Higher Kaplansky Theory

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Abstract

When studying the structure of a valued field (K, v), immediate extensions are of special interest since they have the same value group and the same residue field as the ground field. One immediate extension that is of particular interest is the henselization $(K, v)^h$ of (K, v) as it is a minimal immediate algebraic extension satisfying Hensel's Lemma, which in turn allows us to study the algebraic structure of a valued field through its residue field.

Kaplansky's work, based on earlier work of Ostrowski, laid the foundations for the understanding of immediate extensions. Here we present a continuation of Kaplansky's work, which allows us to determine special properties of elements in immediate extensions. As a tool to study these properties we introduce the notion of approximation types which represent an alternative, and in some sense an improvement, to the pseudo-convergent sequences used by Kaplansky. As a special interest to F.-V. Kuhlmann's work on henselian rationality over tame fields, we will investigate the question when an immediate valued function field of transcendence degree 1 is henselian rational (i.e., generated, modulo henselization, by one element). Henselian rationality is central in F.-V. Kuhlmann's work on local uniformization which is a local form of resolution of singularities.

Every immediate algebraic approximation type $\mathbf{A}$ over a valued field (K, v), has a class of monic polynomials of minimal degree whose value is not fixed by $\mathbf{A}$. Such polynomials are called associated minimal polynomials for $\mathbf{A}$ and Kaplansky in [4] stated a Theorem 10 indicating that easy normal forms can be determined for these polynomials. By generalizing Kaplansky's approach, we will show in Chapter 5 how such forms can be obtained.

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To my son Naum

Contents

Chapter 1

Introduction

In algebraic geometry an algebraic variety is defined to be the solution set of a system of polynomial equations in n variables. The points where such a variety fails to be smooth are called singularities. At those points the Implicit Function Theorem cannot be used to compute the nearby solutions. As a consequence, it is hard (even for computers) to describe correctly the local shape of the variety at its singular points.

Singularities appear in many different forms outside of mathematics as well. They appear in robotics when we consider for example the motions of a robotic arm in a given configuration as the positions the arm would reach if it could cross itself. They also appear in general relativity as spacetime singularities. A spacetime with a singularity is defined to be one that contains geodesics that cannot be extended in a smooth manner. The end of such a geodesic is considered to be the singularity. In an optical field, a phase singularity (or an optical vortex) appears as a point of zero intensity. Light can be twisted like a corkscrew around its axis of travel. Because of the twisting, the light waves at the axis itself cancel each other out. When projected onto a flat surface, an optical vortex looks like a ring of light, with a dark hole in the center. This corkscrew of light, with darkness at the center, is called an optical vortex.

Having established that singularities are a common problem for many areas of research it is important to be able to resolve them mathematically. Resolution of singularities is a method to understand where singularities come from, what they look like and what their internal structure is.

Resolution is well established over fields of characteristic zero, but still unknown in positive characteristic. One reason to study the positive characteristic case is because algebraic number theory, while dealing with the integers which live in characteristic 0, studies them by reduction modulo prime numbers and so brings in positive characteristic. Also many virtual results in number theory and arithmetic are just waiting to become true by having at hand resolution in positive characteristic.

One way of attacking the question of resolution in positive characteristic is to work on solving a local form of resolution of singularities, which is called local uniformization, and this is where valuation theory comes in. A place P of an algebraic function field $F|K$ is said to admit local uniformization if there exists a K-variety X having F as its field of rational functions and such that the center $x \in X$ of P on X is a regular point. In [21], Zariski proved the Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. In [22], he uses this theorem to prove resolution of singularities for algebraic surfaces in characteristic 0, later on generalized to positive characteristic by Abhyankar [1]. So now one is interested in generalizations of the Local Uniformization Theorem to positive characteristic and in this quest the very difficult task of elimination of wild ramification in valued function fields plays a very important role (see [5] and [6]). To be able to study elimination of ramification it is necessary to have a deep knowledge about the structure of valued function fields, for which it is essential to study immediate extensions of valued fields. This thesis presents a continuation of the work of Kaplansky [4] in which, based on earlier work of Ostrowski [15], he laid the foundations for an understanding of immediate extensions of valued fields.

The theory developed by Kaplansky and Ostrowski is very useful for valuations with residue fields of characteristic 0, but its real strength (as well as its limitations) become visible when the residue characteristic is positive.

While Kaplansky was mainly concerned with embeddings in power series fields and the question when maximal immediate extensions are unique up to isomorphism, the above mentioned problems have added new questions to the spectrum. We further developed Kaplansky's tools here in order to be able to answer various questions about the structure of immediate function fields. Several results of [13] (a joint paper with F.-V. Kuhlmann), presented in Chapter 4 of this thesis, are indispensable for the paper [12] on henselian rationality, which is central in F.-V. Kuhlmann's work on elimination of wild ramification and local uniformization (see [6]), as well as the model theory of valued fields (see [11]).

The main theorem of [12] states that every immediate function field $(F|K, v)$ of transcendence degree 1 over a tame field (K, v) is henselian rational (for the definitions of "henselian rational" and "tame" see end of Sections 2.7 and 2.8). For the proof of this theorem, one first reduces the problem to the case of valued fields of rank 1 (i.e., having archimedean ordered value groups), and then starts with an arbitrary element $x \in F$ transcendental over K; it can be chosen such that $F|K(x)$ is separable. If x is not a henselian generator, then $(F^h|K(x)^h, v)$ is a proper finite immediate extension. Let us describe the further steps of the proof in the important special case where $\operatorname{char} K = p > 0$. If one replaces $(F|K, v)$ by the valued function field $(F.K^r|K^r, v)$, which again is immediate, then the extension $((F.K^r)^h|K^r(x)^h, v)$ becomes a tower of Artin-Schreier extensions. (To facilitate notation, let us write again $F|K$ in place of $F.K^r|K^r$.) The lowest of them is shown to be generated by a root y of a polynomial $X^p - X - f(x)$ where p is the residue characteristic and $f(x) \in K[x]$. We observe that $f(x) = y^p - y \in K(y)$, hence if $K(x)^h = K(f(x))^h$, then $K(x)^h \subsetneqq K(y)^h$. Replacing x by y, we have then reduced the degree of $F^h|K(x)^h$ by a factor of p. This shows that it is crucial to determine the degree $[K(x)^h : K(f(x))^h]$ for a given $f(x) \in K[x]$ and to choose $f(x)$ in such a way that the degree becomes 1.

In order to gain insight on the degree $[K(x)^h : K(f(x))^h]$, we study the elements $f(x) \in K[x]$ in (not necessarily transcendental) immediate extensions $(K(x)|K, v)$, through extending Kaplansky's technical lemmas. This study is carried out in Sections 3.3 to 3.6. In Section 3.5, we define the "relative approximation degree of $f(x)$ in x" to be the integer h that appears in Kaplansky's Lemma 8. We then show in Theorem 4.1.1 that under suitable assumptions about the extension $(K(x)|K, v)$ and the element $f(x)$, the degree $[K(x)^h : K(f(x))^h]$ is bounded from above by the relative approximation degree of $f(x)$ in x.

Having proved (in [12]) that the immediate function field $(F.K^r|K^r(x),v)$ is henselian rational, one has to pull this property down to $(F|K,v)$. Observe that if $(F.K^r|K^r(x),v)$ is henselian rational, then the same already holds for $(F.L|L(x),v)$, for a suitable finite subextension $L|K$ of $K^r|K$. Moreover, $L|K$ can be chosen to be Galois since also $K^r|K$ is Galois (we allow Galois extensions to be infinite). An extension of a henselian field (K,v) is called tame if it lies in K^r. Consequently, a Galois extension is tame if and only if its ramification group is trivial. So what we need is a pull down principle for henselian rationality through finite tame extensions of the base field. This is presented in Theorem 4.6.5. More precisely, we show in Section 4.6 that if x is a henselian generator for $(F.L|L,v)$, where $(L|K,v)$ is a finite tame Galois extension, then for a suitable element $d \in L$, the trace $\mathrm{Tr}(d \cdot x)$ is a henselian generator for $(F|K,v)$. We use a valuation theoretical characterization of the Galois groups of tame Galois extensions that is developed in Section 4.5.

Once a henselian generator $x \in F^h$ is found, the question arises whether x can already be chosen in F. We show in Theorem 4.3.1 that this can be done. This result is crucial for the proof given in [6] that local uniformization can always be achieved after a finite Galois extension of the function field. In order to prove Theorem 4.3.1, we generalize the relative approximation degree to other elements $y \in K(x)^h$ in place of $f(x)$ in Section 4.2. We then prove the corresponding generalization of Theorem 4.1.1: Theorem 4.2.7 states that under suitable assumptions, we again have that the degree $[K(x)^h : K(y)^h]$ is smaller than or equal to the relative approximation degree of y in x.

Theorem 4.3.1 can be seen as a special case of a "dehenselization" procedure (analogous to the "decompletion" used by M. Temkin in [18]). If for a given valued function field $(F|K,v)$ there is a finite extension F' of F within its henselization such that $(F'|K,v)$ admits local uniformization, one would like to deduce that also $(F|K,v)$ admits local uniformization. This can be done if Theorem 4.3.1 can be generalized in a suitable way to the case of non-immediate valued function fields.

Our investigation of the properties of elements in immediate extensions is facilitated by the introduction of the notion of "approximation type", which we use in place of Kaplansky's "pseudo-convergent sequences" (also called "pseudo-Cauchy sequences" or "Ostrowski nets" in the literature). This new notion makes computations and the formulation of results easier. For instance, to every element x in an immediate extension $(L|K,v)$, we associate the unique approximation type of x over K, while there are many pseudo-convergent sequences in K that have x as a pseudo-limit, and in addition one needs to require maximality of such sequences (for $x \notin K$ one asks that they do not have a pseudo limit in K). Furthermore, the definition of approximation types is not restricted to immediate extensions only. In fact, approximation types can be further enhanced to a tool for describing properties of elements in non-immediate extensions. In Section 3.4, we take the occasion to show how Kaplansky's fundamental Theorems 2 and 3 can be proved by using approximation types in place of pseudo-convergent sequences.

It is important to mention that as part of the research work I did under the supervision of Prof. F.-V. Kuhlmann, I mainly worked on valuation theoretic results that previously appeared in his doctoral thesis [7] but were in need of revision and improvement in order to be ready for publication. As an initial task I made sure all tools involved were absolutely necessary for the main results to be presented; this resulted in discarding a significant amount of unnecessary material. Afterwards I worked on making the theory more accessible by using the modern version of definitions (where such exist) and by helping to develop the optimal notation. I also helped simplify the proofs of several results. I found several mistakes and gaps in the old proof of Theorem 4.1.1 and helped to develop the new Theorem 4.3.1. I also developed some other results that do not appear in F.-V. Kuhlmann's thesis (at least not in this form) among which are Lemma 5.1.1, Lemma 5.1.3, Lemma 5.2.1, Lemma 5.4.2, Theorem 5.4.9.

Chapter 2

Preliminaries

This chapter gives a short presentation of the basic facts from algebra and valuation theory that are needed for the work presented in the subsequent chapters so as to allow an uninterrupted reading of this thesis. The attempt was to be as explicit as possible without going into too much detail; most of the theorems do not have their proofs included for the sake of brevity, but appropriate references for the interested reader have been made. For a more detailed presentation on valuation theory we refer the reader to [2], [3], [17], [19], [20].

2.1 Ordered sets

Take a linearly ordered set $(S,<)$ and two subsets D and E of S. For an element $a \in S$, we will write $a < E$ if $a < b$ for all $b \in E$, and we will write $a \leq E$ if $a \leq b$ for all $b \in E$. Similarly, we will write $D < a$ if $c < a$ for all $c \in D$ and $D \leq a$ if $c \leq a$ for all $c \in D$. Finally, we will write $D < E$ if $c < E$ for all $c \in D$ (or $D < b$ for all $b \in E$) and $D \leq E$ if $c \leq E$ for all $c \in D$.

A subset T of S is said to be **convex in** $(S,<)$ if for all $a, b \in T$ and $c \in S$, with $a \leq c \leq b$, we have that $c \in T$. A subset D of S is called an **initial segment of** S if for all $a \in D$ and $c \in S$, with $c \leq a$, we have that $c \in D$. Symmetrically, E is called a **final segment of** S if for all $a \in E$ and $c \in S$ with $a \leq c$, we have that $c \in E$. A set F is said to be **cofinal in** S if for every element $a \in S$ there is an element $b \in F$ such that $a \leq b$.

If D and E are subsets of S such that $D < E$ and $D \cup E = S$, we call the pair (D,E) a **cut** in S. Then D is an initial segment of S, E is a final segment of S, and D and E have an empty intersection. If $(S',<)$ is an extension of $(S,<)$ and $a \in S'$ such that $D \leq a \leq E$, then we will say that a **realizes** (D,E) (in $(S',<)$). Often we will identify cuts with the elements realizing them.

If C and C' are two cuts in a linearly ordered set S defined by their lower cut sets D and D', respectively, then $C = C'$ if $D = D'$, and we write $C < C'$ if $D \subsetneq D'$. For an element $a \in S$ we write $a > C$ if $a > b$ for all $b \in D$, and $a \geq C$ if $a \geq b$ for all $b \in D$; note that if D has no last element, then $a > C \Leftrightarrow a \geq C$. We write $a \leq C$ if $a \in D$, and $a < C$ if $a \in D$ but is not the last element of D.

We will say that $(S,<)$ is **dense** if for every two distinct elements of S there is a third element of S strictly between them. This holds if and only if there are no cuts (D,E) in S for which D has a last element and E has a first element. If $a < b$ and there is no element strictly between a and b, then b is called the **immediate successor** of a, and a is called the **immediate predecessor** of b. Further, $(S, <)$ is called **discretely ordered** if for every $a \in S$ there is an immediate successor given that a is not the last element in S, and there is an immediate predecessor of a given that a is not the first element in S. Note that the properties "dense" and "discretely ordered" are mutually exclusive if S has more than one element.

2.2 Ordered abelian groups

Given a linear ordering $<$ on an abelian group G, we call $(G, <)$ an **ordered abelian group** if the ordering is compatible with the group operation, that is, if for every $x, y, z \in G$ with $x < y$ we have that $x + z < y + z$. Now, for a cut $C = (D, E)$ in G, we can define $-C$ to be the cut $(-E, -D)$, where $-D = \{-d \mid d \in D\}$, and similarly for $-E$. For an element $g \in G$, we define $g + C$ to be the cut $(g + D,\ G \setminus g + D)$, where $g + D = \{g + d \mid d \in D\}$. For two cuts $C_1 = (D_1, E_1)$ and $C_2 = (D_2, E_2)$ in G, we define the sum $C_1 + C_2$ to be the cut $(D_1 + D_2, G \setminus (D_1 + D_2))$, where $D_1 + D_2 = \{d_1 + d_2 \mid d_1 \in D_1\ \wedge\ d_2 \in D_2\}$.

If $a \in G$, then a^+ is the cut whose lower cut consists of all elements in G less than or equal to a and a^- is the cut whose lower cut set consists of all elements less than a. Cuts of this form are called **principal cuts**.

Lemma 2.2.1. *Let C be a cut in G. Then C is principal if and only if kC is principal, for $k \in \mathbb{N}$.*

Proof. Let D and D' denote the lower cut sets of C and kC respectively.

If $C = a^+$ for some $a \in G$, then the cofinality of kD implies that $D' = \{g \in G \mid g \leq ka\}$. Indeed, if D' would contain an element $b > ka$ then D would contain an element g such that $kg \geq b$ which would imply that $g > a$ as $k > 0$, a contradiction. If $C = a^-$ then $-C = (-a)^+$ and by what we have already shown, $-kC = k(-C) = (k(-a))^+ = (-ka)^+$. From here we get that $kC = (ka)^-$.

Suppose now that $kC = b^+$ for some $b \in G$. Since kD is cofinal in D', there is an element $a \in D$ such that $b \leq ka \in kD \subseteq D'$, thus also $ka \leq b$. It follows that $b = ka$ and $C = a^+$. If $kC = b^-$ then as above $-kC = (-b)^+$. Then by what we have already shown, there is $c \in G$ such that $-C = (-c)^+$ which in turn implies that $C = c^-$. □

An abelian group $(G, +)$ is said to be **torsion free** if for every natural number n we have that $ng \neq 0$ whenever $g \neq 0$. The group G is said to be **divisible** if for every natural number n and every element $g \in G$ there is an element $h \in G$ such that $nh = g$ and this element h is unique if G is torsion free. If G is divisible and torsion free we will denote the element h by g/n. Every torsion free abelian group G admits a **divisible hull**, that is, a divisible extension group $\tilde{G}$ (i.e., a group in which G embeds) with the universal property that $\tilde{G}$ admits a unique embedding in every other divisible extension group of G. If we agree to identify isomorphic structures (which we will normally do), we may define *the* divisible hull of a torsion free abelian group G to be:

$$\tilde{G} := \left\{ \frac{g}{n} \mid g \in G, n \neq 0 \right\} .$$

Lemma 2.2.2. *Every ordered abelian group is torsion free.*

Proof. Take $0 \neq g \in G$. We have two cases, namely $g > 0$ or $g < 0$. If $g > 0$ and we assume that $ng > 0$ for a natural number n, then we have that $(n+1)g = ng + g > 0 + g > 0$. Similarly, if $g < 0$ and we assume that $ng < 0$, then $(n+1)g = ng + g < 0 + g < 0$. □

If G is an ordered abelian group and $\tilde{G}$ is the divisible hull of G, then there is a unique extension of the ordering from G to $\tilde{G}$ that makes $\tilde{G}$ into an ordered abelian group; namely, for $g, h \in G$ and $m, n > 0$ we set:

$$\frac{g}{m} < \frac{h}{n} \iff ng < mh \; .$$

Given an ordered abelian group G and a subgroup H of G, we call H a **convex subgroup** of G if it is convex as a subset of G. Now suppose that H is a convex subgroup of G. Then the canonical epimorphism $\eta : G \to G/H$ induces an ordering on the quotient group G/H, i.e., for two elements $g, h \in G$ we have that

$$g \leq h \implies g + H \leq h + H \; ,$$

and so, G/H is again an ordered abelian group.

We define the **rank** of an ordered abelian group G to be the order type of the chain of proper convex subgroups of G, ordered by inclusion. In particular, if $\{0\}$ is the only proper convex subgroup of G, we say that G is of rank 1. An ordered abelian group G is called **Archimedean** if for any two positive elements $g, h \in G$ there is a natural number n such that $ng > h$. Otherwise, G is called non-Archimedean. It is clear that an Archimedean ordered group has no non-trivial convex subgroups, thus it is of rank 1. The converse is also true and it follows from the subsequent theorem.

Theorem 2.2.3. (Proposition 2.2.1 of [3])
An ordered abelian group is of rank 1 if and only if it admits an order preserving embedding in $(\mathbb{R}, <)$.

Let G be a torsion-free abelian group and take $g_1, \ldots, g_n \in G$. We call $g_1, \ldots, g_n$ **rationally independent** if for integers $k_1, \ldots, k_n \in \mathbb{Z}$ we have that

$$k_1 g_1 + \ldots + k_n g_n = 0 \implies k_1 = \ldots = k_n = 0 \; .$$

If $H \subset G$ is an extension of torsion-free abelian groups, we call $g_1, \ldots, g_n \in G$ **rationally independent over** H if for integers $k_1, \ldots, k_n \in \mathbb{Z}$ we have that

$$k_1 g_1 + \ldots + k_n g_n \in H \implies k_1 = \ldots = k_n = 0 \; .$$

2.3 Valued fields

Take a field K and suppose that $<$ is a linear ordering on K. We call $(K, <)$ an **ordered field** if $(K, +, <)$ is an ordered abelian group and the ordering is compatible with the multiplication, that is, for each $x, y, z \in K$ with $x < y$ and $z > 0$, we have that $xz < yz$.

Take a field K and an ordered abelian group Γ. We adjoin the symbol ∞ to Γ and extend the ordering on Γ to an ordering of $\Gamma \cup \{\infty\}$ by setting ∞ to be greater than every element in Γ. We also extend the addition by adopting the rule that for every $\gamma \in \Gamma$ we have that $\gamma + \infty = \infty = \infty + \gamma$.

Let $v : K \longrightarrow \Gamma \cup \{\infty\}$ be a surjective map satisfying:

1. $vx = \infty$ if and only if $x = 0$
2. $v(x + y) \geq \min\{vx, vy\}$ (ultrametric triangle law)
3. $v(xy) = vx + vy$ (Homomorphism property).

Then v is called a (field) **valuation** and (K, v) is called a **valued field**.

Proposition 2.3.1. *If (K, v) is a valued field, then:*

a) $v(1) = 0$

b) $v(a^{-1}) = -va$, $a \neq 0$

c) $v(-a) = va$

d) If $vx \neq vy$ then $v(x + y) = \min\{vx, vy\}$.

Proof. a): Using the homomorphism property of valuations, we have that

$$v(1) = v(1 \cdot 1) = v(1) + v(1)$$

which implies that $v(1) = 0$.

b): Using the homomorphism property again, we have that

$$0 = v(1) = v(a \cdot a^{-1}) = va + v(a^{-1}) \ .$$

This implies that $v(a^{-1}) = -va$.

c): In view of

$$0 = v(1) = v((-1) \cdot (-1)) = 2v(-1)$$

we have that $v(-1) = 0$. This implies that

$$v(-a) = v((-1) \cdot a) = v(-1) + va = va \ .$$

d): Suppose that $vx < vy$. Then using c) we have that

$$vx = v(x + y - y) \geq \min\{v(x + y), vy\} \ .$$

Since by assumption $vx < vy$ we must have that $\min\{v(x + y), vy\} = v(x + y)$. Now,

$$vx \geq v(x + y) \geq \min\{vx, vy\} = vx \ .$$

We get that $v(x + y) = vx = \min\{vx, vy\}$, as desired. □

Given a valued field (K, v), we define the **value group** of (K, v) to be

$$vK := \{va \mid a \in K \setminus \{0\}\} .$$

From the definition of a valuation it is clear that vK is an ordered abelian group. If $vK = \{0\}$, we call v the **trivial valuation**.

If vK has rank 1, we call v a **rank 1 valuation**. More generally, we define the **rank of a valued field** to be the rank of its value group.

Next, we define the **valuation ring of** v to be the set

$$\mathcal{O}_v := \{a \in K \mid va \geq 0\} .$$

Proposition 2.3.2. *The valuation ring $\mathcal{O}_v$ is a subring of K with 1.*

Proof. By the previous proposition $v(1) = 0$ and so $1 \in \mathcal{O}_v$. Now take $a, b \in \mathcal{O}_v$. Then $va, vb \geq 0$ which implies that

$$v(a - b) \geq \min\{va, v(-b)\} = \min\{va, vb\} \geq 0 ,$$

and also

$$v(ab) = va + vb \geq 0$$

hence $a - b,\ ab \in \mathcal{O}_v$. □

Proposition 2.3.3. *For every $a \in K$ we have that $a \in \mathcal{O}_v$ or $a^{-1} \in \mathcal{O}_v$. If $va = 0$ then $a \in \mathcal{O}_v^{\times}$.*

Proof. If $a \notin \mathcal{O}_v$ it follows that $va < 0$. Then part b) of Proposition 2.3.1 implies that $v(a^{-1}) = -va > 0$ and so $a^{-1} \in \mathcal{O}_v$. If $va = 0$ then $v(a^{-1}) = -va = 0$, hence both $a, a^{-1} \in \mathcal{O}_v$ and a is a unit in $\mathcal{O}_v$. □

Proposition 2.3.4. *The set*

$$\mathcal{M}_v := \{a \in K \mid va > 0\}$$

is the unique maximal ideal of $\mathcal{O}_v$.

Proof. To show that $\mathcal{M}_v$ is an ideal, take $a, b \in \mathcal{M}_v$. Then we have that $v(a - b) \geq \min\{va, vb\} > 0$ which means that $a - b \in \mathcal{M}_v$. If $a \in \mathcal{O}_v$ and $b \in \mathcal{M}_v$, then $v(ab) = va + vb > 0$, thus $ab \in \mathcal{M}_v$.

Using the preceding proposition we have that

$$\mathcal{O}_v^{\times} = \{a \in K \mid va = 0\} = \mathcal{O}_v \setminus \mathcal{M}_v .$$

In other words, $\mathcal{M}_v$ consists of the non-units of $\mathcal{O}_v$ which implies that $\mathcal{M}_v$ is the unique maximal ideal of $\mathcal{O}_v$. □

From the preceding proposition we have that $\mathcal{O}_v/\mathcal{M}_v$ is a field, which we call the **residue field** of (K, v) and denote by Kv. Also, we call the map

$$\mathcal{O}_v \ni x \longmapsto xv \in \mathcal{O}_v/\mathcal{M}_v$$

the **residue map** of the valuation.

Note that

$$xv = 0 \iff vx > 0$$

and

$$xv \neq 0 \iff vx = 0 \; .$$

Let K be a field and $\mathcal{O}$ a subring of K. We call $\mathcal{O}$ a **valuation ring** if for every $a \in K$ we have that $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$. Again we have that the subset of $\mathcal{O}$ consisting of the non-units of $\mathcal{O}$ is the unique maximal ideal of $\mathcal{O}$. As before, we denote it by $\mathcal{M}$. Then $\mathcal{O}$ defines a valuation v on K. Set

$$\Gamma := K^\times / \mathcal{O}^\times$$

and define addition on it by setting

$$x\mathcal{O}^\times + y\mathcal{O}^\times := xy\mathcal{O}^\times \; .$$

Next we introduce an ordering on Γ by setting

$$x\mathcal{O}^\times \leq y\mathcal{O}^\times \iff \frac{y}{x} \in \mathcal{O}$$

This turns Γ into an ordered abelian group.

Now we define $v : K \longrightarrow \Gamma \cup \{\infty\}$ by setting $v(0) := \infty$ and $vx := x\mathcal{O}^\times$ for $x \in K^\times$. Then v satisfies the homomorphism property:

$$v(xy) = xy\mathcal{O}^\times = x\mathcal{O}^\times + y\mathcal{O}^\times = vx + vy \; .$$

Next we show that v satisfies the ultrametric triangle law. Suppose that $vx \leq vy$, i.e., $\frac{y}{x} \in \mathcal{O}$. Then

$$\frac{x+y}{x} = 1 + \frac{y}{x} \in \mathcal{O} \; .$$

From here,

$$\min\{vx, vy\} = vx = x\mathcal{O}^\times \leq (x+y)\mathcal{O}^\times = v(x+y) \; .$$

Finally,

$$\mathcal{O}_v = \{x \in K^\times \mid vx \geq 0\} = \{x \in K^\times \mid vx \geq v(1)\} = \{x \in K^\times \mid \frac{x}{1} \in \mathcal{O}\} = \mathcal{O} \; .$$

In view of the above we call two valuations **equivalent** if they have the same valuation ring.

If v and w are two valuations on a field K and $\mathcal{O}_v$ and $\mathcal{O}_w$ are their valuation rings, then w is called a **coarsening** of v if $\mathcal{O}_v \subseteq \mathcal{O}_w$. If this holds, then $vc \geq vd$ implies $wc \geq wd$ and in particular, $vc \geq 0$ implies $wc \geq 0$ and $wd > 0$ implies $vd > 0$.

