

Second-Order Contributions to the Non-Exotic  
( $J^{PC} = 1^{--}$ ) Light Hybrid Meson Correlation  
Function in the Chiral Limit

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# ABSTRACT

Elementary particles form hadrons through the strong interaction; one interpretation of a possible hadron bound-state is a hybrid meson which is composed of a quark-antiquark pair and gluonic content. Non-exotic hybrid mesons share spin  $J$ , parity  $P$  and charge conjugation  $C$  quantum numbers with quark-antiquark states while exotic hybrids do not. Aspects of particle physics, strong interactions, and quantum field theory necessary for calculating the correlation function for a hybrid meson will be reviewed. In particular, the perturbative part of the correlation function for a hybrid meson with  $J^{PC} = 1^{--}$  will be formulated in terms of Feynman rules and diagrams and calculated to next-to-leading order in the light (massless) quark case. Assuming the hybrid current renormalizes multiplicatively, the next-to-leading order effects are found to be large, and are potentially important for future determinations of the light-quark non-exotic hybrid meson.

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# CHAPTER 1

## INTRODUCTION TO QUANTUM FIELD THEORY

### 1.1 Particle Physics Overview

From the particle physics perspective, a particle is a very small object ( $\lesssim$  fm) that behaves in a way dictated by fundamental (strong and electroweak) interactions. Particles can be classified by intrinsic properties, conserved quantities and/or quantum numbers including spin  $J$ , mass, parity  $P$ , charge conjugation  $C$ , flavour, isospin, electric charge and colour. A particle's behaviour can be described by a theoretical model of the interactions that represents an approximation of the actual phenomena.

Quantum field theory allows us to describe the properties and interactions of fundamental and composite particles. Quantum field theories are constrained by spacetime symmetries and, as such, must be Lorentz invariant; that is, they describe relativistic systems. Four-momentum conservation and spin are consequences of this symmetry [1]. Quantum mechanics does not allow us to fix a particle to a point, so instead we consider a local quantized field. The field operators represent all possible particle states including multiple particles. Consequently, the fields have statistical properties that are reflected in their mathematical description. From the spin-statistics theorem, particles can be classified by their statistical properties. Bosons have integer spin and obey equal time commutator relations (Bose-Einstein statistics) and fermions have half integer spin and obey equal time anticommutator relations (Fermi-Dirac statistics) [2]. In order to gain information about a particle's properties we need to develop a mathematical framework to describe its field and how it interacts. In a gauge field theory interactions are constrained by a symmetry and interactions may be described by the mutual influence of the fields, where the

interaction between two elementary particles would be mediated by the exchange of a gauge boson. These mediators are represented in field theory by a gauge field. Some of the other properties that are used to distinguish between the different types of particle can be determined from symmetries of the Lagrangian. Other information can be gained directly from the equations of motion.

We know that elementary particles obey relativistic energy-momentum relations from which we can formulate the equations of motion of our fields. In relativistic quantum mechanics (which can be formulated as a classical field theory), spin  $\frac{1}{2}$  particles are described by the Dirac equation. Dirac's formulation required antiparticles; for instance, the corresponding antiparticle to the electron would have the same mass, but opposite charge. Antiparticles were confirmed when a positive particle with mass of the electron (positron) was discovered by Anderson in 1932 [3].

Initially the proton and the electron were both considered to be fundamental particles, and hence their magnetic moments could be calculated from the Dirac equation. In classical field theory, Dirac's magnetic moment is exact. In quantum field theory interactions modify the magnetic moments for this classical prediction and the magnetic moments are parametrized by their  $g$  factors. The Dirac magnetic moment is then the lowest order perturbative approximation. For the electron the experimental value is in good agreement with this prediction, but when the magnetic moment of the proton was first measured by Stern in 1932 it was  $\sim 2.5$  times larger than expected (see, *e.g.* Ref. [4]): the first indicator of proton substructure and an indicator of more fundamental particles as the proton's constituents. The discovery of the neutron and its non-vanishing magnetic moment, and the prediction and later detection of the pion, were also stepping stones for explaining the strong nuclear force. In 1934 Yukawa proposed a particle as the force carrier between the neutrons and the protons in the nucleus as an explanation for how the nucleus is held together. Since the range of the force is about the size of the nucleus, Yukawa calculated the particle's approximate mass which corresponded to the observed mass of the pions. As this mass was  $\sim 300$  times that of the electron it was called a meson meaning "middle-weight" whereas electrons were leptons ("light-weight") and neu-

trons and protons were baryons (“heavy-weight”) [2]. However the detection of more mesons and baryons, not to mention the muon which behaved like a heavy electron, showed the pion-exchange model of strong interactions was flawed and added more complexity to the problem of describing fundamental particles and their interactions. Eventually this led to a theoretical model (the quark model) that could predict some of the properties of mesons and baryons given certain assumptions about the nature of these particles as bound states resulting from the strong force.

As more particles, of all types, were discovered they were then classified by mass, lifetime and various quantum numbers. The particle lifetimes separated weak ( $\sim 10^{-10}$  s) from strong ( $\sim 10^{-23}$  s) decays. Some particles were created readily in  $\sim 10^{-23}$  s and decayed slowly in  $\sim 10^{-10}$  s indicating that different processes were occurring for a particle’s creation versus its decay [2]. Leptons did not interact strongly which separated them into their own category, whereas mesons and baryons do interact strongly so they were jointly classified as hadrons. In collisions and decays, lepton number and baryon number are conserved,<sup>1</sup> but there is no conservation of meson number [2]. With the discovery of these new particles, the idea of strangeness was introduced and later refined. “Strange” particles would be created in pairs by the strong force, but some would then decay weakly. The strange quantum number was introduced, and is only conserved for strong processes [2]. Gell-Mann’s organization of the hadron spectrum into the eightfold way patterns further refined the idea of strangeness. He organized mesons and baryons into groups by spin and then in patterns by mass, strangeness, and charge (see Figure 1). Mesons (and baryons) are classified by the combination of quantum numbers  $J$ ,  $P$  and  $C$  as  $J^{PC}$  where  $J = 0$  would represent the pseudo-scalar mesons (meson octet) and  $J = 1$  would be vector mesons [2].

Later the idea was introduced that hadrons were comprised of constituents that were fundamental particles; Gell-Mann called these particles quarks. These constituents combined to form mesons (which have integer spin) as a quark/antiquark

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<sup>1</sup>Note that some theoretical models permit proton decay and neutrino oscillations represent a lepton number violating process

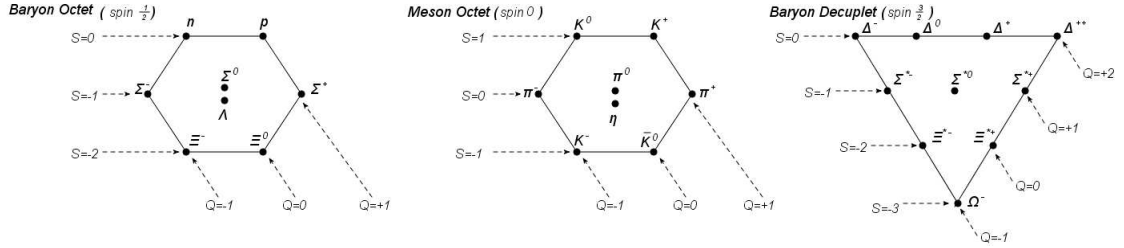


Figure 1.1: The eightfold way represents patterns of the lightest hadrons organized by charge  $Q$  in units of proton charge and strangeness  $S$

pair and baryons (which have half-integer spin) as three quarks and antibaryons as three antiquarks. These quarks had different flavours up ( $u$ ), down ( $d$ ), and strange ( $s$ ) which, when arranged according to certain rules resulted in the eightfold way patterns. In order to predict the proper charges and spins of the hadrons, the quarks needed to be spin  $\frac{1}{2}$  and the different flavours needed to have fractional charges with the  $d$  and  $s$  having charge  $-\frac{1}{3}|e|$  and  $u$  has charge  $+\frac{2}{3}|e|$  where  $|e|$  is the charge of a proton. Since the mass of the  $u$  and  $d$  quarks are approximately equal, there is no strong interaction distinction between them; hence there is an internal symmetry (with mathematical analogies to spin) that can help classify how these quarks form hadrons. This quantum number is called isospin and is conserved in strong processes. The patterns in the meson and baryon spectra can also be determined directly from flavour symmetries in the Lagrangian, that is, they can be confirmed theoretically through calculation if we assume that  $m_d \simeq m_u \simeq m_s$ . The discovery of even heavier particles required the addition of more flavours: charm ( $c$ ), bottom ( $b$ ) and top ( $t$ ); however the flavour symmetry is usually not extended to these particles due to large increases in the quark masses.

The quark model successfully describes the patterns of the hadronic spectrum. However, we still need some particle to act as the force carrier for the strong force, since the discovery of heavier mesons complicated Yukawa's model (which was also non-renormalizable). We want to have a model where at the most fundamental level the interactions are between quarks. We call the mediator for the strong force the gluon, the strong charge is called colour and the resulting theory is called quantum chromodynamics (QCD). However, individually free quarks and colour have not been

observed, so we assume there is some mechanism that confines colour and/or quarks to the bound states of hadrons.

Some scattering processes result in short-lived strongly interacting particles. From experimental results there are meson-like particles that are not readily classified within established quark-antiquark patterns, so we need to consider other possible combinations of states to describe these particles. Within the standard model, there are particles predicted by QCD, other than the conventional quark-antiquark mesons that could describe these particles. Hybrid mesons comprised of a quark-antiquark pair and gluonic content are possible candidates. Hybrid mesons come in two types: exotic and non-exotic. Exotic hybrids do not share combined quantum numbers  $J^{PC}$  with the standard mesons, while non-exotics have the same  $J^{PC}$  as conventional mesons and are therefore hard to isolate from the spectrum. A mathematical description of the hybrid meson is required in order to study these states.

In the following sections some particle physics and field theory background is presented, which leads to a discussion of the basic mathematical procedure and techniques needed to describe elementary particles. In Chapter 2, I will discuss some of the standard methods used to calculate quantities from field theory and reproduce a calculation [5] for a specific non-strange non-exotic hybrid meson candidate with  $J^{PC} = 1^{--}$ . Chapter 3 contains the original work of this thesis, where I will extend this calculation to higher-order in perturbation theory to improve the description of this specific non-exotic hybrid. In Chapter 4, I analyze my results and conclude that the higher-order corrections are substantive and could therefore be important in predictions of the hybrid mass. The Appendices contain conventions, and details of my calculations.

## 1.2 Lagrangian Field Theory

There are mathematical constraints we can use to describe how quark and gluon fields interact via colour and what a bound quark state would look like mathematically; in order to describe this we need Lagrangian mechanics and some group theory concepts. The Lagrangian formulation provides a mathematical way to describe the different types of particles and their interactions (see *e.g.*, Ref. [6]). Information on conserved quantities can be obtained from the action  $S$  or from symmetries of the Lagrangian  $\mathcal{L}$ . The action is given by an integral of the Lagrangian density over all spacetime (normally this density is just referred to as the Lagrangian)

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (1.1)$$

From the principle of least action ( $\delta S = 0$ ), if the action is varied such that the field  $\phi$  is fixed on the boundary of the integration region (that is  $\delta\phi = 0$ ), then

$$\begin{aligned} 0 = \delta S &= \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] \\ &= \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta\phi \right) \right]. \end{aligned} \quad (1.2)$$

The four-divergence piece in the last term in the second line can be written as a surface integral via Gauss's theorem. The surface integral is zero which leaves the Euler-Lagrange equation (see *e.g.*, Ref. [6])

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (1.3)$$

The principle of least action gives the Euler-Lagrange equation (1.3), so this equation is valid for any  $\delta\phi$  that vanishes on the boundary. Symmetry transformations leave the Lagrangian invariant up to a four-divergence and leads to conserved (Noether) currents when the field satisfies the equations of motion. If a particle's equation of motion is known, it is possible to work backwards and determine a Lagrangian that gives the equation. However this approach is not unique and constraints are necessary to specify the Lagrangian that suitably represents a quantum field. From symmetries of the Lagrangian, the conserved quantities of the theory can be found,

and so the Lagrangian should reflect the actual conservation laws. The Lagrangian is also invariant for gauge transformations, so these gauge symmetries should emerge from a conserved quantity. There are also other constraints on the Lagrangian; space-time symmetries need to be preserved, so the Lagrangian also needs to be Lorentz invariant. The Lagrangian is also required to be renormalizable; specifically, any quantum field theoretical divergences should be able to be systematically absorbed into the Lagrangian without introducing extra terms.

### 1.3 Symmetries

Group theory is a valuable mathematical language to describe and compare the symmetries of a theory. Conserved properties of the particles correspond to symmetries of the theory; transformations of the fields that leave the Lagrangian invariant result in conserved quantities. If we want our fields to obey certain conservation laws, the Lagrangian should be invariant under the appropriate transformation. Quantities like electric charge, total angular momentum  $J$ , and for strong interactions colour, parity, flavour and charge conjugation are physically important conserved quantities. Properties associated with conservation of colour and flavour can be determined by considering specific  $SU(N)$  symmetries in the Lagrangian. These global  $SU(N)$  transformations leave the Lagrangian invariant. If we require the Lagrangian to also be invariant locally, we need to add terms to our Lagrangian in order for the invariance to be maintained. A unitary transformation operator  $U$ , where the field is represented by  $\psi$ , has the form

$$U\psi = e^{i\alpha^a(x)t^a}\psi, \tag{1.4}$$

which has the infinitesimal expression

$$U\psi = (1 + i\alpha^a(x)t^a)\psi, \tag{1.5}$$

where  $\alpha^a(x)$  are free local parameters and  $t^a$  are Hermitian generators of the transformations. The generators satisfy a commutator algebra involving a linear combination



of the rest of the generators, so

$$[t^a, t^b] = if^{abc}t^c, \tag{1.6}$$

where  $f^{abc}$  are the structure constants of the group and are totally antisymmetric in their indices [1]. When  $f^{abc} \neq 0$  we have a non-Abelian theory. However, when we go from a global symmetry transformation to a local one we require the Lagrangian to be invariant under the transformation with  $\alpha^a(x)$  as an arbitrary function of  $x$  [6]. As will be shown below, this requires replacement of terms in the Lagrangian that have partial derivatives with covariant derivatives in order to maintain the invariance locally.

Group theory allows us, once we have ascertained that our transformation can be placed in a particular group, to then use the algebraic properties of the group to stream-line our calculations. The strong force has three types of charge or degrees of freedom. This colour degree of freedom allows us to maintain an anti-symmetric quark wave function [1] in otherwise symmetric states like the  $\Delta^{++}$  particle which has a symmetric field in terms of flavour, spin and space, because it is a spin  $\frac{3}{2}$  particle composed of three  $u$  quarks,

$$\Psi_{\Delta^{++}} = \overbrace{\Psi_{spin} \Psi_{SU(3)_{flavour}} \Psi_{space}}^{symmetric} \overbrace{\Psi_{colour}}^{antisymmetric} . \tag{1.7}$$

We postulate<sup>2</sup> that quarks transform under colour symmetries as  $SU(3)$  and that the anti-symmetric colour state is a colour singlet. We refer to a colour singlet as “colourless”. That is, we need the composite particles (baryons and mesons) to transform trivially under  $SU(3)_{colour}$  since we do not observe particles with colour charge. Colour is a useful analogy because referring to the charges as red, blue and green allows us to use the colour theory analogy red + blue +green = white (colourless) for baryons. We also need to consider the possibility anti-red + anti-blue +anti-green = white for anti-baryons and then also anti-red + red = white, and so forth, for mesons. This analogy captures the underlying mathematical properties of  $SU(3)$ .

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<sup>2</sup>This postulate is supported by empirical evidence *e.g.*,  $e^+e^-$  annihilation into hadrons.

Isospin can be represented by  $SU(2)$  flavour symmetry for  $u$  and  $d$  quarks, whereas flavour symmetry for light quarks ( $u, d, s$ ) can be represented by  $SU(3)$ . These flavour symmetries are not exact as the masses of the quarks are not equal, but otherwise the strong interaction is apparently flavour blind. The observed experimental light (ground state where  $\vec{L} = 0$ ) baryon and meson spectra have the eightfold way patterns described in Section 1.1; if we consider only light quark flavours we can use  $SU(3)$  flavour symmetry to predict this pattern. The patterns were predicted assuming that the strong force “sees” the light quarks equally except for small differences in masses (compared to hadron scales). Then a quark/antiquark combination has  $SU(3)$  flavour symmetry of  $3^* \otimes 3 = 1 \oplus 8$  and forms the observed meson patterns. For baryons the three quarks result in the symmetry  $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ . Thus if  $SU(3)$  flavour symmetry was perfect, there would be an 8-fold degeneracy in the masses of the mesons, but the symmetry is broken by the mass terms in a systematic way which allows us to describe it using group theory. By including spin to get an  $SU(6)$  symmetry the Gell-Mann Okubo mass relations can predict the masses of the lightest states to  $\sim 20\%$ , if we consider the quarks as fundamental particles with appropriately chosen masses [4].

The QCD Lagrangian for strong interactions preserves the flavour and the colour symmetries of our quark fields  $q^A(x)$  where  $A$  is a flavour index. Writing the free Dirac Lagrangian for our quark field, we have

$$\mathcal{L}_{quark} = \sum_A \bar{q}^A (i\cancel{\partial} - m_A) q^A, \quad (1.8)$$

where  $\cancel{\partial} = \gamma^\mu \partial_\mu$ ,  $\gamma^\mu$  are matrices that satisfy  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  and  $m_A$  is the quark mass (for details on units and conventions see Appendix A). However, as was noted earlier, in order for our Lagrangian to be invariant locally we need to write it in terms of the covariant derivative  $D_\mu$  defined by

$$D_\mu = \partial_\mu - ig_s A_\mu^a t^a, \quad (1.9)$$

where  $g_s$  is the strong coupling constant and the  $t^a$  are the generators of  $SU(3)$  appropriate to the field on which they are acting and we have a gauge field  $A_\mu^a$  for every

generator (the colour index  $a$  can be considered as a column vector with eight entries  $a \in \{1 \cdots 8\}$  for gluons). This means that if we want our Lagrangian to be invariant under local gauge field transformations, we must replace our partial derivatives with covariant derivatives. That is,  $\partial_\mu \rightarrow D_\mu$  in our quark field Lagrangian, so that

$$\mathcal{L}_{quark} = \bar{q}^A (i\not{D} - m_A) q^A. \quad (1.10)$$

Following Peskin and Schroeder [6], we note that this invariance is contingent upon the following infinitesimal gauge transformation of the gluon field  $A_\mu^a$ , which can be written as

$$A_\mu^a \rightarrow A_\mu^a + \frac{1}{g_s} \partial_\mu \alpha^a + f_{abc} A_\mu^b \alpha^c, \quad (1.11)$$

where  $\alpha^a$  are the transformation parameters introduced in (1.4). The covariant derivative has the algebraic relation [6]

$$[D_\mu, D_\nu] = -ig_s G_{\mu\nu}^a t^a, \quad (1.12)$$

where the strong field strength  $G_{\mu\nu}^a$  can be written as

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f_{abc} A_\mu^b A_\nu^c. \quad (1.13)$$

The last term in (1.13) reflects the non-Abelian nature of  $SU(3)$  and should be contrasted with the analogue in electromagnetic theory. We want our Lagrangian for the strong force to be gauge invariant so

$$\mathcal{L}_{gauge} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}, \quad (1.14)$$

and we can then write down our QCD Lagrangian as

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \bar{q}^A (i\not{D} - m_A) q^A. \quad (1.15)$$

However, like in QED, the gauge fields have states that correspond to non-physical polarizations and so we need to include some form of gauge condition. Unlike in QED, the gauge fixing in itself is not enough to remove the unphysical gluon polarizations. We also require the addition of another (ghost) field to our Lagrangian for which we may use the path integral to formulate.

