Holography for Rotating Black Holes

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Abstract

In 1993, 't Hooft (1999 Nobel Prize winner in physics) proposed that quantum gravity requires that the information in a three dimensional world can be stored on a two dimensional manifold much like a hologram. This is known as the holographic principle, and since then this idea has changed the direction of researches in quantum gravity. A concrete realization of this idea in string theory was first discovered in 1997 by Maldacena in his famous anti de-Sitter/Conformal Field Theory[1] correspondence conjecture. The AdS/CFT correspondence states that some string theories on a certain manifold that contains AdS space, in some limits, are dual to a CFT living on the boundary of this manifold.

Despite the rapid progress in studying the AdS/CFT, this proposal is still away from practical applications. Some of the reasons are the fact that the AdS (anti-de Sitter) spacetime is not likely the spacetime where we are living nowadays and the existence of extra dimensions (as one of the ingredients in string theory) is still under question. The Kerr/CFT correspondence which was proposed in 2008 by Strominger et al appears to be a more "down to earth" duality, compared to the AdS/CFT correspondence. Originally, this new correspondence states that the physics of extremal Kerr black holes which are rotating by the maximal angular velocity can be described by a two dimensional CFT living on the near horizon.

In this thesis, after reviewing some concepts in Kerr/CFT correspondence, I present some of my research results which extend and support the correspondence for non-extremal rotating black holes. I discuss the extension of the Kerr/CFT correspondence for the rotating black holes in string theory, namely Kerr-Sen black holes, and the Kerr/CFT analysis for vector field perturbations near the horizon of Kerr black holes.

It is recently conjectured that a generic non-extremal Kerr black hole could be holographically dual to a hidden conformal field theory in two dimensions. Furthermore, it is known that there are two CFT duals (pictures) to describe the charged rotating black holes which correspond to angular momentum J and electric charge Q of the black hole. Further-

[1]AdS/CFT for short. AdS stands for anti de-Sitter, and CFT is the acronym for Conformal Field Theory.

more these two pictures can be incorporated by the CFT duals (general picture) that are generated by $SL(2,\mathbb{Z})$ modular group. The general conformal structure can be revealed by looking at a charged scalar wave equation with some appropriate values of frequency and charge. In this regard, we consider the wave equation of a charged massless scalar field in the background of Kerr-Sen black hole and show in the "near region", the wave equation can be reproduced by the squared Casimir operator of a local $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ hidden conformal symmetry. We can find the exact agreement between macroscopic and microscopic physical quantities like entropy and absorption cross section of scalars for Kerr-Sen black hole. We then find an extension of the vector fields that in turn yields an extended local family of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ hidden conformal symmetries, parameterized by one parameter. For some special values of the parameter, we find a copy of $SL(2,\mathbb{R})$ hidden conformal algebra for the charged Gibbons-Maeda-Garfinkle-Horowitz-Strominger black hole in the strong deflection limit.

The generic non-extremal Kerr-Newman black holes are holographically dual to hidden conformal field theories in two different pictures. The two pictures can be merged together to the CFT duals in the general picture that are generated by $SL(2,\mathbb{Z})$ modular group. We find some extensions of the conformal symmetry generators that yield an extended local family of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ hidden conformal symmetries for the Kerr-Newman black holes, parameterized by one deformation parameter. The family of deformed hidden conformal symmetry for Kerr-Newman black holes also provides a set of deformed hidden conformal generators for the charged Reissner-Nordstrom black holes. The set of deformed hidden conformal generators reduce to the hidden $SL(2,\mathbb{R})$ conformal generators for the Reissner-Nordstrom black hole for specific value of deformation parameter. We also find agreement between the macroscopic and microscopic entropy and absorption cross section of scalars for the Kerr-Newman black hole by considering the appropriate temperatures and central charges for the deformed CFTs.

Also in this thesis, we derive an appropriate boundary action for the vector fields near the horizon of near extremal Kerr black hole. We then use the obtained boundary action to calculate the two-point function for the vector fields in Kerr/CFT correspondence. In performing this analysis we borrow a formula proposed in AdS/CFT, namely the equality

between the bulk and boundary theories partition functions. We show the gauge-independent part of the two-point function is in agreement with what is expected from CFT.

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To my parents, sister, and brothers.

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List of Abbreviations

A_H	Area of the surface of event horizon
AdS	anti de-Sitter
CFT	Conformal Field Theory
CFT_2	Two Dimensional Conformal Field Theory
GMGHS	Gibbons Maeda Garfinkle Horowitz Strominger
NHEK	Near Horizon of Extremal Kerr
QFT	Quantum Field Theory
SCT	Special Conformal Transformation
S_{BH}	Bekenstein-Hawking entropy
T_H	Hawking temperature
Ω_H	Angular velocity of black holes at the event horizon
KN	Kerr-Newman
KS	Kerr-Sen

Chapter 1

Introduction

One of the most dramatic predictions in general relativity is the existence of black holes. These objects are mysterious and yet very interesting. Even today, we may find a physicist who has such a skepticism of the existence of a black hole. However, at least we have two reasons in believing that black holes exist out there. The first one is that black holes are the unavoidable natural consequences in Einstein general theory of relativity. The success of Einstein's general relativity in predicting some physical phenomenon, which are tested by experimental observations, convinces us that black holes must exist. The second one is we are provided astronomical data that strongly support them to be part of our universe [1, 2]. For many years black holes had been just the objects of science fiction, but there is now a significant body of evidence that supports the existence of black holes, or at least objects very much like them. They are considered as the endpoints of stellar collapse. A strong candidate of black hole can be found in the X-ray binary Cygnus X-1. Quite recently the authors of [3] have studied this black hole candidate to determine it's accurate value of mass.

One way to understand a black hole is describing it as a spacetime singularity surrounded by an event horizon. A classical point particle can enter an event horizon of a black hole with no ill effects, but the spacetime structure inside of a black hole is such that he cannot return from inside of the event horizon. Black holes according to general relativity are characterized by three physical parameters only. They are the black hole mass M, the angular momentum J, and the electric charge Q. The fact that only these three parameters that characterize the most complex black hole in general relativity raises the "no hair" theorem of a black hole. This "no hair" theorem says that black holes have no other distinguishing classical characteristics beyond M, J, and Q.

In statistical mechanics, entropy has an important physical implication as an amount of “disorder” of a system. By considering general relativity only, black holes are “dead” thermodynamical objects, i.e. they have no entropy. Including quantum mechanics in the studies of black holes gives birth to the nonzero black hole entropy. Gedanken experiments carried out since the early 1970s [4, 5, 6] have established that a black hole of horizon area A behaves as if it were a thermodynamic object with an entropy

$$S = \frac{c^3 k_B A}{4\hbar G}\,, \tag{1.0.1}$$

where G is the Newton gravitational constant, $\hbar$ is Planck's constant, k_B is the Boltzmann's constant, and c is the speed of light. It turns out, by using this entropy formula, we find that the entropy of black holes is enormous. One can compute that a solar mass black hole (about 6 kilometers in diameter) should have an entropy that is 22 orders of magnitude greater than the entropy of the sun itself. According to Boltzmann, entropy in a physical system is a manifestation of statistical degeneracy of the underlying states. Then, by following Boltzmann' idea, one big question for a quantum theory of gravity is explaining how black holes can have the statistical degeneracy of $\exp\left(c^3 k_B A/4G\hbar\right)$. In other words, a key challenge to any quantum theory of gravity is to identify the “atoms” of spacetime that can explain such a spectacular amount of entropy for a black hole.

In string theory, for a special class of highly symmetric and near-extremal charged black holes, this problem was solved by Strominger and Vafa [7]. The main idea of their work is that the physics of near-extremal black holes is strongly related to the properties of the spacetime in the vicinity of, but outside, the black hole horizon. This vicinity is called as the black hole “throat”, and has the AdS or AdS like geometrical structure. The AdS geometry itself is a solution to the Einstein's gravitational equation with a negative cosmological constant [8]. In his very famous paper, Maldacena [9] argued that quantum gravity in the AdS spacetime is equivalent or holographically dual to a conformally invariant quantum field theory (CFT) with no gravity in a lower number of spacetime dimensions namely the boundary.

The idea by Maldacena, which later known as the AdS/CFT correspondence, is a concrete example of holographic world proposal by 't Hooft [10] and Susskind [11]. The name “holographic” comes from an analogy to the optical holograms analogy, where the image of a three dimensional object can be stored in a two dimensional piece of film. The image in this film can be recreated by using some light technologies, where the image or information can be rebuilt. According to 't Hooft and Suskind, quantum gravity theory needs that the degrees of freedom a gravitational object in D dimensions are equivalent to the degrees of freedom of matter described by quantum theory in $D-1$ dimensions. In the context of the AdS/CFT correspondence, the conformal field theory living on the boundary of AdS can reproduce the results of computations in the AdS spacetime gravitational theory. A black hole in AdS spacetime is then simply described by the holographic dual CFT as a thermal state (gas with a temperature), whose statistical degeneracy explains the black hole entropy. The AdS/CFT correspondence is found to be such a remarkable discovery, that for about more than a decade since it was proposed, the main concern of many theoretical physicists, not only among string theorists, has been around this subject.

Nevertheless, the universe that we live in does not have a negative cosmological constant, and the evidence of astrophysical black holes that we have now do not hint at the existence of the highly electrically or magnetically charged ones as in [7]. How then can we get the benefits of this new holographic or duality ideas to the "real" world, or to be more specific "real" astrophysical black holes. In a 2008 Physical Review D paper [12], Guica et al propose that the entropy of an extremal Kerr black hole can be described holographically by a non-gravitational two dimensional CFT (CFT_2). The rotational parameter per unit mass a^* for an extremal black hole is unity. In [13], the authors show that the astrophysical object GRS 1915+105, whose mass is about 14 times the mass of the sun, has $a^* > 0.98$. It reflects that near-extreme rotating black holes, or at least a near-extremal ones, certainly occur in nature.

The argument given by Guica et al is free of string theory, or any other specific quantum gravity theory. Following an earlier work by Bardeen and Horowitz [14], the authors observe that an extreme rotating black hole has a near-horizon throat of a certain form that controls the dynamical properties of low-energy objects orbiting the black hole horizon. Examining the properties of this geometry, they argue that the quantum theory of gravity in this space must have the two dimensional conformal symmetry. Guica et al. use this symmetry to count the microstates of an extremal Kerr black hole, i.e. deriving its entropy. This work later is called the Kerr/CFT correspondence. The Kerr/CFT correspondence has been studied extensively for different four and higher dimensional extremal rotating black holes which the dual chiral conformal field theory always contains a left-moving sector[1] [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. For these extremal black holes, the near horizon geometry contains a copy of AdS space with isometries that could be extended to Virasoro algebra, hence it may explain the appearance of conformal structure.

However, the standard techniques of Kerr/CFT correspondence for extremal rotating black holes cannot be applied to non-extremal black holes because there is no simple symmetry near the non-extremal black hole horizon that may point to the conformal structure. Moreover, for the non-extremal black holes, the right-moving sector of dual CFT turns on and there is no consistent boundary conditions that allow for both left and right-moving sectors in CFT. Nevertheless, as it is noted in [28], there is other conformal invariance, known as hidden conformal invariance, in the solution space of the wave equation in the background of rotating non-extremal black holes. This means the existence of conformal invariance in a near horizon geometry is not a necessary condition, and the hidden conformal invariance is sufficient to have a dual CFT description. The idea of hidden conformal symmetry in the solution space of the wave equation for a neutral scalar field in different rotating backgrounds was explored in detail in [29, 30, 31, 32, 33].

For the class of four-dimensional rotating charged black holes such as Kerr-Newman, there are two dual CFTs; one is associated with the rotation of the black hole, while the other one is associated with the electric charge of the black hole. The two different dual pictures of black hole are called J and Q pictures [34]. The angular momentum and the charge of a Kerr-Newman black hole are in correspondence with the rotational symmetry of a black hole in the ϕ direction and the gauge symmetry, respectively. The latter symmetry can be considered geometrically as the rotational symmetry of the uplifted Kerr-Newman black hole in the fifth-direction χ. By doing so, the original four dimensional spacetime with coordinates (t, r, θ, ϕ) is embedded into a five dimensional one with χ as an extra dimension. As a result, the combination of two rotational symmetries of uplifted five-dimensional Kerr-Newman black hole lead to two new CFTs (ϕ' and χ' pictures [23]). These two pictures neatly can be embedded into a general picture by using the torus (ϕ, χ) modular group $SL(2, \mathbb{Z})$.

[1]In CFT_2, we are introduced to the left-moving and right-moving sectors that refer to the holomorphic and anti-holomorphic parts of the theory, respectively.

One can easily obtain the CFT results in J or Q pictures for Kerr-Newman black hole [34] as the special case of CFT results in either ϕ' or χ' pictures after setting some specific values to the components of $SL(2,\mathbb{Z})$ modular group transformation's matrix.

One other class of rotating charged black hole solutions in four dimensions is Kerr-Sen geometry [35]. The solution includes three non gravitational fields: an antisymmetric tensor field, a vector field and a dilaton. The Kerr-Sen solution is an exact solution to the equations of motion of effective action of heterotic string theory in four dimensions. In [16], it was shown that for an extremal Kerr-Sen black hole, the central charge of dual chiral CFT doesn't get any contributions from the non-gravitational fields. Furthermore, the central charge leads to the microscopic entropy of Kerr-Sen black hole that is in perfect agreement with the Bekenstein-Hawking entropy. We also notice that the Kerr-Sen solutions contain a scalar dilatonic field. However, the solution space of dilaton equation does not show any conformal symmetry, which is in agreement with previously observation of no contribution of non-gravitational fields to the central charges of dual CFT in the extremal case [16].

Inspired by the existence of different CFT pictures for the four-dimensional non-extremal rotating charged Kerr-Newman black hole, we investigate the CFT results in a general picture and so the possibility of finding the CFT results in ϕ' and χ' pictures for the generic non-extremal Kerr-Sen black hole. In this regard, we consider the equation of motion for a charged scalar field in the background of Kerr-Sen black hole and look for the hidden conformal symmetry in the general picture. The charge of the scalar field appears to be crucial in determining the existence of the general picture, hence we cannot consider the wave equation of a neutral scalar field as in [29]. We then discuss the absorption of scalar fields in the near region of a non-extremal Kerr-Sen black hole. In addition, we find an extended version of hidden conformal generators [36] that involves one parameter for the class of Kerr-Sen solutions. These conformal generators in the appropriate limits, provide a completely new set of conformal symmetry generators for the charged Gibbons-Maeda-Garfinkle-Horowitz-Strominger black hole, which is the charged black hole solution known in string theory.

The discussion of hidden conformal symmetry for Kerr-Sen black holes is extended to the extremal case also. Interestingly, the deformed version of hidden conformal generators [36] can be found in the Kerr-Newman black holes studies. Here we deform the low energy and near region scalar wave equation in Kerr-Newman background. We find the links between the conformal symmetries of black holes in Einstein-Maxwell theory, which cannot be seen by using the hidden conformal symmetry with no deformation for Kerr-Newman black holes. We find that our studies give us a new method in distinguishing the Kerr-Newman and Kerr-Sen black holes. We know that the Kerr-Newman and Kerr-Sen black holes are characterized by the same physical parameters, i.e. M, J, and Q.

Almost for all generic rotating black holes, the hidden conformal symmetry has been found by looking at the symmetry of the solution space of a scalar test particle. Yet, the higher spin test particles in Kerr/CFT correspondence also were considered recently in [37, 21], though the higher spin test particles in the background of black holes had been considered before [38, 39, 40] with different techniques. Quite recently the authors of [41] found the two-point function of spinor fields in Kerr/CFT correspondence by variation of boundary action for spin-1/2 particles. They determined an appropriate boundary term for the spinors in NHEK geometry and used it to calculate the two-point function of spinors. Moreover, they found a relation between spinors in the four-dimensional bulk and the boundary spinors living in two dimensions. The two-point function of spinor fields is in agreement with the correlation function of a two-dimensional CFT. The variational method in [41] for spinors in NHEK geometry is in the same spirit for spinors in the context of AdS/CFT correspondence [42]. In reference [37], the authors show that the two-point function of an operator at left and right temperatures (T_L, T_R) and with conformal dimensions (h_L, h_R), is

$$G \sim (-1)^{h_L+h_R} \left(\frac{\pi T_L}{\sinh\left(\pi T_L t^-\right)}\right)^{2h_L} \left(\frac{\pi T_R}{\sinh\left(\pi T_R t^+\right)}\right)^{2h_R} \tag{1.0.2}$$

that will be useful later in this thesis.

The outline of this thesis is as follows. Chapter 2 contains some reviews on general relativity and black holes. In this section we discuss the Einstein-Hilbert action, from which the vacuum Einstein equations can be obtained. Then we show quite thoroughly how to get the Kerr solution, starting from a general metric with the axial symmetry. Since the purpose of this thesis is to show that black holes could be studied holographically by using CFT_2, we study in details some properties of Schwarzschild and Kerr black holes. Finally, the thermodynamical aspects of Kerr black holes are given.