2.4 Extensions of valuations

In this section we will show that for every valuation v of a field K and every field extension L of K there is a valuation w of L lying over v, i.e., such that the restriction of w to K equals v. We denote this extension by

$(K, v) \subset (L, w)$ and call it an **extension of valued fields**. That such extensions exist is a direct consequence of Chevalley's Theorem:

Theorem 2.4.1. (Theorem 3.1.1 of [3])
For a field K, let $R \subset K$ be a subring and let $\wp \subset R$ be a prime ideal of R. Then there exists a valuation ring $\mathcal{O}$ of K such that

$$R \subset \mathcal{O} \text{ and } \mathcal{M} \cap R = \wp.$$

Let $L|K$ be a field extension, and $\mathcal{O}_K \subset K$, $\mathcal{O}_L \subset L$ be valuation rings. We say that $\mathcal{O}_L$ is an extension of $\mathcal{O}_K$ if $\mathcal{O}_L \cap K = \mathcal{O}_K$. Since valuation rings define valuations we will also write $(K, \mathcal{O}_K) \subset (L, \mathcal{O}_L)$ to indicate an extension of valued fields.

Take a field extension $L|K$, valuation rings $\mathcal{O}_K \subset K$, $\mathcal{O}_L \subset L$ and let $\mathcal{O}_L$ be an extension of $\mathcal{O}_K$. Let $\mathcal{M}_K$, $\mathcal{M}_L$ be the maximal ideals of $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively. Then, the following hold:

$$\mathcal{M}_L \cap K = \mathcal{M}_L \cap \mathcal{O}_K = \mathcal{M}_K$$

$$\mathcal{O}_L{}^\times \cap K = \mathcal{O}_L{}^\times \cap \mathcal{O}_K = \mathcal{O}_K{}^\times \ .$$

For a field extension $L|K$ and a valuation ring $\mathcal{O}_L$ of L, one also sees that $\mathcal{O}_L \cap K$ is a valuation ring of K. This means that if v is a valuation on L then the restriction of v to K is a valuation on K. Now, by $(L|K, v)$ we will denote a valued field extension where v is the valuation on L and its restriction is the valuation on K which we will again denote by v.

Theorem 2.4.2. (Theorem 3.1.2 of [3])
Let $L|K$ be a field extension, and let $\mathcal{O}_K \subset K$ be a valuation ring. Then there is an extension $\mathcal{O}_L$ of $\mathcal{O}_K$ in L.

Proof. Since $\mathcal{O}_K$ is a subring of L, according to Chevalley's Theorem there exists a valuation ring $\mathcal{O}_L$ of L with $\mathcal{O}_K \subset \mathcal{O}_L$ and $\mathcal{M}_L \cap \mathcal{O}_K = \mathcal{M}_K$ for the maximal ideals. Also, as mentioned above, $\mathcal{O}_L{}^\times \cap K = \mathcal{O}_K{}^\times$. Then since $\mathcal{O}_L \cap K$ and $\mathcal{O}_K$ are valuation rings with the same maximal ideal and the same group of units, the two rings must coincide. □

Lemma 2.4.3. (Lemma 3.1.5 of [3])
Let $L|K$ be a field extension, and let $\mathcal{O}'$ be a valuation ring of L. Then every valuation ring $\mathcal{O} \supseteq \mathcal{O}' \cap K$ of K can be extended to a valuation ring $\mathcal{O}'' \supseteq \mathcal{O}'$ on L.

2.5 Immediate extensions

An extension $(L|K, v)$ is said to be **immediate** if the canonical embeddings $vK \hookrightarrow vL$ of the value groups and $Kv \hookrightarrow Lv$ of the residue fields are onto.

The following lemma gives an alternative definition for immediate extensions of valued fields that in turn can be used to define immediate extensions of other valued structures, such as valued abelian groups and valued vector spaces.

Lemma 2.5.1. *The extension $(L|K,v)$ is immediate if and only if for every element $a \in L$ there is an element $c \in K$ such that $v(a-c) > va$.*

Proof. Suppose that $(L|K,v)$ is immediate and take $a \in L$. Then $va \in vL = vK$ so there is some $b \in K$ such that $va = vb$. Then $v(\frac{a}{b}) = va - vb = 0$ and so $0 \neq (\frac{a}{b})v \in Lv = Kv$. Thus there is $d \in K$ such that $(\frac{a}{b})v = dv$. Then $\left(\frac{a}{b} - d\right)v = \left(\frac{a}{b}\right)v - dv = 0$ which implies that $v\left(\frac{a}{b} - d\right) > 0$, and from here $v(a - bd) > vb = va$. Setting $c := bd$ gives the first direction.

Conversely, suppose the latter holds. Take any $\alpha \in vL$ and pick an $a \in L$ with $va = \alpha$. Also, pick $c \in K$ such that $v(a-c) > va$. Then $vc \geq \min\{v(c-a), va\}$. Since $v(a-c) > va$ it follows that $vc = va$, so $\alpha = va = vc \in vK$ and hence $vL = vK$. To show that the residue fields are the same, take an element $\zeta \in Lv$ and pick an element $a \in L$ with $va = 0$ and $av = \zeta$. Next pick $c \in K$ such that $v(a-c) > va = 0$, implying that $av - cv = (a-c)v = 0$. Hence we have that $\zeta = av = cv \in Kv$ and $Lv = Kv$. □

A valued field is called **maximal** if it has no proper immediate extensions and **algebraically maximal** if it has no proper immediate algebraic extensions.

2.6 Algebraic extensions

By $\tilde{K}$ we will denote the algebraic closure of K. For each extension of v to $\tilde{K}$, we have that $\tilde{K}v = \widetilde{Kv}$, and $v\tilde{K}$ is the divisible hull of vK, which we denote by $\widetilde{vK}$. Throughout this section suppose that $L|K$ is an algebraic field extension and let v be a valuation on K. In this section we look at relations between the possible extensions of v from K to L. We also describe some of the relations between the value groups and residue fields.

Theorem 2.6.1. *Let $L|K$ be an algebraic field extension and let v be a valuation on K. Then all extensions of v from K to L are conjugate. In other words, if $\tilde{v}$ is an extension of v to the algebraic closure $\widetilde{K}$ of K, then the set*

$$\left\{\tilde{v} \circ \sigma_{|_L} \mid \sigma \in \mathrm{Gal}\,(\widetilde{K}|K)\right\}$$

is the set of all valuations on L extending v.

Corollary 2.6.2. *Let $N|K$ be a finite normal extension and let v be a valuation on K. Then all extensions of v to N have the same value group and the same residue field.*

Corollary 2.6.3. *If $L|K$ is a finite extension, then there are at most $[L:K]_{\mathrm{sep}}$ many extensions of the valuation v on K to L.*

Corollary 2.6.4. *If $L|K$ is a purely inseparable extension, then v has a unique extension to L.*

Proof. This corollary follows immediately from the preceding one, and the value of each element in L can easily be computed: As we know that v extends uniquely to a valuation on L, let us use v to denote both the valuation on K and its extension to L. Now take $b \in L$. Then there is some integer $n \geq 0$ such that $b^{p^i} \in K$ for some prime number p. So we have that $vb^{p^i} = p^i vb$ which implies that

$$vb = \frac{1}{p^i} vb^{p^i}$$

. □

Theorem 2.6.5. *Take a finite valued field extension $(L|K, v)$. Then we have that $(vL : vK) < \infty$, $[Lv : Kv] < \infty$ and*

$$[L : K] \geq (vL : vK) \cdot [Lv : Kv] \ .$$

2.7 Henselian fields

We call a valued field (K, v) **henselian** if v has a unique extension to $\widetilde{K}$. If (K, v) is henselian, then certainly v extends uniquely to every field L algebraic over K. Conversely, (K, v) is henselian if v extends uniquely to every finite extension of K. If K^{sep} denotes the separable algebraic closure of K, then $\widetilde{K}|K^{\text{sep}}$ is a purely inseparable extension, and as noted in the previous section, any valuation on K^{sep} extends uniquely to $\widetilde{K}$. In view of this, we see that (K, v) is henselian already if the extension of v to K^{sep} is unique.

Theorem 2.7.1. *Saying that a valued field (K, v) has the property of being henselian is equivalent to the following properties:*

a) Hensel's Lemma: If $f(X)$ is a polynomial with coefficients in the valuation ring $\mathcal{O}$ and $b \in \mathcal{O}$ with $vf(b) > 0$ and $vf'(b) = 0$ then there exists a unique $a \in \mathcal{O}$ such that $f(a) = 0$ and $v(a - b) > 0$.

b) The strong Hensel's Lemma: If $f(X) \in \mathcal{O}[X]$ and $f(X)v = g(X)v \cdot h(X)v$, for some polynomials $g(X), h(X) \in K[X]$, then there are polynomials $g_1(X), h_1(X) \in \mathcal{O}[X]$ such that $g_1(X)v = g(X)v$, $h_1(X)v = h(X)v$, $deg(g_1) = deg(g)$ and $f(X) = g_1(X)h_1(X)$.

c) Krasner's Lemma: If for an element $a \in K^{\text{sep}}$ there exists an element $b \in K^{\text{sep}}$ such that

$$v(a - b) > \max\{v(a - \sigma a) \mid \sigma \in \text{Gal}\,(K^{\text{sep}}|K) \wedge a \neq \sigma a\}$$

then $a \in K(b)$.

Take a valued field (K, v) and a sequence $(a_n)_{n\in\mathbb{N}}$ in K. We say that $(a_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if for each value $\gamma \in vK$ there is a natural number $N \in \mathbb{N}$ such that for every $n > N$ we have that $v(a_n - a_{n+1}) > \gamma$. We call a a **limit** of $(a_n)_{n\in\mathbb{N}}$ if for each value $\gamma \in vK$ there is a natural number $N \in \mathbb{N}$ such that for every $n > N$ we have that $v(a - a_n) > \gamma$. We call a valued field K **complete** if every Cauchy sequence in K has a limit in K and we denote its completion by K^c.

Theorem 2.7.2. *If (K, v) is a complete valued field of rank 1, then it is henselian.*

Theorem 2.7.3. *Every maximal field is henselian.*

Lemma 2.7.4. (Lemma of Ostrowski, cf. [2], [17]) *Suppose that (K, v) is henselian and let $(L|K, v)$ be a finite valued field extension. Then*

$$[L : K] = p^i \cdot (vL : vK) \cdot [Lv : Kv]$$

for $p = \max\{1, \mathrm{char}(Kv)\}$ and i a nonnegative integer.

Corollary 2.7.5. *If $\mathrm{char}(Kv) = 0$ and (K, v) is henselian then we have that*

$$[L : K] = (vL : vK) \cdot [Lv : Kv] \ .$$

Corollary 2.7.6. *If $\mathrm{char}(Kv) = 0$ and (K, v) is henselian then it is algebraically maximal.*

Every valued field (K, v) admits a minimal algebraic immediate extension in which Hensel's Lemma holds, i.e., an extension which embeds in every henselian field extending (K, v). As we have agreed to identify isomorphic structures, we call such a minimal extension the **henselization** of (K, v) and denote it by $(K, v)^h$ or just by K^h. Recall that "Hensel's Lemma" states that a polynomial with coefficients in the valuation ring has a simple root if its reduction has a simple root in the residue field. This lemma allows many algebraic problems of a henselian field to be reduced to problems of its residue field which usually has a simpler algebraic structure. Since the henselization is an immediate extension, which means that it has the same value group and same residue field as K, when studying the algebraic structure of valued fields it is natural to pass to the henselizations to study the structure there.

An immediate function field $(F|K, v)$ of transcendence degree 1 will be called **henselian rational** if there exists an element $x \in F^h$ such that $F^h = K(x)^h$, that is, F^h is the henselization of the rational function field $K(x)$. We then call x a **henselian generator** of F^h.

2.8 Ramification theory

Let $(N|K, v)$ be a normal extension of valued fields and let $L|K$ be the maximal separable subextension of $N|K$. Then the extension has a Galois group $G = \mathrm{Gal}\,(N|K)$. We define the **decomposition group** of $(N|K, v)$ to be

$$G^{\mathrm{dec}} = G^{\mathrm{dec}}(N|K, v) := \{\sigma \in G \mid v \circ \sigma = v \text{ on } N\} \ ,$$

and the **decomposition field** is defined to be the fixed field of the decomposition group within $L|K$ i.e.,

$$(N|K, v)^{\mathrm{dec}} := \mathrm{Fix}_L(G^{\mathrm{dec}}) \ .$$

Next, the **inertia group** of $(N|K, v)$ is defined to be

$$G^i = G^i(N|K, v) := \{\sigma \in G \mid \forall a \in \mathcal{O}_N, \ v(\sigma a - a) > 0\} \ ,$$

and the **inertia field**

$$(N|K,v)^i := \mathrm{Fix}_L(G^i)\ .$$

Finally, we define the **ramification group** of $(N|K,v)$ to be

$$G^r = G^r(N|K,v) := \{\sigma \in G \mid \forall a \in \mathcal{O}_N \setminus \{0\},\ v(\sigma a - a) > va\}\ ,$$

and the **ramification field** to be

$$(N|K,v)^r := \mathrm{Fix}_L(G^r)\ .$$

In the case $N = K^{\mathrm{sep}}$ the group G is called the **absolute Galois group** of (K,v), thus when considering the extension $K^{\mathrm{sep}}|K$ (or $\widetilde{K}|K$) we will talk about the absolute decomposition, absolute inertia and absolute ramification groups and fields.

The field (K,v) is called **tame** if it is henselian and the absolute ramification group G^r of K is trivial, that is, the absolute ramification field K^r is algebraically closed.

2.9 Defectless extensions

An algebraic extension $(L|K,v)$ of henselian fields is called **defectless** if every finite subextension $E|K$ satisfies the fundamental equality $[E:K] = \mathrm{e}\cdot\mathrm{f}$, where $\mathrm{e} = (vE:vK)$ is the ramification index and $\mathrm{f} = [Ev:Kv]$ is the inertia degree. In this case, $(E|K,v)$ admits a **standard valuation basis**, which we construct as follows: we take $a_1,\ldots,a_{\mathrm{e}} \in E$ such that $va_1 + vK, \ldots, va_{\mathrm{e}} + vK$ are the cosets of vK in vE, and $b_1,\ldots,b_{\mathrm{f}} \in E$ such that $b_1v,\ldots,b_{\mathrm{f}}v$ are a basis of $Ev|Kv$. Then a_ib_j , $1 \le i \le \mathrm{e}$, $1 \le j \le \mathrm{f}$, is a basis of $E|K$, and it has the following property: for all choices of $c_{ij} \in K$,

$$v\sum_{i,j} c_{ij}a_ib_j \ = \ \min_{i,j} vc_{ij}a_ib_j \ = \ \min_{i,j} vc_{ij}a_i\ .$$

Note that we can always choose $a_1 = b_1 = 1$ so that $a_1b_1 = 1$.

All tame extensions of henselian fields are defectless, see [11]. The following facts are well known and easy to prove:

Lemma 2.9.1. *Take a defectless extension $(L|K,v)$ of henselian fields and $a \in L$. Then the set $\{v(a-c) \mid c \in K\}$ has a maximum. More precisely, if we choose a standard valuation basis for $E = K(a)$ as above, with $a_1 = b_1 = 1$ and write*

$$a \ = \ \sum_{i,j} c_{ij}a_ib_j\ ,$$

then $v(a - c_{1,1})$ is the maximum of $\{v(a-c) \mid c \in K\}$.

We will also need the following tool (cf. [9, Lemma 2.5]):

Lemma 2.9.2. *Take a henselian field (K,v), a valued field extension $(K'|K,v)$, an immediate subextension $(F|K,v)$, and a defectless algebraic subextension $(L|K,v)$. Then $F|K$ and $L|K$ are linearly disjoint and $F.L|L$ is immediate.*

Chapter 3

Higher Kaplansky Theory

3.1 Approximation types and distances

We will now introduce approximation types, which constitute a suitable structure for dealing with immediate extensions of valued fields.

We define $B_\alpha(c, K) = \{a \in K \mid v(a - c) \geq \alpha\}$ to be the "closed" ultrametric ball in (K, v) of radius $\alpha \in vK\infty := vK \cup \{\infty\}$ centered at $c \in K$. An **approximation type over** (K, v) is a full nest of closed balls in (K, v), that is, a collection

$$\mathbf{A} \;=\; \{B_\alpha(c_\alpha, K) \mid \alpha \in S\}$$

with S an initial segment of $vK\infty$, $c_\alpha \in K$, and the balls $B_\alpha(c_\alpha, K)$ linearly ordered by inclusion. We write $\mathbf{A}_\alpha = B_\alpha(c_\alpha, K)$ for $\alpha \in S$, and $\mathbf{A}_\alpha = \emptyset$ otherwise. We call S the **support** of $\mathbf{A}$ and denote it by $\operatorname{supp} \mathbf{A}$.

Note that if $\beta < \alpha \in \operatorname{supp} \mathbf{A}$, then $\mathbf{A}_\beta = B_\beta(c_\beta, K) = B_\beta(c_\alpha, K)$, i.e., $\mathbf{A}_\beta$ is uniquely determined. Hence, $\mathbf{A}$ is uniquely determined by the balls $\mathbf{A}_\alpha$ where α runs through an arbitrary cofinal sequence in $\operatorname{supp} \mathbf{A}$.

Take any extension $(L|K, v)$ and $x \in L$. For all $\alpha \in vK\infty$, we set

$$\operatorname{appr}(x, K)_\alpha \;:=\; \{c \in K \mid v(x - c) \geq \alpha\} \;=\; B_\alpha(x, L) \cap K \;. \tag{3.1.1}$$

It is easy to check that $\operatorname{appr}(x, K)_\alpha$ is empty or a closed ball of radius α. If $\operatorname{appr}(x, K)_\alpha \neq \emptyset$ and $\beta < \alpha$, then also $\operatorname{appr}(x, K)_\beta \neq \emptyset$. This shows that the set

$$\{\alpha \in vK\infty \mid \operatorname{appr}(x, K)_\alpha \neq \emptyset\}$$

is an initial segment of $vK\infty$ and therefore,

$$\operatorname{appr}(x, K) \;:=\; \{\operatorname{appr}(x, K)_\alpha \mid \alpha \in vK\infty \text{ and } \operatorname{appr}(x, K)_\alpha \neq \emptyset\} \tag{3.1.2}$$

is an approximation type over (K, v). We call $\operatorname{appr}(x, K)$ the **approximation type of** x **over** (K, v).

As the support S of $\operatorname{appr}(x, K)$ is an initial segment of $vK\infty$, $S \cap vK = S \setminus \{\infty\}$ is an initial segment of vK and thus induces a cut in vK with lower cut set $S \setminus \{\infty\}$. Now this cut induces a cut in the divisible hull $\widetilde{vK}$ of vK, where the lower cut set is the smallest initial segment of $\widetilde{vK}$ containing $S \setminus \{\infty\}$. We call this cut the **distance of** x **from** (K, v) and denote it by

$$\operatorname{dist}(x, K) \;.$$

We write $\mathrm{dist}\,(x,K)=\infty$ if the lower cut set is $\widetilde{vK}$, and $\mathrm{dist}\,(x,K)<\infty$ otherwise. Note that $\mathrm{dist}\,(x,K)=\infty$ if and only if S contains vK, which holds if and only if x lies in the completion of (K,v).

For a subset $A\subset K$ we define $\mathrm{dist}_K(x,A)$, the distance of x from A over K, to be the cut in $\widetilde{vK}$ having as lower cut set the smallest initial segment in $\widetilde{vK}$ containing the set $\{v(x-c)\mid c\in A\}\cap vK$.

Note that if $(L|K,v)$ is an algebraic extension of valued fields, then the divisible hull of vK coincides with the divisible hull of vL and so for an element x in an extension of K, we have that $\mathrm{dist}\,(x,K)$ and $\mathrm{dist}\,(x,L)$ are both cuts in the same group. This allows us to compare these distances by set inclusion of the lower cut sets. Another reason to take the distance in the divisible hull is that the classification of Artin–Schreier defect extensions through distances presented in [9] does not work if they are taken in ordered abelian groups with archimedean components which are not dense; this situation does not appear in divisible groups.

If n is a natural number and the lower cut set of $\mathrm{dist}\,(x,K)$ is D, then

$$n\cdot \mathrm{dist}\,(x,K)$$

will denote the cut with lower cut set $nD:=\{n\gamma\mid\gamma\in D\}$; note that nD is again an initial segment of $\widetilde{vK}$ because of divisibility.

If C and C' are two cuts in a linearly ordered set T defined by their lower cut sets D and D', respectively, then $C=C'$ if $D=D'$, and we write $C<C'$ if $D\subsetneqq D'$. For an element $\alpha\in T$ we write $\alpha>C$ if $\alpha>\beta$ for all $\beta\in D$, and $\alpha\geq C$ if $\alpha\geq\beta$ for all $\beta\in D$; note that if D has no last element, then $\alpha>C\Leftrightarrow\alpha\geq C$. We write $\alpha\leq C$ if $\alpha\in D$, and $\alpha<C$ if $\alpha\in D$ but is not the last element of D.

Lemma 3.1.1. *Take an extension $(L|K,v)$ of valued fields, and $x,x'\in L$.*

a) For every α in the support of $\mathrm{appr}\,(x,K)$, $\mathrm{appr}\,(x,K)_\alpha=\mathrm{appr}\,(x',K)_\alpha$ *holds if and only if* $v(x-x')\geq\alpha$.

b) Further,

$$\mathrm{appr}\,(x,K)=\mathrm{appr}\,(x',K)\implies v(x-x')\geq \mathrm{dist}\,(x,K)=\mathrm{dist}\,(x',K)\,, \tag{3.1.3}$$

$$v(x-x')\geq\max\{\mathrm{dist}\,(x,K),\mathrm{dist}\,(x',K)\}\implies \mathrm{appr}\,(x,K)=\mathrm{appr}\,(x',K)\,. \tag{3.1.4}$$

Proof. a): Take $\alpha\in vK\infty$. If $v(x-x')\geq\alpha$, then $B_\alpha(x,L)=B_\alpha(x',L)$, which yields that $\mathrm{appr}\,(x,K)_\alpha=B_\alpha(x,L)\cap K=B_\alpha(x',L)\cap K=\mathrm{appr}\,(x',K)_\alpha$. If $v(x-x')<\alpha$, then $B_\alpha(x,L)\cap B_\alpha(x',L)=\emptyset$, whence $\mathrm{appr}\,(x,K)_\alpha\cap\mathrm{appr}\,(x',K)_\alpha=\emptyset$; for $\mathrm{appr}\,(x,K)_\alpha\neq\emptyset$, this yields that $\mathrm{appr}\,(x,K)_\alpha\neq\mathrm{appr}\,(x',K)_\alpha$.

b): If $\mathrm{dist}\,(x,K)\neq\mathrm{dist}\,(x',K)$, then $\mathrm{appr}\,(x,K)\neq\mathrm{appr}\,(x',K)$. If $v(x-x')\geq\mathrm{dist}\,(x,K)$ does not hold, then there is some α in the support of $\mathrm{appr}\,(x,K)$ such that $\alpha>v(x-x')$. By part a), it follows that $\mathrm{appr}\,(x,K)_\alpha\neq\mathrm{appr}\,(x',K)_\alpha$. This proves (3.1.3).

If $v(x-x')\geq\mathrm{dist}\,(x,K)$ holds, then $v(x-x')\geq\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}\,(x,K)$. Again by part a), it follows that $\mathrm{appr}\,(x,K)_\alpha=\mathrm{appr}\,(x',K)_\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}\,(x,K)$. Similarly, $v(x-x')\geq\mathrm{dist}\,(x',K)$ implies that $\mathrm{appr}\,(x,K)_\alpha=\mathrm{appr}\,(x',K)_\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}\,(x',K)$. If none of the supports contains ∞, then we obtain that $\mathrm{appr}\,(x,K)=\mathrm{appr}\,(x',K)$. If on the other hand, at least one support contains ∞, then the corresponding distance is ∞, whence $v(x-x')=\infty$, i.e., $x=x'$ and again, $\mathrm{appr}\,(x,K)=\mathrm{appr}\,(x',K)$. We have proved (3.1.4). □

If $\mathbf{A}$ is an approximation type over (K,v) and there exists an element x in some valued extension field L such that $\mathbf{A}=\mathrm{appr}\,(x,K)$, then we say that x **realizes** $\mathbf{A}$ (in (L,v)). If $\mathbf{A}$ is realized by some $c\in K$, then $\mathbf{A}$ will be called **trivial**. This holds if and only if $\mathbf{A}_\infty\neq\emptyset$, in which case $\mathbf{A}_\infty=\{c\}$.

We leave the easy proof of the following lemma to the reader.

Lemma 3.1.2. *Take an approximation type* $\mathbf{A}$ *over* (K,v) *and an extension* $(L|K,v)$ *of valued fields. The element* $x\in L$ *realizes* $\mathbf{A}$ *if and only if the following conditions hold:*

1) if $\alpha\in\mathrm{supp}\,\mathbf{A}$, *then* $v(x-c)\geq\alpha$ *for some* $c\in\mathbf{A}_\alpha$,

2) if $\beta\notin\mathrm{supp}\,\mathbf{A}$, *then* $v(x-c)<\beta$ *for all* $c\in K$.

For our work with approximation types, we introduce the following notation which is particularly useful in the immediate case. We introduce it in connection with valued fields, but its application to ultrametric spaces and other valued structures is similar. So take an arbitrary valued field (K,v) and an approximation type $\mathbf{A}$ over (K,v). Further, take a formula φ with one free variable. Then the sentence

$$\varphi(c)\ \text{for}\ c\nearrow\mathbf{A}$$

will denote the assertion

$$\text{there is } \alpha\in vK \text{ such that } \mathbf{A}_\alpha\neq\emptyset \text{ and } \varphi(c) \text{ holds for all } c\in\mathbf{A}_\alpha\,.$$

Note that if $\varphi_1(c)$ for $c\nearrow\mathbf{A}$ and $\varphi_2(c)$ for $c\nearrow\mathbf{A}$, then also $\varphi_1(c)\wedge\varphi_2(c)$ for $c\nearrow\mathbf{A}$.

In the case of $\mathbf{A}=\mathrm{appr}\,(x,K)$, we will also write "$c\nearrow x$" in place of "$c\nearrow\mathbf{A}$".

If $\gamma=\gamma(c)\in vK$ is a value that depends on $c\in K$ (e.g., the value $vf(c)$ for a polynomial $f\in K[X]$), then we will say that γ **increases for** $c\nearrow x$ if there exists some $\alpha\neq\infty$ in the support of $\mathrm{appr}\,(x,K)$ such that for every choice of $c'\in\mathrm{appr}\,(x,K)_\alpha$ with $x\neq c'$,

$$\gamma(c)>\gamma(c')\quad\text{for}\quad c\nearrow x\,.$$

Note that the condition $x\neq c'$ is automatically satisfied if $\mathrm{appr}\,(x,K)$ is nontrivial.

3.2 Immediate approximation types

An approximation type $\mathbf{A}$ with support S will be called **immediate** if its intersection

$$\bigcap\mathbf{A}\;=\;\bigcap_{\alpha\in S}\mathbf{A}_\alpha$$

is empty. If $\mathbf{A}$ is trivial, then $\bigcap\mathbf{A}=\mathbf{A}_\infty\neq\emptyset$; therefore, an immediate approximation type is never trivial.

Lemma 3.2.1. *Let $(L|K,v)$ be an extension of valued fields.*

a) If $x \in L$, then $\mathrm{appr}\,(x,K)$ *is immediate if and only if for every $c \in K$ there is some $c' \in K$ such that $v(x-c') > v(x-c)$, that is, the set*

$$v(x-K) \;:=\; \{v(x-c) \mid c \in K\}$$

has no maximal element.

b) The extension $(L|K,v)$ is immediate if and only if for every $x \in L \setminus K$, its approximation type $\mathrm{appr}\,(x,K)$ *over (K,v) is immediate.*

c) If $\mathrm{appr}\,(x,K)$ *is immediate, then its support is equal to $v(x-K)$.*

Proof. a): Suppose that $\mathrm{appr}\,(x,K)$ is immediate and that c is an arbitrary element of K. Then by definition there is some α such that $c \notin \mathrm{appr}\,(x,K)_\alpha \neq \emptyset$, so $v(x-c) < \alpha$. Choosing some $c' \in \mathrm{appr}\,(x,K)_\alpha$, we obtain that $v(x-c) < \alpha \leq v(x-c')$.