## 1.4 Path Integral in Quantum Field Theory

We want to describe mathematically how fields propagate and interact. Processes in quantum field theory can be calculated from the generating functional  $Z[J]$ . This functional describes the time evolution of the field from an initial state to a final state [7]. For example, a free real scalar field  $\varphi$  would have the vacuum-to-vacuum amplitude  $Z_{free}[J]$  in the presence of a source  $J$

$$\begin{aligned}
 Z_{free}[J] &= \langle 0 | e^{-iHT} | 0 \rangle \\
 &= \int \mathcal{D}\varphi e^{iS(\varphi)} \\
 &= \int \mathcal{D}\varphi e^{i \int d^4x \mathcal{L}(\partial_\mu\varphi, \varphi) + J\varphi} \\
 &= \int \mathcal{D}\varphi e^{i \int d^4x \{ [-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J\varphi] + i\frac{\epsilon}{2}\varphi^2 \}},
 \end{aligned} \tag{1.16}$$

where the integral is over all possible field configurations and depends on the action. The source/sink term  $J(x)\varphi(x)$  in the Lagrangian allows us to describe our free field as propagating in spacetime and being created and annihilated. The  $\frac{i}{2}\epsilon\varphi^2$  term involves an implicit limit  $\epsilon \rightarrow 0^+$ , and is introduced to ensure the path integral is convergent. This integral can be solved by completing the square which gives us a known Gaussian integral, resulting in

$$Z_{free}[J] = e^{-[\frac{i}{2} \int \int d^4x d^4y J(x)D(x-y)J(y)]}, \tag{1.17}$$

where we have applied the normalization condition  $Z_{free}[J = 0] = 1$  and where  $D(x - y)$  is a Green function which satisfies

$$(\partial^2 + m^2 + i\epsilon) D(x - y) = -i\delta^4(x - y). \tag{1.18}$$

Equation (1.18) has the solution

$$D(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)}, \tag{1.19}$$

which is the scalar free field propagator where the  $i\epsilon$  term corresponds to the Feynman prescription for the integration around the poles [6]. If we include a  $\varphi^4$  term

in the Lagrangian, then it corresponds to an interaction. However, we no longer know the solution of the integral, so we use a perturbative expansion and consider the solution term by term. Following [7], the expansion in terms of the number of sources  $J$  is

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\varphi e^{i \int d^4x \left\{ \frac{1}{2} [(\partial\varphi)^2 - m^2\varphi^2] + J\varphi - \frac{\lambda}{4!} \varphi^4 \right\}} \\
&= \sum_{s=0}^{\infty} \frac{i^s}{s!} \int d^4x_1 \cdots \int d^4x_s [J(x_1) \cdots J(x_s)] \\
&\times \int \mathcal{D}\varphi e^{i \int d^4x \left\{ \frac{1}{2} [(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!} \varphi^4 \right\}} \varphi(x_1) \cdots \varphi(x_s) \\
&= Z[0] \sum_{s=0}^{\infty} \frac{i^s}{s!} \int d^4x_1 \cdots \int d^4x_s [J(x_1) \cdots J(x_s)] G_s(x_1, \cdots, x_s),
\end{aligned} \tag{1.20}$$

where the last line is written in terms of the  $s$ -point Green function  $G_s(x_1, \cdots, x_s)$ . These Green functions can be written in terms of functional derivatives with respect to  $J$

$$\begin{aligned}
G_s(x_1, \cdots, x_s) &= \frac{1}{Z[0]} \int \mathcal{D}\varphi e^{i \int d^4x \left\{ \frac{1}{2} [(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!} \varphi^4 \right\}} \varphi(x_1) \cdots \varphi(x_s) \\
&= \frac{1}{Z[J]} e^{-i \frac{\lambda}{4!} \int d^4w \left[ \frac{\delta}{i\delta J(w)} \right]^4} \left[ \frac{\delta}{i\delta J(x_1)} \right] \cdots \left[ \frac{\delta}{i\delta J(x_s)} \right] Z_{free}[J] \Big|_{J=0} \\
&= \frac{1}{Z[J]} e^{-i \frac{\lambda}{4!} \int d^4w \left[ \frac{\delta}{i\delta J(w)} \right]^4} \\
&\times \left[ \frac{\delta}{i\delta J(x_1)} \right] \cdots \left[ \frac{\delta}{i\delta J(x_s)} \right] e^{-\left[ \frac{i}{2} \int d^4x d^4y J(x) D(x-y) J(y) \right]} \Big|_{J=0}.
\end{aligned} \tag{1.21}$$

An event that could be thought of as having two sources and two sinks would be described in terms as a four-point Green function ( $s = 4$ ) corresponding to

$$\begin{aligned}
G_4(x_1, x_2, x_3, x_4) &= \langle T(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)) \rangle \\
&= \frac{1}{Z[J]} \left[ \frac{\delta}{i\delta J(x_1)} \right] \left[ \frac{\delta}{i\delta J(x_2)} \right] \left[ \frac{\delta}{i\delta J(x_3)} \right] \left[ \frac{\delta}{i\delta J(x_4)} \right] Z[J] \Big|_{J=0},
\end{aligned} \tag{1.22}$$

where omitted initial and final states correspond to vacuum expectation values. We can expand the interaction term in terms of the field coupling strength  $\lambda$ . The lowest order in  $\lambda$  term for a four-point Green function would be

$$\begin{aligned}
G_4(x_1, x_2, x_3, x_4) &= \frac{1}{Z[J]} \left[ -\frac{i\lambda}{4!} \right] \int d^4w \left[ \frac{\delta}{i\delta J(w)} \right]^4 \\
&\left[ \frac{\delta}{i\delta J(x_1)} \right] \left[ \frac{\delta}{i\delta J(x_2)} \right] \left[ \frac{\delta}{i\delta J(x_3)} \right] \left[ \frac{\delta}{i\delta J(x_4)} \right] Z_{free}[J] \Big|_{J=0}
\end{aligned} \tag{1.23}$$

$$\begin{aligned}
G_4(x_1, x_2, x_3, x_4) &= -\frac{i\lambda}{4!} \int d^4w \left[ \frac{\delta}{i\delta J(w)} \right]^4 \left[ \frac{\delta}{i\delta J(x_2)} \right] \left[ \frac{\delta}{i\delta J(x_3)} \right] \left[ \frac{\delta}{i\delta J(x_4)} \right] \\
&\times \frac{1}{2} \int d^4y_1 D(x_1 - y_1) J(y_1) \frac{Z_{free}[J]}{Z[J]} \Big|_{J=0} \\
&= -\frac{i\lambda}{4!} \int d^4w \left[ \left\{ D(x_1 - w) D(x_2 - w) D(x_3 - w) D(x_4 - w) \right. \right. \\
&+ D(x_1 - x_2) D(x_3 - w) D(x_4 - w) D(0) \\
&\left. \left. + D(x_1 - x_2) D(x_3 - x_4) D(0) D(0) \right\} \frac{Z_{free}[J]}{Z[J]} \Big|_{J=0} + \text{more terms} \right].
\end{aligned} \tag{1.24}$$

In the first line only one of the functional derivatives has been taken and in the last line all the derivatives have been taken. However, there are many possible combinations that will eventually result in the same contribution so I have only shown several of the possible terms. The first term shown will result in a “connected” diagram. This term corresponds to an interaction, and there are other similar terms that will also contribute. The second and third terms will result in “disconnected” diagrams. These terms do not correspond to an interaction and will not contribute to the process of physical relevance. We can represent these terms diagrammatically in Figure 1.2 where the straight lines represent the propagators  $D(x_1 - w)$  and so on, and the dot represents the interaction  $(-i\lambda) \int d^4w$  in position-space. Diagram (a) is “connected” and corresponds to the first term in the last line of (1.24). Diagrams (b) and (c) are “disconnected” and correspond to terms two and three where the  $D(0)$  is represented as a loop at  $w$ . These terms are vacuum-bubbles which are cancelled by contributions from the  $\frac{1}{Z[0]}$  factor. Each of the terms that is not shown has similar diagrams involving permutations of  $x_1, x_2, x_3$ , and  $x_4$ . Only the connected diagram (a) needs to be calculated; in fact, the connected Green functions can be isolated by the generating functional  $W[J] = \log Z[J]$ . There are  $4!$  possible combinations, so the final result would be

$$G_4(x_1, x_2, x_3, x_4) = (-i\lambda) \int d^4w D(x_1 - w) D(x_2 - w) D(x_3 - w) D(x_4 - w), \tag{1.25}$$

which is in terms of the free field propagators and the coupling constant  $\lambda$ . Also it is more convenient to Fourier transform and work in momentum space.

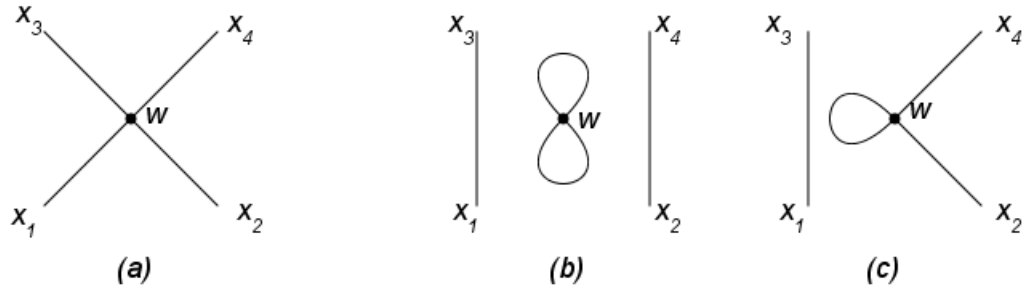


Figure 1.2: Diagrammatic representations for the four-point Green function of scalar fields with a  $\lambda\phi^4$  interaction.

For higher-order in  $\lambda$  terms the four-point function would have more complicated diagrams that would be evaluated following the same process. These diagrams could then be compared to the mathematical results and would have the same basic components, that is, they would be expressed only in terms of powers of the coupling constant and the free field propagators. Note that some of these processes would have loops in the connected diagrams. The loop propagators have momentum  $k$  that is arbitrary, so in order to include all possible combinations we include an integral over the loop momentum:  $\int d^4k$ . We can then formulate Feynman rules for how to calculate any desired term since the interactions are described by vertices  $(-i\lambda)$  in momentum-space and propagating particles are described by the propagator in momentum-space. However, these integrals may introduce divergent quantities into the theory, because very small and large momenta are included within the integration.

This systematic expansion will allow us to set up the calculation for whatever order in the coupling constant we desire and is most useful when the coupling constant is small. In QCD we would like to work in the energy region where the coupling constant is small enough for perturbation theory to be valid; this occurs when there are large momentum transfers because QCD is an asymptotically free theory [8]. However, sometimes the constraints of the problem correspond to energies where perturbation theory is not particularly accurate, and some form of non-perturbative

physics needs to be included to supplement the perturbative calculations. One approach, is to consider condensates within the operator product expansion [9]. We now have a process for describing the correlation function as a perturbative series plus condensates, and now we need to adapt this procedure to describe QCD fields.

The path-integral formalism for QCD requires that we first incorporate the spin statistics of our fermion fields into the path-integral. Then we need to find the free field propagator and expand the generating functional  $Z$  perturbatively for QCD. For free fermions  $Z[0]$  is given by

$$Z_{free}[0] = \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} e^{i \int d^4x \bar{\psi} (i\cancel{\partial} - m + i\epsilon)\psi}, \quad (1.26)$$

where if we consider the Dirac spinors  $\psi$  and  $\bar{\psi}$  to be Grassmann quantities, we can maintain the proper statistical properties of  $\psi$  and  $\bar{\psi}$ . We can then integrate over our fermion fields from the known properties of Grassmann quantities. Then following a similar procedure to the real free scalar field we can find the free field propagator

$$S(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (x-y)}}{\cancel{k} - m + i\epsilon}, \quad (1.27)$$

corresponding to the momentum-space result

$$S(\not{p}) = i \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}. \quad (1.28)$$

Quark fields have an extra index (colour), but quarks can only exchange colour via interactions, and so have the same propagator (1.28) with an implicit identity matrix in colour space. For quark/gluon and gluon/gluon interactions we can use the path integral to determine the Feynman rules in a way analogous to (1.24). We could also describe more complicated objects such as our hybrid using this approach.

For the gluon field we want to quantize the gauge field. Following Section 16.2 of Peskin and Schroeder [6], we have the functional integral for pure gauge theory

$$Z = \int \mathcal{D}A e^{i \int d^4x \left[ -\frac{1}{4} (G_{\mu\nu}^a)^2 \right]}. \quad (1.29)$$

This integral will contain field configurations that do not correspond to physical quantities. These contributions are related to unphysical gluon polarizations. Also



since we integrate over all possible field configurations, we include an infinite number of gauge fields that are related to each other by a gauge transformation. These configurations lead to the same action, and are not independent of each other. We introduce a gauge fixing parameter to remove these configurations from our solution. Following Faddeev and Popov's [10] method as outlined in [6], we constrain our gauge fixing term ( $G(A) = 0$ ) by inserting the following identity into our functional

$$1 = \int \mathcal{D}\alpha(x) \delta(G(A^\alpha)) \det \left[ \frac{\delta(G(A^\alpha))}{\delta\alpha} \right], \quad (1.30)$$

where  $\alpha$  represents a gauge transformation [6]. This constraint allows us to isolate the field configurations that correspond to gauge-transformed terms; we can then represent unphysical field configurations by a new field and finally cancel these states out by adding the appropriate terms to the Lagrangian. The field  $A_\mu^\alpha$  has the transformation properties defined in Section 1.3. The functional integral then becomes

$$\int \mathcal{D}A e^{iS[A]} = \left( \int \mathcal{D}\alpha \right) \int \mathcal{D}A e^{iS[A]} \delta(G(A^\alpha)) \det \left( \frac{\delta(G(A^\alpha))}{\delta\alpha} \right), \quad (1.31)$$

where we choose the generalized Lorentz gauge condition with Gaussian weight  $w^a(x)$

$$G(A) = \partial^\mu A_\mu^a(x) - w^a(x), \quad (1.32)$$

resulting in the gauge field propagator

$$\delta^{ab} D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i\delta^{ab}}{k^2 + i\epsilon} \left\{ g_{\mu\nu} - [1 - \xi] \frac{k_\mu k_\nu}{k^2} \right\} e^{-ik \cdot (x-y)} \quad (1.33)$$

where  $\xi$  is a freely adjustable gauge parameter. However, we often use  $\xi = 1$  which corresponds to the Feynman-'t Hooft gauge. Again using  $G(A)$  as in (1.32) and the infinitesimal form of the gauge transform (1.11), we know

$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{g} \partial^\mu D_\mu. \quad (1.34)$$

The determinant in (1.31) was rewritten by Faddeev and Popov as a functional integral over a new set of anticommuting fields with

$$\det \left( \frac{1}{g} \partial^\mu D_\mu \right) = \int \mathcal{D}c \mathcal{D}\bar{c} e^{[i \int d^4x \bar{c} (-\partial^\mu D_\mu) c]}. \quad (1.35)$$

The fields  $c$  and  $\bar{c}$  are anticommuting but transform as Lorentz scalars, that is, they do not correspond to the usual spin-statistical properties of physical fields. However, their inclusion allows us to remove the non-physical field configurations. These fields are called Faddeev-Popov ghosts, and they have the Lagrangian

$$\mathcal{L}_{ghost} = \bar{c}^a \left( -\partial^2 \delta^{ac} - g \partial^\mu f_{abc} A_\mu^b \right) c^c. \quad (1.36)$$

From (1.36) we can formulate Feynman rules for propagators and the vertices of ghost fields by following the usual procedure.

## 1.5 Basic Feynman Rules in QCD

The Feynman rules that are needed for further calculation in Chapters 2 and 3 are included in Appendix A in Section A.3 in Table A.1. For example the Feynman rule for quarks would be the fermion propagator on the right hand side of (1.28) represented by a line as shown in Table A.1 with the implied colour and flavour structure. In Section 2.3 the left hand side of (1.28) is used for quark fields but with  $S^{\alpha\beta}(\not{p})$  where  $\alpha$  and  $\beta$  are added colour indices. The gluon propagator  $D_{\mu\nu}^{ab}(k)$  in the Feynman-'t Hooft gauge  $\xi = 1$  has Feynman rule given by

$$D_{\mu\nu}^{ab}(k) = \frac{-ig_{\mu\nu}\delta^{ab}}{k^2 + i\epsilon}. \quad (1.37)$$

This is also seen from (1.33) and the diagrammatic expression is given in Table A.1.

The gluon self-energy (see Figure 3.4) which is calculated in Chapter 3 contains a gluon loop and a ghost loop (see Figure 1.3) to maintain the physicality of the process. As a simple example of how the Feynman rules are implemented, the amplitude for the ghost loop is given by

$$\Pi^{\mu\nu}(q) = \frac{1}{\nu^{2\epsilon}} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{i\delta^{gd}}{k^2 + i\epsilon} \right] \left[ -g_s f^{dac} k^\mu \right] \left[ \frac{i\delta^{ce}}{(k-q)^2 + i\epsilon} \right] \left[ -g_s f^{ebg} (k-q)^\nu \right]. \quad (1.38)$$

The first and third terms in (1.38) are ghost propagators chosen such that four-momentum is conserved at each vertex and the second and fourth are ghost-gluon

vertices. There is also an integral over loop momentum  $k$  included. This integral is divergent as can be seen by comparing powers of  $k$  in the numerator to those in the denominator. However, if the divergence is treated in a certain way calculation is still possible. In general, the first step is to regulate the integral to parametrize the divergence; dimensional regularization is used in this thesis for this purpose. The second step is to renormalize the results by systematically adjusting the physical parameters of the theory to absorb the divergent terms.

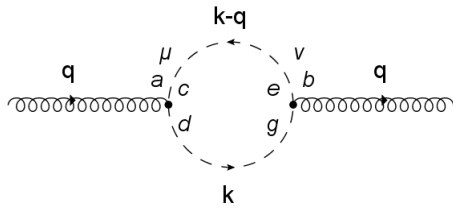


Figure 1.3: Ghost loop contribution to the gluon self-energy.

# CHAPTER 2

## LEADING-ORDER CALCULATION

### 2.1 Hybrid Mesons

Hybrid mesons with some combination of quark-antiquark and gluonic content are predicted by the standard model since there is a hybrid  $SU(3)$  colour singlet in group theory for this combination. Gluons carry one unit of colour and one of anticolour corresponding to a colour octet 8 in  $SU(3)$  [2]. The combinations  $8 \otimes 8 = 1 \oplus \dots$  and  $3^* \otimes 3 \otimes 8 = 1 \oplus \dots$  are product representations decomposed into a direct sum of irreducible representations which include colour singlet 1 states. Thus it is theoretically possible to have glueballs which are particles made entirely of gluons ( $8 \otimes 8$ ) or hybrids that are composed of a quark, an antiquark and a gluon ( $3^* \otimes 3 \otimes 8$ ). For the ground state ( $\vec{L} = 0$ ), hybrid mesons would have total angular momentum  $J = 1 \otimes \frac{1}{2} \otimes \frac{1}{2} = 0, 1, 2$ .

In the charmonium meson spectrum there are particles, with  $J^{PC} = 1^{--}$  [like the  $Y(4260)$ ], that are in excess of the predicted  $q\bar{q}$  meson states. The most attractive explanation for the  $Y(4260)$  is that it is a hybrid meson [11] since Ref. [12] calculated using the flux tube model that the lightest hybrid charmonium state would have a mass of  $\sim 4200$  MeV. The  $Y(2175)$  has been proposed as a strangeonium version of the  $Y(4260)$ , because the  $Y(2175)$  has similar production characteristics and decays [13]. The  $Y(2175)$  has  $J^{PC} = 1^{--}$  [14, 15, 16] and is below the  $c\bar{c}$  threshold, so it is a possible light or strangeonium hybrid ( $q\bar{q}g$ ) where the quarks are some combination of  $u, d$  and  $s$ . Ref. [5] calculated the first-order perturbative correction and the non-perturbative terms and obtained a possible mass from QCD sum-rules of 2.3–2.6 GeV with the range depending on their chosen quark content (massless case

for lowest mass and  $s\bar{s}$  for highest). However the  $Y(2175)$  decays to  $\phi(1020)f_0(980)$  [16], so it is likely to have  $s$  quark content because the  $\phi$  is a  $s\bar{s}$  meson.

Other possible theoretical interpretations for the  $Y(2175)$  have been suggested; Ref. [17] suggests that the tetraquark scenario is unlikely, since the experimental width is narrower than would be expected for a tetraquark state. Other possible explanations for the  $Y(2175)$  include a resonance of  $K\bar{K}$  meson bound state [18]. Refs. [13, 17] calculated the mass and allowed decay products for various hybrid models and concluded that their mass and width predictions were consistent with the  $Y(2175)$  and that a more precise experimental determination of the decays would better identify the best theoretical candidate.

The mass calculated from sum-rules by [5] is higher than the  $Y(2175)$  mass. However the next-to-leading order perturbative correction may be sizeable; calculating this correction will produce a more accurate sum-rule that will then provide a more precise determination of the mass facilitating better comparison with experiment. In Section 2.5, I will reproduce the leading-order perturbative correction in the chiral limit (massless or light quark case) calculated by [5] for a current that represents a vector particle with  $J^{PC} = 1^{--}$ . In Chapter 3, I will calculate the second-order perturbative corrections in the chiral limit for this same current. Including the strange quark mass requires an expansion to order  $m^2$  where  $m$  is the strange quark mass. The chiral limit is the first step in calculating the second-order  $m^2$  corrections, since the three-loop integrals that are produced in the chiral limit form the basis for the entire calculation. Including the  $m^2$  correction at next-to-leading orders requires a drastic increase in the number of integrals computed and the chiral limit has a large number of integrals as it is. However, the integrals required for the  $\mathcal{O}(m^2)$  terms are a basic generalization of those calculated in the chiral limit, so the massless case is the first stage of this calculation and it also provides the necessary  $\mathcal{O}(m^0)$  term in the  $m^2$  expansion.

