A SECOND ORDER SENSITIVITY LOAD FLOW TECHNIQUE

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ABSTRACT

Real and reactive power injections into the buses of an electric power system can be used to control voltages at system buses. The voltage changes caused by incremental changes in power injections can be estimated by using the sensitivities of bus voltages with respect to power injections. Based on this consideration, a second order sensitivity load flow technique has been developed in this thesis. Both the first and second order sensitivities are used. In the proposed technique, the voltage corrections are first computed using the first order sensitivities. The second order corrections are then calculated by using the first order corrections. The real and imaginary components of bus voltages are updated using the calculated first and second order corrections. The procedure is repeated until a converged solution is obtained.

Two modifications for the second order sensitivity load flow technique are presented in the thesis. A constant Jacobian second order sensitivity load flow method has also been developed.

The proposed second order sensitivity load flow techniques have been used for computing load flows of five test systems. The results of these studies are presented and are compared with those of the Newton-Raphson, Second Order and Fast Decoupled Load Flow techniques.

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CHAPTER 1

INTRODUCTION

1.1 Background

Load flows provide steady state solutions of electric power systems. These solutions are used in system planning and operation for checking that the system will operate in stable states and the consumers will receive energy at (approximately) normal voltage levels. Load flows are also used in fault studies and, transient and dynamic stability studies. The basic objective is to ensure that :

1. the system performs its intended function adequately.
2. the service provided to customers is of acceptable quality.
3. the various system operating states are secure.
4. the system operating states are most economical or efficient.

Initially load flow studies were conducted on network analyzers which were later replaced gradually by digital computers [1]. Presently, load flow studies of large power systems are conducted exclusively on digital computers.

Load flow studies of a power system are conducted in two stages. The first stage consists of preparing a model of the system and the second stage consists of obtaining numerical solutions of the mathematical model. In the model, each element is generally represented as a combination of lumped parameter components. Transmission lines and transformers are represented by pi connected equivalents that use lumped resistive, reactive and capacitive branches. The off-nominal tap transformers are also modeled by their equivalent pi circuits. Generators and loads are represented by constant real and reactive generation and consumption.

The operating state of a system is defined by specifying for each system bus four of the following six quantities

1. Real power generation, P_g.
2. Reactive power generation, Q_g.
3. Real power load, P_l.
4. Reactive power load, Q_l.
5. Magnitude of bus voltage, V.
6. Phase angle of bus voltage, θ.

In some techniques, bus voltages are defined by their real and imaginary components (E and F). Depending on the specified operating levels, each system bus is classified in one of the following categories:

1. PV bus: At each voltage controlled bus, net real power (P_g-P_l) and voltage magnitude are specified. These buses are, therefore, referred to as PV buses. It is important to appreciate that the reactive power loads at these buses are also specified.

2. PQ bus: A bus is specified as a PQ bus if the net real and reactive power injections, (P_g-P_l) and (Q_g-Q_l), are specified. All load buses are PQ buses.

3. Slack bus: One of the voltage controlled buses is used as a reference bus. At this bus, voltage is specified and its phase angle is set to zero. This bus is referred to as the swing bus or slack bus.

For an N bus system, power injections are functions of bus voltages and their phase angles. These functions are dictated by the power system network and can, therefore, be defined by equations of the following form [2]:

$$P_i = \Phi_i(|V_1| \;....\; |V_N|,\ \Theta_1 \;....\; \Theta_N) \qquad (1.1a)$$

$$Q_i = \Phi_i'(|V_1| \;....\; |V_N|,\ \Theta_1 \;....\; \Theta_N) \qquad (1.1b)$$

where: the subscripts i represent system busses;

P_i is the net real power injection at bus i;

Q_i is the net reactive power injection at bus i.

If bus voltages are specified in rectangular coordinates, the functions of the real and reactive powers would be of the following form:

$$P_i = \Psi_i(E_1 \;....\; E_N,\ F_1 \;....\; F_N) \qquad (1.2a)$$

$$Q_i = \Psi_i'(E_1 \;....\; E_N,\ F_1 \;....\; F_N) \qquad (1.2b)$$

Equations 1.1 and 1.2 are nonlinear algebraic equations and form the basis for load flow models of power systems.

1.2 Review of Load Flow Techniques

The load flow problem consists of finding a set of voltages that satisfy Equation 1.1 or 1.2 for each bus when real power injections and voltage magnitudes at PV buses and, real and reactive power injections at PQ buses are specified. Several methods for solving load flow problems have been proposed in the past. The generally used load flow techniques are briefly reviewed in this section.

1.2.1 Y and Z matrix methods

The Y-matrix method was the first iterative method that was used for digital load flow solutions [3]. This method uses nodal voltage equations that describe linear relationships between bus voltages and current injections. These equations are of the form,

$$[Y][V] = [I] \tag{1.3}$$

where [Y] is the admittance matrix of the system,

[V] is the vector of bus voltages and

[I] is the vector of bus current injections.

The nodal currents are first calculated from bus voltage estimates and specified power injections. New bus voltage estimates are obtained from the nodal currents and the elements of the Y-matrix using Equation 1.3. Either the Gauss-Jordan or the Gauss-Seidel technique is used to solve these equations. The procedure is repeated until the maximum bus voltage changes in two successive iterations are less than a pre-specified tolerance. This method usually requires several iterations for obtaining converged solutions.

The Z-matrix method [4] was used after the development of the Y-matrix method. In the Z-matrix method, the system equations are expressed in the form

$$[V] = [Z][I] \qquad (1.4)$$

where [Z] is the inverse of the [Y] matrix.

This method takes into consideration couplings of each bus voltage with current injections at all system buses. This method, therefore, provides converged solutions in fewer iterations than those required by the Y-matrix method. However, the computer memory required for solving a load flow using the Z-matrix method can be substantially larger than the computer memory required when using the Y-matrix method.

1.2.2 Newton-Raphson and modified Newton-Raphson methods

Equations 1.1 and 1.2 are nonlinear equations that define power injections in terms of bus voltages and system parameters. The Newton-Raphson (NR) technique has been adopted to solve these equations [5]. The load flow equations are first linearized in the neighborhood of an estimated operating state. New estimates of bus voltages are obtained by solving the linearized load flow equations. Real and reactive power mismatches are then calculated. The procedure is repeated until the power mismatches are less than a pre-specified tolerance. This method provides a converged solution in a few iterations. The optimally ordered elimination technique [6] reduces the required computer storage and solution times. Experience has indicated that the NR method exhibits quadratic convergence characteristics. Also, the number of iterations to convergence is independent of the system size [2].

Several modifications to the NR method have also been suggested [7,8,9,10,11], One of the modifications provides the decoupled methods [12,13]. These approaches make use of the fact that couplings between active powers and bus voltage magnitudes, and between reactive powers and phase angles of bus voltages are relatively weak and can, therefore, be neglected. Combined with other approximations, the decoupled approaches achieve reductions in computer storage and CPU time requirements. However, they do not provide converged solutions in some cases when system R/X ratios are high.

1.2.3 Second order load flow methods

The NR method has also been extended to provide the Second Order Load Flow approaches (SOLF). These methods consider the first three terms of the Taylor series expansions of the load flow equations. The technique can be applied when the system parameters and voltages are expressed in either polar or rectangular coordinates [14,15,16].

In recent publications, the rectangular formulation of the SOLF method has received considerable attention [17,18]. The load flow equations in this technique form a set of quadratic algebraic equations and, therefore, the third and higher order derivatives do not exist. A SOLF method based on this approach has to compute the inverse of the Jacobian matrix only once [16]. This approach generally provides converged solutions faster than the NR method because the Jacobian matrix is not required to be inverted in each iteration. As is shown in Chapter 3, the SOLF technique takes an unacceptably large number of iterations in one case. Moreover, the method requires more computer memory than the NR technique does.

While discussing Reference [16], Mr. H. Duran pointed out that the SOLF method is the same as the constant Jacobian matrix version of the NR method (CNR). It can also be shown that the computations required by the SOLF and CNR methods are identical.

1.3 Basis of the Proposed Load Flow Technique

In the NR method, bus voltages are updated in each iteration until power mismatches are less than a prespecified tolerance. Power mismatches and the Jacobian matrix are used in updating the bus voltages. This process can also be considered from the system control point of view.

Power mismatches observed in each iteration can be considered as disturbances applied to power injections. Voltage changes required to decrease the effects of power injection disturbances can be calculated from the sensitivities of bus voltages with respect to power injections. It is shown later in this thesis that using the first order sensitivities is equivalent to using the NR technique. The load flow technique suggested in this thesis uses the first and second order sensitivities. The voltage changes required in each iteration, as calculated by the proposed approach, provide better estimates than those provided by the NR technique. The proposed method has the property of cubic convergence.

1.4 Scope of The Thesis

The method proposed in this thesis uses the concept of sensitivity analysis. To facilitate understanding of the proposed method, the concepts of sensitivity matrix and Taylor series expansion of a multivariable function are described in Chapter 2.