Now take $x \in L \setminus K$ and suppose that for every $c \in K$ there is $c' \in K$ such that $v(x-c') > v(x-c)$. Then there is also some $c'' \in K$ such that $v(x-c'') > v(x-c')$. By the ultrametric triangle law we obtain that $v(c'-c) = v(x-c) < v(x-c') = v(c''-c')$. Hence $v(c'-c) \in v(x-K)$ and $c \notin \mathrm{appr}\,(x,K)_{v(c''-c')} \neq \emptyset$. As $c \in K$ was arbitrary, this shows that $\mathrm{appr}\,(x,K)$ is immediate.

b): Assume that $(L|K,v)$ is immediate. Take $x \in L \setminus K$ and an arbitrary $c \in K$. Then $v(x-c) \in vL = vK$, i.e., there is $d \in K$ such that $v(x-c) = vd$ so that $vd^{-1}(x-c) = 0$. Then $d^{-1}(x-c)v \in Lv = Kv$, i.e., there is $d' \in K$ such that $d^{-1}(x-c)v = d'v$, which means that $v(d^{-1}(x-c)-d') > 0$. This implies that $v(x-c-dd') > vd = v(x-c)$. Setting $c' = c+dd'$, we obtain $v(x-c') > v(x-c)$. By part a) it now follows that $\mathrm{appr}\,(x,K)$ is immediate.

For the converse, assume that for every $x \in L \setminus K$, $\mathrm{appr}\,(x,K)$ is immediate. By the proof of a), for every $c \in K$ we have that $v(x-c) \in vK$, so in particular, $v(x-0) \in vK$; this shows that $vL|vK$ is trivial. It remains to show that $Lv|Kv$ is trivial. Take any $x \in L \setminus K$ with $vx = 0$. Since $\mathrm{appr}\,(x,K)$ is immediate, there is $c' \in K$ such that $v(x-c') > v(x-0) = vx$. From this we obtain that $xv = c'v \in Kv$. Hence $Lv|Kv$ is trivial.

c): If $\alpha \in vK$ is an element of the support of $\mathrm{appr}\,(x,K)$, then $\mathrm{appr}\,(x,K)_\alpha \neq \emptyset$, and so by (3.1.1), there is $c \in K$ such that $v(x-c) \geq \alpha$. In the case of $v(x-c) = \alpha$, we immediately see that $\alpha \in v(x-K)$. In the case of $v(x-c) > \alpha$, choose some $d \in K$ with $vd = \alpha$; then $v(x-(c+d)) = vd = \alpha$, which again shows that $\alpha \in v(x-K)$.

For the converse inclusion, take $c \in K$. By the proof of part a), there is $c' \in K$ such that $v(x-c) = v(c'-c)$, which shows that $v(x-c) \in vK$. It follows from (3.1.1) that $c \in \mathrm{appr}\,(x,K)_{v(x-c)}$, so $v(x-c)$ is in the support of $\mathrm{appr}\,(x,K)$. □

For immediate approximation types, we can improve part b) of Lemma 3.1.1, and Lemma 3.1.2.

Lemma 3.2.2. *Take an extension $(L|K,v)$ of valued fields, and $x,x'\in L$. If* $\mathrm{appr}\,(x,K)$ *is immediate, then*

$$\mathrm{appr}\,(x,K)=\mathrm{appr}\,(x',K)\iff v(x-x')\geq \mathrm{dist}\,(x,K)\;. \tag{3.2.1}$$

Proof. We only have to prove the implication "$\Leftarrow$". As in the proof of (3.1.4), we deduce from $v(x-x')\geq \mathrm{dist}\,(x,K)$ that $v(x-x')\geq\alpha$ and $\mathrm{appr}\,(x,K)_\alpha=\mathrm{appr}\,(x',K)_\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}\,(x,K)$. Since $\mathrm{appr}\,(x,K)$ is immediate, we also know that ∞ is not in its support. It remains to show that $\mathrm{appr}\,(x',K)_\alpha=\emptyset$ for every α not in the support of $\mathrm{appr}\,(x,K)$. If this were not true, there would be $c\in K$ such that $v(x'-c)>\mathrm{supp\,appr}\,(x,K)$. Since also $v(x-x')>\mathrm{supp\,appr}\,(x,K)$, we would obtain that $v(x-c)>\mathrm{supp\,appr}\,(x,K)$. But then $c\in\bigcap\mathrm{appr}\,(x,K)$, contradicting the assumption that $\mathrm{appr}\,(x,K)$ is immediate. □

Lemma 3.2.3. *Take an immediate approximation type* $\mathbf{A}$ *over (K,v) and an extension $(L|K,v)$. The element $x\in L$ realizes* $\mathbf{A}$ *if and only if for every* $\alpha\in\mathrm{supp}\,\mathbf{A}$*,* $v(x-c)\geq\alpha$ *for some* $c\in\mathbf{A}_\alpha$*.*

Proof. We have to show that for every immediate approximation type $\mathbf{A}$, condition 2) of Lemma 3.1.2 holds if condition 1) holds. Assume that $\beta\notin\mathrm{supp}\,\mathbf{A}$. Since the support is an initial segment of $vK\infty$, this means that $\beta>\mathrm{supp}\,\mathbf{A}$. Take any $c\in K$. Since $\mathbf{A}$ is immediate, there is some $\alpha\in\mathrm{supp}\,\mathbf{A}$ such that $c\notin\mathbf{A}_\alpha$. By condition 1), there is some $c'\in\mathbf{A}_\alpha$ such that $v(x-c')\geq\alpha$. Now $v(x-c)\geq\alpha$ would imply that $v(c-c')\geq\min\{v(x-c),v(x-c')\}\geq\alpha$, whence $c\in\mathbf{A}_\alpha$, a contradiction. It follows that $v(x-c)<\alpha<\beta$. Hence condition 2) holds. □

Corollary 3.2.4. *Take an immediate approximation type* $\mathbf{A}$ *over (K,v), an extension $(L|K,v)$ of valued fields, and $x\in L$. If $v(x-c)$ is not fixed for $c\nearrow\mathbf{A}$, then* $\mathbf{A}=\mathrm{appr}\,(x,K)$*.*

Proof. Our assumption means that for all $\alpha\in\mathrm{supp}\,\mathbf{A}$ there are $c,c'\in\mathbf{A}_\alpha$ such that $v(x-c')>v(x-c)$. This implies that $v(x-c')>\min\{v(x-c),v(c-c')\}$, whence $v(x-c)=v(c-c')\geq\alpha$. Now our assertion follows from the previous lemma. □

In the remainder of this section, we wish to explore how immediate approximation types behave under valued field extensions $(L|K,v)$. Take x in some extension of L such that $x\notin L$ and $\mathrm{appr}\,(x,K)$ is immediate. Obviously,

$$\mathrm{dist}\,(x,L)\;\geq\;\mathrm{dist}\,(x,K)$$

and

$$\mathrm{appr}\,(x,K)_\alpha=B_\alpha(c_\alpha,K)\implies \mathrm{appr}\,(x,L)_\alpha=B_\alpha(c_\alpha,L)\;. \tag{3.2.2}$$

If $\mathrm{dist}\,(x,L)=\mathrm{dist}\,(x,K)$, then by (3.2.2), $\mathrm{appr}\,(x,K)$ fully determines $\mathrm{appr}\,(x,L)$. But if $\mathrm{dist}\,(x,L)>\mathrm{dist}\,(x,K)$, then $\mathrm{appr}\,(x,K)$ does not provide enough information for those $\mathrm{appr}\,(x,L)_\beta$ with $\beta>\mathrm{dist}\,(x,L)$.

Lemma 3.2.5. *If in the above situation $(L|K,v)$ is a defectless extension, then* $\mathrm{dist}\,(x,L)=\mathrm{dist}\,(x,K)$ *and by (3.2.2),* $\mathrm{appr}\,(x,K)$ *fully determines* $\mathrm{appr}\,(x,L)$*.*

Proof. Suppose that $\operatorname{dist}(x,L) > \operatorname{dist}(x,K)$. Then there is some $a \in L$ such that $v(x-a) > \operatorname{dist}(x,K)$, which by (3.1.4) implies that $\operatorname{appr}(a,K) = \operatorname{appr}(x,K)$, which is immediate. But by Lemma 2.9.1, $\{v(a-c) \mid c \in K\}$ has a maximum. This contradicts part a) of Lemma 3.2.1. $\square$

3.3 Polynomials and immediate approximation types

Take an arbitrary polynomial $f \in K[X]$ and an approximation type $\mathbf{A}$ over (K,v). We will say that $\mathbf{A}$ **fixes the value of** f if there is some $\alpha \in vK$ such that $vf(c) = \alpha$ for $c \nearrow \mathbf{A}$. We will call an immediate approximation type $\mathbf{A}$ a **transcendental approximation type** if $\mathbf{A}$ fixes the value of every polynomial $f(X) \in K[X]$. Otherwise, $\mathbf{A}$ is called an **algebraic approximation type**. If there exists any polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, then there exists also a monic polynomial of the same degree having the same property (since this property is not lost by multiplication with nonzero constants from K). If $f(X)$ is a monic polynomial of minimal degree $\mathbf{d}$ such that $\mathbf{A}$ does not fix the value of f, then it will be called an **associated minimal polynomial** for $\mathbf{A}$, and $\mathbf{A}$ is said to be of **degree d**. We define the degree of a transcendental approximation type to be $\mathbf{d} = \infty$. According to this terminology, an approximation type over K of degree $\mathbf{d}$ fixes the value of every polynomial $f \in K[X]$ with $\deg f < \mathbf{d}$. Note that an associated minimal polynomial f for $\mathbf{A}$ is always irreducible over K. Indeed, if the degree of $g, h \in K[X]$ is smaller than $\deg f$, then $\mathbf{A}$ fixes the value of g and h and thus also of $g \cdot h$. Since every polynomial $g \in K[X]$ of degree $\mathbf{d}$ whose value is not fixed by $\mathbf{A}$ is just a multiple cf of an associated minimal polynomial f for $\mathbf{A}$ (with $c \in K^\times$), the irreducibility holds for every such polynomial as well.

We note that an immediate approximation type $\mathbf{A}$ fixes the value of every linear polynomial in $K[X]$. Indeed, for every $c \in K$ there is $\alpha \in \operatorname{supp} \mathbf{A}$ such that $c \notin \mathbf{A}_\alpha$. Hence for all $c', c'' \in \mathbf{A}_\alpha$, $v(c'-c'') > v(c-c')$ and thus $v(c'-c) = v(c''-c)$. This shows that $\mathbf{A}$ fixes the value of $X - c$. We conclude that the degree of an algebraic approximation type is not less than 2.

We will now study the behaviour of polynomials with respect to immediate approximation types $\operatorname{appr}(x,K)$. We need the following lemma for ordered abelian groups, which is a reformulation of Lemma 4 of Kaplansky [4]. For archimedean ordered groups, it was proved by Ostrowski [15].

Lemma 3.3.1. *Take elements $\alpha_1, \dots, \alpha_m$ of an ordered abelian group Γ and a subset $\Upsilon \subset \Gamma$ without maximal element. Let $t_1, \dots, t_m$ be distinct integers. Then there exists an element $\beta \in \Upsilon$ and a permutation σ of the indices $1, \dots, m$ such that for all $\gamma \in \Upsilon$, $\gamma \geq \beta$,*

$$\alpha_{\sigma(1)} + t_{\sigma(1)}\gamma > \alpha_{\sigma(2)} + t_{\sigma(2)}\gamma > \ldots > \alpha_{\sigma(m)} + t_{\sigma(m)}\gamma \,.$$

For an arbitrary polynomial $f(X) = c_n X^n + c_{n-1} X^{n-1} + \ldots + c_0$, we call

$$f_i(X) \;:=\; \sum_{j=i}^{n} \binom{j}{i} c_j X^{j-i} \;=\; \sum_{j=0}^{n-i} \binom{j+i}{i} c_{j+i} X^j \tag{3.3.1}$$

the i**-th formal derivative of** f and

$$f(X) = \sum_{i=0}^{n} f_i(c)(X-c)^i \tag{3.3.2}$$

$$f_i(X) = \sum_{j=i}^{n} \binom{j}{i} f_j(c)(X-c)^{j-i} \tag{3.3.3}$$

the **Taylor expansions** of f and f_i at c.

If the immediate approximation type $\mathbf{A}$ is of degree $\mathbf{d}$ and $f \in K[X]$ is of degree at most $\mathbf{d}$, then $\mathbf{A}$ fixes the value of every formal derivative f_i of f $(1 \le i \le \deg f)$, since every such derivative has degree less than $\mathbf{d}$. So we can define β_i to be the fixed value $vf_i(c)$ for $c \nearrow x$. In certain cases, a derivative may be identically 0. In this case, we have $\beta_i = \infty$. However, the Taylor expansion of f shows that not all derivatives vanish identically, and the vanishing ones will not play a role in our computations.

By use of Lemma 3.3.1, we can now prove:

Lemma 3.3.2. *Take an immediate approximation type* $\mathbf{A} = \operatorname{appr}(x,K)$ *of degree* $\mathbf{d}$ *over* (K,v) *and* $f \in K[X]$ *a polynomial of degree at most* $\mathbf{d}$. *Further, let* β_i *denote the fixed value* $vf_i(c)$ *for* $c \nearrow x$. *Then there is a positive integer* $\mathbf{h} \le \deg f$ *such that*

$$\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) < \beta_i + i \cdot v(x-c) \tag{3.3.4}$$

whenever $i \ne \mathbf{h}$, $1 \le i \le \deg f$ *and* $c \nearrow x$. *Hence,*

$$v(f(x)-f(c)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \quad \text{for } c \nearrow x\ . \tag{3.3.5}$$

Consequently, if $\mathbf{A}$ *fixes the value of* f, *then*

$$v(f(x)-f(c)) > vf(x) = vf(c) \quad \text{for } c \nearrow x\ ,$$

and if $\mathbf{A}$ *does not fix the value of* f, *then*

$$vf(x) > vf(c) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \quad \text{for } c \nearrow x\ .$$

Proof. Set $n = \deg f$. We consider the Taylor expansion

$$f(x) - f(c) = f_1(c)(x-c) + \ldots + f_n(c)(x-c)^n \tag{3.3.6}$$

with $c \in K$. We have that $vf_i(c)(x-c)^i = \beta_i + i \cdot v(x-c)$ for $c \nearrow x$. So we apply the foregoing lemma with $\alpha_i = \beta_i$ and $t_i = i$, and with Υ equal to the support of $\mathbf{A}$ (which has no maximal element since $\mathbf{A}$ is an immediate approximation type). We find that there is an integer $\mathbf{h} \le \deg f$ such that $\beta_{\mathbf{h}} + \mathbf{h}v(x-c) < \beta_i + iv(x-c)$ for $c \nearrow x$ and $i \ne \mathbf{h}$. This is equation (3.3.4), which in turn implies equation (3.3.5).

If $\mathbf{A}$ fixes the value of f, then $vf(x) \ne vf(c)$ is impossible for $c \nearrow x$ since otherwise, the left hand side of (3.3.5) would be equal to $\min\{vf(x), vf(c)\}$ and thus fixed while the right hand side of (3.3.5) increases for

$c \nearrow x$. This proves that $vf(x) = vf(c)$ and thus also $v(f(x) - f(c)) \geq vf(x)$ for $c \nearrow x$. But since the right hand side increases, we find that $v(f(x) - f(c)) > vf(x)$ for $c \nearrow x$.

If $\mathbf{A}$ does not fix the value of f, then $vf(x) \neq vf(c)$ and thus $v(f(x) - f(c)) = \min\{vf(x), vf(c)\}$ for $c \nearrow x$. Since $v(f(x) - f(c))$ increases for $c \nearrow x$ and $vf(x)$ is a constant, the minimum must be $vf(c)$, and $vf(x) = vf(c)$ is impossible. □

If $g \in K[X]$ has a degree smaller than the degree of $\mathbf{A}$, then by the foregoing lemma, the value of $g(x)$ in $(K(x), v)$ is given by $vg(x) = vg(c)$ for $c \nearrow x$. Since $g(c) \in K$, that means that the value of $g(x)$ is uniquely determined by $\mathbf{A}$ and the restriction of v to K. If g is a nonzero polynomial, then $g(c) \neq 0$ for $c \nearrow x$ (since there is a nonempty $\mathbf{A}_\alpha$ which does not contain the finitely many zeros of g, as $\mathbf{A}$ is immediate). Consequently, $g(x) \neq 0$, which shows that the elements $1, x, \ldots, x^{\mathbf{d}-1}$ are K-linearly independent.

We even know that $v(g(x)-g(c)) > vg(x)$ for $c \nearrow x$. This means that $(K, v) \subset (K+Kx+\ldots+Kx^{\mathbf{d}-1}, v)$ is an immediate extension of valued vector spaces. If $\mathbf{d} = [K(x) : K] < \infty$, then $K(x) = K[x] = K + Kx + \ldots + Kx^{\mathbf{d}-1}$, and so the valued field extension $(K(x)|K, v)$ is immediate. If $\mathbf{d} = \infty$, then $(K, v) \subset (K[x], v)$ is immediate. But then again it follows that the valued field extension $(K(x)|K, v)$ is immediate. Indeed, if $v(g(x) - g(c)) > vg(x)$ and $v(h(x) - h(c)) > vh(x)$, then $vg(x) = vg(c)$, $vh(x) = vh(c)$ and

$$\begin{aligned} v\left(\frac{g(x)}{h(x)} - \frac{g(c)}{h(c)}\right) &= v\left[g(x)h(c) - g(c)h(x)\right] - vh(x)h(c) \\ &= v\left[g(x)h(c) - g(c)h(c) + g(c)h(c) - g(c)h(x)\right] - vh(x)h(c) \\ &= v\left[(g(x) - g(c))h(c) + g(c)(h(c) - h(x))\right] - vh(x)h(c) \\ &> vg(x)h(x) - vh(x)h(x) \;=\; v\,\frac{g(x)}{h(x)} \,. \end{aligned}$$

We have proved:

Lemma 3.3.3. *Take an immediate approximation type* $\mathbf{A} = \operatorname{appr}(x, K)$ *of degree* $\mathbf{d}$ *over* (K, v)*. Then the valuation on the valued* (K, v)*-vector subspace* $(K+Kx+\ldots+Kx^{\mathbf{d}-1}, v)$ *of* $(K(x), v)$ *is uniquely determined by* $\mathbf{A}$ *because*

$$vg(x) = vg(c) \quad \text{for } c \nearrow x$$

for every $g(x) \in K+Kx+\ldots+Kx^{\mathbf{d}-1}$*. The elements* $1, x, \ldots, x^{\mathbf{d}-1}$ *are* K*-linearly independent. In particular,* x *is transcendental over* K *if* $\mathbf{d} = \infty$*.*

Moreover, the extension $(K, v) \subset (K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ *of valued vector spaces is immediate. In particular, if* $\mathbf{d} = \infty$ *or if* $\mathbf{d} = [K(x) : K] < \infty$*, then* $(K[x]|K, v)$ *is immediate and the same is consequently true for the valued field extension* $(K(x)|K, v)$*.*

So far we have only considered polynomials of degree at most $\mathbf{d}$; the next lemma will cover the remaining case.

Lemma 3.3.4. *Take an immediate algebraic approximation type* $\mathbf{A} = \operatorname{appr}(x, K)$ *over* (K, v) *and an associated minimal polynomial* $f \in K[X]$ *for* $\mathbf{A}$*. Further, take an arbitrary polynomial* $g \in K[X]$ *and write*

$$g(X) = c_k(X)f(X)^k + \ldots + c_1(X)f(X) + c_0(X) \tag{3.3.7}$$

with polynomials $c_i \in K[X]$ *of degree less than* $\deg f$*. Then there is some integer* m*,* $1 \leq m < k$*, and a value* $\beta \in vK$ *such that with* $\mathbf{h}$ *as in Lemma 3.3.2,*

$$v(g(c) - c_0(c)) = vc_m(c) + m \cdot vf(c) = \beta + m \cdot \mathbf{h} \cdot v(x - c) \quad \textit{for } c \nearrow x \; . \tag{3.3.8}$$

Consequently, if $\mathbf{A}$ *fixes the value of* g*, then*

$$vg(x) = vg(c) = vc_0(c) = vc_0(x) < v(g(c) - c_0(c)) \quad \textit{for } c \nearrow x \; ,$$

and if $\mathbf{A}$ *does not fix the value of* g*, then*

$$vg(x) > vg(c) = \beta + m \cdot \mathbf{h} \cdot v(x - c) \quad \textit{for } c \nearrow x \; .$$

Proof. Since $\deg c_i(X) < \deg f(X) = \deg \mathbf{A}$, we have that $\mathbf{A}$ fixes the value of $c_i(X)$, for $0 \leq i \leq k$. We denote by γ_i the fixed value $vc_i(c)$ for $c \nearrow x$. Since f is an associated minimal polynomial for $\mathbf{A}$, we know that $\mathbf{A}$ does not fix the value of f. From Lemma 3.3.2 we infer that the value of $c_i(c)f(c)^i$ is equal to $\gamma_i + i\beta_{\mathbf{h}} + i\mathbf{h}v(x - c)$. We apply Lemma 3.3.1 with $\alpha_i = \gamma_i + i\beta_{\mathbf{h}}$, $t_i = i\mathbf{h}$ and $\Upsilon = \operatorname{supp} \mathbf{A}$ to deduce that there is an integer m such that $0 \leq m < k$ and $vc_m(c)f(c)^m < vc_i(c)f(c)^i$ for $c \nearrow x$ and $1 < i \neq m$. Consequently,

$$v(g(c) - c_0(c)) = vc_m(c)f(c)^m = \gamma_m + m \cdot \beta_{\mathbf{h}} + m \cdot \mathbf{h} \cdot v(x - c) \; . \tag{3.3.9}$$

We set $\beta := \gamma_m + m\beta_{\mathbf{h}}$.

The value of the right hand side of (3.3.9) is not fixed for $c \nearrow x$. Consequently, if $\mathbf{A}$ fixes the value of g, then from our representation (3.3.7) of g we see that the value $vc_m(c)f(c)^m$ must be greater than the fixed value of $c_0(c)$ for $c \nearrow x$, which yields that $vg(c) = vc_0(c)$. From Lemma 3.3.2, we know that $vc_0(x) = vc_0(c)$ and $vf(x) > vf(c)$ for $c \nearrow x$. Therefore,

$$vc_i(x)f(x)^i > vc_i(c)f(c)^i > vc_0(c) = vc_0(x) \tag{3.3.10}$$

for $1 \leq i \leq k$ and $c \nearrow x$, whence $vg(x) = vc_0(x) = vc_0(c) = vg(c)$.

If $\mathbf{A}$ does not fix the value of g, then $vc_m(c)f(c)^m < vc_0(c)$ and

$$vg(c) = vc_m(c)f(c)^m = \beta + m \cdot \mathbf{h} \cdot v(x - c)$$

for $c \nearrow x$. The inequality $vg(x) > vg(c)$ for $c \nearrow x$, is seen as follows. Using the first inequality of (3.3.10) together with $vc_m(c)f(c)^m < vc_0(c)$, we obtain:

$$\begin{aligned} vg(x) \;\; &\geq \;\; \min\{v(c_k(x)f(x)^k)\, , \ldots ,\, v(c_1(x)f(x))\, ,\, vc_0(x)\} \\ &> \;\; \min\{v(c_k(c)f(c)^k)\, , \ldots ,\, v(c_1(c)f(c))\, ,\, vc_0(c)\} = vg(c) \; . \end{aligned}$$

This completes the proof of our lemma. □

Corollary 3.3.5. *Take an immediate approximation type* $\operatorname{appr}(x, K)$ *over* (K, v)*. If* x *is algebraic over* K *with minimal polynomial* $g \in K[X]$*, then* $\operatorname{appr}(x, K)$ *does not fix the value of* g *and is thus of degree* $\mathbf{d} \leq [K(x) : K]$.

Proof. Since $\operatorname{appr}(x,K)$ is immediate, it is nontrivial, so $x \notin K$ and $g(c) \neq 0$ for all $c \in K$. But by hypothesis, $g(x) = 0$. Hence $vg(x) > vg(c)$ for all $c \in K$. Now the assertion follows by an application of Lemma 3.3.4. □

Unfortunately, $\mathbf{d}$ may be smaller than $[K(x) : K]$, as the following example will show:

Example 3.3.6. We choose (K, v) to be $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{F}_p((t)), v_t)$ or any henselian intermediate field (where $\mathbb{F}_p$ is the field with p elements). We take L to be the perfect hull $K(t^{1/p^i} \mid i \in \mathbb{N})$ of K.

If ϑ is a root of the polynomial

$$X^p - X - \frac{1}{t}$$

then the Artin–Schreier extension $L(\vartheta)|L$ is immediate with $v(\vartheta - L) = \{\alpha \in vL \mid \alpha < 0\}$ (see [10, Example 3.12]). It follows from Proposition 3.4.5 below and the fact that (L, v) is henselian (being an algebraic extension of the henselian field (K, v)) that $\deg \operatorname{appr}(\vartheta, L) = p = [L(\vartheta) : L]$. But an element $x = \vartheta + y$ in some extension of (L, v) has the same approximation type as ϑ over L if $vy \geq 0$ (cf. Lemma 3.1.1). We may take y of arbitrarily high degree over L. Indeed, we may even take y to be transcendental over L to obtain that $\vartheta + y$ is transcendental over L. This shows that a transcendental element may have an algebraic approximation type. Moreover, we may choose y such that $vy \notin vL$ or $yv \notin Lv$ to obtain an extension which is not immediate, although its generating element has an immediate approximation type.

3.4 Realization of immediate approximation types

In this section we will present the two basic theorems due to Kaplansky ([4]) which show that each immediate approximation type can be realized in a simple immediate extension. Kaplansky proved these theorems to derive a characterization of maximal fields, which we will also present here.

Theorem 3.4.1. (Theorem 2 of [4], approximation type version)
For every immediate transcendental approximation type $\mathbf{A}$ *over* (K, v) *there exists a simple immediate transcendental extension* $(K(x), v)$ *such that* $\operatorname{appr}(x, K) = \mathbf{A}$.

If $(K(y), v)$ *is another valued extension field of* (K, v) *such that* $\operatorname{appr}(y, K) = \mathbf{A}$*, then* y *is also transcendental over* K *and the isomorphism between* $K(x)$ *and* $K(y)$ *over* K *sending* x *to* y *is valuation preserving.*

Proof. We take $K(x)|K$ to be a transcendental extension and define the valuation on $K(x)$ as follows. In view of the rule $v(g/h) = vg - vh$, it suffices to define v on $K[x]$. Take $g \in K[X]$. By assumption, $\mathbf{A}$ fixes the value of g, that is, there is $\beta \in vK$ such that $vg(c) = \beta$ for $c \nearrow \mathbf{A}$. We set $vg(x) = \beta$. If g is a constant in K, we just obtain the value given by the valuation v on K. Our definition implies that $vg \neq \infty$ for every nonzero $g \in K[x]$.