Chapter 3 is divided into two parts, the review of CFT and AdS/CFT. On the CFT part, we give a review on some basics in conformal field theory. We discuss the symmetry of conformal field theory, and pay more attention to the two dimensional conformal field theory. After deriving the partition function for the CFT_2 defined on a torus, the Cardy formula which is the entropy formula in CFT_2 can be obtained. Furthermore, we also discuss the scattering computation in CFT_2. In AdS/CFT part, we discuss the prescription by Witten for computing the two point function for scalars and vectors. The AdS/CFT computations in the vector case become a warming up for the study of the Kerr/CFT correspondence using the massless vector fields.

In chapter 4, we review the Kerr/CFT correspondence idea. Here we show how to get the central charge for the Kerr black holes starting from the symmetry of a spacetime in the near horizon of an extremal Kerr black hole. It turns out that we can recover the Bekenstein-Hawking entropy for the Kerr black holes by using the Cardy formula in CFT_2. In the Cardy formula, i.e. the entropy formula in CFT_2, we use the central charge associated to the near horizon of extremal Kerr (NHEK) spacetime and the temperature that is measured by an observer near the black hole's event horizon. In this chapter we also review the proposal by Becker et al [18] on the appropriate bulk-to-boundary propagator in Kerr/CFT correspondence. Furthermore, we also review the hidden conformal symmetry of Kerr black holes which was first proposed in [28], which would lead us the Kerr/CFT correspondence for non-extremal Kerr black holes.

In chapter 5, we consider the wave equation of a massless charged scalar field in the background of Kerr-Sen and Kerr-Newman spacetimes. We show in some appropriate limits of parameters and using the general $SL(2,\mathbb{Z})$ modular transformation, the equations of motion can be simplified in the near region of these black holes. Then we show that the radial part of wave equation in the near region can be rewritten as the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ squared Casimir operators in ϕ' picture. In addition, we find the microscopic entropy of the dual CFT for both black holes and compare them to their macroscopic Bekenstein-Hawking entropies. We also compute the absorption cross section of scalars in the near region of Kerr-Sen black hole and show explicitly that the result is in perfect agreement with the finite temperature absorption cross section for a two-dimensional conformal field theory. Then we introduce the deformed equation of motion for the test field and find explicitly two classes of generators that generate a generalized hidden conformal symmetry for the Kerr-Newman and Kerr-Sen black holes. In Kerr-Sen case, the generators obtained in the deformed case can be used to find the hidden conformal symmetry for the charged Gibbons-Maeda-Garfinkle-Horowitz-Strominger black hole. In this chapter, we also study the hidden conformal symmetry of extremal Kerr-Sen black holes and discuss their absorption cross section. As we have mentioned previously, Kerr-Newman black holes also have the deformed hidden symmetry. We will show that the hidden conformal symmetry generators with κ parameter for the Kerr-Newman black holes can approach to the hidden conformal symmetry generators for Reissner-Nordstrom black holes after setting an appropriate value for κ. Furthermore, these generators can be used to get the conformal hidden symmetry generators for a Schwarzschild black hole after setting the black hole's charge to be zero.

In chapter 6, inspired by Becker et al [41], we derive the two-point function for Maxwell fields in Kerr spacetime by varying the corresponding boundary action. Unlike the analysis of spin-1/2 particles where there is no gauge condition, one needs to perform a more careful treatment for the gauge fields where they are subjected to the gauge condition. In this regard, we use the wave equation for spin-1 objects in Kerr background given in [46]. We note that in [46], Teukolsky derived a set of wave equations for spin-$0, 1/2, 1$, and 2 field perturbations in Kerr background. Furthermore, Chandrasekhar derived the solutions for Maxwell fields in Kerr spacetime in term of Teukolsky radial and angular wave functions [47]. The gauge condition that is used in [47] to get the spin-1 field solutions in Kerr background is quite complicated which encumbers the derivation of the two-dimensional CFT correlators of vector fields. However, we can use (1.0.2) to justify that gauge-independent part of two-point function for Maxwell fields in NHEK geometry is dual to the thermal CFT correlators. We start with the Maxwell action in four-dimensional Kerr background where all four components of Maxwell fields are taken into account. After explicitly calculating the appropriate boundary action for the Maxwell fields, the leading terms in the boundary action contain only the boundary fields corresponding to A_t and A_ϕ. Interestingly enough, this result provides the correct number of degrees of freedom for the boundary fields and yields the corresponding two-point function of spin-1 fields. Both the dimensionless Hawking temperature τ_H and the boundary value of the metric function $\Delta = (r - r_+)(r - r_-)$ are small numbers that play an important role to get the appropriate number of Maxwell fields on the boundary. The smallness of these quantities is a result of considering the near horizon and near extremal limits of Kerr geometry. All the results of this chapter support the Kerr/CFT correspondence where the four-dimensional rotating black hole physics is dual to two-dimensional CFT on the boundary. Finally in chapter 7, we wrap up where some concluding remarks and future possible research problems.

Chapter 2

Black Holes in Einstein gravity

John Wheeler, in the mid 1960s, was the first person who coined the name black hole for a dead star. Before that, this object was known as the collapsed star according to English literature and the frozen one in Russian. The two later names give us a better picture about the origin of black hole. When stars run out their nuclear fuel, there is no more thermal pressure produced by the nuclear reaction inside of a star which balances out the gravitational attraction toward the center of mass. With a sufficient initial amount of mass, this "dead" star cannot withstand to collapse into a singularity. At the singularity, a black hole with a finite mass ceases to fill a volume, therefore the density of mass would become extremely large.

Without general relativity, the concept of black holes can also be understood from the Newtonian gravity. The "escape velocity" v_e in Newtonian gravity is defined as the velocity for an object to escape the orbit of a planet or star and reach a point at infinity. The escape velocity is given by

$$v_e = \sqrt{\frac{2M}{r}}\,, \tag{2.0.1}$$

where M is the mass of planet or star, and r is the distance from the object to the center of mass of a planet or star. In equation (2.0.1) we use the natural unit where $G = c = 1$. We will use this convention for the rest of this thesis unless we need to restore these units. In general, the mass M and radius r may vary, and for $r = 2M$ the velocity (2.0.1) becomes the speed of light. Therefore, when $r = 2M$, only light rays that can escape. We know that there is nothing that moves faster than light except for the hypothetical object tachyon. Nevertheless, we do not consider tachyon as a physical object in this thesis, hence the further contraction of the planet or star beyond the critical radius $r = 2M$ creates an eternal prison for any physical objects that fall into it.

The history of black holes' prediction begins when Pierre Laplace predicted the invisible stars. By using the concept of escape velocity in Newtonian gravity, he concluded that a very dense star does not allow any of its rays to radiate which yields the star to be invisible. A quite similar argument in predicting the existence of black holes was also given by British priest and geologist John Michell. Not only a star that may end as a black hole, even our earth would become one if it is contracted into a size where the corresponding escape velocity is larger than the speed of light. However, the problem is the huge amount of external energy that is needed to perform such contraction. For stars with sufficient initial mass, the story is different. When these stars run out of their nuclear fuel, there is no more thermal pressure that balances the gravitational attraction towards the center of stars, hence the only dominant force in these stars is gravity. It is gravity that shrinks these dead stars which finally end at a singular point.

We understand that the prediction of black holes by using Newtonian gravity would not be quite comprehensive. Einstein gave us a better theoretical framework regarding gravitational interaction, namely the general theory of relativity. This theory is built from a very revolutionary paradigm and provides us with some predictions that survive experimental tests so far. This theory gives a more accurate prediction while Newtonian one fails. Therefore it is natural to expect that prediction of black holes' existence by using Newtonian gravity gets some corrections from the general theory of relativity. Surprisingly, for a collapsing static and electrically neutral massive object, the critical radius where the light rays will be trapped forever is just the same as the one given from Newtonian calculation, i.e. $r = 2M$. All objects that enter the surface with this radius will never get escape, hence one can imagine this surface as a one way membrane.

In this chapter, we briefly review the Einstein theory of general relativity. Then we will show step by step how to obtain the rotating solution of the vacuum Einstein equations, namely Kerr solution, starting from a general form of axial symmetric solution. Subsequently, we review some properties of the static and rotating black holes, and discuss their thermodynamics. The main references in this chapter are [43, 44, 45].

2.1 A brief review on Einstein's gravity

2.1.1 Metric tensor and isometries

In relativity, the spacetime metric or the metric for short, i.e. ds^2, is a squared distance between two spacetime points, say $x^\mu = (t, x, y, z)$ and $x^\mu + dx^\mu = (t+dt, x+dx, y+dy, z+dz)$. For example, the four dimensional flat metric can be written as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \, , \tag{2.1.2}$$

where we have used the natural unit $c = 1$. The metric (2.1.2) is also known as the Minkowski metric, named after mathematician Hermann Minkowski who was the first to formulate the four dimensional spacetime which suits the Einstein's special theory of relativity. The formula (2.1.2) can be written in a more compact expression by using the Einstein summation convention [1],

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \, , \tag{2.1.3}$$

where the tensor $\eta_{\mu\nu}$ is known as the Minkowski metric tensor. As a matrix, the metric tensor $\eta_{\mu\nu}$ can be read

$$\eta = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.1.4}$$

As we have mentioned before, the Minkowski metric (2.1.3) describes a flat spacetime where the associated coordinates are three Cartesian spatial dimensions x, y, z, and one time dimension t. It is clear that the metric tensor components will vary as we change the coordinate system to describe the spacetime. As an example, if we prefer to use the spherical coordinates $\{r, \theta, \phi\}$ instead of the Cartesian ones $\{x, y, z\}$, then the metric (2.1.2) becomes

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, . \tag{2.1.5}$$

[1] The repeated indices are implicitly summed over, $a_\mu a^\mu = \sum\limits_\mu a_\mu a^\mu$.

There is no change for the time component in the metric, hence a matrix expression of the metric tensor $\eta_{\mu\nu}$ associated with the metric (2.1.5) is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} . \tag{2.1.6}$$

When gravity is considered, the special theory of relativity is inadequate. To explain gravity, Einstein introduced his other masterpiece, namely the general theory of relativity. According to Einstein's general relativity, the spacetime is curved by the presence of matter or energy. Indeed, the concept of metric ds^2 is still used in the case of curved spacetime, but in a more general form compared to (2.1.3). We notice that the non-vanishing entries in (2.1.4) are only in the diagonal parts, and the terms that couple to dt^2 and dr^2 are just some constants[2]. It is a natural guess that for a curved spacetime, the corresponding metric tensor contains several (if not all) non-vanishing off-diagonal entries.

Figure 2.1 illustrates this idea: the curved membrane cartoon represents a slice of curved spacetime as a result of the presence of matter on it. We denote the metric tensor for the curved spacetime by $g_{\mu\nu}$. The components of $g_{\mu\nu}$ in general depend on the spacetime coordinates, i.e. $g_{\mu\nu} = g_{\mu\nu}(x)$, which can be obtained by solving the Einstein equations. The metric in a curved spacetime then can be read as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \, . \tag{2.1.7}$$

An important concept that would be useful in the latter discussion is the symmetry of spacetime. It is possible to perform a coordinate transformation, $x^\mu \to x'^\mu$, that leaves the metric tensor $g_{\mu\nu}$ invariant. Such transformation is called an isometry. Accordingly, the infinitesimal coordinate transformation that keeps $g_{\mu\nu}$ invariant is known as the infinitesimal isometry. Consider an infinitesimal coordinate transformation,

$$x^\mu \to x'^\mu = x^\mu + \varepsilon\xi^\mu(x) \, , \tag{2.1.8}$$

[2]Later we will see that in curved spacetime, the functions in front of dt^2 and dr^2 are coordinate dependent.

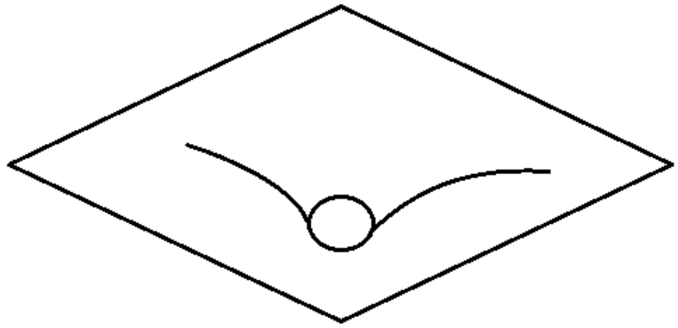

Figure 2.1: The illustration of curved spacetime by the presence of matter.

where ε is an arbitrary small constant and ξ^μ is an arbitrary vector. In general, the vector ξ^μ depends on the spacetime coordinates x^μ. Under the coordinate transformation (2.1.8), the metric tensor $g_{\mu\nu}$ changes as

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \varepsilon \mathcal{L}_\xi g_{\mu\nu} + \mathcal{O}\left(\varepsilon^2\right) , \tag{2.1.9}$$

where

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\beta}\partial_\nu \xi^\beta + g_{\nu\beta}\partial_\mu \xi^\beta . \tag{2.1.10}$$

Equation (2.1.10) is known as the Lie derivative of $g_{\mu\nu}$ with respect to ξ. Neglecting the $\mathcal{O}(\varepsilon^2)$ term in (2.1.9), the vanishing of Lie derivative (2.1.10) yields the metric tensor $g_{\mu\nu}$ to be invariant under the transformation (2.1.8). The Lie derivative $\mathcal{L}_\xi g_{\mu\nu}$ can also be rewritten in terms of covariant derivative operator ∇_μ,

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu . \tag{2.1.11}$$

When $\mathcal{L}_\xi g_{\mu\nu}$ vanishes, we get the equations

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 , \tag{2.1.12}$$

which is known as the Killing equation. The vector ξ^μ which satisfies (2.1.12) is called the Killing vector. In equation (2.1.12), we have used the covariant derivative ∇_μ for ξ_μ that is given by

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\alpha_{\nu\mu} \xi_\alpha . \tag{2.1.13}$$

The followings are some other operations of ∇_μ to some arbitrary scalars and tensors

$$\begin{aligned}
\nabla_\mu \phi &= \partial_\mu \phi \,, \\
\nabla_\mu T^\nu &= \partial_\mu T^\nu + \Gamma^\nu_{\mu\alpha} T^\alpha \,, \\
\nabla_\mu T^{\alpha\beta} &= \partial_\mu T^{\alpha\beta} + \Gamma^\alpha_{\mu\nu} T^{\nu\beta} + \Gamma^\beta_{\mu\nu} T^{\alpha\nu} \,, \\
\nabla_\mu T_{\alpha\beta} &= \partial_\mu T_{\alpha\beta} - \Gamma^\nu_{\mu\alpha} T_{\nu\beta} - \Gamma^\nu_{\mu\beta} T_{\alpha\nu} \,, \\
\nabla_\mu T^\alpha_\beta &= \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\delta\mu} T^\delta_\beta - \Gamma^\delta_{\beta\mu} T^\alpha_\delta \,,
\end{aligned} \tag{2.1.14}$$

which is the generalization of ∂_μ in flat spacetime. Christoffel symbol of the second kind, $\Gamma^\alpha_{\delta\chi}$, is given in (2.1.31).

In general, the Lie derivative of a tensor metric (2.1.11) is not zero. We call the following mapping

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \,, \tag{2.1.15}$$

where

$$h_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \,. \tag{2.1.16}$$

as the diffeomorphism of $g_{\mu\nu}$ with the diffeomorphism parameter ξ. In deriving the central charge for NHEK spacetime discussed in section 4.1, we use the diffeomorphism formula (2.1.16).

2.1.2 Einstein-Hilbert action

An action is found to be a powerful framework in theoretical physics. From an action, a set of equation of motions that describes the dynamics of each fields in the theory can be obtained by using the least action principle. In fact, from an action we can explore the symmetries of the system under consideration.

