Gröbner bases via linkage for classes of generalized determinantal ideals

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Abstract

Gröbner bases are an important tool for working with ideals in polynomial rings. They have both computational and theoretical importance. In this dissertation, we produce Gröbner bases for some families of generalized determinantal ideals. Our main contribution is a Gröbner basis for Schubert patch ideals.

Schubert patch ideals are prime defining ideals of open patches of Schubert varieties in the type A flag variety. We adapt E. Gorla, J. Migliore, and U. Nagel's "Grobner basis via linkage" technique to prove a conjecture of A. Yong, namely, the essential minors of every Schubert patch ideal form a Gröbner basis. Using the same approach, we recover the result of A. Woo and A. Yong that the essential minors of a Kazhdan-Lusztig ideal (and hence, essential minors of a Schubert determinantal ideal) form a Gröbner basis with respect to an appropriate term order. In addition, with respect to the standard grading, we show that homogeneous Schubert patch ideals, homogeneous Kazhdan-Lusztig ideals and Schubert determinantal ideals are glicci. In the last chapter of this dissertation, we briefly discuss some future directions.

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Dedication

This thesis is dedicated to the blessed Trinity: (1) God; the creator of heaven and earth (John 1:1), (2) Jesus; the Son of God and Saviour of the world (John 3:16), and (3) the Holy Spirit; one who will convict the world of sin, of righteousness, and of the coming judgment (John 16:8).

Contents

List of Notations

$\mathbb{K}$	An arbitrary base field, occasionally of characteristic 0
$\mathbb{K}[x_1, \ldots, x_n]$	A polynomial ring in n indeterminates
$\mathbf{x} = \{x_{\alpha\beta}\}_{1\leq\alpha\leq m,\, 1\leq\beta\leq n}$	A sequence of variables in a generic $m \times n$ matrix $\mathbf{X}$
$\mathbb{Z}$	The set of integers
e_i	The i^{th} standard basis vector in $\mathbb{Z}^n$
$\text{in}_\succ(f)$	The initial term of f with respect to a term order $\succ$
$\text{in}_\succ(I)$	The initial ideal of I with respect to a term order $\succ$
$\text{in}_\succ(\mathcal{G})$	The set of initial terms, with respect to $\succ$, of elements in set $\mathcal{G}$
$I : J$	The ideal quotient or colon ideal of I by J
$\text{ht}(I)$	The codimension or height of I
G_0	Generically Gorenstein
$M(-\boldsymbol{e})$	Shifting of degrees in M by $\boldsymbol{e}$
$M_{\boldsymbol{\ell}}$	Degree $\boldsymbol{\ell}$ graded component of M
s_b	An adjacent transposition $(b, b+1)$ that swaps b and $b+1$
$X_{p\times q}$ (resp. $w_{p\times q}$)	The upper left $p \times q$ rectangular submatrix of X (resp. w)
$\text{rank}(w_{p\times q})$	The rank of $w_{p\times q}$
$D(w)$	The Rothe diagram of w
$\ell(w)$	The length of a permutation w
$\mathcal{E}ss(w)$	The essential set of a permutation w
$I_{v,w}$	The Kazhdan-Lusztig ideal associated to the permutations v, w
$\mathbf{X}^{(v)}$	The specialized generic matrix consisting of variables in the coordinate ring for $I_{v,w}$
$\mathbf{x}^{(v)}$	The set of variables in $\mathbf{X}^{(v)}$
x_{last}	The variable in $\mathbf{x}^{(v)}$ that is maximal with respect to a fixed term order on $\mathbb{K}[\mathbf{x}^{(v)}]$
$Q_{v,w}$	The Schubert patch ideal associated to the permutations v, w
$\mathbf{Z}^{(v)}$	The specialized generic matrix consisting of variables in the coordinate ring for $Q_{v,w}$
$\mathbf{z}^{(v)}$	The set of variables in $\mathbf{X}^{(v)}$
z_{last}	The variable in $\mathbf{z}^{(v)}$ that is maximal with respect to a fixed term order on $\mathbb{K}[\mathbf{z}^{(v)}]$
I_w	The Schubert determinantal ideal associated to the permutation w
$\mathbf{y}^{(v)}$	The set of variables in $\mathbf{z}^{(v)}$ but not in $\mathbf{x}^{(v)}$

$\mathbf{Z}^{(v)}s_b$	The resulting matrix from swapping columns b and $b+1$ of $\mathbf{Z}^{(v)}$
$T_{vs_b,w}$	The ideal generated by minors of size $1+\operatorname{rank}(w_{p\times q})$ in $\left(\mathbf{Z}^{(v)}s_b\right)_{p\times q}$, $1\le p,q\le n$
$\mathbf{X}_{[i_1,\dots,i_\ell;j_1,\dots,j_\ell]}$	An $\ell\times\ell$ minor of matrix $\mathbf{X}$ that involves rows $i_1<\cdots<i_\ell$ and columns $j_1<\cdots<j_\ell$
$c_{i_1,\dots,i_{\ell_1};j_1,\dots,j_{\ell_2}}(M)$	The resulting matrix from deleting rows $i_1,\dots,i_{\ell_1}$ and columns $j_1,\dots,j_{\ell_2}$ of M
$c_{i_1,\dots,i_{\ell_1};}(M)$	The resulting matrix from deleting rows $i_1,\dots,i_{\ell_1}$ of M
$c_{;j_1,\dots,j_{\ell_2}}(M)$	The resulting matrix from deleting columns $j_1,\dots,j_{\ell_2}$ of M
$\mathbf{W}^{(vs_b)}$	The resulting matrix from substituting $x_{j,b}$ for $x_{j,b+1}$ in the matrix $\mathbf{X}^{(vs_b)}$, where b is the last descent of v
$L_{vs_b,w}$	An ideal generated by minors of size $1+\operatorname{rank}(w_{p\times q})$ in $\mathbf{W}^{(vs_b)}_{p\times q}$, $1\le p,q\le n$
$I\xrightarrow{N}_h J$	The ideal I can be obtained from another ideal J by an elementary G-biliaison of height h on N

Chapter 1

Introduction

Generalized determinantal ideals are polynomial ideals generated by minors of matrices. They generalize the classical determinantal ideals. In this dissertation, we study three related families of generalized determinantal ideals: Schubert patch ideals, Kazhdan-Lusztig ideals and Schubert determinantal ideals. We use an approach called "Gröbner basis via linkage" [GMN13] to provide Gröbner bases for these ideals. The Gröbner basis for Schubert patch ideals is new and we adapted its proof to recover the known Gröbner bases for Kazhdan-Lusztig ideals [WY12, Theorem 2.1] and Schubert determinantal ideals [KM05, Theorem B].

Let $G = GL_n(\mathbb{K})$ denote the general linear group of invertible $n \times n$ matrices with entries in a field $\mathbb{K}$. Let B_- (resp. B_+) be the subgroup of lower (resp. upper) triangular matrices in G. In chapter 3 of this dissertation, we work with the complete flag variety $B_-\backslash G$. Given a variety $X \subseteq B_-\backslash G$, set $\mathcal{M}_{v,X} := X \cap \big(\Omega^\circ_{v_0}\big)v_0 v$, where $\Omega^\circ_v := B_-\backslash B_- v B_-$ is the opposite Schubert cell associated to v, and $v_0 \in S_n$ is the long word permutation $v_0(i) = n - i + 1$. A. Knutson called this variety $\mathcal{M}_{v,X}$ an X patch in [Knu08] while E. Insko and A. Yong called the defining ideal $Q_{v,X}$ of this X patch a patch ideal in [IY12]. When X is the Schubert variety X_w, $w \in S_n$ (X_w is the Zariski closure of the Schubert cell $X^\circ_w := B_-\backslash B_- w B_+$), the variety $\mathcal{M}_{v,X} = \mathcal{M}_{v,w}$ is called Schubert patch variety and its defining ideal is called Schubert patch ideal.

Kazhdan-Lusztig ideals, introduced in [WY08], are related to Schubert patch ideals; they are prime defining ideals of the varieties $\mathcal{N}_{v,w} := X_w \cap \Omega^\circ_v$, $v, w \in S_n$. To be explicit, given an arbitrary Schubert patch ideal, on setting some specific variables of this ideal to zero, we obtain a Kazhdan-Lusztig ideal. Schubert determinantal ideals, which are special cases of Kazhdan-Lusztig ideals, are defining ideals for sets of rectangular matrices, called matrix Schubert varieties, that satisfy certain conditions on ranks of their submatrices.

Below is a conjecture due to Alexander Yong which was publicized through talks and conversations.

Conjecture (A. Yong)**.** *The essential minors of Schubert patch ideals form a Gröbner basis.*

The main result of this dissertation is:

Main Theorem. *A. Yong's conjecture is true.*

This result with its proof appears in Theorem 3.2.33 of this dissertation. Our proof of this main result uses a technique called "Gröbner basis via linkage" [GMN13] and this technique involves a key lemma that gives a sufficient condition for a set of polynomials to form a Gröbner basis for the ideal it generates. The standard graded version of this lemma is due to Gorla, Migliore and Nagel [GMN13, Lemma 1.12], where it was used to show that the generators of some families of generalized determinantal ideals form Gröbner bases. It was also used in [FK20, Lemma 3.1] to show that the natural generators of a double determinantal ideal form a Gröbner basis. We use a multigraded version of this lemma in this dissertation and this appears as Lemma 3.1.1.

We adapt our proof of the Main Theorem above to give an alternative proof to the main result in [WY12, Theorem 2.1], namely, the essential minors generating a Kazhdan-Lusztig ideal form a Gröbner basis. Since Schubert determinantal ideals are special cases of Kazhdan-Lusztig ideals, we therefore also get an alternative proof to the well-known fact that the essential minors of every Schubert determinantal ideal form a Gröbner basis ([KM05, Theorem B]).

The question "Is every arithmetically Cohen-Macaulay subvariety of projective space glicci?" is one of the open problems in liaison theory. Many families of ideals generated by minors of generic matrices have been shown to be glicci i.e., they belong to the **G**orenstein **li**aison **c**lass of **c**omplete **i**ntersection. Examples include standard determinantal ideals [KMMR$^+$01, chapter 3], symmetric mixed ladder determinantal ideals [Gor10] and mixed ladder determinantal ideals [Gor07]. By slightly adapting our proof of the Main Theorem above, we show in Theorem 3.5.2 of this dissertation that:

Theorem. *With respect to standard grading, homogeneous Schubert patch ideals are glicci.*

We also show that the Kazhdan-Lusztig ideals that are homogeneous with respect to the standard grading are glicci. Consequently, Schubert determinantal ideals are glicci. This glicci result for Schubert determinantal ideals can be inferred from [KR21b], but their proof relies on combinatorial results of Knutson and Miller [KM05].

Knutson showed in his paper [Knu08] that Schubert patches degenerate to a Stanley-Reisner scheme whose underlying simplicial complex is a subword complex, and therefore vertex decomposable [KM05, Theorem E]. In addition to the Gröbner bases results for these generalized determinantal ideals, we give an alternative proof to the known fact that their initial ideals are squarefree and their simplicial complexes are vertex decomposable.

Though the Kazhdan-Lusztig ideals are homogeneous with respect to some positive multigrading ([WY08, Lemma 5.2]) and the Schubert patch ideals are also homogeneous with respect to some positive multigrading (Lemmas 2.8.2 and 2.8.4), not all Schubert patch ideals and Kazhdan-Lusztig ideals are homogeneous with respect to the standard grading. Characterizing the pairs $(v, w) \in S_n \times S_n$ for which the Kazhdan-Lusztig ideal $I_{v,w}$ is homogeneous with respect to the standard grading is an open problem in [WY08, Problem 5.5]. In relation to these homogeneity problems, we obtain some pattern avoidance results on when Kazhdan-Lusztig ideals and Schubert patch ideals are homogeneous with respect to standard grading (see Propositions 3.6.7 and 3.6.8).

In Chapter 2, we give a general commutative and combinatorial commutative algebra background that are relevant and useful for our study. In addition, we give some background and literature review on Gorenstein liaison. Lastly, we give a concise overview of the generalized determinantal ideals that are of interest to us, namely, Schubert determinantal ideals, Kazhdan-Lusztig ideals and Schubert patch ideals.

In Chapter 3, we present the key lemma of this dissertation (Lemma 3.1.1). This lemma is presented in a multigraded setting and it gives a sufficient condition for a set of polynomials to form a Gröbner basis for the ideal it generates. We then prove a conjecture due to Alexander Yong - the essential minors of Schubert patch ideals from a Gröbner basis (Theorem 3.2.33). As a result, we recover and give a new proof of the known Gröbner basis result for Kazhdan-Lusztig ideals [WY12, Theorem 2.1], which generalizes the Gröbner basis result for Schubert determinantal ideals [KM05, Theorem B]. Also in this chapter, under the standard grading, we show that the homogeneous Schubert patch ideals and homogeneous Kazhdan-Lusztig ideals (and hence Schubert determinantal ideals) are glicci. Finally, we give some pattern avoidance results in relation to the problem of when Schubert patch ideals and Kazhdan-Lusztig ideals are homogeneous with respect to the standard grading.

In Chapter 4, we give an introduction to some of our future work. Precisely, we wish to adapt the techniques in Chapter 3 to provide Gröbner bases results for some type C Kazhdan-Lusztig ideals and possibly some type D quiver ideals.

Chapter 2

Background and Literature Review

Throughout this chapter, $S = \mathbb{K}[x_1, \dots, x_n]$ will represent a polynomial ring in n indeterminates $x_1, \dots, x_n$ over an arbitrary field $\mathbb{K}$, unless otherwise stated.

2.1 General Commutative Algebra

Here in this section, we give definitions and some results that will be useful for our study in the subsequent chapters. The primary references for this section are [Pee10], [CLO92], [Eis13] and [Mat89].

Definition 2.1.1. A **monomial** in S is of the form $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where each α_i, $1 \le i \le n$, is a nonnegative integer.

Definition 2.1.2. A **term (or monomial) order** on S is a relation $\succ$ that satisfies the following conditions:

(i) $\succ$ is a total order, i.e., for all monomials $\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}} \in S$, exactly one of the following holds:

$$\mathbf{x}^{\boldsymbol{\alpha}} \succ \mathbf{x}^{\boldsymbol{\beta}}, \quad \mathbf{x}^{\boldsymbol{\beta}} \succ \mathbf{x}^{\boldsymbol{\alpha}}, \quad \text{or} \quad \mathbf{x}^{\boldsymbol{\alpha}} = \mathbf{x}^{\boldsymbol{\beta}}.$$

(ii) $\succ$ is multiplicative, i.e., if $\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\beta}}$ are monomials in S with $\mathbf{x}^{\boldsymbol{\alpha}} \succ \mathbf{x}^{\boldsymbol{\beta}}$ and $\mathbf{x}^{\boldsymbol{\gamma}} \in S$ is another monomial, then $\mathbf{x}^{\boldsymbol{\alpha}} \cdot \mathbf{x}^{\boldsymbol{\gamma}} \succ \mathbf{x}^{\boldsymbol{\beta}} \cdot \mathbf{x}^{\boldsymbol{\gamma}}$.

(iii) $\succ$ is well ordered, i.e., every nonempty set of monomials in S has a least element with respect to $\succ$.

There are many term orders on S. Of most importance to us in our study is the lexicographic term order.

Definition 2.1.3. A **lexicographic term order**, denoted **lex**, on monomials in S is defined as follows: For monomials $\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\beta}}$ in S, we say $\mathbf{x}^{\boldsymbol{\alpha}} \succ_{\text{lex}} \mathbf{x}^{\boldsymbol{\beta}}$ if the leftmost nonzero entry in the vector difference $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{Z}^n$ is positive.

Definition 2.1.4. Let $f = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n_{\ge 0}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ be a nonzero polynomial in S and $\succ$ be a term order on S. The **initial term** of f with respect to $\succ$, denoted $\text{in}_{\succ}(f)$, is the term $c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$, where $c_{\boldsymbol{\alpha}} \neq 0$ and $\mathbf{x}^{\boldsymbol{\alpha}}$ is the maximum monomial, with respect to $\succ$, among the monomials that make up f.

The following example shows how terms of a polynomial can be ordered with respect to a lexicographic term order.

Example 2.1.5. Let $R = \mathbb{K}[x, y, z]$, $f = xy^2z + 6x^2z^2 + z^2 - x^2$ be a polynomial in R. With respect to the lexicographic term order on monomials in R, where $x \succ_{\text{lex}} y \succ_{\text{lex}} z$, the terms of f are reordered in decreasing order as

$$f = 6x^2z^2 - x^2 + xy^2z + z^2,$$

so that $\text{in}_{\succ_{\text{lex}}}(f) = 6x^2z^2$. $\square$

Definition 2.1.6. Let I be a nonzero ideal in S and $\succ$ be a term order on S. The ideal generated by the set of initial terms of all polynomials in I with respect to $\succ$ is called the **initial ideal** of I, and denoted by $\text{in}_{\succ}(I)$. That is,

$$\text{in}_{\succ}(I) = \langle \text{in}_{\succ}(f) \mid f \in I \rangle.$$

An ideal $I \subseteq S$ is called a *monomial ideal* if it is generated by monomials in S. So, with respect to a given term order on S, the initial ideal of an ideal in S is a monomial ideal.

Theorem 2.1.7 (Hilbert's Basis Theorem)**.** *Let I be an arbitrary ideal in S. Then $I = \langle g_1, \dots, g_s \rangle$ for some $g_1, \dots, g_s \in I$.*

Given an ideal in S, Hilbert's basis theorem assures that it has a finite generating set, called a basis. A special basis for every ideal in S is defined below.

Definition 2.1.8. Let I be an ideal in S and $\succ$ be a fixed term order on S. A finite subset $\mathcal{G} = \{g_1, \dots, g_s\} \subseteq I$ is called a **Gröbner basis** for I if

$$\text{in}_{\succ}(I) = \langle \text{in}_{\succ}(g_1), \dots, \text{in}_{\succ}(g_s) \rangle.$$

Gröbner bases are particularly nice bases and their importance cannot be overemphasized; they help in solving many problems that have to do with polynomial ideals. For example, some problems that have to do with ideals in S can be reduced to problems about monomial ideals in S using Gröbner bases. The latter proves to be easier to work with compared to the former (see, for instance, Example 2.1.21).

There are existing algorithms for constructing Gröbner basis. The most common of these algorithms is Buchberger's algorithm (see Theorem 2.1.13). Division algorithm in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ plays a major role in Buchberger's algorithm.

Theorem 2.1.9. [CLO92, Theorem 2.3.2] *Fix a monomial order $\succ$ on S and let $\mathcal{F} = \{f_1, \dots, f_t\}$ be an ordered set of t polynomials in S. Then every $f \in S$ can be written as*

$$f = a_1 f_1 + \cdots + a_t f_t + r,$$

where $a_i, r \in S$ and either $r = 0$ or r is a linear combination, with coefficient in $\mathbb{K}$, of monomials, none of which is divisible by any $in_\succ(f_1), \ldots, in_\succ(f_t)$. Furthermore, if $a_i f_i \neq 0$, then we have

$$in_\succ(f) \succ in_\succ(a_i f_i) \quad or \quad in_\succ(f) = in_\succ(a_i f_i).$$

Definition 2.1.10. The polynomial r in Theorem 2.1.9 is called a **remainder** of f on division by $\mathcal{F} = \{f_1, \ldots, f_t\}$, denoted $\overline{f}^{\mathcal{F}}$.

Given $\mathcal{F} = \{f_1, \ldots, f_t\} \subseteq S$, $f \in S$ and an order $\succ$ on S, below is an algorithm, termed **division algorithm in** S, for construction of the corresponding polynomials $a_1, \ldots, a_t$ and r in Theorem 2.1.9.

Input: $f \in S$, $\mathcal{F} = \{f_1, \ldots, f_t\} \subseteq S$ and an order $\succ$ on S

Output: $a_1, \ldots, a_t, r \in S$ such that $f = \sum_{i=1}^t a_i f_i + r$ and either $r = 0$ or r is a $\mathbb{K}$-linear combination of monomials, none of which is divisible by any of $\text{in}_\succ(f_1), \ldots, \text{in}_\succ(f_t)$

Initialization: $a_1 := 0, \ldots, a_t := 0, r := 0, p := f$

WHILE $p \neq 0$ DO

  $i := 1$

  divisionoccured := False

  WHILE $i \leq t$ AND *divisionoccured* = False DO

    IF $\text{in}_\succ(f_i)$ divides $\text{in}_\succ(p)$ THEN

      $a_i := a_i + \frac{\text{in}_\succ(p)}{\text{in}_\succ(f_i)}$

      $p := p - \frac{\text{in}_\succ(p)}{\text{in}_\succ(f_i)} f_i$

      divisionoccured := True

    ELSE

      $i := i + 1$

  IF *divisionoccured* = False THEN

    $r := r + \text{in}_\succ(p)$

    $p := p - \text{in}_\succ(p)$

Example 2.1.11. Given a lexicographic order $\succ$ on $R := \mathbb{K}[x, y]$, with $x \succ y$, we wish to divide the polynomial $f = x^2y + xy^2 + y^2 \in R$ by the ordered set $\mathcal{F} = \{f_1, f_2\} \subseteq R$, where $f_1 = y^2 - 1$ and $f_2 = xy - 1$. Using the division algorithm in R, we have the following: set $a_1 := 0$, $a_2 := 0$, $r := 0$ and $p := f$.

Since $p \neq 0$, we set $i := 1$ and *divisionoccured* := False. Since $i \leq s$ (where $i = 1$ and $s = 2$), *divisionoccured* = False and $\text{in}_\succ(f_1) = y^2 \nmid x^2y = \text{in}_\succ(p)$, we set $i := i + 1 = 2$. Since $i \leq s$ (where $i = 2$ and $s = 2$), *divisionoccured* = False and $\text{in}_\succ(f_2) = xy \mid x^2y = \text{in}_\succ(p)$, we set $a_2 := a_2 + \frac{\text{in}_\succ(p)}{\text{in}_\succ(f_2)} = 0 + x = x$, $p := p - \frac{\text{in}_\succ(p)}{\text{in}_\succ(f_2)} f_2 = (x^2y + xy^2 + y^2) - x(xy - 1) = xy^2 + x + y^2$ and *divisionoccured* := True.

Since $p \neq 0$, we set $i := 1$ and *divisionoccured* := False. Since $i \leq s$ (where $i = 1$ and $s = 2$), *divisionoccured* = False and $\mathrm{in}_{\succ}(f_1) = y^2 \mid xy^2 = \mathrm{in}_{\succ}(p)$, we set $a_1 := a_1 + \frac{\mathrm{in}_{\succ}(p)}{\mathrm{in}_{\succ}(f_1)} = 0 + x = x$, $p := p - \frac{\mathrm{in}_{\succ}(p)}{\mathrm{in}_{\succ}(f_1)} f_1 = (xy^2 + x + y^2) - x(y^2 - 1) = 2x + y^2$ and *divisionoccured* := True.

Since $p \neq 0$, we set $i := 1$ and *divisionoccured* := False. Since $i \leq s$ (where $i = 1$ and $s = 2$), *divisionoccured* = False and $\mathrm{in}_{\succ}(f_1) = y^2 \nmid 2x = \mathrm{in}_{\succ}(p)$, we set $i := i + 1 = 2$. Since $i \leq s$ (where $i = 2$ and $s = 2$), *divisionoccured* = False and $\mathrm{in}_{\succ}(f_2) = xy \nmid 2x = \mathrm{in}_{\succ}(p)$, we set $i := i + 1 = 3$. Since $i > s$ (where $i = 3$ and $s = 2$) and *divisionoccured* = False, we set $r := r + \mathrm{in}_{\succ}(p) = 0 + 2x = 2x$ and $p := p - \mathrm{in}_{\succ}(p) = (2x + y^2) - 2x = y^2$.

Since $p \neq 0$, we set $i := 1$ and *divisionoccured* := False. Since $i \leq s$ (where $i = 1$ and $s = 2$), *divisionoccured* = False and $\mathrm{in}_{\succ}(f_1) = y^2 \mid y^2 = \mathrm{in}_{\succ}(p)$, we set $a_1 := a_1 + \frac{\mathrm{in}_{\succ}(p)}{\mathrm{in}_{\succ}(f_1)} = x+1$, $p := p - \frac{\mathrm{in}_{\succ}(p)}{\mathrm{in}_{\succ}(f_1)} f_1 = y^2 - 1(y^2 - 1) = 1$ and *divisionoccured* := True.

Since $p \neq 0$, we set $i := 1$ and *divisionoccured* := False. Since $i \leq s$ (where $i = 1$ and $s = 2$), *divisionoccured* = False and $\mathrm{in}_{\succ}(f_1) = y^2 \nmid 1 = \mathrm{in}_{\succ}(p)$, we set $i := i + 1 = 2$. Since $i \leq s$ (where $i = 2$ and $s = 2$), *divisionoccured* = False and $\mathrm{in}_{\succ}(f_2) = xy \nmid 1 = \mathrm{in}_{\succ}(p)$, we set $i := i+1 = 3$. Since $i > s$ (where $i = 3$ and $s = 2$) and *divisionoccured* = False, we set $r := r + \mathrm{in}_{\succ}(p) = 2x + 1$ and $p := p - \mathrm{in}_{\succ}(p) = 1 - 1 = 0$.

Therefore, $f = a_1 f_1 + a_2 f_2 + r$, where $a_1 = x + 1$, $a_2 = x$ and $r = 2x + 1$, i.e., $\overline{f}^{\mathcal{F}} = r = 2x + 1$.

The above procedures can be summarized in the following (multivariate) long division:

$$
\begin{array}{r|l}
a_1 : & x + 1 \\
a_2 : & x \\
\hline
y^2 - 1 & \\
xy - 1 & x^2y + xy^2 - y^2 \\
 & -\ (x^2y - x) \\
\hline
 & \qquad xy^2 + x + y^2 \\
 & \qquad -\ (xy^2 - x) \\
\hline
 & \qquad\qquad 2x + y^2 \\
 & \qquad\qquad -\ (y^2 - 1) \\
\hline
 & \qquad\qquad 2x + 1
\end{array}
$$

□

Fix a term order $\succ$ on S, and let $f, g \in S$ be nonzero with $\mathrm{in}_{\succ}(f) = c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathrm{in}_{\succ}(g) = c_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$. Let $\mathbf{x}^{\gamma}$ be the least common multiple of $\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\beta}}$. The *S-polynomial* of f and g, denoted $\mathrm{S}(f, g)$, is the polynomial

$$S(f, g) = \frac{\mathbf{x}^{\boldsymbol{\gamma}}}{\mathrm{in}_{\succ}(f)} \cdot f - \frac{\mathbf{x}^{\boldsymbol{\gamma}}}{\mathrm{in}_{\succ}(g)} \cdot g.$$

The following result gives a criterion for when a basis of an ideal is a Gröbner basis.

Theorem 2.1.12. [CLO92, Theorem 2.6.6, Buchberger's Criterion] *Let $I \subseteq S$ be an ideal. Then a basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ for I is a Gröbner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by $\mathcal{G}$ (listed in some order) is zero.*

The following is an algorithm for constructing a Gröbner basis for a nonzero ideal in S.

Theorem 2.1.13. [CLO92, Theorem 2.7.2, Buchberger's Algorithm] *Let $I = \langle f_1, \ldots, f_t \rangle$ be a nonzero ideal in S. Then a Gröbner basis for I can be constructed in a finite number of steps by the following algorithm, where $\overline{S(f,g)}^{\mathcal{G}'}$ is the remainder on division of $S(f,g)$ by $\mathcal{G}'$:*

Input: *$\mathcal{F} = \{f_1, \ldots, f_t\} \subseteq S$ and an order $\succ$ on S*

Output: *With respect to $\succ$, a Gröbner basis $\mathcal{G}$ for $I = \langle \mathcal{F} \rangle$, with $\mathcal{F} \subseteq \mathcal{G}$*

Initialization: $\mathcal{G} := \mathcal{F}$

REPEAT

$\mathcal{G}' := \mathcal{G}$

FOR each pair $\{f, g\}$, $f \neq g$ in $\mathcal{G}'$ DO

$\mathcal{S} := \overline{S(f,g)}^{G'}$

IF $\mathcal{S} \neq 0$ THEN $\mathcal{G} := \mathcal{G} \cup \{\mathcal{S}\}$

UNTIL $\mathcal{G} = \mathcal{G}'$.

Example 2.1.14. Consider the ring $R := \mathbb{Q}[x, y]$ with a lexicographic term order $\succ_{\text{lex}}$, where $x \succ_{\text{lex}} y$, and let $I = \langle f, g \rangle \subseteq R$, where $f = x^2 - y$ and $g = x^3 - x$. The S-polynomial $S(f, g)$ equals

$$S(f,g) = \frac{x^3}{x^2} \cdot (x^2 - y) - \frac{x^3}{x^3} \cdot (x^3 - x) = -xy + x.$$

Observe that $\text{in}_{\succ_{\text{lex}}}(S(f,g)) = -xy$ belongs to $\text{in}_{\succ_{\text{lex}}}(I)$, but it does not belong to the monomial ideal $\langle \text{in}_{\succ_{\text{lex}}}(f), \text{in}_{\succ_{\text{lex}}}(g) \rangle = \langle x^2, x^3 \rangle = \langle x^2 \rangle$. It therefore follows that the set $\mathcal{G} = \{f, g\}$ is not a Gröbner basis for I with respect to the given term order $\succ_{\text{lex}}$.

We will now use Buchberger's algorithm (Theorem 2.1.13) to compute a Gröbner basis for I with respect to $\succ_{\text{lex}}$. To this end, we set $\mathcal{G}' := \{f, g\}$. We have $\overline{S(f,g)}^{\mathcal{G}'} = -xy + x$. Since this is nonzero, we set $\mathcal{G} := \{f, g, -xy + x\}$. Since $\mathcal{G} \neq \mathcal{G}'$, we set $\mathcal{G}' := \{f, g, -xy + x\}$. We have the following:

$$S(f, -xy + x) = \frac{x^2 y}{x^2} \cdot (x^2 - y) - \frac{x^2 y}{-xy} \cdot (-xy + x) = x^2 - y^2 = f + (-y^2 + y),$$

so that

$$\overline{S(f, -xy + x)}^{\mathcal{G}'} = -y^2 + y.$$

$$S(g, -xy+x) = \frac{x^3y}{x^3} \cdot (x^3 - x) - \frac{x^3y}{-xy} \cdot (-xy+x) = x^3 - xy = g + (-xy+x),$$

so that

$$\overline{S(g, -xy+x)}^{\mathcal{G}'} = 0.$$

Since $\overline{S(f, -xy+x)}^{\mathcal{G}'}$ is nonzero, we set $\mathcal{G} := \{f, g, -xy+x, -y^2+y\}$. Since $\mathcal{G} \neq \mathcal{G}'$, we set $\mathcal{G}' := \{f, g, -xy+x, -y^2+y\}$. We further have the following:

$$S(f, -y^2+y) = \frac{x^2y^2}{x^2} \cdot (x^2 - y) - \frac{x^2y^2}{-y^2} \cdot (-y^2+y) = x^2y - y^3 = yf + y(-y^2+y),$$

so that

$$\overline{S(f, -y^2+y)}^{\mathcal{G}'} = 0.$$

$$S(g, -y^2+y) = \frac{x^3y^2}{x^3} \cdot (x^3 - x) - \frac{x^3y^2}{-y^2} \cdot (-y^2+y) = x^3y - xy^2 = xyf,$$

so that

$$\overline{S(g, -y^2+y)}^{\mathcal{G}'} = 0.$$

$$S(-xy+x, -y^2+y) = \frac{xy^3}{-xy} \cdot (-xy+x) - \frac{xy^3}{-y^2} \cdot (-y^2+y) = 0,$$

so that

$$\overline{S(-xy+x, -y^2+y)}^{\mathcal{G}'} = 0.$$

Since $\overline{S(h, h')}^{\mathcal{G}'}$ is zero for every pair $\{h, h'\}$, $h \neq h'$ in $\mathcal{G}'$, the set $\mathcal{G}$ remains unchanged, i.e., $\mathcal{G} = \mathcal{G}'$. Since $\mathcal{G} = \mathcal{G}'$, the algorithm terminates and so a Gröbner basis for I with respect to the given term order $\succ_{\text{lex}}$ is $\mathcal{G} = \{f, g, -xy+x, -y^2+y\}$. □

Given a term order $\succ$ on S, there are many Gröbner bases, with respect to $\succ$, for a particular ideal. However, for a fixed term order $\succ$, there is always a type of Gröbner basis that is unique with respect to $\succ$. This is called a reduced Gröbner basis.

Definition 2.1.15. A **reduced Gröbner basis** for an ideal $I \subseteq S$ is a Gröbner basis $\mathcal{G}$ for I such that:

(i) For each $g \in \mathcal{G}$, the coefficient of $\text{in}_{\succ}(g)$ is 1.

(ii) For any $g_i \in \mathcal{G}$, no monomial of g_i lies in $\langle \text{in}_{\succ}(g_1), \ldots, \text{in}_{\succ}(g_{i-1}), \text{in}_{\succ}(g_{i+1}), \ldots, \text{in}_{\succ}(g_s) \rangle$.

Example 2.1.16. The reduced Gröbner basis for the ideal in Example 2.1.14 is $\{x^2 - y, xy - x, y^2 - y\}$. □

Definition 2.1.17. A **graded ring** is a ring R together with a direct sum decomposition

$$R = \bigoplus_{i=0}^{\infty} R_i$$

of additive groups satisfying $R_iR_j \subseteq R_{i+j}$, for all nonnegative integers i and j. The elements of R_i are called **homogeneous elements** of degree i in the grading.

In the following example, it will be shown that the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ over the field $\mathbb{K}$ is a graded ring. In particular, we will see what it means for S to be standard graded.

Example 2.1.18. For each variable $x_i \in S$, if we set $\deg(x_i) = 1$, then $\deg(\boldsymbol{m}) = \alpha_1 + \cdots + \alpha_n$, where $\boldsymbol{m} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For each nonnegative integer i, denote by S_i the $\mathbb{K}$-vector space spanned by all monomials of degree i. Observe that $S_0 = \mathbb{K}$. A polynomial $h \in S$ is called *homogeneous* if h belongs to S_i, for some i, and in this case we say that h has *degree* i, i.e., $\deg(h) = i$. Furthermore, since for every two homogeneous elements h and h' in S, we have $\deg(hh') = \deg(h) + \deg(h')$, it follows that $S_iS_j \subseteq S_{i+j}$, for all nonnegative integers i and j. Every polynomial $f \in S$ can be written uniquely as a finite sum $f = \sum_i f_i$ of nonzero elements $f_i \in S_i$, and in this case, f_i is called *homogeneous component of f of degree i*. Consequently, we have a direct sum decomposition $\bigoplus_{i \geq 0} S_i$ of S as a $\mathbb{K}$-vector space such that $S_iS_j \subseteq S_{i+j}$, for all nonnegative integers i and j. In this case, we say S is *standard graded.* □

Given a graded ring R, there is a function that encodes important information about R. For example, this function measures the growth of the dimension of homogeneous components of R. Since $R_0 = \mathbb{K}$, it follows that R_i is a $\mathbb{K}$-vector space because $R_0R_i \subseteq R_i$. A basis of the $\mathbb{K}$-vector space R_i is called a *basis in degree i*. In the case where each R_i, $i \geq 0$, is finitely generated as $\mathbb{K}$-vector space, i.e., $\dim_{\mathbb{K}}(R_i) < \infty$, we have the following definition.

Definition 2.1.19. Let R be a graded ring. The generating function $i \mapsto \dim_{\mathbb{K}}(R_i)$ is called the **Hilbert function** of R and is studied via the **Hilbert series**

$$\mathrm{Hilb}_R(t) = \sum_{i=0}^{\infty} \dim_{\mathbb{K}}(R_i)t^i.$$

Example 2.1.20. Let $R = \mathbb{K}[x, y]$ and $J = \langle x^2, xy, y^2 \rangle$. Then R/J is graded with basis $\{1\}$ in degree 0 and $\{x, y\}$ in degree 1. Its Hilbert series is therefore

$$\mathrm{Hilb}_{A/J}(t) = 1 + 2t.$$

□

Let I be a homogeneous ideal in S, i.e., I is generated by homogeneous polynomials in S. In [CLO92, Proposition 9.3.4], it is shown that the Hilbert function $\mathrm{Hilb}_{S/\mathrm{in}_\succ(I)}$ and $\mathrm{Hilb}_{S/I}$ are equal. Thus, to obtain Hilbert function of S/I, we may first compute a Gröbner basis $\mathcal{G}$ for I and then compute the Hilbert function $\mathrm{Hilb}_{S/\langle \mathrm{in}_\succ(\mathcal{G}) \rangle}$, since $\mathrm{in}_\succ(I) = \langle \mathrm{in}_\succ(\mathcal{G}) \rangle$ by definition of a Gröbner basis.

Example 2.1.21. Continuing with Example 2.1.14, the Hilbert series for R/I, where $I = \langle x^2 - y, x^3 - x \rangle$ is

$$\mathrm{Hilb}_{A/I}(t) = \mathrm{Hilb}_{A/\mathrm{in}_{\succ_{\mathrm{lex}}}(I)}(t),$$

where $\succ_{\mathrm{lex}}$ is the lexicographic order on R with $x \succ_{\mathrm{lex}} y$. Using the Gröbner basis for I in Example 2.1.16, we have $\mathrm{in}_{\succ_{\mathrm{lex}}}(I) = \langle x^2, xy, y^2 \rangle$, the same ideal in Example 2.1.20. Therefore,

$$\mathrm{Hilb}_{A/I}(t) = \mathrm{Hilb}_{A/\mathrm{in}_{\succ_{\mathrm{lex}}}(I)}(t) = 1 + 2t,$$

as in Example 2.1.20. $\square$

There are other applications of Gröbner bases. For example, they can be used to compute intersection of any two ideals in S which is a useful concept in computing saturation of ideals in S. The following result can be used in the computation of intersection of two ideals in S.

Theorem 2.1.22. [CLO92, Theorem 4.3.11] *Let I and J be ideals in S. Then*

$$I \cap J = (tI + (1-t)J) \cap S.$$

Following Theorem 2.1.22, in order to compute intersection of ideals $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$ in S, it suffices to consider the ideal

$$\tilde{I} := \langle tf_1, \ldots, tf_r, (1-t)g_1, \ldots, (1-t)g_s \rangle \subseteq \mathbb{K}[t, x_1, \ldots, x_n].$$

Computing a Gröbner basis $\mathcal{G}$ with respect to the lexicographic order, where the new variable t is set above all other variables $x_1, \ldots, x_n$ in S, the elements of $\mathcal{G}$ that do not involve the variable t are generators for $I \cap J$. A simple example of this procedure is given below.

Example 2.1.23. Let $J_1 = \langle x \rangle$ and $J_2 = \langle y, z \rangle$ be ideals in $\mathbb{K}[x, y, z]$. We want to compute the intersection of the ideals J_1 and J_2. Set

$$\tilde{I} := \langle tx, (1-t)y, (1-t)z \rangle = \langle tx, y - ty, z - tz \rangle$$

and let $\succ_{\mathrm{lex}}$ be the lexicographic term order on R, where $t \succ_{\mathrm{lex}} x \succ_{\mathrm{lex}} y \succ_{\mathrm{lex}} z$. We have the following S-polynomials: $S(tx, -ty + y) = xy$, $S(tx, -tz + z) = xz$ and $S(-ty + y, -tz + z) = 0$, for which the first two are nonzero and not divisible by any initial terms of generators of $\tilde{I}$. In fact, the set $\{tx, y - ty, z - tz, xy, xz\}$ is a Gröbner basis for $\tilde{I}$ with respect to $\succ_{\mathrm{lex}}$. Hence, $J_1 \cap J_2 = \langle xy, xz \rangle$. $\square$

Definition 2.1.24. Let $I \subseteq S$ be an homogeneous ideal. The **saturation** $\bar{I}$ of I is defined to be the set

$$\bar{I} = \{f \in S \mid \text{for each } i = 1, \ldots, n, \text{ there exists an } m > 0 \text{ such that } x_i^m f \in I\}.$$

I is said to be **saturated** if $I = \bar{I}$.

Definition 2.1.25. Let $I, J \subseteq S$ be ideals. The **ideal quotient** (or **colon ideal**) of I by J, denoted $\boldsymbol{I : J}$, is the set

$$I : J = \{f \in S \mid fg \in I \text{ for all } g \in J\}.$$

In order to show that an ideal $I \subseteq S$ is saturated, it suffices to show that

$$I = I : \langle x_1, \dots, x_n \rangle^\infty,$$

where $I : \langle x_1, \dots, x_n \rangle^\infty$ is defined as the ideal $\bigcup_{l \geq 1} \Big(I : \langle x_1, \dots, x_n \rangle^l \Big)$. Using Gröbner basis, one can compute each colon ideal $I : \langle x_1, \dots, x_n \rangle^l$, $l \geq 1$. Since the inclusion $I : \langle x_1, \dots, x_n \rangle \subseteq I : \langle x_1, \dots, x_n \rangle^2 \subseteq I : \langle x_1, \dots, x_n \rangle^3 \subseteq \cdots$ is an ascending chain of ideals in S, as soon as we obtain an integer $l \geq 1$ such that $I : \langle x_1, \dots, x_n \rangle^l = I : \langle x_1, \dots, x_n \rangle^{l+1}$, we have the equality $\bar{I} = I : \langle x_1, \dots, x_n \rangle^\infty = I : \langle x_1, \dots, x_n \rangle^l$, for such l.

For a fixed $l \geq 1$, if $h_1, \dots, h_k$ are generators for the ideal $\langle x_1, \dots, x_n \rangle^l$, then

$$I : \langle x_1, \dots, x_n \rangle^l = I : \langle h_1, \dots, h_k \rangle = \bigcap_{i=1}^{k} \Big(I : \langle h_i \rangle \Big).$$

Each colon ideal $I : \langle h_i \rangle$, $1 \leq i \leq k$, can be computed by first computing a Gröbner basis $\mathcal{G}_i$ for the ideal $tI + (1-t)\langle h_i \rangle$ with respect to a lexicographic term order, where the new variable t is set above all other variables $x_1, \dots, x_n$ in S. Then by [CLO92, Theorem 4.4.11], the polynomials obtained by dividing each polynomial in $\mathcal{G}_i$ that do not involve t by h_i is a basis for $I : \langle h_i \rangle$, i.e., for $1 \leq i \leq k$,

$$I : \langle h_i \rangle = \left\langle \frac{g_{i_j}}{h_i} \mid g_{i_j} \in \mathcal{G}_i \cap S \right\rangle.$$

Observe from Theorem 2.1.22 that $\mathcal{G}_i \cap S$ is a basis for $I \cap \langle h_i \rangle$, and so each polynomial g_{i_j} in this basis is divisible by h_i.

Example 2.1.26. Let $I = \langle x^2, xy, xz \rangle$ and $J = \langle x^2, y^2 \rangle$ be ideals in $\mathbb{K}[x, y, z]$. We have

$$I : \langle x, y, z \rangle = I : \langle x, y, z \rangle^2 = \langle x \rangle \quad \text{and} \quad J : \langle x, y, z \rangle = J : \langle x, y, z \rangle^2 = \langle x^2, y^2 \rangle = J,$$

and so only the ideal J is saturated. □

Height of an ideal in a polynomial ring is important to our study. The following definitions and examples give a brief background of this concept.

Definition 2.1.27. A chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r$ in a ring R has **length** r.

Definition 2.1.28. The **Krull dimension** (or simply the **dimension**) of a ring R, written $\mathbf{dim}\,\boldsymbol{R}$, is the supremum of the lengths of chains of distinct prime ideals in R.

Example 2.1.29. Below are Krull dimensions of some rings.

(a) $\dim \mathbb{K} = 0$, for any field $\mathbb{K}$. The only prime ideal in a field is the zero ideal.

(b) Let R be a principal ideal domain (an integral domain in which every ideal can be generated by a single element) that is not a field. If $\langle p \rangle$ and $\langle q \rangle$ are prime ideals in R with the strict inclusions $\langle 0 \rangle \subsetneq \langle p \rangle \subsetneq \langle q \rangle$, then p belongs to $\langle q\rangle$, i.e., p is divisible by q. Since p is a prime element in R, it follows that q is a unit, which is not possible since it is a prime element as well. Hence, there is no such chain of prime ideals $\langle 0\rangle \subsetneq \langle p\rangle \subsetneq \langle q\rangle$ in R. Therefore, in R, there cannot be longer chains of prime ideals than the chains $\langle 0\rangle \subsetneq \langle p\rangle$ of prime ideals of length 1, i.e., $\dim R = 1$.

(c) The Krull dimension of the polynomial ring S is n. One increasing sequence of prime ideals in S is

$$\langle 0\rangle \subsetneq \langle x_1\rangle \subsetneq \langle x_1, x_2\rangle \subsetneq \cdots \subsetneq \langle x_1, \ldots, x_n\rangle.$$

□

Definition 2.1.30. If $I \subsetneq R$ is a prime ideal, then the **codimension** I, written **codim $\boldsymbol{I}$** (also called **height** of I, written **ht $\boldsymbol{I}$**), is the Krull dimension of the local ring $R_I := \{\frac{a}{b} \mid a \in R, b \notin I\}$. In other words, $\operatorname{ht} I$ is the supremum of lengths of chains of primes descending from I. If I is not prime, then $\operatorname{ht} I$ is the minimum of the heights of the primes ideals containing I.

There is a formula for computing codimensions of ideals in the polynomial ring S.

Definition 2.1.31. Let $f_1, \ldots, f_s$ be polynomials in $S = \mathbb{K}[x_1, \ldots, x_n]$. The **affine variety** defined by $f_1, \ldots, f_s$ is the set

$$V(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n \mid f_i(a_1, \ldots, a_n) = 0 \text{ for all } i,\ 1 \le i \le s\}.$$

Let I be an ideal in the polynomial ring S in n variables. We have that

$$\operatorname{codim} I = \dim S - \dim S/I = n - \dim S/I, \tag{2.1}$$

where $\dim S/I$ equals the dimension of the affine variety

$$V(I) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

Example 2.1.32. Let $R = \mathbb{K}[x, y]$ and $I = \langle x^2, xy\rangle \subseteq R$. Then, $\dim R/I = 1$, since $V(I) = \{(0, y) \mid y \in \mathbb{K}\}$. Consider the ideal J/I of R/I, where $J = \langle x, y\rangle \subseteq R$. The codimension of J equals 2; one increasing sequence of prime ideals descending from J is $\langle 0\rangle \subsetneq \langle x\rangle \subsetneq J$. If we quotient these prime ideals by I, we obtain an increasing sequence $I \subsetneq \langle x\rangle/I \subsetneq J/I$ of ideals descending from J/I. Since, I is not prime, we have the increasing sequence $\langle x\rangle/I \subsetneq J/I$ of prime ideals descending from J/I. In fact, $\operatorname{codim} J/I = 1$. □

In the subsequent chapters of this dissertation, we will be mostly concerned with some studies relating to some classes of generalized determinantal ideals. Fulton in [Ful92] showed that the codimensions of these determinantal ideals is equal to the *length* of some permutations.

For the rest of this subsection, all rings will be assumed to be Noetherian, i.e., rings for which every ideal in them is finitely generated. A nonzero element r in a ring R is a *zerodivisor* in R if there exists another nonzero element $r' \in R$ such that $rr' = 0$. A nonzero element r in a ring R is a *nonzerodivisor* in R if $rr' = 0$ implies $r' = 0$ for $r' \in R$.

Example 2.1.33. Let $R = \mathbb{K}[x, y]$ and $I = \langle x^2, xy\rangle \subseteq R$. Suppose $\overline{ax + by}$, for some $a, b \in R$, is a nonzero element in R/I. The element $\bar{x}$ is nonzero in R/I and the product $\overline{ax + by \cdot x} = ax^2 + bxy$ belongs to I. Therefore, $\overline{ax + by}$ is a zerodivisor in R/I. □

Definition 2.1.34. A sequence of elements $r_1, \ldots, r_d$ in a ring R is called an ***R*-sequence** (or **regular sequence** on R) if the ideal $\langle r_1, \ldots, r_d\rangle$ is proper and for each i, the image of r_{i+1} is a nonzerodivisor in $R/\langle r_1, \ldots, r_i\rangle$.

Example 2.1.35. Below are some examples and non-examples of regular sequences. The first two examples show that a regular sequence may depend on the order of elements.

(a) Let $R = \mathbb{K}[x, y, z]$. The sequence $x, y(x + 1), z(x + 1)$ is a regular sequence in R. First, observe that the ideal they generate is $\langle x, y, z\rangle \neq R$. Observe also that x is a nonzerodivisor in R, and $R/\langle x\rangle = \mathbb{K}[y, z]$. Since y is a nonzerodivisor in $\mathbb{K}[y, z]$, it follows that $y(x+1)$ is a nonzerodivisor in $R/\langle x\rangle$. Lastly, we have $R/\langle x, y(x+1)\rangle = \mathbb{K}[z]$, and since z is a nonzerodivisor in $\mathbb{K}[z]$, it follows that $z(x+1)$ is a nonzerodivisor in $R/\langle x, y(x + 1)\rangle$.