Take $g, h \in K[X]$. Again by our definition, $vg(x) = vg(c)$, $vh(x) = vh(c)$, and $vg(x)h(x) = v(g \cdot h)(x) = v(g \cdot h)(c) = vg(c)h(c)$ for $c \nearrow \mathbf{A}$. Thus, $vg(x)h(x) = vg(c)h(c) = vg(c) + vh(c) = vg(x) + vh(x)$ and $v(g(x) + h(x)) = v((g + h)(x)) = v((g + h)(c)) = v(g(c) + h(c)) \geq \min\{vg(c), vh(c)\} = \min\{vg(x), vh(x)\}$ for $c \nearrow \mathbf{A}$. So indeed, our definition yields a valuation v on $K(x)$ which extends the valuation v of K. Under this valuation, we have that $\mathbf{A} = \mathrm{appr}\,(x, K)$. This is seen as follows. In view of Lemma 3.2.3, it suffices to prove that for every $\alpha \in \mathrm{supp}\,\mathbf{A}$, we have that $v(x - c_\alpha) \geq \alpha$ for each $c_\alpha \in \mathbf{A}_\alpha$. But this follows directly from our definition of $v(x - c_\alpha)$ because for $c \nearrow \mathbf{A}$, $c \in \mathbf{A}_\alpha$ and thus $v(x - c_\alpha) = v(c - c_\alpha) \geq \alpha$.

From Lemma 3.3.3, we now infer that $(K(x)|K, v)$ is an immediate extension. Given another element y in some valued field extension of (K, v) such that $\mathbf{A} = \mathrm{appr}\,(y, K)$, we want to show that the epimorphism from $K[x]$ onto $K[y]$ induced by $x \mapsto y$ is valuation preserving. For this, we only have to show that $vg(x) = vg(y)$ for every $g \in K[X]$. By hypothesis, the degree of $\mathbf{A}$ is ∞. From Lemma 3.3.3 we can thus infer that $vg(x) = vg(c) = vg(y)$ holds for $c \nearrow \mathbf{A}$; this proves the desired equality. Again from Lemma 3.3.3, we deduce that y is transcendental over K. Hence, the assignment $x \mapsto y$ induces an isomorphism from $K(x)$ onto $K(y)$. Since the valuations of $K(x)$ and $K(y)$ are uniquely determined by its restriction to $K[x]$ and $K[y]$ respectively, it follows from what we have already proved that this isomorphism is valuation preserving. □

Corollary 3.4.2. *Take an extension $(L|K, v)$ of valued fields and $y \in L$. If* $\mathrm{appr}\,(y, K)$ *is an immediate transcendental approximation type, then y is transcendental over K and $(K(y)|K, v)$ is immediate.*

Proof. By the foregoing theorem, there is an immediate extension $(K(x)|K, v)$ such that $\mathrm{appr}\,(x, K) = \mathrm{appr}\,(y, K)$, with x transcendental over K. By the same theorem, there is a valuation preserving isomorphism of $K(x)$ and $K(y)$ over K. This proves our assertions. □

The next lemma will show that every immediate algebraic approximation type is of the form $\mathrm{appr}\,(y, K)$.

Lemma 3.4.3. *Take an immediate algebraic approximation type $\mathbf{A}$ over (K, v), a polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, and a root y of f. Then there is an extension of v from K to $K(y)$ such that* $\mathbf{A} = \mathrm{appr}\,(y, K)$.

Proof. We choose some extension w of v from K to $K(y)$. We write $f(X) = d\prod_{i=1}^{\deg f}(X - a_i)$ with $d \in K$ and $a_i \in \tilde{K}$. If for all i, the values $w(c - a_i)$ would be fixed for $c \nearrow \mathbf{A}$, then $\mathbf{A}$ would fix the value of f, contrary to our assumption. Hence there is a root a of f such that $w(a - c)$ is not fixed for $c \nearrow \mathbf{A}$. Take some automorphism σ of $\tilde{K}|K$ such that $\sigma y = a$ and set $v := w \circ \sigma$. Then v extends the valuation of K, and $v(y - c) = w \circ \sigma(y - c) = w(\sigma y - c) = w(a - c)$ is not fixed for $c \nearrow \mathbf{A}$. By Corollary 3.2.4, $\mathbf{A} = \mathrm{appr}\,(y, K)$. □

The following is the analogue of Theorem 3.4.1 for immediate algebraic approximation types.

Theorem 3.4.4. (Theorem 3 of [4], approximation type version)
For every immediate algebraic approximation type $\mathbf{A}$ over (K, v) of degree $\mathbf{d}$ with associated minimal polynomial $f(X) \in K[X]$ and y a root of f, there exists an extension of v from K to $K(y)$ such that $(K(y)|K, v)$ is an immediate extension and $\mathrm{appr}\,(y, K) = \mathbf{A}$.

If $(K(z), v)$ is another valued extension field of (K, v) such that $\mathrm{appr}\,(z, K) = \mathbf{A}$*, then any field isomorphism between $K(y)$ and $K(z)$ over K sending y to z will preserve the valuation. (Note that there exists such an isomorphism if and only if z is also a root of f.)*

Proof. We take the valuation v of $K(y)$ given by Lemma 3.4.3. Then $\mathrm{appr}\,(y,K) = \mathbf{A}$. The fact that $(K(y)|K,v)$ is immediate follows from Lemma 3.3.3.

The last assertion of our theorem is shown in the same way as the corresponding assertion of Theorem 3.4.1: if $\mathrm{appr}\,(y,K) = \mathrm{appr}\,(z,K)$ and $g \in K[X]$ with $\deg g < \mathbf{d}$ then, again by Lemma 3.3.3, $vg(y) = vg(c) = vg(z)$ for $c \nearrow x$. Hence an isomorphism over K sending y to z will preserve the valuation. □

From this theorem, we can derive important information about the degree of immediate algebraic approximation types.

Proposition 3.4.5. *The degree of an immediate algebraic approximation type over a henselian field (K,v) is a power of the characteristic of the residue field Kv.*

Proof. Take an immediate algebraic approximation type $\mathbf{A}$ over a henselian field (K,v) of degree $\mathbf{d}$. Then by Theorem 3.4.4 there is an immediate extension $(L|K,v)$ of degree $\mathbf{d}$. As (K,v) is henselian, the extension of v from K to L is unique. Hence by the Lemma of Ostrowski, i.e., by Lemma 2.7.4,

$$\mathbf{d} \;=\; [L:K] \;=\; p^{\nu}\cdot(vL:vK)\cdot[Lv:Kv] \;=\; p^{\nu}\;,$$

where $\nu \in \mathbb{N}\cup\{0\}$ and $p = \mathrm{char}\,Kv$. Note that $\nu > 0$ because the degree of $\mathbf{A}$ is not less than 2. □

Theorem 3.4.1 and Theorem 3.4.4 together imply:

Proposition 3.4.6. *Every immediate approximation type is realized in some immediate simple valued field extension.*

We say that a valued field (K,v) is **maximal** if it admits no proper immediate extensions. In this case, by the two theorems, it admits no immediate approximation types. On the other hand, if (K,v) admits no immediate approximation types, then by part b) of Lemma 3.2.1, it admits no proper immediate extensions. This proves:

Theorem 3.4.7. (Theorem 4 of [4], approximation type version)
A valued field (K,v) is maximal if and only if it does not admit immediate approximation types.

Similarly, we say that a valued field (K,v) is **algebraically maximal** if it does not admit proper immediate algebraic extensions. In this case, Theorem 3.4.4 shows that it does not admit immediate algebraic approximation types. On the other hand, if (K,v) admits a proper immediate algebraic extension $(L|K,v)$, and $x \in L\setminus K$, then by part b) of Lemma 3.2.1, $\mathrm{appr}\,(x,K)$ is an immediate approximation type, and by Corollary 3.3.5, it is algebraic. This proves:

Theorem 3.4.8. *A valued field (K,v) is algebraically maximal if and only if it does not admit immediate algebraic approximation types.*

3.5 The relative approximation degree of polynomials

In view of Proposition 3.4.6, we can from now on assume that every immediate approximation type $\mathbf{A}$ is of the form $\mathbf{A} = \mathrm{appr}\,(x, K)$. For the integer $\mathbf{h}$ that appears in Lemma 3.3.2, where $\deg f \leq \deg \mathbf{A}$, we will write $\mathbf{h}_K(x : f)$ or just $\mathbf{h}(x : f)$. We call $\mathbf{h}(x : f)$ the **relative approximation degree of $f(x)$ in x (over K)**. From Lemma 3.3.2 we know that

$$1 \leq \mathbf{h}_K(x : f) \leq \deg f \; .$$

One can extend the definition of the relative approximation degree to polynomials of arbitrary degree as follows. Take any polynomial $g \in K[X]$. Suppose that there exist $\beta \in vK$ and a positive integer k such that

$$v(g(x) - g(c)) = \beta + k \cdot v(x - c)$$

for $c \nearrow x$. Note that β and k are uniquely determined because as $\mathrm{appr}\,(x, K)$ is immediate, there are infinitely many values $v(x - c)$ for $c \nearrow x$. We will call k the **relative approximation degree of $g(x)$ in x**, denoted by $\mathbf{h}_K(x : g)$ as before. Further, we will call β the **relative approximation constant of $g(x)$ in x**, denoted by

$$\beta_K(x : g) \; .$$

By virtue of equation (3.3.5) of Lemma 3.3.2, our new definition of the relative approximation degree coincides with the definition as given for polynomials of degree at most $\mathbf{d}$. On the other hand, our new definition assigns a relative approximation degree to every polynomial of arbitrary degree whose value is not fixed, as Lemma 3.3.4 shows because in this case, $v(g(x) - g(c)) = vg(c)$ for $c \nearrow x$. However, for polynomials of degree bigger than $\mathbf{d}$, the relative approximation degree may not be a power of p. Unfortunately, Lemma 3.3.4 does not give information about the value $v(g(x) - g(c))$ if $\mathbf{A}$ fixes the value of g; this is an open problem.

From Lemma 3.3.4 we derive:

Corollary 3.5.1. *The value of g is fixed by $\mathbf{A}$ if and only if $vg(x) = vg(c)$ for $c \nearrow x$. On the other hand, $\mathbf{A}$ does not fix the value of g if and only if $vg(x) > vg(c)$ for $c \nearrow x$, and this holds if and only if*

$$v(g(x) - g(c)) = vg(c) = \beta_{\mathbf{h}}(x : g) + \mathbf{h}_K(x : g) \cdot v(x - c) \tag{3.5.1}$$

for $c \nearrow x$.

For the distances associated with $g(x)$, the following inequalities will hold in all cases where $\beta_K(x : g)$ and $\mathbf{h}_K(x : g)$ are defined:

$$\mathrm{dist}\,(g(x), K) \geq \mathrm{dist}_K(g(x), g(K)) \geq \beta_K(x : g) + \mathbf{h}_K(x : g) \cdot \mathrm{dist}\,(x, K) \tag{3.5.2}$$

(the first inequality is trivial and the second follows directly from the definition of relative approximation degree and relative approximation constant). In the next section, we will consider various cases where equalities hold.

We will now investigate the relative approximation degree more closely for the case of $\deg f \leq \deg \mathbf{A}$. We will first consider the relation between $\mathbf{h}_K(x:f)$ and the approximation type $\mathrm{appr}\,(f(x),K)$. Then we show that $\mathbf{h}_K(x:f)$ is a power of the characteristic exponent of the residue field, where the **characteristic exponent** of a field is defined to be its characteristic if this is positive, and 1 otherwise. Finally we will give some hints for the computation of $\mathbf{h}_K(x:f)$.

Throughout this and the next two sections, we will assume the following situation:

$$\left\{\begin{array}{ll} \mathbf{A} = \mathrm{appr}\,(x,K) & \text{an immediate approximation type over } (K,v) \\ p & \text{the characteristic exponent of } Kv, \\ \mathbf{d} & \text{the degree of } \mathrm{appr}\,(x,K), \\ f \in K[X] & \text{a nonconstant polynomial of degree } n \leq \mathbf{d}\,, \\ \mathbf{h} & = \mathbf{h}_K(x:f) \\ \beta_i & \text{the fixed value } vf_i(c) \text{ for } c \nearrow x. \end{array}\right. \tag{3.5.3}$$

Lemma 3.5.2. *Take another polynomial $g \in K[X]$ of degree at most $\mathbf{d}$ such that $\mathrm{appr}\,(x,K)$ fixes the value of $f-g$. If $\mathrm{appr}\,(f(x),K) = \mathrm{appr}\,(g(x),K)$, then $\mathbf{h}_K(x:f) = \mathbf{h}_K(x:g)$ and $\beta_K(x:f) = \beta_K(x:g)$.*

Proof. By part b) of Lemma 3.1.1, $\mathrm{appr}\,(f(x),K) = \mathrm{appr}\,(g(x),K)$ implies that

$$v(f(x)-g(x)) \geq \mathrm{dist}\,(f(x),K)\ .$$

By hypothesis, $\mathrm{appr}\,(x,K)$ fixes the value of $f-g$, hence by Lemma 3.3.2,

$$v(f(c)-g(c)) = v(f(x)-g(x)) \geq \mathrm{dist}\,(f(x),K) \geq v(f(x)-f(c)) \quad \text{for} \quad c \nearrow x\ .$$

As (3.3.5) shows that the values $v(f(x)-f(c))$ are increasing for $c \nearrow x$, the last inequality can be replaced by a strict inequality. So we obtain that

$$\begin{aligned} v(g(x)-g(c)) &= \min\{v(g(x)-f(x))\,,\, v(f(x)-f(c))\,,\, v(f(c)-g(c))\} \\ &= v(f(x)-f(c)) \;=\; \beta_K(x:f) + \mathbf{h}_K(x:f)\cdot v(x-c) \end{aligned}$$

for $c \nearrow x$. This implies our assertion. □

To achieve our second goal, we need the following lemma:

Lemma 3.5.3. *If p is prime and r is a positive integer prime to p, $r>1$, then*

$$\binom{p^t r}{p^t}$$

is prime to p, for every integer $t \geq 0$.

Proof. Consider

$$\binom{p^t r}{p^t} = \frac{p^t r(p^t r-1)\cdot\ldots\cdot(p^t r-p^t+1)}{p^t(p^t-1)\cdot\ldots\cdot 1}\ .$$

In the numerator of this fraction, the first factor $p^t r$ is divisible by precisely p^t, while the remaining factors $p^t r - m$, $1 \leq m \leq p^t - 1$, are not divisible by p^t. Hence, for every such factor occurring in the numerator, the corresponding factor $p^t - m = p^t r - m - p^t(r-1)$ which occurs in the denominator will be divisible by p to precisely the same power. This gives the desired result. □

Now we are able to prove:

Proposition 3.5.4. *If $i = p^t$, $j = p^t r \leq n$ with $r > 1$, $(r,p) = 1$, and if $\beta_i \neq \infty$, then for $c \nearrow x$,*

$$\beta_i + i \cdot v(x-c) < \beta_j + j \cdot v(x-c) \; .$$

Consequently, $\mathbf{h}_K(x:f)$ is a power of p (including the case of $\mathbf{h}_K(x:f) = 1 = p^0$).

Proof. We consider the Taylor expansion (3.3.3) for $f_i(x)$:

$$\begin{aligned} & f_i(x) - f_i(c) = \\ & (i+1) f_{i+1}(c)(x-c) + \ldots + \tbinom{j}{i} f_j(c)(x-c)^{j-i} + \ldots + \tbinom{n}{i} f_n(c)(x-c)^{n-i} \; . \end{aligned}$$

For $c \nearrow x$, the values $vf_{i+1}(c), \ldots, vf_n(c)$ will be equal to $\beta_{i+1}, \ldots, \beta_n$ as defined in (3.5.3). We apply Lemma 3.3.1 with $m = n - i$, $t_k = k$ for $1 \leq k \leq m$, and

$$\alpha_1 = v(i+1) + \beta_{i+1}\, , \ldots, \; \alpha_{j-i} = v\binom{j}{i} + \beta_j\, , \ldots, \; \alpha_m = v\binom{n}{i} + \beta_n \; .$$

We find that among the terms on the right hand side of the Taylor expansion, there will be precisely one which has least value for $c \nearrow x$. The value of this term must then equal the value of the left hand side of the Taylor expansion, which yields that the latter increases for $c \nearrow x$. But both values $vf_i(x)$ and $vf_i(c)$ are fixed for $c \nearrow x$. Hence, $v(f_i(x) - f_i(c)) > vf_i(x) = vf_i(c) = \beta_i$ for $c \nearrow x$. It follows that in particular, the term

$$\binom{j}{i} f_j(c)(x-c)^{j-i}$$

on the right hand side of the Taylor expansion will also have value $> \beta_i$ for $c \nearrow x$. But $v\binom{j}{i} = 0$: if $p > 0$, this is shown in Lemma 3.5.3, and if $p = 1$, then $\operatorname{char} Kv = 0$ which means that $\operatorname{char} K = 0$ and v is trivial on the subfield $\mathbb{Q}$ of K. Therefore,

$$\beta_i < \beta_j + (j-i) \cdot v(x-c)$$

for $c \nearrow x$. This yields our assertion. □

The following lemma will give more detailed information on the computation of $\mathbf{h}_K(x:f)$.

Lemma 3.5.5. *Assume that $v(x-c) \geq 0$ for $c \nearrow x$. If i is an integer such that β_i is minimal among all β_j, $j > 0$, then $\mathbf{h}_K(x:f) \leq i$.*

Proof. By assumption, we have that $\beta_j - \beta_i \geq 0$ for all $j > 0$. Further,

$$\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) < \beta_j + j \cdot v(x-c)$$

for $j > 0$, $j \neq \mathbf{h}$, and $c \nearrow x$. Thus,

$$0 \leq \beta_{\mathbf{h}} - \beta_i \leq (i - \mathbf{h}) \cdot v(x - c)$$

for $c \nearrow x$, which in view of $v(x - c) \geq 0$ for $c \nearrow x$ yields that $i - \mathbf{h} \geq 0$, which is the assertion. □

Lemma 3.5.6. *Assume that $p \geq 2$, and write $f(X) = c_n X^n + \ldots + c_0$. Suppose that there exists $i > 0$ such that $vc_i < vc_k$ for all $k > 0$, $j \neq i$, and write $i = p^t r$ with r prime to p. Then $vf_{\mathbf{h}}(c) \geq vc_i$ holds for every c with $vc = 0$. And if $vx = 0$, then*

$$\mathbf{h}_K(x : f) \leq p^t \ .$$

Proof. For $vc = 0$ and $j \geq 1$, by the definition (3.3.1) of the j-th formal derivative,

$$vf_j(c) \ = \ v \sum_{k=j}^{n} \binom{k}{j} c_k c^{k-j} \ \geq \ \min_{j \leq k \leq n} v \binom{k}{j} c_k c^{k-j} \ \geq \ vc_i \ .$$

By Lemma 3.5.3, the binomial coefficient $\binom{p^t r}{p^t}$ is not divisible by p. This shows that $v\binom{p^t r}{p^t} = 0$ and thus,

$$vf_{p^t}(c) = vc_i \ .$$

Now assume in addition that $vx = 0$. Then $vc = 0$ for $c \nearrow x$. This yields that

$$\beta_{p^t} = vc_i \leq \beta_j$$

for all $j > 0$. The foregoing lemma now gives our assertion. □

Corollary 3.5.7. *Assume that $vx = 0$, and take an integer $e \geq 1$. Suppose that all nonzero coefficients c_i of f, $i > 0$, have different values and that for all i with $p^e | i$, the coefficient c_i is equal to zero. Then $\mathbf{h}_K(x : f) < p^e$.*

3.6 Approximation types and distances of polynomials

Recall that throughout this section, we assume the situation of (3.5.3).

Lemma 3.6.1. *The following holds:*

$$c \in \mathrm{appr}\,(x, K)_\gamma \Longleftrightarrow f(c) \in \mathrm{appr}\,(f(x), K)_{\beta_{\mathbf{h}} + \mathbf{h} \cdot \gamma} \text{ for } c \nearrow x \ . \tag{3.6.1}$$

In particular,

$$\mathrm{dist}\,(f(x), K) \geq \mathrm{dist}_K(f(x), f(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}\,(x, K) \ . \tag{3.6.2}$$

Proof. Equation (3.3.5) of Lemma 3.3.2 yields (3.6.1), while $\mathrm{dist}\,(f(x), K) \geq \mathrm{dist}_K(f(x), f(K))$ was already stated in (3.5.2). It remains to prove that

$$\mathrm{dist}_K(f(x), f(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}\,(x, K) \ .$$

If $\mathrm{dist}\,(x,K) = \infty$, this equality follows immediately from (3.5.2). So let us assume from now on that $\mathrm{dist}\,(x,K) < \infty$. In order to deduce a contradiction, assume that there exists an element $c_0 \in K$ such that

$$v(f(x) - f(c_0)) > \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}\,(x,K)\ ,$$

or equivalently,

$$v(f(x) - f(c_0)) > v(f(x) - f(c))$$

for $c \nearrow x$. Hence

$$\begin{aligned} v(f(c_0) - f(c)) &= \min\{v(f(x) - f(c)), v(f(x) - f(c_0))\} \\ &= v(f(x) - f(c)) \ = \ \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \end{aligned}$$

for $c \nearrow x$. Replacing x by c_0 in the Taylor expansion (3.3.6), we find

$$\begin{aligned} v(f_1(c_0)\cdot(c-c_0) + \ldots + f_n(c_0)\cdot(c-c_0)^n) &= v(f(c_0) - f(c)) \\ &= \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \end{aligned}$$

for $c \nearrow x$. As noted already at the beginning of Section 3.2, an immediate approximation type fixes the value of every linear polynomial. Hence, $v(c - c_0)$ will be fixed for $c \nearrow x$. On the other hand, the value $\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)$ is not fixed for $c \nearrow x$, so we conclude that the value

$$v(\, f_1(c_0) + f_2(c_0)\cdot(c-c_0) + \ldots + f_n(c_0)\cdot(c-c_0)^{n-1}\,)$$

is not fixed for $c \nearrow x$. This proves the existence of a polynomial of degree $n-1$ whose value is not fixed by $\mathrm{appr}\,(x,K)$. But $n - 1 = \deg f - 1 < \mathbf{d}$, a contradiction. This proves the desired equality. □

Lemma 3.6.2. *Assume that* $\deg f < \mathbf{d}$. *Then* $\mathrm{appr}\,(f(x),K)$ *is an immediate approximation type over* K *with*

$$\mathrm{dist}\,(f(x),K) = \mathrm{dist}_K(f(x), f(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}\,(x,K), \tag{3.6.3}$$

and $\mathrm{appr}\,(f(x),K)$ *is determined by (3.6.1).*

Proof. In view of (3.6.2), to prove the first equality in (3.6.3) we have to show that for every $b \in K$ there exists an element $c \in K$ such that $v(f(x) - f(c)) \geq v(f(x) - b)$. Since $\deg(f - b) = \deg f < \mathbf{d}$, it follows that $\mathrm{appr}\,(x,K)$ fixes the value of $f - b$. Applying Lemma 3.3.2 to $f - b$ in place of f, we deduce that $v(f(x) - b) = v(f(c) - b)$ for $c \nearrow x$. Consequently, for such an element $c \in K$ we get that

$$v(f(x) - f(c)) \geq \min\{v(f(x) - b), v(f(c) - b)\} = v(f(x) - b)\ ,$$

as desired.

By the second equality of (3.6.3), which has already been proved in Lemma 3.6.1, we know that there exists $c' \in K$ such that $v(f(x) - f(c')) > v(f(x) - f(c)) \geq v(f(x) - b)$. We have proved that for every

$b \in K$ there is $b' = f(c') \in K$ such that $v(f(x) - b') > v(f(x) - b)$. Part a) of Lemma 3.2.1 now shows that appr $(f(x), K)$ is immediate.

By (3.6.3), the values $\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)$ are cofinal in supp appr $(f(x), K)$ for $c \nearrow x$. Therefore, appr $(f(x), K)$ is determined by the balls appr $(f(x), K)_{\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)}$ for those c, which in turn are determined by (3.6.1). □

Corollary 3.6.3. *Assume that* $\deg f < \mathbf{d}$, *and let* $d' \geq 1$ *be a natural number such that* $d' \cdot \deg f \leq \mathbf{d}$. *Then*

$$\deg \operatorname{appr}(f(x), K) \geq d' .$$

In particular, if appr (x, K) *is transcendental, then so is* appr $(f(x), K)$.

Proof. Let g be a polynomial of degree smaller than $d' \leq \mathbf{d}$. Suppose that appr $(f(x), K)$ does not fix the value of g. Then by Lemma 3.3.2,

$$vg(f(x)) \; > \; vg(a)$$

for $a \nearrow f(x)$. Since $\deg f < \mathbf{d}$, Lemma 3.6.2 shows that $\operatorname{dist}_K(f(x), f(K)) = \operatorname{dist}(f(x), K)$, so

$$vg(f(x)) \; > \; vg(f(c))$$

for $c \nearrow x$. But then by Lemma 3.3.2, appr (x, K) does not fix the value of the polynomial $f(g(X))$. This contradicts the fact that its degree is smaller than $\mathbf{d}$. □

Lemma 3.6.4. *Assume that* appr (x, K) *does not fix the value of* f *(hence* $\deg f = \mathbf{d}$*). Then*

$$vf(x) > \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \quad \textit{for } c \nearrow x.$$

Proof. We rewrite (3.3.6) as follows:

$$-f(c) = f_1(c) \cdot (x-c) + \ldots + f_n(c) \cdot (x-c)^n - f(x) .$$

Suppose that $vf(x) < \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)$ for $c \nearrow x$. This in turn implies that the value of the right hand side is equal to $vf(x)$ and hence the value $vf(c)$ is fixed for for $c \nearrow x$, which contradicts our assumption. This proves that $vf(x) \geq \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)$, and since $v(x-K)$ has no maximal element, also $vf(x) > \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)$ for $c \nearrow x$. □

Note that in the case of $\deg f = \mathbf{d}$ we can only say that "appr $(f(x), K)$ is determined by (3.6.1) up to $\operatorname{dist}_K(f(x), f(K))$". But it may happen that

$$\operatorname{dist}(f(x), K) > \operatorname{dist}_K(f(x), f(K)) .$$

This will usually be the case when f is the minimal polynomial of x, which yields that $f(x) = 0$ and hence $\operatorname{dist}(f(x), K) = \operatorname{dist}(0, K) = \infty$.

Example 3.6.5. Take (L, v) and $f(X) = X^p - X - t^{-1}$ with root ϑ as in Example 3.3.6. As noted there, $v(\vartheta - L) = \{\alpha \in vL \mid \alpha < 0\}$, so $\mathrm{dist}\,(\vartheta, L)$ is the cut in $\widetilde{vL}$ whose lower cut set consists of all negative elements. This implies that $\mathrm{dist}\,(\vartheta, L) = p \cdot \mathrm{dist}\,(\vartheta, L)$.

We have that $f(X) - f(c) = X^p - X - (c^p - c) = (X - c)^p - (X - c)$. Since $v(\vartheta - c) < 0$, it follows that $v(\vartheta - c)^p = p \cdot v(\vartheta - c) < v(\vartheta - c)$ and therefore, $v(f(\vartheta) - f(c)) = v((\vartheta - c)^p - (\vartheta - c)) = \min\{v(\vartheta - c)^p, v(\vartheta - c)\} = p \cdot v(\vartheta - c)$. This shows that $\mathbf{h}_L(x : f) = p$ and $\beta_L(x : f) = 0$. We obtain that

$$\mathrm{dist}\,(f(\vartheta), L) \;=\; \infty \;>\; \mathrm{dist}\,(\vartheta, L) \;=\; p \cdot \mathrm{dist}\,(\vartheta, L) \;=\; \mathrm{dist}_{\,L}(f(\vartheta), f(L))\,,$$

where the last equality holds by Lemma 3.6.1.

Chapter 4

The Relative Approximation Degree in Valued Function Fields

4.1 The degree $[K(x)^h : K(f(x))^h]$

In the situation of (3.5.3), we ask for the degree

$$[K(x)^h : K(f(x))^h] \; .$$

This can indeed be calculated by means of $\mathbf{h}_K(x:f)$. Inequality (4.1.1) below will explain the origin of the notation "$\mathbf{h}_K(x:f)$". Note that $[K(x):K(f(x))] = \deg f$, while in general, we may have that $[K(x)^h : K(f(x))^h] < \deg f$.