Before we discuss an action that describes the gravity according to Einstein, let us review the action for free scalar fields in curved spacetime. Let us start by writing an action

$$S = \int_{\mathcal{R}} \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) d^4x \,, \tag{2.1.17}$$

where $\mathcal{L}$ is the Lagrangian density. In the action (2.1.17), we consider the Lagrangian density is a function of the field $\Phi(x)$ and its first derivative, $\partial_\mu \Phi(x)$. The integration in (2.1.17) is

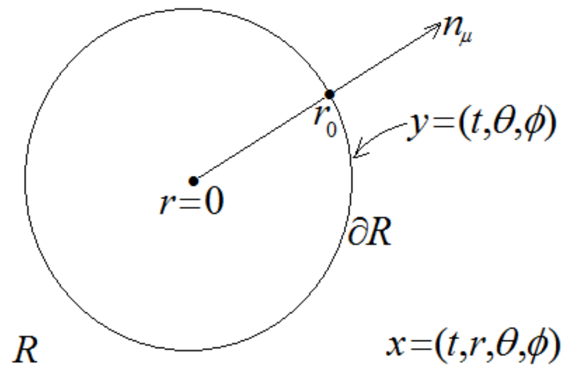

Figure 2.2: The illustration of spacetime $\mathcal{R}$ with coordinates x and the boundary $\partial\mathcal{R}$ with coordinates y. To get y, we fix one of the coordinates in x which in the illustration is the radial coordinate, i.e. r is fixed to be r_0.

over some four dimensional spacetime region $\mathcal{R}$. We consider Φ and $\partial_\mu\Phi$ as two independent variables, hence the variation of the action (2.1.17) can be written as

$$\delta S = \int_R \left(\frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\partial_\mu\left(\delta\Phi\right)\right) d^4x \,. \tag{2.1.18}$$

In the last equation, we have assumed that the operators ∂_μ and δ commute with each other. Furthermore, by using the Leibniz rule, we can rewrite the integrand of (2.1.18) as

$$\delta S = \int_{\mathcal{R}} \left(\left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\right)\right)\delta\Phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\delta\Phi\right)}_{\text{surface term}}\right) d^4x \,. \tag{2.1.19}$$

The variation of the field, $\delta\Phi$, vanishes on the boundary of integration region, $\partial\mathcal{R}$. Thus we can get rid of the surface term in (2.1.19) after applying the divergence theorem,

$$\int_{\mathcal{R}} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\delta\Phi\right) d^4x = \int_{\partial R} \left(n_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\delta\Phi\right) d^3y \,. \tag{2.1.20}$$

In the last equation, n_μ is a normal vector to the boundary $\partial\mathcal{R}$, and y represents the boundary coordinates, which is illustrated in figure 2.2.

Now the variation of action (2.1.19) becomes

$$\delta S = \int_{\mathcal{R}} \left(\left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\right)\right)\delta\Phi\right) d^4x \,. \tag{2.1.21}$$

The principle of least action, or stationary action principle, tells us that the action (2.1.21) must vanish, $\delta S = 0$. It can be fulfilled by the condition

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) = 0 \,. \tag{2.1.22}$$

Equation (2.1.22) is known as the Euler-Lagrange equation for the field Φ. When the Lagrangian density depends on the field differentiations up to the second order, i.e. $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi)$, we can generalize the procedure as shown in (2.1.18) - (2.1.22),

$$\begin{aligned}
\delta S &= \int_{\mathcal{R}} \left(\frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial_\mu (\delta\Phi) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \partial_\mu \partial_\nu (\delta\Phi) \right) d^4x \\
&= \int_{\mathcal{R}} \left(\frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \delta\Phi \right) - \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) \right) \delta\Phi + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \partial_\mu (\delta\Phi) \right) \right. \\
&\quad \left. - \left(\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \right) \right) \partial_\mu (\delta\Phi) \right) d^4x \\
&= \int_{\mathcal{R}} \left(\frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \delta\Phi \right) - \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) \right) \delta\Phi + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \partial_\mu (\delta\Phi) \right) \right. \\
&\quad \left. - \partial_\mu \left(\left(\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \right) \right) \delta\Phi \right) + \left(\partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \right) \right) \delta\Phi \right) d^4x \,.
\end{aligned} \tag{2.1.23}$$

We consider that the field's variation $\delta\Phi$ together with its first derivative with respect to spacetime coordinates $\partial_\mu \delta\Phi$ vanish at the boundary. Hence the vanishing of δS built from the Lagrangian density $\mathcal{L}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi)$ is given by the condition

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \Phi)} \right) = 0 \,. \tag{2.1.24}$$

It is clear that the last equation reduces to (2.1.22) if the Lagrangian density $\mathcal{L}$ does not depend on $\partial_\mu \partial_\nu \phi$.

As an explicit example, consider the action for free massless scalar in curved space

$$S = \int d^4x \sqrt{-g} \partial_\mu \Phi \partial^\mu \Phi \,. \tag{2.1.25}$$

Here g is the determinant of covariant metric tensor $g_{\mu\nu}$. The corresponding Lagrangian density for this action is

$$\mathcal{L} = \sqrt{-g} \partial_\mu \Phi \partial^\mu \Phi \,. \tag{2.1.26}$$

The Euler-Lagrange equation (2.1.22) for this Lagrangian density gives an equation of motion for massless scalar fields in a curved spacetime

$$\partial_\mu \left(\sqrt{-g} \partial^\mu \Phi \right) = \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) = 0 \,. \tag{2.1.27}$$

This massless scalar equation is discussed quite extensively in chapters 4 and 5.

It is interesting to note that one can build an equivalent action by adding a total derivative term, $\partial_\mu Z^\mu$, to the Lagrangian density

$$\mathcal{L}' = \mathcal{L} + \partial_\nu Z^\nu \left(\Phi, \partial_\mu \Phi\right) . \tag{2.1.28}$$

As long as $\delta\Phi$ and $\partial_\mu \delta\Phi$ vanish at the boundary, the divergence theorem again tells us that the extra terms that comes from $\partial_\mu Z^\mu \left(\Phi, \partial_\mu \Phi\right)$ give no contribution to the variation of action. In this sense, we can conclude that the old action $S = \int_{\mathcal{R}} \mathcal{L} d^4x$ and the new one $S' = \int_{\mathcal{R}} \mathcal{L}' d^4x$ are equivalent.

We now can discuss an action that produces the Einstein equations for gravity. To avoid some complications, let us restrict to the vacuum case first, i.e. there is no matter or energy outside of the massive body which curves the spacetime. In this case, the Einstein gravitational equations can be read as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 . \tag{2.1.29}$$

The tensor $R_{\mu\nu}$ is known as Ricci tensor, defined as

$$R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\alpha\beta} - \partial_\beta \Gamma^\rho_{\alpha\rho} + \Gamma^\sigma_{\alpha\beta} \Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\alpha\rho} \Gamma^\rho_{\beta\sigma} , \tag{2.1.30}$$

where $\Gamma^\alpha_{\mu\nu}$ is known as Christoffel symbol of the second kind[3],

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right) . \tag{2.1.31}$$

The Ricci tensor is symmetric under the permutation of its indices, i.e. $R_{\alpha\beta} = R_{\beta\alpha}$. The Christoffel symbol $\Gamma^\alpha_{\mu\nu}$ is symmetric under the permutation of lower indices,

$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} . \tag{2.1.32}$$

The action that produces the vacuum Einstein equations (2.1.29) after using the stationary action principle is known as the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{16\pi} \int R \sqrt{-g} d^4x . \tag{2.1.33}$$

[3]There are two kinds of Christoffel symbol, the first kind and the second kind. Christoffel symbol of the first kind is denoted by $\Gamma_{\alpha\beta\gamma}$, while the second kind has one upper index, $\Gamma^\gamma_{\alpha\beta}$. Both of these Christoffel symbols are related each other by a contraction with the tensor metric, $\Gamma_{\alpha\beta\gamma} = g_{\gamma\rho} \Gamma^\rho_{\alpha\beta}$.

The action (2.1.33) was first proposed by the German mathematician David Hilbert in 1915. The scalar R in Einstein-Hilbert action (2.1.33) is called the Ricci scalar, given by $R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$. The variation of (2.1.33) can be written as

$$\delta S_{EH} = \frac{1}{16\pi}\int \left(\sqrt{-g}\delta R + R\delta\sqrt{-g}\right) d^4x\,. \tag{2.1.34}$$

Now our goal is to factor out the variation of metric tensor, $\delta g^{\mu\nu}$, from the integrand (2.1.34) to get the vacuum Einstein equations (2.1.29) with lower indices.

The variation of Ricci scalar can be examined in the following way,

$$\delta R = \delta\left(g^{\mu\nu}R_{\mu\nu}\right) = g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}\,. \tag{2.1.35}$$

Before we show the expression for $\delta R_{\mu\nu}$, it is useful first to discuss the geodesic coordinate system. In the geodesic coordinate system, one can find a point where all components of Christoffel symbol $\Gamma^{\alpha}_{\mu\nu}$ are vanishing. To do so, let us consider that originally the Christoffel symbol has some non-zero components at a point $\tilde{x}$ in a coordinate system, $\Gamma^{\alpha}_{\mu\nu}(\tilde{x}) \neq 0$. Then we perform the following transformation,

$$x'^{\alpha} = x^{\alpha} - \tilde{x}^{\alpha} + \frac{1}{2}\Gamma^{\alpha}_{\mu\nu}\left(\tilde{x}\right)\left(x^{\mu} - \tilde{x}^{\mu}\right)\left(x^{\nu} - \tilde{x}^{\nu}\right)\,. \tag{2.1.36}$$

Now by using the relation

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$$

where δ^{μ}_{ν} is the Kronecker delta function

$$\delta^{\nu}_{\mu} = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \text{if} \quad \mu \neq \nu \end{cases}\,, \tag{2.1.37}$$

we can verify following results for the transformation (2.1.36),

$$\left.\frac{\partial^2 x'^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}\right|_{x=\tilde{x}} = -\Gamma^{\alpha}_{\mu\nu}\left(\tilde{x}\right)\,. \tag{2.1.38}$$

In the geodesic coordinate system x'^{α}, we can show the transformed Christoffel symbol vanishes at $\tilde{x}$. The transformation of a Christoffel symbol between two different coordinate system, say from x^{μ} to x'^{μ}, is

$$\Gamma'^{\alpha}_{\mu\nu} = \underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\beta}_{\rho\sigma}}_{1st} + \underbrace{\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\frac{\partial^2 x^{\beta}}{\partial x'^{\mu}\partial x'^{\nu}}}_{2nd}\,. \tag{2.1.39}$$

We know that the Christoffel symbol is not a tensor, because it is not transformed in the way a tensor is. The non-tensorial behaviour of Christoffel symbol can be seen from the appearance of the second term in the last equation. If only the second term in (2.1.39) did not appear, then the Christoffel symbol fulfills the requirement to be a tensor. Now we evaluate the transformation of Christoffel symbol at $\tilde{x}$ according to the coordinate transformation (2.1.36),

$$\Gamma'^{\alpha}_{\mu\nu}(\tilde{x}) = \delta^{\alpha}_{\beta}\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\Gamma^{\beta}_{\rho\sigma}(\tilde{x}) - \delta^{\alpha}_{\beta}\Gamma^{\beta}_{\mu\nu}(\tilde{x}) = 0\,. \tag{2.1.40}$$

The result in (2.1.40) can be extended to all points of a curve in the geodesic coordinate system.

We now compute the variation of Ricci tensor. To make it simple, let us perform the calculation in the geodesic coordinate system first, where the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ vanishes, but not its derivative. From the definition of Ricci tensor (2.1.30), one can compute

$$\delta R_{\mu\nu} = \nabla_{\alpha}(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\mu\alpha})\,, \tag{2.1.41}$$

which is known as the Palatini identity. The last expression is a tensor relation, therefore it should be valid in any coordinate system. By using the equation (2.1.41), the term $\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$ in the variation of Ricci scalar (2.1.35) can be written as a total derivative of a vector,

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \partial_{\alpha}V^{\alpha}\,, \tag{2.1.42}$$

where

$$V^{\alpha} = \sqrt{-g}\left(g_{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\alpha\mu}\delta\Gamma^{\tau}_{\mu\tau}\right)\,. \tag{2.1.43}$$

The total derivative $\partial_{\alpha}V^{\alpha}$ gives no contribution to the variation of action. Consequently, we can remove the total derivative term (2.1.42) from the variation of Einstein-Hilbert action. Therefore, the only term that contributes to the variation of Ricci scalar (2.1.35) is $R_{\mu\nu}\delta g^{\mu\nu}$,

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu}\,. \tag{2.1.44}$$

Since g is the determinant of tensor metric $g_{\mu\nu}$, one can show that

$$\frac{\delta g}{\delta g_{\mu\nu}} = g g^{\mu\nu}\,. \tag{2.1.45}$$

Accordingly, by using $g_{\mu\sigma}g^{\sigma\nu} = \delta^{\nu}_{\mu}$, we can write

$$\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}\,. \tag{2.1.46}$$

Hence, the equation (2.1.45) equivalently can be written as

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{2.1.47}$$

Furthermore, we have

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{2.1.48}$$

By combining the results in (2.1.44) and (2.1.47), we can show the variation of Einstein-Hilbert action as

$$\delta S_{EH} = \frac{1}{16\pi}\int\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}d^4x\,. \tag{2.1.49}$$

By looking at the integrand of the last equation, we can see that the vacuum Einstein equations (2.1.29) ensures the Einstein-Hilbert action (2.1.33) is stationary.

2.1.3 Schwarzschild Black Holes

In order to get some insights about black holes in general, first we review some properties of static black holes described by the Schwarzschild metric,

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -\left(1 - \frac{2M}{r}\right)dt^2\left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{2.1.50}$$

Accordingly, the non-zero components of the Schwarzschild metric tensor $g_{\mu\nu}$ are

$$g_{tt} = -1 + \frac{2M}{r}\;,\; g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}\;,\; g_{\theta\theta} = r^2\;,\; g_{\phi\phi} = r^2\sin^2\theta\,. \tag{2.1.51}$$

The Schwarzschild metric becomes singular at two points, $r = 0$ and $r = 2M$. It turns out that the singularity at $r = 2M$ is just the incapability of the coordinate system that is being used to be smooth everywhere except at the origin, i.e. $r = 0$. One can perform some coordinate transformations to remove the singularity at $r = 2M$. However, the singularity at $r = 0$ is special, and there is no coordinate transformation that can sweep away this singularity. The singularity at $r = 2M$ is called the coordinate singularity, due to the fact that it depends on the choice of coordinates. The singularity at $r = 0$ is called the physical or curvature singularity, since it always appear in any coordinate system.

There is analytic way to distinguish the singularity at $r = 0$ and $r = 2M$. The Riemann tensor

$$R^{\delta}_{\alpha\beta\chi} = \partial_{\beta}\Gamma^{\delta}_{\alpha\chi} - \partial_{\chi}\Gamma^{\delta}_{\alpha\beta} + \Gamma^{\varepsilon}_{\alpha\chi}\Gamma^{\delta}_{\varepsilon\beta} - \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\chi}\,, \tag{2.1.52}$$

which is also called as curvature tensor, reflects the curvature of spacetime. According to our tensor notation, the Riemann tensor expressed in (2.1.52) is a $(1, 3)$ type tensor[4]. To have the $(0, 4)$ type Riemann tensor, we just to lower one of its indices by using the covariant[5] metric tensor $g_{\mu\nu}$,

$$R_{\mu\alpha\beta\chi} = g_{\mu\delta}R^{\delta}_{\alpha\beta\chi} = g_{\mu\delta}\left(\partial_{\beta}\Gamma^{\delta}_{\alpha\chi} - \partial_{\chi}\Gamma^{\delta}_{\alpha\beta} + \Gamma^{\varepsilon}_{\alpha\chi}\Gamma^{\delta}_{\varepsilon\beta} - \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\chi}\right)\,. \tag{2.1.53}$$

The vanishing of all components of $R_{\mu\alpha\beta\chi}$ everywhere is the signature of a flat spacetime. If the Riemann tensor of a spacetime contains nonzero components, it means that this spacetime is curved. From the Riemann tensor, we can build some scalar quantities, for example the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}_{\mu\alpha\nu}\,, \tag{2.1.54}$$

and the Kretschmann scalar

$$R^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,. \tag{2.1.55}$$

The reason we look at these scalars is because a scalar quantity is unaffected by any coordinate transformation, while tensors in general change as the coordinates are transformed. We know already that Schwarzschild metric is a solution of the vacuum Einstein field equation $R_{\mu\nu} = 0$, which means the associated Ricci scalar is also zero. Hence, there would be no difference between the Ricci scalars at $r = 0$ and $r = 2M$.

The nonzero components of the Riemann tensor for the Schwarzschild spacetime are

$$\begin{aligned} R_{trtr} = \frac{2M}{r^3}\ ,\ R_{t\theta t\theta} = -\frac{(r-2M)\,M}{r^2}\ ,\ R_{t\phi t\phi} = -\frac{(r-2M)\,M\sin^2\theta}{r^2}\ , \\ R_{r\theta r\theta} = \frac{M}{r-2M}\ ,\ R_{r\phi r\phi} = \frac{M\sin^2\theta}{r-2M}\ ,\ R_{\theta\phi\theta\phi} = -2rM\sin^2\theta\,. \end{aligned} \tag{2.1.56}$$

[4]This tensor has an upper index and three lower ones. In general, a (p, q) type tensor has p upper indices and q lower ones.

[5]A covariant tensor has all of its indices as the lower ones, while a contravariant tensor has all of its indices as the upper ones.