Chapter 3 introduces the Second Order Sensitivity (SOS) method. The characteristics of the SOS technique are reported. The method is also compared with the Newton-Raphson (NR) and the Second Order Load Flow (SOLF) techniques that are commonly used.

For a load flow problem, the SOS method uses more computer storage compared to the NR technique. Some modifications that can reduce the computer storage requirements of the SOS method are discussed in Chapter 4. The suggested modifications also reduce the cpu time required for obtaining load flow solutions. Results of load flows for sample power systems are presented and discussed.

A hybrid approach that combines the advantages of the constant Jacobian matrix technique and the SOS method is presented in Chapter 5. The performance of this technique is investigated and is compared with the performance of the fast decoupled load flow method of Reference [13].

Two appendices are also included. Appendix A describes the development of the SOS technique. A detailed analysis of the Jacobian matrix when rectangular coordinates are used, is provided in Appendix B.

Computer programs for the load flow studies were developed by the author using the VAX-11/780 computer at the College of Engineering, University of Saskatchewan. Cpu times reported in this thesis were recorded by the VAX-11 Run-Time library.

CHAPTER 2

SENSITIVITY MATRICES

2.1 General

The basic load flow equations described in Chapter 1 are multivariable functions. The Taylor series expansion of these functions are used to develop the Newton-Raphson and second order sensitivity methods. The mathematical basis for the SOS method is described in this chapter. The series expansions of single and multivariable functions are first presented and the sensitivity matrices are then defined. Two numerical examples are used to illustrate the SOS technique. Before describing the Taylor series expansion and the sensitivity matrices, the load flow problem is examined from the static control theory point of view.

2.2 Static Control Theory View of the Load Flow Problem

A load flow model is essentially a set of nonlinear algebraic equations. It has not been possible to obtain closed form solutions of these equations and, therefore, iterative methods are used. The most commonly used technique is the Newton-Raphson approach that is developed using the Taylor series expansions of power injection equations in the neighborhood of an estimated solution. The first two terms of the series are used to convert the problem to a set of linear equations.

The NR method can also be explained from the following considerations of a single variable equation

$$y = f(x) . \tag{2.1}$$

It is desired to determine the value of x for a specified value of y, say y_s. The NR method requires that an initial estimate, x_o, be selected and then a new estimate, x_1, be calculated using the following equation.

$$x_1 = x_o + (dy/dx)^{-1}\big|_{x=x_o}(y_s - f(x_o)) \tag{2.2}$$

This equation can be rewritten in the following form.

$$\Delta x = s_1 \Delta y \tag{2.3a}$$

$$x_1 = x_o + \Delta x \tag{2.3b}$$

where: $\Delta x = x_1 - x_o$

$\Delta y = y_s - f(x_o)$

$s_1 = (dx/dy)\big|_{x=x_o}$

In control terminology, this equation implies that a steady state of a system $y_o=f(x_o)$ has been subjected to a disturbance $\Delta y=y_s-y_o$. The required changes for the state variable,

Δx, are approximately equal to $(s_1 \Delta y)$. The derivative $dx/dy(=s_1)$ is generally referred to as the first order sensitivity of x with respect to y. The second order sensitivity, s_2, can be added to equation 2.3a changing it to the following form.

$$\Delta x = s_1 \Delta y + \frac{1}{2} s_2 \Delta y^2 \quad (2.4)$$

The second order sensitivity, s_2, is the derivative of s_1 with respect to y. The second and higher order sensitivities of multivariable systems are not normally used.

2.3 Taylor Series Expansion

The Taylor series of single and multivariable functions have been well known to engineers and mathematicians [31], [32]. These expansions are described in this section.

2.3.1 Taylor series expansion of a single variable function

A single variable function $y=f(x)$ can be expressed in the neighborhood of an estimate x_o by the following series.

$$f(x) = f(x_o) + a_1 \Delta x + \frac{a_2}{2!} \Delta x^2 + \ldots + \frac{a_i}{i!} \Delta x^i + \ldots \quad (2.5)$$

where: a_i is the ith derivative of $f(x)$ at $x=x_o$

$$\Delta x = x_1 - x_o$$

This equation can be rearranged to define Δy as follows.

$$\Delta y = f(x) - f(x_o) = a_1 \Delta x + \frac{a_2}{2!}\Delta x^2 + \ldots \qquad (2.6)$$

2.3.2 Taylor series expansion of a multivariable function

A multivariable function $y=f[X]$ $([X]=[x_1 \ \ldots \ x_n]^T)$ can be expanded in the neighborhood of an estimated solution, $[X_o]=[x_{1o} \ \ldots \ x_{no}]^T$ and rearranged to provide the following equation.

$$\Delta y = \sum_{i=1}^{\infty} \frac{1}{i!}\left(\Delta x_1 \frac{\partial}{\partial x_1} + \ldots + \Delta x_n \frac{\partial}{\partial x_n}\right)^i f[X]\Big|_{[X]=[X_o]} \qquad (2.7)$$

where: $\Delta y = f[X] - f[X_o]$

2.4 Sensitivity Matrices

Two types of sensitivities of $[Y]=f[X]$ are described in this section. One type is the sensitivities of [Y] with respect to [X] and the second type is the sensitivities of [X] with respect to [Y].

2.4.1 Sensitivities of [Y] with respect to [X]

The first and second order sensitivities of a function y with respect to a variable x can be defined as follows.

$$s'_1(y/x) = \frac{dy}{dx} \tag{2.8a}$$

$$s'_2(y/x) = \frac{d^2y}{dx^2} \tag{2.8b}$$

Changes in function y caused by changes in the variable x can be estimated by using the following equations.

$$\Delta y = s'_1 \Delta x + \frac{1}{2} s'_2 \Delta x^2 \tag{2.9}$$

For a multivariable function y=f[X], the first order sensitivities of y with respect to the variables [X], can be expressed as follows.

$$[S'_1(y/[X])] = [\frac{\partial y}{\partial x_1} \quad \frac{\partial y}{\partial x_2} \ \cdots \ \frac{\partial y}{\partial x_n}] \tag{2.10}$$

where: S'_1 is the first order sensitivity of y with respect to [X]

The second order sensitivities of y can be similarly expressed as an nxn matrix of the following form.

$$[S'_2(y/[X])] = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \dfrac{\partial^2 y}{\partial x_i \partial x_j} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{2.11}$$

Equation 2.12 can be derived by using the first three terms of the Taylor series expansion of a multivariable function. Changes in the function y caused by changes in [X] can be estimated from this equation.

$$\Delta y = [S'_1(y/[X])][\Delta X] + \frac{1}{2}[\Delta X]^T[S'_2(y/[X])][\Delta X] \qquad (2.12)$$

where: $[\Delta X] = [X] - [X_o]$

A set of multivariable functions, $y_i = f_i[X]$ may be defined as follows.

$$[Y] = F[X] \qquad (2.13)$$

where: $[Y] = [y_1, y_2, \ldots, y_n]^T$

Using the Taylor series of each equation, changes in the functions y_i (i=1,2, .. n) can be defined as follows.

$$\Delta y_i = [S'_1(y_i/[X])][\Delta X] + \frac{1}{2}[\Delta X]^T[S'_2(y_i/[X])[\Delta X] + \ldots (2.14)$$

where: $i = 1, 2, \ldots n$

This equation can be written in the matrix form as follows.

$$[\Delta Y] = [J][\Delta X] + \frac{1}{2}[\Delta X]^T[H][\Delta X] + \ldots \qquad (2.15)$$

where: $[J] = [S'_1([Y]/[X])] = [\frac{\partial Y}{\partial X}]$ is the well known nxn Jacobian matrix.

$[H] = [S'_2([Y]/[X])] = [\frac{\partial^2 Y}{\partial X \partial X}]$ is nxnxn Hessian matrix.

Equation 2.15 indicates that mismatches in [Y] caused by disturbances in [X] can be approximately obtained by using the first and second order sensitivities of [Y] with respect to [X].

2.4.2 Sensitivities of [X] with respect to [Y]

In several situations, the source of disturbance may be the elements of [Y] and, therefore, changes in [X] would have to be estimated for changes in [Y].

For the single variable case, the first and second order sensitivities of x with respect to y can be defined as follows.

$$s_1(x/y) = \frac{dx}{dy} \qquad (2.16a)$$

$$s_2(x/y) = \frac{d^2x}{dy^2} \qquad (2.16b)$$

The changes in x can now be calculated using the equation:

$$\Delta x = s_1 \Delta y + \frac{1}{2} s_2 \Delta y^2 \qquad (2.17)$$

For a set of multivariable functions, changes in [X] can be estimated using Equation 2.18. This has the same general form as Equation 2.15.

$$[\Delta X] = [S_1([X]/[Y])][\Delta Y] + \frac{1}{2}[\Delta Y]^T[S_2([X]/[Y])][\Delta Y] \qquad (2.18)$$

where: $[S_1([X]/[Y])] = [\frac{\partial X}{\partial Y}]$ is the first order sensitivities of [X] with respect to [Y]

$[S_2([X]/[Y])] = [\frac{\partial^2 X}{\partial Y \partial Y}]$ is the second order sensitivities of [X] with respect to [Y]

Equations 2.18 form the basis of the Second Order Sensitivity method proposed in this thesis.