Theorem 4.1.1. *Assume (3.5.3). Then*

$$[K(x)^h : K(f(x))^h] \leq \mathbf{h}_K(x:f) \; . \tag{4.1.1}$$

Proof. We consider the Taylor expansion (3.3.2) of f for an arbitrary $c \in K$. From Lemma 3.3.2, we know that (3.3.4) holds for $1 \leq i \leq \deg f$, $i \neq \mathbf{h} = \mathbf{h}_K(x:f)$ and $c \nearrow x$. We choose such an element $c \in K$ and also an element $d \in K$ with $vd = -v(x-c)$. We set $x_0 = d \cdot (x-c)$; hence $vx_0 = 0$ and $K(x) = K(x_0)$. Now (3.3.4) takes the form

$$v(f_i(c)d^{-i}) > v(f_{\mathbf{h}}(c)d^{-\mathbf{h}}) \;\text{ for } i \neq \mathbf{h},\, 1 \leq i \leq \deg f, \tag{4.1.2}$$

and (3.3.5) reads as

$$v(f(x) - f(c)) = v f_{\mathbf{h}}(c) d^{-\mathbf{h}} \; . \tag{4.1.3}$$

Further, from (3.3.2), (4.1.2) and (4.1.3) we obtain:

$$\begin{aligned} \left(\frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (f(c) - f(x))\right) v &= \left(-\frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot \sum_{i=1}^{\deg f} f_i(c)(x-c)^i\right) v \\ &= \left(-\sum_{i=1}^{\deg f} \frac{f_i(c)d^{-i}}{f_{\mathbf{h}}(c)d^{-\mathbf{h}}}\, x_0^i\right) v \;=\; -(x_0 v)^{\mathbf{h}} \; . \end{aligned} \tag{4.1.4}$$

Now we set

$$\tilde{f}(Z) = \sum_{i=0}^{\deg f} f_i(c) d^{-i} Z^i \; ;$$

hence $\tilde{f}(x_0) = f(x)$. Let us consider the polynomial

$$F(Z) = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (\tilde{f}(Z) - \tilde{f}(x_0))$$

whose coefficients lie in $K(\tilde{f}(x_0)) = K(f(x))$ and for which x_0 is a zero. Using (4.1.2) and (4.1.3), we compute

$$F(Z) = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (f(c) - f(x)) + \sum_{i=1}^{\deg f} \frac{f_i(c) d^{-i}}{f_{\mathbf{h}}(c) d^{-\mathbf{h}}} Z^i \in \mathcal{O}_{K(f(x))}[Z]$$

and, using also (4.1.4),

$$F(Z)v = Z^{\mathbf{h}} - (x_0 v)^{\mathbf{h}} = (Z - x_0 v)^{\mathbf{h}}$$

(where the latter equation holds because by Proposition 3.5.4, $\mathbf{h}$ is a power of p). Using the strong Hensel's Lemma, i.e., part b) of Theorem 2.7.1, we deduce that there is a factorization

$$F(Z) = G(Z)H(Z)$$

over $K(f(x))^h$ with

$$G(Z)v = Z^{\mathbf{h}} - (x_0 v)^{\mathbf{h}}$$

and

$$\deg G(Z) = \deg G(Z)v = \mathbf{h} .$$

A zero of $F(Z)$ which has residue $x_0 v$ cannot be a zero of $H(Z)$ since $H(Z)v = 1$, hence it must appear as a zero of $G(Z)$. In particular, $G(x_0) = 0$. Since $G(Z) \in K(f(x))^h[Z]$ and $\deg G(Z) = \mathbf{h}$, and since $K(x_0)^h = K(f(x))^h(x_0)$, this shows that

$$[K(x)^h : K(f(x))^h] = [K(x_0)^h : K(f(x))^h] \leq \mathbf{h} = \mathbf{h}_K(x : f) .$$

□

Corollary 4.1.2. *In addition to (3.5.3), assume that* (K, v) *is henselian and* x *is algebraic over* K*. If* $\mathbf{d} = [K(x) : K]$ *and* f *is the minimal polynomial of* x *over* K*, then* $p \geq 2$ *and*

$$[K(x) : K] = \mathbf{h}_K(x : f) = p^t$$

for some integer $t \geq 1$*.*

Proof. By hypothesis, we have $\mathbf{d} = [K(x) : K] = \deg f$. Since K is henselian and x is algebraic over K, we have that $K(x)$ is henselian as well. In view of $f(x) = 0$, an application of the foregoing lemma shows that

$$\deg f = [K(x) : K] \leq \mathbf{h}_K(x : f) \leq \deg f .$$

Consequently, equality holds everywhere.

Since $\mathrm{appr}\,(x, K)$ is immediate by assumption, it is nontrivial, hence $x \notin K$ and $\mathbf{h}_K(x : f) = [K(x) : K] > 1$. Proposition 3.5.4 yields that $p \geq 2$ and $\mathbf{h}_K(x : f) = p^t$ with $t \geq 1$. □

4.2 The degree $[K(x)^h : K(y)^h]$

Throughout this section, we will work with the following situation:

$$\begin{cases} (K,v) \text{ a valued field of rank 1} \\ (K(x)|K,v) \text{ an immediate extension such that } x \notin K^c \\ \text{and appr}\,(x,K) \text{ is transcendental} \\ y \in K(x)^h \text{ transcendental over } K. \end{cases} \tag{4.2.1}$$

Note that by Corollary 3.4.1, the assumption that appr (x, K) is transcendental implies that x is transcendental over K. Furthermore, if (K, v) is algebraically maximal, then appr (x, K) is always transcendental, provided that $(K(x)|K, v)$ is immediate and nontrivial.

We ask for the degree

$$[K(x)^h : K(y)^h] \ .$$

To treat this question and in particular to define the relative approximation degree of x over y, we look for a polynomial $f \in K[X]$ such that

$$v(y - f(x)) \geq \text{dist}\,(y, K) \ . \tag{4.2.2}$$

We need some preparation.

Lemma 4.2.1. *If K is of rank 1 and $K(x)|K$ is immediate, then $K[x]$ is dense in $K(x)^h$.*

Proof. Since any valued field of rank 1 is dense in its henselization, it suffices to show that $K[x]$ is dense in $K(x)$. For this we only have to show that for every $f(x) \in K[x]$ and every $\alpha \in vK$ there exists an element $g(x) \in K[x]$ such that $v(g(x) - 1/f(x)) > \alpha$. Since $K(x)|K$ is immediate there is an element $c \in K$ satisfying $v(c - f(x)) > vf(x) = vc$, which yields that $v(1 - f(x)/c) > 0$. By our hypothesis on the rank which means that the value group vK is archimedian, there exists $j \in \mathbb{N}$ such that $j \cdot v(1 - f(x)/c) > \alpha + vc$. Now we put $h(x) = 1 - f(x)/c \in K[x]$ and compute

$$\begin{aligned} v\left(\frac{1}{f(x)} - c^{-1}\sum_{i=0}^{j-1} h(x)^i\right) &= v\left(\frac{1}{c(1-h(x))} - c^{-1}\sum_{i=0}^{j-1} h(x)^i\right) \\ &= vc^{-1}h(x)^j = j \cdot v(1 - f(x)/c) - vc > \alpha \ . \end{aligned}$$

As the sum is an element of $K[x]$, this proves our lemma. □

Lemma 4.2.2. *Assume (4.2.1). Then $y \in K[x]^c \setminus K^c$ and there exists a polynomial $f \in K[X]$ such that (4.2.2) holds.*

Proof. From Lemma 4.2.1, we infer that $y \in K[x]^c$. Suppose that $y \in K^c$. Then K is dense in $K(y)$ and also in $K(y)^h$ since $K(y)$ is dense in its henselization, being of rank 1 like K. Let $g(X) \in K(y)^h[X]$ be the minimal polynomial of x over $K(y)^h$. We can choose polynomials $\tilde{g}(X) \in K[X]$ with coefficients arbitrarily close to the corresponding coefficients of g. By the continuity of roots (cf. Theorem 4.5 of [PZ]) and our assumption that $x \notin K^c$, i.e., $\mathrm{dist}\,(x,K) < \infty$, we can find a suitable polynomial $\tilde{g}$ with a suitable root $\tilde{x} \in \tilde{K}$ such that

$$v(x-\tilde{x}) \geq \mathrm{dist}\,(x,K)\;.$$

By Lemma 3.1.1 b), this implies that

$$\mathrm{appr}\,(x,K) = \mathrm{appr}\,(\tilde{x},K)\;.$$

Since $\tilde{x}$ is algebraic over K, it follows by Corollary 3.3.5 that $\mathrm{appr}\,(\tilde{x},K)$ and hence $\mathrm{appr}\,(x,K)$ is an algebraic approximation type over K, a contradiction to hypothesis (4.2.1). This shows that $y \notin K^c$, i.e., $\mathrm{dist}\,(y,K) < \infty$. As $y \in K[x]^c$, this shows the existence of a polynomial $f \in K[X]$ such that $v(y - f(x)) \geq \mathrm{dist}\,(y,K)$. □

With f as in this lemma, we define

$$\mathbf{h}_K(x:y) \;:=\; \mathbf{h}_K(x:f) \;\text{ and }\; \beta_K(x:y) \;:=\; \beta_K(x:f)$$

and call $\mathbf{h}_K(x:y)$ the **relative approximation degree of** y **in** x (over K).

Lemma 4.2.3. *The integers $\mathbf{h}_K(x:y)$ and $\beta_K(x:y)$ are well-defined, i.e., they does not depend on the choice of $f(x)$ as long as $v(y-f(x)) \geq \mathrm{dist}\,(y,K)$ is satisfied.*

Proof. If $g(x)$ is another polynomial in $K[x]$ such that $v(y-g(x)) \geq \mathrm{dist}\,(y,K)$, then by Lemma 3.1.1, we have that

$$\mathrm{appr}\,(g(x),K) = \mathrm{appr}\,(y,K) = \mathrm{appr}\,(f(x),K)\;,$$

whence $\mathbf{h}_K(x:g) = \mathbf{h}_K(x:f)$ and $\beta_K(x:g) = \beta_K(x:f)$ by Lemma 3.5.2 since $\mathrm{appr}\,(x,K)$ is transcendental.

□

In the situation described in (4.2.1), we can prove Theorem 4.1.1 also for y in place of $f(x)$ provided that the extension $K(x)^h|K(y)^h$ is separable. For the proof, we need the following lemma:

Lemma 4.2.4. *Assume (4.2.1) and let $v(y-f(x)) \geq \mathrm{dist}\,(y,K)$. Then there exists an element z in the algebraic closure $\widetilde{K(y)}$ of $K(y)$ such that*

$$[K(y,z)^h : K(y)^h] \leq \mathbf{h} = \mathbf{h}_K(x:y)$$

and

$$v(x-z) \geq \frac{1}{\mathbf{h}}\left(v(y-f(x)) - \beta_K(x:f)\right)\;.$$

Proof. Recall that $\mathbf{h} = \mathbf{h}_K(x:y) = \mathbf{h}_K(x:f)$. We put $r := y - f(x)$. We choose $c,d \in K$, x_0 and $F(Z)$ as in the proof of Theorem 4.1.1. Then

$$\begin{aligned} vr \;\geq\; \mathrm{dist}\,(y,K) \;&>\; v(y-f(c)) \;=\; v(f(x)-f(c)) \\ &=\; v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) \;=\; v(f_{\mathbf{h}}(c)d^{-\mathbf{h}})\;. \end{aligned}$$

This shows that

$$F^\circ(Z) := F(Z) - \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot r = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (\tilde{f}(Z) - y) \ \in \ \mathcal{O}_{K(y)}[Z]$$

has the same reduction as $F(Z)$. We find, as for $F(Z)$, that $F^\circ(Z)$ admits a factorization

$$F^\circ(Z) = G^\circ(Z)H^\circ(Z)$$

over $K(y)^h$ with $G^\circ(Z)v = Z^{\mathbf{h}} - (x_0 v)^{\mathbf{h}}$, G° monic, $\deg G^\circ(Z) = \deg G^\circ(Z)v = \mathbf{h}$ and $H^\circ(Z)v = 1$. Note that $vF^\circ(x_0) = vG^\circ(x_0)$ since $x_0 \in \mathcal{O}_{K(x)}$. Recall that $F(x_0) = 0$. Consequently, from

$$F^\circ(x_0) = -\frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot r$$

it follows that, with $\beta_{\mathbf{h}} = vf_{\mathbf{h}}(c) = \beta_K(x:f)$,

$$v(d^{\mathbf{h}} r) - \beta_{\mathbf{h}} = vF^\circ(x_0) = vG^\circ(x_0) \ .$$

Hence there must exist a root z_{j_0} of

$$G^\circ(Z) = \prod_{1 \le j \le \mathbf{h}} (Z - z_j) \ , \quad z_j \in \widetilde{K(y)}$$

with

$$v(x_0 - z_{j_0}) \ge \frac{1}{\mathbf{h}} \left(v(d^{\mathbf{h}} r) - \beta_{\mathbf{h}} \right) \ ,$$

which is equivalent to

$$v(x - (d^{-1} z_{j_0} + c)) \ge \frac{1}{\mathbf{h}} (vr - \beta_{\mathbf{h}}) = \frac{1}{\mathbf{h}} (v(y - f(x)) - \beta_K(x:f)) \ .$$

Now $z := d^{-1} z_{j_0} + c$ is the element of our assertion, since it satisfies $K(y,z) = K(y, z_{j_0})$ and thus $[K(y,z)^h : K(y)^h] \le \mathbf{h}$. □

Proposition 4.2.5. *Assume (4.2.1). If $K(x)^h | K(y)^h$ is separable, then*

$$[K(x)^h : K(y)^h] \ \le \ \mathbf{h}_K(x:y) \ .$$

Proof. Set

$$\alpha := \max\{v(\sigma x - x) \mid \sigma \in \mathrm{Gal}(\widetilde{K(y)} | K(y)^h) \text{ with } \sigma x \ne x\} \ .$$

Then $\alpha < \infty$ since $K(x)^h | K(y)^h$ is separable. Now, by Lemma 4.2.2 we can choose $f(x) \in K[x]$ such that $v(y - f(x)) \ge \mathrm{dist}\,(y, K) = \mathrm{dist}\,(f(x), K)$ and

$$v(y - f(x)) \ > \ \beta_K(x:f) + \mathbf{h}\alpha \ ,$$

where $\mathbf{h} = \mathbf{h}_K(x:y)$. Using the foregoing lemma, we choose $z \in \widetilde{K(y)}$ such that

$$v(x - z) \ \ge \ \frac{1}{\mathbf{h}} (v(y - f(x)) - \beta_K(x:f)) \ > \ \alpha \ ,$$

and $[K(y,z)^h : K(y)^h] \le \mathbf{h}$. In view of our separability condition, we can deduce by Krasner's Lemma (see [3], Theorem 4.1.7) that $x \in K(y)^h(z)$. This yields that $[K(x,y)^h : K(y)^h] \le [K(y,z)^h : K(y)^h] \le \mathbf{h}$. Since $y \in K(x)^h$ by assumption, $K(x,y)^h = K(x)^h$ and thus $[K(x)^h : K(y)^h] \le \mathbf{h}$, as asserted. □

In order to prove the assertion of the proposition without the separability condition, we need the following tool.

Lemma 4.2.6. *Assume that (4.2.1) holds. Then it also holds for y in place of x. So if $z \in K(y)^h$ is transcendental over K, then $\mathbf{h}_K(y:z)$ is defined. In this situation, $\mathbf{h}_K(x:z) = \mathbf{h}_K(x:y) \cdot \mathbf{h}_K(y:z)$.*

Proof. Recall that from Lemma 4.2.2 we have that $y \notin K^c$. Moreover, as $K(y)|K$ is a subextension of the immediate extension $K(x)^h|K$, it is also immediate. For the definition of $\mathbf{h}_K(x:y)$ we have already used the fact that there exists some polynomial $f(x)$ such that $\mathrm{appr}\,(y,K) = \mathrm{appr}\,(f(x),K)$; by Corollary 3.6.3, this approximation type is transcendental since $\mathrm{appr}\,(x,K)$ is. We have proved that (4.2.1) holds for y in place of x.

Let us now prove the multiplicativity. Since $\mathbf{h}_K(y:z) = \mathbf{h}_K(y:g(y))$ whenever $v(z-g(y)) \geq \mathrm{dist}\,(z,K)$, it suffices to show our assertion under the additional assumption $z = g(y) \in K[y]$. Furthermore, because of $y \in K[x]^c \setminus K^c$ we may choose $f(x) \in K[x]$ so that $v(y-f(x)) \geq \mathrm{dist}\,(y,K)$ and $v(g(y)-g(f(x))) \geq \mathrm{dist}\,(g(y),K)$; hence it suffices to show our assertion under the assumption that $y = f(x) \in K[x]$ and $z = g(f(x)) \in K[x]$. Since by hypothesis, $\mathrm{appr}\,(x,K)$ is transcendental, it fixes the value of every polynomial over K, and thus we know from Lemma 3.6.2 that $f(c) \nearrow f(x)$ whenever $c \nearrow x$. Also since $\mathrm{appr}\,(f(x),K)$ is transcendental, it fixes the value of every polynomial over K, and thus for $f(c) \nearrow f(x)$,

$$\begin{aligned} v(g(f(x)) - g(f(c))) &= vg_{\mathbf{h}_1}(f(c)) + \mathbf{h}_1 \cdot v(f(x) - f(c)) \\ &= vg_{\mathbf{h}_1}(f(c)) + \mathbf{h}_1 \cdot (vf_{\mathbf{h}_2}(c) + \mathbf{h}_2 \cdot v(x-c)) \\ &= \beta + \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot v(x-c) \end{aligned}$$

where $\mathbf{h}_1 = \mathbf{h}_K(f(x):g(f(x)))$, $\mathbf{h}_2 = \mathbf{h}_K(x:f)$ and $\beta = vg_{\mathbf{h}_1}(f(c)) + \mathbf{h}_1 \cdot vf_{\mathbf{h}_2}(c)$. This shows that

$$\mathbf{h}_K(x:g(f(x))) = \mathbf{h}_1 \cdot \mathbf{h}_2 = \mathbf{h}_2 \cdot \mathbf{h}_1 = \mathbf{h}_K(x:f) \cdot \mathbf{h}_K(f(x):g(f(x))) \,,$$

as asserted. □

Theorem 4.2.7. *Assume (4.2.1). Then*

$$[K(x)^h : K(y)^h] \;\leq\; \mathbf{h}_K(x:y) \,.$$

Proof. Take p^n to be the inseparable degree of $K(x)^h|K(y)^h$ and $L|K(y)^h$ to be the maximal separable subextension of $K(x)^h|K(y)^h$. Then $[K(x)^h : L] = p^n$. Further, x^{p^n} is separable over $K(y)^h$, so $x^{p^n} \in L$ and $K(x^{p^n})^h \subseteq L$. As $K(x)^h = K(x^{p^n})^h(x)$, we find that

$$p^n \;\geq\; [K(x)^h : K(x^{p^n})^h] \;=\; [K(x)^h : L] \cdot [L : K(x^{p^n})^h] \;=\; p^n \cdot [L : K(x^{p^n})^h] \,,$$

which shows that $[L : K(x^{p^n})^h] = 1$ and in particular, $y \in K(x^{p^n})^h$. So we are able to apply Lemma 4.2.6 to obtain that $\mathbf{h}_K(x:y) = \mathbf{h}_K(x:x^{p^n}) \cdot \mathbf{h}_K(x^{p^n}:y) = p^n \cdot \mathbf{h}_K(x^{p^n}:y)$.

As x^{p^n} is separable over $K(y)^h$, we can infer from Proposition 4.2.5 that $[K(x^{p^n})^h : K(y)^h] \leq \mathbf{h}_K(x^{p^n} : y)$. On the other hand, $[K(x)^h : K(x^{p^n})^h] = [K(x)^h : L] = p^n$. So we get

$$[K(x)^h : K(y)^h] \;=\; p^n \cdot [K(x^{p^n})^h : K(y)^h] \;\leq\; p^n \cdot \mathbf{h}_K(x^{p^n} : y) \;=\; \mathbf{h}_K(x : y)\;,$$

as desired. □

Corollary 4.2.8. *Assume that (4.2.1) holds. Then*

$$K(x)^h = K(y)^h \iff \mathbf{h}_K(x : y) = 1\;.$$

Proof. If $K(x)^h = K(y)^h$, then $x \in K(y)^h$ and $y \in K(x)^h$, and by Lemma 4.2.6 we have that

$$\mathbf{h}_K(x : y) \cdot \mathbf{h}_K(y : x) \;=\; \mathbf{h}_K(x : x) \;=\; 1\;,$$

which yields $\mathbf{h}_K(x : y) = 1$. The reverse implication follows from Theorem 4.2.7. □

4.3 An application to henselian rationality

In this section we will apply Theorem 4.2.7 to immediate valued function fields which are the henselization of a rational function field.

Theorem 4.3.1. *Take a valued field (K, v) of rank 1 and an immediate function field $(F|K, v)$ of transcendence degree 1. Suppose there is some $x \in F^h \setminus K^c$ with transcendental approximation type over K such that $F^h = K(x)^h$. Then there is already some $y \in F$ such that $F^h = K(y)^h$.*

Proof. Since $x \notin K^c$ there is $\gamma \in vK$ such that $\gamma \geq \mathrm{dist}\,(x, K)$. By assumption, the rank of (K, v) is 1, and since $(F|K, v)$ is immediate, also (F, v) has rank 1. Thus, the element x lies in the completion of F. So we may take some $y \in F$ such that $v(x-y) > \mathrm{dist}\,(x, K)$. Hence by Theorem 4.2.7, $[K(x)^h : K(y)^h] \leq \mathbf{h}_K(x : y)$, and by Lemma 4.2.3, $\mathbf{h}_K(x : y) = \mathbf{h}_K(x : x) = 1$. This proves that $K(x)^h = K(y)^h$. □

4.4 Approximation coefficients

Throughout this section, we will assume the situation as described in (4.2.1). As before, take $f(x) \in K[x]$ such that $v(y - f(x)) \geq \mathrm{dist}\,(y, K)$. An element $d \in K$ will be called an **approximation coefficient of** y **in** x (over K), if

$$v(f(x) - f(c)) < v(f(x) - f(c) - d \cdot (x - c)^{\mathbf{h}}) \tag{4.4.1}$$

for $c \nearrow x$, where $\mathbf{h} = \mathbf{h}_K(x : y)$.

Lemma 4.4.1. *If d satisfies (4.4.1) for some $f(x)$ with $v(y - f(x)) \geq \mathrm{dist}\,(y, K)$, then it satisfies (4.4.1) for every such $f(x)$; in other words: approximation coefficients are independent of the choice of $f(x)$. If d satisfies (4.4.1), then it satisfies*

$$v(y - f(c)) < v(y - f(c) - d \cdot (x - c)^{\mathbf{h}}) \quad \textit{for } c \nearrow x\;. \tag{4.4.2}$$

Proof. If $g(x)$ is another element of $K[x]$ with $v(y - g(x)) \geq \mathrm{dist}\,(y, K)$, then

$$v(f(x) - g(x)) \geq \mathrm{dist}\,(y, K) = \mathrm{dist}\,(f(x), K) > v(f(x) - f(c))$$

for all $c \in K$. Since $\mathrm{appr}\,(x, K)$ is transcendental, it fixes the value of the polynomial $f - g$, whence

$$v(f(c) - g(c)) = v(f(x) - g(x)) > v(f(x) - f(c)) \quad \text{for } c \nearrow x\,.$$

Hence by the ultrametric triangle law,

$$\begin{aligned} v(g(x) - g(c)) &= \min\{v(g(x) - f(x)), v(f(x) - f(c)), v(f(c) - g(c))\} \\ &= v(f(x) - f(c)) \end{aligned}$$

and

$$\begin{aligned} & v(g(x) - g(c) - d \cdot (x - c)^{\mathbf{h}}) \\ \geq \ & \min\{v(f(x) - f(c) - d \cdot (x - c)^{\mathbf{h}})\,,\ v(f(x) - g(x))\,,\ v(f(c) - g(c))\} \\ > \ & v(f(x) - f(c)) = v(g(x) - g(c)) \end{aligned}$$

for $c \nearrow x$, which shows that d fulfills equation (4.4.1) also with g in place of f. Replacing $g(x)$ by y and $g(c)$ by $f(c)$ in the above deduction, one obtains a proof of (4.4.2). □

The following lemma proves the existence of approximation coefficients:

Lemma 4.4.2. *The element $d \in K$ is an approximation coefficient of y in x if and only if*

$$vd = vf_{\mathbf{h}}(c) < v(f_{\mathbf{h}}(c) - d) \quad \textit{for} \quad c \nearrow x\,.$$

In particular, there exists an approximation coefficient of y in x. Furthermore,

$$\mathrm{dist}\,(y, K) = vd + \mathbf{h} \cdot \mathrm{dist}\,(x, K) \tag{4.4.3}$$

Proof. By definition of $\mathbf{h} = \mathbf{h}_K(x : y) = \mathbf{h}_K(x : f)$, we have that

$$v(f(x) - f(c) - f_{\mathbf{h}}(c)(x - c)^{\mathbf{h}}) > v(f(x) - f(c)) = v(f_{\mathbf{h}}(c)(x - c)^{\mathbf{h}})$$

for $c \nearrow x$. Hence (4.4.1) holds for $c \nearrow x$ if and only if

$$v(f_{\mathbf{h}}(c)(x - c)^{\mathbf{h}} - d \cdot (x - c)^{\mathbf{h}}) > v(f_{\mathbf{h}}(c)(x - c)^{\mathbf{h}})\,,$$

which is equivalent to

$$vf_{\mathbf{h}}(c) < v(f_{\mathbf{h}}(c) - d) \quad \text{for} \quad c \nearrow x\,.$$

Since $K(x)|K$ is assumed to be an immediate extension, by Lemma 3.2.1 a) there exists some $d \in K$ such that $v(f_{\mathbf{h}}(x) - d) > vf_{\mathbf{h}}(x)$. Since $\mathrm{appr}\,(x, K)$ is transcendental, for $c \nearrow x$ we have that $v(f_{\mathbf{h}}(c) - d) = v(f_{\mathbf{h}}(x) - d)$ and $vf_{\mathbf{h}}(c) = vf_{\mathbf{h}}(x)$ and thus,

$$v(f_{\mathbf{h}}(c) - d) = v(f_{\mathbf{h}}(x) - d) > vf_{\mathbf{h}}(x) = vf_{\mathbf{h}}(c) = vd\,.$$

Hence d is an approximation coefficient for y in x by the first part of our proof.

In view of the hypothesis that $\mathrm{appr}\,(x,K)$ is transcendental, $f(x)$ satisfies equation (3.6.3) of Lemma 3.6.2. From this we obtain:

$$\begin{aligned}\mathrm{dist}\,(y,K) &= \mathrm{dist}\,(f(x),K) = vf_{\mathbf{h}}(c) + \mathbf{h}\cdot\mathrm{dist}\,(x,K)\\ &= vd + \mathbf{h}\cdot\mathrm{dist}\,(x,K)\;.\end{aligned}$$

□

Lemma 4.4.3. *Take elements $y_i \in K[x]^c\backslash K^c$ with common approximation degree $\mathbf{h} = \mathbf{h}_K(x:y_i)$, $1\leq i\leq m$. Assume that $d_i \in K$ is an approximation coefficient of y_i in x and let k_i be elements in K such that*

$$v\sum_{i=1}^{m} k_i d_i \;=\; \min_{1\leq i\leq m} vk_id_i \;<\infty\;. \tag{4.4.4}$$

Then the following will hold:

$$\mathbf{h}_K\left(x:\sum_{i=1}^{m}k_iy_i\right) = \mathbf{h}\;.$$

Proof. We choose polynomials $f^{[i]}(X)\in K[X]$ with $v(y_i - f^{[i]}(x)) \geq \mathrm{dist}\,(y_i,K)$. Then by Lemma 3.1.1 b), we have that $\mathrm{dist}\,(f^{[i]}(x),K) = \mathrm{dist}\,(y_i,K)$. We set

$$g(X) := \sum_{i=1}^{m} k_i f^{[i]}(X)\in K[X]$$

and show that $\mathbf{h}_K(x:g) = \mathbf{h}$.

