the probability of finding the variable in state b at time $(t+\Delta t)$, and

$P_a(t)$ and $P_b(t)$ are the probabilities of finding the discrete process in state a or in state b respectively at time t.

From (3.5),

$$\frac{P_a(t+\Delta t) - P_a(t)}{\Delta t} = -\lambda\, P_a(t) + P_b(t)\,\mu$$

and

$$\lim_{\Delta t\to 0} \frac{P_a(t+\Delta t) - P_a(t)}{\Delta t} = P_a'(t)$$

Thus,

$$P_a'(t) + \lambda\, P_a(t) = P_b(t)\,\mu \qquad 3.7$$

Similarly from (3.6),

$$P_b'(t) + \mu P_b(t) = P_a(t)\lambda \qquad 3.8$$

Taking the Laplace transforms of (3.7) and (3.8) gives

$$SP_a(S) - P_a(0) + \lambda P_a(S) = P_b(S)\mu$$

and

$$SP_b(S) - P_b(0) + P_b(S)\mu = P_a(S)\lambda$$

or in matrix form,

$$\begin{bmatrix} (S+\lambda) & -\mu \\ -\lambda & (S+\mu) \end{bmatrix} \begin{bmatrix} P_a(S) \\ P_b(S) \end{bmatrix} = \begin{bmatrix} P_a(0) \\ P_b(0) \end{bmatrix} \qquad 3.9$$

where $P_a(0)$ and $P_b(0)$ are the probabilities of being in state a or in state b at time t=0. $P_a(S)$ and $P_b(S)$ are the Laplace transforms of $P_a(t)$ and $P_b(t)$. S is the Laplace operator.

The actual time response solutions for $P_a(t)$ from (3.9) depend on the specific initial conditions $P_a(0)$ and $P_b(0)$. Assuming that the process can be in state a or in state b at time t=o, then the condition 1 probabilities are derived by Cramer's Rule and are given by (3.10) to (3.13).

i. For $P_a(0) = 1$, $P_b(0) = 0$,

$$P_{\beta/\alpha}(a/a;\ T) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)T} \tag{3.10}$$

$$P_{\beta/\alpha}(b/a;\ T) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)T} \tag{3.11}$$

ii. Similarly for $P_b(0) = 1$ and $P_a(0) = 0$,

$$P_{\beta/\alpha}(a/b;\ T) = \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)T} \tag{3.12}$$

$$P_{\beta/\alpha}(b/b;\ T) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)T} \tag{3.13}$$

where,

$P_{\beta/\alpha}(a/a;\ T)$ is the conditional probability of finding β=a, provided that at time T earlier α=a,

$P_{\beta/\alpha}(b/a;\ T)$ is the conditional probability of finding β=b, provided that at time T earlier α=a,

$P_{\beta/\alpha}(a/b;\ T)$ is the conditional probability of finding β=a, provided that at time T earlier α=b,

$P_{\beta/\alpha}(b/b;\ T)$ is the conditional probability of finding β=b, provided that at time T earlier α=b.

From (2.6),

$$\phi_{ii}(T) = P_\alpha(a)\, P_{\beta/\alpha}(a/a;\, T)\, a^2 + P_\alpha(a)\, P_{\beta/\alpha}(b/a;\, T)\, a\, b$$
$$+ b\, a\, P_\alpha(b)\, P_{\beta/\alpha}(a/b;\, T) + b^2\, P_\alpha(b)\, P_{\beta/\alpha}(b/b;\, T) \qquad 3.14$$

Substituting the values of conditional probabilities from (3.10) to (3.13), in (3.14) gives the autocorrelation of the input.

$$\begin{aligned}\phi_{ii}(T) = {} & a^2\, P(a)\, \left(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)T}\right) \\ & + ab\, P(a)\, \left(\frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)T}\right) \\ & + ab\, P(b)\, \left(\frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}\, e^{-(\lambda+\mu)T}\right) \\ & + b^2\, P(b)\, \left(\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\, e^{-(\lambda+\mu)T}\right) \qquad 3.15\end{aligned}$$

(3.15) is independent of the sign of the time T and can be written as,

$$\begin{aligned}\phi_{ii}(T) = {} & a^2\, P(a)\, \left(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)|T|}\right) \\ & + ab\, P(a)\, \left(\frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)|T|}\right) \\ & + ab\, P(b)\, \left(\frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}\, e^{-(\lambda+\mu)|T|}\right) \\ & + b^2\, P(b)\, \left(\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\, e^{-(\lambda+\mu)|T|}\right) \qquad 3.16\end{aligned}$$

When $\lambda = \mu = k$, (3.16) is given by,

$$\phi_{ii}(T) = \frac{a^2 P(a)}{2} (1 + e^{-2k|T|}) + \frac{a\,b}{2} (P(a) + P(b)) (1 - e^{-2k|T|}) + \frac{b^2}{2} P(b) (1 + e^{-2k|T|}) \qquad 3.17$$

The constant λ is estimated by dividing the number of departures from state a by the total period of residence in state a. Similarly μ is estimated by dividing the total number of departures from state b by the total period of residence in state b. From (3.17) and (3.4) it can be seen that the expressions for the input autocorrelation $\phi_{ii}(T)$ are the same. This strengthens further the earlier inference that the Markov process can be used to evaluate the conditional probabilities of a discrete random input.

3.4 Experimental Verification

Having established apriori that the Markov process could be used for the evaluation of the input autocorrelation, it was necessary to have an experimental verification. The load on a power system is assumed to be a random process with a Gaussian distribution; therefore the steps outlined in section 3.3 were carried out with an actual signal. A random function with a Gaussian distribution was generated by a signal generator and recorded. It was of the form shown in Figure 3.2a. The range variable X(t) was divided into two and then three equal intervals and the