## 2.2 Current for Hybrid Mesons

Quark confinement necessitates that a particle with quark content is mathematically represented by a current which is constructed as a composite operator with the appropriate field content. Thus, particles with a specific  $J^{PC}$  can be described mathematically via a current where properties of the particle are manifested through the current. The current is constrained to have certain qualities: a Lorentz structure that corresponds to the desired parity and spin, is a colour singlet, and a valance quark/gluon content appropriate to a hybrid state. We can calculate the current's correlation function (both the perturbative terms and Wilson coefficients of condensates) via Feynman diagrams. A correlation function is simply the Green function of the currents (composite operators). In order to do this calculation, the Feynman rules for the vertex function for this current need to be determined. The hybrid current of interest given by [5] is

$$J_\mu(x) = g_s \bar{q}_A^\alpha(x) \gamma^\nu \gamma_5 t_{\alpha\beta}^a \tilde{G}_{\mu\nu}^a q_B^\beta(x), \quad (2.1)$$

where  $t_{\alpha\beta}^a = \frac{\lambda_{\alpha\beta}^a}{2}$  with  $\lambda_{\alpha\beta}^a$  being the Gell-Mann matrices with properties defined in Appendix A.2,  $q_A^\alpha(x)$  and  $q_B^\beta(x)$  are the quark and antiquark field operators,  $\gamma^\nu$  and  $\gamma^5$  are Dirac gamma matrices with relations defined in (A.2), (A.3),  $A$  and  $B$  are flavour indices, and the dual field strength  $\tilde{G}_{\mu\nu}^c$  is

$$\tilde{G}_{\mu\nu}^c = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{c\rho\sigma}, \quad (2.2)$$

where  $\epsilon_{\mu\nu\rho\sigma}$  is the totally antisymmetric tensor in four dimensions. The current (2.1) would permit the study of a hybrid meson with  $J^{PC} = 1^{--}$  or  $0^{--}$ . Parity and charge conjugation are quantum numbers that are conserved in the strong interaction. The parity  $P$  operator describes reflection symmetry; it transforms spatial systems from right-handed to left-handed and vice versa [4]. The charge conjugation operator  $C$  is the formal expression of particle/anti-particle exchange. The parity of this current is

$$-(-1)^\mu (-1)^\nu [-(-1)^\nu] = (-1)^\mu \quad (2.3)$$

following the conventions used by [6] where  $-(-1)^\mu(-1)^\nu$  is from the dual tensor and  $-(-1)^\nu$  piece is from  $\gamma_\nu\gamma_5$ . The  $(-1)^\mu$  indicates that a particle represented by this current has the same parity as a vector meson which is  $P = -$  in  $J^{PC}$ . The charge conjugation of the current has  $+1$  from the  $\gamma_\nu\gamma_5$  part and  $-1$  from  $\tilde{G}$  as [19]

$$C[A_\mu, A_\nu]C^{-1} = [-A_\mu^T, -A_\nu^T] \quad (2.4)$$

$$[A_\mu, A_\nu] = [A_\mu^a\lambda^a, A_\nu^b\lambda^b] = if^{abc}\lambda^a A_\mu^b A_\nu^c \quad (2.5)$$

$$[\lambda_a^T, \lambda_b^T] = -([\lambda_a, \lambda_b T])^T = -if^{abc}\lambda_c^T, \quad (2.6)$$

and therefore  $C = -$  in  $J^{PC}$ . The combination of the Lorentz transformations properties of  $J_\mu$  and its  $PC$  values implies that the current can probe  $1^{--}$  states.

## 2.3 Feynman Rules for the Current

The interactions of the current (2.1) with quarks and gluons would be represented at the order  $g_s$  by Figure 2.1 where  $1 \leq \alpha_2, \alpha_3 \leq 3$  and  $1 \leq a_1 \leq 8$  are colour indices.

The Green function

$$\begin{aligned} & \int d^4x_1 d^4x_2 d^4x_3 d^4y e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 + p \cdot y)} \langle 0 | T \{ q_{\alpha_3}(x_3) \bar{q}_{\alpha_2}(x_2) A_{\mu_1}^{a_1}(x_1) J^\mu(y) \} | 0 \rangle \\ & = g_s [iS^{\alpha_3\alpha}(-\not{p}_3)] \gamma_\nu \gamma_5 t_{\alpha\beta}^a [iS^{\beta\alpha_2}(\not{p}_2)] \epsilon^{\mu\nu\rho\sigma} [iD_{\sigma\mu_1}^{aa_1}(p_1)] [ip_1]_\rho, \end{aligned} \quad (2.7)$$

represents this process. There is also an implicit delta function that enforces four-momentum conservation at the vertex. The flavour indices have been suppressed as flavour will be conserved at the vertex and flavours will enter into the calculation via the quark propagator masses in the Feynman diagrams, since the mass  $m$  in the propagator could be  $m_d$ ,  $m_u$  or  $m_s$  for light quarks. If the propagators  $[iS^{\alpha_3\alpha}(-\not{p}_3)]$ ,  $[iS^{\beta\alpha_2}(\not{p}_2)]$  and  $[iD_{\sigma\mu_1}^{aa_1}(p_1)]$  are removed from the last line we get the Feynman rule for the vertex as shown in Figure 2.2. The diagram for the first-order calculation of the hybrid correlator (see Figure 2.5), will only contain this vertex. In the next section, I will evaluate the correlation function needed to represent a hybrid with  $u$ ,  $d$  quark content in the massless quark case (also referred to as the chiral limit) where  $m_d \simeq m_u \simeq 0$ .

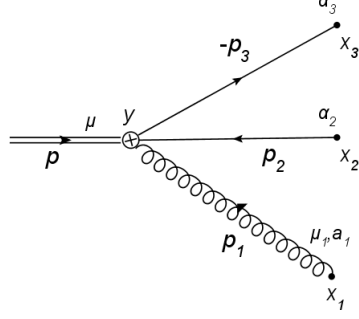


Figure 2.1: Single-gluon vertex for the hybrid current.

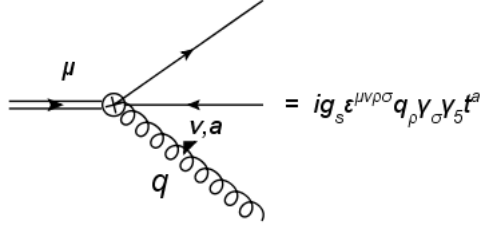


Figure 2.2: Vertex Feynman rule for the hybrid current with a quark, an antiquark and a gluon.

At next-to-leading order, there is another possible vertex which is shown in Figure 2.3, it has the expression

$$\begin{aligned}
 & \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4y e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 + p_4 \cdot x_4 + p \cdot y)} \\
 & \quad \times \langle 0 | T \{ q_{\alpha_4}(x_4) \bar{q}_{\alpha_3}(x_3) A_{\mu_1}^{b_1}(x_1) A_{\mu_2}^{b_2}(x_2) J^\mu(y) \} | 0 \rangle \\
 & = g_s [iS^{\alpha_4\alpha}(-p_4)] \gamma_\nu \gamma_5 t_{\alpha\beta}^a [iS^{\beta\alpha_3}(p_3)] \frac{\epsilon^{\mu\nu\rho\sigma}}{2} [iD_{\mu_1\rho}(p_1)] \\
 & \quad \times [iD_{\mu_2\sigma}(p_2)] [f^{ab_1b_2} - f^{ab_2b_1}],
 \end{aligned} \tag{2.8}$$

where again removing the propagators and simplifying  $f^{ab_2b_1} = -f^{ab_1b_2}$  results in the Feynman rule shown in Figure 2.4.

## 2.4 Correlation Functions and Feynman Integrals

The correlation function  $\Pi^{\mu\nu}(p)$  of hybrid currents

$$\Pi^{\mu\nu}(p) = i \int d^4x e^{ip \cdot x} \langle \Omega | T \{ J^\mu(x) J^\nu(0) \} | \Omega \rangle \tag{2.9}$$

is a Green function of composite operators. The current is not conserved, and so the correlation function has a longitudinal part  $\Pi_0(p^2)$  (representing a spin zero



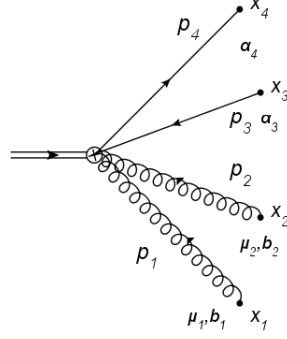


Figure 2.3: Vertex for interaction of hybrid current with a quark, an antiquark and two gluons.

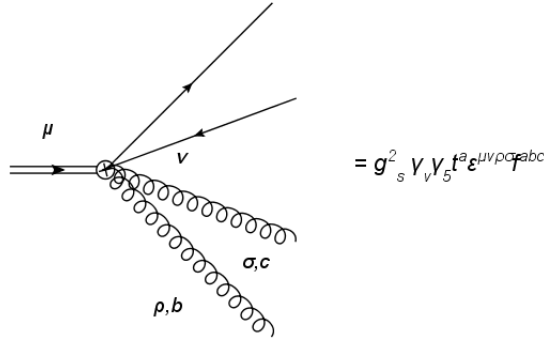


Figure 2.4: Vertex Feynman rule for interaction of hybrid current with a quark, an antiquark and two gluons.

particle) in addition to a transverse part  $\Pi_1(p^2)$  (representing a spin one particle). This decomposition is given by

$$\Pi^{\mu\nu}(p^2) = \left( \frac{p^\mu p^\nu}{p^2} - g^{\mu\nu} \right) \Pi_1(p^2) + \frac{p^\mu p^\nu}{p^2} \Pi_0(p^2). \quad (2.10)$$

The  $\Pi_1(p^2)$  corresponds to the vector part which is the desired state for this calculation.  $\Pi_1(p^2)$  can be extracted from  $\Pi^{\mu\nu}(p)$  through

$$\Pi_1(p^2) = \frac{[p^2 g_{\mu\nu} - p_\mu p_\nu]}{(1-D)p^2} \Pi^{\mu\nu}(p). \quad (2.11)$$

Feynman diagrams allow us to diagrammatically represent the correlation function as a perturbative expansion in the coupling constant where  $\alpha_s = g_s^2/4\pi$ , and (2.11) is used to isolate the desired state. The spectral function has the dispersion relation

$$\frac{d^n \Pi(Q^2)}{(dQ^2)^n} = \frac{1}{\pi} \int_{t_0}^{\infty} dt \frac{\text{Im}\Pi(t)}{(t+Q^2)^{n+1}}, \quad (2.12)$$

so polynomials in  $Q^2 = -p^2$  will be removed by the derivatives of the correlation function.

The correlation function calculated from Feynman rules will be written in terms of divergent integrals, so we need to isolate the divergences that will affect our results by regularizing the integral, and then we can renormalize the correlation function. Dimensional regularization allows us to do this systematically [22]. To regularize the integral we work in  $D$  dimensions, so  $d^4k/(2\pi)^4 \rightarrow d^Dk/(2\pi)^D$ , where  $D = 4 + 2\epsilon$  (the parameter  $\epsilon$  for dimensional regularization should not be confused with the  $\epsilon$  first introduced in (1.19) for propagators) and we take the limit  $\epsilon \rightarrow 0$  after we have calculated physical quantities [20, 22]. This allows us to do the integral and isolate the divergences in terms proportional to  $\frac{1}{\epsilon}, \frac{1}{\epsilon^2}, \dots$  which then cancel other divergent terms we acquire from renormalizing the bare parameters and currents. The divergent terms that are polynomial in  $Q^2$  will be removed by the derivatives of the dispersion relation (2.12), so these divergences do not enter into the physical quantities. Once the divergences have been eliminated,  $\epsilon$  can be set to zero. We also include a factor  $\frac{1}{\nu^{2\epsilon}}$  where  $\nu$  is a renormalization scale with dimensions of mass.

The integrals can be calculated in terms of several component integrals. The most basic dimensional regularization integral is [20]

$$I(\alpha, \beta) = \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\alpha}{(k^2 - a^2 + i\epsilon)^\beta}. \quad (2.13)$$

The  $k_0$  integral has simple poles at  $k_0^2 = |\vec{k}|^2 - i\epsilon$ . The discontinuity can be dealt with by Wick-rotating the integral where  $k_0 \rightarrow ik_{0E}$ . Then, the momentum  $k^2 = k_0^2 - |\vec{k}|^2 = -k_E^2$  becomes Euclidian momentum  $k_E$  and  $d^D k = id^D k_E$ . Since  $\int d\Omega_D = 2\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2})$ , the integral can be written as a  $D$  dimensional volume integral where  $\int d^D k_E = \int d\Omega_D \int_0^\infty dk_E k_E^{D-1}$  and

$$\begin{aligned} I(\alpha, \beta) &= (-1)^{\alpha-\beta} i \int \frac{d^D k_E}{(2\pi)^D} \frac{(k_E^2)^\alpha}{(k_E^2 + a^2)^\beta} \\ &= (-1)^{\alpha-\beta} i \frac{2}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \int_0^\infty dk_E \frac{k_E^{D-1} (k_E^2)^\alpha}{(k_E^2 + a^2)^\beta}. \end{aligned} \quad (2.14)$$

Setting  $y = k_E/a$  the integral can be rewritten as

$$I(\alpha, \beta) = (-1)^{\alpha-\beta} \frac{i}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} \int_0^\infty dy \frac{1}{a^{\beta-\alpha-\frac{D}{2}}} \frac{y^{\alpha+\frac{D}{2}-1}}{(y+1)^\beta}. \quad (2.15)$$

This integral can be compared with the Beta function [21] which is given by

$$B(z, w) = \int_0^\infty dt \frac{t^{z-1}}{(t+1)^{z+w}} = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z+w)}. \quad (2.16)$$

Thus, (2.15) results in a Beta function as defined in (2.16) with  $w = \beta - \alpha - \frac{D}{2}$  and  $z = \alpha + \frac{D}{2}$ . Since  $D = 4 + 2\epsilon$  the integral becomes

$$\begin{aligned} I(\alpha, \beta) &= \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\alpha}{(k^2 - a^2 + i\epsilon)^\beta} \\ &= (-a^2)^{\alpha-\beta+2} \frac{i}{(4\pi)^2} \left(\frac{a^2}{4\pi}\right)^\epsilon \frac{\Gamma(\epsilon+2+\alpha) \Gamma(\beta-\epsilon-2-\alpha)}{\Gamma(\beta) \Gamma(\epsilon+2)}. \end{aligned} \quad (2.17)$$

The result (2.17) can be generalized to solve the integrals which result from the Feynman rules through an analytic continuation of the Gamma functions. In the massless case the component integrals (with  $\gamma$  a constant) are

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\gamma}{k^2 + i\epsilon} = 0. \quad (2.18)$$

This integral is called a massless tadpole and evaluates to zero [22]. The simplest non-zero one loop integral is

$$\begin{aligned} &\frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + i\epsilon][(k-p)^2 + i\epsilon]} \\ &= \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{\Gamma(2)}{\Gamma(1)^2} \int_0^1 dx \frac{1}{\left[[(k-p)^2 + i\epsilon]x + (k^2 + i\epsilon)(1-x)\right]^2} \\ &= \frac{1}{\nu^{2\epsilon}} \int_0^1 dx \int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell^2)^0}{\ell^2 - a^2 + i\epsilon}, \end{aligned} \quad (2.19)$$

where, in the last line, the change of variables  $\ell = k - xp$  and  $-a^2 = p^2 x(1-x)$  have been made. The integral is now in the form of the basic dimensional regularization

result (2.17) with  $\beta = 2$  and  $\alpha = 0$ :

$$\begin{aligned}
& \frac{1}{\nu^{2\epsilon}} \int_0^1 dx \int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell^2)^0}{\ell^2 - a^2 + i\epsilon} \\
&= \int_0^1 dx [x(1-x)]^\epsilon (-1)^\epsilon (-p^2)^0 \frac{i}{(4\pi)^2} \left( \frac{p^2}{4\pi\nu^2} \right)^\epsilon \frac{\Gamma(\epsilon+2)\Gamma(-\epsilon)}{\Gamma(2)\Gamma(\epsilon+2)} \\
&= \frac{i}{(4\pi)^2} \left[ \frac{-p^2}{4\pi\nu^2} \right]^\epsilon \frac{\Gamma(1+\epsilon)\Gamma(1+\epsilon)\Gamma(-\epsilon)}{\Gamma(2+2\epsilon)}.
\end{aligned} \tag{2.20}$$

In the last line, the integral over  $dx$  has been calculated in terms of a Beta function. The generalized one loop integrals have the form and solution

$$\begin{aligned}
& \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{k^{\{\mu_1\mu_2\cdots\mu_u\}}}{[k^2 + i\epsilon]^r [(k-p)^2 + i\epsilon]^s} = \\
&= \frac{i}{(4\pi)^2} \left[ -\frac{p^2}{4\pi\nu^2} \right]^\epsilon \frac{p^{\{\mu_1\mu_2\cdots\mu_u\}}}{p^{2(r+s-2)}} \frac{\Gamma(u+2-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(u+4-r-s+2\epsilon)},
\end{aligned} \tag{2.21}$$

where the term  $k^{\{\mu_1\mu_2\cdots\mu_u\}}$  is a traceless symmetric tensor (for example in the two index case  $k^{\{\mu_1\mu_2\}} = k^{\mu_1}k^{\mu_2} - \frac{k^2}{D}g^{\mu_1\mu_2}$ ). The cases where the number of indices  $u$  are zero, one and two are shown in [20] and are useful for calculating the leading order perturbative result for the hybrid correlator.

## 2.5 First Order Calculation of the Hybrid Correlation Function

Using Feynman rules to construct a mathematical expression for the leading-order diagram depicted in Figure 2.5 gives us the following expression for the first-order

perturbative contribution to the hybrid correlation function

$$\begin{aligned}
\Pi^{\mu\nu}(p) &= -i \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left\{ i g_s \epsilon^{\mu\lambda\rho\sigma} q_\rho \gamma_\sigma \gamma_5 t^a \left[ \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\epsilon} \right] \right. \\
&\quad \times i g_s \epsilon^{\nu\tau\eta\xi} (-q_\eta) \gamma_\xi \gamma_5 t^b \left[ \frac{i(\not{q} - \not{k} + m)}{(q-k)^2 - m^2 + i\epsilon} \right] \left. \left[ -i \frac{g_{\lambda\tau} \delta^{ab}}{q^2 + i\epsilon} \right] \right\} \\
&= -i g_s^2 \frac{1}{\nu^4 \epsilon} \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dq}{(2\pi)^D} \text{Tr} \{ t^b t^a \} \text{Tr} \left\{ \epsilon_{\lambda\ \mu\rho\sigma} \epsilon^{\lambda\nu\eta\xi} i q_\rho \gamma_\sigma \left[ \frac{i(\not{k} - \not{p} - m)}{(p-k)^2 - m^2 + i\epsilon} \right] \right. \\
&\quad \times i (-q_\eta) \gamma_\xi \left. \left[ \frac{i(\not{q} - \not{k} + m)}{(q-k)^2 - m^2 + i\epsilon} \right] \left[ -\frac{i}{q^2 + i\epsilon} \right] \right\}, \tag{2.22}
\end{aligned}$$

where in the last line (A.3) and (A.4) have been used to simplify the expression.

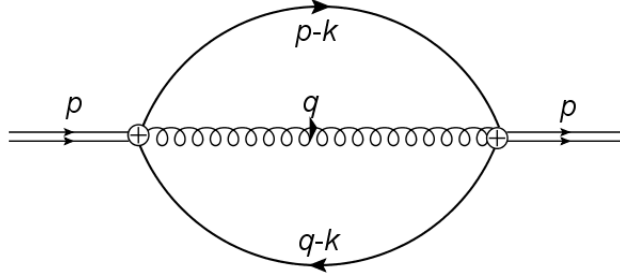


Figure 2.5: Leading order diagram for the correlation function.