(b) Let $R = \mathbb{K}[x, y, z]$. Though $y(x + 1)$ is a nonzerodivisor in R, the element $z(x + 1)$ is a zerodivisor in $R/\langle y(x+1)\rangle$. To see this, observe that the product $z(x+1)\cdot y = z\cdot y(x+1)$ belongs to the ideal $\langle y(x+1)\rangle$, and hence, its image is zero in $R/\langle y(x+1)\rangle$. Therefore, the sequence $y(x+1), z(x+1), x$ is not a regular sequence in R.

(c) Let $R = \mathbb{K}[x, y]$ and $I = \langle x^2, xy\rangle \subseteq R$. The sequence of elements $\bar{x}, \bar{y}$ in R/I is not a regular sequence, since $\bar{x}$ is a zerodivisor in R/I. For instance, the product $x \cdot y$ belongs to I, and hence, its image is zero in R/I. Similarly, the sequence of elements $\bar{y}, \bar{x}$ in R/I is not a regular sequence, since $\bar{y}$ is a zerodivisor in R/I. □

Definition 2.1.36 ([Eis13], Chapter 18)**.** The **depth** of an ideal $I \subseteq R$, written **depth *I***, is the length of any maximal R-sequence in I.

The following definition is yet another concept that is important to our study. It gives a name for a special ideal in R whose codimension and depth are the same.

Definition 2.1.37 ([Mat89], Chapter 17)**.** An ideal $I \subset R$ is called **Cohen-Macaulay ideal** if $I \neq 0$ and $\operatorname{depth} I = \operatorname{codim} I$, or if $I = 0$.

Example 2.1.38. Let $R = \mathbb{K}[x, y]$ and $I = \langle x^2, xy\rangle$ be an ideal in R, as in Example 2.1.32. We wish to show that I is a Cohen-Macaulay ideal. Recall from Example 2.1.32 that $\dim R/I = 1$, and so using equation (2.1), we have $\operatorname{codim} I = \dim R - \dim R/I = 2-1 = 1$. Further, we will find R-sequences in I. For arbitrary elements $a, a', b, b' \in R$, with $(a,b) \neq (a',b') \neq (0,0)$, consider the sequence of elements f, g in I, where $f = ax^2 + bxy$ and $f' = a'x^2 + b'xy$. Observe that by the choice of $a, a', b, b' \in R$, the two polynomials f and f' are nonzero and not equal. Though $\langle f, f' \rangle \neq R$ and f is a nonzerodivisor in R, the element f' is a zerodivisor in $R/\langle f \rangle$. This is true because $f' \cdot (ax+by) = (a'x^2 + b'xy) \cdot (ax+by) = (ax^2+bxy) \cdot (a'x+b'y) = f \cdot (a'x+b'y)$, i.e., $f' \cdot (ax+by)$ belongs to the ideal $\langle f \rangle$, and hence, its image is zero in $R/\langle f \rangle$. Consequently, the sequence f, f' is not an R-regular sequence in I. Similarly, the sequence f', f is not an R-regular sequence in I. Therefore, length of any maximal R-sequence in I is 1; one maximal R-sequence in I is f. Since $\operatorname{codim} I = \operatorname{depth} I$ for this ideal, we conclude that I is a Cohen-Macaulay ideal. □

Definition 2.1.39 ([Eis13], Chapter 18)**.** A ring R such that $\operatorname{depth} \mathfrak{m} = \operatorname{codim} \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R is called a **Cohen-Macaulay** ring.

Example 2.1.40. Below are some Cohen-Macaulay rings.

(a) Every field is Cohen-Macaulay, since the only maximal ideal in a field is the zero ideal $\langle 0 \rangle$, and $\operatorname{codim} \langle 0 \rangle = 0 = \operatorname{depth} \langle 0 \rangle$.

(b) Every polynomial ring over a Cohen-Macaulay ring is Cohen-Macaulay (see [Eis13, Proposition 18.9]). In particular, the polynomial ring S is Cohen-Macaulay.

(c) A non-example is the ring R/I, where $R = \mathbb{K}[x,y]$ and $I = \langle x^2, xy \rangle \subseteq R$. Consider the ideal $J = \langle x, y \rangle \subseteq R$ and observe that $I \subseteq J$. Since J is a maximal in R, it follows that the ideal J/I is maximal in R/I. From Example 2.1.33, none of the nonzero elements of J/I can form an R/I-sequence. So, $\operatorname{depth} J/I = 0$. From Example 2.1.32, we obtain that $\operatorname{codim} J/I = 1$. Since J/I is maximal in R/I and $\operatorname{codim} J/I \neq \operatorname{depth} J/I$, it follows that R/I is not Cohen-Macaulay. □

Definition 2.1.41 ([Mat89], Chapter 2, Section 6)**.** Let R be a ring and M an R-module. A prime ideal $\mathfrak{p}$ of R is called an **associated prime ideal** of M if $\mathfrak{p}$ is the annihilator $\operatorname{ann}(m)$ for some $m \in M$. The set of associated primes of M is written $\operatorname{Ass}(M)$ or $\operatorname{Ass}_R(M)$. For an ideal I of R, the associated primes of the R-module R/I are referred to as the **prime divisors** of I.

Example 2.1.42. Let $R = \mathbb{K}[x,y,z]$ and $I = \langle xy, xz \rangle \subseteq R$ be an ideal. Set $M := R/I$, an R-module. From Example 2.1.23, we have that $\langle x \rangle \cap \langle y, z \rangle = I$. For all elements of the form $(ay+bz)+I \in M$, we have

$$x \cdot \big((ay+bz)+I\big) = (axy+bxz)+I = I,$$

i.e., $\operatorname{ann}\big(\langle y,z \rangle\big) = \langle x \rangle$. Similarly, we have $\operatorname{ann}\big(\langle x \rangle\big) = \langle y,z \rangle$. In fact, $\operatorname{Ass}_R(M) = \{\langle x \rangle, \langle y,z \rangle\}$. In other words, the prime divisors of the ideal I are $\langle x \rangle$ and $\langle y, z \rangle$. □

Definition 2.1.43 ([Mat89], Chapter 6, Section 17)**.** A proper ideal I in a ring R is said to be **unmixed** if the codimensions of its prime divisors are all equal.

Example 2.1.44. Let $R = \mathbb{K}[x, y, z]$ and $I = \langle x, yz \rangle \subseteq R$ be an ideal. We observe here that $I = \langle x, y \rangle \cap \langle x, z \rangle$. The prime divisors of the ideal I are $\langle x, y \rangle$ and $\langle x, z \rangle$, each of codimension 2. Hence, the ideal I is unmixed. $\square$

Example 2.1.45. Let $\mathfrak{p}$ be an arbitrary prime ideal in the polynomial ring S and set $M := S/\mathfrak{p}$. Observe that M is a domain and so it has no zerodivisor. Therefore, $\mathrm{Ass}_S(M) = \{\mathfrak{p}\}$, since $\mathrm{ann}(m) = \mathfrak{p}$ for all nonzero element $m \in M$. Hence, prime ideals in S are unmixed. $\square$

The families of generalized determinantal ideals we discuss in this dissertation are prime, and so unmixed.

2.2 Combinatorial Commutative Algebra

In this section, we give a brief background on the algebraic objects – *Stanley-Reisner ideals* and *Stanley-Reisner rings*, which are very important to our study. The initial ideals of the families of determinantal ideals we consider in this dissertation have a special property – they are *squarefree*. Recall that an ideal $I \subseteq S$ is called a **monomial ideal** if it is generated by monomials $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a vector of nonnegative integers.

The primary reference for this section is [MS04].

Definition 2.2.1. A monomial $\mathbf{x}^{\boldsymbol{\alpha}}$ is **squarefree** if every coordinate of $\boldsymbol{\alpha}$ is 0 or 1. A monomial ideal is **squarefree** if it is generated by squarefree monomials.

Example 2.2.2. Let $R = \mathbb{K}[x_{ij} \,|\, 1 \leq i \leq 2, 1 \leq j \leq 3]$ and $\mathcal{G}$ be the set of minors of size 2 in the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}.$$

That is,

$$\mathcal{G} = \left\{ \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix}, \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \right\}.$$

Let $\succ$ be the lexicographic term order on R, where $x_{23} \succ x_{22} \succ x_{21} \succ x_{13} \succ x_{12} \succ x_{11}$. Then $\mathrm{in}_{\succ}(\mathcal{G}) = \{x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23}\}$. Observe that each monomial in the set $\mathrm{in}_{\succ}(\mathcal{G})$ is squarefree. Therefore, the (monomial) ideal

$$\langle \mathrm{in}_{\succ}(\mathcal{G}) \rangle = \langle x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23} \rangle$$

is squarefree. $\square$

One of the ways of characterizing the information carried by squarefree monomial ideals is through *simplicial complexes.*

Definition 2.2.3. An (abstract) **simplicial complex** $\triangle$ on the vertex set $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ that is closed under taking subsets; that is, if $\sigma \in \triangle$ and $\tau \subseteq \sigma$, then $\tau \in \triangle$. An element $\sigma \in \triangle$ is called a **face**. The **dimension of a face** $\sigma \in \triangle$ is $\dim(\sigma) = |\sigma| - 1$, where $|\sigma|$ is the cardinality of σ. The **maximal faces** of $\triangle$ are the faces that are not subsets of any other faces, and they are called **facets**. A face with dimension i is called an ***i*-face** of $\triangle$. The **dimension of the complex** $\triangle$, denoted $\dim(\triangle)$, is the maximum of the dimensions of its faces, or it is $-\infty$ if $\triangle = \{\}$, i.e., if it has no faces. A simplicial complex with no faces is called a **void complex**. The complex $\triangle = 2^\sigma$ consisting of all subsets of σ, for some $\sigma \subseteq [n]$, is called a **simplex**.

The dimension of the complex $\triangle = \{\emptyset\}$ is -1. This complex is often called *irrelevant complex*, and it is not the same as the void complex.

Example 2.2.4. The collection of all subsets of the sets $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ and $\{5\}$ is a simplicial complex $\triangle$ on the vertex set [5]. For instance, the set $\{2, 3\}$ is a subset of the facet $\{1, 2, 3\} \in \triangle$, and so it is also a face in $\triangle$. Its dimension is 1. The dimension of this complex is 2, i.e., $\dim(\triangle) = 2$. □

A simplicial complex is completely specified by its facets. Observe that in Example 2.2.4, only the facets in the simplicial complex $\triangle$ are specified.

Example 2.2.5. The collection of all subsets of the sets $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$ and $\{3, 4, 5, 6\}$ is a simplicial complex $\triangle$ on the vertex set [6]. □

A simplicial complex determines a squarefree monomial ideal. If we identify each subset $\sigma \subseteq [n]$ with its *squarefree vector* in $\{0, 1\}^n$, which has entry 1 in the ith spot when $i \in \sigma$, and 0 in all other entries, then we can write $\mathbf{x}^\sigma = \prod_{i \in \sigma} x_i$.

Definition 2.2.6. The **Stanley-Reisner ideal** of the simplicial complex $\triangle$ is the squarefree monomial ideal

$$I_\triangle = \langle \mathbf{x}^\tau \mid \tau \notin \triangle \rangle$$

generated by monomials corresponding to **nonfaces** τ of $\triangle$. The **Stanley-Reisner ring** of $\triangle$ is the quotient ring $S/I_\triangle$.

Squarefree monomial ideals can be presented either by its generators or as an intersection of monomial prime ideals generated by subsets of $\{x_1, \ldots, x_n\}$. Let

$$\mathfrak{m}^\tau := \langle x_i \mid i \in \tau \rangle$$

be the monomial prime ideal corresponding to τ.

Theorem 2.2.7. [MS04, Theorem 1.7] *The correspondence $\triangle \leftrightarrow I_\triangle$ constitutes a bijection from simplicial* *complexes on vertices* $[n]$ *to squarefree monomial ideals inside* S. *Furthermore,*

$$I_\triangle = \bigcap_{\sigma \in \triangle} \mathfrak{m}^{\overline{\sigma}},$$

where $\overline{\sigma} = [n] \backslash \sigma$.

Example 2.2.8. After replacing numbers 1, 2, 3, 4, 5 and 6 with variables x_1, x_2, x_3, x_4, x_5 and x_6 respectively, the simplicial complex $\triangle$ on the vertex set [6] in Example 2.2.5 has Stanley-Reisner ideal:

$$\begin{aligned} I_\triangle &= \mathfrak{m}^{\overline{\{1,2,3,4\}}} \cap \mathfrak{m}^{\overline{\{2,3,4,5\}}} \cap \mathfrak{m}^{\overline{\{3,4,5,6\}}} \\ &= \mathfrak{m}^{\{5,6\}} \cap \mathfrak{m}^{\{1,6\}} \cap \mathfrak{m}^{\{1,2\}} \\ &= \langle x_5, x_6 \rangle \cap \langle x_1, x_6 \rangle \cap \langle x_1, x_2 \rangle \\ &= \langle x_1x_5, x_1x_6, x_2x_6 \rangle, \end{aligned}$$

which corresponds to the squarefree monomial ideal $\langle x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23} \rangle$ in Example 2.2.2 by the map $x_{ij} \mapsto x_{j+3(i-1)}$. □

One more notion to be considered in this section in relation to our study is *vertex decomposable* simplicial complex.

Definition 2.2.9. Let $\triangle$ be a simplicial complex on the vertex set $[n]$ and let σ be a face in $\triangle$. The **link** of $\sigma \in \triangle$, denoted $\mathrm{link}_\triangle(\sigma)$, is the set

$$\mathrm{link}_\triangle(\sigma) = \{\tau \in \triangle \,|\, \tau \cup \sigma \in \triangle \text{ and } \tau \cap \sigma = \emptyset\}$$

and the **deletion** of $\sigma \in \triangle$, denoted $\mathrm{del}_\triangle(\sigma)$, is the set

$$\mathrm{del}_\triangle(\sigma) = \{\tau \in \triangle \,|\, \tau \cap \sigma = \emptyset\}.$$

Example 2.2.10. Continuing with Example 2.2.5, the link and deletion of the face $\sigma = \{6\}$ are computed as follows:

$$\begin{aligned} \mathrm{link}_\triangle(\sigma) &= \{\tau \in \triangle \,|\, \tau \cup \sigma \in \triangle \text{ and } \tau \cap \sigma = \emptyset\} \\ &= \{\tau \in \triangle \,|\, \tau \cup \{6\} \in \triangle \text{ and } \tau \cap \{6\} = \emptyset\} \\ &= \text{all subsets of the set } \{3,4,5\}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{del}_\triangle(\sigma) &= \{\tau \in \triangle \,|\, \tau \cap \sigma = \emptyset\} \\ &= \{\tau \in \triangle \,|\, \tau \cap \{6\} = \emptyset\} \\ &= \text{all subsets of the sets } \{1,2,3,4\} \text{ and } \{2,3,4,5\}. \end{aligned}$$

Observe that both $\mathrm{link}_{\triangle}(\sigma)$ and $\mathrm{del}_{\triangle}(\sigma)$ are simplicial complexes in their own right. The Stanley-Reisner ideals associated with these simplicial complexes are:

$$I_{\mathrm{link}_{\triangle}(\sigma)} = \langle x_1, x_2 \rangle \quad \text{and} \quad I_{\mathrm{del}_{\triangle}(\sigma)} = \langle x_5, x_6 \rangle \cap \langle x_1, x_6 \rangle = \langle x_1 x_5, x_6 \rangle = \langle x_1 x_5 \rangle + \langle x_6 \rangle,$$

which corresponds, respectively, to the squarefree monomial ideals $\langle x_{11}, x_{22} \rangle$ and $\langle x_{11}x_{22} \rangle + \langle x_{23} \rangle$, by the substitution $x_{ij} \mapsto x_{j+3(i-1)}$. These two ideals are related in some sense to the monomial ideal in Example 2.2.2. Precisely, if $B := \langle x_{11}, x_{22} \rangle$ and $C := \langle x_{11}x_{22} \rangle$, then

(i) the inclusion $C \subseteq \langle \mathrm{in}_{\succ}(\mathcal{G}) \rangle \subseteq \mathrm{in}_{\succ}(I)$ holds, where $\mathrm{in}_{\succ}(I)$ is the initial ideal, with respect to $\succ$, of the ideal generated by $\mathcal{G}$ (as given in Example 2.2.2), and

(ii) we have the R/C-module isomorphism $A/C \cong B/C$, where $A := \langle \mathrm{in}_{\succ}(\mathcal{G}) \rangle$. □

In general, the inclusion and isomorphism properties stated for monomial ideals A, B and C in Example 2.2.10 play a role in a technique of showing that a set of polynomials forms a Gröbner basis for the ideal it generates. This technique is termed "Gröbner basis via linkage" [GMN13], and will be described in the next section.

Definition 2.2.11. The simplicial complex $\triangle$ on the vertex set $[n]$ is **vertex decomposable** if $\triangle$ is pure (i.e., if dimension of the facets of $\triangle$ are all equal) and either (i) $\triangle$ is a simplex, or (ii) $\triangle = \{\}$, or (iii) for some vertex $\sigma \in \triangle$, both $\mathrm{del}_{\triangle}(\sigma)$ and $\mathrm{link}_{\triangle}(\sigma)$ are vertex-decomposable, and

$$\dim(\triangle) = \dim(\mathrm{del}_{\triangle}(\sigma)) = \dim(\mathrm{link}_{\triangle}(\sigma)) + 1.$$

Example 2.2.12. Continuing with Examples 2.2.5 and 2.2.10, we claim that the given simplicial complex $\triangle$ on the vertex set [6] is vertex decomposable. First, observe that $\triangle$ is pure; all its facets have dimension 3. Furthermore, $\triangle$ is neither a simplex nor a void complex. For the vertex $\sigma = \{6\} \in \triangle$, the complex $\mathrm{link}_{\triangle}(\sigma)$ (as shown in Example 2.2.10) is vertex decomposable, since it is a simplex. In addition, we have $\dim(\triangle) = 3$, $\dim(\mathrm{del}_{\triangle}(\sigma)) = 3$ and $\dim(\mathrm{link}_{\triangle}(\sigma)) = 2$, and so

$$\dim(\triangle) = \dim(\mathrm{del}_{\triangle}(\sigma)) = \dim(\mathrm{link}_{\triangle}(\sigma)) + 1.$$

Lastly, let the simplicial complex $\mathrm{del}_{\triangle}(\sigma)$ be denoted by $\triangle'$, i.e., $\triangle'$ is a simplicial complex on [6] consisting of all subsets of the sets $\{1, 2, 3, 4\}$ and $\{2, 3, 4, 5\}$. This complex $\triangle'$ is pure; its two facets have dimension 3, and it is neither a simplex nor a void complex. For the vertex $\sigma' = \{5\} \in \triangle'$, we have

$$\begin{aligned} \mathrm{link}_{\triangle'}(\sigma') &= \{\tau \in \triangle' \mid \tau \cup \sigma' \in \triangle' \text{ and } \tau \cap \sigma' = \emptyset\} \\ &= \{\tau \in \triangle' \mid \tau \cup \{5\} \in \triangle' \text{ and } \tau \cap \{5\} = \emptyset\} \\ &= \text{all subsets of the set } \{2,3,4\}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{del}_{\triangle'}(\sigma') &= \{\tau \in \triangle' \,|\, \tau \cap \sigma' = \emptyset\} \\ &= \{\tau \in \triangle' \,|\, \tau \cap \{5\} = \emptyset\} \\ &= \text{all subsets of the set } \{1,2,3,4\}. \end{aligned}$$

Since both $\mathrm{link}_{\triangle'}(\sigma')$ and $\mathrm{del}_{\triangle'}(\sigma')$ are simplexes, it follows that they are vertex decomposable, and

$$\dim(\triangle') = \dim(\mathrm{del}_{\triangle'(\sigma')}) = \dim(\mathrm{link}_{\triangle'(\sigma')}) + 1,$$

where $\dim(\triangle') = 3$, $\dim(\mathrm{del}_{\triangle'(\sigma')}) = 3$ and $\dim(\mathrm{link}_{\triangle'(\sigma')}) = 2$. Hence, the claim. □

In this dissertation, it will be shown that the initial ideals of the families of generalized determinantal ideals we consider are squarefree and that their simplicial complexes are vertex decomposable.

2.3 Liaison Theory

Here in this section, we recall some definitions and some results in liaison theory in relation to our study. All of these will be given in the language of ideals. The primary references for this section are [MN01], [KMMR$^+$01], [AAMS11], [Har07] and [GMN13].

Definition 2.3.1. [CS16] An ideal $I \subseteq S$ is called a **complete intersection** (shortened CI) if it is minimally generated by $\operatorname{codim} I$ elements.

Example 2.3.2. Below are an example and non-example of complete intersection ideal.

(a) Recall that S is a polynomial ring in n variables. Let $I = \langle x_{i_1}, \ldots, x_{i_d} \rangle \subseteq S$ be the ideal generated by $d \leq n$ number of variables. Then using equation (2.1), we have $\operatorname{codim} I = \dim S - \dim S/I = n-(n-d) = d$. Observe that $V(I) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n \,|\, a_{i_j} = 0 \text{ for all } j,\ 1 \leq j \leq d\}$. So the ideal I is minimally generated by $d = \operatorname{codim} I$ elements.

(b) A non-example is the twisted cubic ideal $I = \langle ad - bc, b^2 - ac, c^2 - bd \rangle \subseteq \mathbb{K}[a, b, c, d]$. Using Macaulay2, we obtain $\operatorname{codim} I = 2$. □

Definition 2.3.3. [Eis13, Chapter 21, Section 1] Let R be a Noetherian zero-dimensional local ring with (unique) maximal ideal $\mathfrak{m}$. The **socle** of an R-module M is the annihilator in M of the maximal ideal $\mathfrak{m}$.

Definition 2.3.4. [Eis13, Chapter 21, Section 2] A Noetherian zero-dimensional local ring R is said to be **Gorenstein** if it has a simple socle.

Example 2.3.5. An example and non-example of zero-dimensional Gorenstein ring are given below.

(a) The ring $R = \mathbb{K}[x, y]/\langle x^2, y^3\rangle$ is zero-dimensional Gorenstein. It has a simple socle. Precisely, the annihilator in R of the maximal ideal $\langle x, y\rangle$ is generated by xy^2. Observe that the image in R of the product $xy^2 \cdot (ax + by) = ax^2y^2 + bxy^3$ is zero.

(b) The ring $R = \mathbb{K}[x, y]/\langle x^2, xy^2, y^3\rangle$ is zero-dimensional, but not Gorenstein. It does not have a simple socle. Here, the annihilator in R of the maximal ideal $\langle x, y\rangle$ is generated by xy and y^2. Observe that the image in R of the product $(axy + by^2) \cdot (cx + dy) = acx^2y + (ad + bc)xy^2 + bdy^3$ is zero.

Definition 2.3.6. [Eis13, Chapter 21, Section 3] A Noetherian local ring R for which $R/\langle r_1, \ldots, r_d\rangle$ is a zero-dimensional Gorenstein ring for some maximal regular sequence $r_1, \ldots, r_d$ on R is said to be **Gorenstein**.

Definition 2.3.7. Let $I \subseteq S$ be a homogeneous, saturated ideal. We say that I is **Gorenstein in codimension $\leq c$** if the localization $(S/I)_\mathfrak{p}$ is a Gorenstein ring for any prime ideal $\mathfrak{p}$ of S/I of height smaller than or equal to c. In this case, I is said to be G_c. An ideal I which is Gorenstein in codimension 0 is said to be **generically Gorenstein** or G_0.

Definition 2.3.8. Two homogeneous ideals $I, J \subseteq S$ are said to be **Gorenstein linked**, shortened **G-linked**, (in one step) by the Gorenstein ideal $\mathfrak{c} \subseteq S$ if

$$\mathfrak{c} : I = J \quad \text{and} \quad \mathfrak{c} : J = I.$$

In this case, we write $I \sim_\mathfrak{c} J$.

It follows from Definition 2.3.8 that $\mathfrak{c}$ is necessarily contained in $I \cap J$.

Definition 2.3.9. Two ideals $I, J \subseteq S$ are said to be in the same **Gorenstein liaison class** (or **G-liaison class**) if there exists Gorenstein ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_t$ such that

$$I = I_0 \sim_{\mathfrak{c}_1} I_1 \sim_{\mathfrak{c}_2} \cdots \sim_{\mathfrak{c}_t} I_t = J.$$

From Definition 2.3.9, if $t = 3$, then I is G-linked to J in two steps. In this case, we say I and J are *G-bilinked.*

Many families of ideals have been shown to belong to Gorenstein liaison class of a complete intersection.

Definition 2.3.10. A ideal $I \subseteq S$ is said to be **glicci** if it is in the Gorenstein liaison class of a complete intersection. That is, there exists a complete intersection ideal $J \subseteq S$ such that I and J are in the same G-liaison class.

There is a particular kind of liaison that is important to us in our study. This is termed Gorenstein biliaison and it has a connection to Gorenstein liaison by Theorem 2.3.12 below.

Definition 2.3.11. [GMN13] Let $I, J \subset S$ be homogeneous, saturated and unmixed ideals such that $\operatorname{ht} I = \operatorname{ht} J = c$. The ideal I is said to be obtained by an **elementary biliaison** of height h from J if there exists a Cohen-Macaulay ideal N in S of height $c-1$ such that $N \subseteq I \cap J$ and $I/N \cong [J/N](-h)$ as S/N-modules. If in addition the ideal N is G_0, then I is obtained from J via an **elementary G-biliaison**. If $h > 0$, we have an **ascending** elementary G-biliaison.

The following result provides a passage from elementary G-biliaison to G-liaison.

Theorem 2.3.12. [Har07, Theorem 3.5] *Let I_1 be obtained by an elementary G-biliaison from I_2. Then I_2 is G-linked to I_1 in two steps.*

There is yet another concept in liaison that is of importance to us in our study. It is termed Basic Double Gorenstein Link and its connection to Gorenstein liaison is given in [MN01, Proposition 6.3].

Definition 2.3.13 ([GMN13])**.** Let $C \subseteq B \subseteq S$ be homogeneous ideals such that $\operatorname{ht}(C) = \operatorname{ht}(B) - 1$ and S/C is Cohen-Macaulay. Let $f \in R_d$ be a homogeneous element of degree d such that $C : \langle f \rangle = C$. The ideal $A := C + fB$ is called a **Basic Double Link** of degree d of B on C. If moreover C is G_0 and B is unmixed, then A is a **Basic Double G-Link** of B on C.

Define an equivalence relation $\sim$ on the nonzero points of $\mathbb{K}^{n+1}$ as:

$$(x_0, \ldots, x_n) \sim (x'_0, \ldots, x'_n)$$

if there is a nonzero element $\lambda \in \mathbb{K}$ such that $(x_0, \ldots, x_n) = \lambda(x'_0, \ldots, x'_n)$. An **$n$-dimensional projective space** over the field $\mathbb{K}$, denoted $\mathbb{P}^n(\mathbb{K})$, is the set of equivalence classes of $\sim$ on $\mathbb{K}^{n+1} - \{0\}$. That is,

$$\mathbb{P}^n(\mathbb{K}) = (\mathbb{K}^{n+1} - \{0\})/\sim .$$

Each nonzero $(n+1)$-tuple $(x_0, \ldots, x_n) \in \mathbb{K}^{n+1}$ defines a point $\mathbf{p}$ in $\mathbb{P}^n(\mathbb{K})$, and we say that $(x_0, \ldots, x_n)$ are **homogeneous coordinates** of p.

Definition 2.3.14. Let $f_1, \ldots, f_s$ be homogeneous polynomials in $\mathbb{K}[x_0, x_1, \ldots, x_n]$. The **projective variety** defined by $f_1, \ldots, f_s$ is the set

$$V(f_1, \ldots, f_s) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n(\mathbb{K}) \mid f_i(a_0, \ldots, a_n) = 0 \text{ for all } i,\ 1 \leq i \leq s\}.$$

Definition 2.3.15. Let $V \subseteq \mathbb{P}^n(\mathbb{K})$ be a projective variety with homogeneous coordinate ring $R := \mathbb{K}[x_0, x_1, \ldots, x_n]/I(V)$, where

$$I(V) = \{f \in \mathbb{K}[x_0, x_1, \ldots, x_n] \mid f(a_0, \ldots, a_n) = 0 \text{ for all } (a_0, \ldots, a_n) \in V\}$$

is a homogeneous ideal that defines V. The variety V is said to be **arithmetically Cohen-Macaulay** if the ring R is Cohen-Macaulay.

The question "is every arithmetically Cohen-Macaulay subvariety of projective space glicci?" is one of the open problems in liaison theory. Many families of ideals generated by minors of generic matrices have been shown to be glicci. Examples include the standard determinantal ideals [KMMR$^+$01], symmetric mixed ladder determinantal ideals [Gor10] and mixed ladder determinantal ideals [Gor07].

That every standard determinantal ideal is glicci was first proven in [KMMR$^+$01, Theorem 3.6]. Hartshorne, in [Har07, Theorem 4.1], later proved a stronger result about the standard determinantal ideals. In algebraic language, he showed that every standard determinantal ideal is in the G-biliaison class of a complete intersection. This result coupled with [Har07, Lemma 3.3] or [KMMR$^+$01, Proposition 5.2] implies that the standard determinantal ideals are glicci. This approach was later employed in some literature to prove that the aforementioned families of ideals are glicci. It was first shown that these families of ideals belong to G-biliaison class of a complete intersection and then by [Har07, Theorem 3.5], it follows that these families of ideals are glicci. In Section 5 of Chapter 3 of this dissertation, we employ the same technique as above to show that some generalized determinantal ideals are glicci. We first show that they belong to G-biliaison class of a complete intersection and then use Theorem 2.3.12 to conclude they are glicci.

2.4 Flag Varieties and their Schubert Varieties

Here in this section, we give a brief description of flag varieties and their Schubert varieties. We will assume $\mathbb{K}$ to be a field and e_i will be a row vector in $\mathbb{K}^n$ with 1 in its ith entry and 0 elsewhere. The primary references for this section are [Ful92], [Ful97], [Bri05] and [MS04].

Definition 2.4.1. A **complete flag** in a finite dimensional vector space V over $\mathbb{K}$ is an increasing sequence

$$V_\bullet : \{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

of subspaces for which $\dim(V_i) = i$, for all i.

Example 2.4.2. The increasing sequence

$$\epsilon_\bullet : \{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_n \rangle = \mathbb{K}^n$$

is a complete flag. This is called the *standard* (complete) flag. □

Definition 2.4.3. The **complete flag variety** $\mathcal{F}l(V) = \mathcal{F}l(n)$ consists of all complete flags $V_\bullet$ in a vector space V of dimension n.

Recall that B_- is the subgroup of all lower triangular matrices in the general linear group $G := \mathrm{GL}_n(\mathbb{K})$. In what follows, we wish to establish a bijection between $B_-\backslash G$ and $\mathcal{F}l(n)$.

Consider, for example, the matrix

$$g = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{K}).$$

From g, we have the subspaces $V_i = \langle \mathbf{r}_1, \ldots, \mathbf{r}_i \rangle$, $1 \leq i \leq 3$, where $\mathbf{r}_i$ is the ith row vector of g. Since g belongs to $\mathrm{GL}_3(\mathbb{K})$, it follows that $\dim(V_i) = i$, for all i, and so the sequence

$$V_\bullet : \{0\} = V_0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{K}^3$$

is a complete flag. Furthermore, consider the product bg, where

$$b = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Observe that b is a lower triangular matrix in $\mathrm{GL}_3(\mathbb{K})$. In addition, the first, second and third row vectors of $bg \in \mathrm{GL}_3(\mathbb{K})$ are $3\mathbf{r}_1$, $2\mathbf{r}_1 + \mathbf{r}_2$ and $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3$ respectively. Therefore, we have the complete flag

$$\{0\} \subset \langle 3\mathbf{r}_1 \rangle \subset \langle 3\mathbf{r}_1, 2\mathbf{r}_1 + \mathbf{r}_2 \rangle \subset \langle 3\mathbf{r}_1, 2\mathbf{r}_1 + \mathbf{r}_2, \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \rangle = \mathbb{K}^3,$$

which is the same as the complete flag $V_\bullet$ above. In general, for a fixed $g \in G$ with row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$, we have the complete flag

$$V_\bullet : \{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{K}^n, \tag{2.2}$$

where $V_i = \langle \mathbf{r}_1, \ldots, \mathbf{r}_i \rangle$. In addition, for any arbitrary $b = (b_{ij}) \in B_-$, since bg also belongs to G, we have the complete flag

$$\{0\} \subset \langle b_{11}\mathbf{r}_1 \rangle \subset \left\langle b_{11}\mathbf{r}_1, \sum_{j=1}^{2} b_{2j}\mathbf{r}_j \right\rangle \subset \cdots \subset \left\langle b_{11}\mathbf{r}_1, \sum_{j=1}^{2} b_{2j}\mathbf{r}_j, \ldots, \sum_{j=1}^{n} b_{nj}\mathbf{r}_j \right\rangle = \mathbb{K}^n.$$

Observe that each $\sum_{j=1}^{i} b_{ij}\mathbf{r}_j$, $1 \leq i \leq n$, is the ith row vector of bg. Since b is arbitrarily chosen in B_-, it follows that the coset B_-g is determined by the subspaces $\langle \mathbf{r}_1, \ldots, \mathbf{r}_i \rangle$, $1 \leq i \leq n$, and hence, by the complete flag (2.2). Conversely, a complete flag

$$V_\bullet : \{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{K}^n$$

in $\mathcal{F}l(n)$ can be identified with a coset $B_-g \in B_-\backslash G$, where g is an (invertible) matrix whose first i, $1 \leq i \leq n$, row vectors form some basis for V_i. So, we have $B_-\backslash G = \mathcal{F}l(n)$.

Definition 2.4.4. A **partial permutation matrix** $\tilde{w}$ is an $m \times n$ rectangular matrix having all entries equal to 0 except for at most one entry equal to 1 in each row and column.

Example 2.4.5. The matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a 4×3 partial permutation matrix. □

In the case where $m = n$ and $\tilde{w}$ has exactly one 1 in each row and each column, we say $\tilde{w}$ is a *(square) permutation matrix*. In this case, for each permutation $w \in S_n$, we can identify w with the permutation matrix $\tilde{w}$ with a 1 at the position $(w(j), j)$ and zeros elsewhere. So, if w is expressed in one-line notation $w = w_1 \ldots w_n$, where $w_j := w(j)$, then we have the matrix $\tilde{w} = (\tilde{w}_{ij})$, where $\tilde{w}_{ij} = 1$ whenever $w(j) = i$ and zero elsewhere. For instance, consider the permutation $w = 2413 \in S_4$, written in one-line notation, its corresponding permutation matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The *Zariski closure* of a subset of affine (resp. projective) space is the smallest affine (resp. projective) variety containing the set.

Definition 2.4.6. For each permutation $w \in S_n$, a **Schubert cell** X_w° in the flag variety $B_-\backslash G$ is a B_+-orbit for the right action of B_+ on $B_-\backslash G$ by multiplication, i.e., $X_w^\circ := B_-\backslash(B_- w \cdot B_+)$. The Zariski closure of the Schubert cell X_w° is called **Schubert variety**, denoted X_w, i.e., $X_w = \overline{B_-\backslash(B_- w \cdot B_+)}$. An **opposite Schubert cell** is a B_--orbit in $B_-\backslash G$ and an **opposite Schubert variety** is its closure.

By [Ful92, Lemma 6.1], each Schubert variety X_w is an irreducible variety of dimension $\ell(w)$, where $\ell(w) = |\{(i,j) \,|\, i < j \text{ and } w(i) > w(j)\}|$.

2.5 Matrix Schubert Varieties

In this section, M_{mn} will denote the vector space of $m \times n$ matrices over an algebraically closed field $\mathbb{K}$. The coordinate ring of M_{mn} will be denoted by $\mathbb{K}[\mathbf{x}]$, where $\mathbf{x} = \{x_{\alpha\beta}\}_{1 \le \alpha \le m,\, 1 \le \beta \le n}$ is a sequence of variables in a generic $m \times n$ matrix $\mathbf{X}$. The primary references for this section are [KM05] and [MS04, Chapter 15].

The following are some standard terminology about permutations.

Definition 2.5.1. Let v be a permutation in S_n. An integer i, $1 \le i < n$, is an **ascent** of v if $v(i) < v(i+1)$ and a **descent** of v if $v(i) > v(i+1)$.

Definition 2.5.2. The **simple (or adjacent) transposition** s_i, $1 \leq i < n$, is the 2-cycle $(i, i+1)$ which swaps i and $i+1$.

Onward, we will write w to represent both a permutation and its permutation matrix. This will not cause confusion as the intended meaning will be clear from the context.

Definition 2.5.3. Let $w \in M_{mn}$ be a partial permutation. The **matrix Schubert variety** $\overline{X}_w \subseteq M_{mn}$ is the subvariety

$$\overline{X}_w := \{\mathbf{X} \in M_{mn} \mid \operatorname{rank}(\mathbf{X}_{p\times q}) \leq \operatorname{rank}(w_{p\times q}) \text{ for all } p \text{ and } q\},$$

where $\mathbf{X}_{p\times q}$ (resp. $w_{p\times q}$) denotes the upper left $p \times q$ rectangular submatrix of $\mathbf{X}$ (resp. w).

Example 2.5.4. Let $w = 2143 \in S_4$ be a permutation with corresponding matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} \overline{X}_w &= \{\mathbf{X} \in M_{44} \mid \operatorname{rank}(\mathbf{X}_{p\times q}) \leq \operatorname{rank}(w_{p\times q}) \text{ for all } p \text{ and } q\} \\ &= \left\{ \mathbf{X} \in M_{44} \,\middle|\, x_{11} = 0, \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0 \right\}. \end{aligned}$$

□

A relation is called *(non-strict) partial order* if it is reflexive, antisymmetric and transitive, and a *partially ordered set (or poset)* is a set with a partial order on it. The set of matrix Schubert varieties inside M_{mn} forms a poset under inclusion.

Definition 2.5.5. Let v and w be permutations (or permutation matrices). **Bruhat order** is defined as:

$$v \leq w \text{ if } \overline{X}_w \subseteq \overline{X}_v.$$

Remark 2.5.6. Bruhat order can also be defined for Schubert varieties in the flag variety. $v \leq w$ in Bruhat order if $X_w \subseteq X_v$.

Example 2.5.7. Let $v = 2413$ and $w = 3412$ be permutations, with corresponding matrices

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

respectively. Then, we have

$$\overline{X}_v = \{\mathbf{X} \in M_{44} \mid \operatorname{rank}(\mathbf{X}_{p\times q}) \leq \operatorname{rank}(v_{p\times q}) \text{ for all } p \text{ and } q\}$$

$$= \left\{ \mathbf{X} \in M_{44} \;\middle|\; x_{11} = x_{12} = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} = 0 \right\}$$

and

$$\begin{aligned} \overline{X}_w &= \{\mathbf{X} \in M_{44} \mid \text{rank}(\mathbf{X}_{p\times q}) \leq \text{rank}(w_{p\times q}) \text{ for all } p \text{ and } q\} \\ &= \left\{ \mathbf{X} \in M_{44} \;\middle|\; x_{11} = x_{12} = x_{21} = x_{22} = 0 \right\}. \end{aligned}$$

So, if $\mathbf{X}$ is chosen arbitrarily in $\overline{X}_w$, then

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}, \quad \text{with} \quad x_{11} = x_{12} = x_{21} = x_{22} = 0.$$

Since this matrix $\mathbf{X}$ also satisfies: $x_{11} = 0$, $x_{12} = 0$ and $\begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} = 0$, it follows that it belongs to $\overline{X}_v$. Therefore, $\overline{X}_w \subseteq \overline{X}_w$, and so $v \leq w$ in Bruhat order. □

Recall that $w_{p\times q}$ is the upper left $p \times q$ rectangular submatrix of the permutation matrix w. Let $\text{rk}(w)$ be the matrix whose entry at position (p, q) is $\text{rank}(w_{p\times q})$. Then we obtain the following matrices for the permutations $v = 2413$ and $w = 3412$ in the previous example:

$$\text{rk}(v) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \text{rk}(w) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Observe that for any p, q, the entry at position (p, q) of $\text{rk}(v)$ is greater than or equal to the entry at same position (p, q) in $\text{rk}(w)$. That is, $\text{rank}(v_{p\times q}) \geq \text{rank}(w_{p\times q})$, for all p, q. Whenever $v \leq w$ in Bruhat order, this observation is always true, as shown in the following result.

Lemma 2.5.8. [MS04, Lemma 15.19] *The following conditions are equivalent:*

(i) $v \leq w$ *in Bruhat order.*

(ii) w *lies in* $\overline{X}_v$.

(iii) $\text{rank}(v_{p\times q}) \geq \text{rank}(w_{p\times q})$, *for all* p, q.

Proof.

(i) ⇒ (ii) Suppose $v \leq w$ in Bruhat order. Then, by definition of Bruhat order, $\overline{X}_w \subseteq \overline{X}_v$. Since $\operatorname{rank}(w_{p\times q}) \leq \operatorname{rank}(w_{p\times q})$, for all p, q, it follows by definition of $\overline{X}_w$ that the permutation matrix w belongs to $\overline{X}_w$. Therefore, $w \in \overline{X}_w \subseteq \overline{X}_v$, i.e., $w \in \overline{X}_v$.

(ii) ⇒ (iii) Suppose w lies in $\overline{X}_v$. Then, by definition of $\overline{X}_v$, we have the inequality $\operatorname{rank}(w_{p\times q}) \leq \operatorname{rank}(v_{p\times q})$, for all p, q.

(iii) ⇒ (i) Suppose we have the inequality $\operatorname{rank}(v_{p\times q}) \geq \operatorname{rank}(w_{p\times q})$, for all p, q, and let $\mathbf{X}$ be an arbitrary matrix in $\overline{X}_w$. Then, by definition of $\overline{X}_w$, we have the inequality $\operatorname{rank}(\mathbf{X}_{p\times q}) \leq \operatorname{rank}(w_{p\times q})$, for all p, q. Consequently, we have the inequality $\operatorname{rank}(\mathbf{X}_{p\times q}) \leq \operatorname{rank}(v_{p\times q})$, for all p, q, since $\operatorname{rank}(w_{p\times q}) \leq \operatorname{rank}(v_{p\times q})$, for all p, q. Therefore, by definition of $\overline{X}_v$, $\mathbf{X}$ belongs to $\overline{X}_v$, and so, $\overline{X}_w \subseteq \overline{X}_v$. Hence, $v \leq w$ in Bruhat order, by definition. □

Definition 2.5.9. Let $w \in M_{mn}$ be a partial permutation. The **Schubert determinantal ideal** $I_w \subseteq \mathbb{K}[\mathbf{x}]$ is the ideal generated by all minors in $\mathbf{X}_{p\times q}$ of size $1 + \operatorname{rank}(w_{p\times q})$ for all p and q. That is,

$$I_w := \langle \text{minors of size } 1 + \operatorname{rank}(w_{p\times q}) \text{ in } \mathbf{X}_{p\times q} \mid (p, q) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \rangle.$$

Example 2.5.10. Let $w = 2143$, as in Example 2.5.4. Then

$$I_w \;=\; \left\langle x_{11}, \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \right\rangle.$$

□

Every partial permutation matrix can be extended to a square permutation matrix. To see this, let w be an $m \times n$ partial permutation matrix that is not already a square permutation matrix. Then w has at least one zero row or one zero column. Without loss of generality, assume w has at least one zero row. Suppose the first zero row in w is in row m'. Define a new $m \times (n + 1)$ partial permutation matrix w' by adding a new column to w and placing 1 at position $(m', n + 1)$ and 0s elsewhere in that column. If w' is not yet a square permutation matrix, then we continue as before. That is, we add a new column and place a 1 entry in the highest possible row of w'. Continuing this way, we will eventually obtain a partial permutation matrix with no zero row. If this partial permutation matrix is not yet a square permutation matrix, then there must be a zero column in it. This time, adding a new row and placing a 1 entry in its first possible column gives another partial permutation matrix. Continuing in this new way of adding new row, we will eventually obtain a square permutation matrix.

Example 2.5.11. The 4×4 partial permutation matrix $w = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ can be extended to a square permutation matrix as follows:

$$\begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&\mathbf{1}\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&\mathbf{1} \end{pmatrix} \longrightarrow \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&\mathbf{1}&0&0&0&0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&\mathbf{1}&0&0 \end{pmatrix}.$$

Observe that the generators for the Schubert determinantal ideal I_w are the same as the generators for the Schubert determinantal ideal of the square permutation matrix with corresponding permutation $w = 351624$. This square permutation matrix is the first matrix from the right of the above matrices. □

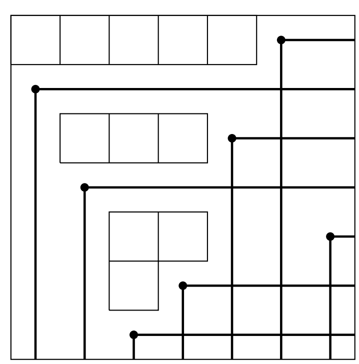
There are some important tools that help in reducing the number of generating minors of I_w – these are Rothe diagram, length, and essential set of permutations.

Definition 2.5.12. The **Rothe diagram** $D(w)$ of a partial permutation matrix $w \in M_{mn}$ consists of all locations (called "boxes") in the $m \times n$ grid neither due south nor due east of a nonzero entry in w.

To obtain the Rothe diagram $D(w)$ for $w \in S_n$, in an $n \times n$ grid, we place a dot • in position $(w(j), j)$ for $1 \leq j \leq n$, and for each dot, we draw the "hook" that extends to the right and below that dot. The boxes that are not in any hook are the boxes of $D(w)$, i.e.,

$$D(w) = \{(p, q) \,|\, p < w(q) \text{ and } q < w^{-1}(p)\}. \tag{2.3}$$

Example 2.5.13. Let $w = 2476315 \in S_7$. The Rothe diagram $D(w)$ for w is given below, where the 1 entries in the matrix of w are indicated by "•", and all locations due south and east of these entries with 1 are crossed out:

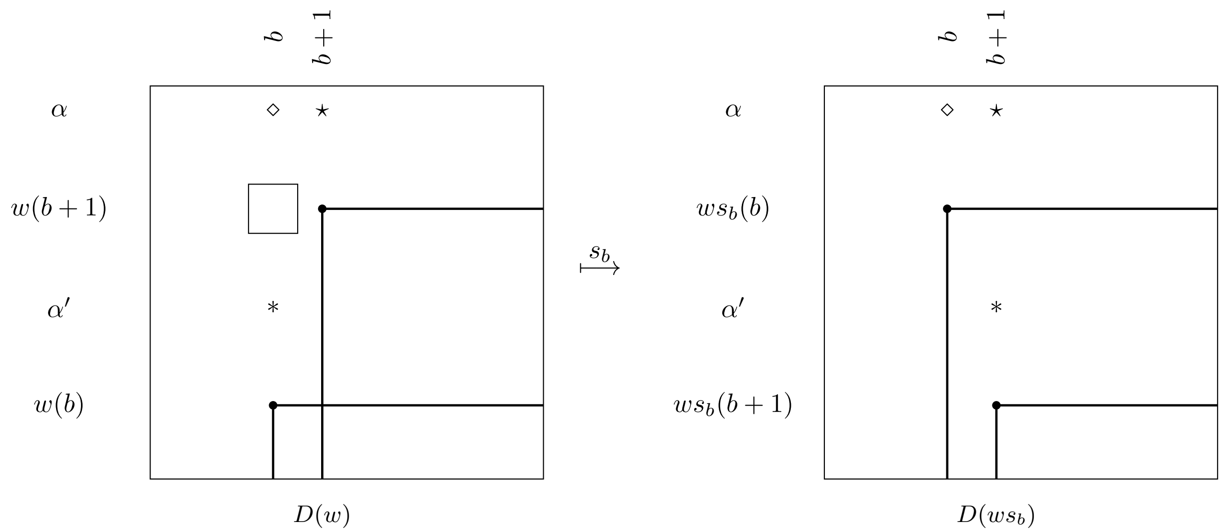
$D(w) =$ 

□

Below is a fact which is closely related to [WY12, Lemma 6.5], and it aids the proof of Lemma 3.2.16 in this dissertation.

Lemma 2.5.14. *Let $w \in S_n$ and b be a descent of w. Then the placement of boxes of $D(w)$ and $D(ws_b)$ agree in all columns except columns b and $b+1$. Moreover, to obtain $D(ws_b)$ from $D(w)$, move all the boxes of $D(w)$ in column b strictly below row $w(b+1)$ one box to the right, and delete the box (that must appear)* *in position* $(w(b+1), b)$ *of* $D(w)$*. The positions of the boxes of* $D(w)$ *in columns* b *and* $b+1$*, and strictly above row* $w(b+1)$ *remain unchanged in* $D(ws_b)$*.*

Proof. Let $w \in S_n$ and b be a descent of w. The pictures below give a brief description of columns b and $b+1$ in the diagrams $D(w)$ and $D(ws_b)$, with an assumption that there are boxes at positions where the symbols $*$, $\diamond$ and $\star$ are located.