2.5 Two Examples

It is shown in the last section that the first and second order sensitivities can be used to estimate changes in a vector caused by changes in another vector. For the function [Y]=F[X], the sensitivities of [Y] with respect to [X] are not difficult to calculate. On the other hand, it is not easy to get the sensitivities of [X] with respect to [Y]. Two examples are used in this section to demonstrate the general approach for deriving the sensitivities and the procedure for the second order sensitivity method.

2.5.1 Single variable example

In this section, a single variable nonlinear equation is solved using the SOS method. Consider that the value of x is required to be calculated for $y_s=2.0$ if

$$y = x^2 \tag{2.19}$$

Assume that the initial estimate of $x^{(0)}=1.0$.

The first order sensitivity of x with respect to y is obtained as follows.

$$\frac{dy}{dy} = 1 = \frac{d(x^2)}{dy}$$

$$= \frac{dx^2}{dx} \cdot \frac{dx}{dy} = 2x\frac{dx}{dy} \qquad (2.20)$$

Rearranging this equation, the first order sensitivity of x with respect to y is given by

$$s_1 = \frac{dx}{dy} = \frac{1}{2x} \; . \qquad (2.21)$$

The second order sensitivity of x with respect to y can be obtained by taking the derivatives of both sides of Equation 2.20 as follows

$$0 = \frac{d}{dy}(2x\frac{dx}{dy})$$

$$= \frac{d(2x)}{dy}\,\frac{dx}{dy} + 2x(\frac{d^2x}{dy^2}) \; . \qquad (2.22)$$

This equation provides the second order sensitivity, s_2, as follows.

$$s_2 = \frac{d^2x}{dy^2}$$

$$= -2(\frac{dx}{dy})^2\frac{1}{2x} \qquad (2.23)$$

Substituting for $\frac{dx}{dy}$ from equation 2.21, the second order sensitivity of the example, that is being considered, is

$$s_2 = -1/(4x^3) \; . \qquad (2.24)$$

The first and second order sensitivities at $x=x^{(0)}$ (=1.0) are

$$s_1(x^{(0)}) = 0.5$$

$$s_2(x^{(0)}) = -0.25$$

The function y at $x=x^{(0)}$ is $(1.0)^2 = 1.0$ and, therefore $\Delta y^{(0)}=2.0-1.0=1.0$. The changes in x are therefore given by:

$$\begin{aligned}\Delta x^{(0)} &= s_1\Delta y^{(0)} + s_2(\Delta y^{(0)})^2/2 \\ &= (0.5)(1.0) + (-0.25)(1.0)^2/2 \\ &= 0.375\end{aligned}$$

Since $\Delta x^{(0)}= x^{(1)}- x^{(0)}$, $x^{(1)}$ is given by:

$$\begin{aligned}x^{(1)} &= x^{(0)}+\Delta x^{(0)} \\ &= 1.0 + 0.375 = 1.375\end{aligned}$$

Using the general form

$$\Delta x^{(k)} = s_1(x^{(k)})\Delta y^{(k)} + \tfrac{1}{2}s_2(x^{(k)})(\Delta y^{(k)})^2 \qquad (2.25)$$

the procedure described above can be repeated until a converged value for x is obtained. For the selected example, the results are summarized in Table 2.1. A summary of the results obtained by the NR approach are also given in this table for comparison.

2.5.2 Two variable example

A two variable example was presented in Reference [16].

Table 2.1 Iterative results of ($y=x^2$)

k	SOS x	SOS Δy	SOS Δx	NR x	NR Δy	NR Δx
0	1.0000	1.0000	0.3750	1.0000	1.0000	0.5000
1	1.3750	0.1094	0.0392	1.5000	-0.2500	-0.0833
2	1.4142			1.4167	-0.0072	-0.0025
3				1.4142		

The problem is solved in this section to illustrate the procedures for calculating the sensitivities and for applying the SOS approach. Consider the non-linear equations:

$$y_1 = f_1(x_1,x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2 = 2.14 \qquad (2.26a)$$

$$y_2 = f_2(x_1,x_2) = -2x_1^2 - x_1x_2 + 2x_2^2 = 0.64 \qquad (2.26b)$$

The first order sensitivities calculated by taking the partial derivatives of these equations are as follows.

$$\frac{\partial f_1}{\partial y_1} = 1$$

$$= \frac{\partial f_1}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial x_2}{\partial y_1}$$

$$= (4x_1 - 2x_2)\frac{\partial x_1}{\partial y_1} + (-2x_1 + 4x_2)\frac{\partial x_2}{\partial y_1} \qquad (2.27a)$$

similarly,

$$\frac{\partial f_1}{\partial y_2} = 0$$

$$= (4x_1 - 2x_2)\frac{\partial x_1}{\partial y_2} + (-2x_1 + 4x_2)\frac{\partial x_2}{\partial y_2} \qquad (2.27b)$$

$$\frac{\partial f_2}{\partial y_1} = 0$$

$$= (-4x_1 - x_2)\frac{\partial x_1}{\partial y_1} + (-x_1 + 4x_2)\frac{\partial x_2}{\partial y_1} \qquad (2.27c)$$

$$\frac{\partial f_2}{\partial y_2} = 1$$

$$= (-4x_1 - x_2)\frac{\partial x_1}{\partial y_2} + (-x_1 + 4x_2)\frac{\partial x_2}{\partial y_2} \qquad (2.27d)$$

These equations can be written in the matrix form as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4x_1-2x_2 & -2x_1+4x_2 \\ -4x_1-x_2 & -x_1+4x_2 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} \qquad (2.28)$$

The first order sensitivities are expressed as

$$[S_1([X]/[Y])] = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} \qquad (2.29)$$

and are as follows:

$$\begin{bmatrix} 4x_1-2x_2 & -2x_1+4x_2 \\ -4x_1-x_2 & -x_1+4x_2 \end{bmatrix}^{-1} = [J]^{-1} = [[A][X] \mid [B][X]]^{-1} \qquad (2.30)$$

where:

$$[A] = \begin{bmatrix} 4 & -2 \\ -4 & -1 \end{bmatrix} \qquad [B] = \begin{bmatrix} -2 & 4 \\ -1 & 4 \end{bmatrix} \qquad [X] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Assuming that the estimated value of [X] is $[1.0,\ 1.0]^T$, the Jacobian matrix and the first order sensitivities are as follows.

$$[J] = \begin{bmatrix} 2 & 2 \\ -5 & 3 \end{bmatrix} \qquad [S_1] = \frac{1}{16}\begin{bmatrix} 3 & -2 \\ 5 & 2 \end{bmatrix}$$

Equation 2.28 can be partitioned in two parts as follows.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [J]\begin{bmatrix} \frac{\partial x_1}{\partial y_1} \\ \frac{\partial x_2}{\partial y_1} \end{bmatrix} = [J]\,[\frac{\partial X}{\partial y_1}] \qquad (2.31a)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [J]\begin{bmatrix} \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_2} \end{bmatrix} = [J]\,[\frac{\partial X}{\partial y_2}] \qquad (2.31b)$$

The first order sensitivity vector can, therefore, be estimated using the following equations.

$$[\frac{\partial X}{\partial y_1}] = [J]^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (2.32a)$$

$$[\frac{\partial X}{\partial y_2}] = [J]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (2.32b)$$

For the selected example, the first order sensitivity vectors are:

$$[\frac{\partial X}{\partial y_1}] = [J]^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3/16\\5/16\end{bmatrix}$$

$$[\frac{\partial X}{\partial y_2}] = [J]^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-2/16\\2/16\end{bmatrix}$$

Taking the partial derivatives of Equations 2.31 with respect to y_1 and y_2, the following equations can be obtained.

$$\begin{bmatrix}0\\0\end{bmatrix} = [\frac{\partial J}{\partial y_1}][\frac{\partial X}{\partial y_1}] + [J][\frac{\partial^2 X}{\partial y_1 \partial y_1}] \qquad (2.33a)$$

$$\begin{bmatrix}0\\0\end{bmatrix} = [\frac{\partial J}{\partial y_2}][\frac{\partial X}{\partial y_1}] + [J][\frac{\partial^2 X}{\partial y_2 \partial y_1}] \qquad (2.33b)$$

$$\begin{bmatrix}0\\0\end{bmatrix} = [\frac{\partial J}{\partial y_1}][\frac{\partial X}{\partial y_2}] + [J][\frac{\partial^2 X}{\partial y_1 \partial y_2}] \qquad (2.33c)$$

$$\begin{bmatrix}0\\0\end{bmatrix} = [\frac{\partial J}{\partial y_2}][\frac{\partial X}{\partial y_2}] + [J][\frac{\partial^2 X}{\partial y_2 \partial y_2}] \qquad (2.33d)$$