Then by using the contravariant Schwarzschild metric tensor

$$g^{tt} = -\left(1 - \frac{2M}{r}\right)^{-1} \ , \ g^{rr} = \left(1 - \frac{2M}{r}\right) \ , \ g^{\theta\theta} = r^{-2} \ , \ g^{\phi\phi} = r^{-2}\sin^{-2}\theta \, , \tag{2.1.57}$$

one can compute the contravariant Riemann tensor

$$R^{\alpha\beta\chi\delta} = g^{\alpha\kappa}g^{\beta\lambda}g^{\chi\mu}g^{\delta\nu}R_{\kappa\lambda\mu\nu} \, , \tag{2.1.58}$$

as follows

$$\begin{aligned} R^{trtr} &= \frac{2M}{r^3} \ , \ R^{t\theta t\theta} = -\frac{1}{r^4\,(r - 2\,M)} \ , \ R^{t\phi t\phi} = -\frac{M}{r^4\,(r-2M)\sin^2\theta} \, , \\ R^{r\theta r\theta} &= \frac{(r-2M)\,M}{r^6} \ , \ R^{r\phi r\phi} = \frac{(r-2M)\,M}{r^6\sin^2\theta} \ , \ R^{\theta\phi\theta\phi} = -\frac{2M}{r^7\sin^2\theta} \, . \end{aligned} \tag{2.1.59}$$

The Kretschmann scalar for Schwarzschild spacetime can be computed using the formula (2.1.55), which is

$$R^2 = \frac{48M^2}{r^6} \, . \tag{2.1.60}$$

In deriving (2.1.60), we have used the identities

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu} \quad , \quad R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda} \, . \tag{2.1.61}$$

We can see the Kretschmann scalar (2.1.60) is singular for $r = 0$, but remains finite at $r = 2M$. This result supports the statement that for Schwarzschild black holes, $r = 0$ is the physical singularity while $r = 2M$ is just a coordinate singularity.

Therefore, the surface $r = 2M$ which is called the event horizon for Schwarzschild plays an important role in black hole physics. Previously, by using the Newtonian gravity, we have shown light rays can't escape from the black hole's horizon. Now, in the framework of Eintein's general relativity, this fact is supported by studying the behavior of light cones in the Schwarzschild geometry.

Using a light cone as depicted in figure 2.3, we can distinguish the timelike (i), null or lightlike (ii), and the spacelike (iii) trajectories. A particle has a timelike path if it moves slower than light. Light propagates in the null or lightlike path, and an object that moves faster than light has a spacelike path which lies outside of the light cone. Definitely the latter case has no physical significance in our present discussion, i.e. we have not observed any object that moves with speed faster than that of light in the laboratory.

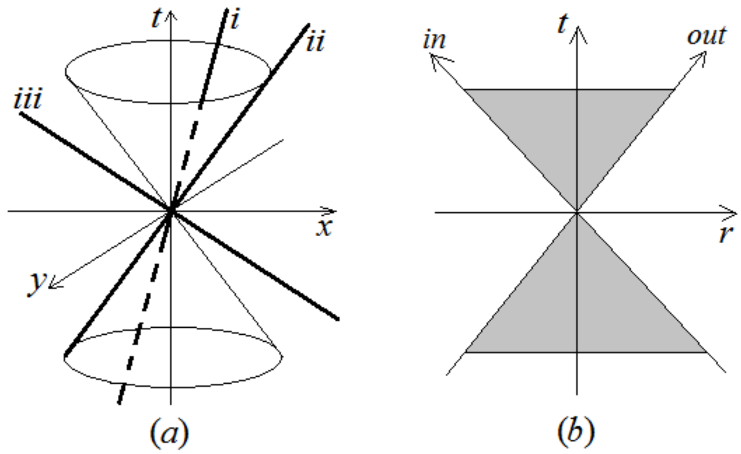

Figure 2.3: (a) Light cone diagram in (2+1) dimensions: (i) inside of the light cone, (ii) on the light cone, (iii) outside of the light cone. (b) "out" and "in" null paths represent the outgoing and ingoing photon in (1+1) dimensions light cone diagram.

In Schwarzschild spacetime, in the case of light rays are moving radially we can set

$$ds^2 = d\theta = d\phi = 0\,. \tag{2.1.62}$$

Hence the Schwarzschild metric (2.1.50) becomes

$$\pm dt = \left(\frac{r}{r-2M}\right) dr\,. \tag{2.1.63}$$

The equation (2.1.63) with (+) sign refers to the outgoing beam of light moving away radially from the black hole, while the other one with (−) sign refers to the ingoing beam of light towards the black hole. For the equation with (+) sign, the solution for equation (2.1.63) is

$$t = r + 2M \ln|r - 2M| + \text{constant}\,. \tag{2.1.64}$$

Originally, the solution of (2.1.63) does not contain the absolute sign | | as what appears in the last expression. Without this absolute sign, we understand that this solution applies only to the region outside the horizon $r = 2M$, otherwise we would get a complex value for t which is not physical. The role of the absolute sign in (2.1.64) is to extend the validity of solution to the region $r < 2M$. Next for the equation (2.1.63) with minus sign in front of dt, one can find the solution to be

$$-t = r + 2M \ln|r - 2M| + \text{constant}\,. \tag{2.1.65}$$

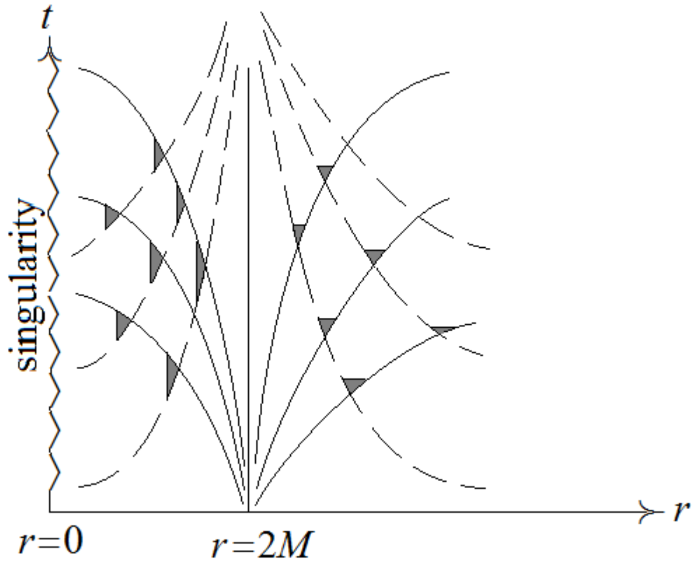

Figure 2.4: The structure of lightcones in Schwarzschild spacetime. The dashed lines represent ingoing photon and the solid lines represent the outgoing one. The shaded triangles represent the two dimensional light cones.

The curves given by these two functions, (2.1.64) and (2.1.65), represent the outgoing and ingoing photon respectively, in the sense of "out" and "in" paths in figure 2.3 (b).

Figure 2.4 tells us that the curves representing the ingoing and outgoing photons are asymptotic at $r = 2M$, which can be understood from the beginning since the coordinate system that we choose in the Schwarzschild metric (2.1.50) fails to be smooth at $r = 2M$. However, applying the Eddington-Finkelstein coordinates transformation

$$t = v - r - 2M \ln \left| \frac{r}{2M} - 1 \right| \,, \tag{2.1.66}$$

to the metric (2.1.50) gives us a new metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{2.1.67}$$

which is known as the Eddington-Finklestein metric. In the transformation (2.1.66), v is called as the advanced time parameter. The Eddington-Finklestein metric (2.1.67) is free from $r = 2M$ pathology, i.e. it is not singular at $r = 2M$. Furthermore, the significance of $r = 2M$ surface can be studied in more details by using the equation (2.1.67). We observe

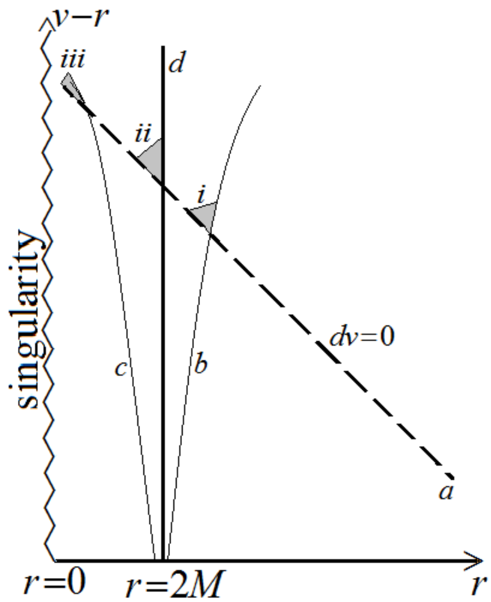

Figure 2.5: Lightcones in Eddington-Finklestein spacetime.

that, even though the Eddington-Finklestein metric (2.1.67) does not diverge at $r = 2M$, we still find that light rays are trapped once they enter the $r = 2M$ surface.

Figure 2.5 is the diagram for the light cones in Eddington-Finkelstein coordinates, where the diagram tells us the behavior of light rays moving in the radial motion,

$$ds^2 = d\theta = d\phi = 0\,. \tag{2.1.68}$$

After employing (2.1.68), the metric (2.1.67) becomes an equation

$$2dvdr = (1 - \frac{2M}{r})dv^2\,, \tag{2.1.69}$$

which has two general solutions. The first one is the constant v, which yields $dv = 0$. When v is constant, equation (2.1.66) tells us that r decreases as t increases. Therefore, the straight dash line (a) in figure 2.5 which represents the solution describe the ingoing light rays.

The second general solution for the equation (2.1.69) is when $dv \neq 0$, where we have

$$dv = \frac{2r}{r - 2M}dr\,. \tag{2.1.70}$$

Upon integration in (2.1.70), we obtain

$$v = \text{constant} + 2\left(r + 2M\ln\left|\frac{r}{2M} - 1\right|\right)\,, \tag{2.1.71}$$

which solves (2.1.69). The solution (2.1.71) diverges at $r = 2M$, which is interpreted as the situation where light will stay forever on the horizon. The lightcone (ii) in figure 2.5 describes the outgoing null path of light rays which coincides with the the curve $r = 2M$. In this sense, sometime horizon is viewed as a null surface, i.e. a surface where the radial light rays neither can escape to infinity or fall into the physical singularity. For the inside and outside of horizon regions, the solution (2.1.71) describes the ingoing and outgoing light rays respectively as illustrated by the curved lines (c) and (b) in figure 2.5.

2.2 Black Hole in Kerr Spacetime and Its Thermodynamics

2.2.1 Kerr Solution

In this subsection, we show how to obtain the Kerr solution starting from a general form of axially symmetric metric. We have seen that the non-vanishing components for the Minkoswki and Schwarzschild metric tensors are the diagonal entries only. In fact, the term

$$r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2.72}$$

that appears in Schwarzschild metric (2.1.50) as well as in flat spacetime (2.1.5) shows that these spacetimes are spherically symmetric[6]. In turns out that the metric for Kerr spacetime in the Boyer-Lindquist coordinates[7] does not contain (2.2.72), which indicates that the Kerr spacetime is no longer spherical symmetric.

Attempts to obtain a solution of the vacuum Einstein equations (2.1.29) which describes an empty spacetime outside a spinning massive object had been started since the discovery of Schwarzschild solution in 1916. In fact, it was Kerr in 1963 [48] who first derived an asymptotically flat solution of vacuum Einstein field equation outside of a spinning massive object. It is such a quite straightforward idea to generalize the static spacetime solution to the spinning one. It took about fifty years to achieve an acceptable solution to describe

[6]All spherical symmetric spacetime solutions in four dimension must have this term [8].

[7]Kerr metric in Boyer-Lindquist coordinates reduces to the Schwarzschild metric (2.1.50) after turning off the rotational parameter.

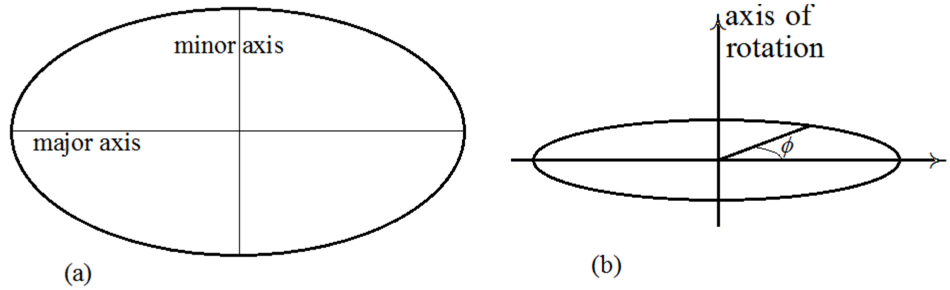

Figure 2.6: (a) oblate spheroid, (b) rotation in ϕ direction.

the empty spacetime outside of a spinning massive body. The problem was that the solution must be asymptotically flat, i.e. the spacetime must be flat at infinity. This constraint comes from the physical necessity that an observer should not feel gravity when he is very far from a gravitational source. In this section, we show in details how to obtain the Kerr solution.

Let us start with a general metric proposed by Lewis [49],

$$ds^2 = -Vdt^2 + 2Wdtd\phi + ((e^Y dx^1)^2 + e^Z (dx^2)^2) + Xd\phi^2 \,. \tag{2.2.73}$$

The functions V, W, X, Y, and Z are in general x_1 and x_2 dependent. Unlike the Schwarzschild spacetime which enjoys the spherical symmetry and contain no cross terms[8] in their metric, we observe a cross term in the Lewis metric (2.2.73), i.e. $2Wdtd\phi$. The cross term appears as the off-diagonal components of the metric tensor, i.e. $g_{t\phi} = g_{\phi t} = W$. The existence of this off-diagonal component signs the lost of spherical symmetry in the spacetime (2.2.73). The metric (2.2.73) was first proposed by Lewis in his effort to find a more general solution of vacuum Einstein field equation which describes an axial symmetric and time independent spacetime. An axial symmetry is expected to be possessed by a spacetime outside of a spinning object. In fact, from a simple mechanical picture, we understand that a spinning object will evolve to have an oblate spheroid configuration provided by its rotation about the minor axis as depicted in 2.6 (a).

Since the rotation is denoted by the rate of ϕ angle as depicted in figure 2.6 (b), the stationary and axially symmetric properties of the metric (2.2.73) can be seen by its t and ϕ

[8]The cross term we are mentioning here is $f(x)dx^\mu dx^\nu$ where $\mu \neq \nu$ and $f(x)$ is the corresponding metric component.

independence. As a matrix, the metric tensor for the metric (2.2.73) can be rewritten as

$$g_{\mu\nu} = \begin{pmatrix} -V & 0 & 0 & W \\ 0 & e^{Y} & 0 & 0 \\ 0 & 0 & e^{Z} & 0 \\ W & 0 & 0 & X \end{pmatrix} . \tag{2.2.74}$$

It follows from (2.2.74) that the contravariant version of (2.2.74) can be read as

$$g^{\mu\nu} = \begin{pmatrix} -\frac{X}{VX+W^2} & 0 & 0 & \frac{W}{VX+W^2} \\ 0 & e^{-Y} & 0 & 0 \\ 0 & 0 & e^{-Z} & 0 \\ \frac{W}{VX+W^2} & 0 & 0 & \frac{V}{VX+W^2} \end{pmatrix} . \tag{2.2.75}$$

We can verify that the covariant metric (2.2.74) and its contravariant version (2.2.75) obey

$$g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}_{\mu} \quad \text{and} \quad g_{\mu\nu}g^{\mu\nu} = 4\,. \tag{2.2.76}$$

Since we are hunting the solution for an empty spacetime outside of a massive rotating body, the explicit expression of each functions $V, W, X, Y,$ and Z in the metric (2.2.74) is dictated by the vacuum Einstein equations (2.1.29). By performing a little bit algebra, the vacuum Einstein equations (2.1.29) can be written as

$$R_{\mu\nu} = 0\,. \tag{2.2.77}$$

Therefore, the functions $V, W, X, Y,$ and Z can be obtained by solving the equation (2.2.77). In fact, this task is not easy to be performed. After computing all components of the Ricci tensor for the metric (2.2.73), one realizes finding these functions by solving (2.2.77) is a complicated task[9]. However, some tricks can be performed to reduce the complexities.

Previously, we have discussed the Einstein-Hilbert action where the Lagrangian density is

$$\mathcal{L} = \sqrt{-g}R\,. \tag{2.2.78}$$

In detail, this Lagrangian density can be written as

$$\mathcal{L} = \sqrt{-g}g^{\mu\nu}\left(\partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right) . \tag{2.2.79}$$

[9]We provide the components of Ricci tensor for the metric (2.2.73) in appendix A.

Interestingly, the Lagrangian density

$$\mathcal{L}' = \frac{1}{2}\left(\Gamma^{\alpha}_{\beta\alpha}\partial_{\gamma}\left(\sqrt{-g}g^{\beta\gamma}\right) - \Gamma^{\alpha}_{\beta\gamma}\partial_{\alpha}\left(\sqrt{-g}g^{\beta\gamma}\right)\right) \tag{2.2.80}$$

is equivalent to (2.2.79), in the sense that both (2.2.79) and (2.2.80) give the same equations of motion after employing the principle of least action. This is due to the fact that the two Lagrangian densities (2.2.80) and (2.2.79) differ by a divergence term,

$$\mathcal{L} = \mathcal{L}' - \partial_{\alpha}\left(\sqrt{-g}\left(g^{\alpha\mu}\Gamma^{\chi}_{\mu\chi} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}\right)\right) . \tag{2.2.81}$$

Therefore, both (2.2.79) and (2.2.80) produce the same vacuum Einstein equations after using the Euler-Lagrange equations.