First, let (p, q) be a box in $D(w)$ that is neither on column b nor on column $b+1$, i.e., $q \neq b, b+1$. We wish to show that (p, q) belongs to $D(ws_b)$. By equation (2.3), (p, q) being in $D(w)$ implies $p < w(q)$ and $q < w^{-1}(p)$. Since $q \neq b, b+1$, we have $s_b(q) = q$, and consequently, $(ws_b)(q) = w(s_b(q)) = w(q)$, i.e., $p < w(q)$ implies $p < (ws_b)(q)$, as required. It is left to show that $q < (ws_b)^{-1}(p)$, in order to conclude that (p, q) belongs to $D(ws_b)$. To do this, we will consider three possible cases: assume $p = w(b)$, or $p = w(b+1)$, or $p \neq w(b), w(b+1)$.

If $p \neq w(b), w(b+1)$, then $w^{-1}(p) \neq b, b+1$, and so $s_b(w^{-1}(p)) = w^{-1}(p)$. Consequently, $q < w^{-1}(p)$ implies $q < s_b(w^{-1}(p))$, i.e., $q < (s_b^{-1}w^{-1})(p)$, i.e., $q < (ws_b)^{-1}(p)$, as required.

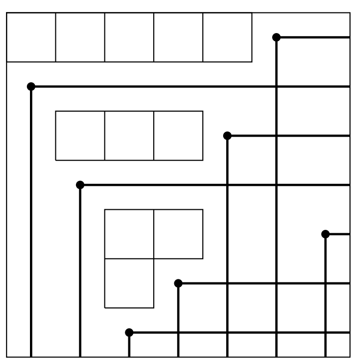
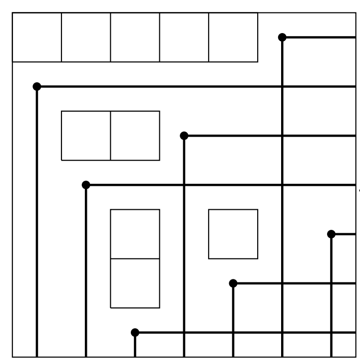
If $p = w(b)$ (resp. $p = w(b+1)$), then $w^{-1}(p) = b$ (resp. $w^{-1}(p) = b+1$), and so $q < w^{-1}(p)$ implies $q < b$ (resp. $q < b+1$), which further implies $q < b+1$ (resp. $q < b$, since $q \neq b$). Since $s_b(b) = b+1$ (resp. $s_b(b+1) = b$), the last inequality implies $q < s_b(b)$ (resp. $q < s_b(b+1)$), i.e., $q < s_b(w^{-1}(w(b)))$ (resp. $q < s_b(w^{-1}(w(b+1)))$), i.e., $q < (s_b^{-1}w^{-1})(w(b))$ (resp. $q < (s_b^{-1}w^{-1})(w(b+1))$), i.e., $q < (ws_b)^{-1}(w(b))$ (resp. $q < (ws_b)^{-1}(w(b+1))$). Since $p = w(b)$ (resp. $p = w(b+1)$), the last inequality implies $q < (ws_b)^{-1}(p)$, as required.

Next, we consider the boxes in columns b and $b+1$ of $D(w)$. Consider first the box $(w(b+1), b) \in D(w)$. This box does not belong to $D(ws_b)$ since the inequality $w(b+1) < (ws_b)(b)$ fails; $(ws_b)(b) = w(s_b(b)) = w(b+1)$.

Furthermore, suppose (α', b) is a box in $D(w)$ for which $w(b+1) < \alpha' < w(b)$ (see, for instance, $*$ in the diagram $D(w)$ above). (α', b) being a box in $D(w)$ implies, by equation (2.3), that $\alpha' < w(b)$ and $b < w^{-1}(\alpha')$. We wish to show that $(\alpha', b+1)$ belongs to $D(ws_b)$ by showing that $\alpha' < (ws_b)(b+1)$ and $b+1 < (ws_b)^{-1}(\alpha')$. Since $s_b(b+1) = b$, the inequality $\alpha' < w(b)$ implies that $\alpha' < w(s_b(b+1))$, i.e., $\alpha' < (ws_b)(b+1)$, as required. In addition, since $w(b+1) < \alpha' < w(b)$, it follows that $\alpha' \neq w(b+1), w(b)$, i.e., $w^{-1}(\alpha') \neq b, b+1$. The inequality $b < w^{-1}(\alpha')$ implies that $b+1 \leq w^{-1}(\alpha')$, i.e., $b+1 < w^{-1}(\alpha')$ since $w^{-1}(\alpha') \neq b+1$. Also, since $w^{-1}(\alpha') \neq b, b+1$, it follows that $s_b(w^{-1}(\alpha')) = w^{-1}(\alpha')$. Therefore, the last inequality $b+1 < w^{-1}(\alpha')$ implies $b+1 < s_b(w^{-1}(\alpha'))$, i.e., $b+1 < (ws_b)^{-1}(\alpha')$, as required.

Lastly, suppose (α, b) is a box in $D(w)$ for which $\alpha < w(b+1) < w(b)$ (see, for instance, $\diamond$ in the diagram $D(w)$ above). (α, b) being a box in $D(w)$ implies, by equation (2.3), that $\alpha < w(b)$ and $b < w^{-1}(\alpha)$. We wish to show that (α, b) belongs to $D(ws_b)$ by showing that $\alpha < (ws_b)(b)$ and $b < (ws_b)^{-1}(\alpha)$. Since $\alpha < w(b+1)$, it follows that $\alpha < w(s_b(b))$, i.e., $\alpha < (ws_b)(b)$, as required. In addition, since $\alpha < w(b+1) < w(b)$, it follows that $\alpha \neq w(b), w(b+1)$, and so $w^{-1}(\alpha) \neq b, b+1$, i.e., $s_b(w^{-1}(\alpha)) = w^{-1}(\alpha)$. Consequently, the inequality $b < w^{-1}(\alpha)$ implies $b < s_b(w^{-1}(\alpha))$, i.e., $b < (ws_b)^{-1}(\alpha)$, as required. Similarly, a box $(\alpha, b+1)$ in $D(w)$ for which $\alpha < w(b+1) < w(b)$ (see, for instance, $\star$ in the diagram $D(w)$ above) is a box in $D(ws_b)$. $\square$

Example 2.5.15. Let $w = 2476315 \in S_7$. This permutation has a descent at $b = 4$. Below are diagrams for $D(w)$ and $D(ws_4)$, to illustrate Lemma 2.5.14.

$D(w) =$  and $D(ws_4) =$ . $\square$

Definition 2.5.16. The **length** of a (partial) permutation $w \in M_{mn}$, denoted by $\ell(w)$, is the cardinality of its diagram $D(w)$.

Definition 2.5.17. The **essential set** of a partial permutation $w \in M_{mn}$, denoted by $\mathcal{E}ss(w)$, consists of the boxes (p, q) in $D(w)$ such that neither $(p, q+1)$ nor $(p+1, q)$ lies in $D(w)$.

Example 2.5.18. Let $w = 2476315 \in S_7$, as in Example 2.5.13. The locations in the Rothe diagram of w are indicated by boxes; precisely $\ell(w) = 11$. Its essential set is $\mathcal{E}ss(w) = \{(1,5), (3,4), (5,4), (6,3)\}$. Following Definition 2.5.9, we have rank $(w_{p\times q}) = 2$, where $(p, q) = (5, 4)$. This is the number of 1s in the upper left 5×4 rectangular submatrix of w. $\square$

It turns out that for any permutation $w \in S_n$, the number of generating minors for the Schubert determinantal ideal I_w given in Definition 2.5.9 can be reduced.

Theorem 2.5.19. [Ful92, Lemma 3.10(a)] *The Schubert determinantal ideal $I_w \subseteq \mathbb{K}[\mathbf{x}]$ is generated by minors coming from ranks in the essential set of w. That is,*

$$I_w = \langle \textit{minors of size } 1 + r_{pq}(w) \textit{ in } \mathbf{X}_{p\times q} \mid (p,q) \in \mathcal{E}ss(w)\rangle.$$

Recall that $\mathbf{x}$ is the sequence of variables in a generic $m \times n$ matrix $\mathbf{X} = (x_{\alpha\beta})$.

Definition 2.5.20. An **antidiagonal** of size r in $\mathbb{K}[\mathbf{x}]$ is the antidiagonal term of a minor of size r, i.e., the product of entries along the antidiagonal of an $r \times r$ submatrix of $\mathbf{X}$. For an $m \times n$ (partial) permutation w, the **antidiagonal ideal** $J_w \subseteq \mathbb{K}[\mathbf{x}]$ is generated by all antidiagonals in $\mathbf{X}_{p\times q}$ of size $1 + \operatorname{rank}(w_{p\times q})$, for all p and q.

Example 2.5.21. Let $v = 2143$, as in Example 2.5.10. Then

$$J_w = \langle x_{11}, x_{13}x_{22}x_{31}\rangle,$$

the ideal generated by the product of entries on the main antidiagonal of the minors that generate I_w. Observe that J_w is squarefree. □

A term order on $\mathbb{K}[\mathbf{x}]$ is called *antidiagonal* if the initial term of every minor of $\mathbf{X}$ is its antidiagonal term. Therefore, if $\mathbf{X}_{[i_1,\ldots,i_r;j_1,\ldots,j_r]}$ denotes an $r\times r$ minor of matrix $\mathbf{X}$ that involves rows $i_1 < \cdots < i_r$ and columns $j_1 < \cdots < j_r$, then

$$\operatorname{in}(\mathbf{X}_{[i_1,\ldots,i_r;j_1,\ldots,j_r]}) = x_{i_rj_1}x_{i_{r-1}j_2}\cdots x_{i_2j_{r-1}}x_{i_1j_r}. \tag{2.4}$$

Theorem 2.5.22. [KM05, Theorem B] *The minors inside the Schubert determinantal ideal I_w constitute a Gröbner basis with respect to any antidiagonal term order:*

$$\textit{in}(I_w) = J_w.$$

Matrix Schubert varieties are closely related to Schubert varieties. For example, Knutson and Miller in [KM05] used them to obtain a geometric explanation for some results in Schubert calculus.

2.6 Kazhdan-Lusztig Ideals

Woo and Yong introduced Kazhdan-Lusztig ideals in [WY08] and studied their Gröbner geometry in [WY12]. In these articles, the authors work with the complete flag variety G/B_+ and Schubert cells $X_w^\circ := B_+wB_+/B_+$. For us in this dissertation, we will stick to the conventions we started using in Section 2.4; we will work with the complete flag variety $B_-\backslash G$ and Schubert cells $X_w^\circ := B_-\backslash B_-wB_+$.

Fix an arbitrary permutation $v \in S_n$ and define a specialized generic matrix $\mathbf{X}^{(v)}$ of size $n \times n$ as follows: for all j, set $\mathbf{X}^{(v)}_{v(j),j} = 1$, and, for all j, set $\mathbf{X}^{(v)}_{v(j),b} = 0$ for $b > j$, and $\mathbf{X}^{(v)}_{a,j} = 0$ for $a > v(j)$. For all other coordinates (i, j), set $\mathbf{X}^{(v)}_{i,j} = x_{ij}$. Let $\mathbf{x}^{(v)} \subseteq \mathbf{x}$ denote the variables appearing in $\mathbf{X}^{(v)}$. Fix the lexicographic term order on $\mathbb{K}[\mathbf{x}^{(v)}]$ which is induced from the following lexicographic order on $\mathbb{K}[\mathbf{x}]$:

$$x_{1n} \succ x_{2n} \succ \cdots \succ x_{nn} \succ x_{1,n-1} \succ x_{2,n-1} \succ \cdots \succ x_{n,n-1} \succ \cdots \succ x_{11} \succ x_{21} \succ \cdots \succ x_{n1}.$$

Let x_{last} be the variable in $\mathbf{X}^{(v)}$, that is maximal with respect to the term order $\succ$; this variable is the rightmost, then uppermost variable appearing in $\mathbf{X}^{(v)}$.

Example 2.6.1. If $n = 5$ and $v = 34512 \in S_n$, then

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix} \quad \text{and} \quad \mathbf{X}^{(v)} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Here, $\mathbf{x}^{(v)} = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}\} \subseteq \mathbf{x}$ and $x_{\text{last}} = x_{13}$. □

Recall that $w_{p\times q}$ denotes the upper left $p \times q$ rectangular submatrix of w.

Definition 2.6.2. Let v, w be arbitrary permutations in S_n. Let $Y^{(v)}$ be the subset of all matrices M_{nn} of the same shape as $\mathbf{X}^{(v)}$ but with entries in $\mathbb{K}$ instead of variables. Then,

$$\mathcal{N}_{v,w} := \{Z \in Y^{(v)} \mid \operatorname{rank}(Z_{p\times q}) \leq \operatorname{rank}(w_{p\times q}),\ 1 \leq p, q \leq n\}$$

is called the **Kazhdan-Lusztig variety**.

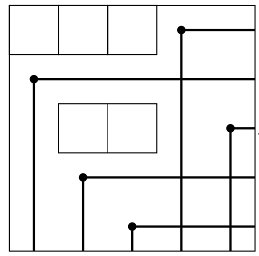
The Kazhdan-Lusztig variety $\mathcal{N}_{v,w}$ is isomorphic to the intersection of the Schubert variety $X_w = \overline{B_-\backslash B_- w B_+}$ with the opposite Schubert cell $\Omega^\circ_v = B_-\backslash B_- v B_-$.

Definition 2.6.3. Let v, w be arbitrary permutations in S_n. The **Kazhdan-Lusztig ideal** $I_{v,w} \subseteq \mathbb{K}[\mathbf{x}^{(v)}]$ is the defining ideal for $\mathcal{N}_{v,w}$. Precisely,

$$I_{v,w} = \langle \text{minors of size } 1 + \operatorname{rank}(w_{p\times q}) \text{ in } \mathbf{X}^{(v)}_{p\times q},\ 1 \leq p, q \leq n \rangle.$$

As stated in [WY12], one consequence of [Ful92, Lemma 3.10] is that for any $w \in S_n$, the minors of size $1 + \operatorname{rank}(w_{p\times q})$ in $\mathbf{X}^{(v)}_{p\times q}$, for all $(p, q) \in \mathcal{E}ss(w)$, are sufficient to generate $I_{v,w}$.

Example 2.6.4. Let $v = 34512$ (as in Example 2.6.1) and $w = 24513$ be permutations in S_5. The Rothe diagram $D(w)$ of w is given below:

$D(w) =$  .

From this diagram, $\ell(w) = 5$, $\mathcal{E}ss(w) = \{(1,3),(3,3)\}$ and $I_{v,w} = \langle x_{11},\, x_{12},\, x_{13},\, x_{22},\, x_{23}\rangle$. □

Every Schubert determinantal ideal is a Kazhdan-Lusztig ideal. To see this, let v be a permutation in S_{2n} such that $v(i) = i + n$ and $v(i + n) = i$ for $1 \le i \le n$, then the matrix $\mathbf{X}^{(v)}$ is of the form

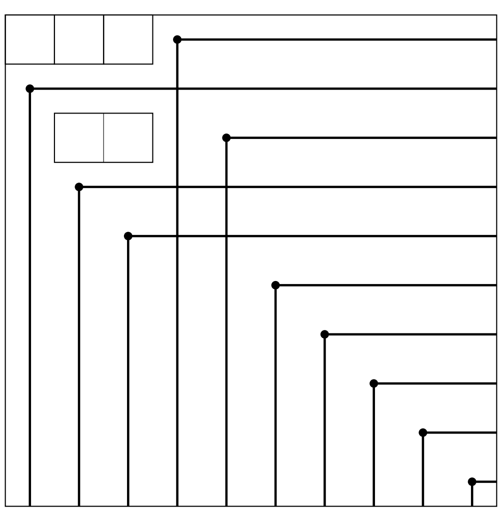
$$\mathbf{X}^{(v)} = \left(\begin{array}{c|c} (x_{ij}) & I_n \\ \hline I_n & 0 \end{array}\right),$$

which only involves the variables x_{ij} for $1 \le i, j \le n$. Given $w \in S_n$, let $w \times 1_n \in S_{2n}$ be the standard embedding into S_{2n}, where $(w \times 1_n)(i) = w(i)$ and $(w \times 1_n)(i+n) = i+n$ for $1 \le i \le n$. The Kazhdan-Lusztig ideal $I_{v,w\times 1_n}$ is equal to the Schubert determinantal ideal I_w. See Example 2.6.5 for an explicit illustration of this idea. Further explanation can be found in [WY12].

Example 2.6.5. Let $w = 24513 \in S_5$, as in Example 2.6.4. The Schubert determinantal ideal corresponding to w is

$$I_w = \left\langle x_{11}, x_{12}, x_{13}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix}, \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} \right\rangle.$$

Since $n = 5$, set $\hat{w} := w \times 1_5 = 2\,4\,5\,1\,3\,6\,7\,8\,9\,10 \in S_{10}$.

$D(\hat{w})=$  and

$$\mathbf{X}^{(v)} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 1 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & 0 & 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & 0 & 0 & 1 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & 0 & 0 & 0 & 1 & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $v = 6\,7\,8\,9\,10\,1\,2\,3\,4\,5 \in S_{10}$, then observe that $I_{v,\hat{w}} = I_w$.

2.7 Schubert patch ideals

Fix an arbitrary permutation $v \in S_n$ and define a specialized generic matrix $\mathbf{Z}^{(v)}$ of size $n \times n$ as follows: for all j, set $\mathbf{Z}^{(v)}_{v(j),j} = 1$, and, for all j, set $\mathbf{Z}^{(v)}_{a,j} = 0$ for $a > v(j)$. For all other coordinates (i,j), set $\mathbf{Z}^{(v)}_{i,j} = z_{ij}$. Let $\mathbf{z}^{(v)} \subseteq \mathbf{z}$ denote the remaining variables appearing in $\mathbf{Z}^{(v)}$. If we define the set

$$\mathbf{y}^{(v)} := \{z_{v(j),b} \in \mathbf{z}^{(v)} \mid b > j,\ 1 \leq j \leq n\},$$

then the resulting set from substituting x_{ij} for z_{ij} in $\mathbf{z}^{(v)} \backslash \mathbf{y}^{(v)}$ coincides with the set $\mathbf{x}^{(v)}$ defined for Kazhdan-Lusztig ideals in Section 2.6. Let $\tilde{\mathbf{x}}^{(v)} := \mathbf{z}^{(v)} \setminus \mathbf{y}^{(v)}$. Fix the lexicographic term order on $\mathbb{K}[\tilde{\mathbf{x}}^{(v)}]$ and $\mathbb{K}[\mathbf{y}^{(v)}]$ which is induced from the following lexicographic order on $\mathbb{K}[\mathbf{z}]$:

$$z_{ij} > z_{i'j'} \text{ if either } j > j', \text{ or } j = j' \text{ and } i < i'.$$

The term order $\succ$ on $\mathbb{K}[\mathbf{z}^{(v)}]$ is defined as: $\tilde{\mathbf{x}}^{(v)} \succ \mathbf{y}^{(v)}$, i.e., using the order on $\mathbb{K}[\mathbf{z}]$ above, we give preference to the variables in $\tilde{\mathbf{x}}^{(v)}$ over the variables in $\mathbf{y}^{(v)}$. In other words, to order the variables in $\mathbb{K}[\mathbf{z}^{(v)}]$, using the order on $\mathbb{K}[\mathbf{z}]$ above, we first order the variables in $\tilde{\mathbf{x}}^{(v)}$ before ordering the variables in $\mathbf{y}^{(v)}$. Since $\mathbf{y}^{(v)} = \emptyset$ for Kazhdan-Lusztig ideals, it follows that the term order defined here coincides, up to setting the variables $\mathbf{y}^{(v)}$ to zero, with the term order defined in Section 2.6. Let $z_{\max}$ be the variable in $\mathbf{z}^{(v)}$, that is maximal with respect to the term order $\succ$. This variable $z_{\max}$ is the same as the variable x_{last} defined in Section 2.6 after the relabelling $z_{ij} \mapsto x_{ij}$.

Example 2.7.1. Revisiting Example 2.6.1, we have the following:

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & z_{23} & 0 & 1 \\ 1 & z_{32} & z_{33} & 0 & 0 \\ 0 & 1 & z_{43} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

$\mathbf{z}^{(v)} = \{z_{11}, z_{12}, z_{13}, z_{15}, z_{21}, z_{22}, z_{23}, z_{32}, z_{33}, z_{43}\} \subseteq \mathbf{z}$, $\tilde{\mathbf{x}}^{(v)} = \{z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23}\}$ (as indicated in blue) and $\mathbf{y}^{(v)} = \{z_{15}, z_{32}, z_{33}, z_{43}\}$ (as indicated in red). Furthermore, we have

$$z_{13} \succ z_{23} \succ z_{12} \succ z_{22} \succ z_{11} \succ z_{21} \succ z_{15} \succ z_{33} \succ z_{43} \succ z_{32}$$

and $z_{\max} = z_{13}$. □

Lemma 2.7.2. *Let v be an arbitrary permutation in S_n and b be the last descent of v. The variable z_{max} is located at the intersection of row $v(b+1)$ and column b of the matrix $\mathbf{Z}^{(v)}$.*

Proof. The variable $z_{\max}$ is the rightmost, then uppermost variable in the resulting matrix from setting to zero the variables $\mathbf{y}^{(v)}$ in the matrix $\mathbf{Z}^{(v)}$. So it must be the variable immediately to the left of the 1 in column $b+1$ of $\mathbf{Z}^{(v)}$, since b is the last descent of v. This 1 on column $b+1$ of $\mathbf{Z}^{(v)}$ is located at row $v(b+1)$, and so $z_{\max}$ is located at position $(v(b+1), b)$. $\square$

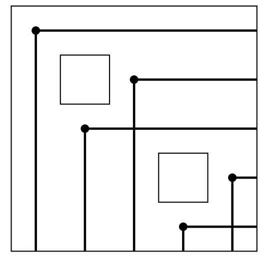
Definition 2.7.3. Let $v, w \in S_n$. The **Schubert patch ideal** $Q_{v,w} \subseteq \mathbb{K}[\mathbf{z}^{(v)}]$ is defined as follows:

$$Q_{v,w} = \langle \text{minors of size } 1 + \operatorname{rank}(w_{p\times q}) \text{ in } \mathbf{Z}^{(v)}_{p\times q}, \ \ 1 \leq p, q \leq n \rangle,$$

where $\mathbf{Z}^{(v)}_{p\times q}$ is the $p \times q$ northwest rectangular submatrix of $\mathbf{Z}^{(v)}$.

Example 2.7.4. Let $v = 24153$ and $w = 13254$. We have:

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{14} & z_{15} \\ 1 & z_{22} & 0 & z_{24} & z_{25} \\ 0 & z_{32} & 0 & z_{34} & 1 \\ 0 & 1 & 0 & z_{44} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D(w) =$$



and

$$Q_{v,w} = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{22} \end{vmatrix}, \begin{vmatrix} z_{32} & z_{34} \\ 1 & z_{44} \end{vmatrix} \right\rangle.$$

$\square$

From Example 2.7.4, the set $\mathbf{y}^{(v)} \subseteq \mathbf{z}^{(v)}$ consists of the variables z_{14}, z_{15}, z_{22}, z_{24}, z_{25} and z_{44}. Observe that out of these variables, the given two generators (essential minors) of the ideal $Q_{v,w}$ above do not involve the variables z_{14} and z_{24} (as indicated in red in $\mathbf{Z}^{(v)}$). These are variables strictly above row 3 and on column 4 of the matrix $\mathbf{Z}^{(v)}$. It turns out that this observation is true in general. See Lemma 2.7.8 for a more general result.

In what follows, we say b is the **last descent** of $v \in S_n$ if

$$b = \max\{i \mid v(i) > v(i+1), \ 1 \leq i < n\}.$$

Given a permutation $v \in S_n$, the matrix $\mathbf{Z}^{(v)}s_b$ is the resulting matrix from swapping columns b and $b+1$ of $\mathbf{Z}^{(v)}$.

Example 2.7.5. Let $v = 45312 \in S_5$ with last descent $b = 3$. $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(v)}s_3$ are given below:

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & z_{23} & 0 & 1 \\ z_{31} & z_{32} & 1 & 0 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15} \\ z_{21} & z_{22} & 0 & z_{23} & 1 \\ z_{31} & z_{32} & 0 & 1 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$

The following result aids the proof of Lemma 2.7.8. Let M be a $\alpha \times \beta$ matrix. Let $\mathcal{R}$ be a set of ℓ strictly increasing integers chosen from $[\alpha]$ and $\mathcal{C}$ be a set of ℓ' strictly increasing integers chosen from $[\beta]$. We denote by $M[\mathcal{R};\mathcal{C}]$ the $\ell \times \ell'$ submatrix of M whose rows are $\mathcal{R}$ and columns are $\mathcal{C}$.

Lemma 2.7.6. *Let $v \in S_n$, b be the last descent of v and M be the $\alpha \times \beta$ northwest rectangular submatrix of $\mathbf{Z}^{(v)}$ (or $\mathbf{Z}^{(v)}s_b$), for some α, β. For some $\ell < \alpha$, consider the submatrix $M[1,\ldots,\ell;1,\ldots,\beta]$ of M which has ℓ 1s, and let I be the ideal generated by the minors of size t in M, for some $t > \ell$. Then, we have the following:*

1. *The ideal I is equal to the ideal generated by the minors of size $t-\ell$ in the submatrix $M[\mathcal{R}';\mathcal{C}']$ of M, where $\mathcal{R}' = [\alpha] \setminus [l]$ and $\mathcal{C} = [\beta] \setminus \{v^{-1}(1),\ldots,v^{-1}(\ell)\}$, both arranged in ascending order.*

2. *If there are m number of minors of size t in M, then the ideal I can be generated by a set $\mathcal{G}$ of fewer than m minors of size t in M.*

3. *This generating set $\mathcal{G}$ for I consists of minors that do not involve the variables in the first ℓ rows of M.*

Proof. Suppose the ℓ 1s in $M[1,\ldots,\ell;1,\ldots,\beta]$ are in columns $j_1,\ldots,j_\ell$ of M. Observe that for $1 \le k \le \ell$, each j_k equals $v^{-1}(k)$. Then, considering the structure of these 1s, by cofactor expansion about the rows and columns where these 1s are located in M, starting from the uppermost 1, we have that the ideal I is equal to the ideal I' generated by the minors of size $t-\ell$ in the submatrix $G := M[\mathcal{R}';\mathcal{C}']$ of M, where $\mathcal{R}' = [\alpha] \setminus [l]$ and $\mathcal{C} = [\beta] \setminus \{v^{-1}(1),\ldots,v^{-1}(\ell)\}$. So, part (1) is verified. Furthermore, given any minor of size $t-\ell$ that involves rows $i_1,\ldots,i_{t-\ell}$ and columns $k_1,\ldots,k_{t-\ell}$ in G, the minor that involves rows $1,\ldots,\ell,i_1,\ldots,i_{t-\ell}$ and columns $j_1,\ldots,j_\ell,k_1,\ldots,k_{t-\ell}$ in M is a minor of size t in M. Hence, each minor of size $t-\ell$ in G is also a minor of size t in M. Hence, part (2) is verified. In addition, none of these minors of size $t-\ell$ in G involves the variables in rows $1,\ldots,\ell$ and columns $j_1,\ldots,j_\ell$ of M, and so their corresponding minors of size t in M do not also involve these variables. Hence, part (3) is verified. $\square$

Example 2.7.7. Let $v = 652143$ and M be the $\alpha \times \beta$ northwest rectangular submatrix of $\mathbf{Z}^{(v)}$, where $(\alpha,\beta) = (4,5)$.

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} & z_{16} \\ z_{21} & z_{22} & 1 & 0 & z_{25} & z_{26} \\ z_{31} & z_{32} & 0 & 0 & z_{35} & 1 \\ z_{41} & z_{42} & 0 & 0 & 1 & 0 \\ z_{51} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & 1 & 0 & z_{25} \\ z_{31} & z_{32} & 0 & 0 & z_{35} \\ z_{41} & z_{42} & 0 & 0 & 1 \end{pmatrix}.$$

If $\ell = 2$, then we have the submatrix

$$M[1,\dots,\ell;1,\dots,\beta] = M[1,2;1,2,3,4,5] = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & 1 & 0 & z_{25} \end{pmatrix}.$$

Observe that $v^{-1} = 436521$ and so the set $\{v^{-1}(1), v^{-1}(\ell)\} = \{4,3\}$. Set $\mathcal{R}' := [\alpha] \setminus [\ell] = \{3,4\}$ and $\mathcal{C}' := [\beta] \setminus \{4,3\} = \{1,2,5\}$. Then we have

$$M[\mathcal{R}';\mathcal{C}'] = M[3,4;1,2,5] = \begin{pmatrix} z_{31} & z_{32} & z_{35} \\ z_{41} & z_{42} & 1 \end{pmatrix}.$$

Let $t = 4$. Then $t - \ell = 2$. By cofactor expansion, observe that the ideal I generated by the 4×4 minors in M is equal to the ideal generated by the 2×2 minors in $M[\mathcal{R}';\mathcal{C}']$. Consequently, out of the $\binom{5}{4} \cdot \binom{4}{4} = 5$ minors of size 4 in M that generates I, only $\binom{3}{2} \cdot \binom{2}{2} = 3$ of them are enough to generate I. These 3 minors are:

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & z_{22} & 1 & 0 \\ z_{31} & z_{32} & 0 & 0 \\ z_{41} & z_{42} & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} z_{11} & z_{13} & 1 & z_{15} \\ z_{21} & 1 & 0 & z_{25} \\ z_{31} & 0 & 0 & z_{35} \\ z_{41} & 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} z_{12} & z_{13} & 1 & z_{15} \\ z_{22} & 1 & 0 & z_{25} \\ z_{32} & 0 & 0 & z_{35} \\ z_{42} & 0 & 0 & 1 \end{vmatrix}.$$

Lastly, after expansions of these 3 minors, observe that none of them involves the variables in the first 2 rows of M. □

Recall from Lemma 2.7.2 that for a permutation $v \in S_n$, the variable $z_{\max}$ in $\mathbf{Z}^{(v)}$ is located at the intersection of row $v(b+1)$ and column b of $\mathbf{Z}^{(v)}$, where b is the last descent of v.

Lemma 2.7.8. *Let v, w be arbitrary permutations in S_n and b be the last descent of v. Set $a := v(b+1)$ and assume $Q_{v,w} \neq \langle 1 \rangle$. The ideal $Q_{v,w}$ can be generated by essential minors that do not involve the variables z_{ij} in $\mathbf{Z}^{(v)}$, for all $1 \leq i < a$ and $b \leq j \leq b+1$; these are variables strictly above row a and on columns b and $b+1$ of the matrix $\mathbf{Z}^{(v)}$.*

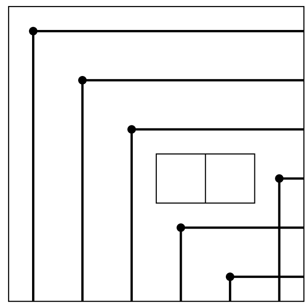
Proof. It suffices to consider the essential boxes (α, β) of $D(w)$ for which $\beta \geq b$. By the setup of $\mathbf{Z}^{(v)}$, the variable $z_{\max}$ is at position (a, b), and consequently, for each of the integers $1 \leq i \leq a-1$, there is a 1 at position $(i, v^{-1}(i))$ of $\mathbf{Z}^{(v)}$, where $v^{-1}(i) < b$. So, the 1s on the first $(a-1)$ rows of $\mathbf{Z}^{(v)}$ are located strictly to the left of column b.

Now, suppose $\beta \geq b$ and $\alpha < a$. Since $\alpha < a$, it follows from above that the 1s on the first α rows of $\mathbf{Z}^{(v)}$ are located strictly to the left of column b. Since $b \leq \beta$, it further follows that, up to rearranging columns, the matrix $\mathbf{Z}^{(v)}_{\alpha \times \beta}$ contains a $t \times t$ upper triangular matrix with 1s on its diagonal, where $t = \min(\alpha, \beta)$. So, $Q_{v,w} = \langle 1 \rangle$, since $1 + \operatorname{rank}(w_{\alpha \times \beta}) \leq \min(\alpha, \beta)$.

On the other hand, suppose $\beta \geq b$, $\alpha \geq a$ and set $M := \mathbf{Z}^{(v)}_{\alpha \times \beta}$. Recall from above that for $1 \leq i \leq a-1$, there is a 1 at position $(i, v^{-1}(i))$ of $\mathbf{Z}^{(v)}$, where $v^{-1}(i) < b$. So, there are $(a-1)$ 1s in the submatrix of M

formed by rows $1, \dots, a-1$. Let $t := 1 + \mathrm{rank}(w_{\alpha \times \beta})$ and assume $t > a-1$. Then, by part (3) of Lemma 2.7.6, the ideal I generated by the minors of size t in M can be generated by some (essential) minors of size t in M, none of which involves, in particular, the variables in rows i, $1 \le i \le a-1$. Observe that if $t \le a-1$, then $Q_{v,w} = \langle 1 \rangle$. $\square$

Example 2.7.9. Let $v = 652143$ (as in Example 2.7.7) and $w = 123564$. We have:

$D(w) =$  .

Following the arguments in the previous example, the 3 minors

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & z_{22} & 1 & 0 \\ z_{31} & z_{32} & 0 & 0 \\ z_{41} & z_{42} & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} z_{11} & z_{13} & 1 & z_{15} \\ z_{21} & 1 & 0 & z_{25} \\ z_{31} & 0 & 0 & z_{35} \\ z_{41} & 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} z_{12} & z_{13} & 1 & z_{15} \\ z_{22} & 1 & 0 & z_{25} \\ z_{32} & 0 & 0 & z_{35} \\ z_{42} & 0 & 0 & 1 \end{vmatrix}.$$

generate $Q_{v,w}$. After expansion, we finally obtain

$$Q_{v,w} = \left\langle \begin{vmatrix} z_{31} & z_{32} \\ z_{41} & z_{42} \end{vmatrix}, \begin{vmatrix} z_{31} & z_{35} \\ z_{41} & 1 \end{vmatrix}, \begin{vmatrix} z_{32} & z_{35} \\ z_{42} & 1 \end{vmatrix} \right\rangle.$$

Observe that none of these generators involves the variables z_{ij} in $\mathbf{Z}^{(v)}$, for all $1 \le i < a$ and $b \le j \le b+1$, where $(a, b) = (3, 5)$. $\square$

It follows from [Ful92, Lemma 3.10] that for any permutation $w \in S_n$, the (essential) minors of size $1 + \mathrm{rank}\,(w_{p \times q})$ in $\mathbf{Z}^{(v)}_{p \times q}$, $(p, q) \in \mathcal{E}ss(w)$, are sufficient to generate $Q_{v,w}$, as justified in the proof of Proposition 2.7.10. Let

$$\mathcal{M}_{v,w} := \mathrm{Spec}\big(\mathbb{K}[\mathbf{z}^{(v)}]/Q_{v,w}\big)$$

be the associated affine scheme.

Proposition 2.7.10. *For permutations $v, w \in S_n$, the Schubert patch variety $\mathcal{M}_{v,w}$ is isomorphic to the intersection of the Schubert variety $X_w = B_- \backslash \overline{B_- w B_+}$ with the permuted opposite big cell $\Omega^\circ_{v_0} v_0 v$, i.e., $\mathcal{M}_{v,w} \cong X_w \cap \big(\Omega^\circ_{v_0} v_0 v\big)$, where $\Omega^\circ_v = B_- \backslash B_- v B_-$ is the opposite Schubert cell associated to v, and $v_0 \in S_n$ is the long word permutation $v_0(i) = n - i + 1$.*

Proof. We proceed in the same way as the proof of [WY08, Proposition 3.1]. Let $\pi : G \to B_- \backslash G$ be the natural quotient map and consider the map $\sigma : \Omega^\circ_{v_0} v_0 v \to G$, where $\sigma(F_\bullet)$ is the unique matrix representative of $F_\bullet$, which, for all i, has 1s at $(v(i), i)$, and for all i, has 0s at (b, i) for $b > v(i)$. It follows that $X_w \cap (\Omega^\circ_{v_0} v_0 v) \cong \pi^{-1}(X_w) \cap \sigma(\Omega^\circ_{v_0} v_0 v)$ since σ is a local section of π. One coordinate ring for $\mathrm{GL}_n(\mathbb{K})$ is $\mathbb{K}[\mathbf{z}, \det^{-1}(\mathbf{Z})]$, where $\mathbf{z} := \{z_{ij} \mid 1 \le i, j \le n\}$, are the entries of a generic matrix $\mathbf{Z}$. With these coordinates, the defining ideal for $\sigma(\Omega^\circ_{v_0} v_0 v)$, denoted by J_v, is generated by the polynomials $z_{v(i),i} - 1$, and monomials of the form $z_{b,i}$ for $b > v(i)$. Fulton in [Ful92, Lemma 3.10] showed that the defining ideal I_w for $\pi^{-1}(X_w)$ is generated by minors of size $1 + \mathrm{rank}\,(w_{p\times q})$ in $\mathbf{Z}^{(v)}_{p\times q}$, for all $(p, q) \in \mathcal{E}ss(w)$. Therefore, $X_w \cap (\Omega^\circ_{v_0} v_0 v) \cong \mathrm{Spec}(\mathbb{K}[\mathbf{z}]/(I_w + J_v))$, noting that $\det(\mathbf{Z}) = \pm 1$ by J_v. To reduce the number of variables, instead of working in a generic matrix $\mathbf{Z}$, we first quotient by J_v and then work in the generic matrix $\mathbf{Z}^{(v)}$. The image of I_w in $\mathbb{K}[\mathbf{z}^{(v)}]$ under this quotient by J_v is precisely $Q_{v,w}$, and hence, $\mathrm{Spec}(\mathbb{K}[\mathbf{z}^{(v)}]/Q_{v,w}) \cong X_w \cap (\Omega^\circ_{v_0} v_0 v)$. □

Corollary 2.7.11 is an immediate consequence of Proposition 2.7.10.

Corollary 2.7.11. *Let $v, w \in S_n$ for which $w \le v$ in Bruhat order. Then $Q_{v,w}$ is prime of codimension $\ell(w)$.*

Proof. If $w \le v$ in Bruhat order, then $Q_{v,w}$ is a proper ideal. From the proof of Proposition 2.7.10, we have that $X_w \cap (\Omega^\circ_{v_0} v_0 v)$ is an irreducible variety of dimension $\ell(w_0) - \ell(w)$ and $\mathrm{Spec}(\mathbb{K}[\mathbf{z}^{(v)}]/Q_{v,w}) \cong X_w \cap (\Omega^\circ_{v_0} v_0 v)$. Hence, $Q_{v,w}$ is prime of codimension $\ell(w)$. □

Example 2.7.12 shows that by setting the variables in $\mathbf{y}^{(v)}$ to zero in a Schubert patch ideal, we obtain a Kazhdan-Lusztig ideal.

Example 2.7.12. Let $v = 34512$ and $w = 24513$, as in Examples 2.7.1 and 2.6.4, respectively. Then

$$Q_{v,w} = \left\langle z_{11},\, z_{12},\, z_{13}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ 1 & z_{33} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} \right\rangle.$$

We observe here that setting the variables $z_{32}, z_{33} \in \mathbf{y}^{(v)} \subseteq \mathbf{z}^{(v)}$ to zero in the Schubert patch ideal $Q_{v,w}$ above, the resulting ideal is the Kazhdan-Lusztig ideal $I_{v,w}$ given in Example 2.6.4. □

2.8 Torus Actions and Multigradings of Schubert Patch Ideals and Kazhdan-Lusztig Ideals

Recall that a monomial in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ is of the form $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{N}$, $1 \le i \le n$. Endow S with a degree map $\mathbb{N}^n \to \mathbb{Z}^n$ that takes each vector $\boldsymbol{\alpha}$ in $\mathbb{N}^n$ (and hence, each monomial $\mathbf{x}^{\boldsymbol{\alpha}}$ in S) to its degree $\deg(\mathbf{x}^{\boldsymbol{\alpha}})$ in $\mathbb{Z}^n$. For each $\boldsymbol{a} \in \mathbb{Z}^n$, let $S_{\boldsymbol{a}}$ denote the $\mathbb{K}$-vector space spanned by homogeneous polynomials of degree $\boldsymbol{a}$. S is *multigraded* by $\mathbb{Z}^n$ since it has a direct sum decomposition

$$S = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} S_{\boldsymbol{a}}$$

as a $\mathbb{K}$-vector space and $S_{\boldsymbol{a}} S_{\boldsymbol{b}} \subseteq S_{\boldsymbol{a}+\boldsymbol{b}}$, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$.

A $\mathbb{K}$-linear basis of $S_{\mathbf{a}}$ could be infinite or even empty. The multigrading of S by $\mathbb{Z}^n$ is said to be *positive* if for all $\mathbf{a} \in \mathbb{Z}^n$, the $\mathbb{K}$-vector space $S_{\mathbf{a}}$ is finite dimensional. As shown in [MS04, Theorem 8.6], this is equivalent to the only polynomials of degree $\mathbf{0} \in \mathbb{Z}^n$ are the constants, i.e., $S_{\mathbf{0}} = \mathbb{K}$. In this special case, for an ideal $I \subseteq S$ that is homogeneous with respect to this positive grading of S by $\mathbb{Z}^n$, we have the *Hilbert series*

$$\mathrm{Hilb}_M(\mathbf{t}) = \sum_{\mathbf{a} \in \mathbb{Z}^n} \dim_{\mathbb{K}}(M_{\mathbf{a}}) \mathbf{t}^{\mathbf{a}},$$

where $M := S/I$. The generating function $\mathbf{a} \mapsto \dim_{\mathbb{K}}(M_{\mathbf{a}})$ is called the **Hilbert function** of M.

We will now describe a positive multigrading of the coordinate ring $\mathbb{K}[\mathbf{z}^{(v)}]$ of $\mathbf{Z}^{(v)}$. Following the same idea in [WY12], consider the right action (torus action) of T on $B_- \backslash G$, where T is the subgroup of diagonal matrices in G. For each $v \in S_n$, this action independently scales the columns of $\mathbf{Z}^{(v)}$, to obtain $\mathbf{Z}^{(v)} \cdot \mathbf{t}$, $\mathbf{t} \in T$. We then carefully choose a matrix $\mathbf{b} \in B_-$ such that the matrix $\mathbf{b} \cdot \mathbf{Z}^{(v)} \cdot \mathbf{t}$ has 1 in position $(v(j), j)$, $1 \leq j \leq n$. By so doing, we dependently rescaled the rows of the matrix $\mathbf{Z}^{(v)} \cdot \mathbf{t}$.

Example 2.8.1. Let $v = 34512$ (as in Example 2.7.1). Setting

$$\mathbf{t} = \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & t_3 & 0 & 0 \\ 0 & 0 & 0 & t_4 & 0 \\ 0 & 0 & 0 & 0 & t_5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} t_4^{-1} & 0 & 0 & 0 & 0 \\ 0 & t_5^{-1} & 0 & 0 & 0 \\ 0 & 0 & t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_3^{-1} \end{pmatrix},$$

we obtain

$$\mathbf{b} \cdot \mathbf{Z}^{(v)} \cdot \mathbf{t} = \begin{pmatrix} z_{11}t_1t_4^{-1} & z_{12}t_2t_4^{-1} & z_{13}t_3t_4^{-1} & 1 & z_{15}t_5t_4^{-1} \\ z_{21}t_1t_5^{-1} & z_{22}t_2t_5^{-1} & z_{23}t_3t_5^{-1} & 0 & 1 \\ 1 & z_{32}t_2t_1^{-1} & z_{33}t_3t_1^{-1} & 0 & 0 \\ 0 & 1 & z_{43}t_3t_2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Writing weights additively, we can assign to the variables z_{11}, z_{12}, z_{13}, z_{15}, z_{21}, z_{22}, z_{23}, z_{32}, z_{33} and z_{43} in $\mathbf{Z}^{(v)}$ the weights $e_4 - e_1$, $e_4 - e_2$, $e_4 - e_3$, $e_4 - e_5$, $e_5 - e_1$, $e_5 - e_2$, $e_5 - e_3$, $e_1 - e_2$, $e_1 - e_3$ and $e_2 - e_3$, respectively, where e_i is the ith standard basis vector in $\mathbb{Z}^5$. □

In general, for a fixed $v \in S_n$, the action described above gives the variable z_{ij} in matrix $\mathbf{Z}^{(v)}$ the weight $e_{v^{-1}(i)} - e_j$. This action therefore yields a $\mathbb{Z}^n$-grading of the coordinate ring $\mathbb{K}[\mathbf{z}^{(v)}]$ of $\mathbf{Z}^{(v)}$ and this multigrading is positive, i.e., the only polynomials in $\mathbb{K}[\mathbf{z}^{(v)}]$ of degree $\mathbf{0}$ are the elements of $\mathbb{K}$ (see Lemma 2.8.4 below for a proof of this statement). In [WY08, Lemma 5.2], the authors showed that every Kazhdan-Lusztig ideal $I_{v,w}$ is homogeneous with respect to the positive multigrading of the underlying ring by $\mathbb{Z}^n$. This same proof can also be used to show the following result.

Lemma 2.8.2. *Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{Z}^n$. Under the multigrading where the variable z_{ij} has degree $e_{v^{-1}(i)} - e_j$, every Schubert patch ideal $Q_{v,w}$ is homogeneous.*

Proof. We assign degree $\mathbf{0}$ to the 1s. This is consistent with $\deg(z_{v(j),j}) = e_{v^{-1}(v(j))} - e_j = \mathbf{0}$, since these 1s are at position $(v(j), j)$, $1 \leq j \leq n$. Observe that for any minor of an arbitrary generic matrix $\mathbf{Z} = (z_{ij})$ that uses rows $i_1, \ldots, i_\ell$ and columns $j_1, \ldots, j_\ell$ of $\mathbf{Z}$, each term $\mathbf{z}^{\boldsymbol{\alpha}}$ of this minor has ℓ variables with row indices $i_1, \ldots, i_\ell$ and column indices $j_1, \ldots, j_\ell$, up to rearranging these indices. Therefore, any minor f of $\mathbf{Z}^{(v)}$ (as a special type of $\mathbf{Z}$) that uses rows $i_1, \ldots, i_\ell$ and columns $j_1, \ldots, j_\ell$ will be homogeneous of degree

$$e_{v^{-1}(i_i)} - e_{j_\ell} + e_{v^{-1}(i_2)} - e_{j_{\ell-1}} + e_{v^{-1}(i_3)} - e_{j_{\ell-2}} + \cdots + e_{v^{-1}(i_\ell)} - e_{j_1} = \sum_{k=1}^{\ell} \left(e_{v^{-1}(i_k)} - e_{j_k}\right).$$

Observe that the antidiagonal initial term of f is $z_{i_1,j\ell} z_{i_2,j_{\ell-1}} \cdots z_{i_\ell,j_1}$, as given in (2.4). Since $I_{v,w}$ is generated by minors of $\mathbf{Z}^{(v)}$, this proves $I_{v,w}$ is homogeneous under this grading. □

Example 2.8.3. Let $v = 34512$ and $w = 24513$. The ideal $Q_{v,w}$ is given in Example 2.7.12. With respect to the above multigrading, we have:

$$\deg(z_{22}) = e_5 - e_2$$

and

$$\deg(z_{21}z_{32}) = (e_5 - e_1) + (e_1 - e_2) \; = \; e_5 - e_2.$$

Thus, the generator $\begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{32} \end{vmatrix}$ of $Q_{v,w}$ is homogeneous. Similarly, the generators $\begin{vmatrix} z_{21} & z_{23} \\ 1 & z_{33} \end{vmatrix}$ and $\begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix}$ of $Q_{v,w}$ are also homogeneous. Therefore, $Q_{v,w}$ is homogeneous with respect to the positive multigrading of $\mathbb{K}[\mathbf{z}^{(v)}]$ by $\mathbb{Z}^5$. □

Recall from the beginning of Subsection 2.7, that for each non-zero variable z_{ij} in the matrix $\mathbf{Z}^{(v)}$, we have the inequality $i < v(j)$.