First, we observe that by the previous lemma together with (4.4.4),

$$\begin{aligned} vg_{\mathbf{h}}(c) &= v\sum_{i=1}^{m}k_if_{\mathbf{h}}^{[i]}(c) \;=\; \min\left\{v\sum_{i=1}^{m}k_id_i\,,\, v\left(\sum_{i=1}^{m}(k_if_h^{[i]}(c)-k_id_i)\right)\right\}\\ &= v\sum_{i=1}^{m}k_id_i \;=\; \min_{1\leq i\leq m} vk_id_i \;=\; \min_{1\leq i\leq m} v\,k_if_{\mathbf{h}}^{[i]}(c)\end{aligned}$$

for $c\nearrow x$ (in particular, $vg_{\mathbf{h}}(c)<\infty$ which implies that g is nonconstant); with $1\leq j\neq \mathbf{h}$ we obtain:

$$\begin{aligned} v\,g_{\mathbf{h}}(c)(x-c)^{\mathbf{h}} &= vg_{\mathbf{h}}(c)+\mathbf{h}\cdot v(x-c) \;=\; \left(\min_{1\leq i\leq m} v\,k_if_{\mathbf{h}}^{[i]}(c)\right)+\mathbf{h}\cdot v(x-c)\\ &= \min_{1\leq i\leq m} v\,k_if_{\mathbf{h}}^{[i]}(c)(x-c)^{\mathbf{h}}\\ &< \min_{1\leq i\leq m} v\,k_if_j^{[i]}(c)(x-c)^j\\ &\leq v\sum_{i=1}^{m}k_if_j^{[i]}(c)(x-c)^j \;=\; v\,g_j(c)(x-c)^j\;.\end{aligned}$$

This proves that $\mathbf{h}_K(x:g)=\mathbf{h}$. It also follows that

$$\begin{aligned}
\operatorname{dist}(g(x),K) &= vg_{\mathbf{h}}(c)+\mathbf{h}\cdot\operatorname{dist}(x,K) \;=\; \left(\min_{1\le i\le m} vk_i f_{\mathbf{h}}^{[i]}(c)\right)+\mathbf{h}\cdot\operatorname{dist}(x,K)\\
&= \min_{1\le i\le m} v\,k_i f_{\mathbf{h}}^{[i]}(c)+\mathbf{h}\cdot\operatorname{dist}(x,K)\\
&= \min_{1\le i\le m} vk_i+vf_{\mathbf{h}}^{[i]}(c)+\mathbf{h}\cdot\operatorname{dist}(x,K)\\
&= \min_{1\le i\le m} vk_i+\operatorname{dist}(f^{[i]}(x),K) \;=\; \min_{1\le i\le m} vk_i+\operatorname{dist}(y_i,K)\\
&\le \min_{1\le i\le m} vk_i+v(y_i-f^{[i]}(x))\le \min_{1\le i\le m} v(k_iy_i-k_if^{[i]}(x))\\
&\le v\sum_{i=1}^{m}(k_iy_i-k_if^{[i]}(x))=v\left(\sum_{i=1}^{m}k_iy_i-g(x)\right)\,,
\end{aligned}$$

where the first equality follows from Lemma 3.6.2 as $\operatorname{appr}(x,K)$ is transcendental. By Lemma 3.1.1 b), this shows that

$$\operatorname{dist}\left(\sum_{i=1}^{m}k_iy_i,K\right) \;=\; \operatorname{dist}(g(x),K)\le v\left(\sum_{i=1}^{m}k_iy_i-g(x)\right)\,.$$

Consequently,

$$\mathbf{h}_K(x:\sum_{i=1}^{m}k_iy_i)=\mathbf{h}_K(x:g)=\mathbf{h}\,.$$

□

4.5 Valuation independence of Galois groups

In this section, we will introduce a valuation theoretical property that characterizes the Galois groups of tame Galois extensions. Take a Galois extension $(L|K,v)$ of henselian fields. Its Galois group $\operatorname{Gal}L|K$ will be called **valuation independent** if for every choice of elements $d_1,\ldots,d_n\in\tilde{L}$ and automorphisms $\sigma_1,\ldots,\sigma_n\in\operatorname{Gal}L|K$ there exists an element $d\in L$ such that (for the unique extension of the valuation v from L to $\tilde{L}$):

$$v\sum_{i=1}^{n}\sigma_i(d)\,d_i \;=\; \min_{1\le i\le n} v\,\sigma_i(d)\,d_i\,. \tag{4.5.1}$$

Since (K,v) is assumed to be henselian, we have that $v\sigma(d)=vd$ for all $\sigma\in\operatorname{Gal}L|K$ and therefore, $v\sigma_i(d)\,d_i=vd+vd_i$. Suppose that $vd_{i_0}=\min_i vd_i$; then (4.5.1) will hold if and only if

$$v\sum_{i=1}^{n}\frac{\sigma_i(d)}{d}\,\frac{d_i}{d_{i_0}} \;=\; 0\,.$$

In this sum, the terms with $v(d_i/d_{i_0})>0$ have no influence, and we can delete the corresponding σ_i from the list. So we see:

Lemma 4.5.1. *Assume that $(L|K,v)$ is a Galois extension of henselian fields. Then $\operatorname{Gal}L|K$ is valuation independent if and only if for every choice of elements $d_i\in\tilde{L}$ with $vd_i=0$ for $1\le i\le n$, and automorphisms*

$\sigma_1,\dots,\sigma_n \in \mathrm{Gal}\, L|K$, *there exists an element* $d \in L$ *such that*

$$v\sum_{i=1}^{n} \frac{\sigma_i(d)}{d}\, d_i \;=\; 0\,. \tag{4.5.2}$$

Theorem 4.5.2. *A Galois extension of henselian fields is tame if and only if its Galois group is valuation independent.*

Proof. Take a Galois extension $(L|K,v)$ of henselian fields, elements $d_i \in \tilde{L}$ with $vd_i = 0$ for $1 \le i \le n$, and automorphisms $\sigma_1,\dots,\sigma_n \in \mathrm{Gal}\, L|K$. For $\sigma \in \mathrm{Gal}\, L|K$ and $d \in L^\times$, we set

$$\chi_\sigma(d) \;:=\; \frac{\sigma(d)}{d}\, v\,.$$

Since $v\sigma(d) = vd$, the right hand side is a nonzero element in Lv. Now equation (4.5.2) is equivalent to

$$\sum_{i=1}^{n} d_i v \cdot \chi_{\sigma_i}(d) \;\neq\; 0\,; \tag{4.5.3}$$

note that $d_i v \neq 0$ since $vd_i = 0$.

We extend the homomorphism

$$G^i(L|K,v) \ni \sigma \;\mapsto\; \chi_\sigma \in \mathrm{Hom}(L^\times, (Lv)^\times)\;,$$

which is well known from ramification theory (see [3], Lemma 5.2.6), to a crossed homomorphism from $\mathrm{Gal}\, L|K$ to $\mathrm{Hom}(L^\times, (Lv)^\times)$. For the definition and an application of crossed homomorphisms, see [8, §6]. As in the case of $\sigma \in G^i(L|K,v)$, it is shown that $\chi_\sigma \in \mathrm{Hom}(L^\times, \overline{L}^\times)$. This group is a right $\mathrm{Gal}\, L|K$-module under the scalar multiplication

$$\chi^\rho \;:=\; \chi \circ \rho\,.$$

We compute:

$$\chi_{\sigma\tau}(d) = \frac{\sigma\tau(d)}{d}\, v = \frac{\sigma\tau(d)}{\tau(d)}\, v \cdot \frac{\tau(d)}{d}\, v = (\chi_\sigma \circ \tau)(d) \cdot \chi_\tau(d)\,.$$

Thus,

$$\chi_{\sigma\tau} \;=\; \chi_\sigma^\tau \cdot \chi_\tau\,.$$

In other words, the map

$$\mathrm{Gal}\, L|K \ni \sigma \;\mapsto\; \chi_\sigma \in \mathrm{Hom}(L^\times, (Lv)^\times) \tag{4.5.4}$$

is a crossed homomorphism. Hence, it is injective if and only if its kernel is trivial. This kernel consists of all $\sigma \in \mathrm{Gal}\, L|K$ for which $\frac{\sigma(d)}{d}\, v = 1$ for all $d \in L^\times$. So the kernel is the ramification group $G^r(L|K,v)$.

The theorem of Artin on linear independence of characters (see [14], VI, §4, Theorem 4.1) tells us that if the χ_{σ_i} are distinct characters, then an element d satisfying (4.5.3) will exist. This shows that G is valuation independent if the map in (4.5.4) is injective. The converse is also true: if $\sigma_1 \neq \sigma_2$ but $\chi_{\sigma_1} = \chi_{\sigma_2}$, then with $n = 2$ and $d_1 = -d_2 = 1$, (4.5.2) does not hold for any d.

Since the kernel is the ramification group of $(L|K,v)$, we conclude that $\mathrm{Gal}\, L|K$ is valuation independent if and only if the ramification group is trivial. This is equivalent to $(L|K,v)$ being a tame extension. □

Note that we could give the above definition and the result of the theorem also for extensions which are not Galois, replacing automorphisms by embeddings; however, the normal hull of an algebraic extension $L|K$ of a henselian field K is a tame extension of K if and only if $L|K$ is a tame extension, so there is no loss of generality in restricting our scope to Galois extensions.

4.6 A pull down principle for henselian rationality through tame extensions

Take a tame extension $(L|K, v)$ of fields of rank 1 and an immediate function field $(F|K, v)$ of transcendence degree 1 with F not contained in the completion K^c of K. By Lemma 2.9.2, the extension $(F^h.L|L, v)$ is again immediate. Since $L|K$ is algebraic, so is $F^h.L|F^h$ and therefore, $F^h.L$ is henselian, so $F^h.L = (F.L)^h$. We consider the following question:

If $F^h.L|L$ is a henselian rational function field, does this imply the same for $F^h|K$?

To start with, we observe that w.l.o.g. we may assume the extension $L|K$ to be finite and Galois. Indeed, if $x \in F^h.L$ such that $F^h.L = L(x)^h$, then x lies already in $F^h.L_1$ for some finite subextension $L_1|K$ of $L|K$. Since x must be transcendental over L_1 , the extension $F^h.L_1|L_1(x)^h$ is finite, generated by finitely many elements that lie in $L(x)^h$. So we can choose a finite subextension $L_2|L_1$ of $L|L_1$ such that these elements already lie in $L_2(x)^h$. Since the normal hull of a tame extension is a tame extension as well, we may replace L_2 by its normal hull L_3 over K because also $L_3(x)^h$ will contain these elements.

From now on we assume that $L|K$ is a finite tame Galois extension and that $F^h.L = L(x)^h$ for some $x \in F^h.L$. In addition, we assume that $\text{appr}\,(x, L)$ is transcendental.

We show that hypothesis (4.2.1) holds with K replaced by L. First, since $(F.L|L, v)$ is an immediate function field, so is $(L(x)|L, v)$. Second, $\text{appr}\,(x, L)$ is transcendental by assumption. Third, we have:

Lemma 4.6.1. *The condition $F \not\subset K^c$ implies that $F.L \not\subset L^c$, hence $x \notin L^c$.*

Proof. Since $F \not\subset K^c$, there exists some $z \in F$ with $z \notin K^c$. By assumption, $(L|K, v)$ is a tame extension, and as remarked in Section 2.9, is therefore defectless. Hence by Lemma 3.2.5, $\text{dist}\,(z, L) = \text{dist}\,(z, K) < \infty$. Consequently, $F.L \not\subset L^c$, as asserted.

Furthermore, $x \in L^c$ would imply that $L(x) \subset L^c$; since the rank of (K, v) is 1 by assumption, the same is true for $(L(x), v)$ and $L(x)$ is thus dense in $L(x)^h$, so we would get that $F.L \subset F^h.L = L(x)^h \subset L^c$, a contradiction. □

Lemma 4.6.2. *If there exists an element $y \in F^h$ such that $L(y)^h = L(x)^h$, then $F^h = K(y)^h$.*

Proof. Since $(F^h|K, v)$ and hence also its subextension $(K(y)^h|K, v)$ are immediate and $(L|K, v)$ is defectless and finite, we obtain from Lemma 2.9.2 that $[F^h.L : F^h] = [L : K] = [K(y)^h.L : K(y)^h]$. On the other hand, $F^h.L = L(x)^h = L(y)^h = K(y)^h.L$, so $F^h = K(y)^h$ must hold, because by assumption on y, $K(y)^h \subseteq F^h$. □

Since $L|K$ is a finite tame Galois extension, also the extension $F^h.L|F^h$ is a finite tame Galois extension. As shown in the preceding proof, it is of degree $n := [L : K]$. We write

$$\mathrm{Gal}\,(F^h.L|F^h) = \{\rho_i \mid 1 \leq i \leq n\}\;.$$

Then $\mathrm{Gal}\,(L|K) = \{\rho_i|_L \mid 1 \leq i \leq n\}$.

The next lemma will help us to determine the relative approximation degrees of the conjugates $\rho_i(x)$.

Lemma 4.6.3. *Assume that ρ is a valuation preserving automorphism of $L(x)^h$ such that $\rho(L) = L$. Then*

$$L(x)^h = L(\rho x)^h\;.$$

Proof. Since $\rho x \in \rho(L(x)^h) = L(x)^h$, we have that $L(\rho x)^h \subseteq L(x)^h$. Further, $L \subseteq \rho^{-1}(L(\rho x)^h) \subseteq L(x)^h$ and $x \in \rho^{-1}(L(\rho x)^h)$. Thus, $L(x) \subseteq \rho^{-1}(L(\rho x)^h)$. Since ρ is valuation preserving and induces an isomorphism from $\rho^{-1}(L(\rho x)^h)$ to the henselian field $L(\rho x)^h$, also $\rho^{-1}(L(\rho x)^h)$ is henselian; it is therefore equal to $L(x)^h$. This shows that its image $L(\rho x)^h$ under the automorphism ρ is also equal to $L(x)^h$. □

The following lemma and theorem make essential use of the valuation independence of Galois groups of tame Galois extensions. Let Tr denote the trace.

Lemma 4.6.4. *There is an element $d \in L$ such that*

$$\mathbf{h}_K(x : \mathrm{Tr}_{F^h.L|F^h}(d \cdot x)) = 1\;.$$

Proof. From the preceding lemma it follows that every $\rho_i(x)$ is transcendental over L and hence over K. Hence by Lemma 4.4.2 we can choose approximation coefficients d_i of $\rho_i(x)$ in x over K for $1 \leq i \leq n$. By Theorem 4.5.2, we have that $\mathrm{Gal}\,(L|K)$ is valuation independent. This means we can choose an element $d \in L$ such that (4.5.1) holds with $\sigma_i = \rho_i|_L$. Then for $k_i := \sigma_i(d) = \rho_i(d)$, the hypothesis (4.4.4) of Lemma 4.4.3 holds. In view of the previous lemma and Corollary 4.2.8 we have that $\mathbf{h}_K(x : \rho_i(x)) = 1$. From Lemma 4.4.3 we can now infer that

$$\begin{aligned}\mathbf{h}_K\left(x : \mathrm{Tr}_{F^h.L|F^h}(d \cdot x)\right) &= \mathbf{h}_K\left(x : \sum_i \rho_i(d \cdot x)\right)\\ &= \mathbf{h}_K\left(x : \sum_i \rho_i(d) \cdot \rho_i(x)\right) = 1\;.\end{aligned}$$

□

Now we are able to answer our question:

Theorem 4.6.5. *Let (K, v) be an algebraically maximal field of rank 1, and let (F, v) be an immediate function field of transcendence degree 1 over (K, v), with $F \not\subseteq K^c$. If $F^h.L$ is a henselian rational function field over L for some tame extension $(L|K, v)$, then F^h is a henselian rational function field over K.*

Proof. As shown in the beginning of this section, we may assume that $L|K$ is finite and Galois. Now the foregoing lemma shows that there is some $d \in L$ such that for $y := \mathrm{Tr}_{F^h.L|F^h}(d \cdot x) \in F^h$ we have $\mathbf{h}_K(x : y) = 1$. By virtue of Corollary 4.2.8, $L(y)^h = L(x)^h$. From Lemma 4.6.2, we can now infer that F^h is henselian rational over K, as asserted. □

Chapter 5

Normal Forms for the Associated Minimal Polynomials

Recall that in Section 3.4 we showed how algebraic approximation types can be realized in certain immediate algebraic extensions of the ground field. To build these extensions an associated minimal polynomial of the approximation type was used. In this way, associated minimal polynomials may be associated to algebraic approximation types in the sense that they are the potential minimal polynomials for those immediate extensions which realize the algebraic approximation types. In [4] Kaplansky stated a Theorem 10 in which he suggests that nice normal forms can be found for the associated minimal polynomials of an algebraic approximation type. In this chapter, by extending Kaplansky's approach and using approximation types instead of pseudo-convergent sequences we completely describe the normal forms of the associated minimal polynomials.

This chapter uses the machinery presented in Chapter 3 but before we can investigate the classes of associated minimal polynomials we need three more notions, namely the invariance group of an approximation type, distinguished cuts and a special coarsening of valuations.

5.1 The invariance group of an approximation type

Throughout this section, we will consider an ordered abelian group Γ and a cut δ in Γ.

If S is a subset of Γ, then S^+ is the cut whose lower cut set is the smallest initial segment which contains S. If Λ is the lower cut set of δ, then $S^+ < \delta$ means that S is a proper subset of Λ, as defined in Section 2.1. Note that for an element $\alpha \in \Gamma$, $\alpha < \delta$ means that $\alpha \in \Lambda$, and similarly $\alpha > \delta$ means that $\alpha \in \Gamma \setminus \Lambda$. Also recall that $-\delta$ is the cut with upper cut set $-\Lambda = \{-\alpha \mid \alpha \in \Lambda\}$.

Lemma 5.1.1. *Let Λ be a an initial segment of Γ. Then the set*

$$\mathcal{I}(\Lambda) := \{\gamma \in \Gamma \mid \gamma + \Lambda = \Lambda\}$$

is a convex subgroup of Γ. Hence, $\Gamma/\mathcal{I}(\Lambda)$ is again an ordered abelian group. Further, $\mathcal{I}(\Lambda)$ is the biggest convex subgroup Δ of Γ satisfying $\Delta + \Lambda \subseteq \Lambda$. If Λ admits a maximal or $\Gamma \setminus \Lambda$ admits a minimal element, then $\mathcal{I}(\Lambda) = \{0\}$.

Proof. Let us show that $\mathcal{I} = \mathcal{I}(\Lambda)$ is convex in Γ. Since Λ is an initial segment of Γ, we have $\Lambda + \alpha \subseteq \Lambda + \beta$ whenever $\alpha \leq \beta$. Hence if $\alpha, \beta \in \mathcal{I}$ and $\gamma \in \Gamma$ with $\alpha \leq \gamma \leq \beta$, then

$$\Lambda = \Lambda + \alpha \subseteq \Lambda + \gamma \subseteq \Lambda + \beta = \Lambda$$

which shows that $\Lambda + \gamma = \Lambda$, i.e., $\gamma \in \mathcal{I}$. This proves that $\mathcal{I}$ is convex in Γ.

Since every convex subgroup $\Delta \leq \Gamma$ contains 0, it satisfies $\Lambda \subseteq \Delta + \Lambda$. Therefore, $\Delta + \Lambda = \Lambda$ if and only if $\Delta + \Lambda \subseteq \Lambda$. This proves that $\mathcal{I}$ coincides with the biggest convex subgroup Δ of Γ that satisfies $\Delta + \Lambda \subseteq \Lambda$.

For the last assertion, suppose that λ is the maximal element of Λ. If $\mathcal{I} \neq \{0\}$ then $\mathcal{I}$ contains a positive element γ. Then by definition of $\mathcal{I}$, $0 < \gamma$ implies that $\lambda < \gamma + \lambda \in \gamma + \Lambda = \Lambda$, thus contradicting the maximality of λ. Finally, if μ is the minimal element of $\mathcal{I} \setminus \Lambda$, then for every $\alpha \in \Gamma$, $\alpha < \mu$ implies that $\alpha \in \Lambda$. If $\mathcal{I}$ contains an element $\gamma' < 0$, then it follows that $\mu > \gamma' + \mu$ and so $\mu = -\gamma' + (\gamma' + \mu) \in -\gamma' + \Lambda = \Lambda$ contradicting our choice of $\mu > \Lambda$. ☐

The group $\mathcal{I}(\Lambda)$ will be called the **invariance subgroup of** Λ. If δ is a cut in Γ with lower cut set Λ, then we call $\mathcal{I}(\Lambda)$ the invariance subgroup of δ and denote it by $\mathcal{I}(\delta)$.

Given a cut δ in Γ with lower cut set Λ and invariance group $\mathcal{I} = \mathcal{I}(\delta)$, let $\delta/\mathcal{I}$ denote the cut in $\Gamma/\mathcal{I}$ with lower cut set $\Lambda/\mathcal{I} := \{\alpha + \mathcal{I} \mid \alpha \in \Lambda\}$. Note that we will write $\alpha/\mathcal{I}$ in place of $\alpha + \mathcal{I}$.

Lemma 5.1.2. *Take a cut δ in Γ with lower cut set Λ. For the invariance subgroup $\mathcal{I} = \mathcal{I}(\delta)$, the following assertions hold:*

a) If $\delta \geq 0^+$, then $\delta \geq \mathcal{I}^+$; if $\delta \leq 0^+$, then $\delta \leq \mathcal{I}^+$.

b) For all cuts δ in Γ, $\mathcal{I}(\delta/\mathcal{I}) = \{0\}$ (in $\Gamma/\mathcal{I}$).

c) If i is a nonzero integer and $\gamma \in \Gamma$, then

$$\mathcal{I}(\gamma + i \cdot \delta) = \mathcal{I}(\delta) \; .$$

d) If i is a positive integer and $\gamma, \gamma_\delta \in \Gamma$ we find that $\gamma_\delta/\mathcal{I}$ is a maximal element of $\Lambda/\mathcal{I}$ if and only if $(\gamma + i \cdot \gamma_\delta)/\mathcal{I}$ is a maximal element of $(\gamma + i \cdot \Lambda)/\mathcal{I}$.

Proof. If $\delta \geq 0^+$ and $\alpha \in \mathcal{I}$, then $\Lambda = \Lambda + \alpha \ni \alpha$. This yields $\delta \geq \mathcal{I}^+$. The proof of the second assertion of a) is similar. If $\gamma/\mathcal{I} + \Lambda/\mathcal{I} = \Lambda/\mathcal{I}$ for an element $\gamma \in \Gamma$, this means that

$$(\gamma + \mathcal{I}) + (\Lambda + \mathcal{I}) = \Lambda + \mathcal{I} \; ,$$

and in view of $\mathcal{I} + \Lambda = \Lambda$, this yields $\gamma + \Lambda = \Lambda$, i.e., $\gamma \in \mathcal{I}$ and thus $\gamma/\mathcal{I} = 0$. This proves part b).

Next, let i be a nonzero integer and $\gamma \in \Gamma$. Since invariance subgroups are convex subgroups, we have that $\alpha \in \mathcal{I}(i \cdot \delta) \Longleftrightarrow i \cdot \alpha \in \mathcal{I}(i \cdot \delta)$. Using this, we compute

$$\begin{aligned} \alpha + \Lambda = \Lambda \quad &\Longleftrightarrow \quad i \cdot \alpha + i \cdot \Lambda = i \cdot \Lambda \\ &\Longleftrightarrow \quad \alpha + i \cdot \Lambda = i \cdot \Lambda \\ &\Longleftrightarrow \quad \alpha + i \cdot \Lambda + \gamma = i \cdot \Lambda + \gamma \; . \end{aligned}$$

For positive i, this proves that $\mathcal{I}(\gamma + i \cdot \delta) = \mathcal{I}(\delta)$. For negative i, we thus have

$$\mathcal{I}(\gamma + i \cdot \delta) = \mathcal{I}(\gamma + (-i) \cdot (-\delta)) = \mathcal{I}(-\delta) ,$$

and it remains to show that

$$\mathcal{I}(\delta) = \mathcal{I}(-\delta) .$$

Note that we have

$$\begin{aligned} \gamma \in \mathcal{I}(\delta) &\iff \gamma + \Lambda = \Lambda \\ &\iff -\gamma + (-\Lambda) = -\Lambda \\ &\iff -\gamma + (\Gamma \setminus -\Lambda) = \Gamma \setminus -\Lambda \\ &\iff -\gamma \in \mathcal{I}(-\delta) \\ &\iff \gamma \in \mathcal{I}(-\delta) \end{aligned}$$

which completes the proof of part c). Now for the proof of d), by part c) we have that

$$(\gamma + i \cdot \Lambda)/\mathcal{I}(\gamma + i \cdot \delta) = \gamma/\mathcal{I}(\delta) + i \cdot (\Lambda/\mathcal{I}(\delta))$$

which for positive i shows that $\Lambda/\mathcal{I}(\delta)$ admits $\gamma_\delta/\mathcal{I}(\delta)$ as its maximal element if and only if $(\gamma+i\cdot\Lambda)/\mathcal{I}(\gamma+i\cdot\delta)$ admits $(\gamma + i \cdot \gamma_\delta)/\mathcal{I}(\delta)$ as its maximal element. □

Lemma 5.1.3. *Take a cut δ in Γ and let Λ be the lower cut set of δ. Suppose that Λ has no maximal element and that $\mathcal{I}(\delta) = \{0\}$. Then for $\alpha \in \Gamma$ and natural numbers $j < k$ we have*

$$\alpha + j\delta = k\delta \iff (k-j)\delta = \alpha^- .$$

Proof. Assume first that $\alpha + j\delta = k\delta$.

Suppose that $(k-j)\delta > \alpha^-$. Then α is in the lower cut set of $(k-j)\delta$, hence there is $\gamma_0 \in \Lambda$ such that $(k-j)\gamma_0 \geq \alpha$. Since Λ has no maximal element, there is also $\gamma_1 \in \Lambda$ such that $(k-j)\gamma_1 > \alpha$. Since $(k-j)\gamma_1 - \alpha > 0$ and $I(j\delta) = I(\delta) = \{0\}$ by assumption, we have that $j\gamma + (k-j)\gamma_1 - \alpha > j\delta$ for all large enough $\gamma \in \Lambda$. Choosing $\gamma \geq \gamma_1$, we obtain $k\gamma - \alpha = j\gamma + (k-j)\gamma - \alpha \geq j\gamma + (k-j)\gamma_1 - \alpha > j\delta$, whence $k\gamma > \alpha + j\delta = k\delta$, a contradiction. This shows that $(k-j)\delta \leq \alpha^-$.

Now suppose that $(k-j)\delta < \alpha^-$. This means that there is some $\beta \in \Gamma$ such that for all $\gamma \in \Lambda$ we have that $(k-j)\gamma < \beta$, or equivalently, $\beta + j\gamma > k\gamma$. This implies that $\beta + j\delta \geq k\delta = \alpha + j\delta$, whence $j\delta \geq \alpha - \beta + j\delta$. Now $\alpha - \beta > 0$ implies $\alpha - \beta + j\delta \geq j\delta$ and thus, $\alpha - \beta + j\delta = j\delta$. But this means that $\alpha - \beta \in I(j\delta) = \{0\}$, a contradiction. This proves that $(k-j)\delta = \alpha^-$.