For the metric (2.2.73), the Lagrangian density (2.2.80) can be written as

$$\mathcal{L}' = \frac{e^{-(Y-Z)/2}}{2\chi}\left(\partial_1 V\partial_1 X + (\partial_1 W)^2 + 2\chi\partial_1\chi\partial_1 Z + \partial_2 V\partial_2 X + (\partial_2 W)^2 + 2\chi\partial_2\chi\partial_2 Y\right) . \tag{2.2.82}$$

Here $\chi^2 = VX + W^2$ and we use the notation $\partial_k \equiv \partial/\partial x^k$ for $k = 1, 2$. The determinant of metric tensor (2.2.74) is $g = -\chi^2 e^{Y+Z}$. The Lagrangian density $\mathcal{L}'$ in (2.2.82) gives the Euler-Lagrange equations for $g_{\mu\nu}$

$$\frac{\partial\mathcal{L}'}{\partial g_{\alpha\beta}} - \partial_{\gamma}\frac{\partial\mathcal{L}'}{\partial\left(\partial_{\gamma}g_{\alpha\beta}\right)} = 0\,, \tag{2.2.83}$$

as

$$2\partial_1^2\chi + (\partial_1\chi\partial_1 Y - \partial_2\chi\partial_2 Z) + \frac{1}{2}\left(\left(\partial_1 V\partial_1 X + (\partial_1 W)^2\right) - \left(\partial_2 V\partial_2 X + (\partial_2 W)^2\right)\right) = 0\,, \tag{2.2.84}$$

$$2\partial_2^2\chi - (\partial_1\chi\partial_1 Y - \partial_2\chi\partial_2 Z) - \frac{1}{2}\left(\left(\partial_1 V\partial_1 X + (\partial_1 W)^2\right) - \left(\partial_2 V\partial_2 X + (\partial_2 W)^2\right)\right) = 0\,, \tag{2.2.85}$$

$$\partial_1\left(\frac{\partial_1 V}{\chi}\right) + \partial_2\left(\frac{\partial_2 V}{\chi}\right) + \frac{V}{2\chi}F = 0\,, \tag{2.2.86}$$

$$\partial_1\left(\frac{\partial_1 X}{\chi}\right) + \partial_2\left(\frac{\partial_2 X}{\chi}\right) + \frac{X}{2\chi}F = 0\,, \tag{2.2.87}$$

and

$$\partial_1\left(\frac{\partial_1 W}{\chi}\right) + \partial_2\left(\frac{\partial_2 W}{\chi}\right) + \frac{W}{2\chi}F = 0\,, \tag{2.2.88}$$

where

$$F \equiv \frac{(\partial_1 V)(\partial_1 X) + (\partial_2 V)(\partial_2 X) + (\partial_1 W)^2 + (\partial_2 W)^2}{\chi^2} + \tilde{\nabla}^2(Y+Z) . \tag{2.2.89}$$

The equations (2.2.84) and (2.2.85) are obtained from (2.2.83) for $g_{22} = e^Z$ and $g_{11} = e^Y$ respectively. The operator $\tilde{\nabla}^2$ is the two dimensional Laplace operator, $\tilde{\nabla}^2 \equiv \partial_1^2 + \partial_2^2$. Adding (2.2.84) and (2.2.85) shows that the function χ must satisfy the two dimensional Laplace equation,

$$\tilde{\nabla}^2 \chi = 0 \,. \tag{2.2.90}$$

Furthermore, equation (2.2.90) says that χ must be a harmonic function of x_1 and x_2. We now assign x^2 to be the axis of rotation z as depicted in Figure 2.6 (b), and x^1 is the function χ itself.

Subsequently we multiply equation (2.2.86) by X, equation (2.2.87) by V, and equation (2.2.88) by $2W$, then add all of these three equations to obtain

$$\partial_\chi \left(\frac{1}{\chi} \partial_\chi \chi^2 \right) + \partial_z \left(\frac{1}{\chi} \partial_z \chi^2 \right) = \chi F \,. \tag{2.2.91}$$

Since the left hand side of (2.2.91) is zero, the last equation can be rewritten as

$$-2\chi^2 \tilde{\nabla}^2 Y = \partial_\chi V \partial_\chi X + \left(\partial_\chi W\right)^2 + \partial_z V \partial_z X + \left(\partial_z W\right)^2 \,, \tag{2.2.92}$$

where we have considered $Z = Y$. By using the last equation, (2.2.86) and (2.2.88) can be simplified to

$$\partial_\chi^2 V + \partial_z^2 V + \chi^{-1} \partial_\chi V = -\chi^{-2} V \left(\partial_\chi V \partial_\chi X + \left(\partial_\chi W\right)^2 + \partial_Z V \partial_z X + \left(\partial_z W\right)^2 \right) \,, \tag{2.2.93}$$

and

$$\partial_\chi^2 W + \partial_z^2 W + \chi^{-1} \partial_\chi W = -\chi^{-2} W \left(\partial_\chi V \partial_\chi X + \left(\partial_\chi W\right)^2 + \partial_Z V \partial_z X + \left(\partial_z W\right)^2 \right) \,. \tag{2.2.94}$$

We now can use a trick to reduce the number of functions that should be handled. We have introduced χ which is defined as

$$\chi = \sqrt{VX + W^2} \tag{2.2.95}$$

which can be considered as a constraint equation for V, X, and W. Therefore, only two out of three functions V, X, and W that are really "free". Accordingly, we could redefine V, X, and W as

$$V = f \quad , \quad W = f\omega \,, \quad \text{and} \quad X = f^{-1}\chi^2 - f\omega^2 \,. \tag{2.2.96}$$

which satisfy (2.2.95). We now see, instead of dealing with three functions X, V, and W, we are left with two functions f and ω which are really "free".

Plugging the equations in (2.2.96) into the metric (2.2.73), we find

$$ds^2 = f\,(dt - \omega d\phi)^2 - f^{-1}e^{2U}\left(d\chi^2 + dz^2\right) + f^{-1}\chi^2 d\phi^2\,, \tag{2.2.97}$$

where $Y = -\ln f + 2U$. From (2.2.93) and (2.2.94) we can have

$$f\left(\partial_\chi^2 f + \partial_z^2 f + \chi^{-1}\partial_\chi f\right) = \left((\partial_\chi f)^2 + (\partial_z f)^2\right) + f^4\chi^{-2}\left((\partial_\chi \omega)^2 + (\partial_z \omega)^2\right)\,, \tag{2.2.98}$$

and

$$f^2\left(\partial_\chi^2 \omega + \partial_z^2 \omega - \chi^{-1}\partial_\chi f\right) + 2f\left(\partial_\chi f \partial_\chi \omega + \partial_z f \partial_z \omega\right) = 0\,. \tag{2.2.99}$$

Interestingly, the last two equations can be rewritten in terms of ∇ operator in the flat three dimensional cylindrical coordinates, where

$$\nabla = \hat{e}_\chi \partial_\chi + \hat{e}_z \partial_z + \hat{e}_\phi \chi^{-1}\partial_\phi\,, \tag{2.2.100}$$

$$\nabla^2 = \partial_\chi^2 + \partial_z^2 + \chi^{-1}\partial_\chi + \chi^{-2}\partial_\phi^2\,, \tag{2.2.101}$$

$$\nabla \cdot \vec{A} = \chi^{-1}\partial_\chi\left(\chi A_\chi\right) + \partial_z A_z\,, \tag{2.2.102}$$

for $\vec{A} = \hat{e}_\chi A_\chi + \hat{e}_z A_z$.

We can rewrite (2.2.98) and (2.2.99) as

$$f\nabla^2 f - \vec{\nabla} f \cdot \nabla f - \chi^{-2} f^4 \nabla\omega \cdot \nabla\omega = 0\,, \tag{2.2.103}$$

and

$$\nabla \cdot \left(\chi^{-2} f^2 \nabla \omega\right) = 0\,. \tag{2.2.104}$$

Following Ernts [50], the equations (2.2.103) and (2.2.104) can be derived by using the Euler-Lagrange equation with the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\chi f^{-2}\nabla f \cdot \nabla f - \chi^{-1} f^2 \nabla\omega \cdot \nabla\omega\right)\,. \tag{2.2.105}$$

Performing the Euler-Lagrange equation for fields f and ω from the Lagrangian density (2.2.105) gives the equations (2.2.103) and (2.2.104) respectively. In this sense, one can say that the equation (2.2.105) is an effective Lagrangian density to get the equations (2.2.103)

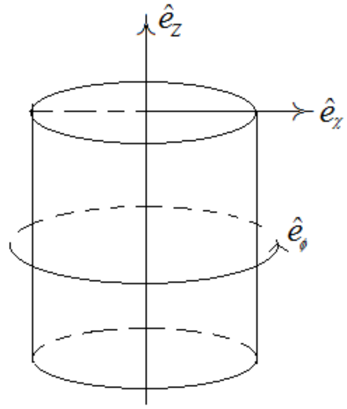

Figure 2.7: The illustration of the unit vectors in cylindrical coordinates related to the operator 2.2.100.

and (2.2.104). Unfortunately, from the Lagrangian density (2.2.105), we are unable to get any information about the dynamics of U with respect to χ and z. However, using equations (2.2.84) and (2.2.85) we can get

$$\begin{aligned} 2\partial_\chi U - f^{-1}\partial_\chi f &= (4\chi)^{-1}\left[f^{-2}\chi^2\left((\partial_\chi f)^2 - (\partial_z f)^2\right) - 2\omega\left(\partial_\chi\omega\partial_\chi f - \partial_z\omega\partial_z f\right)\right. \\ &\quad \left. -\left(f\partial_\chi\omega + \omega\partial_\chi f\right)^2 + \left(f\partial_z\omega + \omega\partial_z f\right)^2\right], \end{aligned} \tag{2.2.106}$$

and

$$\begin{aligned} 2\partial_z U - f^{-1}\partial_z f &= (2\chi)^{-1}\left[f^{-2}\chi^2\partial_\chi f\partial_z f + \omega\left(\partial_z\omega\partial_\chi f - \partial_\chi\omega\partial_z f\right)\right. \\ &\quad \left. -\left(f\partial_\chi\omega + \omega\partial_\chi f\right)\left(f\partial_z\omega + \omega\partial_z f\right)\right]. \end{aligned} \tag{2.2.107}$$

which show how U changes with respect to χ and z coordinates.

Equation (2.2.104) shows the existence of a vector $\vec{A}$ which obeys

$$\chi^{-2}f^2\nabla\omega = \nabla\times\vec{A}. \tag{2.2.108}$$

Since ω is not a function of ϕ, then the vector $\nabla\omega$ will be orthogonal to the unit vector $\hat{e}_\phi$ as illustrated in figure 2.7. Hence, from equation (2.2.108) we can get

$$(\nabla\times\vec{A})\cdot\hat{e}_\phi = 0. \tag{2.2.109}$$

We shall now find that

$$\nabla\times\vec{A} = \chi^{-1}\left(\partial_\chi A_z - \partial_z\left(\chi A_z\right)\right)\hat{e}_\chi + \chi^{-1}\left(\partial_\chi\left(\chi A_\phi\right) - \partial_\phi A_\chi\right)\hat{e}_z + \left(\partial_\chi A_z - \partial_z A_\chi\right)\hat{e}_\phi. \tag{2.2.110}$$

Also it follows from (2.2.109) that we must have

$$\partial_\chi A_z = \partial_z A_\chi \,. \tag{2.2.111}$$

Consequently, in term of a function $\mathcal{F}$, we can write the component of $\vec{A}$ as

$$A_\chi = \partial_\chi \mathcal{F} \;,\; A_z = \partial_z \mathcal{F} \,. \tag{2.2.112}$$

Now we introduce the twist potential

$$\Phi = \partial_\phi F - \chi A_\phi \,. \tag{2.2.113}$$

Hence, the curl of $\vec{A}$ in (2.2.110) can be rewritten as

$$\nabla \times \vec{A} = \chi^{-1} \left(\hat{e}_\chi \partial_z \Phi - \hat{e}_z \partial_\chi \Phi \right) \,. \tag{2.2.114}$$

Furthermore, the last equation finally can be read

$$\nabla \times \vec{A} = \chi^{-1} \hat{e}_\phi \times \nabla \Phi \,. \tag{2.2.115}$$

In [50], Ernst considered that the function Φ is ϕ independent, i.e.

$$\partial_\phi \Phi = 0 \,. \tag{2.2.116}$$

To guarantee that Φ does not depend on ϕ, the $\mathcal{F}$ function must satisfy

$$\partial_\phi^2 \mathcal{F} = \chi \partial_\phi A_\phi \,. \tag{2.2.117}$$

In addition, using the vector relation

$$\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \left(\vec{A} \cdot \vec{C} \right) \vec{B} - \left(\vec{A} \cdot \vec{B} \right) \vec{C} \tag{2.2.118}$$

one can verify that

$$\hat{e}_\phi \times \left(\hat{e}_\phi \times \nabla \Phi \right) = -\nabla \Phi \,. \tag{2.2.119}$$

Plugging the result (2.2.115) to (2.2.108) yields

$$\chi^{-1} f^2 \nabla \omega = \hat{e}_\phi \times \nabla \Phi \,, \tag{2.2.120}$$

from which it follows that

$$-\chi^{-1}\hat{e}_\phi \times \nabla\omega = f^{-2}\nabla\Phi\,. \tag{2.2.121}$$

We have used (2.2.119) to get the last equation. In fact, from the relation (2.2.115) we can write

$$\nabla \cdot (\chi^{-1}\hat{e}_\phi \times \nabla\Phi) = 0 \tag{2.2.122}$$

which can be seen easily from an identity in vector calculus, $\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$. Furthermore the last equation also implies

$$\nabla \cdot (f^{-2}\nabla\Phi) = 0\,. \tag{2.2.123}$$

The last equation gives some benefits later in deriving the Ernst equation, from which the solution for the vacuum Einstein equations (2.1.29) can be obtained.

We now arrive at an important step. Following Ernst [50], we introduce a complex potential $\mathcal{E}$ in term of f and the twist potential Φ,

$$\mathcal{E} = f + i\Phi\,. \tag{2.2.124}$$

Recall that f is just the metric component g_{tt} in (2.2.97). Using this complex potential (2.2.124), we can verify that the equations (2.2.103) and (2.2.104) are equivalent to

$$\mathrm{Re}\,(\mathcal{E})\,\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E}\,. \tag{2.2.125}$$

Proving the last equation is quite simple. First we need to recall that $\nabla\omega$ is orthogonal to the unit vector in azimuth direction $\hat{e}_\phi$. Consequently, by using the vector product identity $\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{A}\times\vec{B}\right) = \left(\vec{A}\cdot\vec{A}\right)\left(\vec{B}\cdot\vec{B}\right) - \left(\vec{A}\cdot\vec{B}\right)^2$, we can have

$$f\nabla^2 f = \nabla f\cdot\nabla f - \nabla\Phi\cdot\nabla\Phi\,. \tag{2.2.126}$$

We have plugged $\nabla\omega$ in (2.2.120) to (2.2.103) to get the last equation. Therefore, by using the definition of complex potential (2.2.124), we can rewrite (2.2.126) as

$$f\nabla^2 f + 2i\nabla f\cdot\nabla\Phi = \nabla\mathcal{E}\cdot\nabla\mathcal{E}\,. \tag{2.2.127}$$

From the equation (2.2.123), we can show that

$$2\nabla f.\nabla\Phi = f\nabla^2\Phi\,. \tag{2.2.128}$$

By plugging the last formula into (2.2.127), we finally recover (2.2.125).