Lemma 2.8.4. *Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{Z}^n$. The multigrading where each variable z_{ij} in $\mathbf{Z}^{(v)}$ has degree $e_{v^{-1}(i)} - e_j$ is positive.*

Proof. It suffices to show that no product $\boldsymbol{m} := z_{i_1,j_1} z_{i_2,j_2} z_{i_3,j_3} \ldots z_{i_\ell,j_\ell} \in R$ has degree $\mathbf{0}$. To this end, suppose there exists a monomial $\boldsymbol{m} := z_{i_1,j_1} z_{i_2,j_2} z_{i_3,j_3} \ldots z_{i_\ell,j_\ell} \in R$ (the variables involved are not necessarily distinct) such that $\deg(\boldsymbol{m}) = \mathbf{0}$, i.e.

$$\sum_{k=1}^{\ell} e_{v^{-1}(i_k)} - e_{j_k} = 0. \tag{2.5}$$

Equation (2.5) implies complete cancellations of vectors; all vectors $e_{v^{-1}(i_k)}$ and $e_{j_{k'}}$ can be paired such that $e_{v^{-1}(i_k)} = e_{j_{k'}}$ with $1 \leq k \neq k' \leq \ell$ i.e., $v^{-1}(i_k) = j_{k'}$, for some $k \neq k'$. If $v^{-1}(i_k) = j_{k'}$, then $i_k = v(j_{k'})$. Thus, $v(j_{k'}) < v(j_k)$, since by definition, $i_k < v(j_k)$ for $z_{i_k,j_k} \in \mathbf{Z}^{(v)}$. In summary, as a result of equation

(2.5), for any vector $e_{v^{-1}(i_k)}$ that cancels out the vector $e_{j_{k'}}$ $(k \neq k')$, we have $v(j_{k'}) < v(j_k)$. Without loss of generality, suppose the variables involved in $\boldsymbol{m}$ are distinct.

If $\ell = 2$, then $e_{v^{-1}(i_2)} = e_{j_1}$ and $e_{v^{-1}(i_1)} = e_{j_2}$. Therefore, by the above argument, it follows that $v(j_1) < v(j_2)$ and $v(j_2) < v(j_1)$, which is a contradiction. Hence, no product $z_{i_1,j_1} z_{i_2,j_2}$ in R with degree $\mathbf{0}$.

Let $\ell > 2$ and consider, for instance, the variable z_{i_1,j_1} with degree $e_{v^{-1}(i_1)} - e_{j_1}$. Equation (2.5) then implies that among the variables that appear in $\boldsymbol{m}$, there exists two other variables, say, $z_{i_{k_1},j_{k_1}}$ (with degree $e_{v^{-1}(i_{k_1})} - e_{j_{k_1}}$) and $z_{i_{k_2},j_{k_2}}$ (with degree $e_{v^{-1}(i_{k_2})} - e_{j_{k_2}}$), both different from z_{i_1,j_1}, such that $e_{v^{-1}(i_1)} = e_{j_{k_1}}$ and $e_{v^{-1}(i_{k_2})} = e_{j_1}$. Then, by the argument in the second paragraph, it follows that

$$v(j_{k_1}) < v(j_1) \qquad \text{and} \qquad v(j_1) < v(j_{k_2}). \tag{2.6}$$

If $k_1 = k_2$, then inequalities (2.6) become a contradiction. Therefore, $k_1 \neq k_2$, and equation (2.5) becomes

$$\sum_{k \neq 1, k_1, k_2} e_{v^{-1}(i_k)} - e_{j_k} = e_{j_{k_2}} - e_{v^{-1}(i_{k_1})}. \tag{2.7}$$

Vectors $e_{v^{-1}(i_{k_1})}$ and $e_{j_{k_2}}$ are canceled out by some other vectors in equation (2.7), thereby canceling out degrees of the variables $z_{i_{k_1},j_{k_1}}$ and $z_{i_{k_2},j_{k_2}}$ completely from equation (2.7) (and hence from equation (2.5)), in addition to the variable z_{i_1,j_1} that has been canceled out. Precisely, if $v^{-1}(i_{k_1}) = j_{k'_1}$ and $v^{-1}(i_{k'_2}) = j_{k_2}$, for some k'_1, k'_2, with $1 < k'_1, k'_2 \leq \ell$ and different from k_1, k_2, then by the argument in the second paragraph, it follows that

$$v(j_{k'_1}) < v(j_{k_1}) \qquad \text{and} \qquad v(j_{k_2}) < v(j_{k'_2}). \tag{2.8}$$

If $k'_1 = k'_2$, then combining inequalities (2.6) and (2.8), we have $v(j_{k_1}) < v(j_1) < v(j_{k_2}) < v(j_{k'_2}) = v(j_{k'_1}) < v(j_{k_1})$, which is a contradiction. Therefore $k'_1 \neq k'_2$, and equation (2.7) becomes

$$\sum_{k \neq 1, k_1, k_2, k'_1, k'_2} e_{v^{-1}(i_k)} - e_{j_k} = e_{j_{k'_2}} - e_{v^{-1}(i_{k'_1})}. \tag{2.9}$$

Observe that, so far, degrees of three variables have been completely eliminated from equation (2.5). We can continue this way by finding next some other vectors that cancel out vectors $e_{v^{-1}(i_{k'_1})}$ and $e_{j_{k'_2}}$ on the right hand side of equation (2.9). Continuing in this way, we always arrive at a contradiction. Hence, the claim, since ℓ is finite. □

Chapter 3

Gröbner Basis via Linkage for Schubert Patch Ideals, Kazhdan-Lusztig Ideals and Schubert Determinantal Ideals

In this chapter, a proof will be given of the fact that the essential minors form Gröbner bases for Schubert patch ideals under the term order $\succ$ defined in Section 2.7. This proof can be easily adapted to give a new proof of the known fact that the essential minors form Gröbner bases for Kazhdan-Lusztig ideals.

The material presented in this chapter is based on the student's work in [Ney21].

3.1 The Key Lemma

To show that the essential minors of a Schubert patch ideal form a Gröbner basis with respect to the term order $\succ$ defined in Section 2.7, we employ a technique called *"Gröbner basis via linkage"* [GMN13]. This technique involves an important lemma that gives a sufficient condition for a set of polynomials to form a Gröbner basis for the ideal it generates. The standard graded version of this lemma was originally given in [GMN13, Lemma 1.12] and can also be found in [FK20, Lemma 3.1]. In Lemma 3.1.1 below, we provide the multigraded version.

Lemma 3.1.1. *Let R be a positively $\mathbb{Z}^d$-graded polynomial ring over an arbitrary field $\mathbb{K}$. Let I, J and N be homogeneous ideals with respect to the positive grading of R by $\mathbb{Z}^d$ such that $N \subseteq I \cap J$. Let A, B and C be monomial ideals of R such that $C \subseteq A \subseteq in_\sigma(I)$, $B = in_\sigma(J)$ and $C = in_\sigma(N)$ for some term order σ. Suppose that there exists $\boldsymbol{e} \in \mathbb{Z}^d$ such that $(I/N)_{\boldsymbol{\ell}} \cong (J/N)_{\boldsymbol{\ell}-\boldsymbol{e}}$ and that $(A/C)_{\boldsymbol{\ell}} \cong (B/C)_{\boldsymbol{\ell}-\boldsymbol{e}}$ for all $\boldsymbol{\ell} \in \mathbb{Z}^d$. Then $A = in_\sigma(I)$.*

Proof. We proceed exactly as in the proof of [FK20, Lemma 3.1]. Consider, for instance, the sequence $0 \to I/N \overset{\phi}{\to} R/N \overset{\varphi}{\to} R/I \to 0$ of module homomorphisms $\phi : I/N \to R/N$ defined by $i + N \mapsto i + N$ and $\varphi : R/N \to R/I$ defined by $r + N \mapsto r + I$. The maps ϕ and φ are injective (one-to-one) and surjective, respectively. So, the image of ϕ is I/N and

$$\ker(\varphi) = \{r+N \mid \varphi(r+N) \in I\} = \{r+N \mid r \in I\} = I/N.$$

Therefore, the sequence $0 \to I/N \xrightarrow{\phi} R/N \xrightarrow{\varphi} R/I \to 0$ is a short exact sequence. Consequently, by the rank-nullity theorem for short exact sequences of vector spaces, we have $\dim(R/N) = \dim(R/I) + \dim(I/N)$. Similarly, we have $\dim(R/C) = \dim(R/A) + \dim(A/C)$. If we therefore set $H_I(\boldsymbol{e}) := \dim_{\mathbb{K}}\big((R/I)_{\boldsymbol{e}}\big)$, then for an arbitrary $\boldsymbol{\ell} \in \mathbb{Z}^d$, we have the following:

$$\begin{aligned}
H_I(\boldsymbol{\ell}) &= H_N(\boldsymbol{\ell}) - \dim_{\mathbb{K}}\big((I/N)_{\boldsymbol{\ell}}\big) && \big(0 \to I/N \to R/N \to R/I \to 0\big)\\
&= H_N(\boldsymbol{\ell}) - \dim_{\mathbb{K}}\big((J/N)_{\boldsymbol{\ell}-\boldsymbol{e}}\big) && \big((I/N)_{\boldsymbol{\ell}} \cong (J/N)_{\boldsymbol{\ell}-\boldsymbol{e}}\big)\\
&= H_N(\boldsymbol{\ell}) - \big(H_N(\boldsymbol{\ell}-\boldsymbol{e}) - H_J(\boldsymbol{\ell}-\boldsymbol{e})\big) && \big(0 \to J/N \to R/N \to R/J \to 0\big)\\
&= H_C(\boldsymbol{\ell}) - \big(H_C(\boldsymbol{\ell}-\boldsymbol{e}) - H_B(\boldsymbol{\ell}-\boldsymbol{e})\big) && \big(B = \mathrm{in}_\sigma(J) \text{ and } C = \mathrm{in}_\sigma(N)\big)\\
&= H_C(\boldsymbol{\ell}) - \dim_{\mathbb{K}}\big((B/C)_{\boldsymbol{\ell}-\boldsymbol{e}}\big) && \big(0 \to B/C \to R/C \to R/B \to 0\big)\\
&= H_C(\boldsymbol{\ell}) - \dim_{\mathbb{K}}\big((A/C)_{\boldsymbol{\ell}}\big) && \big((A/C)_{\boldsymbol{\ell}} \cong (B/C)_{\boldsymbol{\ell}-\boldsymbol{e}}\big)\\
&= H_A(\boldsymbol{\ell}) && \big(0 \to A/C \to R/C \to R/A \to 0\big).
\end{aligned}$$

Since $A \subseteq \mathrm{in}_\sigma(I)$, we have $H_A(\boldsymbol{\ell}) \leq H_{\mathrm{in}_\sigma(I)}(\boldsymbol{\ell})$ for all $\boldsymbol{\ell} \in \mathbb{Z}^d$. So, for all $\boldsymbol{\ell} \in \mathbb{Z}^d$, we have $H_A(\boldsymbol{\ell}) \leq H_{\mathrm{in}_\sigma(I)}(\boldsymbol{\ell}) = H_I(\boldsymbol{\ell}) = H_A(\boldsymbol{\ell})$ Therefore, $H_A(\boldsymbol{\ell}) = H_{\mathrm{in}_\sigma(I)}(\boldsymbol{\ell})$, for all $\boldsymbol{\ell} \in \mathbb{Z}^d$. Hence, $A = \mathrm{in}_\sigma(I)$. □

3.2 Gröbner Basis for Schubert Patch Ideals

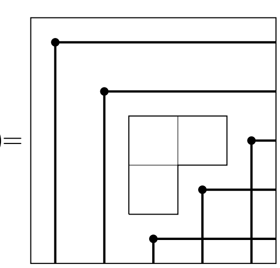
Recall from Section 2.7 that if b is the last descent of a permutation $v \in S_n$, then b is the maximum of all integers i, $1 \leq i < n$, such that $v(i) > v(i+1)$, and the matrix $\mathbf{Z}^{(v)}s_b$ is the resulting matrix from swapping columns b and $b+1$ of matrix $\mathbf{Z}^{(v)}$.

Definition 3.2.1. Let $v, w \in S_n$ and b be the last descent of v. Define an ideal $T_{vs_b,w} \subseteq \mathbb{K}[\mathbf{z}^{(v)}]$ as follows:

$$T_{vs_b,w} = \langle \text{minors of size } 1 + \operatorname{rank}(w_{p\times q}) \text{ in } \big(\mathbf{Z}^{(v)}s_b\big)_{p\times q},\ \ 1 \leq p,q \leq n\rangle.$$

Example 3.2.2. Let $v = 45312$ as in Example 2.7.5 and $w = 12543$. The last descent of v is $b = 3$. Matrices $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(v)}s_3$, the Rothe diagram $D(w)$ and ideal $T_{vs_3,w}$ are given below:

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15}\\ z_{21} & z_{22} & z_{23} & 0 & 1\\ z_{31} & z_{32} & 1 & 0 & 0\\ 1 & z_{42} & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15}\\ z_{21} & z_{22} & 0 & z_{23} & 1\\ z_{31} & z_{32} & 0 & 1 & 0\\ 1 & z_{42} & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},\ D(w) =$$



and

$$T_{vs_3,w} = \left\langle \begin{vmatrix} z_{21} & z_{22}\\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23}\\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23}\\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22}\\ 1 & z_{42} \end{vmatrix}, \begin{vmatrix} z_{31} & z_{32}\\ 1 & z_{42} \end{vmatrix} \right\rangle.$$

□

Definition 3.2.3. Let $v \in S_n$ and b be the last descent of v. Define a map $\varphi : \mathbb{K}[\mathbf{z}^{(vs_b)}] \to \mathbb{K}[\mathbf{z}^{(v)}]$ by $z_{i,b+1} \mapsto z_{i,b}$, $z_{i,b} \mapsto z_{i,b+1}$ and $z_{i,j} \mapsto z_{i,j}$, if $j \neq b, b+1$, i.e., for $f \in \mathbb{K}[\mathbf{z}^{(vs_b)}]$, $\varphi(f)$ is obtained by simultaneous substitution of $z_{i,b}$ for $z_{i,b+1}$ and $z_{i,b+1}$ for $z_{i,b}$, while other variables remain unchanged.

Example 3.2.4. Let $v = 45312$, as in Example 2.7.5. The last descent of v is $b = 3$. Matrices $\mathbf{Z}^{(vs_b)}$ and $\mathbf{Z}^{(v)}s_b$ are given below:

$$\mathbf{Z}^{(vs_3)} = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{14} & z_{15} \\ z_{21} & z_{22} & 0 & z_{24} & 1 \\ z_{31} & z_{32} & 0 & 1 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15} \\ z_{21} & z_{22} & 0 & z_{23} & 1 \\ z_{31} & z_{32} & 0 & 1 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that both matrices are the same up to applying the map φ, in the definition above, to all variables in the matrix $\mathbf{Z}^{(vs_3)}$. □

Example 3.2.5. Continuing with Examples 3.2.2 and 3.2.4, the Schubert patch ideal $Q_{vs_3,w}$ is given below:

$$Q_{vs_3,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{24} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{42} \end{vmatrix}, \begin{vmatrix} z_{31} & z_{32} \\ 1 & z_{42} \end{vmatrix} \right\rangle.$$

Note that the ideal $T_{vs_3,w}$ (as in Example 3.2.2) is the resulting ideal from applying the substitution map in Definition 3.2.3 to each of the generators of $Q_{vs_3,w}$. □

Remark 3.2.6. Following Definition 3.2.3, and observations from Examples 3.2.4 and 3.2.5, we claim that for $v, w \in S_n$, f is a generator of $Q_{vs_b,w}$ (resp. Q_{vs_b,ws_b}) if and only if $\varphi(f)$ is a generator of $T_{vs_b,w}$ (resp. T_{vs_b,ws_b}), where b is the last descent of v. To see this, let $\varphi(M)$ be the resulting matrix from applying φ to all variables in M. Then $\varphi(\mathbf{Z}^{(vs_b)}) = \mathbf{Z}^{(v)}s_b$. So, by definition of $Q_{vs_b,w}$, f is a generator of $Q_{vs_b,w}$ if and only if f is a minor of size $1 + \operatorname{rank}(w_{p\times q})$ in $\mathbf{Z}^{(vs_b)}_{p\times q}$, for some p, q, if and only if $\varphi(f)$ is a minor of size $1 + \operatorname{rank}(w_{p\times q})$ in $\left(\varphi(\mathbf{Z}^{(vs_b)})\right)_{p\times q}$, for some p, q, if and only if $\varphi(f)$ is a minor of size $1 + \operatorname{rank}(w_{p\times q})$ in $\left(\mathbf{Z}^{(v)}s_b\right)_{p\times q}$, for some p, q, if and only if $\varphi(f)$ is a generator of $T_{vs_b,w}$, by definition of $T_{vs_b,w}$. Hence, the claim. Similarly, f is a generator of Q_{vs_b,ws_b} if and only if $\varphi(f)$ is a generator of T_{vs_b,ws_b}. □

Lemma 3.2.7. *Let $v, w \in S_n$ and b be the last descent of v. If b is an ascent of w, then the ideals $Q_{v,w}$ and $T_{vs_b,w}$ are equal.*

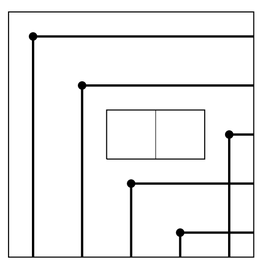
Proof. Consider all locations of essential boxes in $D(w)$. Since b is an ascent of w, it follows that there is no essential box in column b of $D(w)$. For any essential box (p, q), $q \neq b$, in $D(w)$, up to sign, set of minors of size $1 + \operatorname{rank}(w_{p\times q})$ in $\mathbf{Z}^{(v)}_{p\times q}$ is equal to set of minors of size $1 + \operatorname{rank}(w_{p\times q})$ in $\left(\mathbf{Z}^{(v)}s_b\right)_{p\times q}$, for all p, i.e., both ideals $Q_{v,w}$ and $T_{vs_b,w}$ have the same essential minors, up to sign, for the locations of essential boxes not in column b of $D(w)$. This is true since the matrices $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(v)}s_b$ are the same up to rearranging columns. Precisely, columns b (resp. $b+1$) in $\mathbf{Z}^{(v)}$ is the same as column $b+1$ (resp. b) in $\mathbf{Z}^{(v)}s_b$. □

Remark 3.2.8. Following up on the result above, let v, w be permutations for which the last descent b of v is an ascent of w. It turns out that, in general, both ideals $Q_{v,w}$ and $T_{vs_b,w}$ can be generated by essential minors that do not involve the variable $z_{\max}$ at position $(v(b+1), b)$ of $\mathbf{Z}^{(v)}$. To see this, since there is no essential box in column b of $D(w)$ (b being an ascent of w), it suffices to consider the essential boxes (α, β) of $D(w)$ for which $\alpha \geq a$ and $\beta \geq b+1$.

Suppose $\alpha \geq a$, $\beta \geq b+1$ and set $M := \mathbf{Z}^{(v)}_{\alpha\times\beta}$. We have that $M_{a,b+1} = 1$, $M_{i,b+1} = z_{i,b+1}$ for all $1 \leq i < a$ and $M_{i,b+1} = 0$ for all $a < i \leq \alpha$. For each of the variables $z_{i,b+1}$, $1 \leq i \leq a-1$, there is a 1 at position (i, j_i) of $\mathbf{Z}^{(v)}$, for some $j_i < b$. So, there are a 1s in the submatrix of M formed by rows $1, \ldots, a-1, a$ and columns $j_1, \ldots, j_{a-1}, b+1$. Then, by part (3) of Lemma 2.7.6, the ideal I generated by the minors of size $1 + \operatorname{rank}(w_{\alpha\times\beta})$ in M can be generated by a set consisting of some (essential) minors of size $1 + \operatorname{rank}(w_{\alpha\times\beta})$ in M, none of which involves, in particular, the variable $z_{\max} = z_{ab}$. Therefore, $Q_{v,w}$ can be generated by essential minors that do not involve the variable $z_{\max}$.

Suppose $\alpha \geq a$, $\beta \geq b+1$ and set $M := \left(\mathbf{Z}^{(v)} s_b\right)_{\alpha\times\beta}$. We have that $M_{ab} = 1$, $M_{i,b} = z_{i,b+1}$ for all $1 \leq i < a$ and $M_{i,b} = 0$ for all $a < i \geq \alpha$. And similar to above argument, there are a number of 1s in the submatrix of M formed by rows $1, \ldots, a-1, a$ and columns $j_1, \ldots, j_{a-1}, b$, and consequently, $T_{vs_b,w}$ can be generated by essential minors that do not involve the variable $z_{\max}$. □

Example 3.2.9. Let $v = 45312$, as in Example 2.7.5 and $w = 12453$. Here, the last descent of v is $b = 3$, which is an ascent of w. Consequently, the ideals $Q_{v,w}$ and $T_{vs_3,w}$ are equal by Lemma 3.2.7. To see this, the Rothe diagram $D(w)$, and ideals $Q_{v,w}$ and $T_{vs_3,w}$ are given below:

$D(w) =$ 

and

$$Q_{v,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & 1 \end{vmatrix} \right\rangle = T_{vs_3,w}.$$

□

Definition 3.2.10. Let M be an $m \times n$ matrix. The $(m - \ell_1) \times (n - \ell_2)$ submatrix of M formed by deleting rows $i_1, \ldots, i_{\ell_1}$ and columns $j_1, \ldots, j_{\ell_2}$ of M will be denoted by $c_{i_1,\ldots,i_{\ell_1};j_1,\ldots,j_{\ell_2}}(M)$. If only rows $i_1, \ldots, i_{\ell_1}$ (resp. only columns $j_1, \ldots, j_{\ell_2}$) of M are deleted, then we write $c_{i_1,\ldots,i_{\ell_1};}(M)$ (res. $c_{;j_1,\ldots,j_{\ell_2}}(M)$) to represent the resulting matrix.

The following result aids the proof of Lemma 3.2.13 in this dissertation.

Lemma 3.2.11. *Let $v, w \in S_n$ for which the last descent of v is a descent of w. Let b be the last descent of v and (α, b), $\alpha \geq v(b+1)$, be a location for an essential box in column b, on or below row $v(b+1)$ of $D(w)$. Then the ideal generated by minors of size rank$(w_{\alpha\times b})$ in $c_{v(b+1);b}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$ is equal to the ideal generated by minors of size $1 + $rank$(w_{\alpha\times b})$ in $\big(\mathbf{Z}^{(v)} s_b\big)_{\alpha\times b}$.*

Proof. Set $a := v(b+1)$. Let I be the ideal generated by minors of size $t := 1+\text{rank}(w_{\alpha\times b})$ in $\big(\mathbf{Z}^{(v)} s_b\big)_{\alpha\times b}$ and I' be the ideal generated by minors of size $t-1$ in $c_{a;b}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$. We claim that $I = I'$. Set $M := \big(\mathbf{Z}^{(v)} s_b\big)_{\alpha\times b}$. Then we observe that $M_{ab} = 1$, $M_{ib} = z_{i,b+1}$ for all $i < a$ and $M_{ib} = 0$ for all $i > a$. As a result, for some j_is, there are a 1s in the submatrix formed by rows $1, \ldots, a$ and columns $j_1, \ldots, j_{a-1}, b$ of M. Then, by part (1) of Lemma 2.7.6, the ideal I is equal to the ideal generated by minors of size $t-a$ in $G := c_{1,\ldots,a;j_1,\ldots,j_{a-1},b}(M)$. Observe that the matrix M has one row and one column more than the matrix $c_{a,b}(M')$, where $M' := \mathbf{Z}^{(v)}_{\alpha\times b}$; precisely, $c_{a;b}(M') = c_{a;b}(M)$. So there are $(a-1)$ 1s in the submatrix formed by rows $1, \ldots, (a-1)$ and columns $j_1, \ldots, j_{a-1}$ of $c_{a;b}(M')$. Therefore, by part (1) of Lemma 2.7.6, the ideal I' is equal to the ideal generated by minors of size $(t-1)-(a-1) = t-a$ in $G' := c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}\big(c_{a;b}(M')\big) = c_{1,\ldots,a;j_1,\ldots,j_{a-1},b}(M')$. In summary, I is the ideal generated by minors of size $t-a$ in G and I' is the ideal generated by minors of size $t-a$ in G'. Since $G = G'$, it follows that $I = I'$. Note that if $t = a$, then both I and I' are unit ideal. □

Example 3.2.12. Let $v = 45312$ and $w = 12543$, as in Examples 2.7.5 and 3.2.2. The last descent of v is $b = 3$, which is a descent for w, and $v(b+1) = 1$. For the essential box $(4, 3)$ in $D(w)$, consider the upper left 4×3 rectangular submatrix $\mathbf{Z}^{(v)}_{4\times 3}$ of $\mathbf{Z}^{(v)}$:

$$\mathbf{Z}^{(v)}_{4\times 3} = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & 1 \\ 1 & z_{42} & 0 \end{pmatrix}, \qquad \text{where} \qquad \mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & z_{23} & 0 & 1 \\ z_{31} & z_{32} & 1 & 0 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The ideal generated by minors of size $\text{rank}(w_{4\times 3}) = 2$ in $c_{1;3}\big(\mathbf{Z}^{(v)}_{4\times 3}\big) = \begin{pmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \\ 1 & z_{42} \end{pmatrix}$ is equal to the ideal generated by minors of size $1 + \text{rank}(w_{4\times 3}) = 3$ in $\big(\mathbf{Z}^{(v)} s_3\big)_{4\times 3} = \begin{pmatrix} z_{11} & z_{12} & 1 \\ z_{21} & z_{22} & 0 \\ z_{31} & z_{32} & 0 \\ 1 & z_{42} & 0 \end{pmatrix}$. □

The following result gives a connection between the ideals $Q_{v,w}$ and $T_{vs_b,w}$ in terms of their generators.

Lemma 3.2.13. *Let $v, w \in S_n$ for which the last descent b of v is a descent of w. If we write*

$$Q_{v,w} = \langle z_{max} g_1 + r_1, \ldots, z_{max} g_k + r_k, h_1, \ldots, h_\ell \rangle, \tag{3.1}$$

where the set $\mathcal{G} := \{z_{max}g_1 + r_1, \ldots, z_{max}g_k + r_k, h_1, \ldots, h_\ell\}$ is a complete list of all essential minors in $Q_{v,w}$ and z_{max} does not divide any term of g_i or r_i for any $1 \leq i \leq k$ nor any h_j for $1 \leq j \leq \ell$, then

$$T_{vs_b,w} = \langle g_1, \ldots, g_k, h_1, \ldots, h_\ell \rangle .$$

Proof. Set $a := v(b+1)$ and $y := z_{\max}$. The integer a is the row where y is located in $\mathbf{Z}^{(v)}$, by Lemma 2.7.2. Given an arbitrary essential box (α, β) in $D(w)$, the minors of size $1 + \operatorname{rank}(w_{\alpha\times\beta})$ in $\mathbf{Z}^{(v)}_{\alpha\times\beta}$ are in $\mathcal{G}$, and for each of these essential minors $yg_i + r_i$ or h_j of size $1 + \operatorname{rank}(w_{\alpha\times\beta})$ in $\mathcal{G}$, there exists corresponding minors g_i or h_j in the set $\mathcal{G}' := \{g_1, \ldots, g_k, h_1, \ldots, h_\ell\}$. We wish to show that each corresponding minor g_i or h_j belongs to $T_{vs_b,w}$, i.e., we will show that $\langle g_1, \ldots, g_k, h_1, \ldots, h_\ell\rangle$ is contained in $T_{vs_b,w}$. To this end, we will consider all possible locations of essential boxes in $D(w)$.

Let (α, β) be an essential box in $D(w)$ for which $\beta \neq b$.

If $\beta < b$, then for any α, the essential minors of size $1 + \operatorname{rank}(w_{\alpha\times\beta})$ in $\mathbf{Z}^{(v)}_{\alpha\times\beta}$ do not involve the variable y, and so they are some of the h_is in $Q_{v,w}$. These essential minors h_is belong to $Q_{v,w}$ if and only if they belong to $T_{vs_b,w}$, since $\mathbf{Z}^{(v)}_{\alpha\times\beta} = \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times\beta}$ in this case.

Suppose $\beta > b$ and $\alpha < a$. This is similar to the previous case, just that here, we have the equality $\mathbf{Z}^{(v)}_{\alpha\times\beta} = \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times\beta}$, up to swapping of columns b and $b+1$.

Suppose $\beta > b$, $\alpha \geq a$ and set $M := \mathbf{Z}^{(v)}_{\alpha\times\beta}$. Then we observe that $M_{a,b+1} = 1$, $M_{i,b+1} = z_{i,b+1}$ for all $i < a$ and $M_{i,b+1} = 0$ for all $i > a$. So, for some j_is, there are a 1s in the submatrix formed by rows $1, \ldots, a$ and columns $j_1, \ldots, j_{a-1}, b+1$ of M. Then, by part (3) of Lemma 2.7.6, the ideal I generated by the minors of size $1+\operatorname{rank}(w_{\alpha\times\beta})$ in M can be generated by a set $\mathcal{G}$ consisting of some essential minors of size $1+\operatorname{rank}(w_{\alpha\times\beta})$ in M, none of which involves, in particular, the variable y. Therefore, these essential minors in $\mathcal{G}$ are also some of the h_is in $Q_{v,w}$. These essential minors h_is belong to $Q_{v,w}$ if and only if they belong to $T_{vs_b,w}$, since, up to swapping columns b and $b+1$, $\mathbf{Z}^{(v)}_{\alpha\times\beta} = \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times\beta}$ in this case.

On the other hand, let (α, β) be an essential box in $D(w)$ for which $\beta = b$.

If $\alpha < a$, then up to rearranging columns, the submatrix $\mathbf{Z}^{(v)}_{\alpha\times b-1} = \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times b-1}$ contains an $\alpha\times\alpha$ upper triangular matrix with 1s on the diagonal; reason being that each variable z_{ib}, $1 \leq i \leq \alpha$, has 1 to its left in any of these submatrices. Hence, there exists a minor corresponding to this essential box (α, b) in $Q_{v,w}$ and $T_{vs_b,w}$ that is equal to 1.

If $\alpha \geq a$, i.e., (α, b) is an essential box in column b, on or below row a of $D(w)$, and D is any of the essential minors in $Q_{v,w}$ associated to (α, b), then D either involves row a, but not column b of $\mathbf{Z}^{(v)}$, or involves column b, but not row a of $\mathbf{Z}^{(v)}$, or involves both row a and column b of $\mathbf{Z}^{(v)}$, or does not involve both row a and column b of $\mathbf{Z}^{(v)}$. These four subcases are considered below, where $s := 1 + \operatorname{rank}(w_{\alpha\times b})$.

We combine the first and last subcases as one. Suppose M is an $s \times s$ submatrix of $\mathbf{Z}^{(v)}_{\alpha\times b}$ that does not involve column b of $\mathbf{Z}^{(v)}$, and set $D := \det(M)$. Then M is also a submatrix of $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$, and so D is an essential minor in $Q_{v,w}$ if and only if it is an essential minor in $T_{vs_b,w}$; precisely, $D = h_i$, for some i, since it does not involve y.

Suppose M is an $s \times s$ submatrix of $\mathbf{Z}^{(v)}_{\alpha\times b}$ that involves only column b, but not row a of $\mathbf{Z}^{(v)}$, and set $D := \det(M)$. Note that D is one of the h_is in $Q_{v,w}$ since it does not involve y. Expanding D along this column b of $\mathbf{Z}^{(v)}$ in M, it follows that D belongs to the ideal generated by minors of size $s-1 = \operatorname{rank}(w_{\alpha\times b})$ in $c_{;b}(M)$. Since $c_{;b}(M)$ is a submatrix of $c_{a;b}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$, it follows that D belongs to the ideal generated by minors of size $s-1$ in $c_{a;b}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$. Consequently, by Lemma 3.2.11, D belongs to the ideal generated by minors of size $1+\operatorname{rank}(w_{\alpha\times b})$ in $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$, i.e., D belongs to $T_{vs_b,w}$.

Next, suppose M is an $s \times s$ submatrix of $\mathbf{Z}^{(v)}_{\alpha\times b}$ that involves both row a and column b of $\mathbf{Z}^{(v)}$, and set $D := \det(M)$. In other words, D involves the variable y, and so can be written in the form $D = yg + r$, where y does not divide any term of g or r. In this case, since D belongs to the ideal generated by the minors of size $s = 1+\operatorname{rank}(w_{\alpha\times b})$ in $\mathbf{Z}^{(v)}_{\alpha\times b}$, it follows that g belongs to the ideal generated by the minors of size $s-1 = \operatorname{rank}(w_{\alpha\times b})$ in $c_{a;b}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$. Consequently, by Lemma 3.2.11, g belongs to the ideal generated by the minors of size $1+\operatorname{rank}(w_{\alpha\times b})$ in $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$, and so g belongs to $T_{vs_b,w}$.

So far, we have been able to show that the essential minors h_is in $Q_{v,w}$ remain the same in $T_{vs_b,w}$ and g_i belongs to $T_{vs_b,w}$ for each essential minor $yg_i + r_i$ in $Q_{v,w}$, where y does not divide any term of g_i or r_i. Hence, $\langle g_1,\ldots,g_k,h_1,\ldots,h_\ell\rangle \subseteq T_{vs_b,w}$.

To show the other inclusion $T_{vs_b,w} \subseteq \langle g_1,\ldots,g_k,h_1,\ldots,h_\ell\rangle$, we will consider all possible locations of essential boxes in $D(w)$, and compare their corresponding minors in $T_{vs_b,w}$ to the ones in $Q_{v,w}$. To do this, it suffices to consider the last two cases above – we will consider any essential minor D in $T_{vs_b,w}$ associated to an essential box (α,b) in column b, on or below row a of $D(w)$ $(\alpha \geq a)$ and for which D either involves column b, but not row a of $\mathbf{Z}^{(v)}s_b$, or involves both row a and column b of $\mathbf{Z}^{(v)}s_b$. This is enough to be shown since other cases will have same setup as above. Set $s := 1+\operatorname{rank}(w_{\alpha\times b})$.

First, suppose M is an $s \times s$ submatrix of $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$ that involves only column b, but not row a of $\mathbf{Z}^{(v)}s_b$, and set $D := \det(M)$. We claim that D belongs to the ideal generated by the h_is. Assume M uses at least one row from the rows above row a of $\mathbf{Z}^{(v)}s_b$; otherwise, $D = 0$ since the entries below the 1 at position (a,b) of $\mathbf{Z}^{(v)}s_b$ are all zero. Since M is a submatrix of $M' := c_{a;}\big(\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}\big)$, it follows that D belongs to the ideal I generated by minors of size s in M'. Observe that each variable $z_{i,b+1}$, $1 \leq i < a$, above the 1 at position (a,b) of $\mathbf{Z}^{(v)}s_b$ has 1 to its left in $\mathbf{Z}^{(v)}s_b$. So, for some j_is, there are $(a-1)$ 1s in the submatrix formed by rows $1,\ldots,(a-1)$ and columns $j_1,\ldots,j_{a-1}$ of M'. Therefore, by part (1) of Lemma 2.7.6, the ideal I is equal to the ideal generated by the minors of size $s-(a-1)$ in $c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}(M') = c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}\left(c_{a;}\big(\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}\big)\right) = c_{1,\ldots,a;j_1,\ldots,j_{a-1}}\big(\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}\big)$. Even more, this ideal I is equal to the ideal generated by the minors of size $s-(a-1)$ in $G := c_{1,\dots,a;j_1,\dots,j_{a-1},b}\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}\big)$, since the remaining entries in the last column of $c_{1,\dots,a;j_1,\dots,j_{a-1}}\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}\big)$ are all zero. Now, we consider the ideal I' generated by minors of size s in $M'' := c_{a;}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$. The aforementioned $(a-1)$ 1s in $\mathbf{Z}^{(v)}s_b$ remain at the same locations in $\mathbf{Z}^{(v)}$ as $\mathbf{Z}^{(v)}s_b$. Therefore, by part (1) of Lemma 2.7.6, the ideal I' is equal to the ideal generated by the minors of size $s-(a-1)$ in $G' := c_{1,\dots,a-1;j_1,\dots,j_{a-1}}\big(M''\big) = c_{1,\dots,a;j_1,\dots,j_{a-1}}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$. Observe that G is a submatrix of G', which implies $I \subseteq I'$, by definitions of I and I'. Hence, D belongs to I'. Even more, by part (2) of Lemma 2.7.6, the ideal I' can be generated by some (essential) minors of size s in M'', and none of these minors involves y, since row a of $\mathbf{Z}^{(v)}_{\alpha\times b}$ is not involved in the submatrix M''. Therefore, D belongs to the ideal generated by the h_is.

Lastly, suppose M is an $s\times s$ submatrix of $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$ that involves both row a and column b of $\mathbf{Z}^{(v)}s_b$, and set $D := \det(M)$. We claim that D belongs to the ideal generated by the g_is. Since M is a submatrix of $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$, it follows that D belongs to the ideal I generated by minors of size s in $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$. For some j_is, there are a 1s in the submatrix formed by rows $1,\dots,a$ and columns $j_1,\dots,j_{a-1},b$ of $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}$. Therefore, by part (1) of Lemma 2.7.6, the ideal I is equal to the ideal generated by the minors of size $s-a$ in $M' := c_{1,\dots,a;j_1,\dots,j_{a-1},b}\big(\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times b}\big)$. Now, let M'' be any $(s-a)\times(s-a)$ submatrix of M'. Since M' is a submatrix of $G' := c_{1,\dots,a-1;j_1,\dots,j_{a-1}}\big(\mathbf{Z}^{(v)}_{\alpha\times b}\big)$, it follows that M'' is also a submatrix of G'. So, considering an $(s-a+1)\times(s-a+1)$ matrix $M''' = \begin{pmatrix} *_1 & y \\ M'' & *_2 \end{pmatrix}$, with entries $*_1$ and $*_2$ chosen such that M''' is a submatrix of G', we have that $\det(M''')$ belongs to the ideal I' generated by minors of size s in $\mathbf{Z}^{(v)}_{\alpha\times b}$, and hence, $\det(M''')$ belongs to $Q_{v,w}$; reason being that using the 1s in the submatrix formed by rows $1,\dots,a-1$ and columns $j_1,\dots,j_{a-1}$ of $\mathbf{Z}^{(v)}_{\alpha\times b}$, we have from part (1) of Lemma 2.7.6 that the ideal I' is equal to the ideal generated by minors of size $s-a+1$ in G'. Since $\det(M''')$ belongs to $Q_{v,w}$ and $\det(M''') = y\det(M'') + r$, with none of the terms of $\det(M'')$ and r divisible by y, it follows that $\det(M'')$ is one of the g_is. Hence, the claim. $\square$

Example 3.2.14. Continuing with Example 2.7.5 and Example 3.2.2, where $v = 45312$ and $w = 12543$, the last descent of v occurs at $b = 3$, which is a descent for w. The ideal $Q_{v,w}$ is given below:

$$Q_{v,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ 1 & z_{42} & 0 \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{31} & z_{32} & 1 \\ 1 & z_{42} & 0 \end{vmatrix} \right\rangle.$$

The last two generators of $Q_{v,w}$ can be written as $z_{13}g_1 + r_1$ and $z_{13}g_2 + r_2$, where $g_1 = z_{22} - z_{21}z_{42}$, $g_2 = z_{32} - z_{31}z_{42}$, $r_1 = z_{12}z_{23} - z_{11}z_{23}z_{42}$ and $r_2 = z_{12} - z_{11}z_{42}$. Following Lemma 3.2.13, g_1, g_2 and the first three generators of $Q_{v,w}$ generate the ideal $T_{vs_3,w}$. The ideal $T_{vs_3,w}$, as computed in Example 3.2.2, is given below:

$$T_{vs_3,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{42} \end{vmatrix}, \begin{vmatrix} z_{31} & z_{32} \\ 1 & z_{42} \end{vmatrix} \right\rangle.$$

□

By Lemma 2.8.2, $Q_{v,w}$, $v, w \in S_n$, is homogeneous with respect to a positive multigrading of $\mathbb{K}[\mathbf{z}^{(v)}]$ by $\mathbb{Z}^n$, where each variable z_{ij} has degree $e_{v^{-1}(i)} - e_j$. In the following result, $T_{vs_b,w}$ is shown to be homogeneous with respect to this positive multigrading of $\mathbb{K}[\mathbf{z}^{(v)}]$ by $\mathbb{Z}^n$.

Lemma 3.2.15. *Let $v, w \in S_n$ and b be the last descent of v. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{Z}^n$. Under the multigrading where the variable z_{ij} has degree $e_{v^{-1}(i)} - e_j$, the ideal $T_{vs_b,w}$ is homogeneous.*

Proof. Following Lemma 3.2.13, it suffices to show that for each essential minor $z_{\max}g + r \in Q_{v,w}$, the polynomial $g \in T_{vs_b,w}$ is homogeneous with respect to the required multigrading. Let $f = z_{\max}g + r$ be an arbitrary essential minor in $Q_{v,w}$. Then by Lemma 2.8.2, f is homogeneous with respect to the required multigrading. As a result, both $z_{\max}g$ and r are of the same degree under this multigrading, i.e., they are both homogeneous with respect to the required multigrading. Therefore, g is homogeneous with respect to the required multigrading and its degree is $\deg(g) = \deg(f) - \deg(z_{\max})$. □

The following result gives a connection between the ideals $Q_{v,w}$ and T_{vs_b,ws_b} in terms of their generators.

Lemma 3.2.16. *Under the same hypotheses as Lemma 3.2.13, we have*

$$T_{vs_b,ws_b} = \langle h_1, \ldots, h_\ell \rangle .$$

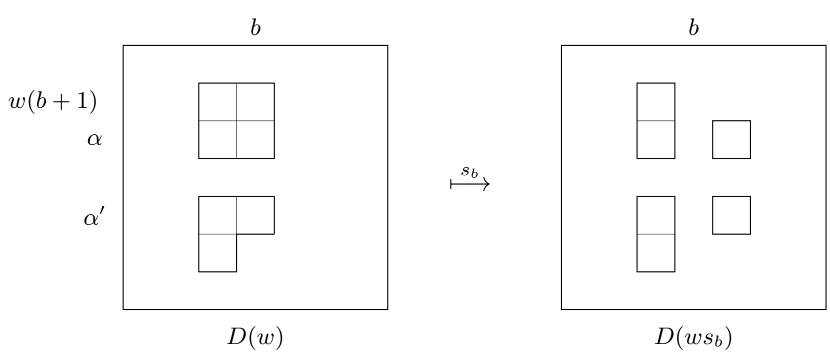
Proof. Set $a := v(b+1)$, $w' := ws_b$ and $y := z_{\max}$. Recall that y is at the position (a, b) of $\mathbf{Z}^{(v)}$. We will first show that $T_{vs_b,w'} \subseteq \langle h_1, \ldots, h_\ell \rangle$ by examining all possible locations of the essential boxes in $D(w')$ and their corresponding minors. Both $D(w)$ and $D(w')$ agree in all columns except for columns b and $b+1$, by Lemma 2.5.14.

If there is an essential box in column b of $D(w')$, then it must be at position (α, b) in $D(w')$, for some $\alpha < w'(b)$. Since $w'(b) = w(b+1)$, it follows that $\alpha < w(b+1)$ and so this box (α, b) in $D(w')$ is the same box at location (α, b) in $D(w)$, by Lemma 2.5.14. If (α, b) is an essential box in $D(w)$, then it follows, by definition of essential boxes, that there is no box at location $(\alpha, b+1)$ of $D(w)$. So at the location $(\alpha, b+1)$ of the Rothe diagram $D(w)$, we expect either a dot • to be at this location or a "hook" to pass down through it. Since b is a descent of w, it follows that there is a dot • in column $b+1$ of $D(w)$ at row $w(b+1)$, and consequently, there cannot be another dot • in column $b+1$ of $D(w)$. Also, there cannot be a "hook" passing down through the location $(\alpha, b+1)$ of $D(w)$. Therefore, there cannot be an essential box in column b of $D(w')$.

Now, we consider the essential boxes that are on column $b+1$ of $D(w')$ strictly below row $w(b+1)$. For any essential box (α, b) in column b of $D(w)$ that is strictly below row $w(b+1)$ (i.e., $\alpha > w(b+1)$), we have $\text{rank}({w'}_{\alpha\times(b+1)}) = 1 + \text{rank}(w_{\alpha\times b})$. Then, since $\mathbf{Z}^{(v)}_{\alpha\times b}$ is a submatrix of $\mathbf{Z}^{(vs_b)}_{\alpha\times(b+1)}$, one observes that the ideal generated by minors of size $t := 1 + \text{rank}({w'}_{\alpha\times(b+1)})$ in $\left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times(b+1)}$ is contained in the ideal generated by minors of size $t-1 = 1+\text{rank}(w_{\alpha\times b})$ in $\mathbf{Z}^{(v)}_{\alpha\times b}$, and consequently, all minors corresponding to the essential boxes on column $b+1$ of $D(w')$ strictly below row $w(b+1)$ belong to $Q_{v,w}$. In particular, suppose the first a (counting from the top) 1s in $M := \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times(b+1)}$ are located in column $j_1,\ldots,j_{a-1},b$ of M. Then by part (3) of Lemma 2.7.6, none of the minors of size $t-a$ in $G := c_{1,\ldots,a;j_1,\ldots,j_{a-1},b}(M)$ involves, in particular, $y = z_{ab}$. Recall that $M' := \mathbf{Z}^{(v)}_{\alpha\times b}$ is a submatrix of M; precisely, M and M' differ by column b of M. The matrix M' therefore has $(a-1)$ 1s in its submatrix formed by rows $1,\ldots,(a-1)$ and columns $j_1,\ldots,j_{a-1}$. So by cofactor expansion about these rows and corresponding columns in M', we obtain minors of size $1+\text{rank}(w_{\alpha\times b})-(a-1) = (t-1)-a+1 = t-a$ in $G' := c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}(M')$. Observe that G' differs from G by its first row, and so, G is a submatrix of G'. Consequently, since the same size of minors is taken in both matrices G and G', it follows that the set of all minors of size $t-a$ in G is contained in the set of all minors of size $t-a$ in G'. Therefore, these minors of size $t-a$ in G are essential minors in $Q_{v,w}$ and since they do not involve y, we conclude that they are some of the h_is in $Q_{v,w}$.

Next, we consider any essential box, say (α,β), in $D(w')$ that is different from any of the essential boxes on column $b+1$ strictly below row $w(b+1)$. For any such essential box (α,β) in $D(w')$, we have, up to swapping columns, that the matrices $\mathbf{Z}^{(v)}_{\alpha\times\beta}$ and $\left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times\beta}$ are the same and $\text{rank}(w_{\alpha\times\beta}) = \text{rank}({w'}_{\alpha\times\beta})$. If $\beta < b$ with any α or $\beta > b$ and $\alpha < a$, the essential minors of size $1+\text{rank}\left(w'_{\alpha\times\beta}\right) = 1+\text{rank}\left(w_{\alpha\times\beta}\right)$ in $\left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times\beta} = \mathbf{Z}^{(v)}_{\alpha\times\beta}$ do not involve the variable y, and so they are some of the h_is in $Q_{v,w}$. If $\beta > b$ and $\alpha \geq a$, then just like this case in the proof of Lemma 3.2.13, by cofactor expansion, the ideal generated by the minors of size $1+\text{rank}(w_{\alpha\times\beta}) = 1+\text{rank}({w'}_{\alpha\times\beta})$ in $\mathbf{Z}^{(v)}_{\alpha\times\beta} = \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times\beta}$ (up to swapping columns b and $b+1$) can be generated by a set consisting of some essential minors in $\mathbf{Z}^{(v)}_{\alpha\times\beta}$, none of which involves, in particular, the variable y, and hence, are the some of the h_is in $Q_{v,w}$. Consequently, these essential minors h_is belong to $Q_{v,w}$ if and only if they belong to $T_{vs_b,w'}$. Therefore, in all, we obtain $T_{vs_b,ws_b} \subseteq \langle h_1,\ldots,h_\ell\rangle$.

To show the other inclusion $\langle h_1,\ldots,h_\ell\rangle \subseteq T_{vs_b,ws_b}$, we will consider all essential minors in $Q_{v,w}$ that do not involve y. To do this, it suffices to consider the essential boxes in column b, on or below row a of $D(w)$, and whose essential minors do not involve y. The diagrams below show locations of boxes in columns $b-1$ and b of a typical $D(w)$, and their corresponding locations in $D(ws_b)$.



Assume $w(b+1) \geq a$ and let (α,b) be the first essential box, from the top, in column b of $D(w)$. First, we claim that any essential minor in $Q_{v,w}$ associated to this essential box $(\alpha, b) \in D(w)$ and that does not involve y belongs to the set of all essential minors in T_{vs_b,ws_b} associated to the boxes $(\alpha, b-1), (\alpha, b+1) \in D(ws_b)$. To see this, we first observe that there are $(a-1)$ 1s in the submatrix formed by rows $1, \ldots, (a-1)$ and columns $j_1, \ldots, j_{a-1}$ of $M := \mathbf{Z}^{(v)}_{\alpha\times b}$, and so, by part (1) of Lemma 2.7.6, the minors of size $t := 1 + \text{rank}(w_{\alpha\times b})$ in M equals minors of size $t-(a-1) = t-a+1$ in $G := c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}(M)$.

The essential box $(\alpha, b+1)$ is an essential box in column $b+1$ of $D(w')$ strictly below row $w(b+1)$. Consequently, by cofactor expansion about the first a rows of $M' := \left(\mathbf{Z}^{(v)} s_b\right)_{\alpha\times(b+1)}$ and the columns where 1s are located in these first a rows, we have that the ideal I' generated by minors of size $1+\text{rank}({w'}_{\alpha\times(b+1)}) = 1+(1+\text{rank}(w_{\alpha\times b})) = 1+t$ in M' is equal to the ideal generated by minors of size $(1+t)-a = t-a+1$ in $G' := c_{1,\ldots,a;j_1,\ldots,j_{a-1},b}(M')$. Here, G differs from G' by its first row, which is row a of M. Consequently, any minor of size $t-a+1$ in G (which is an essential minor of size t in M, by part (2) of Lemma 2.7.6) and that does not involve row a (i.e., does not involve y, and hence, it is one of the h_is) is one of the minors of size $t-a+1$ in G', and hence, belongs to $I' \subseteq T_{vs_b,ws_b}$.