We can choose to work in the Feynman-'t Hooft gauge ( $\xi = 1$ ) as the current is gauge invariant. The numerator in Eq. (2.22) can be expanded and simplified using the computer program REDUCE which results in a number of two-loop integrals. The antisymmetric tensor  $\epsilon^{\nu\tau\eta\xi}$  is defined in four dimensions and dimensional regularization requires us to work in  $D$  dimensions. Therefore  $\epsilon_\nu^{\lambda\rho\sigma} \epsilon^{\nu\tau\eta\xi}$  should be replaced by the D-dimensional continuation of the contraction identities for the antisymmetric tensor. These identities are [23]

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\ \nu\eta\lambda} &= -(D-3) [g^{\nu\nu} (g^{\rho\eta} g^{\sigma\lambda} - g^{\rho\lambda} g^{\sigma\eta}) + g^{\nu\eta} (g^{\rho\lambda} g^{\sigma\nu} - g^{\rho\nu} g^{\sigma\lambda}) \\
&\quad + g^{\nu\lambda} (g^{\rho\nu} g^{\sigma\eta} - g^{\rho\eta} g^{\sigma\nu})] \tag{2.23}
\end{aligned}$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\ \eta\lambda} = -(D-3)(D-2) (g^{\rho\eta} g^{\sigma\lambda} - g^{\rho\lambda} g^{\sigma\eta}). \tag{2.24}$$

REDUCE calculates the D-dimensional trace, so the identities (2.23) and (2.24) used in REDUCE represent the antisymmetric tensor. Using the computer program

REDUCE to simplify the Dirac trace in (2.22) and calculating the resulting Feynman integrals produces

$$\Pi_1(p^2) = \frac{\alpha_s}{\pi^3} p^6 \left( \frac{-p^2}{4\pi\nu^2} \right)^{2\epsilon} \frac{2(4\epsilon^3 + 12\epsilon^2 + 11\epsilon + 3) [\Gamma(3 + \epsilon)]^3 \Gamma(-4 - 2\epsilon)}{(2 + \epsilon)^2 \Gamma(6 + 3\epsilon)}, \quad (2.25)$$

which has the  $\epsilon$  expansion

$$\begin{aligned} \Pi_1(p^2) &= -\frac{\alpha_s}{\pi^3} \left[ \frac{1}{480\epsilon} + \left( \frac{\gamma_E}{240} - \frac{77}{9600} \right) + \mathcal{O}(\epsilon) \right] p^6 \left( \frac{-p^2}{4\pi\nu^2} \right)^{2\epsilon} \\ &= -\frac{\alpha_s}{\pi^3} p^6 \left[ \frac{1}{480\epsilon} + \left( \frac{\gamma_E}{240} - \frac{77}{9600} \right) \right] \left[ 1 + 2\epsilon \log \left( \frac{-p^2}{4\pi\nu^2} \right) \right] \\ &= -\frac{\alpha_s}{\pi^3} p^6 \left( \underbrace{\frac{1}{240} \log \left( \frac{-p^2}{\nu^2} \right)}_{\text{physical part}} \underbrace{-\frac{1}{240} \log \left( \frac{1}{4\pi} \right) + \frac{1}{480\epsilon} + \left( \frac{\gamma_E}{240} - \frac{77}{9600} \right)}_{\text{removed by derivatives of the dispersion relation}} \right), \end{aligned} \quad (2.26)$$

where  $\gamma_E$  is Euler's constant. In the  $\overline{\text{MS}}$  renormalization scheme  $\nu^2$  is rescaled such that terms with  $\gamma_E$  and  $\log \left( \frac{1}{4\pi} \right)$  are removed. Although this rescaling is not important at this order, it is at next-to-leading order. The part of the perturbative first order correction that contributes to further analysis is

$$\Pi_1^{l.o.}(p^2) = -\frac{\alpha_s}{\pi^3} p^6 \frac{1}{240} \log \left( \frac{-p^2}{\nu^2} \right) \quad (2.27)$$

which agrees with the result calculated in [5]. This is also the only piece that will contribute to the sum-rules, since the  $\frac{1}{\epsilon}$  divergent terms are polynomials in  $p^2$ , and can be removed by derivatives of  $\Pi_1^{l.o.}(p^2)$ .

# CHAPTER 3

## NEXT-TO-LEADING ORDER CALCULATION OF THE HYBRID CORRELATION FUNCTION

### 3.1 Overview of Diagrams

At next-to-leading order  $\alpha_s^2$ , there are 14 diagrams as shown in Figure 3.1 where all the gluon self-energy contributions are summed in diagram 4 (see Section 3.4). These diagrams have distinct topologies although some of them (those that are very similar topologically) will have similar if not exact expressions resulting from Feynman rules.

### 3.2 Most Complicated Topology Diagram

I calculated the diagram of Figure 3.2 first as it has the most complicated topology (the diagrams in Section 3.6 have similar topology), and therefore the integrals calculated for this diagram should constitute most of the integrals needed to calculate the remaining diagrams. In general I will use Latin indices for colour ( $a, b, a_1, a_2$ ) and Greek for space-time indices ( $\mu, \nu, \nu_1, \nu_2, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3$ ). The variables  $k, \ell$  and  $q$  are internal momenta, and  $p$  is the external momenta and I have used these assignments for internal and external momenta through-out the calculation. Feynman rules for this diagram give  $\Pi_{(1)}^{\mu\nu}(p)$  corresponding to the amplitude for Figure 3.2, where the subscript (1) represents diagram number as labelled in Figure 3.1.

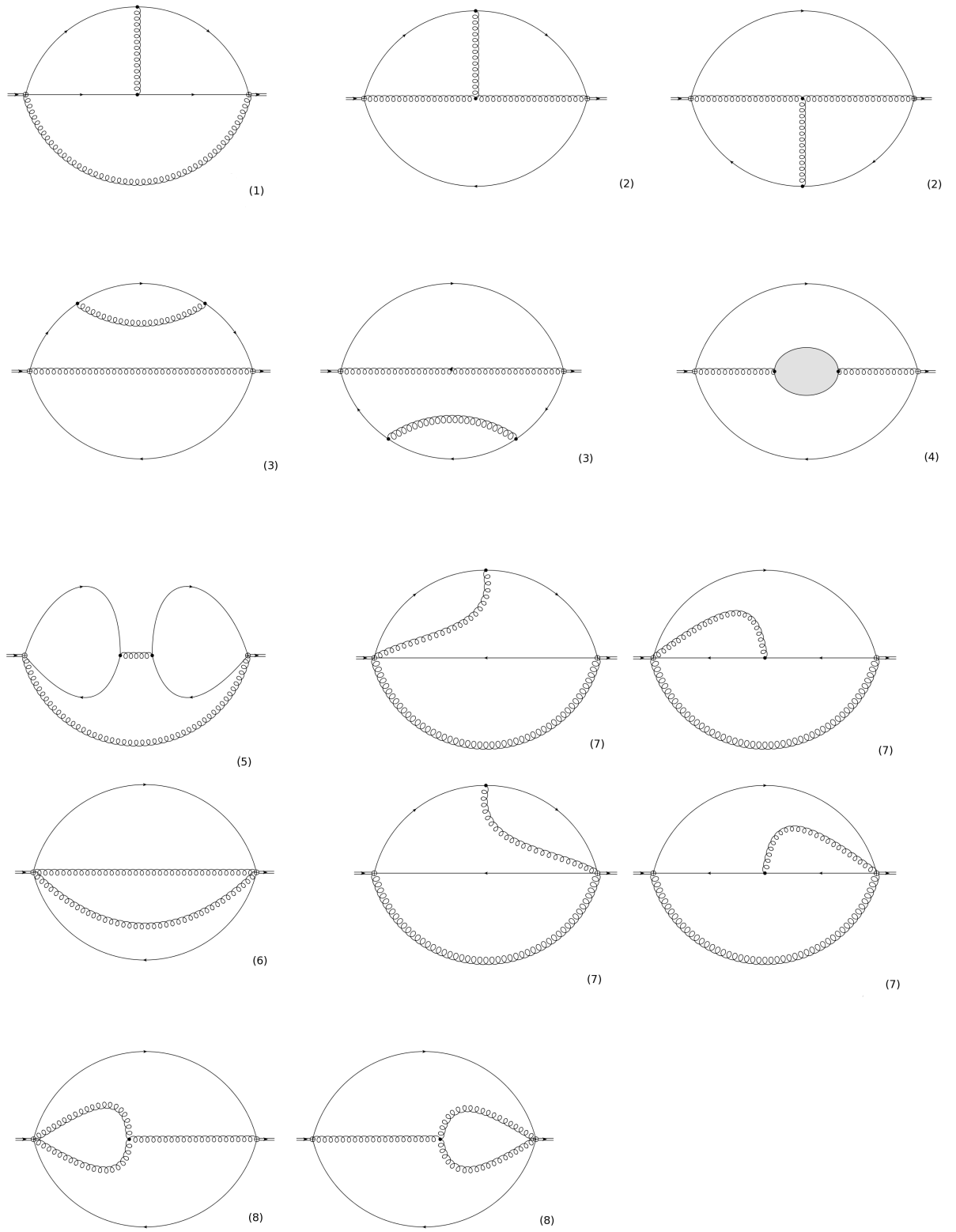


Figure 3.1: Second order diagrams for the two-current correlation function of hybrid currents.



The expression for the diagram is

$$\begin{aligned}
\Pi_{(1)}^{\mu\nu}(p) = & -i \int \frac{d^4\ell}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left\{ \left[ \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} \right] i g_s \gamma^{\nu_2} t^{a_2} \right. \\
& \times \left[ \frac{i(\not{q} + m)}{q^2 - m^2 + i\epsilon} \right] i g_s \epsilon^{\mu\rho_1\rho_2\rho_3} [- (p - \ell)]_{\rho_2} \gamma_{\rho_3} \gamma_5 t^a \left[ \frac{i(\not{q} - \not{\ell} + m)}{(q - \ell)^2 - m^2 + i\epsilon} \right] \\
& \times i g_s \gamma^{\nu_1} t^{a_1} \left[ \frac{i(\not{k} - \not{\ell} + m)}{(k - \ell)^2 - m^2 + i\epsilon} \right] i g_s \epsilon^{\nu\sigma_1\sigma_2\sigma_3} [p - \ell]_{\sigma_2} \gamma_{\sigma_3} \gamma_5 t^b \\
& \left. \times \left[ \frac{-i g_{\nu_2\nu_1} \delta^{a_2 a_1}}{(k - q)^2 + i\epsilon} \right] \left[ \frac{-i g_{\rho_1\sigma_1} \delta^{ab}}{(p - \ell)^2 + i\epsilon} \right] \right\}. \tag{3.1}
\end{aligned}$$

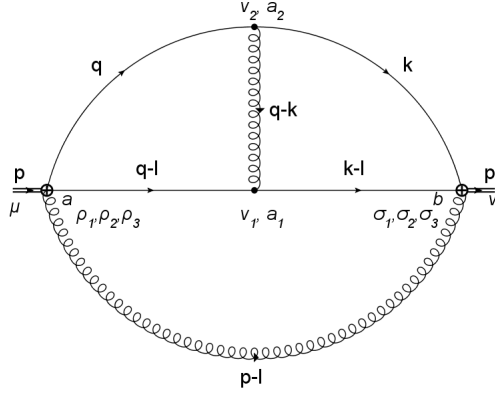


Figure 3.2: Diagram with the most complicated topology, labelled as (1) in Figure 3.1.

There are some simplifications that can be made with the indices and by using the anti-commutation relations between  $\gamma^\lambda$  and  $\gamma_5$  (A.3) as well as (A.4). Also in dimensional regularization the integrals go from 4 to  $D$  dimensions. For the remaining diagrams, I will write down the expression in  $D$  dimensions and drop the  $i\epsilon$  terms for simplicity. Then

$$\begin{aligned}
\Pi_{(1)}^{\mu\nu}(p) = & -i g_s^4 \frac{C_1}{\nu^{6\epsilon}} \int \frac{d^D\ell}{(2\pi)^D} \int \frac{d^Dq}{(2\pi)^D} \int \frac{d^Dk}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k} + m)}{k^2 - m^2} \right] i \gamma_{\nu_1} \left[ \frac{i(\not{q} + m)}{q^2 - m^2} \right] \right. \\
& \times i [- (p - \ell)]_{\rho_2} \gamma_{\rho_3} \left[ \frac{i(\not{q} - \not{\ell} - m)}{(q - \ell)^2 - m^2} \right] i \gamma_{\nu_1} \left[ \frac{i(\not{k} - \not{\ell} + m)}{(k - \ell)^2 - m^2} \right] \\
& \left. \times i [p - \ell]_{\sigma_2} \gamma_{\sigma_3} \left[ \frac{-i}{(k - q)^2} \right] \left[ \frac{-i}{(p - \ell)^2} \right] \epsilon^{\mu \sigma_1 \rho_2 \rho_3} \epsilon^{\nu \sigma_1 \sigma_2 \sigma_3} \right\}, \tag{3.2}
\end{aligned}$$

where the constant  $C_1 = \text{Tr} \{t^{a_1} t^{b_1} t^{a_1} t^{b_1}\}$  is the colour factor for this diagram. As in the first-order calculation it is the vector part of  $\Pi_{(1)}^{\mu\nu}(p)$  that is relevant for the calculation so we want to calculate  $\Pi_1^{(1)}(p^2)$ .

The Dirac trace in the  $\Pi_1(p^2)$  projection of (3.2) was calculated using the program REDUCE in the chiral limit ( $m = 0$ ). The result from REDUCE is in terms of four-momentum dot products, and there are some useful changes of variables that convert these dot products into a form that allows for a systematic classification of the resulting integrals. Since

$$p \cdot q = \frac{1}{2} [p^2 + q^2 - (q - p)^2], \quad (3.3)$$

all of the dot products can be redefined this way in REDUCE. The advantage of this change of variables is that the result is in terms of integrals that are easier to classify. The integrals can be classified into those that result in massless tadpoles which are zero and nonzero integrals which were given numeric designation  $n1$  through  $n66$  in my REDUCE code (see Appendix C for sample code). It also makes it easier to identify when two integrals are the same under a change of variable like  $k \leftrightarrow q$ . Once the above procedure was applied to my results, I had three basic types of integrals, two of which could be calculated from iterated one-loop integrals and a third which could be calculated from recursion relations (see Appendix B.3). Type one, which I called  $n_1$ , looks like

$$n_1(b, c, a) = \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q]^{2c} [q - \ell]^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k]^{2b} [k - q]^2}, \quad (3.4)$$

where  $a$  and  $c$  are integers, and  $b$  is a positive integer. Different combinations of  $b$ ,  $c$  and  $a$  correspond to individual numeric designations as described above; for example  $n1 = n_1(1, 1, 0)$ . The solution for (3.4) is obtained in Eq. (B.1) of Appendix B. The second type of integral has the form

$$n_2(b, c, a) = \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q]^{2c} [q - \ell]^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k]^{2b} [k - \ell]^2}, \quad (3.5)$$

where the solution for (3.5) is obtained in Eq. (B.2) of Appendix B. The third type of integral is

$$n_3(a) = \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[q]^2 [q - \ell]^2 [k - q]^2 [k]^2 [k - \ell]^2}, \quad (3.6)$$

where the solution is again outlined in Appendix B and is given by Eq. (B.5).

Some of the classified integrals have numerator factors such as  $(q-p)^2(k-\ell)^2$ . These integrals can be expanded and the result written in terms of  $n_1$ ,  $n_2$  and  $n_3$  integrals. However integrals that contain a factor like  $k^\mu$  require the vector form of the integral and integrals with terms like  $k^\mu k^\nu$  the appropriate tensor form of  $n_1$ ,  $n_2$  or  $n_3$ . These integrals are discussed in more detail in Appendix B. The numerically classified integrals from the trace calculated in REDUCE can then be replaced by their scalar, vector and tensor components. The trace over the colour algebra is  $C_1$  where the  $a_1 = a$  index has been relabelled

$$\begin{aligned}
C_1 &= \text{Tr} \{t^a t^b t^a t^b\} \\
&= \frac{1}{2^4} \left[ \frac{4}{N} (\delta^{ab} \delta^{ab} - \delta^{aa} \delta^{bb} + \delta^{ab} \delta^{ba}) + 2 (d^{abc} d^{abc} - d^{aac} d^{bbc} + d^{abc} d^{bac}) + \right. \\
&\quad \left. + 2i (d^{abc} f^{abc} - d^{aac} f^{bbc} + d^{abc} f^{bac}) \right] \\
&= \frac{1}{2^4} \left[ \delta^{aa} \left( 4N - \frac{4}{N} \delta^{aa} - \frac{8}{N} \right) \right] = -\frac{2}{3}.
\end{aligned} \tag{3.7}$$

Note that  $d^{aac} = 0$ ,  $d^{abc}$  is real and totally symmetric,  $N = 3$  in  $SU(3)$ ,  $\delta^{aa} = 8$ , and the identity (A.9) has been used to simplify (3.7). The final expression for the vector part is

$$\begin{aligned}
\Pi_1^{(1)}(p^2) &= \frac{32}{243} \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 \frac{(1+\epsilon)\epsilon\Gamma[(\epsilon)]^2\Gamma(1+2\epsilon)\pi^2}{\Gamma(7+4\epsilon)\Gamma(\frac{1}{2}-\epsilon)} \\
&\quad \times \frac{\Gamma(-2\epsilon)\Gamma(2+3\epsilon)\Gamma(-1-3\epsilon)}{\Gamma(\frac{4}{3}+\epsilon)\Gamma(\frac{5}{3}+\epsilon)\Gamma(\frac{3}{2}+\epsilon)} \left\{ 54\sqrt{\pi} 4^{-\epsilon} \Gamma\left(\frac{2}{3}+\epsilon\right) \Gamma\left(\frac{1}{3}+\epsilon\right) \right. \\
&\quad \times \left( \frac{-32\epsilon - 81\epsilon^6 - 166\epsilon^5 - 103\epsilon^2 - 184\epsilon^3 - 4 - 212\epsilon^4 - 18\epsilon^7}{[\Gamma(\frac{1}{2}+\epsilon)]^2 \Gamma(\frac{1}{2}-\epsilon)} \right) + 2\sqrt{3} \times \\
&\quad \left. \times 27^{-\epsilon} \left( \frac{1226\epsilon^5 + 1635\epsilon^2 + 2331\epsilon^4 + 2588\epsilon^3 + 372\epsilon^6 + 56\epsilon^7 + 540\epsilon + 72}{\Gamma(1-\epsilon)} \right) \right\}.
\end{aligned} \tag{3.8}$$

This expression still contains  $\epsilon$ , so it needs to be expanded in a Laurent series (see Table 4.1).

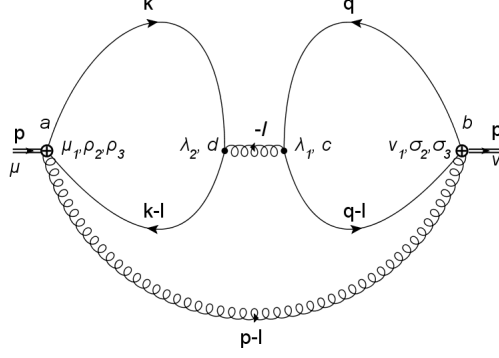


Figure 3.3: The Ravenous Bugbladder Beast of Traal diagram.

### 3.3 The Ravenous Bugbladder Beast of Traal

Despite its intimidating appearance, the result for the diagram in Figure 3.3 turns out to be very simple. Note there is a trace for each fermion loop in the diagram and a factor of  $-1$  for each fermion loop, so the Feynman rules for this diagram give

$$\begin{aligned}
\Pi_{(5)}^{\mu\nu}(p) &= \frac{i}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(k - \ell + m)}{(k - \ell)^2 - m^2} \right] i g_s \gamma^{\lambda_2} t^d \right. \\
&\quad \times \left. \left[ \frac{i(k + m)}{k^2 - m^2} \right] i g_s \epsilon^{\mu\mu_1\rho_2\rho_3} [-(p - \ell)]_{\rho_2} \gamma_{\rho_3} \gamma_5 t^a \right\} \text{Tr} \left\{ \left[ \frac{i(\not{q} - \ell + m)}{(q - \ell)^2 - m^2} \right] \right. \\
&\quad \times i g_s \gamma^{\lambda_1} t^c \left. \left[ \frac{i(\not{q} + m)}{q^2 - m^2} \right] i g_s \epsilon^{\nu\nu_1\sigma_2\sigma_3} [p - \ell]_{\sigma_2} \gamma_{\sigma_3} \gamma_5 t^b \right\} \left[ -i \frac{g_{\mu_1\nu_1} \delta^{ab}}{(-\ell)^2} \right] \\
&\quad \times \left[ -i \frac{g_{\lambda_1\lambda_2} \delta^{cd}}{(p - \ell)^2} \right].
\end{aligned} \tag{3.9}$$

The traces in (3.9) are straightforward; setting  $m = 0$ , and doing some simplification we get

$$\begin{aligned}
\Pi_{(5)}^{\mu\nu}(p) &= \frac{-g_s^4}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k - \ell)^2 k^2 (q - \ell)^2 q^2 \ell^2 (p - \ell)^2} \\
&\quad \times \text{Tr} \{ t^c t^a \} \text{Tr} \{ t^c t^a \} \text{Tr} \left\{ (\not{k} - \ell) \gamma^{\lambda_2} (\not{k}) \epsilon^{\mu \nu_1 \rho_2 \rho_3} [-(p - \ell)]_{\rho_2} \gamma_{\rho_3} \gamma_5 \right\} \\
&\quad \times \text{Tr} \left\{ (\not{q} - \ell) \gamma_{\lambda_2} (\not{q}) \epsilon^{\nu\nu_1\sigma_2\sigma_3} [p - \ell]_{\sigma_2} \gamma_{\sigma_3} \gamma_5 \right\}.
\end{aligned} \tag{3.10}$$

Each trace contains  $\gamma_5$ , and from [24] we have an identity (3.11) which will allow the trace to be calculated in D-dimensions

$$\text{Tr} \{ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 \} = 4i \epsilon^{\mu\nu\rho\sigma}, \tag{3.11}$$

so the traces reduce to momenta times antisymmetric tensors. Some terms, such as  $q_\rho q_\sigma \epsilon^{\mu\nu\rho\sigma}$ , are zero due to symmetric contraction on an antisymmetric tensor, and finally

$$\begin{aligned} \Pi_{(5)}^{\mu\nu}(p) &= g_s^4 \frac{4^2}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k-\ell)^2 k^2 (q-\ell)^2 q^2 \ell^2 (p-\ell)^2} \\ &\times \left[ \text{Tr} \{t^c t^a\} \right]^2 \left( [- (p-\ell)]_{\rho_2} [p-\ell]_{\sigma_2} \epsilon^{\mu \nu_1 \rho_2 \rho_3} k_\lambda \ell_\rho \epsilon^{\lambda \lambda_2 \rho} \rho_3 \epsilon^{\nu \nu_1 \sigma_2 \sigma_3} q_\sigma \ell_\eta \epsilon^\sigma \lambda_2 \eta \sigma_3 \right). \end{aligned} \quad (3.12)$$

The identities (2.23) and (2.24) can be used to expand the expression and calculate  $\Pi_1^{(5)}(p^2)$  in terms of our integrals where neither the colour traces or the algebra is trivially zero. The colour factor  $C_5$  is equal to

$$C_5 = [\text{Tr} \{t^c t^a\}]^2 = \frac{1}{2^4} [4\delta^{ca}\delta^{ca}] = \frac{1}{2^4} [4\delta^{aa}] = 2. \quad (3.13)$$

However  $\Pi_1^{(5)}(p^2) = 0$  when the resulting integrals are added together and simplified.