For the converse, suppose $(k-j)\delta = \alpha^-$ holds. Then by Lemma 2.2.1 δ is a principal cut, i.e., there is $\beta \in \Gamma$ such that $\delta = \beta^-$. By the proof of Lemma 2.2.1 we infer that $\alpha = (k-j)\beta$ and for $\gamma \in \Gamma$

$$\begin{aligned} \{\alpha + \gamma \mid \gamma < j\beta\} &= \{(k-j)\beta + \gamma \mid \gamma < j\beta\} \\ &= \{\gamma \mid \gamma < j\beta + (k-j)\beta\} \\ &= \{\gamma \mid \gamma < k\beta\} . \end{aligned}$$

Since δ is principal this implies that $\alpha + j\delta = k\delta$. □

5.2 Distinguished cuts

Let δ be a cut in Γ with lower cut set Λ. If $\mathcal{I} = \mathcal{I}(\delta)$ is non-trivial and $\Lambda/\mathcal{I}$ admits a maximal element, then δ will be called a **weakly distinguished cut**. If in addition, $\Lambda/\mathcal{I}$ admits $0/\mathcal{I}$ as its maximal element then δ will be called a **distinguished cut**. If the approximation type $\mathrm{appr}\,(x, K)$ is immediate and $\delta = \mathrm{dist}\,(x, K)$ is weakly distinguished, then $\mathrm{appr}\,(x, K)$ will be called a **weakly distinguished approximation type**, and if δ is distinguished, $\mathrm{appr}\,(x, K)$ will be called a **distinguished approximation type**. This name is chosen since distinguished approximation types correspond to distinguished pseudo-convergent sequences in the sense of Ribenboim [17], p. 105. Note that by the above definition, a distinguished cut δ satisfies $\delta > 0^+$, because $0/\mathcal{I} \in \Lambda/\mathcal{I}$ implies that $\{0\} \neq \mathcal{I} \subseteq \Lambda$.

The following lemma gives an important characterization of distinguished and weakly distinguished cuts, and the relation between the two.

Lemma 5.2.1. *Take a cut δ in Γ with lower cut set Λ and let $\mathcal{I} = \mathcal{I}(\delta)$. Further, let i be a positive integer and take $\gamma \in \Gamma$. Then δ is weakly distinguished if and only if $\gamma + i \cdot \delta$ is.*

If δ is weakly distinguished, and $\gamma_\delta \in \Gamma$ such that $\gamma_\delta/\mathcal{I}$ is the maximal element of $\Lambda/\mathcal{I}$, then $\delta - \gamma_\delta$ is a distinguished cut and $\mathcal{I}$ is cofinal in $\Lambda - \gamma_\delta$.

Conversely, if there exists an element $\gamma_\delta \in \Gamma$ and a non-trivial convex subgroup Δ of Γ such that Δ is cofinal in $\Lambda - \gamma_\delta$, then δ is weakly distinguished with $\Delta = \mathcal{I}$ and $\gamma_\delta/\mathcal{I}$ is the maximal element of $\Lambda/\mathcal{I}$.

Proof. The first assertion follows immediately from part d) of Lemma 5.1.2.

Now assume δ to be a weakly distinguished cut and $\gamma_\delta \in \Gamma$ such that $\gamma_\delta/\mathcal{I}$ is the maximal element of $\Lambda/\mathcal{I}$. Consequently, $(\Lambda - \gamma_\delta)/\mathcal{I}$ admits $0/\mathcal{I}$ as its maximal element. This shows that $\delta - \gamma_\delta$ is distinguished. In particular, $(\Lambda - \gamma_\delta)/\mathcal{I}$ contains no positive elements, i.e., there are no elements $\beta \in \Lambda - \gamma_\delta$ with $\beta > \mathcal{I}$. Now let $\beta \in \mathcal{I}$. Note that w.l.o.g., we may choose γ_δ to be an element of Λ. Then $0 = \gamma_\delta - \gamma_\delta \in \Lambda - \gamma_\delta$, and since $\beta + \Lambda = \Lambda$ we find that

$$\beta = \beta + 0 \;\in\; \beta + (\Lambda - \gamma_\delta) = (\beta + \Lambda) - \gamma_\delta = \Lambda - \gamma_\delta \;.$$

Since $\beta \in \mathcal{I}$ was arbitrary, we deduce that $\mathcal{I} \subseteq \Lambda - \gamma_\delta$. This, together with the fact that $(\Lambda - \gamma_\delta)/\mathcal{I}$ admits $0/\mathcal{I}$ as its maximal element, proves that $\mathcal{I}$ is a final segment of $\Lambda - \gamma_\delta$ and hence cofinal in $\Lambda - \gamma_\delta$.

Now assume that there exists an element $\gamma_\delta \in \Gamma$ and a non-trivial convex subgroup Δ of Γ such that Δ is cofinal in $\Lambda - \gamma_\delta$. In view of the first assertion of our lemma, and in view of the equality $\mathcal{I}(\delta - \gamma_\delta) = \mathcal{I}(\delta)$, we may assume w.l.o.g. that $\gamma_\delta = 0$. Since Δ is cofinal in Λ, it follows that $0/\Delta$ is the maximal element of Λ/Δ. Thus, it remains to show that Δ is the invariance subgroup of Λ. To show that $\Delta \subseteq \mathcal{I}$ take $\alpha \in \Delta$ and $\beta \in \Lambda$. Since Δ is cofinal in Λ there is $\alpha' \in \Delta$ such that $\beta \leq \alpha'$. Then $\alpha + \beta \leq \alpha + \alpha' \in \Delta \subseteq \Lambda$. Since Λ is an initial segment, it follows that $\Delta + \Lambda \subseteq \Lambda$. For the other direction, take a non-negative $\gamma \in \mathcal{I}$ and $\mu \in \Delta$.

Then $\mu \leq \gamma + \mu = \eta$ for some $\eta \in \Lambda$. Again by the cofinality of Δ there is $\mu' \in \Delta$ such that $\eta \leq \mu'$. From here we have $0 \leq \gamma \leq \mu' - \mu \in \Delta$. Since Δ is a convex subgroup, it follows that $\gamma \in \Delta$ thus implying that $\mathcal{I} \subseteq \Delta$. □

As a corollary, we obtain the following characterization of distinguished cuts:

Corollary 5.2.2. *The following assertions are equivalent:*

a) δ is distinguished

b) $\delta = \Delta^+$ for some non-trivial convex subgroup Δ of Γ

c) $\delta > 0^+$ and δ is idempotent (that is, $\delta + \delta = \delta$).

If these assertions hold, then $\Delta = \mathcal{I}(\delta)$.

Proof. The equivalence of a) and b) together with the assertion that $\Delta = \mathcal{I}(\delta)$ follow from the foregoing lemma, where we take $\gamma_\delta = 0$. Since $\Delta + \Delta = \Delta$ for every subgroup Δ of Γ, assertion b) implies c). Now assume c). Let Λ be the lower cut set of δ, and set $\Upsilon = \{\alpha \in \Lambda \mid \alpha \geq 0\}$. Since $\delta > 0^+$, Υ is nonempty. Since $\delta + \delta = \delta$, we also have that $\Upsilon + \Upsilon = \Upsilon$. This shows that $\Delta := \Upsilon \cup -\Upsilon$ is a convex subgroup of Γ with $\delta = \Delta^+$. □

Let us remark that in assertion b), the condition "convex" may as well be omitted. Indeed, the convex hull Δ_1 of an arbitrary subgroup $\Delta \subseteq \Gamma$ (i.e., the intersection of all convex subsets of Γ containing Δ) is a convex subgroup with $\Delta^+ = \Delta_1^+$.

Lemma 5.2.3. *Take a distance δ in Γ which is not weakly distinguished. Let Λ be the lower cut set of δ and assume that Λ has no maximal element. Let i, j be natural numbers with $j > i > 0$, and take $\alpha_i, \alpha_j \in \Gamma$. If there exists $\beta \in \Lambda$ such that*

$$\alpha_j + j \cdot \beta > \alpha_i + i \cdot \beta ,$$

then there exists an element $\beta_0 \in \Lambda$ such that for all $\beta \geq \beta_0$ we have that

$$\alpha_j + j \cdot \beta > \alpha_i + i \cdot \Lambda .$$

Proof. First, we show that there exist $\alpha, \eta \in \Gamma$, with $\alpha \in \Lambda$ and $\eta > \mathcal{I} = \mathcal{I}(\delta)$, such that

$$\alpha_j - \alpha_i + (j - i) \cdot \alpha \geq i \cdot \eta > \mathcal{I} .$$

Indeed, by assumption there exists $\beta \in \lambda$ such that

$$\alpha_j - \alpha_i + (j - i) \cdot \beta > 0 .$$

Since δ is assumed to be not weakly distinguished, we have that $\mathcal{I} = \{0\}$ or that $\Lambda/\mathcal{I}$ has no maximal element. By hypothesis, Λ admits no maximal element, so we find that in both cases, $\Lambda/\mathcal{I}$ has no maximal element. Hence, there exists an element $\alpha \in \Lambda$ such that $\alpha/\mathcal{I} > \beta/\mathcal{I}$, i.e., $\alpha - \beta > \mathcal{I}$, whence

$$\alpha_j - \alpha_i + (j - i) \cdot \alpha > (j - i) \cdot (\alpha - \beta) > \mathcal{I} .$$

If $\Lambda/\mathcal{I}$ has no smallest positive element, then $\Gamma/\mathcal{I}$ is dense in $\Gamma/\mathcal{I}$ and thus, there exists an element $\eta > \mathcal{I}$ such that $(j-i)\cdot(\alpha-\beta) > i\cdot\eta > \mathcal{I}$. If on the other hand, $\Lambda/\mathcal{I}$ admits a smallest positive element, say $\eta/\mathcal{I}$, then by our hypothesis that $\Lambda/\mathcal{I}$ has no maximal element, we may choose $\alpha\in\Lambda$ such that $\alpha-\beta > i\cdot\eta$, whence again, $(j-i)\cdot(\alpha-\beta) > i\cdot\eta > \mathcal{I}$. This proves the existence of the required element η.

Since $\eta > \mathcal{I}$, there exists $\gamma\in\Lambda$ such that $\gamma+\eta > \Lambda$. Putting $\beta_0 = \max\{\gamma,\alpha\}$, we get $\beta_0+\eta \geq \gamma+\eta > \Lambda$ and consequently, $i\cdot\beta_0 + i\cdot\eta > i\cdot\Lambda$. Hence, for all $\beta \geq \beta_0$ we have $\beta \geq \alpha$ and

$$\begin{aligned} \alpha_j + j\cdot\beta &= \alpha_j + i\cdot\beta + (j-i)\cdot\beta \\ &\geq \alpha_i + \alpha_j - \alpha_i + i\cdot\beta_0 + (j-i)\cdot\alpha \\ &\geq \alpha_i + i\cdot\beta_0 + i\cdot\eta > \alpha_i + i\cdot\Lambda\ , \end{aligned}$$

as asserted. □

5.3 Coarsening of valuations modulo convex subgroups

Before we apply what we have established for invariance subgroups, we need to define the coarsening of a valuation induced by a convex subgroup of the value group.

Take a convex subgroup Δ of vK. The coarsening v_Δ of v is the valuation whose valuation ring is $\{c\in K \mid vc \in vK^+ \cup \Delta\}$; this contains the valuation ring $\mathcal{O}_v$ of v. The value group of v_Δ on K is canonically isomorphic to vK/Δ. Note that

$$v_\Delta c > 0 \Longleftrightarrow vc > \Delta\ .$$

The valuation v also induces a valuation $\overline{v}_\Delta$ on the residue field Kv_Δ such that v is (equivalent to) the composition $v_\Delta \circ \overline{v}_\Delta$. If $\mathcal{O}_{v_\Delta}$ and $\mathcal{M}_{v_\Delta}$ denote the valuation ring and valuation ideal of v_Δ, then the valuation ring of $\overline{v}_\Delta$ is the image of $\mathcal{O}_v$ under the canonical epimorphism $\mathcal{O}_{v_\Delta} \to \mathcal{O}_{v_\Delta}/\mathcal{M}_{v_\Delta} = Kv_\Delta$. The value group of $\overline{v}_\Delta$ on Kv_Δ is canonically isomorphic to Δ via the map

$$\overline{v}_\Delta(a+\mathcal{M}_{v_\Delta}) \mapsto va \text{ for } a \notin \mathcal{M}_{v_\Delta}\ .$$

If $(L|K,v)$ is an arbitrary extension of valued fields, then the convex hull Δ' of Δ in vL is a convex subgroup of vL, and $v_{\Delta'}$ is an extension of v_Δ from K to L. If vL/vK is a torsion group (which is the case if $L|K$ is algebraic), then taking convex hulls induces a bijective inclusion preserving mapping from the chain of convex subgroups of vK to the chain of convex subgroups of vL, and $v_{\Delta'}$ is the unique coarsening of v on L which extends v_Δ.

Next we will apply invariance subgroups as a tool in the following general situation. Given an immediate algebraic approximation type $\operatorname{appr}(x,K)$ of degree $\mathbf{d}$, let S denote the support of $\operatorname{appr}(x,K)$, $\delta = \operatorname{dist}(x,K)$, and $\mathcal{I} = \mathcal{I}(\delta)$. Recall that distances in vK are cuts in the divisible hull $\widetilde{vK}$ of vK, so their invariance subgroups are subgroups of $\widetilde{vK}$, not of vK. Now, take the convex hull $\mathcal{I}'$ of $\mathcal{I}$ in $\widetilde{vK(x)}$ and let $v_\delta = v_{\mathcal{I}'}$ be the corresponding coarsening of the valuation v on $K(x)$ as described above. Its restriction to K is the coarsening of v on K which corresponds to $\mathcal{I} \cap vK$, and it is the finest extension of v_δ from K to $K(x)$. Note that v_δ is the trivial valuation on K if and only if $\delta = \infty$. Further, $\delta/\mathcal{I}$ is the distance in $v_\delta K$ induced by δ. We will write $\overline{v}_\delta$ for $\overline{v}_{\mathcal{I}'}$, such that $v = v_\delta \circ \overline{v}_\delta$. The following lemma illustrates the special role of the coarsening v_δ.

Lemma 5.3.1. *If* appr (x, K) *is distinguished, with* $\delta = \operatorname{dist}(x, K)$*, then for* $c \nearrow x$*, the* v_δ*-residue* $(x-c)v_\delta$ *does not lie in the residue field* Kv_δ *but is an element of the completion* $(Kv_\delta)^{c(\overline{v}_\delta)}$ *of* Kv_δ *with respect to the induced valuation* $\overline{v}_\delta$*:*

$$(x-c)v_\delta \in Kv_\delta^{c(\overline{v}_\delta)} \setminus Kv_\delta \tag{5.3.1}$$

(in particular, this holds for every $c \in K$ *with* $v_\delta(x-c) = 0$*, and there is no* $c \in K$ *with* $v_\delta(x-c) > 0$*).*

Conversely, if there exists an element $c \in K$ *and a decomposition* $v = w \circ \overline{w}$ *such that (5.3.1) holds for* w *in the place of* v_δ*, then* appr (x, K) *is distinguished with* $w = v_\delta$ *(on* $K(x)$ *).*

Proof. Let Δ be a convex subgroup of vK and let $v = v_\Delta \circ \overline{v}_\Delta$ be the corresponding decomposition of v on $K(x)$. Let S denote the support of appr (x, K). We will show: Δ is cofinal in S if and only if

$$(x-c)v_\Delta \in Kv_\Delta^{c(\overline{v}_\Delta)} \setminus Kv_\Delta \tag{5.3.2}$$

holds for some $c \in K$.

The element $(x-c)v_\Delta$ lies in the completion of Kv_Δ with respect to the induced valuation $\overline{v}_\Delta$ if and only if for every $\alpha \in \overline{v}_\Delta(Kv_\Delta)$ there is $c' \in K$ such that $\overline{v}_\Delta((x-c)v_\Delta - c'v_\Delta) > \alpha$. This in turn holds if and only if the set

$$\{\overline{v}_\Delta((x-a)v_\Delta) \mid (x-a) \in \mathcal{O}^\times_{(K,v_\Delta)}\}$$

is cofinal in $\overline{v}_\Delta(Kv_\Delta)$. Via the isomorphism $v_\Delta K \cong vK/\Delta$, this is equivalent to

$$\Delta \subseteq S.$$

If Δ is not cofinal in S, then there exists an element $\beta \in S$ with $\beta > \Delta$. This means that there exists an element $c_\beta \in K$ with $v_\Delta(x - c_\beta) > 0$, i.e., $(x - c - (c_\beta - c))v_\Delta = (x - c_\beta)v_\Delta = 0$, thus $(x-c)v_\Delta = (c_\beta - c)v_\Delta \in Kv_\Delta$. Together with what we have proved already, this shows that Δ is cofinal in S if and only if (5.3.2) holds for some $c \in K$.

Now let $\delta = \operatorname{dist}(x, K)$. If δ is distinguished, then by virtue of Lemma 5.2.1, the convex subgroup $\mathcal{I} = \mathcal{I}(\delta) \cap vK$ is cofinal in S. Then by the above, there exists $c \in K$ such that (5.3.1) holds. The same must be true for every c' in place of c if $v(x-c') \geq v(x-c)$ or $v_\delta(x-c') \geq 0$. This is seen as follows: If $v(x-c') > v(x-c)$ then $v_\delta(x-c') \geq v_\delta(x-c) = 0$. Thus, we have $v_\delta(x-c') \geq 0$ in both cases, and this implies that $v_\delta(c-c') \geq 0$. Consequently, $(x-c')v_\delta = (x-c)v_\delta + (c-c')v_\delta \in (Kv_\delta)^{c(\overline{v}_\Delta)}$. On the other hand, $(x-c')v_\delta$ cannot be an element of Kv_δ since otherwise, this would also hold for $(x-c)v_\delta$. We have herewith proved that the above property $(x-c)v_\Delta = 0$ in Kv_δ and every $c \in K$ with $v_\delta(x-c) \geq 0$. But $v_\delta(x-c) > 0$ means that $xv_\delta = cv_\delta \in Kv_\delta$ and therefore, (5.3.1) implies that there is no such $c \in K$.

For the converse, assume that there exists an element $c \in K$ and a decomposition $v = w \circ \overline{w}$ such that $(x-c)w \in Kw^{c(\overline{w})} \setminus Kw$. Let Δ be the convex subgroup of vK for which $w = v_\Delta$ on K. By what we showed at the beginning, Δ is cofinal in S. Then Lemma 5.2.1 shows that δ is distinguished with invariance subgroup $\mathcal{I}(\delta) = \Delta'$ (the convex hull of Δ in $\widetilde{K(x)}$), proving moreover that $v_\delta = v'_\Delta$. This completes the proof of our lemma. □

5.4 Classes of associated minimal polynomials

Given an immediate algebraic approximation type $\mathbf{A} = \mathrm{appr}\,(x, K)$ of degree $\mathbf{d}$, we want to consider the **class of all associated minimal polynomials for $\mathbf{A}$**. Take an associated minimal polynomial $f(X) \in K[X]$. The class of all associated minimal polynomials depends only on the approximation type $\mathbf{A}$ and not on the elements which realize this approximation type. If we assume that $(K, v) \prec_\exists (K(x), v)$, i.e., that (K, v) is **existentially closed** in $(K(x), v)$ in the language of valued fields, then it turns out (see Lemma 5.4.3) that the class of all associated minimal polynomials for $\mathbf{A}$ consists precisely of all monic polynomials $g(X) \in K[X]$ of degree $\mathbf{d}$ for which $g(x)$ has the same approximation type as $f(x)$ over K. For a submodel $\mathcal{N}$ of a model $\mathcal{M}$, the notion "$\mathcal{N} \prec_\exists \mathcal{M}$" means that every existential sentence with parameters from $\mathcal{N}$, which holds in $\mathcal{M}$, will also hold in $\mathcal{N}$.

The following lemma indicates that to determine the class of all associated minimal polynomials for $\mathbf{A}$, we may w.l.o.g. assume that $(K, v) \prec_\exists (K(x), v)$.

Lemma 5.4.1. (cf. Lemma 11.13 of [7])
For every immediate approximation type $\mathbf{A}$ over K, there exists a simple valued field extension $(K(x)|K, v)$ that realizes $\mathbf{A}$ and $(K, v) \prec_\exists (K(x), v)$. The latter property implies that x is transcendental over K.

From now on, throughout the entire section, we will fix $\mathbf{A} = \mathrm{appr}\,(x, K)$ to be an immediate algebraic approximation type of degree $\mathbf{d}$ and we will often assume that $(K, v) \prec_\exists (K(x), v)$. When referring to $f \in K[X]$ we will always have in mind a fixed associated minimal polynomial of $\mathbf{A}$. Also, S will denote the support of $\mathbf{A}$ and δ will denote its distance.

Recall that at the end of Section 3.6 we showed how if the degree of a polynomial g equals the degree of the approximation type $\mathbf{A}$, then it can happen that $\mathrm{dist}\,(g(x), K) > \mathrm{dist}_K(g(x), g(K))$. But if we assume that (K, v) is existentially closed in $(K(x), v)$, then these difficulties do not appear:

Lemma 5.4.2. *Take a polynomial $g \in K[X]$ of degree $\leq \mathbf{d}$. If (K, v) is existentially closed in $(K(x), v)$, then*

$$\mathrm{dist}\,(g(x), K) = \mathrm{dist}_K(g(x), g(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}\,(x, K) \tag{5.4.1}$$

for $\mathbf{h} = \mathbf{h}_K(x : g)$.

Proof. If $\mathrm{dist}\,(g(x), K) \neq \mathrm{dist}_K(g(x), g(K))$ then from the definition of distances we infer $\mathrm{dist}\,(g(x), K) > \mathrm{dist}_K(g(x), g(K))$, which by definition means that there is a value in vK bigger that $\mathrm{dist}_K(g(x), g(K))$. So assume that there are elements $c, d \in K$ such that

$$\infty > vd > \text{dist}_K(g(x), g(K)) \text{ and } v(g(x) - c) \geq vd \; .$$

Now the existential sentence

$$\exists Y \; : \; v(g(Y) - c) \geq vd \; ,$$

with constants from K, holds in $(K(x), v)$, and by hypothesis it must hold in (K, v) as well. Hence there is $c' \in K$ with $v(g(c') - c) \geq vd$ which yields $v(g(x) - g(c')) \geq \min\{v(g(x) - c), v(g(c') - c)\} \geq vd > \text{dist}_K(g(x), g(K))$ giving a contradiction. This contradiction shows that $\text{dist}\,(g(x), K) = \text{dist}_K(g(x), g(K))$, while the second equality follows from Lemma 3.6.1. □

Lemma 5.4.3. *Let $g \in K[X]$ be a monic polynomial with $\deg(f) = \deg(g)$. Also, assume that $(K, v) \prec_\exists (K(x), v)$. Then g is an associated minimal polynomial for $\mathbf{A}$ if and only if $\text{appr}\,(f(x), K) = \text{appr}\,(g(x), K)$.*

Proof. Since both f and g are monic and of the same degree, we have that $\deg(g - f) < \deg(f) = \mathbf{d}$ and thus, $\mathbf{A}$ fixes the value of $g - f$. By Corollary 3.5.1, this implies that $v(g(x) - f(x)) = v(g(c) - f(c))$ for $c \nearrow x$. We see that $\mathbf{A}$ does not fix the value of $g = f + (g - f)$ if and only if $v(g(x) - f(x)) > vf(c)$ for $c \nearrow x$. In view of $vf(x) > vf(c)$ for $c \nearrow x$ (which again holds by Corollary 3.5.1), the latter is equivalent to: $v(g(x) - f(x)) > v(f(x) - f(c))$ for $c \nearrow x$. This in turn is equivalent to $v(g(x) - f(x)) \geq \text{dist}\,(f(x), K)$ by Lemma 5.4.2. Then from Lemma 3.1.1 b) we get that $\text{appr}\,(f(x), K) = \text{appr}\,(g(x), K)$. □

Now we want to determine easy normal forms for associated minimal polynomials. The idea is to generalize Kaplansky's Lemma 10 (cf. [4], p. 311) to general rank, using the fact that henselian fields are separable-algebraically closed in their completion:

Lemma 5.4.4. *If K is a henselian field of arbitrary rank, then $K^c \cap \tilde{K}$ is purely inseparable over K.*

Proof. Assume that $K^c \cap \tilde{K}$ contains a nontrivial finite separable extension L of K. Take N to be the normal hull of L over K. Let v be extended to $N.K^c$. Take $a \in L \setminus K$ and let $b \neq a$ be a conjugate of a over K. Then

$$\infty \neq v(a - b) \in v(N.K^c) \subseteq v\widetilde{K^c} = \widetilde{vK} \; ,$$

hence there is an element $\alpha \in vK$ with $\alpha \geq v(a - b)$, and since $a \in K^c$, there is an element $c \in K$ with $v(a - c) > \alpha \geq v(a - b)$. For $\sigma \in \text{Gal}\,(L|K)$ with $\sigma a = b$, this yields

$$v\sigma(a - c) = v(b - c) = \min\{v(a - b), v(a - c)\} = v(a - b) < v(a - c) \; ,$$

showing that v and $v \circ \sigma$ are two distinct extensions of the valuation v from K to N, a contradiction to our hypothesis that K should be henselian. This contradiction proves that $K^c \cap \tilde{K}$ is purely inseparable over K. □

Theorem 5.4.5. *Suppose that* $\mathbf{A}$ *is weakly distinguished. Set* $\mathbf{h} = \mathbf{h}_K(x : f)$*. Let* $v = v_\delta \circ \overline{v}_\delta$ *be the decomposition of* v *on* $K(x)$ *which corresponds to the convex hull* $\mathcal{I}'$ *(in* $\widetilde{K(x)}$*) of the invariance subgroup* $\mathcal{I} = \mathcal{I}(\delta)$*. Then there exists* $b \in K$ *such that for* $c \nearrow x$ *we have:*

$$\omega_c := (b(x-c))v_\delta$$

is finite, it is an element of $(Kv_\delta)^{c(\overline{v}_\delta)} \setminus Kv_\delta$*, and it is a zero of the polynomial*

$$\tilde{f}(X) = (a_c f(c))v_\delta + X^{\mathbf{h}} + \sum_{i=\mathbf{h}+1}^{\mathbf{d}} \left(a_c b^{-i} f_i(c)\right) v_\delta \cdot X^i \ ,$$

where $a_c = b^{\mathbf{h}} f_{\mathbf{h}}(c)^{-1}$*, and the residues* $(a_c b^{-i} f_i(c))v_\delta$ *and* $(a_c f(c))v_\delta$ *are finite. The polynomial* $\tilde{f}$ *is of degree* $\mathbf{d}$ *and irreducible over* Kv_δ*. Its monic multiple*

$$\left(a_c^{-1} b^{\mathbf{d}} f_{\mathbf{d}}(c)^{-1}\right) v_\delta \cdot \tilde{f}$$

is the unique associated minimal polynomial for $\operatorname{appr}(\omega_c, Kv_\delta)$*. We may put* $b = 1$ *if* $\mathbf{A}$ *is distinguished.*

Furthermore,

$$g(X) = f(c) + \sum_{i=\mathbf{h}}^{\mathbf{d}} f_i(c)(X-c)^i$$

is an associated minimal polynomial for $\mathbf{A}$ *whenever* $c \nearrow x$*.*