Furthermore, for the box $(\alpha, b-1)$ in $D(w)$ and $D(w')$, we have that $\mathbf{Z}^{(v)}_{\alpha\times b-1} = \left(\mathbf{Z}^{(v)} s_b\right)_{\alpha\times b-1}$ and $1 + \text{rank}\left(w'_{\alpha\times b-1}\right) = 1+\text{rank}\left(w_{\alpha\times b-1}\right) = 1+\text{rank}\left(w_{\alpha\times b}\right) = t$. Set $M'' := \mathbf{Z}^{(v)}_{\alpha\times b-1} = \left(\mathbf{Z}^{(v)} s_b\right)_{\alpha\times b-1}$. Using the $(a-1)$ 1s in the first $a-1$ rows of M'', it follows from part (1) of Lemma 2.7.6 that the ideal I'' generated by minors of size t in M'' is equal to the ideal generated by minors of size $t-a+1$ in $G'' := c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}(M'')$. Here, G differs from G'' by its last column, which is column b of M. Consequently, any minor of size $t-a+1$ in G (which is an essential minor of size t in M, by part (2) of Lemma 2.7.6) and that does not involve column b (i.e., does not involve y, and hence, it is one of the h_is) is one of the minors of size $t-a+1$ in G'', and hence, belongs to $I'' \subseteq T_{vs_b,ws_b}$. Hence, the first claim.

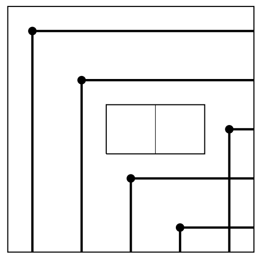
Second, we claim that any essential minor in $Q_{v,w}$ associated to the essential box $(\alpha', b) \in D(w)$ and that does not involve y belongs to the set of all minors in T_{vs_b,ws_b} associated to the boxes $(\alpha', b-1), (\alpha', b+1) \in D(ws_b)$. This is shown exactly the same way as the first claim. Repeating this same procedure for all other essential boxes in column b of $D(w)$, we obtain the desired inclusion: $\langle h_1, \ldots, h_\ell\rangle \subseteq T_{vs_b,ws_b}$. Therefore, the equality $T_{vs_b,ws_b} = \langle h_1, \ldots, h_\ell\rangle$. □

Example 3.2.17. Continuing with Examples 2.7.5 and 3.2.2, where $v = 45312$ and $w = 12543$, the last descent $b = 3$ of v is a descent for w. $Q_{v,w}$ is computed in Example 3.2.14:

$$Q_{v,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ 1 & z_{42} & 0 \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{31} & z_{32} & 1 \\ 1 & z_{42} & 0 \end{vmatrix} \right\rangle.$$

Following Lemma 3.2.16, the first three generators of $Q_{v,w}$ (generators that do not involve the variable

$z_{\max} = z_{13}$) generate the ideal T_{vs_3,ws_3}. The diagram $D(ws_3)$ and matrix $\mathbf{Z}^{(v)}s_3$ are given below:



and

$$\mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15} \\ z_{21} & z_{22} & 0 & z_{23} & 1 \\ z_{31} & z_{32} & 0 & 1 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Following Definition 3.2.1, the ideal T_{vs_3,ws_3} is generated by $\begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}$, $\begin{vmatrix} z_{21} & z_{23} \\ z_{31} & 1 \end{vmatrix}$ and $\begin{vmatrix} z_{22} & z_{23} \\ z_{32} & 1 \end{vmatrix}$. □

The following result is an immediate consequence of Lemma 3.2.13 and Lemma 3.2.16.

Corollary 3.2.18. *Let v, w be permutations in S_n for which the last descent of v is a descent of w. If b is the last descent of v, then $T_{vs_b,ws_b} \subseteq Q_{v,w} \cap T_{vs_b,w}$.*

By Lemma 2.8.2, the ideal $Q_{v,w}$, $v, w \in S_n$, is homogeneous with respect to a positive multigrading of $\mathbb{K}[\mathbf{z}^{(v)}]$ by $\mathbb{Z}^n$. If we set $y := z_{\max}$ and write $Q_{v,w}$ in the form $Q_{v,w} = \langle yg_1 + r_1, \dots, yg_k + r_k, h_1, \dots, h_\ell \rangle$, where the set $\{yg_1 + r_1, \dots, yg_k + r_k, h_1, \dots, h_\ell\}$ is a complete list of all essential minors in $Q_{v,w}$ and y does not divide any term of g_i or r_i for any $1 \le i \le k$ nor any h_j for $1 \le j \le \ell$, then each generator h_j is homogeneous with respect to this positive multigrading of $\mathbb{K}[\mathbf{z}^{(v)}]$ by $\mathbb{Z}^n$. Hence, the following result is a direct consequence of Lemma 2.8.2 and Lemma 3.2.16.

Corollary 3.2.19. *Let $v, w \in S_n$ and b be the last descent of v. Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{Z}^n$. Under the multigrading where the variable z_{ij} has degree $e_{v^{-1}(i)} - e_j$, the ideal T_{vs_b,ws_b} is homogeneous.*

Let P be a $k \times n$ matrix. For any ordered sequence $1 \le i_i < \cdots < i_k \le n$ of k integers, let $P_{[\,;i_1,\dots,i_k]}$ be the determinant of the $k \times k$ matrix whose columns are the $i_i, \dots, i_k$ columns of P. For any two ordered sequences $i_1 < i_2 < \cdots < i_{k-1}$ and $j_1 < j_2 < \cdots < j_{k+1}$ of positive integers $1 \le i_\ell, j_m \le n$, we have

$$\sum_{\ell=1}^{k+1} (-1)^\ell P_{[\,;i_1,\dots,i_{k-1},j_\ell]} P_{[\,;j_1,\dots,\widehat{j_\ell},\dots,j_{k+1}]} = 0, \tag{3.2}$$

where $j_1, \dots, \widehat{j_\ell}, \dots, j_{k+1}$ denotes the sequence $j_1, \dots, j_{k+1}$ with the term j_ℓ omitted. Equation (3.2) is called **Grassmann-Plücker relations**.

The following result aids the proof of Lemma 3.2.22.

Lemma 3.2.20. *Let $P := [M \,|\, I_k]$ be a $k \times (m+k)$ block matrix formed by horizontal concatenation of a $k \times m$ matrix M and a $k \times k$ identity matrix I_k. For $\ell \le k$, let $M_{[p_1,\dots,p_\ell;q_1,\dots,q_\ell]}$ denote an $\ell \times \ell$ minor of matrix M that involves rows $1 \le p_1 < \cdots < p_\ell \le k$ and columns $1 \le q_1 < \cdots < q_\ell \le m$. Set $\{q_{\ell+1}, \dots, q_k\} := \{m + r \,|\, 1 \le r \le k,\ r \notin \{p_1, \dots, p_\ell\}\}$, ordered. Then $M_{[p_1,\dots,p_\ell;q_1,\dots,q_\ell]} = P_{[\,;q_1,\dots,q_\ell,q_{\ell+1},\dots,q_k]}$.*

Proof. By definition of the matrix P, there is a 1 at position $(i, m+i)$ of P and all other entries in column $m+i$ of P is 0. Let $\mathcal{R}$ be the set of row indices of M and consider the ordered set $\mathcal{R} \setminus \{p_1, \ldots, p_\ell\}$. Set the least index in this ordered set to be r_1, the next least index to be r_2, and so on. Going on this way, since the cardinality of $\mathcal{R} \setminus \{p_1, \ldots, p_\ell\}$ is $k - \ell$, we set the maximum index in this ordered set to be $r_{k-\ell}$. For each of these r_is, $1 \leq i \leq k - \ell$, set $q_{\ell+i} := m + r_i$. Then the matrix P has 1s at positions $(r_i, q_{\ell+i})$ and zero elsewhere in columns $q_{\ell+1}, q_{\ell+2}, \ldots, q_{\ell+(k-\ell)} = q_k$. Observe that for any given column index q_j, $1 \leq j \leq \ell$, of M, where $1 \leq q_1 < \cdots < q_\ell \leq m$, each column index $q_{\ell+i} = m + r_i$, $1 \leq i \leq k - \ell$, is strictly greater than q_j, for all $1 \leq j \leq \ell$. In addition, by choice of P, since each row p_j, $1 \leq j \leq \ell$, has a 1 at column $m + p_j$ and none of the r_is, $1 \leq i \leq k - \ell$, is equal to the p_js, it follows that $m + p_j \neq m + r_i$ (i.e., $m + p_j \neq q_{\ell+i}$), for any i, j. By cofactor expansion, the determinant

$$P_{[\,;q_1,\ldots,q_\ell,q_{\ell+1},\ldots,q_k]} = \begin{array}{c|cccccccc|} & q_1 & q_2 & \cdots & q_\ell & q_{\ell+1} & q_{\ell+2} & \cdots & q_k \\ p_1 & * & * & \cdots & * & 0 & 0 & \cdots & 0 \\ p_2 & * & * & \cdots & * & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_\ell & * & * & \cdots & * & 0 & 0 & \cdots & 0 \\ r_1 & * & * & \cdots & * & 1 & 0 & \cdots & 0 \\ r_2 & * & * & \cdots & * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{k-\ell} & * & * & \cdots & * & 0 & 0 & \cdots & 1 \end{array}$$

is therefore equal to the minor

$$M_{[p_1,\ldots,p_\ell;q_1,\ldots,q_\ell]} = \begin{array}{c|cccc|} & q_1 & q_2 & \cdots & q_\ell \\ p_1 & * & * & \cdots & * \\ p_2 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_\ell & * & * & \cdots & * \end{array}.$$

□

Definition 3.2.21. Let v, w be permutations in S_n for which the last descent b of v is a descent of w and (α, b) be a location for an essential box in column b for which $\alpha \geq a$, where $a := v(b+1)$. Define $N_{(\alpha,b)}$ to be the ideal generated by the minors of size $1 + \text{rank}(w_{\alpha\times b})$ in $c_{a;}(\mathbf{Z}^{(v)}_{\alpha\times b})$ and $c_{;b}(\mathbf{Z}^{(v)}_{\alpha\times b})$.

Lemma 3.2.22. *Let v, w be permutations in S_n for which the last descent b of v is a descent of w. Let (α, b) and (α', b), where $\alpha' \geq \alpha \geq a$, $a := v(b+1)$, be locations of any two essential boxes in column b, on or below row a of $D(w)$. Set $M := \mathbf{Z}^{(v)}$, $t := 1 + rank(w_{\alpha\times b})$ and $t' := 1 + rank(w_{\alpha'\times b})$. Then for any*

$1 \le p_1 < \cdots < p_{i-1} < a < p_{i+1} < \cdots < p_t \le \alpha$ *and* $1 \le p'_1 < \cdots < p'_{i'-1} < a < p'_{i'+1} < \cdots < p'_{t'} \le \alpha'$, *we have that*

$$M_{[p_1,\dots,p_{i-1},p_{i+1},\dots,p_t;q_1,\dots,q_{t-1}]} \cdot M_{[p'_1,\dots,p'_{i'-1},a,p'_{i'+1},\dots,p'_{t'};q'_1,\dots,q'_{t'-1},b]} - M_{[p'_1,\dots,p'_{i'-1},p'_{i'+1},\dots,p'_{t'};q'_1,\dots,q'_{t'-1}]} \cdot M_{[p_1,\dots,p_{i-1},a,p_{i+1},\dots,p_t;q_1,\dots,q_{t-1},b]} \tag{3.3}$$

belongs to $N_{(\alpha,b)} + N_{(\alpha',b)}$, *where* $1 \le q_1 < \cdots < q_{t-1} < b$ *and* $1 \le q'_1 < \cdots < q'_{t'-1} < b$.

Proof. Set $k := \alpha'$, $\mathbf{p} := \{p_1,\dots,p_{i-1},p_{i+1},\dots,p_t\}$ and $\mathbf{p}' := \{p'_1,\dots,p'_{i'-1},p'_{i'+1},\dots,p'_{t'}\}$. Recall that $M = \mathbf{Z}^{(v)}$. Let $M_{k\times b}$ be the northwest $k\times b$ submatrix of M. Let P be the $k\times(b+k)$ block matrix formed by a horizontal concatenation of $M_{k\times b}$ and a $k\times k$ identity matrix. If $\{q_t, q_{t+1}\dots, q_{k-1}\} := \{b+r \,|\, 1 \le r \le k,\, r \notin \big(\mathbf{p}\cup\{a\}\big)\}$ and $\{q'_{t'}, q'_{t'+1},\dots,q'_{k-1}\} := \{b+r' \,|\, 1 \le r' \le k,\, r' \notin \big(\mathbf{p}'\cup\{a\}\big)\}$, then we have the following equality by Lemma 3.2.20:

$$\begin{aligned}
f_1 &:= M_{[p_1,\dots,p_{i-1},p_{i+1},\dots,p_t;q_1,\dots,q_{t-1}]} = P_{[\,;q_1,\dots,q_{t-1},b+a,q_t,q_{t+1},\dots,q_{k-1}]},\\
f_2 &:= M_{[p_1,\dots,p_{i-1},a,p_{i+1},\dots,p_t;q_1,\dots,q_{t-1},b]} = P_{[\,;q_1,\dots,q_{t-1},b,q_t,q_{t+1},\dots,q_{k-1}]},\\
f'_1 &:= M_{[p'_1,\dots,p'_{i'-1},p'_{i'+1},\dots,p'_{t'};q'_1,\dots,q'_{t'-1}]} = P_{[\,;q'_1,\dots,q'_{t'-1},b+a,q'_{t'},q'_{t'+1},\dots,q'_{k-1}]},\\
f'_2 &:= M_{[p'_1,\dots,p'_{i'-1},a,p'_{i'+1},\dots,p'_{t'};q'_1,\dots,q'_{t'-1},b]} = P_{[\,;q'_1,\dots,q'_{t'-1},b,q'_{t'},q'_{t'+1},\dots,q'_{k-1}]}.
\end{aligned}$$

Assume $q_1 < \cdots < q_{t-1} < q_t < q_{t+1} < \cdots < q_{k-1}$ and $q'_1 < \cdots < q'_{t'-1} < b < b+a < q'_{t'} < q'_{t'+1} < \cdots < q'_{k-1}$. Then we have the following Grassmann-Plücker relations:

$$\sum_{\ell=1}^{k+1} (-1)^{\ell} P_{[\,;q_1,\dots,q_{t-1},q_t,q_{t+1},\dots,q_{k-1},q'_{j_\ell}]}\, P_{[\,;\underbrace{q'_1,\dots,q'_{t'-1},b,b+a,q'_{t'},q'_{t'+1},\dots,q'_{k-1}}_{q'_{j_\ell}\ \text{omitted}}]} = 0 \tag{3.4}$$

From the equation above, observe that when $\ell = t'$, we have

$$\begin{aligned}
&(-1)^{t'} P_{[\,;q_1,\dots,q_{t-1},q_t,q_{t+1},\dots,q_{k-1},b]}\, P_{[\,;q'_1,\dots,q'_{t'-1},b+a,q'_{t'},q'_{t'+1},\dots,q'_{k-1}]}\\
= \;&(-1)^{t'}(-1)^{k-t} P_{[\,;q_1,\dots,q_{t-1},b,q_t,q_{t+1},\dots,q_{k-1}]}\, P_{[\,;q'_1,\dots,q'_{t'-1},b+a,q'_{t'},q'_{t'+1},\dots,q'_{k-1}]} = (-1)^{k+t'-t} f_2 f'_1,
\end{aligned}$$

and when $\ell = t'+1$, we have

$$\begin{aligned}
&(-1)^{t'+1} P_{[\,;q_1,\dots,q_{t-1},q_t,q_{t+1},\dots,q_{k-1},b+a]}\, P_{[\,;q'_1,\dots,q'_{t'-1},b,q'_{t'},q'_{t'+1},\dots,q'_{k-1}]}\\
= \;&(-1)^{t'+1}(-1)^{k-t} P_{[\,;q_1,\dots,q_{t-1},b+a,q_t,q_{t+1},\dots,q_{k-1}]}\, P_{[\,;q'_1,\dots,q'_{t'-1},b,q'_{t'},q'_{t'+1},\dots,q'_{k-1}]} = (-1)^{k+t'-t+1} f_1 f'_2.
\end{aligned}$$

Therefore, for $q'_{j_\ell} \in \{q'_1,\dots,q'_{t'-1},q'_{t'},q'_{t'+1},\dots,q'_{k-1}\}$ with $q'_{j_\ell} \ne b, b+a$, the polynomial $f_1 f'_2 - f_2 f'_1$ can be isolated from equation (3.4) and then written as sum of the products $h\big(j'_\ell\big)\cdot h'\big(j'_\ell\big)$, where $h\big(j'_\ell\big) := P_{[\,;q_1,\dots,q_{t-1},q_t,q_{t+1},\dots,q_{k-1},q'_{j_\ell}]}$ and $h'\big(j'_\ell\big) := P_{[\,;\underbrace{q'_1,\dots,q'_{t'-1},b,b+a,q'_{t'},\dots,q'_{k-1}}_{q'_{j_\ell}\ \text{omitted}}]}$.

First, we claim that if q'_{j_ℓ} belongs to $\{q'_1,\dots,q'_{t'-1}\}$, then $h\big(j'_\ell\big)$ is one of the minors of size t in $c_{;b}(\mathbf{Z}^{(v)}_{\alpha\times b})$. To see this, we will first rewrite the minors of size t in $c_{;b}(\mathbf{Z}^{(v)}_{\alpha\times b})$ as minors of size k in P. An arbitrary minor of size t in $c_{;b}(\mathbf{Z}^{(v)}_{\alpha\times b})$ is of the form $M_{[\mathbf{p}'';\mathbf{q}'']}$, where $\mathbf{p}''$ is some t-element subset of $[\alpha] = \{1,\dots,\alpha\}$ and $\mathbf{q}''$

is some t-element subset of $[b-1]$. If we set $\mathcal{C} := \{b+r \,|\, 1 \leq r \leq k,\ r \notin \mathbf{p}''\}$, then it follows from Lemma 3.2.20 that $M_{[\mathbf{p}'';\mathbf{q}'']} = P_{[\,;\mathbf{q}''\cup\mathcal{C}]}$. Since $\mathbf{p}''$ and $\mathbf{q}''$ are arbitrary, it follows that the set of minors of size t in $c_{\,;b}(\mathbf{Z}^{(v)}_{\alpha\times b})$ is equal to the set of all determinants $P_{[\,;\mathbf{q}''\cup\{b+r\,|\,1\leq r\leq k,\, r\notin\mathbf{p}''\}]}$ in P, where $\mathbf{p}''$ and $\mathbf{q}''$ are any t-element subsets of $[\alpha]$ and $[b-1]$ respectively. Now, suppose q'_{j_ℓ} belongs to $\{q'_1,\ldots,q'_{t'-1}\}$. Then $q'_{j_\ell} \leq b-1$, since $1 \leq q'_1 < \cdots < q'_{t'-1} < b$. Therefore, in this case, the set $\{q_1,\ldots,q_{t-1},q'_{j_\ell}\}$ is a t-element subset of $[b-1]$. In addition, by definition, the set $\{q_t, q_{t+1}\ldots,q_{k-1}\}$ equals $\{b+r \,|\, 1\leq r\leq k,\ r\notin \big(\mathbf{p}\cup\{a\}\big)\}$. Since $1 \leq p_1 < \cdots < p_{i-1} < a < p_{i+1} < \cdots < p_t \leq \alpha$, it follows that $\mathbf{p}\cup\{a\}$ is a t-element subsets of $[\alpha]$. Hence, the determinant $h\big(j'_\ell\big) = \pm P_{[\,;q_1,\ldots,q_{t-1},q'_{j_\ell},q_t,q_{t+1},\ldots,q_{k-1}]}$ is one of the minors of size t in $c_{\,;b}(\mathbf{Z}^{(v)}_{\alpha\times b})$.

Conversely, we claim that if q'_{j_ℓ} belongs to $\{q'_{t'},\ldots,q'_{k-1}\}$, then $h'\big(j'_\ell\big)$ is one of the minors of size t' in $c_{a;}(\mathbf{Z}^{(v)}_{\alpha'\times b})$. Just like the previous paragraph, by Lemma 3.2.20, the set of minors of size t' in $c_{a;}(\mathbf{Z}^{(v)}_{\alpha'\times b})$ is equal to the set of all determinants $P_{[\,;\mathbf{q}''\cup\{b+r'\,|\,1\leq r'\leq k,\, r'\notin\mathbf{p}''\}]}$ in P, where $\mathbf{p}''$ and $\mathbf{q}''$ are any t'-element subsets of $[\alpha']\setminus\{a\}$ and $[b]$ respectively. Suppose q'_{j_ℓ} belongs to $\{q'_{t'},\ldots,q'_{k-1}\}$ and consider the set $\{b+a, q'_{t'},\ldots,\widehat{q'}_{j_\ell},\ldots,q'_{k-1}\}$ with $k-t'$ elements. Corresponding to this set by translation is the set $\mathcal{R} := \{a, q'_{t'}-b,\ldots,\widehat{q'}_{j_\ell},\ldots,q'_{k-1}-b\}$ which is a subset of $[k]$, by definition. Consequently, the set $[k]\setminus\mathcal{R}$ is a t'-element subset of $[\alpha']\setminus\{a\}$, since $k=\alpha'$. In addition, the set $\{q'_1,\ldots,q'_{t'-1},b\}$ is a t'-element subset of $[b]$, since $1\leq q'_1<\cdots<q'_{t'-1}<b$. Hence, the determinant $h'\big(j'_\ell\big) = P_{[\,;q'_1,\ldots,q'_{t'-1},b,b+a,q'_{t'},\ldots,\widehat{q'}_{j_\ell},\ldots,q'_{k-1}]}$ is one of the minors of size t' in $c_{a;}(\mathbf{Z}^{(v)}_{\alpha'\times b})$. □

Example 3.2.23. Continuing with Example 3.2.25, let $\alpha = 2$ and $\alpha' = 4$, so that $t := 1+\mathrm{rank}(w_{\alpha\times b}) = 2$ and $t' := 1+\mathrm{rank}(w_{\alpha'\times b}) = 3$, where $b = 4$ is the last descent of v. Set $a := v(b+1) = 1$, $f_1 := M_{[p_1;q_1]} = M_{[2;1]}$, $f_2 := M_{[a,p_1;q_1,b]} = M_{[1,2;1,4]}$ $f'_1 := M_{[p'_1,p'_2;q'_1,q'_2]} = M_{[3,4;1,2]}$ and $f'_2 := M_{[a,p'_1,p'_2;q'_1,q'_2,b]} = M_{[1,3,4;1,2,4]}$, where $M = \mathbf{Z}^{(v)}$. We wish to write the difference $f_1f'_2 - f_2f'_1$ as sum of some products of determinants. We proceed as follows: set $k := \alpha' = 4$, $\mathbf{p} := \{2\}$ and $\mathbf{p}' := \{3,4\}$. If we set

$$P := \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & z_{23} & z_{24} & 0 & 1 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & 1 & 0 & 0 & 1 & 0 \\ z_{41} & z_{42} & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\{q_t, q_{t+1}\ldots,q_{k-1}\} = \{q_2,q_3\} := \{b+r \,|\, 1\leq r\leq k,\ r\notin\big(\mathbf{p}\cup\{a\}\big)\} = \{7,8\}$ (i.e., $q_2 = 7$ and $q_3 = 8$) and $\{q'_{t'}, q'_{t'+1},\ldots,q'_{k-1}\} = \{q'_3\} := \{b+r' \,|\, 1\leq r'\leq k,\ r'\notin\big(\mathbf{p}'\cup\{a\}\big)\} = \{6\}$, then we have the following:

$$f_1 = M_{[2;1]} = \begin{vmatrix} z_{21} \end{vmatrix} = P_{[\,;q_1,b+a,q_2,q_3]} = P_{[\,;1,5,7,8]} = \begin{vmatrix} z_{11} & 1 & 0 & 0 \\ z_{21} & 0 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ z_{41} & 0 & 0 & 1 \end{vmatrix},$$

$$f_2 = M_{[1,2;1,4]} = \begin{vmatrix} z_{11} & z_{14} \\ z_{21} & z_{24} \end{vmatrix} = P_{[\,;q_1,b,q_2,q_3]} = P_{[\,;1,4,7,8]} = \begin{vmatrix} z_{11} & z_{14} & 0 & 0 \\ z_{21} & z_{24} & 0 & 0 \\ z_{31} & 1 & 1 & 0 \\ z_{41} & 0 & 0 & 1 \end{vmatrix},$$

$$f_1' = M_{[3,4;1,2]} = \begin{vmatrix} z_{31} & z_{32} \\ z_{41} & z_{42} \end{vmatrix} = P_{[\,;q_1',q_2',b+a,q_3']} = P_{[\,;1,2,5,6]} = \begin{vmatrix} z_{11} & z_{12} & 1 & 0 \\ z_{21} & z_{22} & 0 & 1 \\ z_{31} & z_{32} & 0 & 0 \\ z_{41} & z_{42} & 0 & 0 \end{vmatrix},$$

$$f_2' = M_{[1,3,4;1,2,4]} = \begin{vmatrix} z_{11} & z_{12} & z_{14} \\ z_{31} & z_{32} & 1 \\ z_{41} & z_{42} & 0 \end{vmatrix} = P_{[\,;q_1',q_2',b,q_3']} = P_{[\,;1,2,4,6]} = \begin{vmatrix} z_{11} & z_{12} & z_{14} & 0 \\ z_{21} & z_{22} & z_{24} & 1 \\ z_{31} & z_{32} & 1 & 0 \\ z_{41} & z_{42} & 0 & 0 \end{vmatrix}.$$

Hence, we have the following Grassmann-Plücker relations:

$$\sum_{\ell=1}^{4+1} (-1)^{\ell} P_{[\,;1,7,8,q'_{j_\ell}]}\, P_{[\,;\underbrace{1,2,4,5,6}_{q'_{j_\ell} \text{ omitted}}]} = 0, \qquad \text{where } q'_{j_\ell} \in \{1,2,4,5,6\},$$

i.e.,

$$0 = -P_{[\,;1,7,8,1]}\, P_{[\,;2,4,5,6]} + P_{[\,;1,7,8,2]}\, P_{[\,;1,4,5,6]} - P_{[\,;1,7,8,4]}\, P_{[\,;1,2,5,6]}$$
$$+P_{[\,;1,7,8,5]}\, P_{[\,;1,2,4,6]} - P_{[\,;1,7,8,6]}\, P_{[\,;1,2,4,5]}.$$

Rearranging the column indices in ascending order, we have

$$f_1 f_2' - f_2 f_1' = P_{[\,;1,5,7,8]}\, P_{[\,;1,2,4,6]} - P_{[\,;1,4,7,8]}\, P_{[\,;1,2,5,6]} = -P_{[\,;1,2,7,8]}\, P_{[\,;1,4,5,6]} + P_{[\,;1,6,7,8]}\, P_{[\,;1,2,4,5]}.$$

By the latter arguments in the proof of Lemma 3.2.22, we expect $P_{[\,;1,7,8,1]}$ and $P_{[\,;1,7,8,2]}$ to be one of the minors of size $t = 2$ in $c_{\,;b}(\mathbf{Z}^{(v)}_{\alpha\times b}) = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix}$; here, $q'_{j_\ell} = 1,2 \leq b-1$, where $\alpha = 2$ and $b = 4$. While the first determinant is zero, the second determinant is equal to $M_{[1,2;1,2]} = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}$, which is indeed one of the minors of size t in $c_{\,;4}(\mathbf{Z}^{(v)}_{2\times 4})$. Furthermore, we expect $P_{[\,;1,2,4,5]}$ to be one of the minors of size $t' = 3$ in $c_{a;}(\mathbf{Z}^{(v)}_{\alpha'\times b}) = \begin{pmatrix} z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & 1 \\ z_{41} & z_{42} & 1 & 0 \end{pmatrix}$; here, $q'_{j_\ell} = 6 > b + a$, where $a = 1$ and $\alpha' = 4$. The determinant $P_{[\,;1,2,4,5]}$ is equal to $M_{[2,3,4;1,2,4]} = \begin{pmatrix} z_{21} & z_{22} & z_{24} \\ z_{31} & z_{32} & 1 \\ z_{41} & z_{42} & 0 \end{pmatrix}$, which is indeed one of the minors of size t' in $c_{1;}(\mathbf{Z}^{(v)}_{4\times 4})$. □

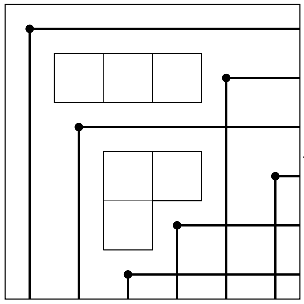
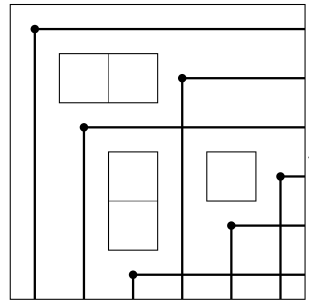
The following result aids the proof of Lemma 3.2.26 in this dissertation.

Lemma 3.2.24. *Let v, w be permutations in S_n for which the last descent b of v is a descent of w. If (α, b) is an essential box for which $\alpha \geq a$, where $a := v(b+1)$, then $N_{(\alpha,b)} \subseteq T_{vs_b,ws_b}$.*

Proof. Set $a := v(b+1)$ and let (α, b), $\alpha \geq a$, be an essential box in column b, on or below row a of $D(w)$. Set $M := \mathbf{Z}^{(v)}_{\alpha\times b}$ and $t := 1+\mathrm{rank}(w_{\alpha\times b})$. The submatrix $c_{;b}(M)$ of M equals $\mathbf{Z}^{(v)}_{\alpha\times b-1} = \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times b-1}$. If there is a box immediately to the left of essential box (α, b) in $D(w)$, then we have that $\mathrm{rank}(w_{\alpha\times b-1}) = \mathrm{rank}(w'_{\alpha\times b-1})$, where $w' := ws_b$. Hence, the set of minors of size t in $c_{;b}(M) = \mathbf{Z}^{(v)}_{\alpha\times b-1}$ is equal to the set of minors of size $1 + \mathrm{rank}(w'_{\alpha\times b})$ in $\left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times b-1}$, and so these minors belong to T_{vs_b,ws_b}. Furthermore, if there is an essential box at location $(\alpha, b+1)$ in $D(w')$, then we have that $\mathrm{rank}(w'_{\alpha\times b+1}) = t$. Set $M' := \left(\mathbf{Z}^{(v)}s_b\right)_{\alpha\times(b+1)}$ and $t' := 1 + \mathrm{rank}(w'_{\alpha\times b+1}) = 1 + t$. The matrix M' has a 1s in its submatrix formed by rows $1, \ldots, a$ and columns $j_1, \ldots, j_{b-1}, b$, for some j_is. Therefore, by parts (1) and (2) of Lemma 2.7.6, the ideal generated by minors of size t' in M' is equal to the ideal generated by minors of size $t' - a$ in $G' := c_{1,\ldots,a;j_1,\ldots,j_{a-1},b}(M')$ and each of these minors of size $t' - a$ in G' can actually be realized as a minor of size t' in M', and hence, belongs to T_{vs_b,ws_b}. Similarly, by cofactor expansion about rows $1, \ldots, (a-1)$ of M and the columns where 1s are located in these rows, we have that the ideal generated by minors of size t in M is equal to the ideal generated by minors of size $t - (a-1)$ in $c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}(M)$. Consequently, we have that the ideal generated by the minors of size t in $c_{a;}(M)$ is equal to the ideal generated by the minors of size $t - (a-1)$ in $G := c_{a;}\left(c_{1,\ldots,a-1;j_1,\ldots,j_{a-1}}(M)\right) = c_{1,\ldots,a;j_1,\ldots,j_{a-1}}(M)$, and each of these minors of size $t - (a-1)$ in G can be realized as minors of size t in $c_{a;}(M)$. Since M' differs from M by its column b; precisely, M is the resulting matrix from deleting column b of M', it follows that $G' = G$. We also have that $t - (a-1) = t' - 1 - (a-1) = t' - a$. Hence, each of the minors of size $t - (a-1) = t' - a$ in $G = G' = c_{1,\ldots,a;j_1,\ldots,j_{a-1},b}(M')$ can be realized as minors of size t' in M', and hence, belongs to T_{vs_b,ws_b}. $\square$

Example 3.2.25. Let $v = 654312$ and $w = 136524$. The last descent $b = 4$ of v is a descent of w. We have:

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & 1 & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & 0 & 1 \\ z_{31} & z_{32} & z_{33} & 1 & 0 & 0 \\ z_{41} & z_{42} & 1 & 0 & 0 & 0 \\ z_{51} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{Z}^{(v)}s_b = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{14} & z_{16} \\ z_{21} & z_{22} & z_{23} & 0 & z_{24} & 1 \\ z_{31} & z_{32} & z_{33} & 0 & 1 & 0 \\ z_{41} & z_{42} & 1 & 0 & 0 & 0 \\ z_{51} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$D(w) =$ , and $D(ws_b) =$ .

Set $a := v(b+1) = 1$, the row at which the variable $z_{\max} = z_{14}$ is located in $\mathbf{Z}^{(v)}$. For a box (α, β) in $D(ws_b)$, let $N'_{(\alpha,\beta)}$ denote the ideal generated by minors of size $1 + \operatorname{rank}((ws_b)_{\alpha\times\beta})$ in $\big(\mathbf{Z}^{(v)}s_b\big)_{\alpha\times\beta}$. For the essential box $(2,4)$ in $D(w)$, while the minors of size $t := 1 + \operatorname{rank}(w_{2\times 4}) = 2$ in $c_{a;}\big(\mathbf{Z}^{(v)}_{2\times 4}\big) = \begin{pmatrix} z_{21} & z_{22} & z_{23} & z_{24}\end{pmatrix}$ equals zero, the minors of size t in $c_{;b}\big(\mathbf{Z}^{(v)}_{2\times 4}\big) = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23}\end{pmatrix}$ is equal to the ideal $N'_{(2,3)}$ which is contained in T_{vs_4,ws_4}. Furthermore, for the essential box $(4,4)$ in $D(w)$, the minors of size $t' := 1 + \operatorname{rank}(w_{4\times 4}) = 3$ in $c_{;b}\big(\mathbf{Z}^{(v)}_{4\times 4}\big) = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \\ z_{41} & z_{42} & 1 \end{pmatrix}$ is equal to the ideal $N'_{(4,3)} \subseteq T_{vs_4,ws_4}$ and the minors of size t' in $c_{a;}\big(\mathbf{Z}^{(v)}_{4\times 4}\big) = \begin{pmatrix} z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & 1 \\ z_{41} & z_{42} & 1 & 0 \end{pmatrix}$ is equal to the ideal $N'_{(4,5)} \subseteq T_{vs_4,ws_4}$, where by definition, the ideal $N'_{(4,5)}$ is the ideal generated by minors of size $1 + \operatorname{rank}((ws_4)_{4\times 5}) = 4$ in $\big(\mathbf{Z}^{(v)}s_4\big)_{4\times 5} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{14} \\ z_{21} & z_{22} & z_{23} & 0 & z_{24} \\ z_{31} & z_{32} & z_{33} & 0 & 1 \\ z_{41} & z_{42} & 1 & 0 & 0 \end{pmatrix}$. Hence, $N_{(2,4)} + N_{(4,4)}$ is contained in T_{vs_4,ws_4}. ☐

Recall that $M_{[p_1,\ldots,p_\ell;q_1,\ldots,q_\ell]}$ denotes an $\ell\times\ell$ minor of matrix M that involves rows $p_1 < \cdots < p_\ell$ and columns $q_1 < \cdots < q_\ell$.

Lemma 3.2.26. *Let v, w be permutations in S_n for which the last descent of v is also a descent of w, and consider the positive multigrading of $R := \mathbb{K}[\mathbf{z}^{(v)}]$ by $\mathbb{Z}^n$. Let b be the last descent of v and assume the variable z_{max} does not belong to $\langle in_{\succ}(\mathcal{G}_I)\rangle$, where $in_{\succ}(\mathcal{G}_I)$ is the set of initial terms of essential minors generating the ideal $I := Q_{v,w}$. Set $J := T_{vs_b,w}$, $N := T_{vs_b,ws_b}$, $A := \langle in_{\succ}(\mathcal{G}_I)\rangle$, $B := \langle in_{\succ}(\mathcal{G}_J)\rangle$ and $C := \langle in_{\succ}(\mathcal{G}_N)\rangle$. Then there exists $\boldsymbol{e} \in \mathbb{Z}^n$ such that there is an R/N-module isomorphism $I/N \cong (J/N)(-\boldsymbol{e})$ and an R/C-module isomorphism $A/C \cong (B/C)(-\boldsymbol{e})$.*

Proof. Set $y := z_{\max}$. First, we will show that $I/N \cong [J/N](-\boldsymbol{e})$, as R/N-modules. Write I in the form $\langle yg_1 + r_1, \ldots, yg_k + r_k, h_1, \ldots, h_\ell\rangle$, where the set $\{yg_1 + r_1, \ldots, yg_k + r_k, h_1, \ldots, h_\ell\}$ is a complete list of all essential minors in I and y does not divide any term of g_i or r_i for any $1 \leq i \leq k$ nor any h_j for $1 \leq j \leq \ell$. Then it follows from Lemmas 3.2.13 and 3.2.16 that $J = \langle g_1, \ldots, g_k, h_1, \ldots, h_\ell \rangle$ and $N = \langle h_1, \ldots, h_\ell \rangle$. Consequently, the mapping $I/N \longrightarrow [J/N](-\boldsymbol{e})$ defined by $\overline{f} \longmapsto \frac{g}{yg+r} \cdot \overline{f}$, for some essential minor $yg+r \in I$ of size t, is an (R/N)-module isomorphism, where $\boldsymbol{e} \in \mathbb{Z}^n$ is the degree of the variable y. Indeed, if $yg' + r'$ is an arbitrary essential minor in I of size t', and set $a := v(b+1)$ and $M := \mathbf{Z}^{(v)}$, then $\frac{g}{yg+r} \cdot (yg' + r') = g'$, since $g(yg' + r') - (yg + r)g'$ equals

$$M_{[p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_t;q_1,\ldots,q_{t-1}]} \cdot M_{[p'_1,\ldots,p'_{i'-1},a,p'_{i'+1},\ldots,p'_{t'};q'_1,\ldots,q'_{t'-1},b]}$$
$$- M_{[p'_1,\ldots,p'_{i'-1},p'_{i'+1},\ldots,p'_{t'};q'_1,\ldots,q'_{t'-1}]} \cdot M_{[p_1,\ldots,p_{i-1},a,p_{i+1},\ldots,p_t;q_1,\ldots,q_{t-1},b]},$$

for some $p_1 < \cdots < p_{i-1} < a < p_{i+1} < \cdots < p_t$, $p'_1 < \cdots < p'_{i'-1} < a < p'_{i'+1} < \cdots < p'_{t'}$, $q_1 < \cdots < q_{t-1} < b$ and $q'_1 < \cdots < q'_{t'-1} < b$, and so belongs to N, by Lemmas 3.2.22 and 3.2.24. Even more, since N is prime, it follows that neither g nor $yg + r$ is a zero-divisor in R/N.

Next, we will show that $A/C \cong [B/C](-\boldsymbol{e})$, as R/C-modules. Observe that

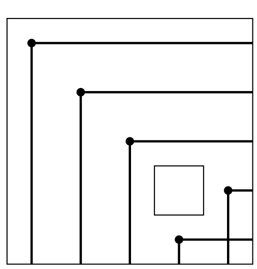
$$\begin{aligned}
\langle \mathrm{in}_\succ(\mathcal{G}_N)\rangle + y \cdot \langle \mathrm{in}_\succ(\mathcal{G}_J)\rangle &= \langle \mathrm{in}_\succ(h_1), \ldots, \mathrm{in}_\succ(h_\ell)\rangle + \langle y\,\mathrm{in}_\succ(g_1), \ldots, y\,\mathrm{in}_\succ(g_k), y\,\mathrm{in}_\succ(h_1), \ldots, \\
&\qquad y\,\mathrm{in}_\succ(h_\ell)\rangle \\
&= \langle \mathrm{in}_\succ(h_1), \ldots, \mathrm{in}_\succ(h_\ell)\rangle + \langle y\,\mathrm{in}_\succ(g_1), \ldots, y\,\mathrm{in}_\succ(g_k)\rangle + \\
&\qquad \langle y\,\mathrm{in}_\succ(h_1), \ldots, y\,\mathrm{in}_\succ(h_\ell)\rangle \\
&= \langle \mathrm{in}_\succ(h_1), \ldots, \mathrm{in}_\succ(h_\ell)\rangle + \langle y\,\mathrm{in}_\succ(g_1), \ldots, y\,\mathrm{in}_\succ(g_k)\rangle \\
&= \langle \mathrm{in}_\succ(h_1), \ldots, \mathrm{in}_\succ(h_\ell), y\,\mathrm{in}_\succ(g_1), \ldots, y\,\mathrm{in}_\succ(g_k)\rangle \\
&= \langle \mathrm{in}_\succ(\mathcal{G}_I)\rangle,
\end{aligned}$$

i.e., $C + y \cdot B = A$, and so we have the R/C-module isomorphism

$$A/C = (y \cdot B + C)/C \cong [B/C](-\boldsymbol{e}).$$

The last isomorphism is true since the generators of C do not involve y and the map $B \to (y \cdot B + C)/C$ defined by $b \mapsto yb + C$ is a surjective homomorphism with kernel C. For the kernel, observe that $b \mapsto C$ if and only if $yb + C = C$ if and only if $yb \in C$ if and only if $b \in C$. $\square$

Example 3.2.27. Let $v = 34512$ (as in Example 2.7.1) and $w = 12354$. The last descent $b = 3$ of v is an ascent of w, and so we set both J and N to be $T_{vs_3,w}$. The Rothe diagram $D(w)$ and matrices $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(v)}s_3$ are given below:

$D(w) =$  ,

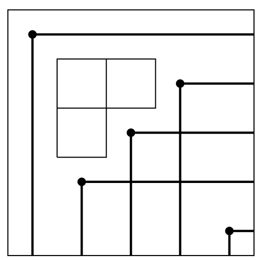
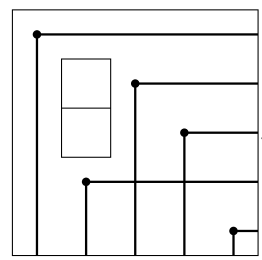
$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & z_{23} & 0 & 1 \\ 1 & z_{32} & z_{33} & 0 & 0 \\ 0 & 1 & z_{43} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15} \\ z_{21} & z_{22} & 0 & z_{23} & 1 \\ 1 & z_{32} & 0 & z_{33} & 0 \\ 0 & 1 & 0 & z_{43} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here, $T_{vs_3,w} = Q_{v,w}$, and so if we set $I := Q_{v,w}$, then $I/N \cong J/N$. □

Example 3.2.28. Let $v = 34512$ (as in Example 3.2.27) and $w = 14325$. The last descent of v is $b = 3$ which is a descent of w. The Rothe diagrams $D(w)$ and $D(ws_3)$ are given below:

$D(w) =$  and $D(ws_3) =$ .

Set

$$J := T_{vs_3,w} = \left\langle z_{21},\, z_{22},\, \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{32} \end{vmatrix} \right\rangle$$

and

$$N := T_{vs_3,ws_3} = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{32} \end{vmatrix} \right\rangle.$$

If we set

$$I := Q_{v,w} = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{32} \end{vmatrix} \right\rangle,$$

then

$$I/N = \left\langle \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \right\rangle \qquad \text{and} \qquad J/N = \langle z_{21},\, z_{22} \rangle.$$

Define the map $I/N \longrightarrow (J/N)(-\boldsymbol{e})$ by

$$\overline{g} \mapsto \frac{f_1}{f_2} \cdot \overline{g},$$

where $f_1 = z_{21}$, $f_2 = \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}$ and $\boldsymbol{e} = e_{v^{-1}(1)} - e_3 = e_4 - e_3$ is the degree of the variable $z_{\max} = z_{13}$. Under this mapping, we have

$$\begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} \longrightarrow \frac{z_{21}}{\begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}} \cdot \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} = z_{21}$$

and

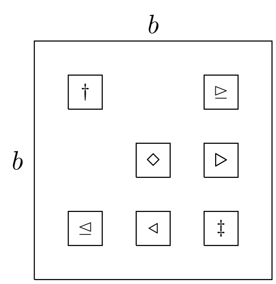
$$\begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \longrightarrow \frac{z_{21}}{\begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}} \cdot \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} = \frac{z_{22}\begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} - z_{23}\color{red}{\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}}}{\begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}} = z_{22},$$

where the determinant (minor) indicated in red is zero modulo N, since it belongs to N. Therefore, $I/N \cong (J/N)(-\boldsymbol{e})$, as $\mathbb{K}[\mathbf{z}^{(v)}]/N$-module. $\square$

The following result aids the proof of Theorem 3.2.33 (Gröbner basis result for Schubert patch ideals) in this dissertation.

Lemma 3.2.29. *Let $v \in S_n$ for which $\ell(v) = 1$. For any w, if $Q_{v,w}$ is a proper ideal, then v equals w and $Q_{v,w} = \langle z_{bb} \rangle$, where b is the last (and only) descent of v.*

Proof. Since $\ell(v) = 1$, it follows that v has only one descent, say b, and v is the simple transposition s_b. In this case, the variable $z_{\max}$ in $\mathbf{Z}^{(v)}$ is z_{bb}. Fix a $w \in S_n$ and consider all possible locations of essential boxes relative to the bth row and bth column of $D(w)$. See below for an illustration; all possible cases are shown in one diagram.



Suppose that there is an essential box at location (i, j) in $D(w)$ where either $i < b$ and $j < b$ (see †), or $i < b$ and $j \geq b$ (see ⊵), or $i \geq b$ and $j < b$ (see ⊴), or $i = b$ and $j > b$ (see ▷), or $i > b$ and $j = b$ (see ◁), or $i > b$ and $j > b$ (see ‡). For any of these pairs (i, j), the corresponding rectangular submatrix $\mathbf{Z}^{(v)}_{i \times j}$ of $\mathbf{Z}^{(v)}$ contains a square submatrix, of size $\min(i, j)$, whose main diagonal entries are all 1s, up to rearranging rows or columns. So $Q_{v,w} = \langle 1 \rangle$ in this case, since $1 + \operatorname{rank}(w_{i \times j}) \leq \min(i, j)$. Suppose that there is an essential box at location (b, b) in $D(w)$ (see ◇). The first $b - 1$ entries on the main diagonal of the submatrix $\mathbf{Z}^{(v)}_{b \times b}$ of $\mathbf{Z}^{(v)}$ are all equal to 1. So any ideal generated by minors of size $t \times t$ in $\mathbf{Z}^{(v)}_{b \times b}$, where $t \leq b - 1$, equals $\langle 1 \rangle$. If $t = b$, then by cofactor expansion, minor of size $t \times t$ in $\mathbf{Z}^{(v)}_{b \times b}$ equals z_{bb}. In summary, we have the following:

- If there is an essential box (i, j) in $D(w)$ for which $(i, j) \neq (b, b)$, then $Q_{v,w} = \langle 1 \rangle$.
- If there is an essential box (b, b) in $D(w)$ for which $\operatorname{rank}(w_{b \times b}) < b - 1$, then $Q_{v,w} = \langle 1 \rangle$, since $1 + \operatorname{rank}(w_{b \times b}) \leq b - 1$.
- If $D(w)$ has only one essential box (b, b) for which $\operatorname{rank}(w_{b \times b}) = b - 1$, then $Q_{v,w} = \langle z_{bb} \rangle$, since $1 + \operatorname{rank}(w_{b \times b}) = b$. Observe that, in this case, w equals s_b.