### 3.4 Gluon Self-Energy Diagram

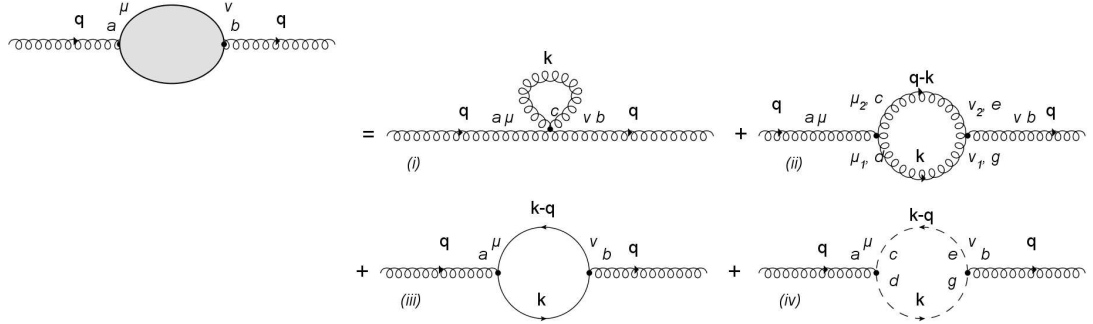


Figure 3.4: Gluon self-energy diagrams.

Diagram (4) as shown in Figure 3.5 contains the gluon self-energy (see Figure 3.4) which I will calculate in terms of integrals over loop momentum  $k$  in the massless case. I will then insert this expression as the Feynman rule for the gluon self-energy in diagram (4) as this will allow me to classify the results in terms of loop integrals in the same way as in previous diagrams. The gluon self-energy  $\Pi_{\mu\nu}(q)$  is the total

of the diagrams (i), (ii), (iii), and (iv). Diagram (i) is proportional to

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} = 0, \quad (3.14)$$

so this diagram is zero. Diagram (iii) has

$$\delta^{ab} \Pi_{(iii)}^{\mu\nu}(q) = \frac{-1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k} + m)}{k^2 - m^2} \right] i g_s \gamma^\mu t^a \left[ \frac{i(\not{k} - \not{q} + m)}{(k - q)^2 - m^2} \right] i g_s \gamma^\nu t^b \right\}. \quad (3.15)$$

However we want  $\Pi(q^2)$  in the massless case so I will calculate<sup>1</sup>

$$\Pi(q^2) = \frac{1}{q^2 (D - 1)} g_{\mu\nu} \Pi^{\mu\nu}(q). \quad (3.16)$$

Then, for the fermion loop, the expression we want is

$$\delta^{ab} \Pi_{(iii)}(q^2) = \frac{-3}{\nu^{6\epsilon}} \frac{4g_s^2}{q^2 (D - 1)} \text{Tr} \{ t^a t^b \} \int \frac{d^D k}{(2\pi)^D} \left[ \frac{-(D - 2)k^2 + (D - 2)k \cdot q}{k^2 (k - q)^2} \right], \quad (3.17)$$

where the factor 3 appears because there are three light quark flavours that can occur. For the ghost loop (iv), the Feynman rules give

$$\delta^{ab} \Pi_{(iv)}^{\mu\nu}(q) = \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \left[ \frac{i\delta^{gd}}{k^2} \right] \left[ -g_s f^{dac} k^\mu \right] \left[ \frac{i\delta^{ce}}{(k - q)^2} \right] \left[ -g_s f^{ebg} (k - q)^\nu \right] \quad (3.18)$$

$$\delta^{ab} \Pi_{(iv)}(q^2) = \frac{g_s^2}{\nu^{2\epsilon}} f^{dae} f^{ebd} \frac{1}{q^2 (D - 1)} \int \frac{d^D k}{(2\pi)^D} \frac{-k^2 + k \cdot q}{[k^2] [(k - q)^2]}, \quad (3.19)$$

---

<sup>1</sup>The gluon self-energy is transverse hence the projection  $\Pi(q^2)$  used in this section of the calculation.

and for the gluon loop (ii) the Feynman rules give

$$\begin{aligned}
\delta^{ab}\Pi_{(ii)}^{\mu\nu}(q) = & \frac{1}{2\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{-ig_{\nu_1\mu_1}\delta^{dg}}{k^2} \right] \left[ \frac{-ig_{\nu_2\mu_2}\delta^{ec}}{(k-q)^2} \right] \left[ g_s f^{adc} \left[ g^{\mu\mu_1} (q+k)^{\mu_2} \right. \right. \right. \\
& + g^{\mu_1\mu_2} (-2k+q)^\mu + g^{\mu_2\mu} (k-2q)^{\mu_1} \left. \left. \left. \right] \right] \left[ g_s f^{egb} \left[ g^{\nu_2\nu_1} (q-2k)^\nu + \right. \right. \right. \\
& + g^{\nu_1\nu} (k+q)^{\nu_2} + g^{\nu\nu_2} (k-2q)^{\nu_1} \left. \left. \left. \right] \right] \right\}
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\delta^{ab}\Pi_{(ii)}(q^2) = & \frac{1}{2\nu^{2\epsilon}} g_s^2 f^{agc} f^{cgb} \frac{1}{q^2 (D-1)} \int \frac{d^D k}{(2\pi)^D} \left[ \frac{(2k^2 + k \cdot q + 5q^2) D +}{[k^2]} \right. \\
& \left. + \frac{(4D-6)k^2 - (2D-3)(2k \cdot q) + (D-6)q^2}{[(k-q)^2]} \right].
\end{aligned} \tag{3.21}$$

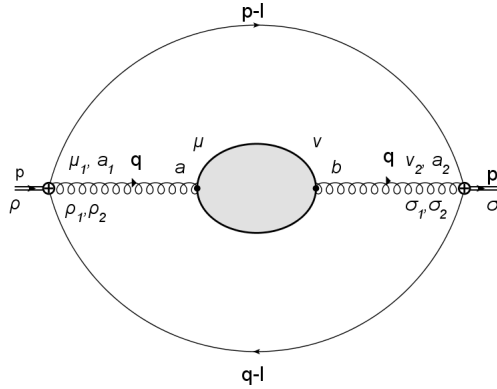


Figure 3.5: Next-to-leading order diagram containing the gluon self-energy.

Figure 3.5 contains the gluon self-energy and is equal to

$$\begin{aligned}
\Pi_{(4)}^{\rho\sigma}(p) &= \frac{-i}{\nu^{4\epsilon}} \int \frac{d^D\ell}{(2\pi)^D} \int \frac{d^Dq}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{p} - \not{\ell} + m)}{(\ell - p)^2 - m^2} \right] i g_s \epsilon^{\rho\mu_1\rho_1\rho_2} [-q_{\rho_1}] \gamma_{\rho_2} \gamma_5 t^{a_1} \right. \\
&\times \left[ \frac{i(\not{q} - \not{\ell} + m)}{(q - \ell)^2 - m^2} \right] i g_s \epsilon^{\sigma\nu_1\sigma_1\sigma_2} [q_{\sigma_1}] \gamma_{\sigma_2} \gamma_5 t^{a_2} \left[ -i \frac{g_{\mu_1\nu} \delta^{a_1 a}}{q^2} \right] \delta^{ab} \Pi^{\mu\nu}(q) \\
&\times \left. \left[ -i \frac{g_{\nu\nu_1} \delta^{b a_2}}{k^2} \right] \right\} \\
&= \frac{-i}{\nu^{4\epsilon}} g_s^2 \int \frac{d^D\ell}{(2\pi)^D} \int \frac{d^Dq}{(2\pi)^D} \text{Tr} \{ t^a t^b \} \text{Tr} \left\{ \left[ \frac{i(\not{p} - \not{\ell} + m)}{(\ell - p)^2 - m^2} \right] i \gamma_{\rho_2} \right. \\
&\times \left[ \frac{i(\not{q} - \not{\ell} - m)}{(q - \ell)^2 - m^2} \right] i \gamma_{\sigma_2} \left[ \frac{-i}{q^2} \right] \left[ \frac{-i}{k^2} \right] \\
&\times \left. \epsilon^{\rho\mu\rho_1\rho_2} [-q_{\rho_1}] \epsilon^{\sigma\nu\sigma_1\sigma_2} [q_{\sigma_1}] [q_\mu q_\nu - q^2 g_{\mu\nu}] \delta^{ab} \Pi(q^2) \right\}.
\end{aligned} \tag{3.22}$$

In the last line, the term proportional to  $q_\mu q_\nu$  is zero due to symmetry arguments. The colour identities used are  $f^{agc} f^{cgb} = -N \delta^{ab}$  and  $\text{Tr} \{ t^a t^b \} = \frac{1}{2^2} 2 \delta^{ab}$ . Including the colour factors, the final expression for the vector part is

$$\begin{aligned}
\Pi_1^{(4)}(p^2) &= -64 \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 (1 + \epsilon)^3 \epsilon^3 (1 + 2\epsilon) (2 + \epsilon) \\
&\times \frac{[\Gamma(\epsilon)]^4 \Gamma(2 + 3\epsilon) \Gamma(-1 - 3\epsilon)}{\Gamma(3 + 3\epsilon) \Gamma(6 + 4\epsilon)}.
\end{aligned} \tag{3.23}$$

### 3.5 Diagrams with Complicated Topology and Three-Gluon Vertex

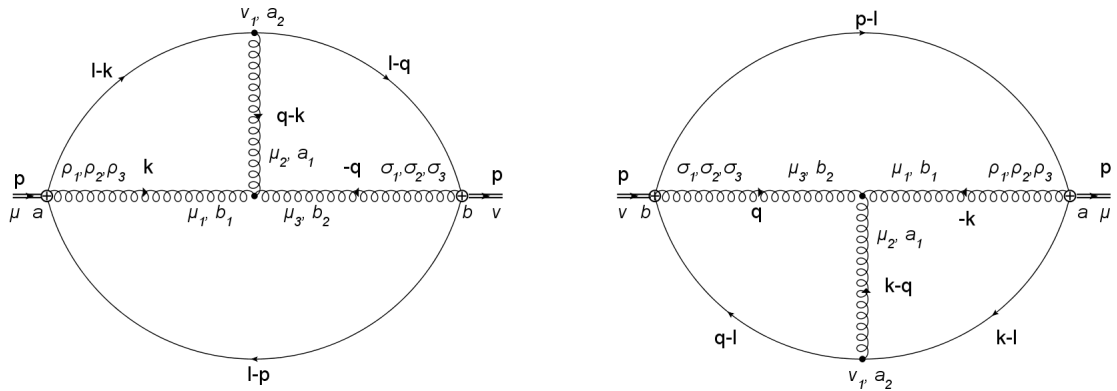


Figure 3.6: Diagrams with complicated topology and three-gluon vertex.



These diagrams shown in Figure 3.6 have the same topology as Figure 3.2, however there are complications that do not occur in that diagram, so I have not used the same momentum routing in Figure 3.6. Both the diagrams in Figure 3.6 have the same expression in the massless case, so we can add them together

$$\begin{aligned}
\Pi_{(2)}^{\mu\nu}(p) = & \frac{-2i}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{q} - \not{\ell} + m)}{(q - \ell)^2 - m^2} \right] i g_s \gamma^{\nu_1} t^{a_2} \right. \\
& \times \left[ \frac{i(\not{k} - \not{\ell} + m)}{(k - \ell)^2 - m^2} \right] i g_s \epsilon^{\mu\rho_1\rho_2\rho_3} [-k]_{\rho_2} \gamma_{\rho_3} \gamma_5 t^a \left[ \frac{i(\not{\ell} - \not{p} + m)}{(p - \ell)^2 - m^2} \right] \\
& \times i g_s \epsilon^{\nu\sigma_1\sigma_2\sigma_3} [-(q)]_{\sigma_2} \gamma_{\sigma_3} \gamma_5 t^b \left[ \frac{-i g_{\sigma_1\mu_3} \delta^{bb_2}}{(-q)^2} \right] \left[ \frac{-i g_{\nu_1\mu_2} \delta^{a_2 a_1}}{(q - k)^2} \right] \left[ \frac{-i g_{\rho_1\mu_1} \delta^{ab_1}}{k^2} \right] \\
& \times \left. \left[ g_s f^{b_1 a_1 b_2} (g^{\mu_1\mu_2} [2k - q]^{\mu_3} + g^{\mu_2\mu_3} [2q - k]^{\mu_1} + g^{\mu_3\mu_1} [-k - q]^{\mu_2}) \right] \right\}.
\end{aligned} \tag{3.24}$$

The vector part  $\Pi_1^{(2)}(p^2)$  for this diagram is

$$\begin{aligned}
\Pi_1^{(2)}(p^2) = & \frac{-2i}{\nu^{6\epsilon}} g_s^4 \text{Tr} \{ t^{a_1} t^{b_1} t^{b_2} f^{b_1 a_1 b_2} \} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{-i}{(q - k)^2} \right] \right. \\
& \times \left[ \frac{-i}{k^2} \right] \left[ \frac{i(\not{q} - \not{\ell} + m)}{(q - \ell)^2 - m^2} \right] i \gamma_{\mu_2} \left[ \frac{i(\not{k} - \not{\ell} + m)}{(k - \ell)^2 - m^2} \right] i [-k]_{\rho_2} \gamma_{\rho_3} \\
& \times \left[ \frac{i(\not{p} - \not{\ell} - m)}{(p - \ell)^2 - m^2} \right] i q_{\sigma_2} \gamma_{\sigma_3} \left[ \frac{-i}{q^2} \right] \frac{[p^2 g_{\mu\nu} - p_\mu p_\nu]}{(1 - D) p^2} \epsilon^{\mu \mu_1 \rho_2 \rho_3} \epsilon^{\nu \mu_3 \sigma_2 \sigma_3} \\
& \times \left. \left[ (g^{\mu_1\mu_2} [2k - q]^{\mu_3} + g^{\mu_2\mu_3} [2q - k]^{\mu_1} + g^{\mu_3\mu_1} [-k - q]^{\mu_2}) \right] \right\}.
\end{aligned} \tag{3.25}$$

There are two terms in  $\Pi_1^{(2)}(p^2)$  that will not have any contractions between indices in the antisymmetric tensors; consequently, the identities (2.23) and (2.24) can not be used to simplify these terms with the expression in its current form. The other terms can be simplified directly with REDUCE. The remaining two terms require several steps of simplification to get them into this form. For these two terms, instead of replacing the product of the antisymmetric tensors by its D-dimensional extension in REDUCE, I used REDUCE to calculate the trace with a constant in place of the one of the antisymmetric tensors. This produced a result with momenta

times antisymmetric tensors which could then be simplified by using the properties  $q_\mu q_\nu \epsilon^{\mu\nu\sigma\rho} = 0$  and  $\epsilon_\mu^{\mu\sigma\rho} = 0$ . The results were then in a form where the identities (2.23) and (2.24) could be used to convert the result to integrals in my classification scheme with the exception of one term which is given by (B.32). The colour factor  $C_2$  where I am relabelling  $a_1 = a$ ,  $b_1 = b$  and  $b_2 = c$  indices to simplify, is

$$C_2 = \text{Tr} \{ t^a t^b t^c f^{bac} \} = \frac{1}{2^3} f^{bac} 2 [d^{abc} + i f^{abc}] = -\frac{2}{2^3} \text{Ni} \delta^{aa} = -6i. \quad (3.26)$$

Finally assembling these three calculations gives

$$\begin{aligned} \Pi_1^{(2)}(p^2) = & \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 \left[ -\frac{4}{27} \frac{(1+\epsilon)\epsilon^2\pi\Gamma(5+3\epsilon)\Gamma(-4-3\epsilon)[\Gamma(\epsilon)]^3}{(7+4\epsilon)\Gamma(\frac{5}{3}+\epsilon)\Gamma(\frac{4}{3}+\epsilon)\Gamma(\frac{5}{2}+\epsilon)} \right. \\ & \left. \left\{ -4^{-\epsilon}\sqrt{\pi} \left( \frac{\Gamma(\frac{2}{3}+\epsilon)\Gamma(\frac{1}{3}+\epsilon)\Gamma(1-\epsilon)}{\Gamma(\frac{1}{2}+\epsilon)\Gamma(\frac{1}{2}-\epsilon)} \right) \left( 5184\epsilon^8 + 467793\epsilon^4 + \right. \right. \right. \\ & \left. \left. \left. + 187776\epsilon^6 + 265419\epsilon^3 + 37908\epsilon^7 + 8712\epsilon + 424989\epsilon^5 + 75231\epsilon^2 + 108 \right) \right. \right. \\ & \left. \left. + 27^{-\epsilon}\sqrt{3}\Gamma\left(\frac{1}{2}+\epsilon\right) \left( 72 + 227600\epsilon^5 + 241248\epsilon^4 + 3252\epsilon + 127140\epsilon^3 \right. \right. \right. \\ & \left. \left. \left. + 98848\epsilon^6 + 17152\epsilon^7 + 2048\epsilon^8 + 32250\epsilon^2 \right) \right\} - 384I(p^2) \right], \end{aligned} \quad (3.27)$$

where  $I(p^2)$  is given by (B.32) in Appendix B.

### 3.6 Quark Self-Energy Diagrams

Both diagrams in Figure 3.7 have the same expression which is

$$\begin{aligned} \Pi_{(3)}^{\mu\nu}(p) = & \frac{-2i}{\nu^{6\epsilon}} \int \frac{d^D\ell}{(2\pi)^D} \int \frac{d^Dq}{(2\pi)^D} \int \frac{d^Dk}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k}+m)}{k^2-m^2} \right] i g_s \gamma^{\mu_1} t^a \left[ \frac{i(\not{q}+m)}{q^2-m^2} \right] \right. \\ & \times i g_s \gamma^{\mu_2} t^b \left[ \frac{i(\not{k}+m)}{k^2-m^2} \right] i g_s \epsilon^{\mu\rho\rho_1\rho_2} [\ell-p]_{\rho_1} \gamma_{\rho_2} \gamma_5 t^c \left[ \frac{i(\not{k}-\not{\ell}+m)}{(k-\ell)^2-m^2} \right] \\ & \left. \times i g_s \epsilon^{\nu\sigma\sigma_1\sigma_2} [-(\ell-p)]_{\sigma_1} \gamma_{\sigma_2} \gamma_5 t^d \left[ \frac{-i g_{\mu_2\mu_1} \delta^{ba}}{(k-q)^2} \right] \left[ \frac{-i g_{\sigma\rho} \delta^{dc}}{(\ell-p)^2} \right] \right\}. \end{aligned} \quad (3.28)$$

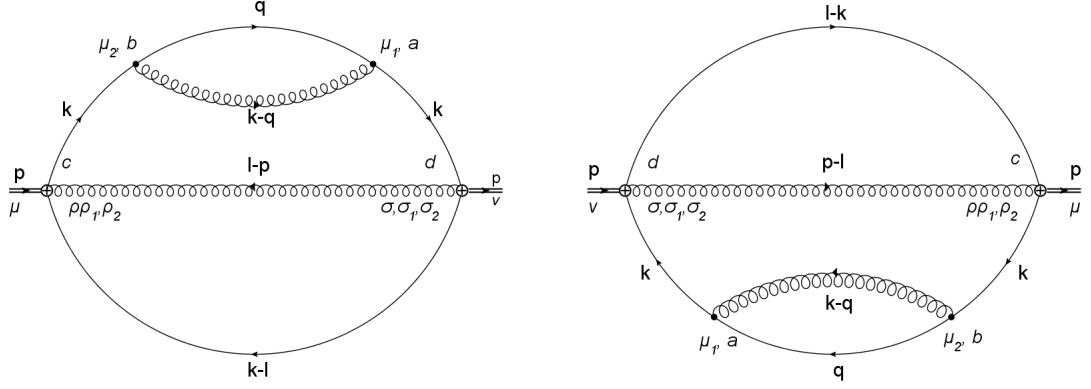


Figure 3.7: Quark self-energy diagrams.