Since the Jacobian matrix [J] is equal to [[A][X]|[B][X]], the first derivatives of the Jacobian matrix are as follows:

$$[\frac{\partial J}{\partial y_1}] = \frac{\partial}{\partial y_1}[[A][X] \;|\; [B][X]]$$

$$= [[A][\frac{\partial X}{\partial y_1}] \,|\, [B][\frac{\partial X}{\partial y_1}]] = \left[[A]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix} \;\middle|\; [B]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix}\right] \qquad (2.34a)$$

$$[\frac{\partial J}{\partial Y_2}] = [[A][\frac{\partial X}{\partial Y_2}] \mid [B][\frac{\partial X}{\partial Y_2}]] = \left[[A]\begin{bmatrix} \frac{\partial x_1}{\partial Y_2} \\ \frac{\partial x_2}{\partial Y_2} \end{bmatrix} \Bigg| [B]\begin{bmatrix} \frac{\partial x_1}{\partial Y_2} \\ \frac{\partial x_2}{\partial Y_2} \end{bmatrix} \right] \quad (2.34b)$$

By rearranging Equation 2.33 (a), (b), (c), and (d), the second order sensitivities are obtained as follows:

$$[\frac{\partial^2 X}{\partial Y_1 \partial Y_1}] = \begin{bmatrix} \frac{\partial^2 x_1}{\partial Y_1 \partial Y_1} \\ \frac{\partial^2 x_2}{\partial Y_1 \partial Y_1} \end{bmatrix} = -[S_1][\frac{\partial J}{\partial Y_1}][\frac{\partial X}{\partial Y_1}] \quad (2.35a)$$

$$[\frac{\partial^2 X}{\partial Y_1 \partial Y_2}] = \begin{bmatrix} \frac{\partial^2 x_1}{\partial Y_1 \partial Y_2} \\ \frac{\partial^2 x_2}{\partial Y_1 \partial Y_2} \end{bmatrix} = -[S_1][\frac{\partial J}{\partial Y_1}][\frac{\partial X}{\partial Y_2}] \quad (2.34b)$$

$$[\frac{\partial^2 X}{\partial Y_2 \partial Y_1}] = \begin{bmatrix} \frac{\partial^2 x_1}{\partial Y_2 \partial Y_1} \\ \frac{\partial^2 x_2}{\partial Y_2 \partial Y_1} \end{bmatrix} = -[S_1][\frac{\partial J}{\partial Y_2}][\frac{\partial X}{\partial Y_1}] \quad (2.34c)$$

$$[\frac{\partial^2 X}{\partial Y_2 \partial Y_2}] = \begin{bmatrix} \frac{\partial^2 x_1}{\partial Y_2 \partial Y_2} \\ \frac{\partial^2 x_2}{\partial Y_2 \partial Y_2} \end{bmatrix} = -[S_1][\frac{\partial J}{\partial Y_2}][\frac{\partial X}{\partial Y_2}] \quad (2.34d)$$

Substituting for $[\frac{\partial J}{\partial Y_1}]$ and $[\frac{\partial J}{\partial Y_2}]$ from Equations 2.34, the following equations are obtained.

$$\left[\frac{\partial^2 X}{\partial y_1 \partial y_1}\right] = -[S_1]\left[[A]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix} \;\vdots\; [B]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix}\right]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix} \qquad (2.35a)$$

$$\left[\frac{\partial^2 X}{\partial y_1 \partial y_2}\right] = -[S_1]\left[[A]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix} \;\vdots\; [B]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix}\right]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_2}\end{bmatrix} \qquad (2.35b)$$

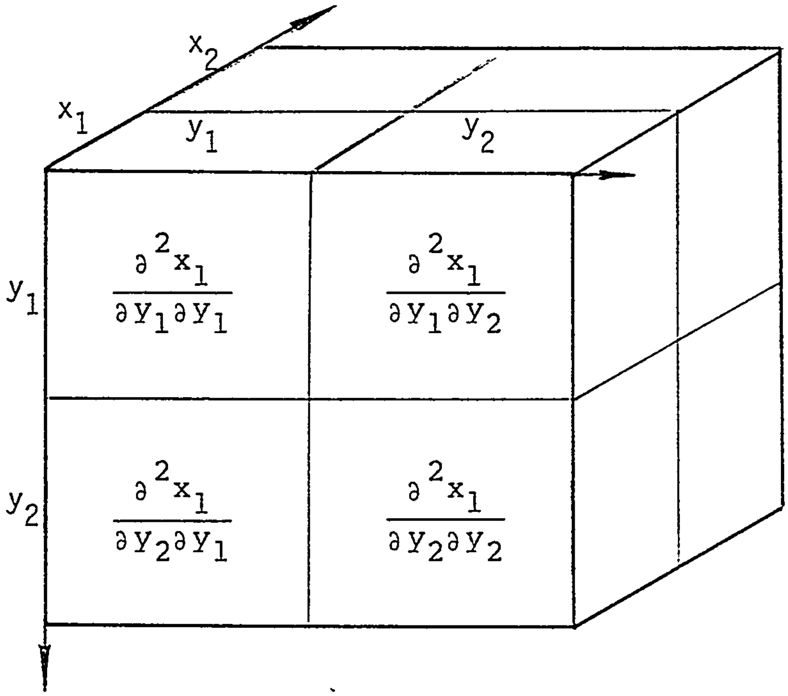
$$\left[\frac{\partial^2 X}{\partial y_2 \partial y_1}\right] = -[S_1]\left[[A]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_2}\end{bmatrix} \;\vdots\; [B]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_2}\end{bmatrix}\right]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\end{bmatrix} \qquad (2.35b)$$

$$\left[\frac{\partial^2 X}{\partial y_2 \partial y_2}\right] = -[S_1]\left[[A]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_2}\end{bmatrix} \;\vdots\; [B]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_2}\end{bmatrix}\right]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_2}\end{bmatrix} \qquad (2.35d)$$

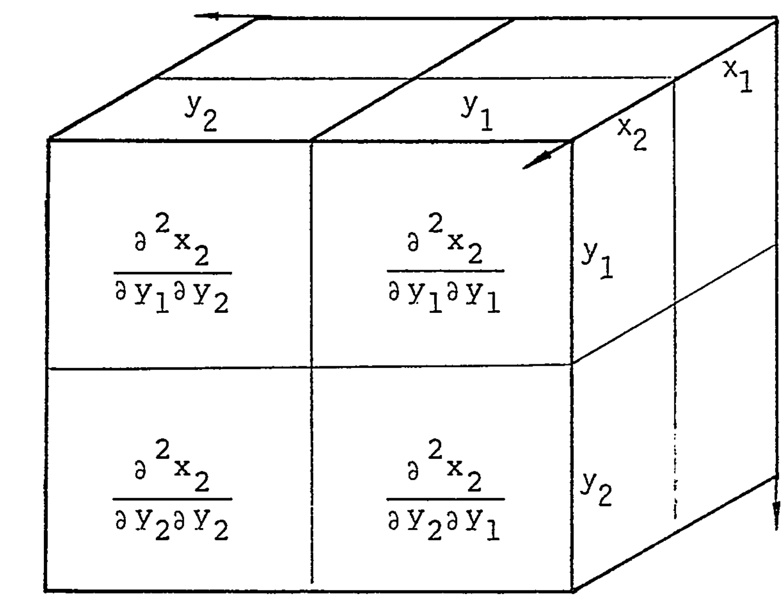
These are the second order sensitivities of [X] with respect to [Y]. These sensitivities can be expressed in the form of a cubic matrix, $[S_2]$, as shown in Figure 2.1.