Thus, $Q_{v,w}$ is a proper ideal provided $D(w)$ has a unique essential box at (b,b) and $\mathrm{rank}(w_{b\times b}) = b-1$. In this case, we have $Q_{v,w} = \langle z_{bb}\rangle$ and $w = s_b = v$. $\square$

Example 3.2.30. There are 3 simple transpositions in S_4, which are: 2134, 1324 and 1243, with permuted opposite big cells

$$\begin{pmatrix} z_{11} & 1 & z_{13} & z_{14} \\ 1 & 0 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & z_{22} & 1 & z_{24} \\ 0 & 1 & 0 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & z_{33} & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively. Let $v = 1243$, for instance. The last (and only) descent of v is $b = 3$. Following the summary in the proof of Lemma 3.2.29, given any $w \in S_4$, if, in $D(w)$, there is an essential box $(i,j) \neq (3,3)$ or an essential box $(i,j) = (3,3)$ with $1 + \mathrm{rank}(w_{3\times 3}) \leq 2$, then $Q_{v,w} = \langle 1\rangle$. However, if $D(w)$ has only one essential box at $(3,3)$ with $1 + \mathrm{rank}(w_{3\times 3}) = 3$, then $Q_{v,w} = \langle z_{33}\rangle$. In this last case, we have $w = 1243 = v$. Similarly, $Q_{2134,2134} = \langle z_{11}\rangle$ and $Q_{1324,1324} = \langle z_{22}\rangle$. $\square$

In what follows, let $\succ_v$ be the term order that we use on $\mathbb{K}[\mathbf{z}^{(v)}]$ in Section 2.7 and $\succ_{vs_b}$ be the corresponding term order on $\mathbb{K}[\mathbf{z}^{(vs_b)}]$, where b is the last descent of $v \in S_n$. The substitution map φ defined in Definition 3.2.3 is not order-preserving, in the sense that, if $z_{ij} \succ_{vs_b} z_{i'j'}$ in $\mathbb{K}[\mathbf{z}^{(vs_b)}]$, then the order $\varphi(z_{ij}) \succ_v \varphi(z_{i'j'})$ in $\mathbb{K}[\mathbf{z}^{(v)}]$ does not necessarily hold. However, φ can be restricted to some variables so that it becomes order-preserving. The following example illustrates this order-preserving concept.

Example 3.2.31. Set $a := v(b+1)$. From Lemma 2.5.14, we infer that the placement of variables of $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(vs_b)}$ agree in all columns except columns b and $b+1$. In addition, the positions of the variables of $\mathbf{Z}^{(v)}$ in columns b and $b+1$, and strictly above row a remain unchanged in $\mathbf{Z}^{(vs_b)}$. See below for a typical description of this placement of variables in some submatrices of $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(vs_b)}$, for some v.

$$\begin{array}{c} \\ \alpha \\ \alpha' \\ a \\ a' \\ \alpha'' \\ a'' \\ \alpha''' \\ v(b) \end{array} \begin{array}{c} \begin{array}{cccccc} & & b & b+1 & & \end{array} \\ \begin{pmatrix} & & *_1 & *_2 & & \\ & & *_3 & *_4 & & \\ & & z_{a,b} & 1 & & \\ & & *_5 & 0 & 1 & \\ 1 & & *_6 & 0 & & \\ & & *_7 & 0 & & 1 \\ & 1 & *_8 & 0 & & \\ & & 1 & 0 & & \end{pmatrix} \end{array} \mapsto \begin{array}{c} \\ \alpha \\ \alpha' \\ a \\ a' \\ \alpha'' \\ a'' \\ \alpha''' \\ v(b) \end{array} \begin{array}{c} \begin{array}{cccccc} & & b & b+1 & & \end{array} \\ \begin{pmatrix} & & *_1 & *_2 & & \\ & & *_3 & *_4 & & \\ & & 1 & z_{a,b+1} & & \\ & & 0 & *'_5 & 1 & \\ 1 & & 0 & *'_6 & & \\ & & 0 & *'_7 & & 1 \\ & 1 & 0 & *'_8 & & \\ & & 0 & 1 & & \end{pmatrix} \end{array}$$

$$\mathbf{Z}^{(v)} \qquad\qquad\qquad \mathbf{Z}^{(vs_b)}$$

The sets $\mathbf{y}^{(v)}$ and $\tilde{\mathbf{x}}^{(v)} = \mathbf{z}^{(v)} \backslash \mathbf{y}^{(v)}$ are defined in Section 2.7. We observe from the second matrix above (the one on the right) that while the variables $*_5'$ and $*_7'$ belong to $\tilde{\mathbf{x}}^{(vs_b)}$, the variables $*_1, *_2, *_3, *_4, z_{a,b+1}, *_6'$ and $*_8'$ belong to $\mathbf{y}^{(vs_b)}$. Consequently, we have the following ordering of variables with respect to $\succ_{vs_b}$:

$$*_5' \succ_{vs_b} *_7' \succ_{vs_b} *_2 \succ_{vs_b} *_4 \succ_{vs_b} z_{a,b+1} \succ_{vs_b} *_6' \succ_{vs_b} *_8' \succ_{vs_b} *_1 \succ_{vs_b} *_3,$$

i.e., using their respective row and column positions, we have

$$z_{a',b+1} \succ_{vs_b} z_{a'',b+1} \succ_{vs_b} z_{\alpha,b+1} \succ_{vs_b} z_{\alpha',b+1} \succ_{vs_b} z_{a,b+1} \succ_{vs_b} z_{\alpha'',b+1} \succ_{vs_b} z_{\alpha''',b+1} \succ_{v} z_{\alpha,b} \succ_{vs_b} z_{\alpha',b}. \quad (3.5)$$

Similarly, we observe from the first matrix above (the one on the left) that while the variables $*_5$ and $*_7$ belong to $\tilde{\mathbf{x}}^{(v)}$, the variables $*_1, *_2, *_3, *_4, *_6$ and $*_8$ belong to $\mathbf{y}^{(v)}$. Consequently, we have the following ordering of variables with respect to $\succ_v$:

$$z_{a,b} \succ_v *_5 \succ_v *_7 \succ_v *_2 \succ_v *_4 \succ_v *_1 \succ_v *_3 \succ_v *_6 \succ_v *_8,$$

i.e.,

$$z_{a,b} \succ_v z_{a',b} \succ_v z_{a'',b} \succ_v z_{\alpha,b+1} \succ_v z_{\alpha',b+1} \succ_v z_{\alpha,b} \succ_v z_{\alpha',b} \succ_v z_{\alpha'',b} \succ_v z_{\alpha''',b}. \quad (3.6)$$

Applying φ to the variables in the ordering (3.5) above, we obtain

$$z_{a',b} \succ_v z_{a'',b} \succ_v z_{\alpha,b} \succ_v z_{\alpha',b} \succ_v z_{a,b} \succ_v z_{\alpha'',b} \succ_v z_{\alpha''',b} \succ_v z_{\alpha,b+1} \succ_v z_{\alpha',b+1}, \quad (3.7)$$

which is not the same ordering in (3.6). However, removing the variables $*_1 = z_{\alpha,b}$, $*_2 = z_{\alpha,b+1}$, $*_3 = z_{\alpha',b}$, $*_4 = z_{\alpha',b+1}$ and $z_{a,b+1}$ from the ordering (3.5), we have

$$z_{a',b+1} \succ_{vs_b} z_{a'',b+1} \succ_{vs_b} z_{\alpha'',b+1} \succ_{vs_b} z_{\alpha''',b+1},$$

and on applying φ to this last ordering, we obtain

$$z_{a',b} \succ_v z_{a'',b} \succ_v z_{\alpha'',b} \succ_v z_{\alpha''',b},$$

which is consistent with the ordering (3.6), after removing the variables $*_1 = z_{\alpha,b}$, $*_2 = z_{\alpha,b+1}$, $*_3 = z_{\alpha',b}$, $*_4 = z_{\alpha',b+1}$ and $z_{a,b}$ from it. In summary, restricting the term order $\succ_{vs_b}$ to those variables in $\mathbb{K}[\mathbf{z}^{(vs_b)}]$ that are different from the variables on or above (i.e., weakly above) row $a = v(b+1)$, and on columns b and $b+1$ of $\mathbf{Z}^{(vs_b)}$, the map φ preserves the term orders $\succ_{vs_b}$ and $\succ_v$. □

We therefore have the following result which aids the proof of Theorem 3.2.33.

Lemma 3.2.32. *Let $v, w \in S_n$, b be the last descent of v and φ be the substitution map defined in Definition 3.2.3. Then we have the following:*

1. $in_{\succ_v}(\varphi(f)) = \varphi(in_{\succ_{vs_b}}(f))$*, for any $f \in \mathbb{K}[\mathbf{z}^{(vs_b)}]$ that does not involve the variables z_{ij}, where $1 \leq i \leq v(b+1)$ and $b \leq j \leq b+1$.*

2. *The essential minors of* $Q_{vs_b,w}$ *(resp.* Q_{vs_b,ws_b}*) form a Gröbner basis with respect to* $\succ_{vs_b}$ *if and only if the essential minors of* $T_{vs_b,w}$ *(resp.* T_{vs_b,ws_b}*) form a Gröbner basis with respect to* $\succ_v$*.*

Proof.

1. The placement of variables of $\mathbf{Z}^{(v)}$ and $\mathbf{Z}^{(vs_b)}$ agree in all columns except columns b and $b+1$. In addition, suppose we order, with respect to $\succ_{vs_b}$, all the variables z_{ij} in $\mathbf{Z}^{(vs_b)}$ for which $i > v(b+1)$ and $j = b+1$, i.e., the variables strictly below row a and on column $b+1$ of $\mathbf{Z}^{(vs_b)}$. Then after applying φ to these variables, since they are all in the same column in $\mathbf{Z}^{(vs_b)}$, they will all be in the same column in $\mathbf{Z}^{(v)}$, and therefore, none of them will lose its order with respect to $\succ_v$. So for any variables z_{ij}, $z_{i'j'}$ in $\mathbb{K}[\mathbf{z}^{(vs_b)}]$, different from the ones that are being avoided, we have: $z_{ij} \succ_{vs_b} z_{i'j'}$ implies $\varphi(z_{ij}) \succ_v \varphi(z_{i'j'})$. So if $\text{in}_{\succ_{vs_b}}(f)$ is the maximum (initial) term of f with respect to $\succ_{vs_b}$, then $\varphi(\text{in}_{\succ_{vs_b}}(f))$ is the maximum (initial) term of $\varphi(f)$ with respect to $\succ_v$, i.e., $\text{in}_{\succ_v}\big(\varphi(f)\big) = \varphi\big(\text{in}_{\succ_{vs_b}}(f)\big)$.

2. It follows from Lemmas 2.7.8, 3.2.13 and 3.2.16 that the ideals $T_{vs_b,w}$ and T_{vs_b,ws_b} can be generated by essential minors that do not involve the variables z_{ij} in $\mathbf{Z}^{(v)}$ for which $1 \le i \le v(b+1)$ and $b \le j \le b+1$, and hence, the ideals $Q_{vs_b,w}$ and Q_{vs_b,ws_b} can also be generated by essential minors that do not involve the variables z_{ij} in $\mathbf{Z}^{(vs_b)}$ for which $1 \le i \le v(b+1)$ and $b \le j \le b+1$. If $\mathcal{G}_{Q_{vs_b,w}} := \{g_1, \dots, g_t\}$ is the set of essential minors of $Q_{vs_b,w}$, then by Remark 3.2.6, the set $\mathcal{G}_{T_{vs_b,w}} := \{\varphi(g_1), \dots, \varphi(g_t)\}$ is the set of essential minors of $T_{vs_b,w}$. To prove the required result, using the Buchberger's criterion in Theorem 2.1.12, it suffices to show that for all pairs $i \neq j$, the remainder $\overline{S(g_i,g_j)}^{\mathcal{G}_{Q_{vs_b,w}}}$ on division of $S(g_i,g_j)$ by $\mathcal{G}_{Q_{vs_b,w}}$ is zero if and only if the remainder $\overline{S\big(\varphi(g_i),\varphi(g_j)\big)}^{\mathcal{G}_{T_{vs_b,w}}}$ on division of $S\big(\varphi(g_i),\varphi(g_j)\big)$ by $\mathcal{G}_{T_{vs_b,w}}$ is zero. To this end, let g_i, g_j be arbitrary elements of $\mathcal{G}_{Q_{vs_b,w}}$ with $\text{in}_{\succ_{vs_b}}(g_i) = c_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}}$ and $\text{in}_{\succ_{vs_b}}(g_j) = c_{\boldsymbol{\beta}}\mathbf{z}^{\boldsymbol{\beta}}$, and let $\mathbf{z}^{\boldsymbol{\gamma}}$ be the least common multiple of $\mathbf{z}^{\boldsymbol{\alpha}}$ and $\mathbf{z}^{\boldsymbol{\beta}}$. Then, the S-polynomial S(g_i,g_j) of g_i and g_j is:
$$S(g_i,g_j) = \frac{\mathbf{z}^{\boldsymbol{\gamma}}}{\text{in}_{\succ_{vs_b}}(g_i)} \cdot g_i - \frac{\mathbf{z}^{\boldsymbol{\gamma}}}{\text{in}_{\succ_{vs_b}}(g_j)} \cdot g_j.$$
Applying the substitution map φ to both sides yields the following:
$$\begin{aligned}\varphi\big(S(g_i,g_j)\big) &= \frac{\varphi\big(\mathbf{z}^{\boldsymbol{\gamma}}\big)}{\varphi\big(\text{in}_{\succ_{vs_b}}(g_i)\big)} \cdot \varphi(g_i) - \frac{\varphi\big(\mathbf{z}^{\boldsymbol{\gamma}}\big)}{\varphi\big(\text{in}_{\succ_{vs_b}}(g_j)\big)} \cdot \varphi(g_j) \\ &= \frac{\varphi\big(\mathbf{z}^{\boldsymbol{\gamma}}\big)}{\text{in}_{\succ_v}\big(\varphi(g_i)\big)} \cdot \varphi(g_i) - \frac{\varphi\big(\mathbf{z}^{\boldsymbol{\gamma}}\big)}{\text{in}_{\succ_v}\varphi\big((g_j)\big)} \cdot \varphi(g_j). \end{aligned} \tag{3.8}$$
Since $\mathbf{z}^{\boldsymbol{\gamma}}$ is the least common multiple of $\mathbf{z}^{\boldsymbol{\alpha}}$ and $\mathbf{z}^{\boldsymbol{\beta}}$, it follows by definition of least common multiple that $\mathbf{z}^{\boldsymbol{\gamma}} = \mathbf{z}^{\boldsymbol{\alpha}}\, h_i = \mathbf{z}^{\boldsymbol{\beta}}\, h_j$, for some h_i, h_j, and if there is another common multiple $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}$ of $\mathbf{z}^{\boldsymbol{\alpha}}$ and $\mathbf{z}^{\boldsymbol{\beta}}$, then $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}} \succ_{vs_b} \mathbf{z}^{\boldsymbol{\gamma}}$. Since $\text{in}_{\succ_{vs_b}}(g_i) = c_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}}$ and $\text{in}_{\succ_{vs_b}}(g_j) = c_{\boldsymbol{\beta}}\mathbf{z}^{\boldsymbol{\beta}}$, it follows that $\text{in}_{\succ_v}\big(\varphi(g_i)\big) = \varphi\big(\text{in}_{\succ_{vs_b}}(g_i)\big) = c_{\boldsymbol{\alpha}}\varphi(\mathbf{z}^{\boldsymbol{\alpha}})$ and $\text{in}_{\succ_v}\big(\varphi(g_j)\big) = \varphi\big(\text{in}_{\succ_{vs_b}}(g_j)\big) = c_{\boldsymbol{\beta}}\varphi(\mathbf{z}^{\boldsymbol{\beta}})$. We claim that $\varphi(\mathbf{z}^{\boldsymbol{\gamma}})$ is the least common multiple of $\varphi(\mathbf{z}^{\boldsymbol{\alpha}})$ and $\varphi(\mathbf{z}^{\boldsymbol{\beta}})$. Since $\mathbf{z}^{\boldsymbol{\gamma}} = \mathbf{z}^{\boldsymbol{\alpha}}\, h_i = \mathbf{z}^{\boldsymbol{\beta}}\, h_j$, it follows that
$$\varphi(\mathbf{z}^{\boldsymbol{\gamma}}) = \varphi(\mathbf{z}^{\boldsymbol{\alpha}})\,\varphi(h_i) = \varphi(\mathbf{z}^{\boldsymbol{\beta}})\,\varphi(h_j),$$

i.e., $\varphi(\mathbf{z}^{\boldsymbol{\gamma}})$ is a common multiple of $\varphi(\mathbf{z}^{\boldsymbol{\alpha}})$ and $\varphi(\mathbf{z}^{\boldsymbol{\beta}})$. Suppose $\mathbf{z}^{\boldsymbol{\gamma}'}$ is another common multiple of $\varphi(\mathbf{z}^{\boldsymbol{\alpha}})$ and $\varphi(\mathbf{z}^{\boldsymbol{\beta}})$. Then $\mathbf{z}^{\boldsymbol{\gamma}'} = \varphi(\mathbf{z}^{\boldsymbol{\alpha}})\, h_i' = \varphi(\mathbf{z}^{\boldsymbol{\beta}})\, h_j'$, for some h_i', h_j'. Choose $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}$, $\tilde{h}_i$ and $\tilde{h}_j$ such that $\varphi(\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}) = \mathbf{z}^{\boldsymbol{\gamma}'}$, $\varphi(\tilde{h}_i) = h_i'$ and $\varphi(\tilde{h}_j) = h_j'$. Then, we have $\varphi(\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}) = \varphi(\mathbf{z}^{\boldsymbol{\alpha}})\, \varphi(\tilde{h}_i) = \varphi(\mathbf{z}^{\boldsymbol{\beta}})\, \varphi(\tilde{h}_j)$, i.e., $\varphi(\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}) = \varphi(\mathbf{z}^{\boldsymbol{\alpha}}\, \tilde{h}_i) = \varphi(\mathbf{z}^{\boldsymbol{\beta}}\, \tilde{h}_j)$, i.e., $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}} = \mathbf{z}^{\boldsymbol{\alpha}}\, \tilde{h}_i = \mathbf{z}^{\boldsymbol{\beta}}\, \tilde{h}_j$, i.e., $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}$ is a common multiple of $\mathbf{z}^{\boldsymbol{\alpha}}$ and $\mathbf{z}^{\boldsymbol{\beta}}$. Therefore, we have $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}} \succ_{vs_b} \mathbf{z}^{\boldsymbol{\gamma}}$, and so by the argument in the proof of part (1), we have $\varphi(\mathbf{z}^{\tilde{\boldsymbol{\gamma}}}) \succ_v \varphi(\mathbf{z}^{\boldsymbol{\gamma}})$, i.e., $\mathbf{z}^{\tilde{\boldsymbol{\gamma}}'} \succ_v \varphi(\mathbf{z}^{\boldsymbol{\gamma}})$. So, $\varphi(\mathbf{z}^{\boldsymbol{\gamma}})$ is the least common multiple of $\varphi(\mathbf{z}^{\boldsymbol{\alpha}})$ and $\varphi(\mathbf{z}^{\boldsymbol{\beta}})$. The expression

$$\frac{\varphi(\mathbf{z}^{\boldsymbol{\gamma}})}{\mathrm{in}_{\succ_v}\big(\varphi(g_i)\big)} \cdot \varphi(g_i) - \frac{\varphi(\mathbf{z}^{\boldsymbol{\gamma}})}{\mathrm{in}_{\succ_v}\varphi\big((g_j)\big)} \cdot \varphi(g_j)$$

in equation (3.8) above is therefore equal to the S-polynomial $S\big(\varphi(g_i), \varphi(g_j)\big)$, i.e.,

$$S\big(\varphi(g_i), \varphi(g_j)\big) = \varphi\big(S(g_i, g_j),$$

and so, we have the following: $\overline{S(g_i, g_j)}^{\mathcal{G}_{Q_{vs_b,w}}} = 0$ if and only if $S(g_i, g_j) = \sum h_k g_k$ for some essential minors $g_k \in \mathcal{G}_{Q_{vs_b,w}}$ and some h_k if and only if $S\big(\varphi(g_i), \varphi(g_j)\big) = \varphi\big(S(g_i, g_j)\big) = \sum \varphi(h_k)\varphi(g_k)$ for some essential minors $\varphi(g_k) \in \mathcal{G}_{T_{vs_b,w}}$ and some $\varphi(h_k)$ if and only if $\overline{S\big(\varphi(g_i), \varphi(g_j)\big)}^{\mathcal{G}_{T_{vs_b,w}}} = 0$. Hence, the result.

□

Below is a Gröbner basis result for Schubert patch ideals. Before stating this result, we want to point out that only the Gröbner basis part of this result is new. Knutson showed in his paper [Knu08] that Schubert patches degenerate to a Stanley-Reisner scheme whose underlying simplicial complex is a subword complex, which are vertex decomposable [KM05, Theorem E].

Theorem 3.2.33. *Let $w \in S_n$ be an arbitrary permutation and $v \in S_n$ be fixed. Under the term order $\succ_v$ on $\mathbb{K}[\mathbf{z}^{(v)}]$, as defined in Section 2.7, the essential minors form a Gröbner basis for $Q_{v,w} \subseteq \mathbb{K}[\mathbf{z}^{(v)}]$. In addition, the initial ideal $in_{\succ_v}(Q_{v,w})$ of $Q_{v,w}$ with respect to $\succ_v$ is squarefree and the simplicial complex associated to $in_{\succ_v}(Q_{v,w})$ is vertex decomposable.*

Proof. Let $v \in S_n$ be fixed and $R := \mathbb{K}[\mathbf{z}^{(v)}]$. We proceed by induction on $\ell(v)$. If $\ell(v) = 0$, then $v = \mathrm{id}$; the identity permutation, and so $Q_{v,w}$ is the unit ideal. If $\ell(v) = 1$, then by Lemma 3.2.29, $Q_{v,w}$ is generated by an indeterminate.

Suppose the hypotheses are true for all Schubert patch ideals $Q_{v',w}$, $v', w \in S_n$ with $v' \leq v$ in Bruhat order. If b is the last descent of v, then $vs_b \leq v$ in Bruhat order and $\ell(vs_b) = \ell(v) - 1$. Set $I := Q_{v,w}$, $J := T_{vs_b,w}$ and $N := T_{vs_b,ws_b}$. Under the positive grading of R by $\mathbb{Z}^n$, the ideals I, J and N are homogeneous by Lemma 2.8.2, Lemma 3.2.15 and Corollary 3.2.19 respectively. Let $\mathcal{G}_I$, $\mathcal{G}_J$, $\mathcal{G}_N$, $\mathcal{G}'_{Q_{vs_b,w}}$ and $\mathcal{G}'_{Q_{vs_b,ws_b}}$ be the sets of essential minors generating I, J, N, $Q_{vs_b,w}$ and Q_{vs_b,ws_b} respectively.

Suppose the variable $y := z_{\max}$ belongs to I. Then J is a unit ideal and $I = N + \langle y \rangle$. By the induction hypothesis, $\mathcal{G}'_{Q_{vs_b,ws_b}}$ is a Gröbner basis for the ideal Q_{vs_b,ws_b} with respect to the term order $\succ_{vs_b}$. Consequently, by Lemma 3.2.32, $\mathcal{G}_N$ is a Gröbner basis for the ideal N with respect to the term order $\succ_v$. Since essential minors in $\mathcal{G}_N$ do not involve y, it follows that the set $\mathcal{G}_I = \{y\} \cup \mathcal{G}_N$ of essential minors generating I is a Gröbner basis for I. On the other hand, suppose the variable y does not belong to I. Then J is proper. By the induction hypothesis, the sets $\mathcal{G}'_{Q_{vs_b,w}}$ and $\mathcal{G}'_{Q_{vs_b,ws_b}}$ are, respectively, Gröbner bases for the ideals $Q_{vs_b,w}$ and Q_{vs_b,ws_b} with respect to the term order $\succ_{vs_b}$. Consequently, by Lemma 3.2.32, $\mathcal{G}_J$ and $\mathcal{G}_N$ are, respectively, Gröbner bases for the ideals J and N with respect to the term order $\succ_v$. Therefore, if B and C are the ideals generated by the initial terms, with respect to $\succ_v$, of the elements of $\mathcal{G}_J$ and $\mathcal{G}_N$ respectively, i.e. if $B := \langle in_{\succ_v}(\mathcal{G}_J)\rangle$ and $C := \langle in_{\succ_v}(\mathcal{G}_N)\rangle$, then $B = \mathrm{in}_{\succ_v}(J)$ and $C = \mathrm{in}_{\succ_v}(N)$. Furthermore, if $A := \langle in_{\succ_v}(\mathcal{G}_I)\rangle$, then it follows from Lemma 3.2.26 that $I/N \cong [J/N](-\boldsymbol{e})$ as R/N-modules and $A/C \cong [B/C](-\boldsymbol{e})$ as R/C-modules, where $\boldsymbol{e} \in \mathbb{Z}^n$ is the degree of the variable y. Hence, by Lemma 3.1.1, $A = \mathrm{in}_{\succ_v}(I)$, i.e., the set $\mathcal{G}_I$ of essential minors generating I is a Gröbner basis for I.

From the proof of Lemma 3.2.26, we establish that $A = C + y \cdot B$. Therefore, A is squarefree since both B and C are squarefree by induction hypotheses. Lastly, let $\triangle_A$, $\triangle_B$ and $\triangle_C$ be the simplicial complexes associated to A, B and C respectively, and $\gamma \in \triangle_A$ be the vertex corresponding to the variable y. Since $A = C + y \cdot B$, it follows that $\triangle_B = \mathrm{link}_{\triangle_A}(\gamma)$ and $\triangle_C = \mathrm{del}_{\triangle_A}(\gamma)$. Therefore, $\triangle_A$ is vertex decomposable, since both $\triangle_B$ and $\triangle_C$ are vertex decomposable by the induction hypotheses. □

Example 3.2.34. Continuing with Example 3.2.28, if $\mathcal{G}_I$, $\mathcal{G}_J$, $\mathcal{G}_N$, $\mathcal{G}'_{Q_{vs_b,w}}$ and $\mathcal{G}'_{Q_{vs_b,ws_b}}$ are the sets of essential minors generating I, J, N, $Q_{vs_b,w}$ and Q_{vs_b,ws_b} respectively, then

$$\mathcal{G}_I = \left\{ \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{32} \end{vmatrix} \right\},$$

$$\mathcal{G}_J = \left\{ z_{21},\ z_{22}, \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{32} \end{vmatrix} \right\} = \mathcal{G}_{Q_{vs_3,w}},$$

and

$$\mathcal{G}_N = \left\{ \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{12} \\ 1 & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{32} \end{vmatrix} \right\} = \mathcal{G}_{Q_{vs_3,ws_3}}.$$

Set

$$A := \langle in_{\succ_v}(\mathcal{G}_I)\rangle = \langle z_{12}z_{21}, z_{13}z_{21}, z_{13}z_{22}, z_{12}, z_{22}\rangle = \langle z_{12}, z_{22}, z_{13}z_{21}\rangle,$$

$$B := \langle in_{\succ_v}(\mathcal{G}_J)\rangle = \langle z_{21}, z_{22}, z_{12}\rangle \quad \text{and} \quad C := \langle in_{\succ_v}(\mathcal{G}_N)\rangle = \langle z_{12}z_{21}, z_{12}, z_{22}\rangle.$$

Then $A/C = \langle z_{13}z_{21}\rangle$ and $B/C = \langle z_{21}\rangle$. Hence, $A/C \cong (B/C)(-\boldsymbol{e})$, as $\mathbb{K}[\mathbf{z}^{(v)}]/C$-module, where $\boldsymbol{e} = e_{v^{-1}(1)} - e_3 = e_4 - e_3$ is the degree of the variable $z_{\max} = z_{13}$. Furthermore, observe that $N \subseteq I \cap J$ and $A \subseteq \mathrm{in}_{\succ_v}(I)$. Assume that $B = \mathrm{in}_{\succ_v}(J)$ and $C = \mathrm{in}_{\succ_v}(N)$. Since $I/N \cong (J/N)(-\boldsymbol{e})$ as $\mathbb{K}[\mathbf{z}^{(v)}]/N$-module and $A/C \cong (B/C)(-\boldsymbol{e})$ as $\mathbb{K}[\mathbf{z}^{(v)}]/C$-module, it follows that $(I/N)_{\boldsymbol{\ell}} \cong (J/N)_{\boldsymbol{\ell}-\boldsymbol{e}}$ and $(A/C)_{\boldsymbol{\ell}} \cong (B/C)_{\boldsymbol{\ell}-\boldsymbol{e}}$, for all $\boldsymbol{\ell} \in \mathbb{Z}^5$, and so $A = \mathrm{in}_{\succ_v}(I)$, by Lemma 3.1.1. □

Note that the assumptions $B = \mathrm{in}_{\succ_v}(J)$ and $C = \mathrm{in}_{\succ_v}(N)$ in Example 3.2.34 mean that the sets $\mathcal{G}_J$ and $\mathcal{G}_N$ form Gröbner bases for the ideals J and N respectively. These assumptions can be easily verified either by a direct S-polynomial computation or by repeating the same procedures in Example 3.2.34 for essential minors in J and N.

3.3 Gröbner Basis for Kazhdan-Lusztig Ideals and Schubert Determinantal Ideals

In this section, we give a new proof of the known result that essential minors form Gröbner basis for Kazhdan-Lusztig ideals, and hence, for Schubert determinantal ideals.

Let $v \in S_n$ be a permutation. We recall from the beginning of Section 2.7 that when the variables in the set $\mathbf{y}^{(v)}$ are all excluded from the set $\mathbf{z}^{(v)}$, the resulting set $\tilde{\mathbf{x}}^{(v)}$, after applying the substitution $z_{ij} \mapsto x_{ij}$, equals the set $\mathbf{x}^{(v)}$ defined in 2.6. Consequently, when the variables in $\mathbf{y}^{(v)}$ are all set to zero in the matrix $\mathbf{Z}^{(v)}$ in the previous section and remaining variables (i.e., the variables in $\tilde{\mathbf{x}}^{(v)}$) are all relabeled from z_{ij} to x_{ij}, the resulting matrix is the matrix $\mathbf{X}^{(v)}$ defined in Section 2.6. The maximum variable x_{last} in $\mathbf{X}^{(v)}$, as defined in Section 2.6, is at position $(v(b+1), b)$, where b is the last descent of v. The maximum variable x_{last} in $\mathbf{X}^{(v)}$, as defined in Section 2.6, is at position $(v(b+1), b)$, where b is the last descent of v, and as a result, there are no variables in columns b and $b+1$, and strictly above row $v(b+1)$ of $\mathbf{X}^{(v)}$.

Lemma 3.3.1. [WY12, Lemma 6.13] *Given $f \in \mathbb{K}[\mathbf{x}^{(vs_b)}]$, let f' be obtained by the substitution $x_{i,b+1} \mapsto x_{i,b}$. Then $f' \in \mathbb{K}[\mathbf{x}^{(v)}]$.*

Proof. Set $a := v(b+1)$. From Lemma 2.5.14, the placement of variables of $\mathbf{X}^{(v)}$ and $\mathbf{X}^{(vs_b)}$ agree in all columns except columns b and $b+1$, and to obtain $\mathbf{X}^{(vs_b)}$ from $\mathbf{X}^{(v)}$, we move all the variables $x_{i,b}$ of $\mathbf{X}^{(v)}$ for which $i > a$ (i.e., variables in column b strictly below row a) one step to the right (i.e., to column $b+1$), and delete the variable x_{last} which appears at position (a, b) of $\mathbf{X}^{(v)}$. Consequently, these variables $x_{i,b}$ of $\mathbf{X}^{(v)}$ for which $i > a$ now become variables $x_{i,b+1}$ in $\mathbf{X}^{(vs_b)}$, for all $i > a$. Hence, the result. □

Let $v \in S_n$ be a permutation and b be the last descent of v. We observe from the proof of the result above that the number of variables in $\mathbf{X}^{(vs_b)}$ is one less than the number of variables in $\mathbf{X}^{(v)}$. Let $\mathbf{W}^{(vs_b)}$ be the resulting matrix from applying the substitution map $x_{i,b+1} \mapsto x_{i,b}$ to $\mathbf{X}^{(vs_b)}$. Below is an analogue of Definition 3.2.1 for Kazhdan-Lusztig ideals.

Definition 3.3.2. Let $v, w \in S_n$ and b be the last descent of v. Define an ideal $L_{vs_b,w} \subseteq \mathbb{K}[\mathbf{x}^{(v)}]$ as follows:

$$L_{vs_b,w} = \langle \text{minors of size } 1 + \operatorname{rank}(w_{p\times q}) \text{ in } \mathbf{W}^{(vs_b)}_{p\times q},\ \ 1 \le p, q \le n \rangle.$$

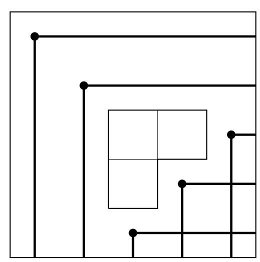
Example 3.3.3. Let $v = 45312$ and $w = 12543$. The last descent of v is $b = 3$. Matrices $\mathbf{X}^{(v)}$, $\mathbf{X}^{(vs_3)}$ and $\mathbf{W}^{(vs_3)}$, and the ideal $L_{vs_3,w}$ are given below:

$$\mathbf{X}^{(v)}=\begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 1 \\ x_{31} & x_{32} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{X}^{(vs_3)}=\begin{pmatrix} x_{11} & x_{12} & 1 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{24} & 1 \\ x_{31} & x_{32} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{W}^{(vs_3)}=\begin{pmatrix} x_{11} & x_{12} & 1 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{23} & 1 \\ x_{31} & x_{32} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$L_{vs_3,w} = \left\langle x_{22},\, x_{32},\, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{23} \\ x_{31} & 1 \end{vmatrix}, \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & 1 \end{vmatrix} \right\rangle,$$

where

$D(w) =$  .

In Example 3.2.2, recall that

$$\mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15} \\ z_{21} & z_{22} & 0 & z_{23} & 1 \\ z_{31} & z_{32} & 0 & 1 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T_{vs_3,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{42} \end{vmatrix}, \begin{vmatrix} z_{31} & z_{32} \\ 1 & z_{42} \end{vmatrix} \right\rangle.$$

Setting to zero in $T_{vs_3,w}$ the set of variables $\{z_{13}, z_{15}, z_{42}\}$ that are due east of the 1s in $\mathbf{Z}^{(v)}s_3$ and relabeling remaining variables $z_{ij} \mapsto x_{ij}$, the resulting ideal is $L_{vs_3,w}$. Furthermore, in Example 3.2.5, recall that

$$\mathbf{Z}^{(vs_3)} = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{14} & z_{15} \\ z_{21} & z_{22} & 0 & z_{24} & 1 \\ z_{31} & z_{32} & 0 & 1 & 0 \\ 1 & z_{42} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$Q_{vs_3,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{24} \\ z_{31} & 1 \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{32} & 1 \end{vmatrix}, \begin{vmatrix} z_{21} & z_{22} \\ 1 & z_{42} \end{vmatrix}, \begin{vmatrix} z_{31} & z_{32} \\ 1 & z_{42} \end{vmatrix} \right\rangle.$$

So, setting to zero in $Q_{vs_3,w}$ the set of variables $\{z_{14}, z_{15}, z_{42}\}$ that are due east of the 1s in $\mathbf{Z}^{(vs_3)}$ and relabeling remaining variables $z_{ij} \mapsto x_{ij}$, the resulting ideal is the ideal $I_{vs_3,w}$ given below:

$$I_{vs_3,w} = \left\langle x_{22},\, x_{32},\, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{24} \\ x_{31} & 1 \end{vmatrix}, \begin{vmatrix} x_{22} & x_{24} \\ x_{32} & 1 \end{vmatrix} \right\rangle.$$

□

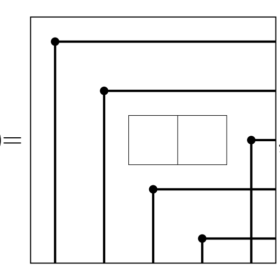
By definition, while the ideal $I_{v,w}$ is a subset of the polynomial ring $\mathbb{K}[\mathbf{x}^{(v)}]$, the ideal $L_{vs_b,w}$ is a subset of the polynomial ring $\mathbb{K}[\mathbf{x}^{(v)} \backslash \{x_{\text{last}}\}]$. The result below is an analogue of Lemma 3.2.7.

Corollary 3.3.4. *Let $v, w \in S_n$ and b be the last descent of v. If b is an ascent of w, then $I_{v,w} = L_{vs_b,w}\mathbb{K}[\mathbf{x}^{(v)}]$.*

Proof. Recall that $\mathbf{y}^{(v)}$ is the set of variables that are due east of the 1s in $\mathbf{Z}^{(v)}$. Therefore, by definition of $\mathbf{Z}^{(v)}s_b$, the set of variables that are due east of the 1s in $\mathbf{Z}^{(v)}s_b$ equals $\mathbf{y}^{(v)} \cup \{z_{ab}\}$, where $a := v(b+1)$. The ideal $I_{v,w}$ is the ideal obtained by setting to zero in $Q_{v,w}$ the variables in the set $\mathbf{y}^{(v)}$ and then relabeling remaining variables from z_{ij} to x_{ij}. Similarly, the ideal $L_{vs_b,w}$ is the ideal obtained by setting to zero in $T_{vs_b,w}$ the variables in the set $\mathbf{y}^{(v)} \cup \{z_{ab}\}$ and then relabeling remaining variables from z_{ij} to x_{ij}. From Lemma 3.2.7, we have the equality $Q_{v,w} = T_{vs_b,w}$. If same variables are therefore set to zero in this ideals, then the equality will be preserved for the resulting ideals. By Remark 3.2.8, the ideal $T_{vs_b,w}$, in particular, can be generated by essential minors that do not involve the variable z_{ab}. So for the ideals $Q_{v,w} = T_{vs_b,w}$, setting to zero in them the variables in the set $\mathbf{y}^{(v)}$ and then relabeling remaining variables from z_{ij} to x_{ij}, we will have that $I_{v,w} = L_{vs_b,w}\mathbb{K}[\mathbf{x}^{(v)}]$. In reality, the previous ideal $L_{vs_b,w}$ is a subset of $\mathbb{K}[\mathbf{x}^{(v)} \backslash \{x_{\text{last}}\}]$ since we set to zero the variable z_{ab} that correspond to the variable $x_{\text{last}} = x_{ab}$. □

Example 3.3.5. Let $v = 34512$ and $w = 12453$. Here, the last descent of v is $b = 3$, which is an ascent of w. We have the following:

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 & z_{15} \\ z_{21} & z_{22} & z_{23} & 0 & 1 \\ 1 & z_{32} & z_{33} & 0 & 0 \\ 0 & 1 & z_{43} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{Z}^{(v)}s_3 = \begin{pmatrix} z_{11} & z_{12} & 1 & z_{13} & z_{15} \\ z_{21} & z_{22} & 0 & z_{23} & 1 \\ 1 & z_{32} & 0 & z_{33} & 0 \\ 0 & 1 & 0 & z_{43} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, D(w) =$$



,

$$\mathbf{X}^{(v)} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{W}^{(vs_3)} = \begin{pmatrix} x_{11} & x_{12} & 1 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{23} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$Q_{v,w} = \left\langle \begin{vmatrix} z_{21} & z_{22} \\ 1 & \color{red}{z_{32}} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ 1 & \color{red}{z_{33}} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ \color{red}{z_{32}} & \color{red}{z_{33}} \end{vmatrix} \right\rangle = T_{vs_3,w},$$

and

$$I_{v,w} = \langle x_{22}, x_{23} \rangle = L_{vs_3,w}\mathbb{K}[\mathbf{x}^{(v)}],$$

where the variables indicated in red are the variables that are set to zero. □

Let $v, w \in S_n$ for which the last descent of v is a descent of w. Define a map

$$\phi : \mathbb{K}[\mathbf{z}^{(v)}] \to \mathbb{K}[\mathbf{x}^{(v)}] \tag{3.9}$$

by $z_{ij} \mapsto 0$, for each $z_{ij} \in \mathbf{y}^{(v)}$, and $z_{ij} \mapsto x_{ij}$, for other variables. It follows that if D is an essential minor of $T_{vs_b,w}$ (resp. T_{vs_b,ws_b}), then $\phi(D)$ is an essential minor of $L_{vs_b,w}$ (resp. L_{vs_b,ws_b}). Consequently, we have that $L_{vs_b,ws_b} \subseteq L_{vs_b,w}$, since $T_{vs_b,ws_b} \subseteq T_{vs_b,w}$ by Corollary 3.2.18. Similarly, if D is an essential minor of $Q_{v,w}$, then $\phi(D)$ is an essential minor of $I_{v,w}$. Consequently, we have that $L_{vs_b,ws_b} \subseteq I_{v,w}$, since $T_{vs_b,ws_b} \subseteq Q_{v,w}$ by Corollary 3.2.18. Hence, the following result.

Corollary 3.3.6. *Let $v, w \in S_n$ for which the last descent of v is a descent of w. If b is the last descent of v, then $L_{vs_b,ws_b} \subseteq I_{v,w} \cap L_{vs_b,w}$.*

By changing the underlying ring $\mathbb{K}[\mathbf{z}^{(v)}]$ for Schubert patch ideals in Lemma 3.2.26 and Theorem 3.2.33 to the corresponding ring $\mathbb{K}[\mathbf{x}^{(v)}]$ for Kazhdan-Lusztig ideals, we obtain proofs for the following results.

Below is a result that is an analogue of Lemma 3.2.26 for Kazhdan-Lusztig ideals.

Corollary 3.3.7. *Let $v, w \in S_n$ for which the last descent b of v is also a descent of w, and consider the positive multigrading of $R := \mathbb{K}[\mathbf{x}^{(v)}]$ by $\mathbb{Z}^n$. Assume the variable x_{last} does not belong to $\langle in_{\succ}(\mathcal{G}_I)\rangle$, where $in_{\succ}(\mathcal{G}_I)$ is the set of initial terms of essential minors generating the ideal $I := I_{v,w}$. Set $J := L_{vs_b,w}$, $N := L_{vs_b,ws_b}$, $A := \langle in_{\succ}(\mathcal{G}_I)\rangle$, $B := \langle in_{\succ}(\mathcal{G}_J)\rangle$ and $C := \langle in_{\succ}(\mathcal{G}_N)\rangle$. Then there exists $\boldsymbol{e} \in \mathbb{Z}^n$ such that there is an R/N-module isomorphism $I/N \cong (J/N)(-\boldsymbol{e})$ and an R/C-module isomorphism $A/C \cong (B/C)(-\boldsymbol{e})$.*

Proof. Same as the proof of Lemma 3.2.26, replacing mainly Corollary 3.2.18 with Corollary 3.3.6. Note that an analogue of Lemma 3.2.13 for Kazhdan-Lusztig ideals is obtained by setting to zero the variables $\mathbf{y}^{(v)} \subseteq \mathbf{z}^{(v)}$. □

In what follows, let $\succ_v$ be the term order that we use on $\mathbb{K}[\mathbf{x}^{(v)}]$ in Section 2.6 and $\succ_{vs_b}$ be the term order on $\mathbb{K}[\mathbf{x}^{(vs_b)}]$, where b is the last descent of $v \in S_n$. Unlike the substitution map φ defined in the previous section, the substitution map $x_{i,b+1} \mapsto x_{i,b}$ in Lemma 3.3.1 is order-preserving, with no restrictions. In other words, if $x_{ij} \succ_{vs_b} x_{i'j'}$ in $\mathbb{K}[\mathbf{x}^{(vs_b)}]$, then $(x_{ij})' \succ_v (x_{i'j'})'$ in $\mathbb{K}[\mathbf{x}^{(v)}]$. This is mainly because $\mathbf{X}^{(vs_b)}$ does not involve the variable $x_{\text{last}} = x_{ab}$. Observe here that $\mathbf{X}_{ij}^{(vs_b)} = 0$, for all i, j, $1 \leq i < a$ and $b \leq j \leq b+1$, i.e., the submatrix of $\mathbf{X}^{(vs_b)}$ formed by rows $1, \ldots, a-1$ and columns $b, b+1$ is a zero matrix. Observe further that when this submatrix (with rows $1, \ldots, a-1$ and columns $b, b+1$) is regarded as a submatrix of $\mathbf{Z}^{(vs_b)}$, it is not a zero matrix, and in fact, the variables in it, in addition to the variable $z_{a,b+1}$ in $\mathbf{Z}^{(vs_b)}$, are the variables we excluded for the map φ to be order-preserving. The following result is therefore an analogue of Remark 3.2.6 and Lemma 3.2.32 for Kazhdan-Lusztig ideals.

Corollary 3.3.8. *Let $v, w \in S_n$ and b be the last descent of v. Given $f \in \mathbb{K}[\mathbf{x}^{(vs_b)}]$, let f' be as in Lemma 3.3.1. Then we have the following:*

1. *$in_{\succ_v}(f') = \left(in_{\succ_{vs_b}}(f)\right)'$, for any $f \in \mathbb{K}[\mathbf{x}^{(vs_b)}]$.*
2. *f is a generator of $I_{vs_b,w}$ (resp. I_{vs_b,ws_b}) if and only if f' is a generator of $L_{vs_b,w}$ (resp. L_{vs_b,ws_b}).*
3. *The essential minors of $I_{vs_b,w}$ (resp. I_{vs_b,ws_b}) form a Gröbner basis with respect to $\succ_{vs_b}$ if and only if the essential minors of $L_{vs_b,w}$ (resp. L_{vs_b,ws_b}) form a Gröbner basis with respect to $\succ_v$.*

Recall that Schubert determinantal ideals are special case of the Kazhdan-Lusztig ideals. The next result is the Gröbner basis result for Kazhdan-Lusztig ideals [WY12, Theorem 2.1], and hence, a Gröbner basis result for Schubert determinantal ideals [KM05, Theorem B].

Corollary 3.3.9. *Let $w \in S_n$ be an arbitrary permutation and $v \in S_n$ be fixed. Under the term order $\succ_v$, the essential minors form a Gröbner basis for $I_{v,w} \subseteq \mathbb{K}[\mathbf{x}^{(v)}]$. In addition, the initial ideal $in_{\succ_v}(I_{v,w})$ of $I_{v,w}$ with respect to $\succ_v$ is squarefree and the simplicial complex associated to $in_{\succ_v}(I_{v,w})$ is vertex decomposable.*

Proof. Same as the proof of Theorem 3.2.33, replacing mainly Lemma 3.2.32, Corollary 3.2.18 and Lemma 3.2.26 by Corollary 3.3.8, Corollary 3.3.6 and Corollary 3.3.7 respectively. Note that Kazhdan-Lusztig ideals are homogeneous under the positive grading of $\mathbb{K}[\mathbf{x}^{(v)}]$ by $\mathbb{Z}^n$ (see [WY08, Lemma 5.2]). □

3.4 Further Remarks on Initial Ideals and K-Polynomials

We begin this section with the definition of K-polynomials.

Definition 3.4.1. [MS04, Definition 8.21] The Hilbert series of a finitely generated graded module M over the polynomial ring $S = [x_1, \ldots, x_n]$, positively multigraded by $\mathbb{Z}^n$, can be uniquely expressed as the quotient function

$$\mathrm{Hilb}_M(\boldsymbol{t}) = \sum_{\boldsymbol{e}\in\mathbb{Z}^n} \dim_{\mathbb{K}}(M_{\boldsymbol{e}})\boldsymbol{t}^{\boldsymbol{e}} = \frac{\mathcal{K}(M;\boldsymbol{t})}{\prod_{i=1}^{n}\left(1-\boldsymbol{t}^{\mathbf{deg}(x_i)}\right)}.$$

The numerator $\mathcal{K}(M;\boldsymbol{t})$ is called the ***K*-polynomial** of M.

Here in this section, we give a proof of a special case of the Kostant-Kumar K-polynomial recursion for Kazhdan-Lusztig ideals [WY12, Theorem 6.12].

Proposition 3.4.3 shows that most of the results given in the paper [WY12] about the initial ideals of Kazhdan-Lusztig ideals are also true for the initial ideals of Schubert patch ideals. In particular, the K-polynomials for these ideals agree.

Below is a result that partly aids the proof of Proposition 3.4.3.

Corollary 3.4.2. *Let $v \in S_n$ and b be the last descent of v. Under the term order $\succ_v$ on $\mathbb{K}[\mathbf{x}^{(v)}]$, we have the equality*

$$in_{\succ_v}(I_{v,w}) = in_{\succ_v}(L_{vs_b,ws_b}) + x_{last}\cdot in_{\succ_v}(L_{vs_b,w}),$$

of initial ideals.