Calculating  $\Pi_1^{(3)}(p^2)$  is then straightforward as it can be calculated from integrals directly as there are no new integrals introduced by this diagram. The colour factor  $C_3$  is

$$\begin{aligned}
C_3 &= \text{Tr} \{t^a t^a t^b t^b\} \\
&= \frac{1}{2^4} \left[ \frac{4}{N} (\delta^{aa} \delta^{bb} - \delta^{ab} \delta^{ab} + \delta^{ab} \delta^{ba}) + 2 (d^{aabc} d^{bbc} - d^{abc} d^{bac} + d^{abc} d^{abc}) + \right. \\
&\quad \left. + 2i (d^{aac} f^{bbc} - d^{abc} f^{bac} + d^{abc} f^{abc}) \right] \\
&= \frac{1}{2^4} \left[ \frac{4}{N} (\delta^{aa})^2 \right] = \frac{16}{3},
\end{aligned} \tag{3.29}$$

and the final unexpanded amplitude is

$$\begin{aligned}
\Pi_1^{(3)}(p^2) &= \frac{128}{9} \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 (11\epsilon + 12) (1 + 2\epsilon) \\
&\quad \times \epsilon^3 (1 + \epsilon)^3 \frac{[\Gamma(\epsilon)]^4 \Gamma(2 + 3\epsilon) \Gamma(-1 - 3\epsilon)}{\Gamma(3 + 3\epsilon) \Gamma(6 + 4\epsilon)}.
\end{aligned} \tag{3.30}$$

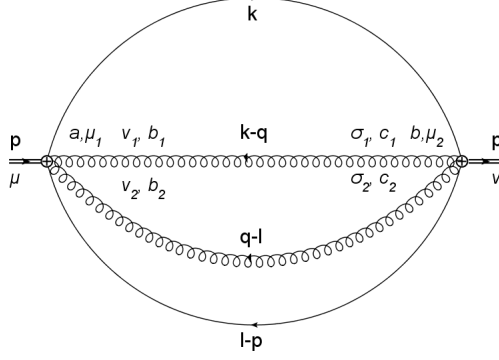


Figure 3.8: Diagram with vertices with a quark, an antiquark and two gluons.

### 3.7 Diagram with the Two-Gluon Vertex

Figure 3.8 has vertices with the two-gluon Feynman rule given in Figure 2.4, so the amplitude for this diagram is

$$\begin{aligned}
\Pi_{(6)}^{\mu\nu}(p) &= \frac{-i}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k} + m)}{k^2 - m^2} \right] g_s^2 \gamma_{\mu_1} \gamma_5 t^a \epsilon^{\mu\mu_1\nu_1\nu_2} f^{ab_1b_2} \right. \\
&\times \left. \left[ \frac{i(\not{\ell} - \not{p} + m)}{(p - \ell)^2 - m^2} \right] g_s^2 \gamma_{\mu_2} \gamma_5 t^b \epsilon^{\nu\mu_2\sigma_1\sigma_2} f^{ac_1c_2} \right\} \left[ \frac{-ig_{\nu_1\sigma_1} \delta^{b_1c_1}}{(k - q)^2} \right] \left[ \frac{-ig_{\nu_2\sigma_2} \delta^{b_2c_2}}{(q - \ell)^2} \right] \\
&= -\frac{ig_s^4}{\nu^{6\epsilon}} \text{Tr} \left\{ t^a f^{ac_1c_2} t^b f^{ac_1c_2} \right\} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k} + m)}{k^2 - m^2} \right] \gamma_{\mu_1} \right. \\
&\times \left. \left[ \frac{i(\not{\ell} - \not{p} - m)}{(p - \ell)^2 - m^2} \right] \gamma_{\mu_2} \left[ \frac{-i}{(k - q)^2} \right] \left[ \frac{-i}{(q - \ell)^2} \right] \epsilon^{\mu\mu_1}{}_{\sigma_1\sigma_2} \epsilon^{\nu\mu_2\sigma_1\sigma_2} \right\}.
\end{aligned} \tag{3.31}$$

Note there are contractions between two sets of indices in the antisymmetric tensor here, so the term in  $\Pi_1^{(6)}(p^2)$  that is proportional to  $g_{\mu\nu}$  will have three sets of contracted indices. Calculating  $g_{\mu\nu}$  times (2.24) gives

$$g_{\mu\nu} \epsilon^{\mu\mu_1}{}_{\sigma_1\sigma_2} \epsilon^{\nu\mu_2\sigma_1\sigma_2} = -(D - 3)(D - 2)(D - 1) g^{\mu_1\mu_2}, \tag{3.32}$$

which will replace the antisymmetric tensors in the trace for this term. The colour factor  $C_6$  for (3.31) is

$$C_6 = \text{Tr} \{ t^a t^b f^{ac_1c_2} f^{bc_1c_2} \} = \text{Tr} \{ t^a t^b N \delta^{ab} \} = \frac{N}{2^2} 2\delta^{aa} = 12. \tag{3.33}$$

The result  $\Pi_1^{(6)}(p^2)$  for this diagram is

$$\Pi_1^{(6)}(p^2) = 384 \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 (1 + 2\epsilon) \frac{[\Gamma(2 + \epsilon)]^4 \Gamma(3 + 3\epsilon) \Gamma(-2 - 3\epsilon)}{\Gamma(4 + 3\epsilon) \Gamma(5 + 4\epsilon)}. \quad (3.34)$$

### 3.8 Four Diagrams with the Same Result

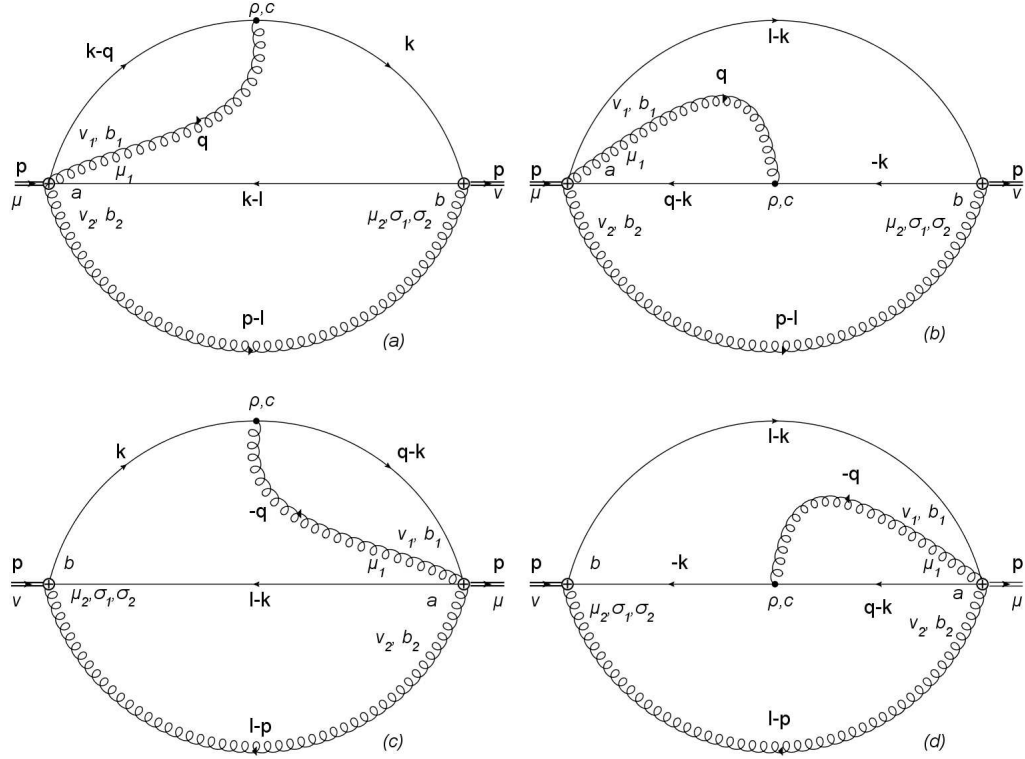


Figure 3.9: Diagrams with one quark, an antiquark and two-gluon vertex.

The four diagrams in Figure 3.9 should have the same  $\Pi_{(7)}^{\mu\nu}(p)$  in the massless case, since topologically they mirror each other. Diagrams (a) and (d) in Figure 3.9 have the expression

$$\begin{aligned} \Pi_{(7a)}^{\mu\nu}(p) &= \frac{-i}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k} + m)}{k^2 - m^2} \right] i g_s \gamma^{\rho t c} \right. \\ &\quad \times \left[ \frac{i(\not{k} - \not{q} + m)}{(k - q)^2 - m^2} \right] \left[ g_s^2 \gamma_{\mu_1} \gamma_5 t^a \epsilon^{\mu\mu_1\nu_1\nu_2} f^{ab_1 b_2} \right] \left[ \frac{i(\not{k} - \not{\ell} + m)}{(k - \ell)^2 - m^2} \right] \\ &\quad \left. \times i g_s [(p - \ell)_{\sigma_1}] \gamma_{\sigma_2} \gamma_5 t^b \epsilon^{\nu\mu_2\sigma_1\sigma_2} \left[ \frac{-i g_{\nu_2 \mu_2} \delta^{b_2 b}}{(p - \ell)^2} \right] \left[ \frac{-i g_{\nu_1 \rho} \delta^{b_1 c}}{q^2} \right] \right\}, \end{aligned} \quad (3.35)$$

and diagrams (b) and (c) have the expression

$$\begin{aligned}
\Pi_{(7b)}^{\mu\nu}(p) &= \frac{-i}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\not{k} - \not{q} + m)}{(k-q)^2 - m^2} \right] i g_s \gamma^\rho t^c \right. \\
&\times \left[ \frac{i(-\not{k} + m)}{k^2 - m^2} \right] i g_s [(p-\ell)_{\sigma_1}] \gamma_{\sigma_2} \gamma_5 t^b \epsilon^{\nu\mu_2\sigma_1\sigma_2} \left[ \frac{i(\not{\ell} - \not{k} + m)}{(k-\ell)^2 - m^2} \right] \\
&\times \left[ g_s^2 \gamma_{\mu_1} \gamma_5 t^a \epsilon^{\mu\mu_1\nu_1\nu_2} f^{ab_1b_2} \right] \left[ \frac{-i g_{\nu_2\mu_2} \delta^{b_2b}}{(p-\ell)^2} \right] \left. \left[ \frac{-i g_{\nu_1\rho} \delta^{b_1c}}{q^2} \right] \right\}. \tag{3.36}
\end{aligned}$$

Eqs. (3.35) and (3.36) are very similar and their colour factors have the relation  $C_{7a} = -C_{7b}$ . Then using REDUCE to calculate the amplitudes divided by the respective colour factors  $\Pi_1^{(7a)}(p^2)/C_{7a}$  and  $\Pi_1^{(7b)}(p^2)/C_{7b}$ , in the chiral limit, these expressions have the relation

$$\frac{\Pi_1^{(7a)}(p^2)}{C_{7a}} = -\frac{\Pi_1^{(7b)}(p^2)}{C_{7b}} \tag{3.37}$$

and hence  $\Pi_1^{(7)}(p^2) = 4\Pi_1^{(7a)}(p^2)$ . The colour factor is the same as in diagram (2) (see Figure 3.6) so  $C_{7a} = C_2 = -6i$ . The final expression is

$$\begin{aligned}
\Pi_1^{(7)}(p^2) &= 32 \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 \epsilon^3 (1+\epsilon)^3 (8\epsilon+9)(2\epsilon+1) \\
&\times \frac{[\Gamma(\epsilon)]^4 \Gamma(2+3\epsilon) \Gamma(-1-3\epsilon)}{\Gamma(3+3\epsilon) \Gamma(6+4\epsilon)}. \tag{3.38}
\end{aligned}$$

### 3.9 Diagrams with One Quark, Antiquark and Two-Gluon Vertex and a Three-Gluon Vertex

Both diagrams in Figure 3.10 have the same expression deriving from the Feynman rules, doubling the amplitude. Also there is a symmetry factor of  $\frac{1}{2}$ , since the gluon lines connecting the current and the three gluon vertex are interchangeable. For

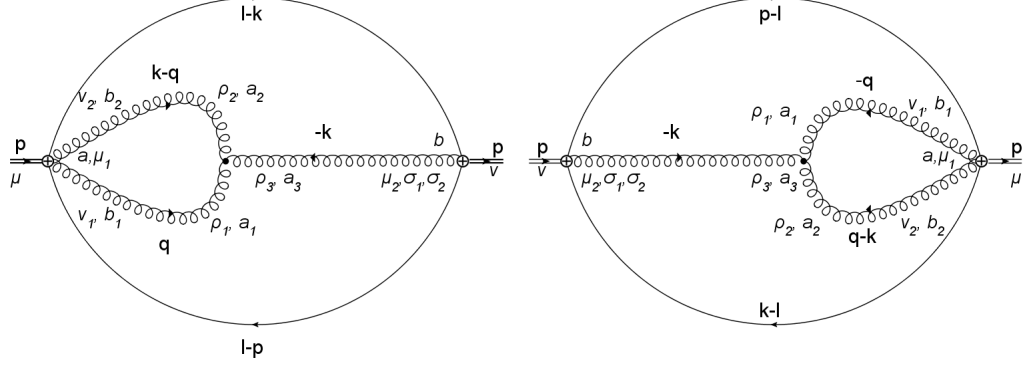


Figure 3.10: Diagrams With one quark, antiquark and two-gluon vertex and a three-gluon vertex.

diagram 8,  $\Pi_{(8)}^{\mu\nu}(p)$  is

$$\begin{aligned}
\Pi_{(8)}^{\mu\nu}(p) &= \frac{-2i}{\nu^{6\epsilon}} \frac{1}{2} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \text{Tr} \left\{ \left[ \frac{i(\ell - \not{k} + m)}{(k - \ell)^2 - m^2} \right] \left[ g_s^2 \gamma_{\mu_1} \gamma_5 \epsilon^{\mu\mu_1\nu_1\nu_2} \right. \right. \\
&\times \left. \left. t^a f^{ab_1b_2} \right] \left[ \frac{i(\ell - \not{p} + m)}{(p - \ell)^2 - m^2} \right] i g_s [ -(-k)_{\sigma_1} ] \gamma_{\sigma_2} \gamma_5 t^b \epsilon^{\nu\mu_2\sigma_1\sigma_2} \left[ \frac{-i g_{\nu_1\rho_1} \delta^{b_1a_1}}{q^2} \right] \right. \\
&\times \left. \left[ \frac{-i g_{\nu_2\rho_2} \delta^{b_2a_2}}{(k - q)^2} \right] \left[ \frac{-i g_{\mu_2\rho_3} \delta^{ba_3}}{(-k)^2} \right] \left[ g_s f^{a_1a_2a_3} (g^{\rho_1\rho_2} [2q - k]^{\rho_3} + g^{\rho_2\rho_3} [2k - q]^{\rho_1} + \right. \right. \\
&\left. \left. + g^{\rho_3\rho_1} [-k - q]^{\rho_2}) \right] \right\} \\
&= \frac{i g_s^4}{\nu^{6\epsilon}} \text{Tr} \left\{ t^a f^{aa_1a_2} t^{a_3} f^{a_1a_2a_3} \right\} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \\
&\times \text{Tr} \left\{ \left[ \frac{i(\ell - \not{k} + m)}{(k - \ell)^2 - m^2} \right] \gamma_{\mu_1} \left[ \frac{i(\ell - \not{p} + m)}{(p - \ell)^2 - m^2} \right] i \gamma_{\sigma_2} \left[ \frac{-i}{q^2} \right] \right. \\
&\times \left. \left[ \frac{-i}{(k - q)^2} \right] \left[ \frac{-i}{k^2} \right] (k)_{\sigma_1} (3k)_{\eta} \epsilon^{\mu\lambda\mu_1\eta} \epsilon^{\nu\lambda\sigma_1\sigma_2} \right\}.
\end{aligned} \tag{3.39}$$

In the last line some of the indices have been relabelled in order to simplify the expression. The colour factor is  $C_8 = \text{Tr} \left\{ t^a f^{aa_1a_2} t^{a_3} f^{a_1a_2a_3} \right\} = -C_6 = -12$ . Then the final expression is

$$\begin{aligned}
\Pi_1^{(8)}(p^2) &= -192 \left( \frac{-\alpha_s^2}{[4\pi]^4} \right) \left( \frac{-p^2}{4\pi\nu^2} \right)^{3\epsilon} p^6 (1 + \epsilon)^3 \epsilon^3 (4\epsilon^2 + 8\epsilon + 3) \\
&\times \frac{[\Gamma(\epsilon)]^4 \Gamma(2 + 3\epsilon) \Gamma(-1 - 3\epsilon)}{\Gamma(3 + 3\epsilon) \Gamma(6 + 4\epsilon)}.
\end{aligned} \tag{3.40}$$

# CHAPTER 4

## RESULTS AND CONCLUSION

In order to isolate the divergent quantities,  $\Pi_1(p^2)$  is series expanded in  $\epsilon$ . Table 4.1 has the expressions for each diagram as a series in  $\epsilon$ . Each diagram has an expansion in the form

$$\begin{aligned} \Pi_1^{(n)}(p^2) = & -\frac{\alpha_s^2 p^6}{(4\pi)^4} \left\{ \frac{a_1}{\epsilon^2} + \frac{1}{\epsilon} \left[ a_2 + b_1 \log\left(\frac{-p^2}{\nu^2}\right) \right] + a_3 + b_2 \log\left(\frac{-p^2}{\nu^2}\right) \right. \\ & \left. + b_3 \log^2\left(\frac{-p^2}{\nu^2}\right) \right\}, \end{aligned} \quad (4.1)$$

where  $n$  is diagram number and  $a_1, a_2, a_3, b_1, b_2,$  and  $b_3$  are constants. Table 4.1 contains the numeric values of the specific values of  $n, a_1$  etc. appearing in (4.1) for the individual diagrams.

The total correlation function for our  $1^{--}$  hybrid meson is the total for all of the diagrams

$$\Pi_1^{(n.l.)}(p^2) = \sum_{n=1}^8 \Pi_1^{(n)}(p^2). \quad (4.2)$$

The dimensionally regularized series expansion of  $\Pi_1^{(n.l.)}(p^2)$  about  $\epsilon = 0$  for the second order correction is

$$\begin{aligned} \Pi_1^{(n.l.)}(p^2) = & \frac{\alpha_s^2}{\pi^4} \left[ \left( -\frac{83}{46080} \pi^2 - \frac{6959}{38400} L + \frac{1}{2880} \zeta(3) + \frac{83}{2560} L^2 + \frac{3708623}{10368000} \right) p^6 \right. \\ & \left. + \left( -\frac{6959}{115200} + \frac{83}{3840} L \right) \frac{p^6}{\epsilon} + \frac{83}{11520} \frac{p^6}{\epsilon^2} \right] + O(\epsilon), \end{aligned} \quad (4.3)$$

where  $L = \log\left(\frac{-p^2}{\nu^2}\right)$ . This result (as well as those above) has been converted to the modified minimal subtraction ( $\overline{\text{MS}}$ ) scheme whereby substituting  $\nu^2 \rightarrow \nu^2 e^{\gamma_E}/4\pi$ ,



$n$	$a_1$	$a_2$	$b_1$	$a_3$	$b_2$	$b_3$
1	$\frac{4}{135}$	$-\frac{553}{2025}$	$\frac{4}{45}$	$-\frac{1}{135}\pi^2 - \frac{113647}{60750} - \frac{16}{45}\zeta(3)$	$-\frac{553}{675}$	$\frac{2}{15}$
2	$-\frac{23}{15}$	$\frac{18541}{1350}$	$-\frac{23}{5}$	$\frac{23}{60}\pi^2 - \frac{1126153}{13500} + \frac{4}{15}\zeta(3)$	$\frac{18541}{450}$	$-\frac{69}{10}$
3	$\frac{32}{135}$	$-\frac{3704}{2025}$	$\frac{32}{45}$	$-\frac{8}{135}\pi^2 + \frac{310948}{30375}$	$-\frac{3704}{675}$	$\frac{16}{15}$
4	$-\frac{8}{45}$	$\frac{976}{675}$	$-\frac{8}{15}$	$\frac{2}{45}\pi^2 - \frac{84212}{10125}$	$\frac{976}{225}$	$-\frac{4}{5}$
5	0	0	0	0	0	0
6	0	$-\frac{8}{9}$	0	$\frac{188}{27}$	$-\frac{8}{3}$	0
7	$\frac{2}{5}$	$-\frac{697}{225}$	$\frac{6}{5}$	$-\frac{1}{10}\pi^2 + \frac{117253}{6750}$	$-\frac{697}{75}$	$\frac{9}{5}$
8	$-\frac{4}{5}$	$\frac{478}{75}$	$-\frac{12}{15}$	$\frac{1}{5}\pi^2 - \frac{40811}{1125}$	$\frac{478}{25}$	$-\frac{18}{5}$

Table 4.1: Numeric values for constants as defined in (4.1).