We may rewrite a new complex potential $\mathcal{E}$ as [50]

$$\mathcal{E} = \frac{\xi - 1}{\xi + 1}\,. \tag{2.2.129}$$

Related to (2.2.129), the following identities

$$\mathrm{Re}\left(\frac{\xi - 1}{\xi + 1}\right) = \frac{\xi\xi^* - 1}{(\xi + 1)(\xi^* + 1)}\,, \tag{2.2.130}$$

$$\nabla\left(\frac{\xi - 1}{\xi + 1}\right)\cdot\nabla\left(\frac{\xi - 1}{\xi + 1}\right) = \frac{4}{(\xi + 1)^4}\left(\nabla\xi\cdot\nabla\xi\right)\,, \tag{2.2.131}$$

and

$$\nabla^2\left(\frac{\xi - 1}{\xi + 1}\right) = \frac{2}{(\xi + 1)^2}\left(\nabla^2\xi - \frac{2}{(\xi + 1)}\nabla\xi\cdot\nabla\xi\right)\,. \tag{2.2.132}$$

can be obtained, and are found to be useful in writing the Ernst equation later. Here we need to recall that ξ is also independent of ϕ. By using the last three equations, it is easy to show that the equation (2.2.125) can be rewritten in terms of ξ as

$$\left(\xi\xi^* - 1\right)\nabla^2\xi = 2\xi\nabla\xi\cdot\nabla\xi\,, \tag{2.2.133}$$

which is known as the Ernst equation. This equation is quite important in general relativity. From this equation we can derive some solutions of the vacuum Einstein equations [45]. In terms of ξ, we can show several equations related to the functions in the metric (2.2.97)

$$f = \mathrm{Re}\left(\frac{\xi - 1}{\xi + 1}\right)\,, \tag{2.2.134}$$

$$\nabla\omega = \frac{2\chi}{(\xi\xi^* - 1)^2}\,\mathrm{Im}\left((\xi^* + 1)^2\,\hat{e}_\phi\times\nabla\xi\right)\,, \tag{2.2.135}$$

$$\frac{\partial U}{\partial\chi} = \frac{\chi}{(\xi\xi^* - 1)^2}\left(\frac{\partial\xi}{\partial\chi}\frac{\partial\xi^*}{\partial\chi} - \frac{\partial\xi}{\partial z}\frac{\partial\xi^*}{\partial z}\right)\,, \tag{2.2.136}$$

and

$$\frac{\partial U}{\partial z} = \frac{2\chi}{(\xi\xi^* - 1)^2}\,\mathrm{Re}\left(\frac{\partial\xi}{\partial\chi}\frac{\partial\xi^*}{\partial z}\right)\,. \tag{2.2.137}$$

It is clear that the relation (2.2.129) gives us (2.2.134), and the equation (2.2.135) is related to the result in (2.2.120). The last two equations, (2.2.136) and (2.2.137), are obtained from (2.2.106) and (2.2.107).

For the latter benefits, it is useful to introduce the prolate spheroidal coordinates $\{x, y\}$ where the transformation can be read as

$$\chi = k\sqrt{(x^2-1)(1-y^2)}\,, \tag{2.2.138}$$

and

$$z = kxy\,, \tag{2.2.139}$$

where $|y| < 1 < |x|$ and k is a constant scale factor. The variables x and y in terms of k, z, and χ obtained from (2.2.138) and (2.2.139) are

$$x = \frac{1}{2k}\left(\sqrt{(z+k)^2+\chi^2} + \sqrt{(z-k)^2+\chi^2}\right)\,, \tag{2.2.140}$$

and

$$y = \frac{1}{2k}\left(\sqrt{(z+k)^2+\chi^2} - \sqrt{(z-k)^2+\chi^2}\right)\,. \tag{2.2.141}$$

In the x and y coordinates, the Ernst equation (2.2.133) can be read as

$$(\xi\xi^*-1)\left[\partial_x\left(\left(x^2-1\right)\partial_x\xi\right)+\partial_y\left(\left(1-y^2\right)\partial_y\xi\right)\right] = 2\xi^*\left(\left(x^2-1\right)(\partial_x\xi)^2+\left(1-y^2\right)(\partial_y\xi)^2\right)\,. \tag{2.2.142}$$

The next step is solving the equation (2.2.142) for the complex scalar ξ. Once we get the solution for ξ, the next step is finding the functions f, ω, and U in terms of this complex scalar. Working out the functions f and ω is quite straightforward, but dealing with U is quite delicate since this function is dictated by the two coupled equations (2.2.106) and (2.2.107). Here we get the benefit of working out the Ernst equation in the prolate spheroid coordinate. In this coordinate, equations (2.2.106) and (2.2.107) can be respectively rewritten as

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{(1-y^2)}{\left(|\xi|^2-1\right)(x^2-y^2)}\left[(x-1)^2\left(x\left|\frac{\partial\xi}{\partial x}\right|^2 - y\left(\frac{\partial\xi}{\partial x}\frac{\partial\xi^*}{\partial y}+\frac{\partial\xi^*}{\partial x}\frac{\partial\xi}{\partial y}\right)\right)\right.\\ &\quad\left. -x\left(1-y^2\right)\left|\frac{\partial\xi}{\partial y}\right|^2\right]\,,\end{aligned} \tag{2.2.143}$$

and

$$\begin{aligned}\frac{\partial U}{\partial y} &= \frac{(x^2-1)}{\left(|\xi|^2-1\right)(x^2-y^2)}\left[\left(x^2-1\right)y\left|\frac{\partial\xi}{\partial x}\right|^2\right.\\ &\quad\left. -\left(1-y^2\right)\left(y\left|\frac{\partial\xi}{\partial y}\right|^2 + x\left(\frac{\partial\xi}{\partial x}\frac{\partial\xi^*}{\partial y}+\frac{\partial\xi^*}{\partial x}\frac{\partial\xi}{\partial y}\right)\right)\right]\,.\end{aligned} \tag{2.2.144}$$

Later we find out that the real part of ξ is coupled linearly to x while the imaginary part is coupled linearly to y. Therefore, we can restrict the case where $\frac{\partial \xi}{\partial x}$ is completely real and $\frac{\partial \xi}{\partial x}$ is completely imaginary. Now, in general we can write the complex potential ξ as

$$\xi = \frac{u + iv}{m + in} \,. \tag{2.2.145}$$

Hence the equations (2.2.143) and (2.2.144) become

$$\frac{\partial U}{\partial x} = \frac{x\left(1-y^2\right)}{A^2\left(x^2-y^2\right)}\left[\left(x^2-1\right)\left(\frac{\partial u}{\partial x}m - \frac{\partial v}{\partial x}n - y\frac{\partial m}{\partial x} + v\frac{\partial n}{\partial x}\right)^2 - \left(1-y^2\right)\left(\frac{\partial u}{\partial y}n - \frac{\partial v}{\partial y}m - u\frac{\partial n}{\partial y} - v\frac{\partial m}{\partial y}\right)^2\right] , \tag{2.2.146}$$

and

$$\frac{\partial U}{\partial y} = \frac{y\left(x^2-1\right)}{A^2\left(x^2-y^2\right)}\left[\left(x^2-1\right)\left(\frac{\partial u}{\partial x}m - \frac{\partial v}{\partial x}n - u\frac{\partial m}{\partial x} + v\frac{\partial n}{\partial x}\right)^2 - \left(1-y^2\right)\left(\frac{\partial u}{\partial y}n + \frac{\partial v}{\partial y}m - u\frac{\partial n}{\partial y} - v\frac{\partial m}{\partial y}\right)^2\right] , \tag{2.2.147}$$

where $A = u^2 + v^2 - m^2 - n^2$. The solution for Z after integration of the last two equations are

$$U = \frac{1}{2}\ln\left(\frac{CA}{\left(x^2-y^2\right)^{\alpha}}\right) . \tag{2.2.148}$$

Here, C is an integration constant, and the boundary condition $e^{2U} \to 1$ as $x \to \infty$ determines both C and α in equation (2.2.148).

Following Ernst, if the new complex scalar ξ is expressed in a linear combination

$$\xi = px - iqy \,, \tag{2.2.149}$$

the equation (2.2.142) can be solved exactly when

$$p^2 + q^2 = 1 \,. \tag{2.2.150}$$

By using this solution for ξ, we have the explicit expression of each functions in the metric (2.2.97)

$$f = \frac{p^2x^2 + q^2y^2 - 1}{q^2y^2 + \left(px+1\right)^2} \,, \tag{2.2.151}$$

$$\omega = -\frac{2q\left(1-y^2\right)\left(px+1\right)}{q^2y^2 + p^2x^2 - 1} \,, \tag{2.2.152}$$

and

$$U=\frac{1}{2}\ln\left(\frac{p^2x^2+q^2y^2-1}{(x^2-y^2)\,p^2}\right)\,. \tag{2.2.153}$$

Finally, using the solutions (2.2.151), (2.2.152), and (2.2.153), as well as the transformations (2.2.140) and (2.2.141), the reading of metric (2.2.97) becomes

$$\begin{aligned} ds^2 &= k^2\left(-\frac{p^2x^2+q^2y^2-1}{(px+1)^2+q^2y^2}\left(dt-\frac{2q\,(1-y^2)\,(px+1)}{p^2x^2+q^2y^2-1}d\phi\right)^2\right.\\ &\left.+\left((px+1)^2+q^2y^2\right)\left(\frac{dx^2}{p^2\,(x^2-1)}+\frac{dy^2}{p^2\,(1-y^2)}+\frac{(x^2-1)\,(1-y^2)\,d\phi^2}{p^2x^2+q^2y^2-1}\right)\right) \end{aligned} \tag{2.2.154}$$

The metric (2.2.154) is not in the form as originally proposed by Kerr in [48], but we can map it to the original form by using some transformations. The original form Kerr metric proposed in [48] is less popular in the literature due to the fact it has no close appearance to the Schwarzschild metric. The more familiar one is the metric derived by Boyer and Lindquist which is also the rotating solution of vacuum Einstein equations (2.2.77). The Boyer-Lindquist metric is used more frequently since it has an appearance like the Schwarzschild metric. To get the Boyer-Lindquist form from the metric (2.2.154), we need to perform the following transformations

$$x=\frac{r-M}{\sqrt{M^2-a^2}}\quad,\quad y=\cos\theta\,, \tag{2.2.155}$$

where ϕ and t are unchanged, as well as to set the parameters

$$p=\frac{k}{M}\quad,\quad q=\frac{a}{M}\quad,\quad k^2=M^2-a^2\,. \tag{2.2.156}$$

The resulting metric now can be read as

$$ds^2=-dt^2+\varrho^2\left(d\theta^2+\frac{dr^2}{\Delta}\right)+\left(r^2+a^2\right)\sin^2\theta d\phi^2+\frac{2Mr}{\varrho^2}\left(dt-a\sin^2\theta d\phi\right)^2 \tag{2.2.157}$$

where $\varrho=r^2+\cos^2\theta$ and $\Delta=r^2-2Mr+a^2$. The mass of black hole is M, and its angular momentum J is given by $J=Ma$ where a is called as the rotational parameter. Taking $a=0$ in (2.2.157), we recover the Schwarzschild metric (2.1.50) as we have mentioned at the beginning.

2.2.2 Kerr Black Holes

In subsection 2.1.3, we studied some aspects of static black holes by using the Schwarzschild metric (2.1.50). A more general case for black holes which are still in the framework of the vacuum Einstein gravitational system is the rotating black holes described by the Kerr solution (2.2.157) in Boyer-Lindquist coordinate,

$$ds^2 = -\frac{\left(\Delta - a^2\sin^2\theta\right)}{\varrho^2}dt^2 - \frac{4Mar\sin^2\theta}{\varrho^2}dtd\phi + \frac{\varrho^2}{\Delta}dr^2 + \varrho^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\varrho^2}d\phi^2\,, \quad (2.2.158)$$

where

$$\Delta = r^2 - 2Mr + a^2\ ,\ \varrho^2 = r^2 + a^2\cos^2\theta\ ,\ \Sigma^2 = \left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\,. \quad (2.2.159)$$

The components of metric tensor $g_{\mu\nu}$ associated to the metric (2.2.158) are

$$\begin{aligned} g_{tt} &= -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\ ,\ g_{t\phi} = g_{\phi t} = -\frac{2Mar}{\rho^2}\sin^2\theta\,, \\ g_{rr} &= \frac{\rho^2}{\Delta}\ ,\ g_{\theta\theta} = \rho^2\ ,\ g_{\phi\phi} = \frac{\Sigma^2\sin^2\theta}{\rho^2}\,, \end{aligned} \quad (2.2.160)$$

and the contravariant version are

$$\begin{aligned} g^{tt} &= -\frac{\left(r^2+a^2\right)^2 - a^2\sin^2\theta}{\Delta\rho^2}\ ,\ g^{\phi t} = g^{t\phi} = -\frac{2Mar}{\Delta\rho^2}\,, \\ g^{rr} &= \frac{\Delta}{\rho^2}\ ,\ g^{\theta\theta} = \frac{1}{\rho^2}\ ,\ g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Delta\rho^2\sin^2\theta}\,. \end{aligned} \quad (2.2.161)$$

Solving $\Delta = 0$ gives the locations of event horizons of Kerr black holes, which are

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \quad (2.2.162)$$

The radius r_+ and r_- are called the outer and inner event horizons respectively, since $r_+ > r_-$. When the rotation of black hole stops, the inner horizon vanishes and the outer one becomes $2M$ which is the location of Schwarzschild black hole's event horizon. In fact, only the outer horizon which behaves like the event horizon of Schwarzschild black holes. The area of Kerr black holes is the surface at $r = r_+$, which can be computed by an integration over proper lengths $\sqrt{|g_{\theta\theta}|}d\theta$ and $\sqrt{|g_{\phi\phi}|}d\phi$,

$$A_H = \Sigma_H \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\phi = 8\pi M r_+\,. \quad (2.2.163)$$

Here Σ_H is the Σ in (2.2.159) evaluated at the event horizon radius r_+.

When $a^2 > M^2$, the outer and inner horizons (2.2.162) become complex functions, which are considered to be non-physical. In such situation, the naked singularity is created, i.e. there is not event horizon that covers the physical singularity of black hole. We exclude the existence of the black hole's naked singularity, hence there is a constraint for the maximum value of rotational parameter, i.e. $a \leq M$. When a Kerr black hole is rotating with the rotational parameter $a = M$, the black hole is said to be in the extremal condition. Furthermore, in the extremal case, the inner and outer horizons of Kerr black holes coincide, $r_+ = r_- = M$.

In classical mechanics, we are familiar with some constants of motion associated to the dynamics of particles such as the energy and momentum. In general relativity, we also can find the constants of motion related to the dynamics of a particle in curved space. In section 2.1.1, we have discussed the Killing vectors ξ_μ which obey the Killing equation (2.1.12). For Kerr spacetime, the associated Killing vectors are ξ_t and ξ_ϕ. These vectors indicate that the Kerr spacetime is stationary and invariant under ϕ angle rotation. In other words, an observer sees no difference when he moves from a point $(t_1, r_1, \theta_1, \phi_1)$ to another one at $(t_2, r_1, \theta_1, \phi_2)$. The stationary behavior of Kerr spacetime is obvious from the fact that the metric tensor components describing it are time dependent.

Let us consider a particle which is moving along the path $x^\mu(\tau)$, where τ is the proper time[10]. The tangent vector to this path is $u^\mu = dx^\mu/d\tau$. Suppose that the path x^μ is a geodesic, thus this path obeys the geodesic equation

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{2.2.164}$$

Therefore, the tangent vector u^μ satisfies

$$u^\alpha \nabla_\alpha u^\mu = 0\,. \tag{2.2.165}$$

Moreover, we consider the scalar $u_\mu \xi^\mu$. In the u^α direction, the rate of change $u_\mu \xi^\mu$ is

$$u^\alpha \nabla_\alpha \left(u_\mu \xi^\mu\right) = \xi^\mu u^\alpha \nabla_\alpha u_\mu + \frac{1}{2} u^\alpha u^\mu \left(\nabla_\alpha \xi_\mu + \nabla_\mu \xi_\alpha\right)\,. \tag{2.2.166}$$

[10]By restoring the speed of light c in the metric, the proper time can be understood from the equation $ds^2 = c^2 d\tau^2$.

It is clear that the right hand side of the last equation vanishes provided by the geodesic equation (2.2.165) and the Killing equation (2.1.12).

We now examine the scalar $u_\mu \xi^\mu$ for Schwarzschild and Kerr spacetimes. Recall that both Kerr and Schwarzschild spacetimes have the ξ^t Killing vector. Let us focus on the Schwarzschild spacetime first. A particle that is moving along a geodesic path in Schwarzschild spacetime has a constant energy,

$$\xi_t u^t = g_{tt}\xi^t u^t = g_{tt}u^t = E = \text{const}\,, \tag{2.2.167}$$

where we have used $\xi^t = (1, 0, 0, 0)$. Furthermore, from (2.2.167) we can write

$$u_t = E\,. \tag{2.2.168}$$

We now show the constant of motion associated to the Killing vector ξ^ϕ. For Schwarzschild spacetime, whose all off-diagonal components in the metric are vanishing, the contraction of the Killing vector ξ^ϕ and u^ϕ can be read as

$$\xi_\phi u^\phi = g_{\phi\phi}\xi^\phi u^\phi = g_{\phi\phi}u^\phi = L = \text{constant}\,. \tag{2.2.169}$$

From the last equation we may write

$$u_\phi = L\,. \tag{2.2.170}$$

Consider a particle which is orbiting around a Schwarzschild black hole on the xy-plane as illustrated in figure 2.8. Hence, the particle's rotation is about the z-axis. Since the particle is moving on the xy-plane where the associated θ angle would be $\pi/2$, by using $u^\phi = d\phi/d\tau$ we can get from (2.2.169) that asymptotically

$$r^2\frac{d\phi}{d\tau} = L \tag{2.2.171}$$

after plugging $g_{\phi\phi} = r^2\sin^2\theta$.

Equation (2.2.171) reminds us the angular momentum in classical mechanics, $m\vec{r}\times\vec{v}$. This fact leads us to the interpretation u_ϕ as the angular momentum per unit mass of a test particle observed at infinity. From the results (2.2.168) and (2.2.170) we learn that the constants of motion in Schwarzschild spacetime related to the Killing vectors ξ^t and ξ^ϕ are the energy u_t and angular momentum per unit mass u_ϕ respectively.