Proof. From the proof of Lemma 3.2.26, we have

$$\left\langle \mathrm{in}_{\succ_v}(\mathcal{G}_{Q_{v,w}})\right\rangle = \left\langle \mathrm{in}_{\succ_v}(\mathcal{G}_{T_{vs_b,ws_b}})\right\rangle + y\cdot\left\langle \mathrm{in}_{\succ_v}(\mathcal{G}_{T_{vs_b,w}})\right\rangle,$$

where $\mathrm{in}_{\succ_v}(\mathcal{G}_{Q_{v,w}})$, $\mathrm{in}_{\succ_v}(\mathcal{G}_{T_{vs_b,w}})$ and $\mathrm{in}_{\succ_v}(\mathcal{G}_{T_{vs_b,ws_b}})$ are the sets of initial terms, with respect to $\succ_v$, of essential minors generating the ideals $Q_{v,w}$, $T_{vs_b,w}$ and T_{vs_b,ws_b} respectively. Applying the map ϕ in (3.9) to essential minors of these ideals, we have

$$\left\langle \mathrm{in}_{\succ_v}(\mathcal{G}_{I_{v,w}})\right\rangle = \left\langle \mathrm{in}_{\succ_v}(\mathcal{G}_{L_{vs_b,ws_b}})\right\rangle + y\cdot\left\langle \mathrm{in}_{\succ}(\mathcal{G}_{L_{vs_b,w}})\right\rangle,$$

where $\mathrm{in}_{\succ_v}(\mathcal{G}_{I_{v,w}})$, $\mathrm{in}_{\succ_v}(\mathcal{G}_{L_{vs_b,w}})$ and $\mathrm{in}_{\succ_v}(\mathcal{G}_{L_{vs_b,ws_b}})$ are the sets of initial terms, with respect to $\succ_v$, of essential minors generating the ideals $I_{v,w}$, $L_{vs_b,w}$ and L_{vs_b,ws_b} respectively. Therefore,

$$\mathrm{in}_{\succ_v}(I_{v,w}) = \mathrm{in}_{\succ_v}(L_{vs_b,ws_b}) + x_{\mathrm{last}}\cdot \mathrm{in}_{\succ_v}(L_{vs_b,w}).$$

□

Proposition 3.4.3. *Let v, w be permutations in S_n. Let $\tilde{I}_{v,w}$ be the ideal generated by resulting minors from applying the substitution map $x_{ij}\mapsto z_{ij}$, for all i,j, to essential minors of $I_{v,w}$. Then $in_{\succ_v}(Q_{v,w}) = in_{\succ_v}\left(\tilde{I}_{v,w}\mathbb{K}[\mathbf{z}^{(v)}]\right)$.*

Proof. Set $R := \mathbb{K}[\mathbf{z}^{(v)}]$. We proceed by induction on $\ell(v)$. If $\ell(v) = 0$, then both ideals $Q_{v,w}$ and $\tilde{I}_{v,w}$ are unit ideal. If $\ell(v) = 1$ and $Q_{v,w} \neq \mathbb{K}[\mathbf{z}^{(v)}]$, then by Lemma 3.2.29, $Q_{v,w}$ is generated by the variable z_{bb}. Recall from Section 2.7 that by setting the variables $\mathbf{y}^{(v)} \subseteq \mathbf{z}^{(v)}$ to zero in $Q_{v,w}$ and relabeling other variables from z_{ij} to x_{ij}, we obtain $I_{v,w}$. For this particular v, where $\ell(v) = 1$ and $Q_{v,w} \neq \mathbb{K}[\mathbf{z}^{(v)}]$, the only variable in $\mathbf{z}^{(v)}$ that is not in $\mathbf{y}^{(v)}$ is z_{bb}, where b is the last (and only) descent of v. It therefore follows that by setting the variables in $\mathbf{y}^{(v)}$ to zero in $Q_{v,w} = \langle z_{bb} \rangle$, we obtain $I_{v,w} = \langle x_{bb} \rangle$, and so $\tilde{I}_{v,w} = \langle z_{bb} \rangle$.

Suppose the hypothesis is true for all ideals $Q_{v',w}$ and $\tilde{I}_{v',w}$, $v', w \in S_n$ with $v' \leq v$ in Bruhat order. Let b be the last descent of v. Then $vs_b \leq v$ in Bruhat order and $\ell(vs_b) = \ell(v) - 1$. By the induction hypothesis, we have $\text{in}_{\succ_{vs_b}}(Q_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(\tilde{I}_{vs_b,w})R$ and $\text{in}_{\succ_{vs_b}}(Q_{vs_b,ws_b}) = \text{in}_{\succ_{vs_b}}(\tilde{I}_{vs_b,ws_b})R$. Let $\tilde{L}_{vs_b,w}$ be the ideal generated by resulting minors from applying the substitution map $x_{ij} \mapsto z_{ij}$, for all i, j, to essential minors of $L_{vs_b,w}$. Then by Lemma 3.2.32 and Corollary 3.3.8, we have the following equality of ideals: $\text{in}_{\succ_{vs_b}}(\hat{T}_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(Q_{vs_b,w})$, $\text{in}_{\succ_{vs_b}}(\hat{\tilde{L}}_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(\tilde{I}_{vs_b,w})$, $\text{in}_{\succ_{vs_b}}(\hat{T}_{vs_b,ws_b}) = \text{in}_{\succ_{vs_b}}(Q_{vs_b,ws_b})$ and $\text{in}_{\succ_{vs_b}}(\hat{\tilde{L}}_{vs_b,ws_b}) = \text{in}_{\succ_{vs_b}}(\tilde{I}_{vs_b,ws_b})$, where the ideals $\hat{T}_{vs_b,w}$, $\hat{T}_{vs_b,ws_b}$, $\hat{\tilde{L}}_{vs_b,w}$ and $\hat{\tilde{L}}_{vs_b,ws_b}$ are the ideals generated by minors obtained from applying the substitution map $z_{i,b+1} \mapsto z_{i,b}$ and $z_{i,b} \mapsto z_{i,b+1}$, for all i, to (essential) minors of $T_{vs_b,w}$, T_{vs_b,ws_b}, $\tilde{L}_{vs_b,w}$ and $\tilde{L}_{vs_b,ws_b}$ respectively. Therefore, $\text{in}_{\succ_{vs_b}}(\hat{T}_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(\hat{\tilde{L}}_{vs_b,w})R$, since $\text{in}_{\succ_{vs_b}}(Q_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(\tilde{I}_{vs_b,w})R$, $\text{in}_{\succ_{vs_b}}(\hat{T}_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(Q_{vs_b,w})$ and $\text{in}_{\succ_{vs_b}}(\hat{\tilde{L}}_{vs_b,w}) = \text{in}_{\succ_{vs_b}}(\tilde{I}_{vs_b,w})$. Similarly, $\text{in}_{\succ_{vs_b}}(\hat{T}_{vs_b,ws_b}) = \text{in}_{\succ_{vs_b}}(\hat{\tilde{L}}_{vs_b,ws_b})R$. Undoing the last substitution map, we have $\text{in}_{\succ_v}(T_{vs_b,w}) = \text{in}_{\succ_v}(\tilde{L}_{vs_b,w})R$ and $\text{in}_{\succ_v}(T_{vs_b,ws_b}) = \text{in}_{\succ_v}(\tilde{L}_{vs_b,ws_b})R$, noting that moving from the order $\succ_{vs_b}$ to $\succ_v$ will not pose a problem as the ideals involved can be generated by essential minors that do not involve the variables z_{ij}, for all $1 \leq i \leq v(b+1)$ and $b \leq j \leq b+1$. Consequently, $\text{in}_{\succ}(Q_{v,w}) = \text{in}_{\succ}(\tilde{I}_{v,w})R$, since we have that $\text{in}_{\succ_v}(Q_{v,w}) = \text{in}_{\succ_v}(T_{vs_b,ws_b}) + z_{\max} \cdot \text{in}_{\succ_v}(T_{vs_b,w})$ from the last part of the proof of Lemma 3.2.26 and $\text{in}_{\succ_v}(\tilde{I}_{v,w}) = \text{in}_{\succ_v}(\tilde{L}_{vs_b,ws_b}) + z_{\max} \cdot \text{in}_{\succ_v}(\tilde{L}_{vs_b,w})$ from Corollary 3.4.2. Note that $z_{\max} = z_{v(b+1),b}$ in $\mathbf{Z}^{(v)}$ and $x_{\text{last}} = x_{v(b+1),b}$ in $\mathbf{X}^{(v)}$, by Lemma 2.7.2. $\square$

Let $v \in S_n$ and b be the last descent of v. For Kostant-Kumar recursion for Kazhdan-Lusztig ideals to be well defined, it is necessary that degrees of corresponding variables in $\mathbb{K}[\mathbf{x}^{(v)}]$ and $\mathbb{K}[\mathbf{x}^{(vs_b)}]$ are preserved. See the example below for what we mean by degree preservation.

Example 3.4.4. Let $v = 34512$. The last descent of v is $b = 3$, so that $vs_3 = 34152$. Then

$$\mathbf{X}^{(v)} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{X}^{(vs_3)} = \begin{pmatrix} x_{11} & x_{12} & 1 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{24} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Using the same multigrading in Lemma 2.8.2, while the variable x_{22} in both matrices has the degree $e_{v^{-1}(2)} - e_2 = e_{(vs_3)^{-1}(2)} - e_2 = e_5 - e_2$, the variable x_{12} has degree $e_{v^{-1}(1)} - e_2 = e_4 - e_2$ in $\mathbf{X}^{(v)}$ and degree

$e_{(vs_3)^{-1}(1)} - e_2 = e_3 - e_2$ in $\mathbf{X}^{(vs_3)}$. So, in this example, degrees of some variables in both matrices are not preserved. □

We do not encounter this problem of degree preservation when we change our conventions and work in the complete flag variety G/B_+, as in the paper [WY12]. In G/B_+, the Kazhdan-Lusztig variety $\mathcal{N}_{u,w}$ is isomorphic to the intersection of the Schubert variety $X_w = \overline{B_+wB_+/B_+}$ with the opposite Schubert cell $\Omega_v^\circ := B_-vB_+/B_+$. For a choice of coordinate system, as in the same paper [WY12], fix an arbitrary permutation $v \in S_n$ and define a specialized generic matrix $\mathbf{X}^{(v)}$ of size $n \times n$ as follows: for all j, set $\mathbf{X}^{(v)}_{n-v(j)+1,j} = 1$, and, for all j, set $\mathbf{X}^{(v)}_{n-v(j)+1,c} = 0$ and $\mathbf{X}^{(v)}_{d,j} = 0$ for $c > j$ and $d > n - v(j)+1$. For all other coordinates (i, j), set $\mathbf{X}^{(v)}_{i,j} = x_{ij}$.

Example 3.4.5. Let $v = 32154$. The last ascent (as opposed to last descent in the previous section) of v is $b = 3$, so that $vs_3 = 32514$. Then

$$\underset{\mathbf{X}^{(v)}}{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 1 \\ x_{11} & x_{12} & x_{13} & 1 & 0 \end{pmatrix}}, \quad \underset{\mathbf{X}^{(vs_3)}}{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{24} & 1 \\ x_{11} & x_{12} & 1 & 0 & 0 \end{pmatrix}}, \quad \underset{\mathbf{X}^{(v)}s_3}{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{23} & 1 \\ x_{11} & x_{12} & 1 & x_{13} & 0 \end{pmatrix}}.$$

□

Note from the above example that the matrix $\mathbf{X}^{(vs_3)}$ is equal to matrix $\mathbf{X}^{(v)}s_3$, up to relabeling the variable x_{24} to x_{23} ($x_{j,b+1} \mapsto x_{j,b}$) and setting $x_{\text{last}} = x_{13} \in \mathbf{x}^{(v)}$ to zero. Also observe that on multiplying (left multiplication) the matrices $\mathbf{X}^{(v)}$ and $\mathbf{X}^{(vs_3)}$ above by the permutation matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

corresponding to the long word $w_0 \in S_n$, the resulting matrices are the matrices $\mathbf{X}^{(v)}$ and $\mathbf{X}^{(vs_3)}$ in Example 3.4.4.

Example 3.4.4 suggests we change our conventions. The following are established in [WY12] in relation to degree preservation of the variables in $\mathbf{X}^{(v)}$ and $\mathbf{X}^{(vs_b)}$, where b is the last ascent of v. Given a permutation $v \in S_n$, the variables x_{ij} in the matrix $\mathbf{X}^{(v)}$ has degree $e_{v(j)} - e_{n-i+1}$, where e_i is the i^{th} standard basis vector in $\mathbb{Z}^n$. Setting $R_v := \mathbb{K}[\mathbf{x}^{(v)}]$ and $R_{vs_b} := \mathbb{K}[\mathbf{x}^{(vs_b)}]$, we have: (i) in both R_v and R_{vs_b}, the variables in $\mathbf{X}^{(v)}$ which are not in column b or $b+1$ have the same degrees as the corresponding variables in $\mathbf{X}^{(vs_b)}$

which are not in column b or $b+1$, (ii) the variables $x_{j,b}$ (except $j = v(b+1)$) in R_v have the same degrees as the variable $x_{j,b+1}$ in R_{vs_b}, and (iii) the rightmost and lowermost variable of $\mathbf{X}^{(v)}$, denoted x_{last}, does not appear in $\mathbf{X}^{(vs_b)}$, and it has degree $e_{v(b)} - e_{v(b+1)}$.

Remark 3.4.6. Let $v \in S_n$ and b be the last ascent of v. Set $A' := \langle f' \mid f \in A \rangle$ (as in Lemma 3.3.1), $I := I_{v,w}$, $J := I_{vs_b,w}$ and $N := I_{vs_b,ws_b}$. The ideals I, J' and N' satisfy the hypothesis of Corollary 3.3.7 and consequently, there exists $\boldsymbol{e} \in \mathbb{Z}^n$ such that there is an R_v/N'-module isomorphism $I/N' \cong (J'/N')(-\boldsymbol{e})$. □

Below is a Kostant-Kumar K-polynomial recursion for the Kazhdan-Lusztig ideals [WY12, Theorem 6.12] for last ascents, originally found in [KK90, Proposition 2.4]. The characteristic of a ring is the least positive integer n such that $n \cdot 1 = 0$ in this ring, and if no such n exists for a ring, then the ring has characteristic 0. We note that our proof of Kostant-Kumar K-polynomial recursion for the Kazhdan-Lusztig ideals for special case of last ascents is not dependent on the characteristic of the underlying field.

Proposition 3.4.7. *Let v and w be permutations in S_n and let b be the last ascent of v, so that $vs_b > v$ in Bruhat order. Set $I := I_{v,w}$, $J := I_{vs_b,w}$ and $N := I_{vs_b,ws_b}$. Then*

1. *If b is a descent of w, so that $ws_b < w$, then*

$$\mathcal{K}(R_v/I; \boldsymbol{t}) = \mathcal{K}(R_{vs_b}/J; \boldsymbol{t}).$$

2. *If b is an ascent of w, so that $ws_b > w$, then*

$$\mathcal{K}(R_v/I; \boldsymbol{t}) \quad = \quad \mathcal{K}(R_{vs_b}/J; \boldsymbol{t}) + (1 - \boldsymbol{t}^{\boldsymbol{e}})\mathcal{K}(R_{vs_b}/N; \boldsymbol{t}) - (1 - \boldsymbol{t}^{\boldsymbol{e}})\mathcal{K}(R_{vs_b}/J; \boldsymbol{t}),$$

where $\boldsymbol{e}$ is the degree (weight) of the variable x_{last} in $\mathbf{X}^{(v)}$.

Proof. The first part follows immediately from a version of Corollary 3.3.4 in G/B_+ setting. For the second part, we proceed as follows: first, since $0 \to I/N' \to R_v/N' \to R_v/I \to 0$, where $N' = \langle f' \mid f \in N \rangle$, we have

$$\text{Hilb}_{R_v/I}(\boldsymbol{t}) = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \dim_{\mathbb{K}} \left((R_v/I)_{\boldsymbol{\ell}} \right) \boldsymbol{t}^{\boldsymbol{\ell}} = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \left[\dim_{\mathbb{K}} \left((R_v/N')_{\boldsymbol{\ell}} \right) - \dim_{\mathbb{K}} \left((I/N')_{\boldsymbol{\ell}} \right) \right] \boldsymbol{t}^{\boldsymbol{\ell}}.$$

Next, since $(I/N')_{\boldsymbol{\ell}} \cong (J'/N')_{\boldsymbol{\ell}-\boldsymbol{e}}$ by Remark 3.4.6, we have

$$\text{Hilb}_{R_v/I}(\boldsymbol{t}) = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \left[\dim_{\mathbb{K}} \left((R_v/N')_{\boldsymbol{\ell}} \right) - \dim_{\mathbb{K}} \left((J'/N')_{\boldsymbol{\ell}-\boldsymbol{e}} \right) \right] \boldsymbol{t}^{\boldsymbol{\ell}}.$$

Furthermore, since $0 \to J'/N' \to R_v/N' \to R_v/J' \to 0$, we have

$$\begin{aligned}
\text{Hilb}_{R_v/I}(\boldsymbol{t}) \quad &= \quad \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \dim_{\mathbb{K}} \left((R_v/N')_{\boldsymbol{\ell}} \right) \boldsymbol{t}^{\boldsymbol{\ell}} - \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \left[\dim_{\mathbb{K}} \left((R_v/N')_{\boldsymbol{\ell}-\boldsymbol{e}} \right) - \dim_{\mathbb{K}} \left((R_v/J')_{\boldsymbol{\ell}-\boldsymbol{e}} \right) \right] \boldsymbol{t}^{\boldsymbol{\ell}} \\
&= \quad \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \dim_{\mathbb{K}} \left((R_v/N')_{\boldsymbol{\ell}} \right) \boldsymbol{t}^{\boldsymbol{\ell}} - \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \left[\dim_{\mathbb{K}} \left((R_v/N')_{\boldsymbol{\ell}-\boldsymbol{e}} \right) - \dim_{\mathbb{K}} \left((R_v/J')_{\boldsymbol{\ell}-\boldsymbol{e}} \right) \right] \boldsymbol{t}^{\boldsymbol{\ell}-\boldsymbol{e}} \boldsymbol{t}^{\boldsymbol{e}} \\
&= \quad \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \dim_{\mathbb{K}} \left((R_{vs_b}/N)_{\boldsymbol{\ell}} \right) \boldsymbol{t}^{\boldsymbol{\ell}} - \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n} \left[\dim_{\mathbb{K}} \left((R_{vs_b}/N)_{\boldsymbol{\ell}-\boldsymbol{e}} \right) \right.
\end{aligned}$$

$$- \ \dim_{\mathbb{K}} \left((R_{vs_b}/J)_{\boldsymbol{\ell}-\boldsymbol{e}}\right)\Big] \boldsymbol{t}^{\boldsymbol{\ell}-\boldsymbol{e}}\boldsymbol{t}^{\boldsymbol{e}}$$

$$= \ \mathrm{Hilb}_{R_{vs_b}/N}(\boldsymbol{t}) - \boldsymbol{t}^{\boldsymbol{e}}\big[\mathrm{Hilb}_{R_{vs_b}/N}(\boldsymbol{t}) - \mathrm{Hilb}_{R_{vs_b}/J}(\boldsymbol{t})\big],$$

i.e.,

$$\frac{\mathcal{K}(R_v/I;\boldsymbol{t})}{\prod_{i,j}(1-\boldsymbol{t}^{\boldsymbol{a}_{ij}})} = \frac{\mathcal{K}(R_{vs_b}/N;\boldsymbol{t})}{\prod_{i,j}(1-\boldsymbol{t}^{\boldsymbol{a}_{ij}})} - \boldsymbol{t}^{\boldsymbol{e}}\left[\frac{\mathcal{K}(R_{vs_b}/N;\boldsymbol{t})}{\prod_{i,j}(1-\boldsymbol{t}^{\boldsymbol{a}_{ij}})} - \frac{\mathcal{K}(R_{vs_b}/J;\boldsymbol{t})}{\prod_{i,j}(1-\boldsymbol{t}^{\boldsymbol{a}_{i,j}})}\right],$$

where each $\boldsymbol{a}_{ij} \in \mathbb{Z}^n$ is the degree of variable x_{ij} in $\mathbf{X}^{(v)}$. Thus,

$$\begin{aligned}\mathcal{K}(R_v/I;\boldsymbol{t}) &= \mathcal{K}(R_{vs_b}/N;\boldsymbol{t}) - \boldsymbol{t}^{\boldsymbol{e}}\left[\mathcal{K}(R_{vs_b}/N;\boldsymbol{t}) - \mathcal{K}(R_{vs_b}/J;\boldsymbol{t})\right] \\ &= \mathcal{K}(R_{vs_b}/J;\boldsymbol{t}) + (1-\boldsymbol{t}^{\boldsymbol{e}})\mathcal{K}(R_{vs_b}/N;\boldsymbol{t}) - (1-\boldsymbol{t}^{\boldsymbol{e}})\mathcal{K}(R_{vs_b}/J;\boldsymbol{t}). \qquad \square\end{aligned}$$

3.5 G-Biliaison of Homogeneous Schubert Patch Ideals

Not all Schubert patch ideals are homogeneous with respect to the standard grading of assigning degree 1 to each variable in $R := \mathbb{K}[\mathbf{z}^{(v)}]$. Here, we say a Schubert patch ideal $Q_{v,w}$ is **standardly homogeneous** if it is homogeneous with respect to the standard grading of assigning degree 1 to each variable in R. Suppose $Q_{v,w}$ is standardly homogeneous and let b be the last descent of v. If b is an ascent of w, then, by Lemma 3.2.7, $T_{vs_b,w}$ is also standardly homogeneous. On the other hand, if b is a descent of w, then it follows from Lemma 3.2.13 and Lemma 3.2.16 that both ideals $T_{vs_b,w}$ and T_{vs_b,ws_b} are also standardly homogeneous. Furthermore, with respect to this standard grading, the isomorphisms in Lemma 3.2.26 are standard graded isomorphisms specifically for the ideals involved, where $n = 1$ and $\boldsymbol{e} = 1$. In summary, if $I := Q_{v,w}$ is standardly homogeneous, then $I/N \cong (J/N)(-1)$ is an R/N module isomorphism, where $J := T_{vs_b,w}$ and $N := T_{vs_b,ws_b}$.

In this chapter, we will show that every standardly homogeneous Schubert patch ideal is glicci. With a minor modification to the proof of this glicci result, we can show that every standardly homogeneous Kazhdan-Lusztig ideal, and hence every Schubert determinantal ideal, is glicci. Here forward, we will write $I \xrightarrow{N}_h J$ to mean that the ideal I can be obtained from another ideal J by an elementary G-biliaison of height h on N. So if a homogeneous, saturated and unmixed ideal $I \subseteq R$ is obtained from a complete intersection generated by linear forms by a finite sequence of ascending elementary G-biliaisons of height h, then there exists Cohen-Macaulay and generically Gorenstein ideals $N_1, \ldots, N_t$ in R such that $I = I_0 \xrightarrow{N_1}_h I_1 \xrightarrow{N_2}_h I_2 \xrightarrow{N_3}_h \cdots \xrightarrow{N_t}_h I_t$, where $I_t \subseteq R$ is a complete intersection generated by linear forms.

Lemma 3.5.1. *Let $v \in S_n$ and b be the last descent of v. If $T_{vs_b,w}$ can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons, then $Q_{v,w}$ can also be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons.*

Proof. We prove this lemma in two different cases, namely when b is an ascent of w and when b is a descent of w.

Case 1 Suppose b is an ascent of w. Then by Lemma 3.2.7, the ideals $Q_{v,w}$ and $T_{vs_b,w}$ are equal. So if $\mathcal{F}: T_{vs_b,w} = I_0 \overset{N_1}{\nearrow}_h I_1 \overset{N_2}{\nearrow}_h I_2 \overset{N_3}{\nearrow}_h \cdots \overset{N_t}{\nearrow}_h I_t$, is a finite sequence of ascending elementary G-biliaisons of height $h = 1$ for $T_{vs_b,w}$ up to an ideal generated by linear forms, then the same sequence $\mathcal{F}$ is a finite sequence of ascending elementary G-biliaisons of height 1 for $Q_{v,w}$ up to an ideal generated by linear forms.

Case 2 Suppose b is a descent of w and

$$\mathcal{F}: \qquad T_{vs_b,w} = I_0 \overset{N_1}{\nearrow}_h I_1 \overset{N_2}{\nearrow}_h I_2 \overset{N_3}{\nearrow}_h \cdots \overset{N_t}{\nearrow}_h I_t,$$

is a finite sequence of ascending elementary G-biliaisons of height $h = 1$ for $T_{vs_b,w}$ up to aan ideal generated by linear forms. Then we claim that the sequence

$$\mathcal{G}: \qquad Q_{v,w} \overset{T_{vs_b,ws_b}}{\nearrow}_h T_{vs_b,w} \overset{N_1}{\nearrow}_h I_1 \overset{N_2}{\nearrow}_h I_2 \overset{N_3}{\nearrow}_h \cdots \overset{N_t}{\nearrow}_h I_t,$$

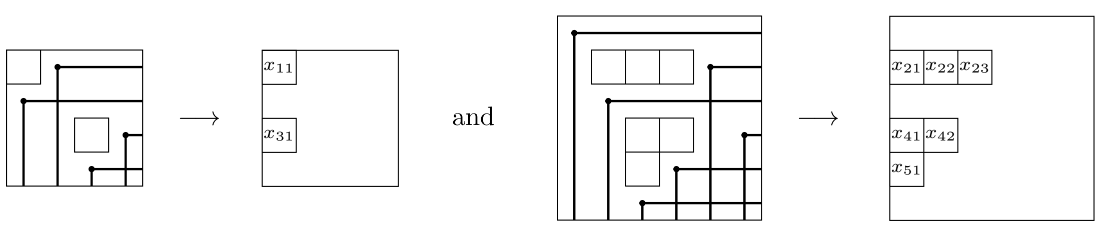
is a finite sequence of ascending elementary G-biliaisons of height $h = 1$ for $Q_{v,w}$ up to an ideal generated by linear forms. In order to establish this, it suffices to only show that $Q_{v,w}$ can be obtained from $T_{vs_b,w}$ by an ascending elementary G-biliaison of height 1 on T_{vs_b,ws_b}, i.e., $Q_{v,w} \overset{T_{vs_b,ws_b}}{\nearrow}_1 T_{vs_b,w}$. To this end, we will verify the statements in Definition 2.3.11. First, we infer from Lemma 3.2.32 that $\operatorname{ht} T_{vs_b,w} = \operatorname{ht} Q_{vs_b,w}$ and $\operatorname{ht} T_{vs_b,ws_b} = \operatorname{ht} Q_{vs_b,ws_b}$, i.e., $\operatorname{ht} T_{vs_b,w} = \ell(w) = \operatorname{ht} Q_{v,w}$ and $\operatorname{ht} T_{vs_b,ws_b} = \ell(ws_b) = \ell(w) - 1$. From Corollary 3.2.18, we have that $T_{vs_b,ws_b} \subseteq Q_{v,w} \cap T_{vs_b,w}$. In addition, since the prime ideal Q_{vs_b,ws_b} is Cohen-Macaulay and generically Gorenstein, we also infer from Lemma 3.2.32 that T_{vs_b,ws_b} is also Cohen-Macaulay and generically Gorenstein. Lastly, it follows from the arguments at the very beginning of this section that $Q_{v,w}/T_{vs_b,ws_b} \cong (T_{vs_b,w}/T_{vs_b,ws_b})(-1)$ as $\left(\mathbb{K}[\mathbf{z}^{(v)}]/T_{vs_b,ws_b}\right)$-module. Hence, there is an elementary G-biliaison. □

We will now consider the linkage process of the Schubert patch ideals that are homogeneous with respect to the standard grading.

Theorem 3.5.2. *Every standardly homogeneous Schubert patch ideal $Q_{v,w}$, $v, w \in S_n$, belongs to the G-liaison class of a complete intersection. That is, $Q_{v,w}$ is glicci.*

Proof. It suffices to show that any standardly homogeneous ideal $Q_{v,w}$ can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons, and so by Theorem 2.3.12, it will follow that $Q_{v,w}$ is glicci. We proceed by induction on $\ell(v)$. If $\ell(v) = 1$, then by Lemma 3.2.29, $Q_{v,w}$ is generated by an indeterminate and hence, the result. Suppose the hypothesis is true for all standardly homogeneous Schubert patch ideals $Q_{v',w}$, $v', w \in S_n$, with $v' \leq v$ in Bruhat order. Let b be the last descent of v. Then $vs_b \leq v$ in Bruhat order and $\ell(vs_b) = \ell(v) - 1$. Suppose the variable $z_{\max}$ belongs to $Q_{v,w}$. Then $Q_{vs_b,w}$ is a unit ideal and $Q_{v,w} = Q_{vs_b,ws_b} + \langle z_{\max} \rangle$. By the induction hypothesis, Q_{vs_b,ws_b} can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons. Consequently, since generators of $Q_{v,w}$ has one more generator (an indeterminate) than generators of Q_{vs_b,ws_b}, it follows that $Q_{v,w}$ can also be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons. Therefore, by Theorem 2.3.12, the ideal $Q_{v,w}$ is in the G-liaison class of a complete intersection. On the other hand, suppose the variable $z_{\max}$ does not belong to $Q_{v,w}$. Then $Q_{vs_b,w}$ is proper and so by the induction hypothesis, it follows that $Q_{vs_b,w}$ can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons. Consequently, we can infer from Remark 3.2.6 and Lemma 3.2.32 that $T_{vs_b,w}$ can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons. Therefore, by Lemma 3.5.1, the ideal $Q_{v,w}$ can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons. Thus, by Theorem 2.3.12, $Q_{v,w}$ is in the G-liaison class of a complete intersection. □

The proof for Theorem 3.5.2 can be easily adapted to show that standardly homogeneous Kazhdan-Lusztig ideals are glicci. Consequently, the Schubert determinantal ideals are also glicci, since they are a special case of the standardly homogeneous Kazhdan-Lusztig ideals. In particular, the final ideal generated by the linear forms in the linkage process of a Schubert determinantal ideal I_w is explicitly given as $\big\langle x_{i,(j-\text{rank}(w_{i\times j}))} \,|\, (i,j) \text{ is a box in } D(w) \big\rangle$, which is a complete intersection. This is true since the linkage process of the ideal I_w basically involves "movement" of essential boxes (or boxes in general) one step at a time to the left until they are no longer "movable". The set of corresponding variables at each of the final destinations of these boxes in $D(w)$ gives the generator for final ideal in the linkage process of I_w. For example, for the permutation $w = 2143 \in S_4$, the ideal $\langle x_{11}, x_{31} \rangle$ is the last ideal in the linkage process of I_w. Another example is $w = 136524 \in S_6$; the final ideal in the linkage process of I_w is $\langle x_{21}, x_{22}, x_{23}, x_{41}, x_{42}, x_{51} \rangle$. See below for illustrations of these movements.



Though we can infer from the paper [KR21b] that Schubert determinantal ideals are glicci, their approach relies on combinatorial results of Knutson-Miller [KM05] to deduce this.

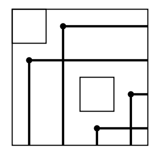
Corollary 3.5.3. *Every standardly homogeneous Kazhdan-Lusztig ideal is glicci. In particular, the Schubert determinantal ideals are glicci.*

3.6 On when some Schubert Patch Ideals are Homogeneous with respect to the Standard Grading

Homogeneity with respect to the standard grading is needed for our results in Section 3.5. Not all Schubert patch ideals are standardly homogeneous as illustrated in Example 3.6.1. So it is natural to ask for which pairs $(v, w) \in S_n \times S_n$ is the ideal $Q_{v,w}$ standardly homogeneous (homogeneous with respect to the standard grading of assigning degree 1 to each variable in $\mathbb{K}[\mathbf{z}]$). The analogue of this problem for Kazhdan-Lusztig ideals is [WY08, Problem 5.5].

Example 3.6.1. Let $v = 4231$ and $w = 2143$. Then we have

$$\mathbf{Z}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & z_{23} & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad D(w) =$$



and $Q_{v,w} = \langle z_{11},\ z_{13}z_{31} + z_{12}z_{21} - z_{12}z_{31}z_{23} \rangle$, which is inhomogeneous. □

Proposition 3.6.2. *Let $v, w \in S_n$ be fixed. If $Q_{v,w}$ is homogeneous with respect to the standard grading on $\mathbb{K}[\mathbf{z}^{(v)}]$, then $I_{v,w}$ is homogeneous with respect to the standard grading on $\mathbb{K}[\mathbf{x}^{(v)}]$.*

Proof. Setting some variables to zero in $Q_{v,w}$ will not affect homogeneity of the resulting ideal. □

The converse of Proposition 3.6.2 is not always true. To see this, let $v = 4231$ and $w = 2143$ (as in Example 3.6.1). We have

$$\mathbf{X}^{(v)} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

While $Q_{v,w}$ is not homogeneous, the ideal $I_{v,w} = \langle z_{11},\ z_{13}z_{31} + z_{12}z_{21} \rangle$ is homogeneous.

In this section, we wish to classify some Schubert patch ideals and Kazhdan-Lusztig ideals that are standardly homogeneous.

Definition 3.6.3. A permutation $w \in S_n$ is called **321-avoiding** if there do not exist three integers $i_1 < i_2 < i_3$ with $w(i_3) < w(i_2) < w(i_1)$.

Definition 3.6.4. A permutation $w \in S_n$ is called **231-avoiding** if there do not exist three integers $i_1 < i_2 < i_3$ with $w(i_3) < w(i_1) < w(i_2)$.

Definition 3.6.5. A permutation $w \in S_n$ is called **132-avoiding** if there do not exist three integers $i_1 < i_2 < i_3$ with $w(i_1) < w(i_3) < w(i_2)$.

Example 3.6.6. Below are some examples and non-examples of *321*, *231* and *132*-avoiding permutations.

Permutation	**Type**	**Permutation**	**Type**	**Permutation**	**Type**
1**243**	Non 132-avoiding	**5132**4	Non 321-avoiding	**41**2**53**	Non 231-avoiding
34215	132-avoiding	4612735	321-avoiding	51324	231-avoiding

□

If a permutation is non *132*-avoiding, then we say it has at least one *132* pattern. Similarly, if a permutation is non *321*-avoiding (resp. *231*-avoiding), then we say it has at least one *321* (resp. *231*) pattern. For instance, while the non *132*-avoiding permutation $1243 \in S_4$ has only one *132* pattern, which is *243*, the non *321*-avoiding permutation $52143 \in S_5$ has only two *321* patterns, which are *521* and *543*.

Proposition 3.6.7. *Let $(v, w) \in S_n \times S_n$ for which $w \leq v$ in Bruhat order.*

a. If w is 132-avoiding, then $Q_{v,w}$ is standardly homogeneous.

b. If v is both 321-avoiding and 231-avoiding, then $Q_{v,w}$ is standardly homogeneous.

c. Suppose v is either non 321-avoiding or non 231-avoiding, and w is non 132-avoiding. Then $Q_{v,w}$ is standardly homogeneous if

- *for all pairs of patterns $(a_1\, a_2\, a_3, c_1\, c_2\, c_3)$, either*

$$c_3 < a_2 \qquad \text{or} \qquad w^{-1}(c_2) < v^{-1}(a_2), \tag{3.10}$$

  and

- *for all pairs of patterns $(b_1\, b_2\, b_3, c_1\, c_2\, c_3)$, either*

$$c_3 < b_1 \qquad \text{or} \qquad w^{-1}(c_2) \leq v^{-1}(b_1), \tag{3.11}$$

where $a_1\, a_2\, a_3$ and $b_1\, b_2\, b_3$ are 321 and 231 patterns in v respectively, and $c_1\, c_2\, c_3$ is a 132 pattern in w.

Proof.

a. If w is 132-avoiding, then from [Sta11, Section 1.5], the diagram $D(w)$ forms a partition, and hence, the ideal $Q_{v,w}$, for any v, is generated by indeterminate(s). Therefore, $Q_{v,w}$ is homogeneous with respect to the standard grading.

b. Suppose $Q_{v,w}$ is not standardly homogeneous. Then we will show that v is either non *321*-avoiding or non *231*-avoiding. In other words, we will show that v either has a *321* or a *231* pattern. Since $Q_{v,w}$ is not standardly homogeneous, it follows that that there is at least an essential minor f in $Q_{v,w}$ that is not standardly homogeneous. Suppose the corresponding essential box of f is (p,q). Set $M := \mathbf{Z}^{(v)}_{p\times q}$, where $\mathbf{Z}^{(v)}_{p\times q}$ is the upper left $p\times q$ rectangular submatrix of $\mathbf{Z}^{(v)}$. We note that f is one of the minors of size $(1+\text{rank}(w_{p\times q}))$ in M. We will show that a matrix of the form

$$N = \begin{bmatrix} z_{ik} & z_{il} \\ z_{jk} & 1 \end{bmatrix} \qquad \text{or} \qquad N' = \begin{bmatrix} z_{ik} & z_{il} \\ 1 & z_{jl} \end{bmatrix} \tag{3.12}$$

is a submatrix of M. We show this by induction on the degree of f. If the degree of f is 2, then there exists a 2×2 submatrix N'' of M whose determinant is f. We first note that since f is not standardly homogeneous, it follows that $\det(N'')$ is not equal to a constant (0 or ± 1). The matrix N'' must therefore take one of the following forms:

$$\begin{bmatrix} z_{ik} & z_{il} \\ z_{jk} & z_{jl} \end{bmatrix}, \begin{bmatrix} 1 & z_{il} \\ 0 & z_{jl} \end{bmatrix}, \begin{bmatrix} z_{ik} & 1 \\ z_{jk} & 0 \end{bmatrix}, \begin{bmatrix} z_{ik} & z_{il} \\ 1 & z_{jl} \end{bmatrix}, \begin{bmatrix} z_{ik} & z_{il} \\ z_{jk} & 1 \end{bmatrix},$$

for which the determinant of the last two matrices are the only polynomials that are not standardly homogeneous. Now, suppose $\deg(f) \geq 2$ and let M' be the corresponding (square) submatrix of M whose determinant is f. We note that M' can take two forms:

Case 1 M' does not have a row or a column with exactly one 1 and the rest of the entries in either of this row or column are 0. The polynomial f is not standardly homogeneous in the first place due to at least one 1 at a strategic position in M; otherwise, all the terms of f will be of same degree, and hence f will be homogeneous. Precisely, M' has at least one 1 among its entries. Therefore, in this case, both the row of this 1 and the column of this 1 contain some variables. In fact, if the 1 is in position (j',l'), then there must be a variable in either position (j',k'), where $k' < l'$, or position (j',k''), where $l' < k''$, and another variable in position (i,l'), where $i < j'$. Hence, there is also a variable in either position (i,k') or position (i,k''), as desired (see below).

$$\begin{bmatrix} z_{ik'} & z_{il'} \\ z_{j'k'} & 1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} z_{il'} & z_{ik''} \\ 1 & z_{j'k''} \end{bmatrix}.$$

Case 2 M' has a row or a column with exactly one 1 and the rest of the entries in either of this row or column are 0. In this case, using cofactor expansions along that row or column, we see that up to sign, $\det(M') = \det(M'')$, where M'' is the resulting matrix from deleting a row and a column from M'. By induction, M'' has a submatrix of the form N or N', and therefore, M also has either of these submatrices.

So, in either case, a matrix of the form N or N' is a submatrix of M. If N is a submatrix of M, then the 1 in column k of $\mathbf{Z}^{(v)}$ is in row $v(k)$ and the 1 in row i of $\mathbf{Z}^{(v)}$ is in column $v^{-1}(i)$. Thus, there exists three integers $k < l < v^{-1}(i)$ such that $v(v^{-1}(i)) < v(l) < v(k)$, i.e., $i < j < v(k)$, i.e., we have explicitly obtained a *321* pattern in v, say $v(k)\ v(l)\ v(v^{-1}(i))$ (or $v(k)\ j\ i$). Similarly, if N' is a submatrix of M, then there exists three integers $k < l < v^{-1}(i)$ such that $v(v^{-1}(i)) < v(k) < v(l)$, i.e., $i < j < v(l)$, i.e., we have explicitly obtained a *231* pattern in v, say $v(k)\ v(l)\ v(v^{-1}(i))$ (or $j\ v(l)\ i$). Hence, the result.

c. Suppose $Q_{v,w}$ is not standardly homogeneous. Then we will find a *321* or a *231* pattern in v, and a *132* pattern in w for which the inequalities 3.10 and 3.11 fail. If $Q_{v,w}$ is not standardly homogeneous, then there is at least an (essential) minor f in $Q_{v,w}$, that is not standardly homogeneous. Suppose the corresponding essential box of this determinant f is (p, q). Then $\text{rank}(w_{p\times q}) > 0$. Consequently, it follows that there is at least a 1 strictly northwest of the location (p, q) in $D(w)$. Suppose one of these 1s is in position (α, β). That is, $\alpha < p$ and $\beta < q$. We also note that in this case, $\alpha = w(\beta)$. In the diagram $D(w)$ of w, let the 1 to the right of box (p, q) be in column γ. Then $q < \gamma$, since the box (p, q) is in column q. Since $w(\gamma)$ is the row for which the box (p, q) is located, it follows that $w(\gamma) = p$. So, we have three integers β, q and γ with the property that $1 \le \beta < q < \gamma \le n$. Since the 1 in column q of diagram $D(w)$ is strictly below the box (p, q), it follows that $p < w(q)$, where $w(q)$ is the row where this 1 is located. Therefore, we have the inequality $\alpha < p < w(q)$, which can also be written as $w(\beta) < w(\gamma) < w(q)$. Thus, we have explicitly obtained a *132* pattern in w.

Furthermore, consider the submatrix $M := \mathbf{Z}^{(v)}_{p\times q}$. From the proof of part (b), we have that a matrix of the form N or N', as in (3.12), is a submatrix of M. The corresponding *321* pattern for N is $v(k)\ v(l)\ v(v^{-1}(i))$ (or $v(k)\ j\ i$) and the corresponding *231* pattern for N' is $v(k)\ v(l)\ v(v^{-1}(i))$ (or $j\ v(l)\ i$). If N is a submatrix of M, then $j \le p$ and $v^{-1}(j) \le w^{-1}(w(q))$ (or $l \le q$), which contradicts inequality 3.10. Furthermore, if N' is a submatrix of M, then $j \le p$ and $v^{-1}(j) < w^{-1}(w(q))$ (or $k < q$), which contradicts inequality 3.11. $\square$

The analogue of Proposition 3.6.7 for Kazhdan-Lusztig ideals is the following result.

Proposition 3.6.8. *Let $(v, w) \in S_n \times S_n$ for which $w \le v$ in Bruhat order.*

(a) If v is 321-avoiding or w is 132-avoiding, then $I_{v,w}$ is standardly homogeneous.

(b) Suppose v is non 321-avoiding and w is non 132-avoiding. Then $I_{v,w}$ is standardly homogeneous if for

all pairs $(a_1\,a_2\,a_3, c_1\,c_2\,c_3)$*, either* $c_3 < a_2$ *or* $w^{-1}(c_2) < v^{-1}(a_2)$*, where* $a_1\,a_2\,a_3$ *is a 321 pattern in* v *and* $c_1\,c_2\,c_3$ *is a 132 pattern in* w*.*

Proof.

(a) If v is 321-avoiding, then $I_{v,w}$, for all w, is known to be homogeneous (see for example the footnote on page 25 of [Knu09]). If w is 132-avoiding, then the ideal $I_{v,w}$, for any v, is generated by indeterminate(s), and hence, homogeneous with respect to the standard grading.

(b) This is a special case of part (c) of Proposition 3.6.7. Recall from Proposition 3.6.2 that $I_{v,w}$ is standardly homogeneous whenever $Q_{v,w}$ is standardly homogeneous. □

Chapter 4

Some Computations and Future Directions

4.1 Gröbner Basis via Linkage for Type C Kazhdan-Lusztig Ideals

In this section, a sketch of a new proof will be given of the known fact that the essential minors form Gröbner bases for some type C Kazhdan-Lusztig ideals under the diagonal term order: $z_{ij} > z_{i'j'}$ if and only if $i > i'$, or $i = i'$ and $j > j'$. We note that we have not yet worked out all the details; this is future work.

The main reference of this section is [EFRW21]. In this dissertation, we will stick to the conventions of the authors of the aforementioned paper, as not so much research has been done by us in relation to giving a new proof for this known result.

4.1.1 Type C Kazhdan-Lusztig Varieties

Kazhdan-Lusztig ideals, as discussed in Section 2.6, are prime defining ideals of Kazhdan-Lusztig varieties in the type A flag variety. Here in this subsection, we will consider type C Kazhdan-Lusztig ideals, which are the defining ideals of Kazhdan-Lusztig ideals in type C flag varieties.

Fix an integer $n \geq 1$ and let E be the $2n \times 2n$ matrix

$$E = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix},$$

where J_n is the $n \times n$ antidiagonal matrix with antidiagonal entries 1. The **symplectic group** $Sp_{2n}(\mathbb{K})$ is

$$Sp_{2n}(\mathbb{K}) := \{M \in SL_{2n}(\mathbb{K}) \,|\, E(M^t)^{-1}E^{-1} = M\}.$$

Set $G := Sp_{2n}(\mathbb{K})$ and let B_+^G (resp. B_-^G) be the subgroup of upper (resp. lower) triangular matrices in G. The type C flag variety is G/B_+^G. A *Schubert cell* in this flag variety is a B_+^G-orbit for the left action of B_+^G on G/B_+^G by multiplication, and a *Schubert variety* is its closure. An *opposite Schubert cell* is a B_-^G-orbit in G/B_+^G and an *opposite Schubert variety* is its closure. The Weyl group C_n of G can be identified with the set of permutations

$$C_n = \{v_1 \cdots v_{2n} \in S_{2n} \,|\, v_i = 2n + 1 - v_{2n+1-i} \text{ for } i = 1, \ldots, n\}.$$

Equivalently, C_n consists of the permutations $v \in S_{2n}$ such that $w_0 v w_0 = v$, where w_0 is the long word permutation in S_{2n}. In the type C flag variety G/B_+^G, Schubert and opposite Schubert cells are denoted by $X_w^\circ := B_+^G \cdot wB_+^G/B_+^G$ and $\Omega_v^\circ := B_-^G \cdot vB_+^G/B_+^G$ respectively, for $w, v \in C_n$.

The intersection of a Schubert variety $X_w := \overline{X_w^\circ}$ with an opposite Schubert cell Ω_v° is referred to as a Kazhdan-Lusztig variety. The **type C Kazhdan-Lusztig variety** is then $\overline{B_+^G \cdot wB_+^G/B_+^G} \cap \left(B_-^G \cdot vB_+^G/B_+^G\right)$.

4.1.2 Rank Conditions on Type C Kazhdan-Lusztig Varieties

Given $w \in S_{2n}$, let $r_w : [2n] \times [2n] \to [2n]$ be the **rank function** of w, defined by

$$r_w(p, q) = |\{i \leq q \,|\, w(i) \geq p\}|,$$

so that $r_w(p, q)$ is the number of entries of w weakly southwest of (p, q).

In type A flag variety G/B_+, Fulton [Ful92] defined the essential set of a permutation $w \in S_m$ as

$$E^A(w) := \{(p, q) \in D(w) \,|\, (p-1, q), (p, q+1) \notin D(w)\},$$

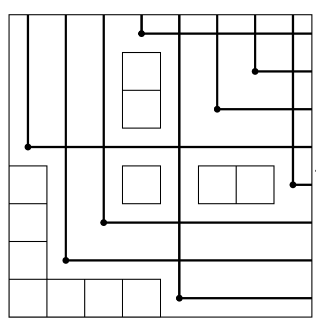
where the Rothe diagram $D(w)$, in this convention, is the set

$$D(w) = \{(p, q) \,|\, w(q) < p \text{ and } q < w^{-1}(p)\}.$$

Note that this Rothe diagram $D(w)$ defined here is different from the one defined in (2.3). This is due to change in convention, as mentioned at the beginning of this section.

Given a permutation matrix $w \in S_m$, the diagram $D(w)$ is drawn as follows: one replaces each 1 by a $\bullet$, deletes all 0s, and draws at each $\bullet$ the "hook" that extends to the east and north of the $\bullet$. The entries of the matrix that no hook passes through are the elements of $D(w)$.

Example 4.1.1. Consider the permutation $w = 47618325$. Its diagram $D(w)$ is given below:

$D(w) =$  .

From the diagram, we have that

$$D(w) = \{(2,4), (3,4), (5,1), (5,4), (5,6), (5,7), (6,1), (7,1), (8,1), (8,2), (8,3), (8,4)\}$$

and

$$E^A(w) = \{(2,4), (5,1), (5,4), (5,7), (8,4)\}.$$

In addition, for instance, if $(p, q) = (3, 4)$, then $r_w(3, 4) = 3$. □

Definition 4.1.2. [EFRW21, Definition 3.10] Define the type C essential set $E(w)$ of w as the subset of $E^A(w)$ consisting of $(p,q) \in E^A(w)$ that satisfy the following conditions:

(i) $p \geq n+1$,

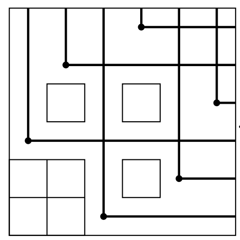
(ii) if $q \geq n+1$ and $(p, 2n-q) \in E^A(w)$, then $r_w(p, 2n-q) > r_w(p,q) + n - q$.

Example 4.1.3. Continuing with Example 4.1.1, the (type C) essential set $E(w)$ for w is

$$E(w) = \{(5,1), (5,4), (8,4)\}.$$

Here, $n = 4$. □

Example 4.1.4. Consider the permutation $w = 426153$. Its diagram $D(w)$ is given below:

$D(w) =$  .