$\gamma_E$  and  $\log(4\pi)$  are eliminated. The polynomial divergences (appearing with  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon^2}$ ) will be eliminated via derivatives in the dispersion relation (2.12), but the divergent  $\frac{1}{\epsilon}L$  term remains and can only be removed by renormalization. Therefore the renormalization properties of the current need to be determined. Once these terms are dealt with the remaining terms are the  $\alpha_s^2$  perturbative correction. In the limit  $\epsilon \rightarrow 0$  the  $\mathcal{O}(\epsilon)$  term in (4.3) goes to zero, however it cannot be ignored in (2.25) because  $\epsilon \frac{1}{\epsilon} \rightarrow 1$ . The first step in renormalization is the replacement of the bare coupling  $\alpha_s = \alpha_b$  appearing in (4.3) and (2.25), with the renormalized coupling constant  $\alpha_r$  related to  $\alpha_b$  by [20]

$$\frac{\alpha_b}{\pi} = \frac{\alpha_r}{\pi} \left( 1 + \frac{9}{4} \frac{\alpha_r}{\pi} \frac{1}{\epsilon} + \dots \right). \quad (4.4)$$

Since I am working to second-order in  $\alpha_r$  this simply replaces  $\alpha_b \rightarrow \alpha_r$  in (4.3), but generates second order terms through a modification of the first-order result (2.25).

Renormalizing the coupling constant to order  $\alpha_r^2$  modifies the first order term as

$$\Pi_1^{(l.o.)}(p^2) = \frac{1}{\pi^2} \frac{\alpha_r}{\pi} \left( 1 + \frac{9}{4} \frac{\alpha_r}{\pi} \frac{1}{\epsilon} \right) \left[ -\frac{1}{480\epsilon} p^6 - \frac{(40L - 77)}{9600} p^6 - \frac{1}{576000} p^6 (14997 + \right. \\ \left. -9240L - 200\pi^2 + 2400L^2) \epsilon \right]. \quad (4.5)$$

Although this renormalization does alter the  $\frac{1}{\epsilon}L$  term, it is not enough to remove the divergent  $\frac{1}{\epsilon}L$  term from (4.3), the correlation function also needs to be renormalized. Although the current's renormalization properties have not been determined in this work we can still explore how they might affect the results. The exotic hybrid current renormalizes multiplicatively [26], so if we assume the renormalization of the non-exotic current is also multiplicative  $J_\mu^R = ZJ_\mu^b$  it would also renormalize the correlation function by

$$\Pi_R = Z^2 \Pi_b = \left( 1 + 2E \frac{\alpha_r}{\pi} \frac{1}{\epsilon} \right) \Pi_b, \quad (4.6)$$

where  $E$  is some renormalization constant that could be determined by assessing the renormalization properties of the current. The expected value of  $E$  can be determined by assuming that the divergent term  $\frac{1}{\epsilon}L$  is removed by this renormalization. The necessary value for the constant is  $E = \frac{47}{32}$  and

$$\Pi_1(p^2) = -\frac{1}{240\pi^2} \frac{\alpha_r}{\pi} \left[ p^6 \left( L - \frac{77}{40} + \frac{1}{2\epsilon} \right) + \frac{\alpha_r}{\pi} p^6 \left( \frac{7527}{320} L - \frac{1}{12} \zeta(3) - \frac{738649}{13824} \right. \right. \\ \left. \left. - \frac{83}{32} L^2 + \frac{8663}{1920\epsilon} + \frac{83}{96\epsilon^2} \right) \right]. \quad (4.7)$$

The polynomial terms in  $p^2$  are removed by derivatives of the dispersion relation and the resulting expression for the correlation function of hybrid meson with  $1^{--}$  in the chiral limit is

$$\Pi_1(p^2) = -\frac{1}{240\pi^2} \frac{\alpha_r}{\pi} \left[ p^6 L + \frac{\alpha_r}{\pi} p^6 \left( \frac{7527}{320} L - \frac{83}{32} L^2 \right) \right]. \quad (4.8)$$

The coefficient of  $L$  at next-to-leading order  $\alpha^2$  is  $\approx 2.3$  times that of the leading order  $\alpha$  coefficient, so on the scale where  $\frac{\alpha}{\pi} \approx 0.1$  the next-to-leading order perturbative correction should be an important contribution.

The result (4.8) would describe a hybrid meson with light quark content and an improved mass for this state could be calculated from sum-rules using this correlation function. Because the second-order  $\alpha^2$  term is substantial compared with the first-order term, the  $1^{--}$  non-strange hybrid mass prediction of Ref. [5] could be altered significantly. However a definitive study of the renormalization of the hybrid current would be a necessary step, and is beyond the scope of this thesis.

The techniques developed in this thesis will provide the foundation to study hybrids with strange quarks, which requires order  $m^2$  effects to the perturbative expansion. An improved determination of the strangeonium hybrid mass could then be obtained and compared with the  $Y(2175)$  particle.

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# APPENDIX A

## CONVENTIONS

The conventions used in this thesis follow those defined in [6].

### A.1 Units

Convenient units are  $\hbar = c = 1$ . Then energy and momentum are in mass units eV or more usually MeV.

### A.2 Dirac and Colour Algebra

The relevant metric of Minkowski spacetime is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.1})$$

The Dirac matrices  $\gamma^\mu$  have the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (\text{A.2})$$

$$[\gamma^\mu, \gamma^5] = 0 \quad (\text{A.3})$$

$$(\gamma^5)^2 = 1, \quad (\text{A.4})$$

and in  $D$  dimensions

$$g_\mu^\mu = D. \quad (\text{A.5})$$

The  $SU(N)$  colour algebra is defined in [20] where the  $t^a = \frac{\lambda^a}{2}$  are defined in terms of the Gell-Mann matrices with the following properties

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c \quad (\text{A.6})$$

$$\{\lambda_a, \lambda_b\} = \frac{4}{N}\delta_{ab}1 + 2d_{abc}\lambda_c \quad (\text{A.7})$$

$$f_{abc}f_{dbc} = N\delta_{ad} \quad (\text{A.8})$$

$$d_{abc}d_{dbc} = \left(N - \frac{N}{4}\right)\delta_{ad} \quad (\text{A.9})$$

$$\text{Tr}\{\lambda_a\lambda_b\} = 2\delta_{ad} \quad (\text{A.10})$$

$$\text{Tr}\{\lambda_a\lambda_b\lambda_c\} = 2(d_{abc} + if_{abc}) \quad (\text{A.11})$$

$$\begin{aligned} \text{Tr}\{\lambda_a\lambda_b\lambda_c\lambda_d\} &= \frac{4}{N}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \\ &+ 2(d_{abr}d_{cdr} - d_{acr}d_{dbr} + d_{adr}d_{bcr}) \\ &+ 2i(d_{abr}f_{cdr} - d_{acr}f_{dbr} + d_{adr}f_{bcr}). \end{aligned} \quad (\text{A.12})$$

### A.3 Feynman Rules for QCD

The QCD Feynman rules as defined in [6] are shown in Table A.1. Four-Momentum is implicitly conserved at every vertex (*e.g.*  $p + k + q = 0$  in the three-gluon vertex).

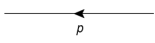
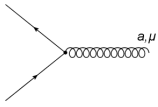
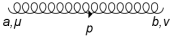
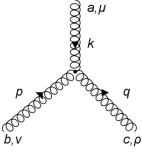
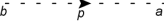
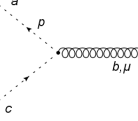
Fermion Propagator	Fermion Vertex	Gluon Propagator
$\frac{i}{p^2 - m^2 + i\epsilon}$ 	$ig\gamma^\mu t^a$ 	$\frac{-ig^{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$ 
Three Gluon Vertex	Ghost Propagator	Ghost Vertex
$g_s f^{abc} [g^{\mu\nu} (k - p)^\rho + g^{\nu\rho} (p - q)^\mu + g^{\rho\mu} (q - k)^\nu]$ 	$\frac{i\delta^{ab}}{p^2 + i\epsilon}$ 	$-gf^{abc}p^\mu$ 

Table A.1: Relevant Feynman rules for QCD. The gluon propagator is given in the Feynman gauge  $\xi = 1$

# APPENDIX B

## FEYNMAN INTEGRALS

### B.1 Relevant Integrals

vector		tensor		more complicated	
$v_1$	$k \cdot q$	$t_1$	$k \cdot pq \cdot p$	$T_1$	$(q \cdot p)^2 k \cdot p$
$v_2$	$q \cdot p$	$t_2$	$(q \cdot p)^2$	$T_2$	$(q \cdot p)^3$
$v_3$	$k \cdot p$	$t_3$	$(k \cdot p)^2$	$T_3$	$(q \cdot p)^2 k \cdot \ell$
$v_4$	$k \cdot \ell$	$t_4$	$(k \cdot \ell)^2$	$T_4$	$(\ell \cdot p)^2 k \cdot \ell$
$v_5$	$q \cdot \ell$	$t_5$	$k \cdot \ell k \cdot p$	$T_5$	$(\ell \cdot p)^3 k \cdot \ell$
$v_6$	$\ell \cdot p$	$t_6$	$(\ell \cdot p)^2$	$T_6$	$(\ell \cdot p)^3$
$t_7$	$q \cdot \ell p \cdot \ell$				
		$t_8$	$q \cdot pk \cdot \ell$		
		$t_9$	$q \cdot pq \cdot \ell$		
		$t_{10}$	$k \cdot pq \cdot \ell$		
		$t_{11}$	$k \cdot \ell q \cdot \ell$		
$t_{12}$	$k \cdot \ell p \cdot \ell$				
		$t_{13}$	$k \cdot pp \cdot \ell$		

Table B.1: Vector and tensor structure of classified integrals.

The three types of integrals have basic scalar, vector, and tensor forms. These integrals are listed in Table B.1. I am using an exponent,  $v_n$  for vector integrals,  $t_n$  for rank two tensor integrals and  $T_n$  for more complicated tensor integrals, to label and differentiate between the different integral forms in my notation. For example,  $t_1$  indicates a tensor integral containing  $k \cdot pq \cdot p$ , or in other words, an integral containing a tensor  $k^\mu q^\nu$ . However, the solution of this integral  $t_1$  is different from  $t_8$  with  $k \cdot \ell q \cdot p$  which also contains an integral over  $k^\mu q^\nu$ . Note that many of the integrals share components and that for  $n_2$  integrals  $n_2^{v_1}(b, c, a) = n_2^{v_2}(c, b, a)$  and for  $n_3$  type integrals  $n_3^{v_1}(a) = n_3^{v_2}(a)$ . Also some of the integrals falsely appear to be tensors at first glance. For example,  $n_1^{t_{12}}$ , is really a vector integral due to the integration order.

### B.2 Basic Integrals

#### B.2.1 Scalar Results

Type  $n_1$  integrals can be calculated using the basic one loop results iteratively. I first calculated the  $d^D k$  integral followed by the  $d^D q$  and  $d^D \ell$  integrals. In (B.1), the



scalar case, each loop integral has the scalar form of (2.21), with the final result

$$\begin{aligned}
n_1(b, c, a) &= \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q]^{2c} [q - \ell]^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k]^{2b} [k - q]^2} \\
&= \left[ \frac{-i}{(4\pi)^4} \right] \left[ \frac{-1}{4\pi\nu^2} \right]^{3\epsilon} \frac{1}{p^{2(b+c-a-3-3\epsilon)}} \frac{[\Gamma(1+\epsilon)]^3 \Gamma(2-b+\epsilon) \Gamma(b-1-\epsilon)}{\Gamma(b) \Gamma(3-b+2\epsilon) \Gamma(c+b-1-\epsilon)} \\
&\times \frac{\Gamma(3-b-c+2\epsilon) \Gamma(b+c-2-2\epsilon) \Gamma(4-b-c+a+3\epsilon)}{\Gamma(4-b-c+3\epsilon) \Gamma(c+b-a-2-2\epsilon)} \\
&\times \frac{\Gamma(b+c-a-3-3\epsilon)}{\Gamma(5-b-c+a+4\epsilon)}.
\end{aligned} \tag{B.1}$$

The  $n_2$  type integrals can also be done iteratively like  $n_1$ , however the  $d^D k$  integral does not depend on  $q$  so the order of integration doesn't matter and of course the  $d^D \ell$  integral is done last, with the scalar result

$$\begin{aligned}
n_2(b, c, a) &= \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q]^{2c} [q - \ell]^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k]^{2b} [k - \ell]^2} \\
&= \left[ \frac{-i}{(4\pi)^6} \right] \left[ \frac{-1}{4\pi\nu^2} \right]^{3\epsilon} \frac{1}{p^{2(b+c-a-3-3\epsilon)}} \frac{[\Gamma(1+\epsilon)]^3 \Gamma(2-b+\epsilon) \Gamma(b-1-\epsilon)}{\Gamma(b) \Gamma(3-b+2\epsilon) \Gamma(c)} \\
&\times \frac{\Gamma(2-c-\epsilon) \Gamma(c-1-\epsilon) \Gamma(4-b-c+a+3\epsilon)}{\Gamma(3-c+2\epsilon) \Gamma(c+b-a-2-2\epsilon)} \\
&\times \frac{\Gamma(b+c-a-3-3\epsilon)}{\Gamma(5-b-c+a+4\epsilon)}.
\end{aligned} \tag{B.2}$$

The  $d^D k$  and  $d^D q$  integrals in  $n_3$  cannot be done iteratively; however they can be calculated via a recursion relation, and then the integral over  $d^D \ell$  can be calculated to get the final result. Following [25] we can determine the recursion relation by using

$$0 = \int d^D q \int d^D k \left( \frac{\partial}{\partial k^\mu} \right) \left[ (k_\mu - q_\mu) \frac{1}{k^2 q^2 (k - q)^2 (k - \ell)^2 (q - \ell)^2} \right]. \tag{B.3}$$

In order to simplify further calculations let  $(k - q)^2 = z^2$ ,  $(k - \ell)^2 = x^2$  and  $(q - \ell)^2 = y^2$ . Then the dot products between the loop momenta can be written as  $k \cdot q = \frac{1}{2}(k^2 + q^2 - z^2)$ ,  $k \cdot \ell = \frac{1}{2}(k^2 + \ell^2 - x^2)$  and  $q \cdot \ell = \frac{1}{2}(q^2 + \ell^2 - y^2)$  and then used to

simplify (B.3):

$$\begin{aligned}
0 &= \int d^D q \int d^D k \left[ \frac{D}{k^2 q^2 z^2 x^2 y^2} - 2(k^\mu - q^\mu) \left\{ \frac{k^\mu}{k^4 q^2 z^2 x^2 y^2} + \frac{(k^\mu - q^\mu)}{k^2 q^2 z^4 x^2 y^2} \right. \right. \\
&\quad \left. \left. + \frac{(k^\mu - \ell^\mu)}{k^2 q^2 z^2 x^4 y^2} \right\} \right] \\
0 &= \int d^D q \int d^D k \left[ \frac{D-2}{k^2 q^2 z^2 x^2 y^2} - 2 \left\{ \frac{k^2 - \frac{1}{2}(k^2 + q^2 - z^2)}{k^4 q^2 z^2 x^2 y^2} \right. \right. \\
&\quad \left. \left. + \frac{(k^2 - \frac{1}{2}(k^2 + q^2 - z^2) - \frac{1}{2}(k^2 + \ell^2 - x^2) + \frac{1}{2}(q^2 + \ell^2 - y^2))}{k^2 q^2 z^2 x^4 y^2} \right\} \right], \\
0 &= \int d^D q \int d^D k \left[ \frac{D-4}{k^2 q^2 z^2 x^2 y^2} - 2 \left\{ \frac{1}{k^4 q^2 x^2 y^2} - \frac{l}{k^2 q^2 z^2 y^4} \right\} \right].
\end{aligned} \tag{B.4}$$

In the first line, the derivative has been computed. In the second, the dot products have been expanded using the above relations, and, in the last, has been simplified. So I now write  $n_3$ 's solution

$$\begin{aligned}
n_3(a) &= \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 q^2 z^2 x^2 y^2} \\
&= \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{2}{D-4} \left[ \frac{1}{k^4 q^2 x^2 y^2} - \frac{l}{k^2 q^2 z^2 y^4} \right] \\
&= \left[ \frac{-i}{(4\pi)^6} \right] \left[ \frac{-1}{4\pi\nu^2} \right]^{3\epsilon} \frac{1}{\epsilon p^{2(-a-3\epsilon)}} \left[ \frac{[\Gamma(1+\epsilon)]^3 \Gamma(\epsilon) \Gamma(-\epsilon) \Gamma(1+a+3\epsilon)}{\Gamma(2+2\epsilon) \Gamma(1-a-2\epsilon)} \right] \\
&\times \frac{\Gamma(-a-3\epsilon)}{\Gamma(2+a+4\epsilon)} \left[ \frac{\Gamma(1-\epsilon) \Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} - \frac{\Gamma(1+2\epsilon) \Gamma(1-2\epsilon)}{\Gamma(1+3\epsilon) \Gamma(1-\epsilon)} \right].
\end{aligned} \tag{B.5}$$

## B.2.2 Tensor and Vector Integrals

The tensor and vector forms of  $n_1$  and  $n_2$  can be calculated by using the tensor and vector forms of (2.21), in some cases repeatedly. For type  $n_3$  a different approach is required. For  $n_3$  tensor type integrals which are

$$n_3^{t_1}(a) = \frac{p_\mu p_\nu}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu q^\nu}{[q]^2 [q - \ell]^2 [k - q]^2 [k]^2 [k - \ell]^2}, \tag{B.6}$$

and

$$n_3^{t_3}(a) = \frac{p_\mu p_\nu}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{[q]^2 [q - \ell]^2 [k - q]^2 [k]^2 [k - \ell]^2}, \tag{B.7}$$

the result of  $\int d^D q$  and  $\int d^D k$  integrals can only be a combination of momenta  $\ell$  and has the form

$$\int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{[q]^2 [q - \ell]^2 [k - q]^2 [k]^2 [k - \ell]^2} = A \ell^\mu \ell^\nu + B g^{\mu\nu} \ell^2. \quad (\text{B.8})$$

As the integration must be relativistically covariant so the right hand side of (B.8) must transform as a rank-two tensor of this form, with  $A$  and  $B$  are scalar functions of  $\ell$ . These functions will be defined in terms of scalar integrals over  $d^D k$  and  $d^D q$  and since  $p \cdot \ell = \frac{1}{2} [\ell^2 + p^2 - (\ell - p)^2]$  the tensor integral above (B.7) can be written in terms of the scalar results from  $n_3(a)$  (B.5).