The second order sensitivities for the selected problem are obtained from Equations 2.35 as follows.

$$\left[\frac{\partial^2 X}{\partial y_1 \partial y_1}\right] = \frac{-1}{16}\begin{bmatrix}3 & -2\\ 5 & 2\end{bmatrix}\left[\begin{bmatrix}4 & -2\\ -4 & -1\end{bmatrix}\begin{bmatrix}3/16\\ 5/16\end{bmatrix} \;\vdots\; \begin{bmatrix}-2 & 4\\ -1 & 4\end{bmatrix}\begin{bmatrix}3/16\\ 5/16\end{bmatrix}\right]\begin{bmatrix}3/16\\ 5/16\end{bmatrix} = \begin{bmatrix}-0.039\\ -0.109\end{bmatrix}$$



Front view of $[S_2]$ matrix



Back view of $[S_2]$ matrix

Figure 2.1 $[S_2]$ matrix

$$\left[\frac{\partial^2 X}{\partial y_1 \partial y_2}\right] = \frac{-1}{16}\begin{bmatrix}3 & -2\\ 5 & 2\end{bmatrix}\left[\begin{bmatrix}4 & -2\\ -4 & -1\end{bmatrix}\begin{bmatrix}3/16\\ 5/16\end{bmatrix} \Bigg| \begin{bmatrix}-2 & 4\\ -1 & 4\end{bmatrix}\begin{bmatrix}3/16\\ 5/16\end{bmatrix}\right]\begin{bmatrix}-2/16\\ 2/16\end{bmatrix} = \begin{bmatrix}0.016\\ -0.063\end{bmatrix}$$

$$\left[\frac{\partial^2 X}{\partial y_2 \partial y_1}\right] = \frac{-1}{16}\begin{bmatrix}3 & -2\\ 5 & 2\end{bmatrix}\left[\begin{bmatrix}4 & -2\\ -4 & -1\end{bmatrix}\begin{bmatrix}-2/16\\ 2/16\end{bmatrix} \Bigg| \begin{bmatrix}-2 & 4\\ -1 & 4\end{bmatrix}\begin{bmatrix}-2/16\\ 2/16\end{bmatrix}\right]\begin{bmatrix}3/16\\ 5/16\end{bmatrix} = \begin{bmatrix}0.016\\ -0.063\end{bmatrix}$$

$$\left[\frac{\partial^2 X}{\partial y_2 \partial y_2}\right] = \frac{-1}{16}\begin{bmatrix}3 & -2\\ 5 & 2\end{bmatrix}\left[\begin{bmatrix}4 & -2\\ -4 & -1\end{bmatrix}\begin{bmatrix}-2/16\\ 2/16\end{bmatrix} \Bigg| \begin{bmatrix}-2 & 4\\ -1 & 4\end{bmatrix}\begin{bmatrix}-2/16\\ 2/16\end{bmatrix}\right]\begin{bmatrix}-2/16\\ 2/16\end{bmatrix} = \begin{bmatrix}-0.031\\ -0.063\end{bmatrix}$$

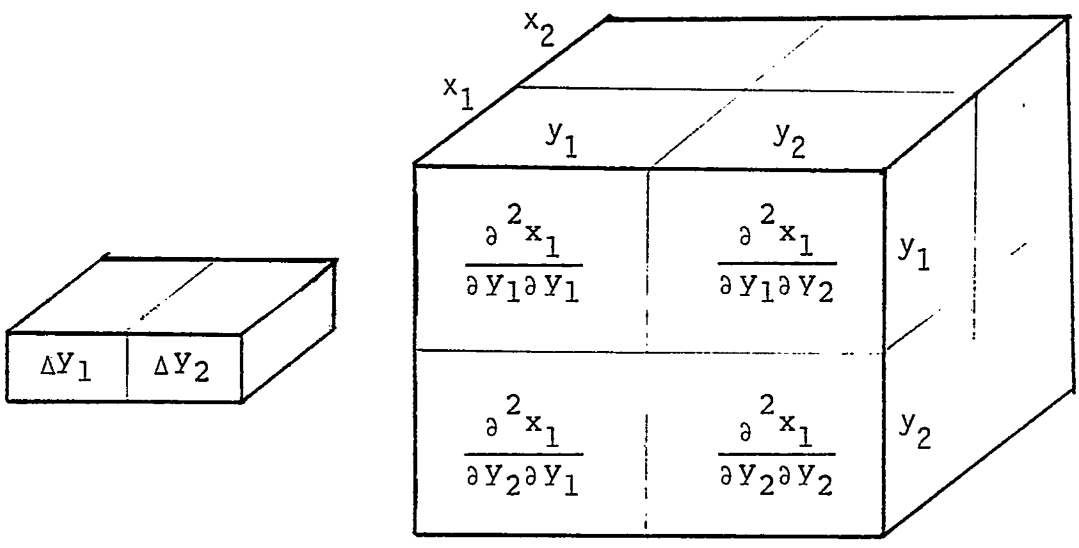
Changes in vector [X] due to changes in [Y] are estimated from the sensitivity vectors as follows.

$$[\Delta X] = [S_1][\Delta Y] + \frac{1}{2}[\Delta Y]^T[S_2][\Delta Y] \qquad (2.36)$$

Where: $[S_1][\Delta Y] = [J]^{-1}[\Delta Y]$

$$[\Delta Y] = [\Delta y_1, \Delta y_2]^T = [y_{1s} - f_1, y_{2s} - f_2]^T \Big|_{[X]=[X_o]}$$

The term $[\Delta Y]^T[S_2][\Delta Y]$ is a vector that is obtained in the following manner. The product $[\Delta Y]^T[S_2]$ is as follows.



$$= \begin{bmatrix} \frac{\partial^2 x_2}{\partial y_1 \partial y_1}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_1}\Delta y_2 & \frac{\partial^2 x_2}{\partial y_1 \partial y_2}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_2}\Delta y_2 \\ \frac{\partial^2 x_1}{\partial y_1 \partial y_1}\Delta y_1 + \frac{\partial^2 x_1}{\partial y_2 \partial y_1}\Delta y_2 & \frac{\partial^2 x_1}{\partial y_1 \partial y_2}\Delta y_1 + \frac{\partial^2 x_1}{\partial y_2 \partial y_2}\Delta y_2 \end{bmatrix}$$

This matrix can be rewritten in the following form.

$$\begin{bmatrix} \frac{\partial^2 x_1}{\partial y_1 \partial y_1}\Delta y_1 + \frac{\partial^2 x_1}{\partial y_2 \partial y_1}\Delta y_2 & \frac{\partial^2 x_1}{\partial y_1 \partial y_2}\Delta y_1 + \frac{\partial^2 x_1}{\partial y_2 \partial y_2}\Delta y_2 \\ \frac{\partial^2 x_2}{\partial y_1 \partial y_1}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_1}\Delta y_2 & \frac{\partial^2 x_2}{\partial y_1 \partial y_2}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_2}\Delta y_2 \end{bmatrix} \quad (2.37)$$

The product $[\Delta Y]^T[S_2][\Delta Y]$ is now obtained as follows.

$[\Delta Y]^T[S_2][\Delta Y] =$

$$= \begin{bmatrix} \frac{\partial^2 x_1}{\partial y_1 \partial y_1}\Delta y_1 + \frac{\partial^2 x_1}{\partial y_2 \partial y_1}\Delta y_2 & \frac{\partial^2 x_1}{\partial y_1 \partial y_2}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_2}\Delta y_2 \\ \frac{\partial^2 x_2}{\partial y_1 \partial y_1}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_1}\Delta y_2 & \frac{\partial^2 x_2}{\partial y_1 \partial y_2}\Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_2}\Delta y_2 \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} \quad (2.38)$$

$$= \begin{bmatrix} \frac{\partial^2 x_1}{\partial y_1 \partial y_1}\Delta y_1 \Delta y_1 + \frac{\partial^2 x_1}{\partial y_2 \partial y_1}\Delta y_2 \Delta y_1 + \frac{\partial^2 x_1}{\partial y_1 \partial y_2}\Delta y_1 \Delta y_2 + \frac{\partial^2 x_1}{\partial y_2 \partial y_2}\Delta y_2 \Delta y_2 \\ \\ \frac{\partial^2 x_2}{\partial y_1 \partial y_1}\Delta y_1 \Delta y_1 + \frac{\partial^2 x_2}{\partial y_2 \partial y_1}\Delta y_2 \Delta y_1 + \frac{\partial^2 x_2}{\partial y_1 \partial y_2}\Delta y_1 \Delta y_2 + \frac{\partial^2 x_2}{\partial y_2 \partial y_2}\Delta y_2 \Delta y_2 \end{bmatrix} \quad \ldots\ldots (2.39)$$

For the selected example, $[\Delta Y]$, $[S_1][\Delta Y]$ and $[\Delta Y]^T[S_2][\Delta Y]$ are as follows.

$$[\Delta Y] = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} 2.24 \\ 0.64 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.24 \\ 1.64 \end{bmatrix}$$

$$[S_1][\Delta Y] = \frac{1}{16}\begin{bmatrix} 3 & -2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 0.24 \\ 1.64 \end{bmatrix} = \begin{bmatrix} -0.16 \\ 0.28 \end{bmatrix}$$

$$[\Delta Y]^T[S_2][\Delta Y] = \begin{bmatrix} (-0.039)(0.24)(0.24) + (0.016)(1.64)(0.24) + (0.016)(0.24)(1.64) + (-0.031)(1.64)(1.64) \\ (-0.109)(0.24)(0.24) + (-0.063)(1.64)(0.24) + (-0.063)(1.64)(0.24) + (-0.063)(1.64)(1.64) \end{bmatrix}$$

$$= \begin{bmatrix} -0.074 \\ -0.224 \end{bmatrix}$$

The correction vector $[\Delta x_1, \Delta x_2]$ is, therefore, given by:

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -0.16 \\ 0.28 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -0.074 \\ -0.224 \end{bmatrix} = \begin{bmatrix} -0.197 \\ 0.168 \end{bmatrix}$$

The updated estimates of the vector $[x_1, x_2]$ are:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} + \begin{bmatrix} -0.197 \\ 0.168 \end{bmatrix} = \begin{bmatrix} 0.803 \\ 1.168 \end{bmatrix}$$

This procedure can be repeated until $\max(|\Delta Y|)$ is within a pre-specified tolerance.