Proof. $\mathbf{A}$ being weakly distinguished by hypothesis, we choose $\gamma_\delta \in \Gamma$ according to Lemma 5.2.1, such that $\mathcal{I}$ is cofinal in $S - \gamma_\delta$. Take $b \in K$ such that $vb = -\gamma_\delta$ (we may put $b = 1$ if we are allowed to choose $\gamma_\delta = 0$, i.e., if $\mathbf{A}$ is distinguished). Then $\operatorname{dist}(bx, K) = \operatorname{dist}(x, K) + vb = \delta - \gamma_\delta$ is distinguished, and for $c \nearrow x$ we will have $v(b(x-c)) \in \mathcal{I}$, thus $v_\delta(b(x-c)) = 0$ and $\omega_c = (b(x-c))v_\delta \neq 0, \infty$. Since $\operatorname{appr}(bx, K)$ is distinguished, we infer from Lemma 5.3.1 that ω_c is an element of $(Kv_\delta)^{c(\overline{v}_\delta)} \setminus Kv_\delta$. Furthermore, putting $a_c = b^{\mathbf{h}} f_{\mathbf{h}}(c)^{-1}$, from equation (3.3.4) we get that

$$v\left((b(x-c))^i a_c b^{-i} f_i(c)\right) > v(b(x-c))^{\mathbf{h}}$$

for all $i \neq \mathbf{h}$, $1 \leq i \leq \mathbf{d}$, and $c \nearrow x$. Then this implies that all residues $\omega_c^i \cdot \left(a_c b^{-i} f_i(c)\right) v_\delta$ and thus all residues $\left(a_c b^{-i} f_i(c)\right) v_\delta$ are finite for $c \nearrow x$. By Lemma 3.6.4, we know that $vf(x) > vf(c) = vf_{\mathbf{h}}(c) + \mathbf{h} \cdot v(x-c)$ for $c \nearrow x$. Hence,

$$v(a_c f(c)) = \mathbf{h} \cdot vb + \mathbf{h} \cdot v(x-c) = \mathbf{h} \cdot v(b(x-c)) \ .$$

Firstly, this shows that $(a_c f(c))v_\delta \neq 0, \infty$. Secondly, since the values $v(b(x-c))$ are cofinal in the convex subgroup $\mathcal{I}$ for $c \nearrow x$, the same holds for the values $\mathbf{h} \cdot v(b(x-c))$. Consequently, $v(a_c f(x)) > \mathcal{I}$, i.e., $(a_c f(x))v_\delta = 0$. On the other hand, multiplication of the Taylor expansion by a_c gives that

$$a_c f(x) = a_c f(c) + (b(x-c))^{\mathbf{h}} + \sum_{\substack{1 \leq i \leq \mathbf{d} \\ i \neq \mathbf{h}}} (b(x-c))^i \cdot a_c b^{-i} f_i(c) \ ,$$

thus, by the finiteness that we have shown above, we get that

$$0 = (a_c f(c))v_\delta + \omega_c^{\mathbf{h}} + \sum_{\substack{1 \le i \le \mathbf{d} \\ i \ne \mathbf{h}}} \omega_c^i \cdot (a_c b^{-i} f_i(c))v_\delta \,. \tag{5.4.2}$$

We want to deduce from this that $\tilde{f}(\omega_c) = 0$, where $\tilde{f}$ is defined as in the assertion of our theorem. If $\mathbf{h} = 1$, then this is already the assertion. Suppose now that $\mathbf{h} > 1$. By the definition of $\mathbf{h}$, we have that

$$v f_{\mathbf{h}}(c) + \mathbf{h} \cdot v(x - c) < v f_i(c) + i \cdot v(x - c)$$

for $i \ne \mathbf{h}$ and $c \nearrow x$. Consequently, if $i < \mathbf{h}$, then

$$\begin{aligned} v(a_c b^{-i} f_i(c)) &= (\mathbf{h} - i) \cdot vb + v f_i(c) - v f_{\mathbf{h}}(c) \\ &> (\mathbf{h} - i) \cdot vb + (\mathbf{h} - i) \cdot v(x - c) \\ &= (\mathbf{h} - i) \cdot v(b(x - c)) \end{aligned}$$

for $c \nearrow x$. The latter values are cofinal in $\mathcal{I}$. On the other hand, the values of $a_c = b^{\mathbf{h}} f_{\mathbf{h}}(c)$ and of $f_i(c)$ are fixed for $c \nearrow x$ ($f_{\mathbf{h}}$ and f_i having degrees $< \mathbf{d}$), hence

$$v(a_c b^{-i} f_i(c)) > \mathcal{I} \tag{5.4.3}$$

for $i < \mathbf{h}$ and $c \nearrow x$, i.e., $(a_c b^{-i} f_i(c))v_\delta = 0$, which proves that the sum in (5.4.2) has to range only over $i > \mathbf{h}$, as asserted in the theorem.

To show the irreducibility of the polynomial $\tilde{f}(X)$, assume that there is a factorization $\tilde{f} = \widetilde{h_1} \cdot \widetilde{h_2}$ over Kv_δ and let $h_1, h_2 \in K[X]$ be preimages of $\widetilde{h_1}, \widetilde{h_2}$ with respect to the residue map modulo v_δ such that $\deg h_j = \deg \widetilde{h_j}$ for $j = 1, 2$. Now we have that

$$\begin{aligned} (a_c f(x) - h_1(b(x-c))h_2(b(x-c)))\, v_\delta &= (a_c f(x))v_\delta - (h_1(b(x-c))h_2(b(x-c)))\, v_\delta \\ &= 0 - \widetilde{h_1}(\omega_c)\widetilde{h_2}(\omega_c) = -\tilde{f}(\omega_c) = 0 \end{aligned}$$

for $c \nearrow x$. Hence,

$$\begin{aligned} v(a_c f(x) - h_1(b(x-c))h_2(b(x-c))) &> \mathcal{I} = \mathbf{h} \cdot \mathcal{I} \\ &= \mathbf{h} \cdot (\gamma_\delta + \mathcal{I}) + v f_{\mathbf{h}}(c) - \mathbf{h} \cdot \gamma_\delta - v f_{\mathbf{h}}(c) \\ &= \mathbf{h} \cdot (\gamma_\delta + \mathcal{I}) + v f_{\mathbf{h}}(c) + v a_c \end{aligned}$$

for $c \nearrow x$. As $(\gamma_\delta + \mathcal{I}) \cap vK$ is cofinal in $v(x - K)$, we obtain:

$$\begin{aligned} v(a_c f(x) - h_1(b(x-c))h_2(b(x-c))) &> \mathbf{h} \cdot \operatorname{dist}(x, K) + v f_{\mathbf{h}}(c) + v a_c \\ &= \operatorname{dist}(f(x), K) + v a_c \\ &= \operatorname{dist}(a_c f(x), K) \end{aligned}$$

for $c \nearrow x$. By Lemma 3.1.1, this shows that $\mathrm{appr}\,(h_1(b(x-c))h_2(b(x-c)), K) = \mathrm{appr}\,(a_c f(x), K)$. Since by hypothesis $\mathrm{appr}\,(x,K)$ does not fix the value of $f(X)$, it also does not fix the value of $a_c f(X)$, and by the equality of the approximation types it follows that it does not fix the value of $h_1(b(X-c))h_2(b(X-c))$ either. But the degree of this polynomial is $\leq \mathbf{d}$, hence it must be equal to $\mathbf{d}$ since $\mathrm{appr}\,(x,K)$ is of degree $\mathbf{d}$, and the polynomial must be irreducible over K (as remarked at the beginning of Section 3.3). This shows that one of the polynomials h_1, h_2 must be a constant. The same must hold for $\widetilde{h_1},\ \widetilde{h_2}$ which proves that $\tilde{f}$ is irreducible over Kv_δ. This is also true for its monic multiple, so it is an associated minimal polynomial for $\mathrm{appr}\,(\omega_c, Kv_\delta)$. As shown above, ω_c is an element of the completion of Kv_δ, thus $\mathrm{appr}\,(\omega_c, Kv_\delta)$ has distance ∞. From this, together with Lemma 5.4.3, we infer that $\left(a_c^{-1}b^{\mathbf{d}}f_{\mathbf{d}}(c)^{-1}\right)v_\delta \cdot \tilde{f}$ is the unique associated minimal polynomial for ω_c over Kv_δ.

Now let $g \in K[X]$ be as in the assertion of the theorem. According to Lemma 5.4.3, to prove that g is an associated minimal polynomial for $\mathbf{A}$, it suffices to prove that $\mathrm{appr}\,(f(x),K) = \mathrm{appr}\,(g(x),K)$, and by Lemma 3.1.1, this is equivalent to

$$v(f(x)-g(x)) \geq \mathrm{dist}\,(f(x),K)\;.$$

Using again the Taylor expansion (3.3.2) for f, we find that

$$f(X)-g(X) = \sum_{1\leq i<\mathbf{h}} f_i(c)(X-c)^i\;.$$

From (5.4.3) and the fact that $v(b(x-c)) \in \mathcal{I}$ for $c \nearrow x$, it follows that also $v((b(x-c))^i a_c b^{-i} f_i(c)) > \mathcal{I}$ for $i < \mathbf{h}$ and $c \nearrow x$. From here it follows that

$$\begin{aligned}
v(f(x)-g(x)) &= v\left(a_c^{-1}\sum_{1\leq i<\mathbf{h}}(b(x-c))^i a_c b^{-i} f_i(c)\right)\\
&= v\left(\sum_{1\leq i<\mathbf{h}}(b(x-c))^i a_c b^{-i} f_i(c)\right) - va_c\\
&> \mathcal{I} - va_c = \mathcal{I} - \mathbf{h}\cdot vb + vf_{\mathbf{h}}(c)\\
&= \mathcal{I} + \mathbf{h}\cdot\gamma_\delta + vf_{\mathbf{h}}(c) = \mathbf{h}\cdot(\gamma_\delta + \mathcal{I}) + vf_{\mathbf{h}}(c)\;,
\end{aligned}$$

whence

$$v(f(x)-g(x)) > vf_{\mathbf{h}}(c) + \mathbf{h}\cdot\mathrm{dist}\,(x,K) = \mathrm{dist}\,(f(x),K)$$

for $c \nearrow x$. This completes the proof of our theorem. □

Corollary 5.4.6. *Suppose that* $\mathbf{A}$ *is distinguished. For* $c \nearrow x$*,* $\mathbf{A}$ *induces a distinguished approximation type*

$$\mathbf{A}_{v_\delta} = \mathrm{appr}\,((x-c)v_\delta, Kv_\delta)$$

which is of the same degree $\mathbf{d}$ *and has distance* ∞*.*

For henselian ground fields, the assertion of Theorem 5.4.5 may be supplemented as follows:

Theorem 5.4.7. *Let the situation and notation be as in the foregoing theorem and assume in addition that K is henselian. Let $p_\delta = \max\{1, \mathrm{char}(Kv_\delta)\}$ be the characteristic exponent of Kv_δ. Then $\mathbf{h} = \mathbf{d} = p_\delta^e$ for some integer $e \geq 0$. Thus,*

$$(b^{\mathbf{h}} f(c))v_\delta + \omega_c^{\mathbf{d}} = 0$$

for $c \nearrow x$, and

$$g(X) = f(c) + (X - c)^{\mathbf{d}}$$

is an associated minimal polynomial for $\mathbf{A}$ for $c \nearrow x$. In particular, if $\mathbf{d} > 1$, then Kv_δ has positive characteristic.

Proof. With b and c as in Theorem 5.4.5, we have that $\omega_c = (b(x - c))v_\delta$ is an element of $(Kv_\delta)^{c(\overline{v}_\delta)}$ for $c \nearrow x$, and it is algebraic over Kv_δ. Since K is assumed to be henselian, $(Kv_\delta, \overline{v}_\delta)$ is also henselian (cf. Ribenboim [17], p. 211, Proposition 10). By Lemma 5.4.4, ω_c must be purely inseparable over K. Since the irreducibility assertion of Theorem 5.4.5 for $\tilde{f}$ shows that $\tilde{f}(X)$ is the minimal polynomial of ω_c over Kv_δ, we find that $\tilde{f}(X)$ must be equal to $(a_c f(c))v_\delta + X^{\mathbf{h}}$, with $\mathbf{h}$ a power of p_δ according to Lemma 3.5.4. But Theorem 5.4.5 also asserts that its degree is $\mathbf{d}$, hence $\mathbf{d} = \mathbf{h}$ is a power of p_δ. Note that as f was chosen to be an associated minimal polynomial for appr (x, K), it is monic and so equation (3.3.1) implies that $f_{\mathbf{d}}(X) = 1$. The special form of $g(X)$ now follows immediately from Theorem 5.4.5. □

We turn to the remaining case of $\mathbf{A}$ not being weakly distinguished.

Theorem 5.4.8. *Assume that $\mathbf{A}$ is not weakly distinguished and let p be the characteristic exponent of Kv. Then $\mathbf{h} = \mathbf{h}(x : f) = \mathbf{d} = p^e$ for some integer $e \geq 0$, and*

$$g(X) = f(c) + \sum_{i=0}^{e} f_{p^i}(c)(X - c)^{p^i}$$

is an associated minimal polynomial for $\mathbf{A}$ whenever $c \nearrow x$. (Note that after deleting the constant term, g is additive if the characteristic of K is positive.)

But also

$$g(X) = f(c) + \sum_{i=0}^{e} \epsilon_i f_{p^i}(c)(X - c)^{p^i}$$

is an associated minimal polynomial for $\mathbf{A}$ whenever $c \nearrow x$, where

$$\epsilon_i = \begin{cases} 1 & \text{if } v_\delta f_{p^i}(c)^- = (\mathbf{h} - p^i) \cdot (\delta/\mathcal{I}(\delta)) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $vf_{p^i}(c) = vf_{p^i}(x)$ and hence also $v_\delta f_{p^i}(c) = v_\delta f_{p^i}(x)$ for $c \nearrow x$. In particular, if the cut $\delta/\mathcal{I}(\delta)$ is not principal, then for $c \nearrow x$,

$$g(X) = f(c) + (X - c)^{p^e}$$

is an associated minimal polynomial for $\mathbf{A}$.

Proof. As in the proof of Lemma 3.6.1, we consider the Taylor expansion (3.3.2) of f, keeping in mind that $\mathbf{A}$ fixes the value of all derivatives f_i, for $i > 0$, since their degree is $< \mathbf{d}$. By Lemma 3.5.4 we know that if $i = p^t$ and $j = p^t r$ with $r > 1$ and $(r,p) = 1$, then

$$vf_i(c) + i \cdot v(x-c) < vf_j(c) + j \cdot v(x-c)$$

for $c \nearrow x$. Since $j > i$, it now follows from Lemma 5.2.3 that there exists an element $\beta_0 \in S$ such that

$$vf_j(c) + j \cdot v(x-c) > vf_i(c) + i \cdot \delta$$

for $c \nearrow x$ and $v(x-c) \geq \beta_0$. On the other hand, if $i \neq \mathbf{h}$ then $vf_i(c) + i \cdot v(x-c) > vf_{\mathbf{h}}(c) + \mathbf{h} \cdot v(x-c)$ for $c \nearrow x$. This yields $vf_i(c) + i \cdot \delta \geq vf_{\mathbf{h}}(c) + \mathbf{h} \cdot \delta$ and thus by (5.4.1) we have that

$$vf_j(c) + j \cdot v(x-c) > vf_{\mathbf{h}}(c) + \mathbf{h} \cdot \delta = \operatorname{dist}(f(x), K) \tag{5.4.4}$$

for $c \nearrow x$. Furthermore, for every j such that $\mathbf{h} < j \leq \mathbf{d}$, by the definition of $\mathbf{h}$ we have that

$$vf_j(c) + j \cdot v(x-c) > vf_{\mathbf{h}}(c) + \mathbf{h} \cdot v(x-c)$$

for $c \nearrow x$, and in the same way as above, (5.4.4) may also be deduced for such j. Altogether, if we put $\mathbf{h} = p^e$ ($\mathbf{h}$ is a power of p by Lemma 3.5.4), by virtue of the Taylor expansion (3.3.2) we obtain that

$$\begin{aligned} v\left(f(x) - f(c) - \sum_{i=0}^{e} f_{p^i}(c)(x-c)^{p^i}\right) &= v\left(\sum_{j \neq p^\nu \,\vee\, j > \mathbf{h}} f_j(c)(x-c)^j\right) \\ &\geq \operatorname{dist}(f(x), K)\ . \end{aligned}$$

We infer from Lemma 3.1.1 that $\operatorname{appr}(f(x), K) = \operatorname{appr}(g(x), K)$ for

$$g(X) = f(c) + \sum_{i=0}^{e} f_{p^i}(c)(X-c)^{p^i}\ .$$

In view of Lemma 5.4.3, this shows that g is an associated minimal polynomial for $\mathbf{A}$ whenever $c \nearrow x$ (recall that w.l.o.g. we may assume that $(K,v) \prec_\exists (K(x),v)$). In particular, $\mathbf{A}$ does not fix the value of this polynomial, hence its degree cannot be less than $\mathbf{d}$. This proves that $\mathbf{d} = \mathbf{h} = p^e$. Note that this yields $f_{\mathbf{h}}(c) = 1$ for all $c \in K$ and thus, $\beta_K(x : f) = 0$.

Now let $\mathcal{I} = \mathcal{I}(\delta)$. Assume that $j = p^i < \mathbf{h}$, $i \geq 0$. Then by the choice of $\mathbf{h}$, we have that

$$vf_j(c) + j \cdot v(x-c) > vf_{\mathbf{h}}(c) + \mathbf{h} \cdot v(x-c) = \mathbf{h} \cdot v(x-c)$$

and thus also

$$v_\delta f_j(c) + j \cdot v_\delta(x-c) \geq \mathbf{h} \cdot v_\delta(x-c)$$

for $c \nearrow x$. This shows that

$$v_\delta f_j(c) + j \cdot (\delta/\mathcal{I}) \geq \mathbf{h} \cdot (\delta/\mathcal{I})\ .$$

If "$>$" holds here, then we know that

$$v_\delta f_j(c) + j \cdot v_\delta(x-c) > \mathbf{h} \cdot (\delta/\mathcal{I})$$

for $c \nearrow x$ and thus also

$$vf_j(c) + j \cdot v(x-c) > \mathbf{h} \cdot \delta = \mathrm{dist}\,(f(x),K)$$

for $c \nearrow x$. As we have shown before in this proof, for such indices j we may omit the summand $f_j(c)(X-c)^j$ from the polynomial $g(X)$ without losing the property $\mathrm{appr}\,(f(x),K) = \mathrm{appr}\,(g(x),K)$. But for $j \neq \mathbf{h}$, Lemma 5.1.3 implies that the equation

$$v_\delta f_j(c) + j \cdot (\delta/\mathcal{I}) = \mathbf{h} \cdot (\delta/\mathcal{I})$$

is equivalent to

$$v_\delta f_j(c)^- = (\mathbf{h}-j)\delta/\mathcal{I}\,. \tag{5.4.5}$$

Consequently, if $\delta/\mathcal{I}$ is not a principal cut, then all summands

$$(X-c)^j f_j(c)$$

for $j \neq \mathbf{h}$ may be omitted from the polynomial $g(X)$ without losing the property that $f(x)$ and $g(x)$ have the same approximation type over K. But if $\delta/\mathcal{I}$ is principal, then the above equation yields the criterion that we have used in the formulation of our theorem for those summands that have to appear in $g(X)$. □

In Corollary 5.4.6, we stated a correlation between a given distinguished approximation type over K and certain approximation types on the residue field Kv_δ. This also covers the case of a weakly distinguished approximation type because it is always connected to a distinguished approximation type through multiplication by a constant. In the case where the given approximation type is not weakly distinguished, there is a correlation with certain approximation types on the valued field (K, v_δ), as we will see below.

Theorem 5.4.9. *Let $\mathbf{A}_v = \mathrm{appr}_{\,v}(x,K)$ be an immediate approximation type w.r.t. the valuation v of K, of degree $\mathbf{d}$ and distance δ. Assume that $\mathbf{A}_v$ is not weakly distinguished, and let as usual v_δ denote the coarsening of v (on $K(x)$) corresponding to the convex hull $\mathcal{I}'$ of $\mathcal{I} = \mathcal{I}(\delta)$ on $K(x)$. Then the approximation type $\mathbf{A}_{v_\delta} = \mathrm{appr}_{\,v_\delta}(x,K)$ of x over K w.r.t. the valuation v_δ is also an immediate approximation type of degree $\mathbf{d}$ which is not weakly distinguished. More precisely, given a polynomial $g \in K[X]$, $\mathbf{A}_v$ fixes the value of g w.r.t. v if and only if $\mathbf{A}_{v_\delta}$ fixes the value of g w.r.t. v_δ. In particular, g is an associated minimal polynomial for $\mathbf{A}_v$ if and only if it is an associated minimal polynomial for $\mathbf{A}_{v_\delta}$. Further, if $1 < \mathbf{d} < \infty$, then* $\mathrm{char}\, Kv_\delta > 0$.

On the other hand, v_δ is the coarsest coarsening of v with the above properties. If w is a proper coarsening of v_δ, then $\mathbf{A}_w = \mathrm{appr}_{\,w}(x,K)$ is not immediate. For the induced valuation $\bar{w}$ on Kw defined by the decomposition $v = w \circ \bar{w}$, there are $b_w, c_w \in K$ such that $\mathbf{A}_{\bar{w}} = \mathrm{appr}_{\,\bar{w}}((b_w(x-c_w))w, Kw)$ is an immediate approximation type and is not weakly distinguished.

Proof. Let S_v and S_{v_δ} denote the supports of $\mathbf{A}_v$ and $\mathbf{A}_{v_\delta}$, respectively. By hypothesis, $\mathbf{A}_v$ is not weakly distinguished, i.e., $S_v/\mathcal{I}$ has no maximal element. But $S_{v_\delta} = S_v/\mathcal{I}$, so S_v has no maximal element as well. Then by Lemma 3.2.1 we have that $\mathbf{A}_{v_\delta}$ is immediate. Since by part b) of Lemma 5.1.2 we have that $\mathcal{I}(S_{v_\delta}) = \mathcal{I}(S_v/\mathcal{I}) = \{0\}$, the approximation type $\mathbf{A}_{v_\delta}$ is not weakly distinguished.

Since S_{v_δ} has no maximal element, we have that $x \nearrow \mathbf{A}_v$ whenever $x \nearrow \mathbf{A}_{v_\delta}$, and the converse is true just because $va \leq vb$ implies $v_\delta a \leq v_\delta b$. Therefore, for an arbitrary polynomial g, if $\mathbf{A}_{v_\delta}$ does not fix the value of g, then neither does $\mathbf{A}_v$. We have to prove the converse; take a polynomial g and assume that $\mathbf{A}_v$ does not fix its value. Then by Lemma 3.3.4, $vg(c) = \beta + k \cdot v(x-c)$ for $c \nearrow \mathbf{A}_v$, for some $\beta \in vK$ and some integer $k \geq 1$. It follows that $v_\delta g(c) = \beta/\mathcal{I} + k \cdot v_\delta(x-c)$ for $c \nearrow \mathbf{A}_{v_\delta}$. As $v_\delta(x-c)$ is not fixed for $c \nearrow \mathbf{A}_{v_\delta}$, this shows that $\mathbf{A}_{v_\delta}$ does not fix the value of g.

The assertion on the characteristic of Kv_δ follows by an application of the equation $\mathbf{d} = p^e$ of Theorem 5.4.8 to the approximation type $\mathbf{A}_{v_\delta}$ which is an approximation type over the valued field (K, v_δ) whose residue field is just Kv_δ and p is its residue characteristic.

Now assume that w is a proper coarsening of v_δ (on $K(x)$). Let Δ be the convex subgroup of $vK(x)$ such that $w = v_\Delta$. Then there is a positive value $\alpha \in \Delta \setminus \mathcal{I}(\delta)$. By the definition of $\mathcal{I}(\delta)$, there is some $c_w \in K$ such that

$$v(x-c_w) + \alpha > \delta \, . \tag{5.4.6}$$

Suppose that $\mathbf{A}_w$ is immediate. Then there is $c \in K$ such that $w(x-c) > w(x-c_w)$. This implies that $w(x-c)(x-c_w)^{-}1 > 0$, whence $v(x-c)(x-c_w)^{-}1 > \Delta$. It follows that $\alpha < v(x-c)(x-c_w)^{-}1 = v(x-c) - v(x-c_w) < \delta - v(x-c_w)$. This contradicts (5.4.6), showing that $\mathbf{A}_w$ is not immediate.

Since $\mathbf{A}_v$ is immediate, there exists some $b_w \in K$ such that $vb_w = -v(x-c_w)$. In particular, $w(b_w(x-c_w)) = 0$. But there is no $c \in K$ such that $0 < w(b_w(x-c_w)-c)$ because otherwise, we would have $0 < w((x-c_w-b_w^{-1}c)(x-c_w)^{-1})$, which implies that

$$\alpha < v\left((x-c_w-b_w^{-1}c)(x-c_w)^{-1}\right) = v(x-c_w-b_w^{-1}c) - v(x-c_w) < \delta - v(x-c_w) \, ,$$

contradicting (5.4.6). Since $w(b_w(x-c_w)-c) \leq 0$ for every $c \in K$,

$$\{b_w(x-c_w) - c \mid c \in K\} \cap \mathcal{M}_w = \emptyset \, ,$$

where $\mathcal{M}_w$ denotes the maximal ideal of w. Then by definition of $\bar{w}$

$$\bar{w}((b_w(x-c_w)-c)w) = v(b_w(x-c_w)-c)$$

for every $c \in K$. This implies that

$$\bar{w}((b_w(x-c_w))w - Kw) = v(b_w(x-c_w)-K) = v(b_w x - K) = vb_w + v(x-K) \, .$$

Since $\mathbf{A}_v$ is immediate by assumption, $v(x-K)$ has no maximal element, whence $\bar{w}((b_w(x-c_w))w - Kw)$ has no maximal element. By Lemma 3.2.1 it follows that $\mathbf{A}_{\bar{w}}$ is immediate. Furthermore, this implies that $\operatorname{dist}_{\bar{w}}((b_w(x - c_w))w, Kw) = vb_w + \operatorname{dist}_v(x, K)$, which by Lemma 5.2.1, shows that $\mathbf{A}_{\bar{w}}$ is not weakly distinguished. $\square$

Chapter 6

Outlook

As already explained in the Introduction, the main results of this thesis are very important for solving local uniformization in positive characteristic. Using Theorem 4.3.1, Knaf and F.-V. Kuhlmann show in [6], that local uniformization can always be achieved after a certain finite extension of the function field. The goal is to do local uniformization without any extension of the function field, and the hope is to achieve this by refining the valuation theoretical methods. For this reason it is of interest to be able to conclude that for a given valued function field $(F|K,v)$ if there is a finite extension F' of F within its henselization such that $(F'|K,v)$ admits local uniformization, then also $(F|K,v)$ admits local uniformization. Now this can be done if Theorem 4.3.1 can be generalized in a suitable way to the case of non-immediate valued function fields. To prove Theorem 4.3.1 we first proved Lemma 4.1.1 which gives us an upper bound for the degree $[K(x)^h : K(f(x))^h]$ under the assumption that the extension $(K(x)|K,v)$ is immediate. Later we generalized this result to Lemma 4.2.7 to find an upper bound for the degree $[K(x)^h : K(y)^h]$ under the assumption that the approximation type of x is transcendental. To generalize this result it is an important task to extend to the case of algebraic approximation types as well. Furthermore we can ask for the degree $[K(x)^h : K(f(x))^h]$ in the case that $(K(x)|K,v)$ is not immediate.

To study the structure of non-immediate extensions one can develop the theory of approximation types to the non-immediate case as well. A natural start would be to study approximation types realized in non-immediate simple extensions over the ground field and then to try to find simple extensions that realize non-trivial non-immediate approximation types.

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