### B.3 Complicated Tensor Examples

In the case of more complicated tensor structure I will show an outline of a typical calculation. For example, consider

$$\begin{aligned} n_1^{T_2}(b, c, a) &= p_\mu p_\nu p_\lambda \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu q^\nu q^\lambda}{[q]^{2c} [q - \ell]^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k]^{2b} [k - q]^2} \\ &= p_\mu p_\nu p_\lambda \frac{1}{\nu^{4\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu q^\nu q^\lambda}{[q]^{2(b+c-1-\epsilon)} [q - \ell]^2} A_o. \end{aligned} \quad (\text{B.9})$$

The integral  $d^D k$  can be done straightforwardly like the  $d^D k$  integral in (B.1) and the result is  $q^2$  to some power times a constant (for simplicity denoted by  $A_o$ ) in this calculation. The  $d^D q$  integral  $I_0$  is

$$\begin{aligned} I_0 &= \frac{1}{\nu^{2\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu q^\nu q^\lambda}{[q]^{2(b+c-1-\epsilon)} [q - \ell]^2} \\ &= A \ell^\mu \ell^\nu \ell^\lambda + B (g^{\mu\nu} \ell^\lambda \ell^2 + g^{\mu\lambda} \ell^\nu \ell^2 + g^{\nu\lambda} \ell^\mu \ell^2) \end{aligned} \quad (\text{B.10})$$

where we can determine the coefficients  $A$  and  $B$  by solving the following set of simultaneous equations

$$I_1 = \ell_\mu \ell_\nu \ell_\lambda I_0 = A \ell^6 + B (3\ell^6), \quad (\text{B.11})$$

$$I_2 = g_{\mu\nu} \ell_\lambda I_0 = A \ell^4 + B (D + 2) \ell^4. \quad (\text{B.12})$$

In terms of the integrals  $I_1$  and  $I_2$  the constants  $A$  and  $B$  are

$$A = -\frac{1}{\ell^6 (D + 1)} [\ell^2 I_2 - (D + 2) I_1] \quad (\text{B.13})$$

$$B = \frac{1}{\ell^6 (D + 1)} [\ell^2 I_2 - I_1], \quad (\text{B.14})$$

where the integrals  $I_1$  and  $I_2$  can be written as

$$I_1 = \int \frac{d^D q}{(2\pi)^D} \frac{(q \cdot \ell)^3}{q^{2(b+c-1-\epsilon)} (q-\ell)^2} = \int \frac{d^D q}{(2\pi)^D} \frac{1}{8} \frac{q^6 + 3q^4 \ell^2 + 3q^2 \ell^4 + \ell^6}{q^{2(b+c-1-\epsilon)} (q-\ell)^2}, \quad (\text{B.15})$$

$$I_2 = \int \frac{d^D q}{(2\pi)^D} \frac{(q \cdot \ell) q^2}{q^{2(b+c-1-\epsilon)} (q-\ell)^2} = \int \frac{d^D q}{(2\pi)^D} \frac{[\frac{1}{2}(q^2 + \ell^2)]}{q^{2(b+c-2-\epsilon)} (q-\ell)^2}. \quad (\text{B.16})$$

The integral (B.10) with the above values for  $A$  and  $B$  with  $I_1$  and  $I_2$  is

$$\begin{aligned} n_1^{T_2}(b, c, a) &= A_o \frac{1}{\nu^{2\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell-p]^2} [A(p \cdot \ell)^3 + B(3p^2 p \cdot \ell \ell^2)] \\ &= \left[ \frac{-i}{(4\pi)^6} \right] \left[ \frac{-1}{4\pi\nu^2} \right]^{3\epsilon} \frac{1}{p^{2(b+c-a-6-3\epsilon)}} \frac{1}{8} \left[ \frac{[\Gamma(1+\epsilon)]^3 \Gamma(2-b+\epsilon)}{\Gamma(b) \Gamma(3-b+2\epsilon) (3+2\epsilon)} \right. \\ &\quad \times \frac{\Gamma(b-1-\epsilon)}{\Gamma(3-b-c+2\epsilon) \Gamma(b+c-2-2\epsilon) \Gamma(6-b-c+a+3\epsilon)} \\ &\quad \times \frac{1}{\Gamma(c+b-a-4-3\epsilon) \Gamma(7-b-c+3\epsilon) \Gamma(b+c-1-\epsilon)} \\ &\quad \times \left\{ \frac{-12(4-b-c+a+2\epsilon)(3+2\epsilon)(1+\epsilon)}{\Gamma(7-b-c+a+4\epsilon) \Gamma(c+b-a-3-2\epsilon)} \right. \\ &\quad \left. - \frac{4(5-b-c+a+2\epsilon)(2a^2-4ac+12a\epsilon+20a-4ba+45+2b^2)}{\Gamma(8-b-c+a+4\epsilon)} \right. \\ &\quad \times (-12c\epsilon-12b\epsilon+57\epsilon+18\epsilon^2+2c^2+4cb-20c)(8\epsilon^3+40\epsilon^2-8b\epsilon^2) \\ &\quad \times (-8c\epsilon^2+4cb\epsilon+2b^2\epsilon-26c\epsilon+66\epsilon-26b\epsilon+2c^2\epsilon+36-21b-21c) \\ &\quad \left. \left. \times \frac{(+3c^2+6cb+3b^2)}{\Gamma(b+c-a-2-2\epsilon)} \right\} \right]. \end{aligned} \quad (\text{B.17})$$

The expression in the first line for the  $n_1^{T_1}(b, c, a)$  integral would only differ from this integral by  $A_o$  which would be the coefficient emerging from a vector rather than scalar one loop integral. Calculating the corresponding  $n_2$  and  $n_3$  integrals follows the same procedure.

## B.4 Integral Without Contractions Between the Antisymmetric Tensors

The integral  $I(p^2)$  from Figure 3.6, which has no contractions between the indices on the antisymmetric tensors as discussed in the corresponding section, is

$$I(p^2) = p_\mu p_\lambda \epsilon^{\alpha\beta\lambda\chi} \epsilon^{\nu\mu\rho\sigma} \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell_\sigma \ell_\beta}{[\ell-p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{q_\rho q_\chi k_\mu k_\alpha}{k^2 q^2 z^2 x^2 y^2}. \quad (\text{B.18})$$

Note that many of the indices have been relabelled, so there is very little correspondence between the labelling here and in (3.25). In order to calculate the  $d^D k$  and  $d^D q$  integrals I used the same method as for the last section with

$$\begin{aligned}
I(\ell^2) &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{q^\rho q^\chi k^\mu k^\alpha}{k^2 q^2 z^2 x^2 y^2} \\
&= A \ell^\mu \ell^\rho \ell^\chi \ell^\alpha + B g^{\rho\chi} \ell^\mu \ell^\alpha \ell^2 + C [g^{\rho\mu} \ell^\chi \ell^\alpha \ell^2 + g^{\chi\mu} \ell^\rho \ell^\alpha \ell^2] + F [g^{\rho\alpha} \ell^\chi \ell^\mu \ell^2 + \\
&\quad + g^{\chi\alpha} \ell^\rho \ell^\mu \ell^2] + G g^{\mu\alpha} \ell^\rho \ell^\chi \ell^2 + H g^{\rho\chi} g^{\mu\alpha} \ell^4 + J [g^{\rho\mu} g^{\chi\alpha} \ell^4 + g^{\rho\alpha} g^{\chi\mu} \ell^4].
\end{aligned} \tag{B.19}$$

To simplify notation I have ignored indices on  $I(\ell^2)$  in (B.19). There are seven coefficients so I need at most seven equations to solve this system. However, if the tensor structure of the integral (B.18) is considered the only terms that will ultimately be nonzero are the those with  $H$  and  $J$  coefficients; the rest of the terms are zero following from the contraction of a symmetric with an antisymmetric tensor. The integral above (B.19) I will refer to as  $I$  in the following equations

$$I_1 = \ell_\mu \ell_\rho \ell_\chi \ell_\alpha I = \ell^8 [A + B + 2C + 2F + G + H + 2J] \tag{B.20}$$

$$I_2 = g_{\rho\chi} \ell_\mu \ell_\alpha I = \ell^6 [A + DB + 2C + 2F + G + DH + 2J] \tag{B.21}$$

$$I_3 = g_{\rho\mu} \ell_\chi \ell_\alpha I = \ell^6 [A + B + (1 + D)C + 2F + G + H + (1 + D)J] \tag{B.22}$$

$$I_4 = g_{\rho\alpha} \ell_\chi \ell_\mu I = \ell^6 [A + B + 2C + (1 + D)F + G + H + (1 + D)J] \tag{B.23}$$

$$I_5 = g_{\mu\alpha} \ell_\rho \ell_\chi I = \ell^6 [A + B + 2C + 2F + DG + DH + 2J] \tag{B.24}$$

$$I_6 = g_{\rho\chi} g_{\mu\alpha} I = \ell^4 [A + DB + 2C + 2F + DG + D^2 H + 2DJ] \tag{B.25}$$

$$I_7 = g_{\rho\mu} g_{\chi\alpha} I = \ell^4 [A + B + (1 + D)C + (1 + D)F + G + DH + D(1 + D)J]. \tag{B.26}$$

Before solving the system of equations it is useful to look at the forms of the integrals  $I_1$  through  $I_7$  with the following simplifications  $I_2 = I_5$ ,  $I_3 = I_4$  and  $I_6 = 0$ . If  $I$  in (B.18) is replaced by the right hand side of (B.19) then

$$\begin{aligned}
I(p^2) &= p_\mu p_\lambda \epsilon^{\alpha\beta\lambda\chi} \epsilon^{\nu\mu\rho\sigma} \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell_\sigma \ell_\beta}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{q_\rho q_\chi k_\mu k_\alpha}{k^2 q^2 z^2 x^2 y^2} \\
&= p_\mu p_\lambda \epsilon^{\alpha\beta\lambda\chi} \epsilon^{\nu\mu\rho\sigma} \frac{1}{\nu^{2\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell_\sigma \ell_\beta}{[\ell - p]^2} \left[ H g^{\rho\chi} g^{\mu\alpha} \ell^4 + J [g^{\rho\mu} g^{\chi\alpha} \ell^4 + g^{\rho\alpha} g^{\chi\mu} \ell^4] \right] \\
&= \frac{p_\mu p_\lambda}{\nu^{2\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{2a}}{[\ell - p]^2} \ell_\sigma \ell_\beta \ell^4 \left[ H \epsilon^{\alpha\beta\lambda\chi} \epsilon^\nu{}_{\alpha\chi}{}^\sigma + J [\epsilon^{\alpha\beta\lambda}{}_\alpha \epsilon^{\nu\mu}{}_\mu{}^\sigma + \epsilon^{\alpha\beta\lambda\chi} \epsilon^\nu{}_{\chi\alpha}{}^\sigma] \right] \\
&= \frac{p_\mu p_\lambda}{\nu^{2\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell_\sigma \ell_\beta}{[\ell - p]^2} \ell^4 \epsilon^{\alpha\beta\lambda\chi} \epsilon^\nu{}_{\alpha\chi}{}^\sigma [H - J].
\end{aligned} \tag{B.27}$$

In the last line the contractions between the antisymmetric tensors allow us to replace that tensor product. Also solving for the constants with the simplifications above

results in

$$H - J = -\frac{2I_2 + I_7\ell^2 - 2I_3}{\ell^6 (D^3 - 2D^2 - D + 2)}, \quad (\text{B.28})$$

where  $I_1$  cancels which simplifies the results, so I only need  $I_2$ ,  $I_3$ , and  $I_7$ :

$$\begin{aligned} I_2 &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{q^2 (k \cdot \ell)^2}{k^2 q^2 z^2 x^2 y^2} \\ &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{4} \frac{[\ell^2 - x^2]^2}{k^2 z^2 x^2 y^2} \end{aligned} \quad (\text{B.29})$$

$$\begin{aligned} I_3 &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{(k \cdot q) (k \cdot \ell) (q \cdot \ell)}{k^2 q^2 z^2 x^2 y^2} \\ &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{8} \left\{ \frac{2\ell^2 x^2}{k^2 q^2 z^2 y^2} - \frac{2x^2}{k^2 z^2 y^2} + \frac{2\ell^4}{k^2 q^2 z^2 y^2} \right. \\ &\quad \left. - \frac{2\ell^2}{k^2 z^2 y^2} - \frac{\ell^4}{k^2 q^2 x^2 y^2} \right\} \end{aligned} \quad (\text{B.30})$$

$$\begin{aligned} I_7 &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{(k \cdot q)^2}{k^2 q^2 z^2 x^2 y^2} \\ &= \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{4} \left\{ \frac{2x^2}{k^2 q^2 z^2 y^2} + \frac{z^2}{k^2 q^2 x^2 y^2} \right\}. \end{aligned} \quad (\text{B.31})$$

Therefore (B.27) can be further simplify and then solved

$$\begin{aligned} I(p^2) &= \frac{p_\mu p_\lambda}{\nu^{2\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell_\sigma \ell_\beta \ell^4}{[\ell - p]^2} [-\{-(D-3)(D-2)[g^{\beta\sigma} g^{\lambda\nu} - g^{\lambda\sigma} g^{\beta\nu}]\}] [H - J] \\ &= \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[\ell - p]^2} \left\{ (D-3)(D-2) [\ell^2 p^2 - (p \cdot \ell)^2] \right\} \\ &\quad \times \left\{ \frac{-(D+1)\ell^4}{4(D+1)(D-1)(D-2)} \left[ \frac{z^2}{k^2 q^2 x^2 y^2} + \frac{\ell^2}{k^2 q^2 x^2 y^2} + \frac{4x^2}{\ell^2 k^2 z^2 y^2} - \frac{2}{k^2 z^2 y^2} \right] \right\} \\ &= \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[\ell - p]^2} \left\{ [2\ell^2 p^2 - p^4 - \ell^4] \right\} \\ &\quad \times \left\{ \frac{-(D-3)}{16(D-1)} \left[ \frac{z^2 \ell^4}{k^2 q^2 x^2 y^2} + \frac{\ell^6}{k^2 q^2 x^2 y^2} + \frac{4x^2 \ell^2}{k^2 z^2 y^2} - \frac{2\ell^2}{k^2 z^2 y^2} \right] \right\} \\ &= \left[ \frac{-i}{(4\pi)^6} \right] \left[ \frac{-1}{4\pi\nu^2} \right]^{3\epsilon} \frac{1}{p^{2(-4-3\epsilon)}} \frac{1}{64} \left[ \frac{[\Gamma(1+\epsilon)]^4}{(5+4\epsilon)\Gamma(4+4\epsilon)} \right] \\ &\quad \times \left[ \frac{\Gamma(1+\epsilon)[\Gamma(-\epsilon)]^2}{\Gamma(2+2\epsilon)\Gamma(1+2\epsilon)\Gamma(-1-2\epsilon)} + \frac{[4+12\epsilon+8\epsilon^2]}{\Gamma(4+3\epsilon)} \right]. \end{aligned} \quad (\text{B.32})$$

# APPENDIX C

## REDUCE CODE

This appendix is the REDUCE code for Figure 3.7. The operators  $fp(p)$ ,  $mfp(p)$  and  $sgp(p)$ , below are the propagators and identities for the contractions of the antisymmetric tensors are the operators  $pl(p, mu, nu, u, r, v, s)$  and  $pt(p, u, r, v, s)$ .

```

off allfac;
vecdim d;
vector p,k,q,o,u,r,v,s,mu,nu,vv;
operator fp,pl,pt,sgp,mfp,pvertex;
for all p let fp(p)=i*(g(l,p)+m)/(p.p-m*m);
for all p let mfp(p)=i*(g(l,p)-m)/(p.p-m*m);
for all p,u,r,v,s,mu,nu let pl(p,mu,nu,u,r,v,s)=
-(d-3)*p.mu*p.nu*(mu.nu*(u.r*v.s-v.r*s.u)+mu.r*(u.s*v.nu-v.s*nu.u)
+mu.s*(u.nu*v.r-v.nu*r.u));
for all p,u,r,v,s let pt(p,u,r,v,s)=-(d-3)*(d-2)*(u.r*v.s-v.r*s.u);
for all p let sgp(p)=-i/(p.p);

index u,r,v,s,mu,nu,vv;
let m=0;

amp:=(p.p*pt(p,u,r,v,s)-pl(p,mu,nu,u,r,v,s))
*fp(k)*i*g(l,vv)*fp(q)*i*g(l,vv)*fp(k)*i*((p-o).u)*g(l,v)*mfp(k-o)
*i*(-(p-o)).r*g(l,s)*sgp(p-o)*sgp(k-q);

let k.k=k2, p.p=p2, q.q=q2, o.o=o2;
amp;

let k.p=1/2*(k2+p2-x);
let p.q=1/2*(q2+p2-y);
let q.k=1/2*(q2+k2-z);
let o.k=1/2*(o2+k2-t);
let o.q=1/2*(o2+q2-h);
let o.p=1/2*(o2+p2-w);

amp;

denp:=x**8*k2**8*y**8*q2**8*z**8*h**8*t**8*w**8*o2**8;
amps:=amp*denp;

for all n1,n2,n3,n4,n5,n6,n7,n8,n9

```

```

match k2**n1*q2**n2*o2**n3*x**n4*y**n5*w**n6*z**n7*t**n8*h**n9
=fi(8-n1,8-n2,8-n3,8-n4,8-n5,8-n6,8-n7,8-n8,8-n9);
let p2^2=p4;
let d^2=d2;
amp:=amps;

```

The  $fi(8-n1,8-n2,8-n3,8-n4,8-n5,8-n6,8-n7,8-n8,8-n9)$  is REDUCE's classifications for the integrals the nonzero integrals in this diagram are classified in REDUCE as

```

let fi(1,0,0,0,0,1,1,-1,1)=n4;
let fi(0,1,0,0,0,1,1,1,-1)=n4;
let fi(1,0,0,0,-1,1,1,0,1)=n5;
let fi(0,1,0,-1,0,1,1,1,0)=n5;
let fi(1,0,0,-1,0,1,1,0,1)=n6;
let fi(0,1,0,0,-1,1,1,1,0)=n6;
let fi(1,1,0,0,0,1,1,1,-1)=n10;
let fi(1,1,0,0,0,1,1,-1,1)=n10;
let fi(1,1,0,0,-2,1,1,0,1)=n132;
let fi(1,1,0,-2,0,1,1,1,0)=n132;
let fi(1,0,-1,0,0,1,1,-1,1)=n19;
let fi(0,1,-1,0,0,1,1,1,-1)=n19;
let fi(1,0,-1,0,-1,1,1,0,1)=n20;
let fi(0,1,-1,-1,0,1,1,1,0)=n20;
let fi(1,0,-1,-1,0,1,1,0,1)=n22;
let fi(0,1,-1,0,-1,1,1,1,0)=n22;
let fi(1,1,0,0,-1,1,1,-1,1)=n24;
let fi(1,1,0,-1,0,1,1,1,-1)=n24;
let fi(1,1,0,-1,-1,1,1,1,0)=n25;
let fi(1,1,0,-1,-1,1,1,0,1)=n25;
let fi(1,1,-1,0,0,1,1,-1,1)=n27;
let fi(1,1,-1,0,0,1,1,1,-1)=n27;
let fi(1,1,-2,0,0,1,1,-1,1)=n272;
let fi(1,1,-2,0,0,1,1,1,-1)=n272;
let fi(1,1,-1,0,-1,1,1,1,0)=n29;
let fi(1,1,-1,-1,0,1,1,0,1)=n29;
let fi(1,1,-2,0,-1,1,1,1,0)=n292;
let fi(1,1,-2,-1,0,1,1,0,1)=n292;
let fi(1,1,-1,-1,0,1,1,1,0)=n30;
let fi(1,1,-1,0,-1,1,1,0,1)=n30;
let fi(1,1,-2,-1,0,1,1,1,0)=n302;
let fi(1,1,-2,0,-1,1,1,0,1)=n302;
let fi(1,1,-1,-2,0,1,1,1,0)=n32;
let fi(1,1,-1,0,-2,1,1,0,1)=n32;
let fi(1,1,-1,-1,0,1,1,1,-1)=n40;
let fi(1,1,-1,0,-1,1,1,-1,1)=n40;

```



let fi(1,1,-1,-1,-1,1,1,1,0)=n43;  
let fi(1,1,-1,-1,-1,1,1,0,1)=n43;

and the final out put is

$$\begin{aligned}
& (d^2 n_{10} p^4 - 4 d^2 n_{132} p^2 - 4 d^2 n_{24} p^2 + 4 d^2 n_{25} p^2 \\
& - 2 d^2 n_{27} p + d^2 n_{272} + 26 d n_{10} p^4 - 64 d n_{132} p^2 \\
& + 10 d n_{19} + 20 d n_{20} - 10 d n_{22} - 74 d n_{24} p^2 + 64 d n_{25} p^2 \\
& - 42 d n_{27} p^2 + 16 d n_{272} - 10 d n_{29} p^2 + 10 d n_{292} + 10 d n_{30} p^2 \\
& - 10 d n_{302} - 20 d n_{32} - 10 d n_{4} p^2 - 10 d n_{40} + 20 d n_{43} \\
& - 20 d n_5 p^2 + 10 d n_6 p^2 - 9 d^2 n_{10} p^4 + 28 d^2 n_{132} p^2 - 2 d^2 n_{19} \\
& - 4 d^2 n_{20} + 2 d^2 n_{22} + 30 d^2 n_{24} p^2 - 28 d^2 n_{25} p^2 + 16 d^2 n_{27} p^2 \\
& - 7 d^2 n_{272} + 2 d^2 n_{29} p^2 - 2 d^2 n_{292} - 2 d^2 n_{30} p^2 + 2 d^2 n_{302} \\
& + 4 d^2 n_{32} + 2 d^2 n_4 p^2 + 2 d^2 n_{40} - 4 d^2 n_{43} + 4 d^2 n_5 p^2 \\
& - 2 d^2 n_6 p^2 - 24 n_{10} p^4 + 48 n_{132} p^2 - 12 n_{19} - 24 n_{20} + 12 n_{22} \\
& + 60 n_{24} p^2 - 48 n_{25} p^2 + 36 n_{27} p^2 - 12 n_{27} + 12 n_{29} p^2 - 12 n_{292} \\
& - 12 n_{30} p^2 + 12 n_{302} + 24 n_{32} + 12 n_4 p^2 + 12 n_{40} - 24 n_{43} \\
& + 24 n_5 p^2 - 12 n_6 p^2) / 8
\end{aligned}$$

where  $p^2 = p^2$ ,  $d^2 = d^2$ . Note that for example  $n_{10}$  would be the integral

$$\begin{aligned}
& \frac{1}{\nu^{6\epsilon}} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{[\ell - p]^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q]^2 [q - \ell]^2} \int \frac{d^D k}{(2\pi)^D} \frac{(k - \ell)^2}{[k]^2 [k - q]^2} \\
& = 0 - 2 n_1^{\nu_4} (1, 1, 0) + n_1 (1, 1, 1).
\end{aligned} \tag{C.1}$$