2.5.3 A simplified procedure

The second order sensitivity method described in the last paragraph is complicated. Some simplifications are possible; these are now described.

Corrections due to the second order sensitivity, $[\Delta Y]^T[S_2][\Delta Y]$, can be rearranged. Equation 2.38 can be rewritten as follows.

$$[\Delta Y]^T[S_2][\Delta Y] =$$

$$= [[\frac{\partial^2 X}{\partial Y_1 \partial Y_1}]\Delta Y_1 + [\frac{\partial^2 X}{\partial Y_2 \partial Y_1}]\Delta Y_2 \,|\, [\frac{\partial^2 X}{\partial Y_1 \partial Y_2}]\Delta Y_1 + [\frac{\partial^2 X}{\partial Y_2 \partial Y_2}]\Delta Y_2][\Delta Y] \quad ......(2.40)$$

Substituting from Equations 2.35, $[\Delta Y]^T[S_2]$ can be rewritten as follows.

$$[\frac{\partial^2 X}{\partial Y_1 \partial Y_1}]\Delta Y_1 + [\frac{\partial^2 X}{\partial Y_2 \partial Y_1}]\Delta Y_2$$

$$= -[S_1][[A][\frac{\partial X}{\partial Y_1}] \;|\; [B][\frac{\partial X}{\partial Y_1}]][\frac{\partial X}{\partial Y_1}]\Delta Y_1$$

$$-[S_1][[A][\frac{\partial X}{\partial Y_2}] \;|\; [B][\frac{\partial X}{\partial Y_2}]][\frac{\partial X}{\partial Y_1}]\Delta Y_2$$

$$= -[S_1][[A][\frac{\partial X}{\partial Y_1}]\Delta Y_1 \;|\; [B][\frac{\partial X}{\partial Y_1}]\Delta Y_1][\frac{\partial X}{\partial Y_1}]$$

$$-[S_1][[A][\frac{\partial X}{\partial Y_2}]\Delta Y_2 \;|\; [B][\frac{\partial X}{\partial Y_2}]\Delta Y_2][\frac{\partial X}{\partial Y_1}]$$

$$= -[S_1][[A]([\frac{\partial X}{\partial Y_1}]\Delta Y_1 + [\frac{\partial X}{\partial Y_2}]\Delta Y_2$$

$$|[B]([\frac{\partial X}{\partial Y_1}]\Delta Y_1 + [\frac{\partial X}{\partial Y_2}]\Delta Y_2)][\frac{\partial X}{\partial Y_1}]$$

$$= -[S_1][[A][D] \;|\; [B][D]][\frac{\partial X}{\partial Y_1}]$$

$$= -[S_1][JD][\frac{\partial X}{\partial Y_1}] \qquad (2.41a)$$

where:

$$[D] = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = [\frac{\partial X}{\partial Y_1}]\Delta Y_1 + [\frac{\partial X}{\partial Y_2}]\Delta Y_2$$

But

$$[D] = \begin{bmatrix} \frac{\partial x_1}{\partial Y_1} & \frac{\partial x_1}{\partial Y_2} \\ \frac{\partial x_2}{\partial Y_1} & \frac{\partial x_2}{\partial Y_2} \end{bmatrix} \begin{bmatrix} \Delta Y_1 \\ \Delta Y_2 \end{bmatrix}$$

$$= [S_1][\Delta Y]$$

Similarly,

$$[\frac{\partial^2 X}{\partial Y_1 \partial Y_2}]\Delta Y_1 + [\frac{\partial^2 X}{\partial Y_2 \partial Y_2}]\Delta Y_2 = -[S_1][JD][\frac{\partial X}{\partial Y_2}] \qquad (2.41b)$$

Using Equations 2.41(a) and 2.41(b), Equation 2.40 can be rewritten as

$$[\Delta Y]^T[S_2][\Delta Y] =$$

$$= [-[S_1][JD][\frac{\partial X}{\partial Y_1}] \mid -[S_1][JD][\frac{\partial X}{\partial Y_2}]][\Delta Y]$$

$$= -[S_1][JD][[\frac{\partial X}{\partial Y_1}] \mid [\frac{\partial X}{\partial Y_2}]][\Delta Y]$$

$$= -[S_1][JD][D]$$

$$= [S_1][MD] \tag{2.42}$$

where: $[MD] = [m_1 \ , \ m_2]^T$

$$= -[JD][D]$$

$$= -[[A][D] \mid [B][D]][D]$$

The simplified equation for the application of the second order sensitivity method is:

$$[\Delta X] = [S_1][\Delta Y] + 0.5[S_1][MD] \tag{2.43}$$

For the selected example

$$[D] = [S_1][\Delta Y] = \begin{bmatrix} -0.16 \\ 0.28 \end{bmatrix}$$

$$[MD] = -\left[\begin{bmatrix} 4 & -2 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \;\middle|\; \begin{bmatrix} -2 & 4 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\right]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$= -\begin{bmatrix} 4d_1-2d_2 & -2d_1+4d_2 \\ -4d_1-d_2 & -d_1+4d_2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$= \begin{bmatrix} 4d_1d_2-4(d_1^2+d_2^2) \\ 2d_1d_2-4(d_1^2-d_2^2) \end{bmatrix} \qquad (2.44)$$

$$= \begin{bmatrix} 4(-0.16)(0.28)-4[(-0.16)^2+(0.28)^2] \\ 2(-0.16)(0.28)-4[(0.28)^2-(-0.16)^2] \end{bmatrix}$$

$$= \begin{bmatrix} -0.595 \\ -0.301 \end{bmatrix}$$

$$0.5[S_1][MD] = 0.5 \begin{bmatrix} 3/16 & -2/16 \\ 5/16 & 2/16 \end{bmatrix} \begin{bmatrix} -0.595 \\ -0.301 \end{bmatrix}$$

$$= \begin{bmatrix} -0.037 \\ -0.112 \end{bmatrix}$$

$$[\Delta X] = \begin{bmatrix} -0.16 \\ 0.28 \end{bmatrix} + \begin{bmatrix} -0.037 \\ -0.112 \end{bmatrix} = \begin{bmatrix} -0.197 \\ 0.168 \end{bmatrix}$$

The simplified procedure for the second order sensitivity technique can be summarized as follows.

$$[\Delta Y]^{(k)} = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix}^{(k)} = \begin{bmatrix} y_{1s} \\ y_{2s} \end{bmatrix} - \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}^{(k)} \qquad (2.45a)$$

$$[D]^{(k)} = ([J]^{-1})^{(k)} [\Delta Y]^{(k)} \qquad (2.45b)$$

$$[\Delta X]^{(k)} = [D]^{(k)} + 0.5([J]^{-1})^{(k)} [MD]^{(k)} \qquad (2.45c)$$

$$[X]^{k+1} = [X]^{(k)} + [\Delta X]^{(k)} \qquad (2.45d)$$

where: $[J]^{-1}$ is the inverse of the Jacobian matrix defined in Equation 2.30.

[MD] is the vector defined in Equation 2.44.

k is the iteration number.

This algorithm was programmed on the VAX-11/780 to solve Equation 2.26. The results are shown in Table 2.2. For comparison, the problem was also solved using the NR method and the second order load flow method of Reference [16]. The results are shown in Tables 2.3 and 2.4 respectively.

Table 2.2 Solution of Equation 2.26 by the SOS method.

k	x_1	Δy_1	Δx_1	x_2	Δy_2	Δx_2
0	1.0000	0.2400	-0.1970	1.0000	1.6400	0.1682
1	0.8030	0.0971	-0.0030	1.1682	0.1383	0.0318
2	0.8000	0.0000		1.2000	0.0000	

Table 2.3 Solution of Equation 2.26 by the NR Technique.

k	x_1	Δy_1	Δx_1	x_2	Δy_2	Δx_2
0	1.0000	0.2400	-0.1600	1.0000	1.6400	0.2800
1	0.8400	-0.2976	-0.0390	1.2800	-0.1504	-0.0774
2	0.8010	-0.0090	-0.0010	1.2026	-0.0059	-0.0026
3	0.8000	0.0000		1.2000	0.0000	

Table 2.4 Solution of Equation 2.26 by the SOLF Method of Reference 16.

k	x_1	Δx_1	x_2	Δx_2	Δy_2	Δy_2
0	1.0000	-0.1600	1.0000	0.2800	0.2400	1.6400
1	0.8400	-0.1970	1.2800	0.1682	-0.2976	-0.1504
2	0.8030	-0.1961	1.1682	0.2158	0.0971	0.1383
3	0.8039	-0.2004	1.2158	0.1931	-0.0542	-0.0465
4	0.7996	-0.1994	1.1931	0.2033	0.0224	0.0257
5	0.8006	-0.2002	1.2033	0.1985	-0.0110	-0.0107
6	0.7998	-0.1999	1.1985	0.2007	0.0049	0.0052
7	0.8001	-0.2000	1.2007	0.1997	-0.0023	-0.0023
8	0.8000	-0.2000	1.1997	0.2001	0.0010	0.0011
9	0.8000	-0.2000	1.2001	0.1999	-0.0005	-0.0005
10	0.8000	-0.2000	1.1999	0.2000	0.0002	0.0002
11	0.8000	-0.2000	1.2000	0.2000	-0.0001	-0.0001

2.6 Summary

The first and second order sensitivity matrices have been introduced in this chapter. It has been shown that the NR method uses the first order sensitivities. The second order sensitivities can be described by a three dimensional matrix and can be used to solve simultaneous non-linear equations. Two examples have been discussed in detail to demonstrate the underlying concepts and procedures.