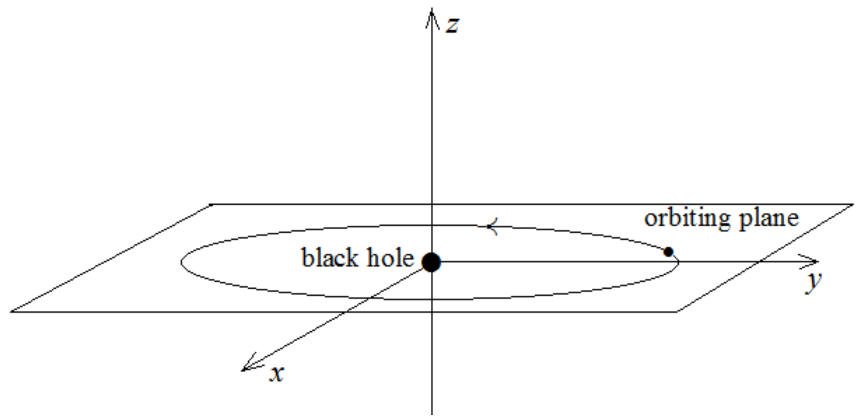

Figure 2.8: An illustration of an orbiting particle on the xy-pane in Schwarzschild spacetime.

We now turn to the problem in identifying the constants of motion in the Kerr spacetime by employing the similar prescriptions that we have performed in the Schwarzschild case. The Killing vectors for Kerr spacetime are ξ^t and ξ^ϕ, just like in the Schwarzschild case. Hence, u_t and u_ϕ would be the constants of motion for Kerr spacetime. In fact, there is a slight difference for Kerr spacetime related to the presence of $g_{t\phi}$ in the corresponding metric tensor. Related to the axial symmetry of Kerr spacetime, the constant of motion associated to the Killing vector ξ^ϕ would be

$$u_\phi = L_z \tag{2.2.172}$$

where L_z is the projection of angular momentum per unit mass with respect to the rotational axis, i.e. z-axis. The existence of the off-diagonal metric components $g_{t\phi} = g_{\phi t}$ leads to

$$\frac{d\phi}{d\tau} = u^\phi = g^{\phi t} u_t + g^{\phi\phi} u_\phi \,. \tag{2.2.173}$$

Interestingly, from the last equation we observe that it is possible for a distant observer to measure a vanishing angular momentum per unit mass u_ϕ for a freely falling test particle while u^ϕ is not zero. This is interpreted as the frame dragging effect, where a particle with initial zero angular momentum acquires some angular velocities as it gets closer to the Kerr black hole. Such effect doesn't exist in the Schwarzschild spacetime. The illustration of this process is given in the figure 2.9.

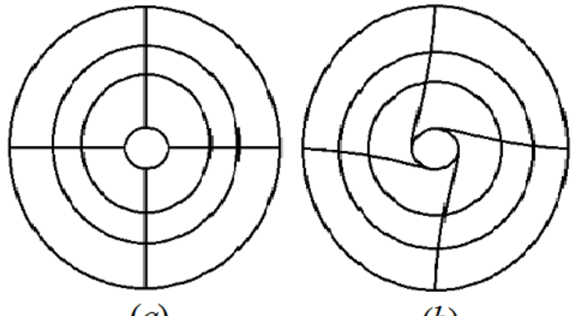

Figure 2.9: (a) Radially falling photon into a static black hole, (b) Dragging effect for photons which initially fall radially into a rotating black hole.

2.2.3 Surface gravity of Kerr black holes

Surface gravity of a black hole is the gravitational strength at the horizon measured by an observer at infinity. It is constant everywhere on the surface of black hole's horizon. This behavior resembles the temperature in thermodynamics, which is also constant at all points of a body in thermal equilibrium. It turns out that the surface gravity and temperature of radiating black holes are closely related.

In this subsection, we review the computation of Kerr black holes's surface gravity by studying the four-velocity and four-acceleration of an observer with zero angular momentum observed from infinity, i.e. $u_\phi = 0$. This observer sits at a fixed radial position outside of the black hole, and also at a fixed θ coordinate. Recall that this observer may feel the dragging effect as we have showed in the previous subsection.

Using the rule of lowering indices for a vector in general relativity, we have

$$u_\phi = g_{\phi\phi}u^\phi + g_{t\phi}u^t \,. \qquad (2.2.174)$$

It can be seen if only $g_{t\phi} = 0$, as in the case of Schwarzschild spacetime, then the zero angular momentum observer in Kerr background will really have no angular speed. The vanishing of u_ϕ in (2.2.174) gives us

$$u^\phi = -\frac{g_{t\phi}}{g_{\phi\phi}}u^t \,. \qquad (2.2.175)$$

For the Kerr spacetime, one can verify the identity

$$\frac{g_{t\phi}}{g_{\phi\phi}} = -\frac{g^{t\phi}}{g^{tt}} \tag{2.2.176}$$

which is obvious from the relation $g_{\mu\alpha}g^{\nu\alpha} = \delta^{\nu}_{\mu}$. Furthermore, we can define the angular speed of observer as

$$\Omega = \frac{p^{\phi}}{p^{t}}\,. \tag{2.2.177}$$

Again, by using the rule of raising indices we can do some algebraic manipulations

$$p^{\phi} = g^{\phi\mu}p_{\mu} = g^{\phi t}p_t + g^{\phi\phi}p_{\phi} \tag{2.2.178}$$

and

$$p^{t} = g^{t\mu}p_{\mu} = g^{tt}p_t + g^{t\phi}p_{\phi}\,. \tag{2.2.179}$$

However, since $p_{\phi} = 0$, it follows that

$$\Omega = \frac{g^{t\phi}}{g^{tt}}\,. \tag{2.2.180}$$

The last formula is clearly related the frame dragging effect in the Kerr spacetime. Consider a test particle that approaches a Kerr black hole from infinity with an initial zero angular momentum. As the particle gets closer to the black hole horizon, it is dragged in the direction of black hole's rotation with the angular velocity is given in (2.2.180).

Explicitly, inserting the components of contravariant metric tensor for Kerr spacetime in (2.2.161) to the formula (2.2.180) yields the reading of angular velocity (2.2.180) as

$$\Omega\left(r,\theta\right) = \frac{2Mar}{\left(\Delta + 2Mr\right)^2 - a^2\Delta\sin^2\theta}\,. \tag{2.2.181}$$

The last equation agrees with our physical intuition. Taking the limit $r \to \infty$, this angular velocity vanishes, which means an observer does not feel the dragging effect at this distance. Indeed, the vanishing of this angular velocity at infinity is also due to the asymptotically flatness of Kerr solution. In fact, the increasing of Ω as the radius r gets smaller agrees with our classical mechanics picture on the conservation of angular momentum. Finally, at the event horizon, the angular velocity (2.2.180) becomes

$$\Omega_H = \frac{a}{2Mr_+}\,. \tag{2.2.182}$$

Plugging (2.2.181) to (2.2.175) gives $u^\phi = \Omega u^t$.

For u^t, the lowering indices rule reads

$$u_t = g_{tt}u^t + g_{\phi t}u^\phi = \left(g_{tt} - \frac{(g_{\phi t})^2}{g_{\phi\phi}}\right) u^t\,. \tag{2.2.183}$$

Therefore, from the fact that u^μ is normalized as $u^\mu u_\mu = 1$, we can compute explicitly u^t,

$$u^t = \left(\frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - (g_{\phi t})^2}\right)^{\frac{1}{2}}\,. \tag{2.2.184}$$

Accordingly, combining the results in (2.2.184) and (2.2.175), the four-velocity describing the zero angular momentum observer being discussed in this subsection reads

$$\begin{aligned} u^\alpha &= \left(u^t, u^r, u^\theta, u^\phi\right) = \left(u^t, u^r, u^\theta, \Omega u^t\right) \\ &= \left(\frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - (g_{\phi t})^2}\right)^{\frac{1}{2}} (1, 0, 0, \Omega)\,. \end{aligned} \tag{2.2.185}$$

We now need to verify whether the four-velocity we just derived in (2.2.185) is normal to a surface of constant time t or not. The displacement on the "constant time" surface can be read as

$$d\tilde{x}^\alpha = (0, dr, d\theta, d\phi)\,. \tag{2.2.186}$$

Accordingly, the covariant counterpart of the displacement (2.2.186) is

$$d\tilde{x}_\alpha = (g_{t\phi}d\phi, g_{rr}dr, g_{\theta\theta}d\theta, g_{\phi\phi}d\phi)\,. \tag{2.2.187}$$

The contraction between the displacement (2.2.187) and the four velocity (2.2.185) can be computed as

$$\begin{aligned} d\tilde{x}_\alpha u^\alpha &= \left(g_{t\phi}u^t + g_{t\phi}u^\phi\right) d\phi \\ &= \left(g_{t\phi} + g_{t\phi}\Omega\right) u^t d\phi \\ &= \left(g_{t\phi} + g_{t\phi}\frac{g^{t\phi}}{g^{tt}}\right) u^t d\phi \\ &= \left(g_{t\phi} + g_{t\phi}\left(-\frac{g_{t\phi}}{g_{\phi\phi}}\right)\right) u^t d\phi \\ &= 0\,, \end{aligned} \tag{2.2.188}$$

which leads us to the conclusion that u^α is normal to the surface with constant time.

We now turn to the discussion to obtain the four-acceleration which is related to the four-velocity (2.2.185),

$$a^{\alpha} = u^{\nu}\left(\partial_{\nu}u^{\alpha} + \Gamma^{\alpha}_{\mu\nu}u^{\mu}\right) . \tag{2.2.189}$$

From the last equation, one can obtain the only non-zero component of four-acceleration, which is

$$a^{r} = \left(u^{t}\right)^{2}\left(\Gamma^{r}_{tt} + 2\Gamma^{r}_{t\phi}\Omega + \Gamma^{r}_{\phi\phi}\Omega^{2}\right) \tag{2.2.190}$$

where Ω is given in (2.2.181). Each of the second kind Christoffel symbols which appear in the last equation are

$$\Gamma^{r}_{tt} = -\frac{g^{rr}M}{\rho^{4}}\left(r^{2} - a^{2}\cos^{2}\theta\right) , \tag{2.2.191}$$

$$\Gamma^{r}_{\phi t} = \frac{g^{rr}M}{\rho^{4}}a\sin^{2}\theta\left(r^{2} - a^{2}\cos^{2}\theta\right) , \tag{2.2.192}$$

and

$$\Gamma^{r}_{\phi\phi} = \frac{g^{rr}M}{2\rho^{4}}\sin^{2}\theta\left(\rho^{2}\frac{\partial\Sigma}{\partial r} - 2\Sigma r\right) . \tag{2.2.193}$$

We know that the proper acceleration is given by

$$g(r)^{2} = a_{\mu}a^{\mu} = g_{\mu\nu}a^{\mu}a^{\nu} = g_{rr}\left(a^{r}\right)^{2} . \tag{2.2.194}$$

Plugging the last formula to (2.2.190) gives us

$$g(r) = \frac{MZ}{\rho^{7}\Delta^{1/2}\Sigma} \tag{2.2.195}$$

where

$$Z = \left(\Sigma^{2} - 4\Sigma M a^{2} r\sin^{2}\theta\right)\left(r^{2} - a^{2}\cos^{2}\theta\right) - 2Ma^{2}r^{2}\sin^{2}\theta\left(\rho^{2}\frac{\partial\Sigma}{\partial r} - 2r\Sigma\right) . \tag{2.2.196}$$

We know in Newtonian gravity, there is a relation between the potential energy E and gravitational acceleration g for a particle with mass m from the center of gravitational attraction with distance h,

$$E = mgh . \tag{2.2.197}$$

It follows from the last equation we can find the ratio of the gravitational acceleration at a finite radius from the center of black hole $g(r)$ to the gravitational acceleration at infinity $g(\infty)$.

Consider the following gedanken experiment. Let an observer at infinity is pulling a particle with unit mass away from a black hole. Initially this particle is sitting somewhere at the radius r from the center of the black hole. The observer at infinity is using an non-extensible massless rope and pulls the particle with a distance dr. For the particle, this pulling yields an increasing local potential energy as

$$dE(r) = g(r)dr \,. \tag{2.2.198}$$

In doing this work, the observer at infinity must provide some energy, that is

$$dE(\infty) = g(\infty)dr \,. \tag{2.2.199}$$

Equating the last two equations gives

$$g\left(\infty\right) = \frac{E\left(\infty\right)}{E\left(r\right)} g\left(r\right) \,. \tag{2.2.200}$$

We understand, for the unit mass particle we are discussing here, the associated energy is the "t" component of u^{μ} given in (2.2.185). Hence, the ratio $E(r)/E(\infty)$ reads

$$\frac{E\left(r\right)}{E\left(\infty\right)} = \sqrt{\frac{\Sigma}{\rho^2 \Delta}} \,. \tag{2.2.201}$$

Plugging (2.2.195) and (2.2.201) into (2.2.200) gives the acceleration observed at infinity as

$$g(\infty) = \frac{MZ}{\rho^6 \Sigma^{3/2}} \,, \tag{2.2.202}$$

where for $r = r_+$ explicitly reads

$$g(\infty) = \frac{r_+ - r_-}{4Mr_+} \equiv \kappa \,. \tag{2.2.203}$$

after we make use of the following equations,

$$\Sigma\left(r = r_+\right) = 4M^2 r_+^2 \,, \tag{2.2.204}$$

and

$$\left.\frac{\partial \Sigma}{\partial r}\right|_{r=r_+} = 8Mr_+^2 - 2a^2\left(r_+ - M\right)\sin^2\theta \,. \tag{2.2.205}$$

The result (2.2.203) is called the surface gravity, i.e. the gravitational acceleration on the event horizon measured by an observer at infinity. The derivation of surface gravity given here is less sophisticated compared to the one performed in [51] which needs some differential geometry knowledge. In fact, the result (2.2.203) agrees with that which is derived in [51], and also we find that the method that we use in this subsection is more intuitive.

2.2.4 Eddington-Finklestein coordinates for Kerr

The Kerr black holes have two event horizons instead of one as a Schwarzschild black hole has. The two horizons, r_+ and r_-, again show the incapability of coordinate system we are using to allow the non-divergent metric (2.2.157) at all points except $r = 0$. Again, computing the Kretschmann scalar for Kerr spacetime is an analytic way to make sure that there are no singularities in the Kerr spacetime at r_+ and r_-.

Using all non-vanishing components of the covariant and contravariant Riemann tensor, as given in the appendix B to the formula

$$K = R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu}\,, \tag{2.2.206}$$

one can get the Kretschmann scalar for Kerr spacetime, which reads

$$K = \frac{48M^2\left(r^2 - a^2\cos\theta\right)\left(\varrho^4 - 16r^2a^2\cos^2\theta\right)}{\varrho^{12}}\,. \tag{2.2.207}$$

The Kretschmann scalar for Kerr (2.2.207) reduces to the one for Schwarzschild (2.1.60) when $a = 0$ as it should. The finite values of Kretschmann scalar (2.2.207) at r_+ and r_- show that the singularities we encountered in the metric (2.2.157) is just some coordinate singularities, the consequences of coordinate system we choose in expressing (2.2.157). Interestingly, the physical singularity provided by $K \to \infty$ is given by two situations at once, $r = 0$ and $\theta = \pi/2$.

It turns out that the singularity for Kerr spacetime has a form as a circle rather than a point as in Schwarzschild case. It can be seen from the mapping between the Kerr metric in Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$ and the Kerr metric in Cartesian coordinates $\{t, x, y, z\}$. To get the mapping in a simple way, we can write down the Kerr metric (2.2.157) at the limit $M \to 0$,

$$ds^2 = -dt^2 + \frac{\varrho^2}{r^2 + a^2}dr^2 + \varrho^2 d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 \tag{2.2.208}$$

where $\varrho^2 = r^2 + a^2 \cos^2\theta$. The metric (2.2.208) is just the flat Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.2.209}$$

in "oblate spheroidal" coordinates. The mapping between the spatial components in (2.2.208) and (2.2.209) are

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi\,, \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi\,, \\ z &= r\cos\theta\,. \end{aligned} \tag{2.2.210}$$

Setting $r = 0$ and $\theta = \pi/2$ in (2.2.210) gives us the relation

$$x^2 + y^2 = a^2\,, \tag{2.2.211}$$

which is just the equation for a circle with radius a on the xy plane of Cartesian coordinate. The Kretschmann scalar for kerr (2.2.207) diverges along this circle which indicates that the singularity for Kerr spacetime has a form as a ring with the radius a, which yields the singularity for Kerr spacetime is called ring singularity.