From this diagram, we have that

$$D(w) = \{(3,2), (3,4), (5,1), (5,2), (5,4), (6,1), (6,2)\},$$

$$E^A(w) = \{(3,2), (3,4), (5,2), (5,4)\} \quad \text{and} \quad E(w) = \{(5,2)\}.$$

□

4.1.3 Type C Kazhdan-Lusztig Ideals

Let $v_\square \in C_n$ denote the **square word** permutation whose permutation matrix is

$$\begin{pmatrix} J_n & 0 \\ 0 & J_n \end{pmatrix}.$$

Here, just as in [EFRW21], specific coordinates on type C opposite Schubert cells Ω_v° will be defined, whenever v is *123*-avoiding. A permutation $v \in S_m$ is *123*-avoiding if there do not exist $i < j < k$ such that $v(i) < v(j) < v(k)$. Onward, we will denote by $\overline{v}$ a factorization $v = u_l v_\square u_r$ such that $\ell(v) = \ell(u_l) + \ell(v_\square) + \ell(u_r)$. Given $v \in S_m$, $\ell(v)$, here, is defined as the number of **inversions** of v, i.e., the number of pairs (i,j) with $1 \leq i < j \leq m$ such that $v(i) > v(j)$. Let

$$R_{\overline{v}} = \mathbb{K}[z_{ij} \,:\, i \leq j,\, u_r^{-1}(i) < u_r^{-1}(2n+1-j),\, u_l(n+1-i) < u_l(n+j)].$$

Furthermore, let $M_{\overline{v}}$ be the matrix with z_{ij} as the entries at $(u_l(n+j), u_r^{-1}(i))$ and $(u_l(n+i), u_r^{-1}(j))$ whenever $u_r^{-1}(i) < u_r^{-1}(2n+1-j)$ and $u_l(n+1-i) < u_l(n+j)$, 1s at $(v(i), i)$ for all i, and 0s at all other positions.

Example 4.1.5. Let $n = 4$, so that $v_\square = 43218765$. If $v = 64218753 \in C_4$, then we have that $v = u_l v_\square u_r$, where $u_l = 12463578$ and $u_r = 12345678$. So the matrix $M_{\overline{v}}$ corresponding to this factorization is:

$$M_{\overline{v}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & z_{14} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & z_{34} & 0 & 1 & 0 & 0 \\ z_{14} & z_{24} & z_{34} & z_{44} & 1 & 0 & 0 & 0 \end{pmatrix}$$

□

Example 4.1.6. This example shows that the labeling of the coordinates in $R_{\overline{v}}$ is dependent on the choice of factorization $\overline{v} = u_l v_\square u_r$ of v. Let $n = 3$, so that $v_\square = 321654$. If $v = 632541 \in C_3$, then we have the following:

(i) $\overline{v} = u_l v_\square u_r$, where $u_l = 123456$ and $u_r = 412563$. So the matrix $M_{\overline{v}}$ corresponding to this factorization is given below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & z_{11} & z_{12} & 0 & 1 & 0 \\ 0 & z_{12} & z_{22} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) $\overline{v} = u_l v_\square u_r$, where $u_l = 236145$ and $u_r = 123456$. So the matrix $M_{\overline{v}}$ corresponding to this factorization is given below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & 0 & 1 & 0 \\ 0 & z_{23} & z_{33} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

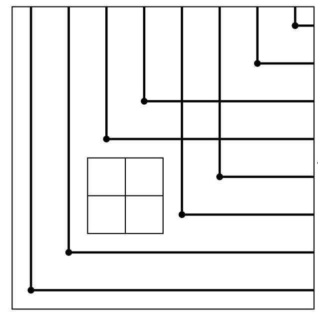
Definition 4.1.7. Let $v, w \in C_n$. The **(type C) Kazhdan-Lusztig ideal** $I_{\overline{v},w} \subseteq R_{\overline{v}}$ is defined as follows:

$$I_{\overline{v},w} = \langle \text{minors of size } 1 + r_w(p,q) \text{ in } \tau_{p,q}(M_{\overline{v}}), \text{ for all } (p,q) \in E(w) \rangle,$$

where $\tau_{p,q}(M_{\overline{v}})$ denote the submatrix of entries of $M_{\overline{v}}$ weakly southwest of position (p, q).

Example 4.1.8. Let $v = 64218753$ (as in Example 4.1.5) and $w = 87436521$. We have:

$$M_{\overline{v}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & z_{14} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & z_{34} & 0 & 1 & 0 & 0 \\ z_{14} & z_{24} & z_{34} & z_{44} & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D(w) =$$



.

Here, $E(w) = \{(5, 4)\}$. The submatrix $\tau_{5,4}(M_{\overline{v}})$ is

$$\tau_{5,4}(M_{\overline{v}}) = \begin{pmatrix} 0 & z_{22} & z_{23} & z_{24} \\ 1 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & z_{34} \\ z_{14} & z_{24} & z_{34} & z_{44} \end{pmatrix}$$

and $r_w(5, 4) = 2$. Therefore,

$$I_{\overline{v},w} = \left\langle \begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{23} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix} \begin{vmatrix} z_{23} & z_{24} \\ z_{34} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{33} & z_{34} \\ z_{34} & z_{44} \end{vmatrix} \right\rangle.$$

□

Define a lexicographic term order $\succ_{\text{lex}}$ as: $z_{ij} \succ_{\text{lex}} z_{i'j'}$ if either $i > i'$, or $i = i'$ and $j > j'$. It is shown in [EFRW21, Proposition 4.14] that this term order $\succ_{\text{lex}}$ is a diagonal term order on $R_{\overline{v}}$.

Theorem 4.1.9. *Let w, v be permutations in C_n for which v is 123-avoiding. Under the term order $\succ_{lex}$ on $R_{\overline{v}}$, the minors of size $1 + r_w(p, q)$ in $\tau_{p,q}(M_{\overline{v}})$ over all $(p, q) \in E(w)$ form a Gröbner basis for $I_{\overline{v},w} \subseteq R_{\overline{v}}$.*

This Gröbner basis result is stated and proven in [EFRW21, Theorem 4.15]. Their technique involves showing that K-polynomials of (type C) subword complexes, suitably weighted, satisfy the Kostant-Kumar recursion. We wish to prove this result using the "Gröbner basis via linkage" approach. This is the same technique we used in Chapter 3 to show that the essential minors form a Gröbner basis for Schubert patch ideals.

4.1.4 Torus Actions and Multigradings of Type C Kazhdan-Lusztig Ideals

Here in this subsection, we describe a positive multigrading of the coordinate ring $R_{\overline{v}}$, when v is *123*-avoiding. Following the same idea in [EFRW21], consider the left multiplication action (torus action) of T on $\Omega_v^\circ = B_-^G \cdot vB_+^G/B_+^G$, where T is the subgroup of diagonal matrices in $G = Sp_{2n}(\mathbb{K})$. For each $v \in C_n$, this action independently scales the rows of $M_{\overline{v}}$, to obtain $\mathbf{t} \cdot M_{\overline{v}}$, $\mathbf{t} \in T$. We then carefully choose a matrix $\mathbf{b}$ such that the matrix $\mathbf{t} \cdot M_{\overline{v}} \cdot \mathbf{b}$ has 1 in position $(v(i), i)$, $1 \leq i \leq 2n$. By so doing, we dependently rescaled the columns of the matrix $\mathbf{t} \cdot M_{\overline{v}}$.

For convenience, denote by $(t_1, \ldots, t_n)$ the element of T where $(t_1, \ldots, t_n) := \operatorname{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1})$, the diagonal matrix with diagonal entries $t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1}$.

Example 4.1.10. Let $v = 64218753$, as in Example 4.1.5. We have

$$(t_1, t_2, t_3, t_4) \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & z_{14} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & z_{34} & 0 & 1 & 0 & 0 \\ z_{14} & z_{24} & z_{34} & z_{44} & 1 & 0 & 0 & 0 \end{pmatrix} \cdot (t_4^{-1}, t_3^{-1}, t_1^{-1}, t_2)$$

$$= \begin{pmatrix} 0 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_2 z_{13} & t_2 z_{14} & 0 & 0 & 0 & t_2 \\ 0 & t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1^{-1} z_{22} & t_1^{-1} z_{23} & t_1^{-1} z_{24} & 0 & 0 & t_1^{-1} & 0 \\ t_2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_3^{-1} z_{13} & t_3^{-1} z_{23} & t_3^{-1} z_{33} & t_3^{-1} z_{34} & 0 & t_3^{-1} & 0 & 0 \\ t_4^{-1} z_{14} & t_4^{-1} z_{24} & t_4^{-1} z_{34} & t_4^{-1} z_{44} & t_4^{-1} & 0 & 0 & 0 \end{pmatrix} \cdot (t_4^{-1}, t_3^{-1}, t_1^{-1}, t_2)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_2 t_3^{-1} z_{13} & t_2 t_4^{-1} z_{14} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1^{-2} z_{22} & t_1^{-1} t_3^{-1} z_{23} & t_1^{-1} t_4^{-1} z_{24} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_2 t_3^{-1} z_{13} & t_1^{-1} t_3^{-1} z_{23} & t_3^{-2} z_{33} & t_3^{-1} t_4^{-1} z_{34} & 0 & 1 & 0 & 0 \\ t_2 t_4^{-1} z_{14} & t_1^{-1} t_4^{-1} z_{24} & t_3^{-1} t_4^{-1} z_{34} & t_4^{-2} z_{44} & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Writing weights additively, we can assign to the variables z_{13}, z_{14}, z_{22}, z_{23}, z_{24}, z_{33}, z_{34} and z_{44} in $M_{\overline{v}}$ the weights $e_3 - e_2$, $e_4 - e_2$, $e_1 + e_1$, $e_3 + e_1$, $e_4 + e_1$, $e_3 + e_3$, $e_4 + e_3$ and $e_4 + e_4$, respectively, where e_i is the i^{th} standard basis vector in $\mathbb{Z}^8$. □

In general, for a given $v \in C_n$ that is *123*-avoiding with $\overline{v} = u_l v_{\square} u_r$, the torus action described above gives the variable z_{ij} in matrix $M_{\overline{v}}$ the weight $u_l \cdot e_i + u_l \cdot e_j$, where for $i \leq n$,

$$u_l \cdot e_i = \begin{cases} -e_{n+1-u_l(n+i)} & \text{if } u_l(n+i) \leq n \\ e_{u_l(n+i)-n} & \text{if } u_l(n+i) \geq n+1. \end{cases}$$

This weight result is stated and proven in [EFRW21, Proposition 4.18]. Even more, the authors showed that this weight does not depend on the choice of factorization $\overline{v}$ of v. See the example below for an explicit illustration of this.

Example 4.1.11. This example shows that the weights of variables in $M_{\overline{v}}$ is not dependent on the choice of factorization $\overline{v} = u_l v_\square u_r$ of v, but rather dependent on positions of variables in any $M_{\overline{v}}$. Let $v = 632541 \in C_3$, as in Example 4.1.6. We have the following:

(i) $\overline{v} = u_l v_\square u_r$, where $u_l = 123456$ and $u_r = 412563$. The matrix $M_{\overline{v}}$ corresponding to this factorization is given in Example 4.1.6. Using the weight formula, as stated above, consider, for example, the variable z_{12}. Since $u_l(3+1) = 4 > 3$, it follows that $u_l \cdot e_1 = e_{u_l(3+1)-3} = e_1$. Also, since $u_l(3+2) = 5 > 3$, it follows that $u_l \cdot e_2 = e_{u_l(3+2)-3} = e_2$. Therefore, the variable z_{12} has the weight $u_l \cdot e_i + u_l \cdot e_j = u_l \cdot e_1 + u_l \cdot e_2 = e_1 + e_2$. Similarly, the variables z_{11} and z_{22} have the weights $e_1 + e_1$ and $e_2 + e_2$ respectively.

(ii) $\overline{v} = u_l v_\square u_r$, where $u_l = 236145$ and $u_r = 123456$. The matrix $M_{\overline{v}}$ corresponding to this factorization is also given in Example 4.1.6. Here, consider, for example, the variable z_{23}. Since $u_l(3+2) = 4 > 3$, it follows that $u_l \cdot e_2 = e_{u_l(3+2)-3} = e_1$. Also, since $u_l(3+3) = 5 > 3$, it follows that $u_l \cdot e_3 = e_{u_l(3+3)-3} = e_2$. Therefore, the variable z_{23} has the weight $u_l \cdot e_i + u_l \cdot e_j = u_l \cdot e_2 + u_l \cdot e_3 = e_1 + e_2$. Similarly, the variables z_{22} and z_{33} have the weights $e_1 + e_1$ and $e_2 + e_2$ respectively.

□

Fix a permutation $v \in C_n$ that is *123*-avoiding. The torus action described above yields a $\mathbb{Z}^n$-grading of the coordinate ring $R_{\overline{v}}$ of $M_{\overline{v}}$ and this multigrading is positive. In addition, the type C Kazhdan-Lusztig ideal $I_{\overline{v},w}$ is homogeneous with respect to the positive multigrading of the underlying ring by $\mathbb{Z}^n$.

Example 4.1.12. Let $v = 64218753$ and $w = 87436521$ be permutations in C_4. The ideal $I_{\overline{v},w}$ is given in Example 4.1.8. With respect to the above multigrading, we have:

$$\deg(z_{22}z_{33}) = (e_1 + e_1) + (e_3 + e_3) = 2(e_1 + e_3)$$

and

$$\deg(z_{23}^2) = (e_1 + e_3) + (e_1 + e_3) = 2(e_1 + e_3).$$

Thus, the generator $\begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix}$ of $I_{\overline{v},w}$ is homogeneous. In a similar fashion, the other generators of $I_{\overline{v},w}$ are also homogeneous. Therefore, $I_{\overline{v},w}$ is homogeneous with respect to the positive multigrading of $R_{\overline{v}}$ by $\mathbb{Z}^4$. □

4.1.5 Sketch of our Proof

Here in this subsection, we provide a sketch of a proof of Theorem 4.1.9 using the “Gröbner basis via linkage” approach in Chapter 3 of this dissertation. Before doing this, we give few more definitions, terminologies and examples that will aid the sketch of this proof.

Definition 4.1.13. The **simple reflections in $\boldsymbol{S}_m$** are the simple transpositions $s_1, \ldots, s_{m-1}$.

Definition 4.1.14. The **simple reflections in $\boldsymbol{C}_n$** are the permutations $c_0, c_1, \ldots, c_{n-1}$, where $c_0 \in C_n$ is the permutation that transposes n and $n+1$, and for $i = 1, \ldots, n-1$, $c_i \in C_n$ is the permutation that transposes $n+i$ and $n+i+1$, and hence, transposes $n-i$ and $n-i+1$.

Definition 4.1.15. A simple reflection $c_k \in C_n$ is a (right) **ascent** of $v \in C_n$ if $v(n+k) < v(n+k+1)$ and $v(n-k) < v(n-k+1)$. A simple reflection $c_k \in C_n$ is a (right) **descent** of $v \in C_n$ if $v(n+k) > v(n+k+1)$ and $v(n-k) > v(n-k+1)$.

In what follows, we say c_k is the **last ascent** of $v \in C_n$ if

$$k = \max\{i \mid v(n+i) < v(n+i+1) \text{ and } v(n-i) < v(n-i+1),\ 1 \leq i \leq n-1\}.$$

Let $z_{\max}$ be the variable in $M_{\overline{v}}$ that is maximal with respect to the diagonal term order $\succ_{\text{lex}}$: $z_{ij} \succ_{\text{lex}} z_{i'j'}$ if either $i > i'$, or $i = i'$ and $j > j'$. If we fix a factorization $\overline{v} = u_l v_\square u_r$ of $v \in C_n$, then we can infer from the following result that the variable $z_{\max}$ is in positions $(v(n \pm k+1), n \pm k)$ of $M_{\overline{v}}$, where k is chosen such that c_k is the last ascent of v.

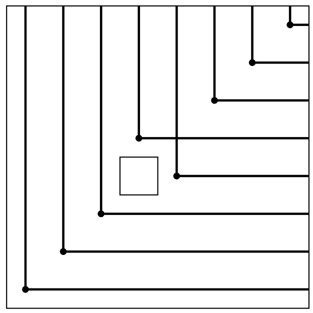
Corollary 4.1.16. [EFRW21, Corollary 4.11] *Let $v \in C_n$ be a permutation that is 123-avoiding and let $V_{\overline{v}}$ be the set of variables of $R_{\overline{v}}$, i.e., the set of variables that appear as entries of $M_{\overline{v}}$. Fix a factorization $\overline{v} = u_l v_\square u_r$ of v. If c_k is an ascent of v and we set $\overline{vc_k} = (u_l)v_\square(u_r c_k)$, then $V_{\overline{vc_k}} \subseteq V_{\overline{v}}$ and $V_{\overline{v}} \setminus V_{\overline{vc_k}} = \{z_{ij}\}$, where z_{ij} is the entry of $M_{\overline{v}}$ in positions $(v(n \pm k+1), n \pm k)$.*

Below is a conjecture (an analogue of Corollary 3.3.7) that will aid our proof of Theorem 4.1.9.

Conjecture 4.1.17. *Let $v, w \in C_n$ for which the last ascent of v is also an ascent of w, and consider the positive multigrading of $R_{\overline{v}}$ by $\mathbb{Z}^n$. Let c_k be the last ascent of v and assume the variable z_{max} does not belong to $\langle in_{\succ_{lex}}(\mathcal{G}_I)\rangle$, where $in_{\succ_{lex}}(\mathcal{G}_I)$ is the set of initial terms of minors generating the ideal $I := I_{\overline{v},w}$. Set $J := I_{\overline{vc_k},w}$, $N := I_{\overline{vc_k},wc_k}$, $A := \langle in_{\succ_{lex}}(\mathcal{G}_I)\rangle$, $B := \langle in_{\succ_{lex}}(\mathcal{G}_J)\rangle$ and $C := \langle in_{\succ_{lex}}(\mathcal{G}_N)\rangle$. Then there exists $\boldsymbol{e} \in \mathbb{Z}^n$ such that there is an $R_{\overline{v}}/N$-module isomorphism $I/N \cong (J/N)(-\boldsymbol{e})$ and an $R_{\overline{v}}/C$-module isomorphism $A/C \cong (B/C)(-\boldsymbol{e})$.*

Example 4.1.18. Let $v = 64281753$ and $w = 87645321$. The last ascent c_1 of v is a descent of w. The

Rothe diagram $D(w)$ and matrices $M_{\overline{v}}$ and $M_{\overline{vc_1}}$ are given below:

$D(w) =$  ,

$$M_{\overline{v}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & 0 & z_{14} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & 0 & z_{24} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & 0 & z_{34} & 1 & 0 & 0 \\ z_{14} & z_{24} & z_{34} & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

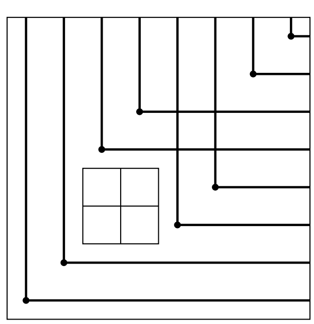
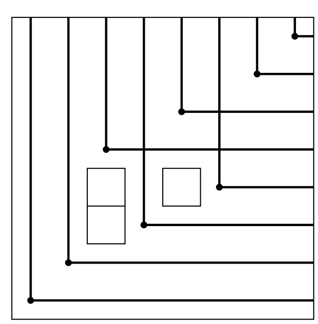
$$M_{\overline{vc_1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{13} & 0 & z_{14} & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & 0 & z_{23} & 0 & z_{24} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & 0 & z_{33} & 1 & 0 & 0 & 0 \\ z_{14} & z_{24} & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that

$$I_{\overline{v},w} = \left\langle \begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix} \right\rangle = I_{\overline{vc_1},w}.$$

So if we set both J and N to be $I_{\overline{vc_1},w}$ and set $I := I_{\overline{v},w}$, then $I/N \cong J/N$. □

Example 4.1.19. Let $v = 64218753$ and $w = 87436521$, as in Example 4.1.8. The last ascent c_0 of v is also an ascent of w. The Rothe diagrams $D(w)$ and $D(wc_0)$, and matrices $M_{\overline{v}}$ and $M_{\overline{vc_0}}$ are given below:

$D(w) =$  , $D(wc_0) =$ 

$$M_{\overline{v}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & z_{14} & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & z_{34} & 0 & 1 & 0 & 0 \\ z_{14} & z_{24} & z_{34} & z_{44} & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_{\overline{vc_0}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & 0 & z_{14} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & 0 & z_{24} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{13} & z_{23} & z_{33} & 0 & z_{34} & 1 & 0 & 0 \\ z_{14} & z_{24} & z_{34} & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with (type C) essential sets $E(w) = \{(5,4)\}$ and $E(wc_0) = \{(5,3)\}$. Set

$$J := I_{\overline{vc_0},w} = \langle z_{22},\, z_{23},\, z_{33} \rangle$$

and

$$N := I_{\overline{vc_0},wc_0} = \left\langle \begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{23} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{vmatrix} \right\rangle.$$

If we set

$$I := I_{\overline{v},w} = \left\langle \begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{23} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{34} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{33} & z_{34} \\ z_{34} & z_{44} \end{vmatrix} \right\rangle,$$

then

$$I/N = \left\langle \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{34} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{33} & z_{34} \\ z_{34} & z_{44} \end{vmatrix} \right\rangle \qquad \text{and} \qquad J/N = \langle z_{22},\, z_{23},\, z_{33} \rangle.$$

Define the map $I/N \longrightarrow (J/N)(-\boldsymbol{e})$ by

$$\overline{g} \mapsto \frac{f_1}{f_2} \cdot \overline{g},$$

where $f_1 = z_{22},\ f_2 = \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}$ and $\boldsymbol{e} = u_l \cdot e_4 + u_l \cdot e_4 = e_4 + e_4$ is the degree of the variable $z_{\max} = z_{44}$. Under this mapping, we have

$$\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix} \longrightarrow \frac{z_{22}}{\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}} \cdot \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix} = z_{22}$$

$$\begin{vmatrix} z_{23} & z_{24} \\ z_{34} & z_{44} \end{vmatrix} \longrightarrow \frac{z_{22}}{\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}} \cdot \begin{vmatrix} z_{23} & z_{24} \\ z_{34} & z_{44} \end{vmatrix} = \frac{z_{23} \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix} - z_{24} \begin{vmatrix} z_{22} & z_{23} \\ z_{24} & z_{34} \end{vmatrix}}{\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}} = z_{23},$$

and

$$\begin{vmatrix} z_{33} & z_{34} \\ z_{34} & z_{44} \end{vmatrix} \longrightarrow \frac{z_{22}}{\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}} \cdot \begin{vmatrix} z_{33} & z_{34} \\ z_{34} & z_{44} \end{vmatrix} = \frac{z_{33}\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix} - z_{24}\begin{vmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{vmatrix} - z_{34}\begin{vmatrix} z_{22} & z_{24} \\ z_{23} & z_{34} \end{vmatrix}}{\begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}} = z_{33},$$

where the determinants indicated in red are zero modulo N, since they belong to N. Therefore, $I/N \cong (J/N)(-\boldsymbol{e})$, as $R_{\overline{v}}/N$-module. □

Proof of Theorem 4.1.9 *(Sketch).* Let $v \in C_n$ be fixed. We proceed by induction on $|V_{\overline{v}}|$. If $|V_{\overline{v}}| = 0$, then $v = w_0$, the long word permutation in S_{2n}, and so $I_{\overline{v},w}$ is the unit ideal.

Suppose the hypothesis is true for all type C Kazhdan-Lusztig ideals $I_{\overline{v'},w}$, $v', w \in C_n$ with $v' \geq v$ in Bruhat order. Let c_k be the last ascent of v. Then $vc_k \geq v$ in Bruhat order and $|V_{\overline{vc_k}}| = |V_{\overline{v}}| - 1$. Set $I := I_{\overline{v},w}$, $J := I_{\overline{vc_k},w}$ and $N := I_{\overline{vc_k},wc_k}$. Under the positive grading of $R_{\overline{v}}$ by $\mathbb{Z}^n$, the ideals I, J and N are homogeneous. Let $\mathcal{G}_I$, $\mathcal{G}_J$ and $\mathcal{G}_N$ be the sets of minors generating I, J and N respectively.

Suppose the variable $y := z_{\max}$ belongs to I. Then J is a unit ideal and $I = N + \langle y \rangle$. By the induction hypothesis, $\mathcal{G}'_N$ is a Gröbner basis for the ideal N with respect to the term order $\succ_{\text{lex}}$ restricted to $R_{\overline{vc_k}}$. Since the minors in $\mathcal{G}_N$ do not involve y, it follows that the set $\mathcal{G}_I = \{y\} \cup \mathcal{G}_N$ of minors generating I is a Gröbner basis for I. On the other hand, suppose the variable y does not belong to I. Then J is non-trivial. By the induction hypothesis, the sets $\mathcal{G}_J$ and $\mathcal{G}_N$ are respectively Gröbner bases for the ideals J and N with respect to the term order $\succ_{\text{lex}}$ restricted to $R_{\overline{vc_k}}$. Therefore, if B and C are the ideals generated by the initial terms, with respect to $\succ_{\text{lex}}$, of the elements of $\mathcal{G}_J$ and $\mathcal{G}_N$ respectively, i.e. if $B := \langle in_{\succ_{\text{lex}}}(\mathcal{G}_J)\rangle$ and $C := \langle in_{\succ_{\text{lex}}}(\mathcal{G}_N)\rangle$, then $B = \text{in}_{\succ_{\text{lex}}}(J)$ and $C = \text{in}_{\succ_{\text{lex}}}(N)$. Furthermore, if $A := \langle in_{\succ_{\text{lex}}}(\mathcal{G}_I)\rangle$, then it will follow from Conjecture 4.1.17 that $I/N \cong [J/N](-\boldsymbol{e})$ as $R_{\overline{v}}/N$-modules and $A/C \cong [B/C](-\boldsymbol{e})$ as $R_{\overline{v}}/C$-modules, where $\boldsymbol{e} \in \mathbb{Z}^n$ is the degree of the variable y. Hence, by Lemma 3.1.1, $A = \text{in}_{\succ_{\text{lex}}}(I)$, i.e., the set $\mathcal{G}_I$ of minors generating I is a Gröbner basis for I. □

Example 4.1.20. Let $v = 64218753$ and $w = 87436521$, as in Examples 4.1.8 and 4.1.19. If $\mathcal{G}_I$, $\mathcal{G}_J$ and $\mathcal{G}_N$ are the sets of minors generating I, J and N respectively, then

$$\mathcal{G}_I = \left\{ \begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{23} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{24} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{34} & z_{44} \end{vmatrix}, \begin{vmatrix} z_{33} & z_{34} \\ z_{34} & z_{44} \end{vmatrix} \right\},$$

$$\mathcal{G}_J = \{z_{22},\, z_{23},\, z_{33}\} \quad \text{and} \quad \mathcal{G}_N = \left\{ \begin{vmatrix} z_{22} & z_{23} \\ z_{23} & z_{33} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{24} \\ z_{23} & z_{34} \end{vmatrix}, \begin{vmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{vmatrix} \right\}.$$

Set

$$A := \langle in_{\succ_{\text{lex}}}(\mathcal{G}_I)\rangle = \langle z_{22}z_{33}, z_{22}z_{34}, z_{23}z_{34}, z_{22}z_{44}, z_{23}z_{44}, z_{33}z_{44}\rangle,$$

$$B := \langle in_{\succ_{\text{lex}}}(\mathcal{G}_J)\rangle = \langle z_{22}, z_{23}, z_{33}\rangle \quad \text{and} \quad C := \langle in_{\succ_{\text{lex}}}(\mathcal{G}_N)\rangle = \langle z_{22}z_{33}, z_{22}z_{34}, z_{23}z_{34}\rangle.$$

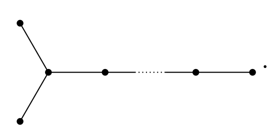
Then $A/C = \langle z_{22}z_{44}, z_{23}z_{44}, z_{33}z_{44}\rangle$ and $B/C = \langle z_{22}, z_{23}, z_{33}\rangle$. Hence, $A/C \cong (B/C)(-\boldsymbol{e})$, as $R_{\overline{v}}/C$-module, where $\boldsymbol{e} = e_4 + e_4$ is the degree of the variable $z_{\max} = z_{44}$. Furthermore, observe that $N \subseteq I \cap J$ and $A \subseteq \text{in}_{\succ_{\text{lex}}}(I)$. It is easy to see that the set $\mathcal{G}_J$ form a Gröbner basis for J, i.e., $B = \text{in}_{\succ_{\text{lex}}}(J)$. Assume that $C = \text{in}_{\succ_{\text{lex}}}(N)$. Since $I/N \cong (J/N)(-\boldsymbol{e})$ as $R_{\overline{v}}/N$-module and $A/C \cong (B/C)(-\boldsymbol{e})$ as $R_{\overline{v}}/C$-module, it follows that $(I/N)_{\boldsymbol{\ell}} \cong (J/N)_{\boldsymbol{\ell}-\boldsymbol{e}}$ and $(A/C)_{\boldsymbol{\ell}} \cong (B/C)_{\boldsymbol{\ell}-\boldsymbol{e}}$, for all $\boldsymbol{\ell} \in \mathbb{Z}^4$, and so $A = \text{in}_{\succ_{\text{lex}}}(I)$, by Lemma 3.1.1. □

Note that the assumption $C = \text{in}_{\succ_{\text{lex}}}(N)$ in Example 4.1.20 above mean that the set $\mathcal{G}_N$ form a Gröbner basis for the ideal N. This assumption can be easily verified either by a direct S-polynomial computation or by repeating the same procedures in Example 4.1.20 for minors in $N := I_{\overline{vc_0},wc_0}$. For the latter option, we set $I := I_{\overline{vc_0},wc_0}$, search for appropriate J and N as in Example 4.1.19, and finally apply the procedures in Example 4.1.20 to these new ideals I, J and N.

4.2 Gröbner Basis via Linkage for Type D Quiver Ideals

In this section, we give a brief introduction of loci and ideals of the bipartite type D quivers. The main reference of this section is [KR21a].

Type D quivers are some directed graph whose underlying graph (when the directions of the arrows are ignored) is the type D Dynkin diagram



Definition 4.2.1. A **quiver** Q is a finite directed graph. The set of vertices of Q is denoted by Q_0 and the set of arrows is denoted by Q_1. The vertex at the tail (starting point) of an arrow $a \in Q_1$ is denoted by ta and the vertex at the head (ending point) is denoted by ha.

Definition 4.2.2. A **representation** V **of a quiver** Q consists of a family of finite dimensional vector spaces V_i indexed by the vertices $i \in Q_0$, together with a family of linear maps $f_a : V_{ta} \to V_{ha}$ indexed by the arrows $a \in Q_1$.

Definition 4.2.3. A **dimension vector** $\boldsymbol{d} : Q_0 \to \mathbb{Z}_{\geq 0}$ for a quiver Q is an assignment of a nonnegative integer to each vertex of Q.

Definition 4.2.4. For a fixed dimension vector $\boldsymbol{d}$ of a quiver Q, define the associated **representation space** to be

$$\text{rep}_Q(\boldsymbol{d}) := \prod_{a \in Q_1} \text{Mat}_{\boldsymbol{d}(ha),\boldsymbol{d}(ta)}(\mathbb{K}),$$

where $\mathrm{Mat}_{m,n}(\mathbb{K})$ denotes the algebraic variety of $m \times n$ matrices with entries in $\mathbb{K}$.

Fix a dimension vector $\boldsymbol{d}$ for Q and let $\mathrm{rep}_Q(\boldsymbol{d})$ be the associated representation space. The *base change group*

$$\mathbf{GL}(\boldsymbol{d}) := \prod_{i \in Q_0} \mathbf{GL}_{\boldsymbol{d}(i)}(\mathbb{K})$$

acts on the quiver representations $\mathrm{rep}_Q(\boldsymbol{d})$ as follows: for $g = (g_i)_{i \in Q_0} \in \mathbf{GL}(\boldsymbol{d})$ and $V = (V_a)_{a \in Q_1} \in \mathrm{rep}_Q(\boldsymbol{d})$, we have

$$g \cdot V = \left(g_{ha} V_a g_{ta}^{-1}\right)_{a \in Q_1}.$$

Definition 4.2.5. The closure of a $\mathbf{GL}(\boldsymbol{d})$-orbit in $\mathrm{rep}_Q(\boldsymbol{d})$ is called a **quiver locus**.

Of interest to us in this section are the bipartite type D quiver loci with representation space of the form:

$$\begin{array}{ccccc} k^a & \overset{\alpha}{\searrow} & & & \\ & & k^n & \xleftarrow[\gamma]{} & k^c \ , \\ k^b & \underset{\beta}{\nearrow} & & & \end{array} \tag{4.1}$$

where α, β and γ are, respectively, $n \times a$, $n \times b$ and $n \times c$ arbitrary matrices with entries in $\mathbb{K}$. Kinser and Rajchgot[KR21a] are interested in knowing if the defining ideals for these type D quiver loci are prime and they are also interested in obtaining Gröbner basis for this family of ideals.

Definition 4.2.6. A **type D quiver ideal**, denoted by $I_{\alpha,\beta,\gamma}$, is the ideal that defines a type D quiver orbit closure.

Generators of the ideal $I_{\alpha,\beta,\gamma}$ are polynomials that vanish at every point in the orbit through $V = (\alpha, \beta, \gamma)$. To be explicit, up to radical, every type D quiver ideal $I_{\alpha,\beta,\gamma}$ is defined by the following minors:

$$\mathrm{minors}\big(1 + \mathrm{rank}\,(\alpha), A\big),\ \mathrm{minors}\big(1 + \mathrm{rank}\,(\beta), B\big),\ \mathrm{minors}\big(1 + \mathrm{rank}\,(\gamma), C\big) \tag{4.2}$$

$$\mathrm{minors}\left(1 + \mathrm{rank}\left(\begin{bmatrix}\alpha \\ \beta\end{bmatrix}\right), \begin{bmatrix}A \\ B\end{bmatrix}\right),\ \mathrm{minors}\left(1 + \mathrm{rank}\left(\begin{bmatrix}\alpha \\ \gamma\end{bmatrix}\right), \begin{bmatrix}A \\ C\end{bmatrix}\right), \tag{4.3}$$

$$\mathrm{minors}\left(1 + \mathrm{rank}\left(\begin{bmatrix}\beta \\ \gamma\end{bmatrix}\right), \begin{bmatrix}B \\ C\end{bmatrix}\right),\ \mathrm{minors}\left(1 + \mathrm{rank}\left(\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}\right), \begin{bmatrix}A \\ B \\ C\end{bmatrix}\right) \tag{4.4}$$

and

$$\mathrm{minors}\left(1 + \mathrm{rank}\left(\begin{bmatrix}0 & \alpha \\ \beta & \beta \\ \gamma & 0\end{bmatrix}\right), \begin{bmatrix}0 & A \\ B & B \\ C & 0\end{bmatrix}\right), \tag{4.5}$$

where $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are $n \times a$, $n \times b$ and $n \times c$ matrices of variables, respectively.

Using the liaison theory approach in Chapter 3 of this dissertation, we plan to show that these minors form a Gröbner basis for the ideal $I_{\alpha,\beta,\gamma}$.

Example 4.2.7. Consider a bipartite type D quiver of the form (4.1), where

$$\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then the type D quiver ideal corresponding to this representation is

$$\begin{aligned} I_{\alpha,\beta,\gamma} &= \operatorname{minors}(2, A) + \operatorname{minors}(2, B) + \operatorname{minors}(2, C) + \operatorname{minors}\left(2, \begin{bmatrix} A \\ B \end{bmatrix}\right) + \operatorname{minors}\left(2, \begin{bmatrix} A \\ C \end{bmatrix}\right) \\ &\quad + \operatorname{minors}\left(2, \begin{bmatrix} B \\ C \end{bmatrix}\right) + \operatorname{minors}\left(2, \begin{bmatrix} A \\ B \\ C \end{bmatrix}\right) + \operatorname{minors}\left(3, \begin{bmatrix} 0 & A \\ B & B \\ C & 0 \end{bmatrix}\right) \\ &= \langle a_{11}b_{12} - a_{12}b_{11}, a_{11}c_{12} - a_{12}c_{11}, b_{11}c_{12} - b_{12}c_{11} \rangle, \end{aligned}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \end{pmatrix}.$$

Identify $\mathbb{K}[\operatorname{rep}_Q(\boldsymbol{d})]$ with the polynomial ring $R := \mathbb{K}[A, B, C]$, where $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are $n \times a$, $n \times b$ and $n \times c$ generic matrices respectively. Define a lexicographic term order $\succ_{\mathrm{X}}$ as: $x_{ij} \succ_{\mathrm{X}} x_{i'j'}$ if either $i > i'$, or $i = i'$ and $j > j'$. Let $\succ$ be a lexicographic term order on R which is induced from the following lexicographic term order on $\mathbb{K}[A]$, $\mathbb{K}[B]$ and $\mathbb{K}[C]$:

$$c_{n,c} \succ_C \cdots \succ_C c_{11} \succ b_{n,b} \succ_B \cdots \succ_B b_{11} \succ a_{n,a} \succ_A \cdots \succ_A a_{11}. \tag{4.6}$$

Conjecture 4.2.8. *Let α, β and γ be $n \times a$, $n \times b$ and $n \times c$ arbitrary matrices respectively, with entries in $\mathbb{K}$. Under the term order $\succ$ on $R := \mathbb{K}[A, B, C]$, as defined in (4.6), the minors (4.2), (4.3), (4.4) and (4.5) form a Gröbner basis for $I_{\alpha,\beta,\gamma} \subseteq R$.*

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Appendix A

Code

Below are some code, written in Macaulay2, to aid some of the computations in this work.

A.1 Computing a Schubert Patch Ideal

```
--prints out permutation matrix corresponding to a permutation in S_n
PermMatrix = L -> (
        mm = length L;
        N = mutableMatrix(ZZ,mm,mm);
        for j from 0 to mm-1 list N_(L_j-1,j) = 1;
        return N
)

--v and w belong to S_n
SchPatIdeal = (v,w) -> (
        --getting rank matrix for w
        oldw = w; m = length oldw; neww = oldw;
        W = PermMatrix(neww);
        N = mutableMatrix(W);
        L1 = for i from 0 to m-1 list
        for j from 0 to m-1 list (if W_(i,j) != 1 then continue; (i,j));
        L2 = flatten L1;

        below_assW = method();
        below_assW (ZZ,ZZ,MutableMatrix) :=
                (x,y,N) -> for i from x+1 to m-1 list N_(i,y) = 2;

        right_assW = method();
        right_assW (ZZ,ZZ,MutableMatrix) :=
                (x,y,N) -> for j from y+1 to m-1 list N_(x,j) = 2;

        for pos in L2 list below_assW (first pos,last pos,N);
        for pos in L2 list right_assW (first pos,last pos,N);

        --Diagram of w
        --DiagW = copy(N);

        L3 = for i from 0 to m-1 list
                for j from 0 to m-1 list (if N_(i,j) != 0 then continue; (i,j));
        L3 = flatten L3;

        L4 = for vv in L3 list
                (if N_(vv_0,vv_1+1) == 0 or N_(vv_0+1,vv_1) == 0 then continue; vv);

        L5 = for pos in L4 list
                ((first pos, last pos), rank(submatrix(W,{0..first pos},{0..last pos})));
```

```
--Constructing schubert cell (variable matrix) corresponding to v
V = PermMatrix(v);
L6 = for i from 0 to m-1 list
for j from 0 to m-1 list (if V_(i,j) != 1 then continue; (i,j));

L7 = flatten L6;
myRingVar = flatten for i from 1 to m list
        for j from 1 to m list
                value concatenate("z","_",toString i, toString ((m+1)-j));

R = QQ[myRingVar,MonomialOrder=>Lex];

myMatVar = flatten for i from 1 to m list
        for j from 1 to m list
                value concatenate("z","_",toString i, toString j);

S = QQ[myMatVar,MonomialOrder=>Lex];

matvar = transpose(genericMatrix(S,z_11,m,m));

Zv = mutableMatrix(matvar);

exact_assV = method();
exact_assV (ZZ,ZZ,MutableMatrix) := (x,y,Zv) -> Zv_(x,y) = 1;

below_assV = method();
below_assV (ZZ,ZZ,MutableMatrix) := (x,y,Zv) ->
        for i from x+1 to m-1 list Zv_(i,y) = 0;

--if released, we obtain a (Type A) Kazhdan-Lusztig Ideal
--right_assV = method();
--right_assV (ZZ,ZZ,MutableMatrix) := (x,y,Zv) ->
--for j from y+1 to m-1 list Zv_(x,j) = 0;


for pos in L7 list exact_assV (first pos,last pos,Zv);
for pos in L7 list below_assV (first pos,last pos,Zv);
--for pos in L7 list right_assV (first pos,last pos,Zv);

Zv = substitute(matrix(Zv),R);

-- computing the req. ideal
I = ideal ();
for pos in L5 do (
        I = I + minors((last pos)+1,submatrix(Zv,{0..first first pos},
        {0..last first pos}));
);

return (v,w,Zv,I,W)

)
```

A.2 Computing a Type C Kazhdan-Lusztig Ideal

```
--prints out permutation matrix corresponding to a permutation in S_n
PermMatrix = L -> (
        mm = length L;
        N = mutableMatrix(ZZ,mm,mm);
        for j from 0 to mm-1 list N_(L_j-1,j) = 1;
        return N
)

--v and w belong to C_n
KazLuzIdealTypeC = (v,w) -> (
        --getting rank matrix for w
        oldw = w;
        m = length oldw;
        mTC = m/2;
        neww = oldw;
        W = PermMatrix(neww);
        N = mutableMatrix(W);

        --positions of the 1s in the permutation matrix for w
        L1 = for i from 0 to m-1 list
                for j from 0 to m-1 list (if W_(i,j) != 1 then continue; (i,j));
        L2 = flatten L1;

        above_ass = method();
        above_ass (ZZ,ZZ,MutableMatrix) :=
                (x,y,N) -> for i from 0 to x-1 list N_(i,y) = 2;

        right_ass = method();
        right_ass (ZZ,ZZ,MutableMatrix) :=
                (x,y,N) -> for j from y+1 to m-1 list N_(x,j) = 2;

        for pos in L2 list above_ass (first pos,last pos,N);
        for pos in L2 list right_ass (first pos,last pos,N);

        --prints the diagram of w;
        --DiagW = copy(N);

        --positions of boxes in D(w)
        L3 = for i from 0 to m-1 list
                for j from 0 to m-1 list (if N_(i,j) != 0 then continue; (i,j));
        L3 = flatten L3;

        L4 = for vv in L3 list
                (if N_(vv_0,vv_1+1) == 0 or N_(vv_0-1,vv_1) == 0 then continue; vv);

        L4TC1 =  for vv in L4 list (if vv_0 < mTC then continue; vv);

        L4TC2 = L4TC1;
        for vv in L4TC1 do (
                tnmq = substitute(2*(mTC-1)-vv_1,ZZ);
                rptnq = rank(submatrix(W,{vv_0..m-1},{0..tnmq}));
                rpq = rank(submatrix(W,{vv_0..m-1},{0..vv_1}));
```

```
        if vv_1 >= mTC and member((vv_0, substitute(2*(mTC-1)-vv_1,ZZ)),L3)==
        true and rptnq <= rpq + (mTC-1) - vv_1  then L4TC2 = delete(vv,L4TC2);
);


--Type C essential set
L5 = for pos in L4TC2 list ((first pos, last pos),
        rank(submatrix(W,{first pos..m-1},{0..last pos})));
print ("L5",L5);

--Constructing matrix variable corresponding to v
V = PermMatrix(v);

--position of 1s in V
L6 = for i from 0 to m-1 list
        for j from 0 to m-1 list (if V_(i,j) != 1 then continue; (i,j));
L7 = flatten L6;

V1 = mutableMatrix(V);

for pos in L7 list above_ass (first pos,last pos,V1);
for pos in L7 list right_ass (first pos,last pos,V1);

DiagV = copy(V1);

L8 = for i from 0 to m-1 list
        for j from 0 to m-1 list (if V1_(i,j) != 0 then continue; (i,j));
L8 = flatten L8;

DistinctR = {};
DistinctC = {};
for rc in L8 do (
        if member(first rc,DistinctR) == false then
        DistinctR = DistinctR|{first rc};
        if member(last rc,DistinctC) == false then
        DistinctC = DistinctC|{last rc};
);

DistinctC = sort(DistinctC);

mrc = length(DistinctR);

myRingVar1 = flatten for i from 1 to mrc list
        for j from 1 to mrc list value concatenate("z",
                "_",toString ((mrc+1)-i),toString ((mrc+1)-j));


R1 = QQ[myRingVar1,MonomialOrder=>Lex];

myMatVar1 = flatten for i from 1 to mrc list
        for j from 1 to mrc list value concatenate("z","_",
                toString i, toString j);

S1 = QQ[myMatVar1,MonomialOrder=>Lex];
```

```
        matvar1 = transpose(genericMatrix(S1,z_11,mrc,mrc));

        Mv1 = mutableMatrix(matvar1);
        for ii from 0 to mrc-1 do (
                for jj from 0 to mrc-1 do (
                        if ii > jj then Mv1_(ii,jj) = Mv1_(jj,ii);
                );
        );

        Mv2 = mutableMatrix(S1,m,m);

        for pp in L7 do (Mv2_(pp_0,pp_1) = 1);

        for ii from 0 to mrc-1 do (
                rr = DistinctR_ii;
                for jj from 0 to mrc-1 do (
                        cc = DistinctC_jj;
                        if DiagV_(rr,cc) == 0 then Mv2_(rr,cc) = Mv1_(ii,jj)
                        else Mv2_(rr,cc) = 0;
                );
        );

        Mv = substitute(matrix(Mv2),R1);

        -- computing the req. ideal
        I = ideal ();
        for pos in L5 do (
                I = I + minors((last pos)+1,submatrix(Mv,{first first pos..m-1},
                {0..last first pos}));
        );

        return (v,w,Mv,I);
        )
```

A.3 Computing a Type D Quiver Ideal

```
--alpha, beta and gamma are matrices of appropriate sizes
QuiverIdealTypeD1 = (alpha,beta,gamma) ->(
        L = {matrix(alpha),matrix(beta),matrix(gamma)};
        Ma = L_0; Mb = L_1; Mc = L_2;
        o = numColumns Ma; l = numRows Ma; m = numRows Mb; n = numRows Mc;

        Z1 = for i from 1 to l list for j from 1 to o list 0;
        Z1 = matrix(Z1);

        Z2 = for i from 1 to n list for j from 1 to o list 0;
        Z2 =  matrix(Z2);

        NMat = {Ma,Mb,Mc,Ma||Mb,Ma||Mc,Mb||Mc,Ma||Mb||Mc,((Z1|Ma)||(Mb|Mb))||(Mc|Z2)};

        LL1o = for i from 1 to l list
        for j from 1 to o list concatenate {toString a,
```

```
        toString i,toString j};
LL1 = flatten LL1o;

LL2o = for i from 1 to m list
for j from 1 to o list concatenate {toString b,
        toString i,toString j};
LL2 = flatten LL2o;

LL3o = for i from 1 to n list
for j from 1 to o list concatenate {toString c,
        toString i,toString j};
LL3 = flatten LL3o;

varRing = LL1 | LL2 | LL3;

S = QQ[varRing, MonomialOrder=>Lex];

A =  transpose genericMatrix(S,(vars S)_(0,0),o,l);
B = transpose genericMatrix(S,(vars S)_(0,l*o),o,m);
C = transpose genericMatrix(S,(vars S)_(0,l*o+m*o),o,n);

NMatVar = {A,B,C,A||B,A||C,B||C,A||B||C,((Z1|A)||(B|B))||(C|Z2)};

--relevant ones
NMatVarRel = {};
for i from 0 to 7 do (
        temp111 = NMat_i;
        if (1+rank(temp111)) <= min(numRows temp111,numColumns temp111) then
        NMatVarRel = NMatVarRel|{(1+rank(temp111),mutableMatrix(NMatVar_i))}
        else continue;
);

nn1 = length(NMatVar);
print("*********************************************************");
print ("Generic matrices printed with size of minors to be taken on their left");
for j from 0 to (nn1//3)-1 do (
        Emp1 = {};
        for i from 0 to 2 do (
                i2 = i+3*j;
                Emp1 = Emp1|{(1+rank(NMat_i2),mutableMatrix(NMatVar_(i2)))};
        );
        print Emp1;
        print ("");
);

Emp1 = {};
if nn1%3 != 0 then (
        for i from 0 to (nn1%3)-1 do (
                i2 = i+3*(nn1//3);
                Emp1 = Emp1|{(1+rank(NMat_i2),mutableMatrix(NMatVar_(i2)))};
        );
        print Emp1;
);
print("*********************************************************");
```

```
print("**********************************************************");

NMatMaxMinors = {};
for i from 0 to 7 do (
        M = NMat_i;
        r = rank(M);
        if r < min(numRows M, numColumns M) then
                (
                NMatMaxMinors = NMatMaxMinors|{minors(1+r,NMatVar_i)};
                ) else NMatMaxMinors = NMatMaxMinors|{0};
);

I = sum(NMatMaxMinors);

return (I)
)


--e.g. QuiverIdealTypeD1(matrix{{1,0}},matrix{{1,0}},matrix{{1,0}})
```