CHAPTER 3

SECOND ORDER SENSITIVITY LOAD FLOW TECHNIQUE

3.1 General

The mathematical basis of a second order sensitivity approach for the solution of nonlinear algebraic equations has been developed in the last chapter. Its application to the load flow problem is described in this Chapter. An algorithm for realizing this approach is also described. The effectiveness of the proposed technique is investigated and its performance is compared with the NR and SOLF methods.

For load flow studies, voltages at power system buses are generally used to describe the system states. Real and reactive power injections at each bus are considered as control inputs. Their adjustments provide effective measures to regulate system voltages. This concept has been extensively used in power system state estimation [25] and secure operation [29].

Mathematical models used for power system studies express power injections in terms of bus voltages. The voltages are initially estimated and are upgraded using the Taylor series expansions (of power injections) in the neighborhood of the estimated values. The NR approach uses the first two terms of the series and provides a linear model for the load flow problem. The second order load flow technique proposed in References [14], [15] and [16] uses the first three terms of the Taylor series and provides a second order model of the problem.

In the second order sensitivity technique, presented in this chapter, changes in the system states are estimated by using the sensitivities of the states with respect to the real and reactive power injections. Since second order sensitivities are used, the proposed method is classified as the Second Order Sensitivity (SOS) Load Flow technique.

3.2 The Second Order Sensitivity Technique

The Second Order Sensitivity load flow technique is presented in this section. The load flow model is first presented. The second order sensitivity technique described in Chapter 2 is then applied to develop the technique that can be used to obtain load flow solutions.

3.2.1 Load flow model

Consider a power system that has 'L' PQ buses and 'K' PV buses. Also consider that the admittance matrix representing the system network is known. It is well known that the current injections into the system buses can be expressed by the following equations [2].

$$[I] = [Y][V] \tag{3.1}$$

where: [I] is the vector of currents injected into the system buses.

[V] is the vector of bus voltages.

[Y] is the admittance matrix.

Current injection into bus i can be expressed as follows.

$$I_i = \sum_{j=1}^{N} Y_{ij} V_j \qquad N=L+K+1 \tag{3.2}$$

where: N is the total number of buses in the system.

The power injection into bus i can now be expressed by

$$S_i = P_i + jQ_i = V_i \sum_{j=1}^{N} \overset{*}{Y}_{ij} \overset{*}{V}_j \tag{3.3}$$

where S_i is the complex power injection into bus i.

P_i is the active power injection into bus i.

Q_i is the reactive power injection into bus i.

* denotes conjugate of a complex number or phasor.

Defining [V] as [E]+j[F] and [Y] as [G]+j[B], substituting in Equation 3.3 and equating the real and imaginary parts the following equations can be obtained.

$$P_i = E_i \sum_{j=1}^{N} (G_{ij} E_j - B_{ij} F_j) + F_i \sum_{j=1}^{N} (G_{ij} F_j + B_{ij} E_j) \tag{3.4}$$

$$Q_i = F_i \sum_{j=1}^{N} (G_{ij}E_j - B_{ij}F_j) - E_i \sum_{j=1}^{N} (G_{ij}F_j + B_{ij}E_j) \qquad (3.5)$$

$i = 1,2,....,N$

Since the voltage of the slack bus is known, 2(N-1) equations of the types 3.4 and 3.5 can be established in 2(N-1) unknowns. At each voltage controlled bus, the magnitude of the voltage is specified and can be expressed by the equation

$$|V_i|^2 = E_i^2 + F_i^2 \ . \qquad (3.6)$$

The state variables (unknowns) and the power injections can be expressed as [X] and [U] vectors that are defined as follows.

$$[X] = [\ F_2 \ \dots \ F_N \ ,\ E_2 \ \dots \ E_N \ ]^T$$

$$[U] = [\ P_2 \ \dots \ P_N \ ,\ Q_2 \ \dots \ Q_{L+1} \ ,\ |V_{L+2}|^2 \ \dots \ |V_N|^2 \ ]^T$$

Equations 3.4, 3.5 and 3.6 can be combined to define the load flow model of a power system. Equation 3.7 defines the model in the mathematical form. P and Q are specified for the PQ buses, and P and |V| are specified for the PV buses.

$$[U] = \Psi[X] \qquad (3.7)$$

3.2.2 Sensitivity approach

Real and reactive power injections and bus voltage mismatches in a power system can be expressed as follows.

$$[\Delta U] = \begin{bmatrix} \Delta P \\ \Delta Q \\ \Delta V \end{bmatrix} = \begin{bmatrix} P_s \\ Q_s \\ |V_s|^2 \end{bmatrix} - \begin{bmatrix} P(X) \\ Q(X) \\ V(X) \end{bmatrix}_{[X]=[X_o]} \quad (3.8)$$

Changes in system states can be estimated using the first order sensitivities as follows.

$$[\Delta X] = [S_1(\text{Voltage/Injection})][\Delta U] \quad (3.9)$$

If the second order sensitivities are also used, the changes in the system states are given by the equation

$$[\Delta X] = [S_1(\text{Voltage/Injection})][\Delta U] + 0.5[\Delta U]^T[S_2(\text{Voltage/Injection})][\Delta U]. \quad (3.10)$$

The first order sensitivity matrix $[S_1]$ is square and the second order sensitivity $[S_2]$ is cubic. They are the first and second order derivatives of [X] with respect to [U].

As explained in Chapter 2, $[S_1]$ is the inverse of the Jacobian matrix that is made up of elements representing derivatives of [U] with respect to [X]. Equation 3.9 uses the first order derivatives and describes the NR technique. Equation 3.10 describes the Second Order Sensitivity method that is the subject of this thesis. Because Equation 3.10 provides estimates of the changes in the state vector, some mismatches remain after the first iteration. An iterative procedure should, therefore, be used until the mismatches are within prespecified tolerances.

3.3 Algorithm

It has been illustrated in Chapter 2 that the second order sensitivity equations can be simplified. Using a similar approach, the load flow model of Equation 3.7 can be reduced to the iterative procedure described by Equations 3.11 to 3.16.

$$[\Delta U]^{(k)} = [U_s] - [\Psi(X)]^{(k)} \tag{3.11}$$

$$[\Delta X1]^{(k)} = \begin{bmatrix} \Delta F1 \\ \Delta E1 \end{bmatrix}^{(k)} = ([J]^{-1})^{(k)} [\Delta U]^{(k)} \tag{3.12}$$

$$\begin{aligned} [M]^{(k)} &= [M(\Delta X1)]^{(k)} \\ &= -0.5[J(\Delta X1)]^{(k)} [\Delta X1]^{(k)} \end{aligned} \tag{3.13}$$

$$[\Delta X2]^{(k)} = \begin{bmatrix} \Delta F2 \\ \Delta E2 \end{bmatrix}^{(k)} = ([J]^{-1})^{(k)} [M]^{(k)} \tag{3.14}$$

$$[\Delta X]^{(k)} = [\Delta X1]^{(k)} + [\Delta X2]^{(k)} \tag{3.15}$$

$$[X]^{(k+1)} = \begin{bmatrix} F \\ E \end{bmatrix}^{(k+1)} = [X]^{(k)} + [\Delta X]^{(k)} \tag{3.16}$$

where: [J] is the Jacobian matrix of the NR approach and is defined as follows.

$$[J] = \left[\frac{\partial U}{\partial X}\right]$$

[ΔU] is the mismatch vector.

[ΔX1] is the correction vector obtained by using the first order sensitivities only.

[ΔX2] is the correction vector due to the second order sensitivities.

[ΔX] is the correction vector of the SOS method.

k is the iteration counter.

A load flow algorithm that uses the SOS method can be described by the following discrete steps.

1. Compute the real and reactive power mismatches for the PQ and the real power and the voltage squared mismatches for PV busses.

2. If the maximum absolute mismatch is less than a prespecified tolerance 'TOL', the procedure has converged, proceed to step 8; otherwise continue to step 3.

3. Compute the elements of the Jacobian matrix, factorize the matrix using one of the well developed techniques and store the elements of the factors.

4. Calculate the elements of the vector [ΔX1] using Equation 3.12.

5. Using the vector [ΔX1], calculate the elements of the [M] vector using Equation 3.13.

6. Calculate the elements of [ΔX2] and then the elements of the correction vector [ΔX].

7. Update the real and imaginary parts of the voltages of the PQ and PV busses. Proceed to step 1.

/

8. Calculate the power flows and print out the required load flow information.

3.4 Testing The Proposed SOS Technique

The SOS technique described in the last section was tested by computing load flows of the following five test systems:

1. IEEE 14 bus test system [14,28].
2. Chinese 22 bus test system [20].
3. IEEE 30 bus test system [14,28].
4. IEEE 57 bus test system [14,22,28].
5. IEEE 118 bus test system [14,29,30].

The convergence characteristics of the SOS technique, and the contributions of the second order corrections ([X2] in Equation 3.14) are examined in each case. All load flows were started with bus voltage estimates set at $1.0\angle 0.0^{\circ}$ pu (flat start). A solution was assumed to have converged when the magnitudes of all mismatches were less than 0.0001 pu.

3.4.1 Convergence of load flows

The convergence of the load flows was checked by examining the maximum power and voltage square mismatches. The maximum absolute value of the elements of each correction vector was also determined. These values are denoted as DUmax and DXmax and are defined as

$$DUmax = Maximum(|\Delta P_i|, |\Delta Q_i|, |\Delta |V_i|^2|)$$

$$DXmax = Maximum(|\Delta E_i|, |\Delta F_i|)$$

Table 3.1 lists the maximum absolute mismatches (DUmax) at the beginning of each iteration of the load flows for the selected systems. The mismatches are plotted in Figure 3.1. Table 3.2 shows the maximum corrections (DXmax) at the end of each iteration. It is obvious that, in each system, the solution converges rapidly.

Table 3.1 Maximum mismatches at the beginning of each iteration of the load flows by the SOS technique.

Iteration #	No. of busses in the system				
	14	22	30	57	118
1	0.94200	26.2901	0.94200	3.00000	6.07000
2	0.02100	4.02610	0.02738	0.05371	0.20800
3	0.00000	0.02298	0.00000	0.00000	0.00015
4		0.00000			0.00000

Table 3.2 Maximum corrections at the end of each iteration of the load flows by the SOS technique.

Iteration #	No. of buses in the system				
	14	22	30	57	118
1	0.20492	0.61271	0.31017	0.31749	0.41969
2	0.00235	0.06326	0.00435	0.00646	0.02202
3	0.00000	0.00023	0.00000	0.00000	0.00002
4		0.00000			0.00000

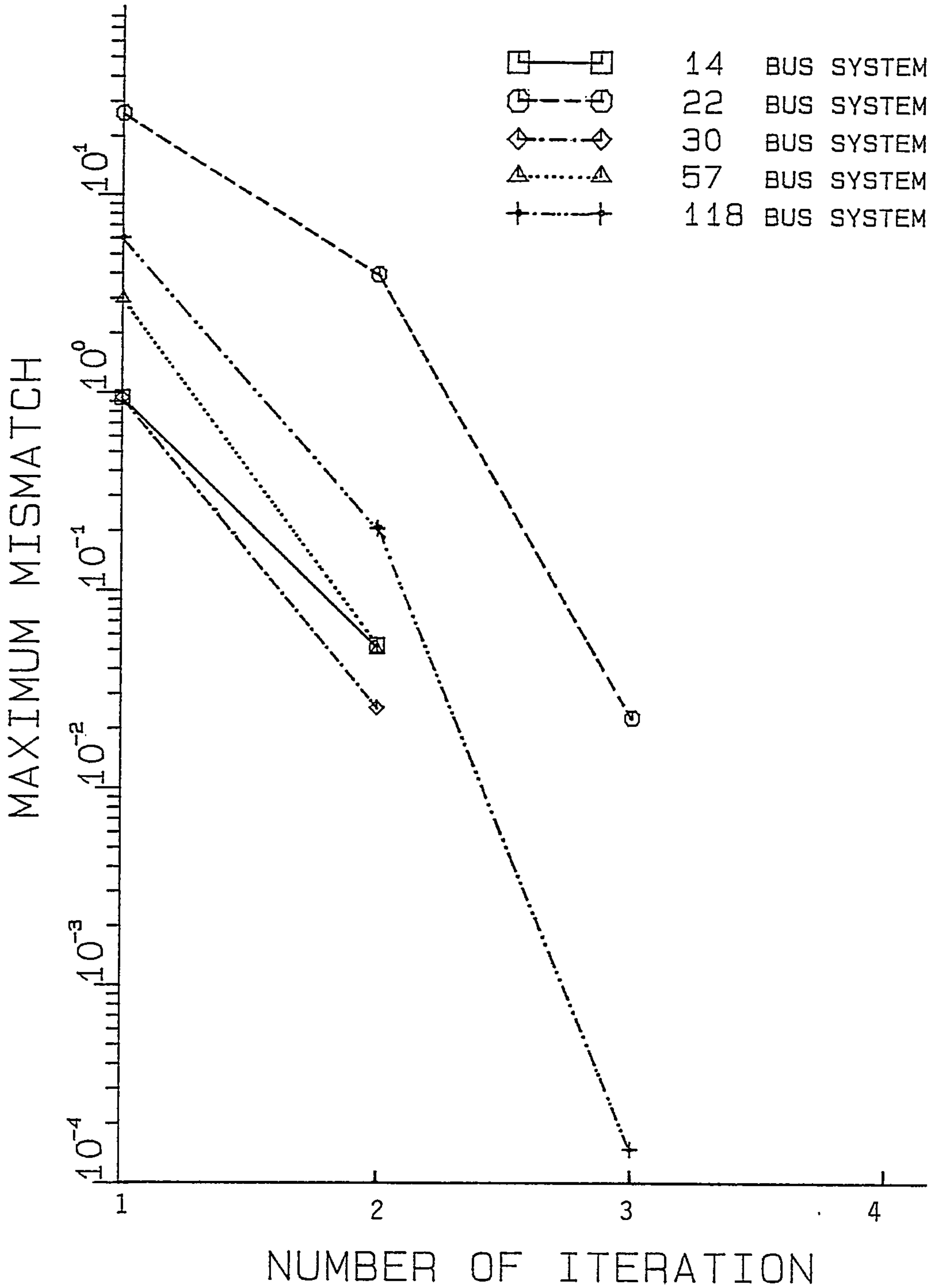


Figure 3.1 Maximum mismatches at the beginning of each iteration of the load flows by the SOS technique.

3.4.2 Comparison of the first and second order corrections

The difference between the proposed and the NR techniques is that the SOS technique uses second order sensitivities in addition to the first order sensitivities used in the NR technique. The corrections due to the second order sensitivities were examined and were compared with those calculated by using the first order sensitivities only. Table 3.3 lists the maximum second order corrections, DDXmax, applied at the end of each iteration; DDXmax is defined as

$$DDXmax = Maximum(|\Delta F2_i|, |\Delta E2_i|)$$

Table 3.3 Corrections due to the second order sensitivities.

Iteration #	No. of buses in the system				
	14	22	30	57	118
1	0.02698	0.33444	0.06324	0.08823	0.08637
2	0.00004	0.00561	0.00004	0.00009	0.00046
3	0.00000	0.00000	0.00000	0.00000	0.00000

The maximum value of the corrections due to the second order sensitivities were compared with those due to the first order sensitivities. The results are listed in Table 3.4. The symbols used in the table are defined as follows.

$$DE = \text{Maximum} (|\Delta E1_i|)$$

$$DDE = \text{Maximum} (|\Delta E2_i|)$$

$$DF = \text{Maximum} (|\Delta F1_i|)$$

$$DDF = \text{Maximum} (|\Delta F2_i|)$$

Table 3.4 Comparison between the corrections due to the first and second order sensitivities.

System	Iter.	DE	DDE	DF	DDF
14	1	0.09405	0.02698	0.21213	0.00720
	2	0.00236	0.00004	0.00031	0.00000
22	1	0.16764	0.33444	0.63312	0.03800
	2	0.01633	0.00561	0.06231	0.00194
	3	0.00023	0.00000	0.00016	0.00000
30	1	0.08536	0.06324	0.32399	0.01383
	2	0.00439	0.00004	0.00428	0.00001
57	1	0.08383	0.08823	0.33121	0.01664
	2	0.00195	0.00005	0.00646	0.00009
118	1	0.05538	0.07132	0.37589	0.08637
	2	0.02169	0.00034	0.01017	0.00046
	3	0.00001	0.00000	0.00002	0.00000

This Table indicates that the order of the magnitudes is the same for the first and second order corrections applied to the real parts of the bus voltages. However the order of the magnitudes is quite different for the first and second order corrections applied to the imaginary parts of the bus