To show that there is no singularities at r_+ and r_-, again we can use the Eddington-Finkelstein coordinate transformation. The Eddington-Finkelstein transformation for the Kerr spacetime can be read as

$$dv = dt + \frac{r^2 + a^2}{\Delta}dr\,, \tag{2.2.212}$$

$$d\tilde{\phi} = d\phi + \frac{a}{\Delta}dr\,. \tag{2.2.213}$$

From equations (2.2.212) and (2.2.213), we can show that

$$dt^2 = dv^2 + \left(\frac{r^2 + a^2}{\Delta}\right)^2 dr^2 - 2\left(\frac{r^2 + a^2}{\Delta}\right) dvdr\,, \tag{2.2.214}$$

and

$$d\phi^2 = d\tilde{\phi}^2 + \frac{a^2}{\Delta^2}dr^2 - 2\frac{a}{\Delta}drd\tilde{\phi}\,. \tag{2.2.215}$$

Inserting the last two expressions into (2.2.157) yields the reading of Kerr spacetime becomes

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dv^2 - 2dvdr + \rho^2 d\theta^2 + \left(\frac{4Mra}{\rho^2}dv + 2adr\right)\sin^2\theta d\tilde{\phi} - \frac{\Sigma}{\rho^2}\sin^2\theta d\tilde{\phi}^2\,. \tag{2.2.216}$$

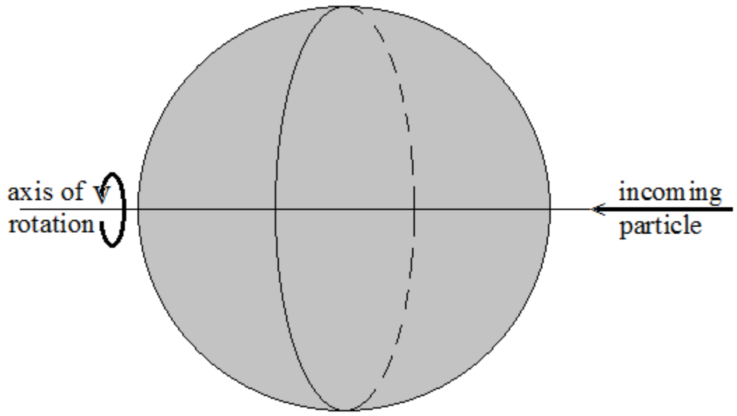

Figure 2.10: Illustration of an outgoing particle along the rotational axis.

The last equation is known as the Eddington-Finkelstein metric for Kerr spacetime, which is regular at r_+ and r_-.

We now analyze the light cones behavior in the spacetime (2.2.216). In contrast to the Schwarzschild case, the dragging effect of Kerr black holes yields only the particle that moves along the rotational axis which is not affected by the black hole's rotation. We understand that the dragging effect of Kerr black holes is due to the off-diagonal terms in Kerr metric,

$$g_{t\phi} = g_{\phi t} = -\frac{\Sigma \sin^2\theta}{\rho^2}\,. \tag{2.2.217}$$

At $\theta = 0$, $g_{t\phi}$ vanishes, thus there is no dragging effect along the $\theta = 0$ trajectory, which coincides with the axis of black hole's rotation. Therefore, the situation is simpler when discuss the ingoing light rays along the black hole's rotational axis. In such case, we can set $d\theta = d\phi = 0$. We would like to see the difference between the inner and outer horizons by studying the phenomena of light around these two horizons.

Consider the purely radial lightlike trajectories, $d\theta = d\phi = 0$, where the metric (2.2.216) reduces to an equation

$$dv\left(\Delta dv - 2\left(\Delta - 2Mr\right)dr\right) = 0\,. \tag{2.2.218}$$

There are two general solutions to this equation. The first one is $dv = 0$, and the second one can be obtained after solving

$$\Delta dv - 2\left(\Delta - 2Mr\right)dr = 0\,. \tag{2.2.219}$$

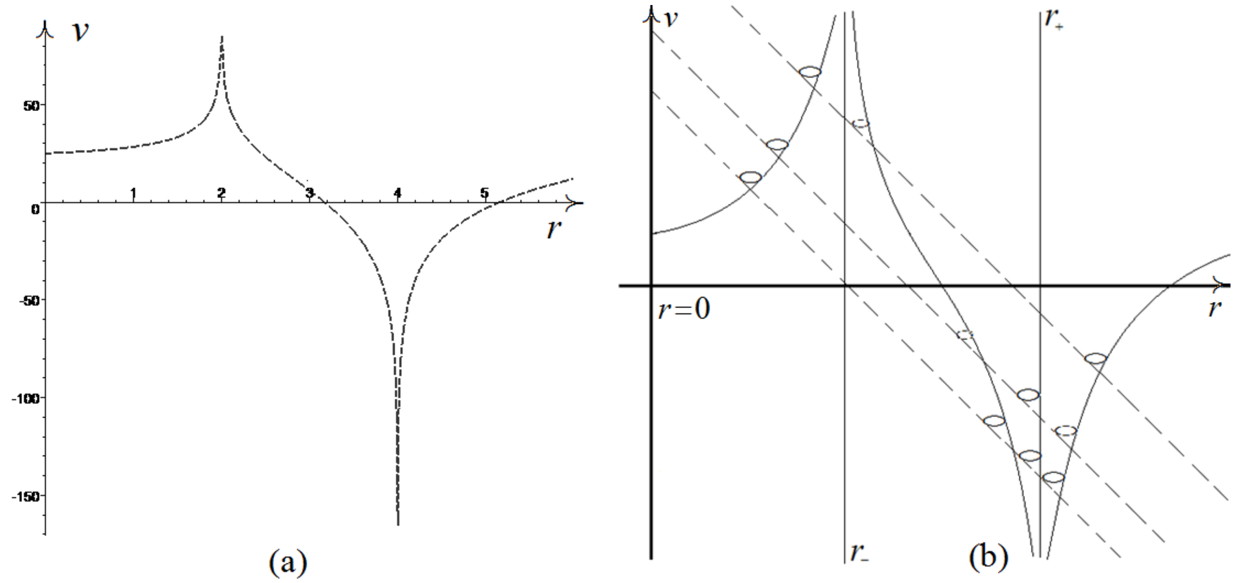

Figure 2.11: (a) The sketch of solution (2.2.221) with numerical value setups $r_+ = 4$ and $r_- = 2$. (b) Lightcones behavior in Kerr spacetime.

The last equation enables one to write an integration

$$v = \int \left(2 + \frac{4Mr}{(r - r_+)(r - r_-)} \right) dr \,, \tag{2.2.220}$$

which yields to the solution

$$v = 2r + \frac{4Mr_+ \ln |r - r_+|}{r_+ - r_-} - \frac{4Mr_- \ln |r - r_-|}{r_+ - r_-} + \text{constant} \,. \tag{2.2.221}$$

Figure 2.11 (a) illustrates the solution (2.2.221). Combining the first and second solution for v gives us an illustration on how light rays behave in a spacetime that contains a Kerr black hole, as sketched in figure 2.11 (b). From figure 2.11 (b), we understand that only the outer horizon of Kerr black hole that behaves just like the event horizon of Schwarzschild black hole. Outside the outer horizon, light rays propagate to the infinity, which is also the case in the Schwarzschild case. Once the light rays touch or enter the outer horizon, they will never escape from the black hole. However, the light rays will be infinitely red shifted when they approach the inner horizon. That is why only the outer horizon of Kerr black hole which behaves as a one way membrane, analogously to the event horizon of Schwarzschild black hole.

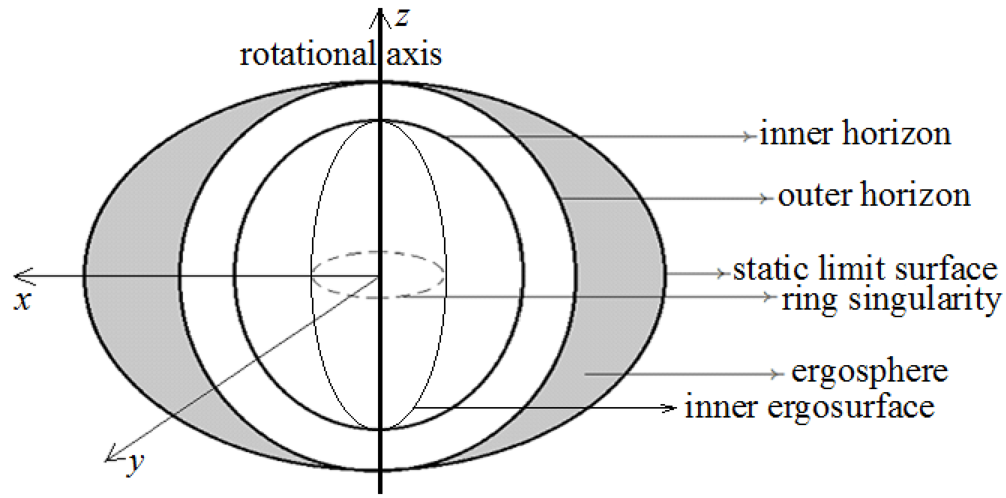

Figure 2.12: Schematic of Kerr black holes.

Schematically, some regions of Kerr black holes are described in figure 2.12. Some new terminologies appear in this figure. They are the ergoregion, inner ergosurface, and outer ergosurface. To discuss these new three objects, let's start with a question: are there any points in Kerr spacetime where it is impossible for a timelike object to stand still?

Suppose that this object is trying to keep its position at

$$X^{\mu} = (t, r_0, \theta_0, \phi_0) \tag{2.2.222}$$

where r_0, θ_0, and ϕ_0 are some fixed r, θ and ϕ in the Boyer-Lindquist coordinate system. Hence, the corresponding tangent vector to the position (2.2.222) is

$$U^{\mu} = \frac{dX^{\mu}}{dt} = (1, 0, 0, 0)\,. \tag{2.2.223}$$

Since we are using the metric convention $(-, +, +, +)$, the following timelike condition for the tangent vector must be satisfied,

$$g_{\mu\nu} U^{\mu} U^{\nu} < 0\,. \tag{2.2.224}$$

Plugging the vector (2.2.223) into the last equation gives us an inequality $g_{tt} < 0$ which yields

$$\Delta - a^2 \sin^2 \theta > 0\,. \tag{2.2.225}$$

Here we have used the g_{tt} from the metric (2.2.160). Nevertheless, this inequality is violated when

$$r^2 + a^2 \cos^2\theta - 2Mr < 0\,. \tag{2.2.226}$$

From the last inequality, one can tell that between the radius

$$M - \sqrt{M^2 - a^2\cos^2\theta} < r < M + \sqrt{M^2 - a^2\cos^2\theta}\,, \tag{2.2.227}$$

it is not possible for a physical object to stand still. The lower and upper bounds in the expression (2.2.227), $M - \sqrt{M^2 - a^2\cos^2\theta}$ and $M + \sqrt{M^2 - a^2\cos^2\theta}$, are the inner and outer ergosurfaces as depicted in figure 2.12. In the region between the inner and outer ergosurfaces, a physical object must be co-rotating with the black holes. Note that as long as the object is still outside the outer horizon, it is still possible for this object to escape from the black hole's gravitational attraction, including if it has entered the outer ergosurface. The fact that the black hole "forces" a particle, once it enters the outer ergosurface, to co-rotates with the black hole becomes a mechanism of the particle to acquire energy from the black hole. Therefore the region between the outer ergosurface and the outer horizon,

$$M + \sqrt{M^2 - a^2} < r < M + \sqrt{M^2 - a^2\cos^2\theta}\,, \tag{2.2.228}$$

is called the ergoregion. The word ergo is a derivative from the Greek word Ergon which means work. A particle that can escape from the ergoregion may get some extra energy from the black hole, i.e. the final energy of particle is larger compared to the initial one.

2.3 Thermodynamics of Kerr black holes

2.3.1 Black Holes Mechanics and Thermodynamics

The area of a Kerr black hole is given in equation (2.2.163)

$$A_{BH} = 8\pi M r_+\,. \tag{2.3.229}$$

Varying the area (2.3.229) with respect to the changes in black hole mass δM and rotation δa gives

$$\begin{aligned}
\delta A_{BH} &= 8\pi\left(r_+\delta M + M\delta r_+\right) \\
&= 8\pi\left(\left(M+\sqrt{M^2-a^2}\right)\delta M + M\left(\delta M + \frac{M\delta M - a\delta a}{\sqrt{M^2-a^2}}\right)\right) \\
&= \frac{8\pi}{\sqrt{M^2-a^2}}\left(\sqrt{M^2-a^2}\left(M+\sqrt{M^2-a^2}\right)\delta M + \sqrt{M^2-a^2}M\delta M + M^2\delta M - aM\delta a\right) \\
&= \frac{8\pi}{\sqrt{M^2-a^2}}\left(2M\left(M+\sqrt{M^2-a^2}\right)\delta M - a^2\delta M - aM\delta a\right) \\
&= \frac{8\pi}{\sqrt{M^2-a^2}}\left(2Mr_+\delta M - a^2\delta M - aM\delta a\right) . \qquad (2.3.230)
\end{aligned}$$

From the definition $a = J/M$, we have $\delta J = a\delta M + M\delta a$. The angular velocity of the black hole at the horizon is $\Omega_H = a/(2Mr_+)$. Consequently, we now can rewrite the variation of black hole area as

$$\delta A_{BH} = \frac{32\pi M r_+}{r_+ - r_-}\left(\delta M - \Omega_H \delta J\right) , \qquad (2.3.231)$$

or equivalently

$$\delta M = \frac{\kappa}{8\pi}\delta A_{BH} + \Omega_H \delta J \, . \qquad (2.3.232)$$

In the formula above, the surface gravity κ is defined in (2.2.203), and as usual r_+ and r_- are the outer and inner horizons of Kerr black hole respectively.

Related to the black hole area A_{BH}, there is a theorem proposed by Hawking which says that the area of black holes cannot decrease classically,

$$\delta A_B H > 0 \, . \qquad (2.3.233)$$

For Schwarzschild black hole, it is easy to figure that, classically, this black hole is a perfect absorber but completely does not emit. When a particle falls into a Schwarzschild black hole, this black hole acquires mass and consequently it yields the increasing area of the black hole. The relation between the variation of mass and area of Schwarzschild black holes is given by

$$\delta A_{BH} = \frac{8\pi\delta M}{\kappa} \, . \qquad (2.3.234)$$

Equation (2.3.233) applies for all classical stationary black holes. Classically, there is no mechanism for a black hole to release some parts of its mass. Since the Kerr black holes are

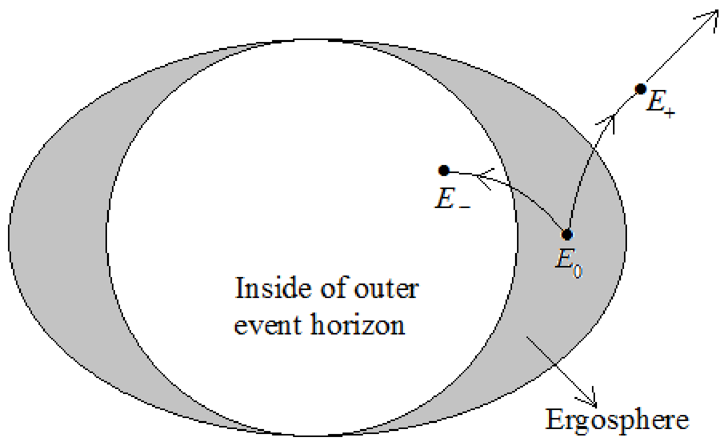

Figure 2.13: Penrose process.

also stationary, they obey the theorem (2.3.234). Let us collect several important facts about Kerr black holes:

- The surface gravity κ is constant on the horizon.
- The conservation of energy, $\delta M = (\kappa/8\pi)\delta A_H + \Omega_H \delta J$.
- The non-decreasing area of black holes, $\delta A_B H > 0$.

However, it is not so easy to see the non-decreasing area theorem works for Kerr black holes whose changes in area is given in (2.3.231). The negative sign in front of the angular momentum variation gives a possibility of the the Kerr black hole's area to decrease. Penrose proposed a gendaken experiment, namely the Penrose process, which may help to understand how Kerr black holes obey the non-decreasing of area theorem (2.3.233). The Penrose process, which is illustrated in figure 2.13, can be described as follows. A particle inside of the ergosphere with intial energy E_0 decays into a particle with positive energy E_+ and a particle with negative energy E_-. The particle with positive energy escapes to infinity and the one with negative energy falls into the black holes.

We now show that in the Kerr spacetime, the particle with postive energy has a negative angular momentum, while the particle with negative energy has a positive angular momentum. Let us start by considering the normalized four-velocity,

$$u_\mu u^\mu = g^{\mu\nu} u_\mu u_\nu = 1\,. \tag{2.3.235}$$

Plugging the corresponding $g^{\mu\nu}$ for Kerr spacetime, the last equation reads

$$g^{tt}\left(u_t\right)^2 + 2g^{t\phi}u_t u_\phi + g^{rr}\left(u_r\right)^2 + g^{\theta\theta}\left(u_\theta\right)^2 + g^{\phi\phi}\left(u_\phi\right)^2 - 1 = 0\,. \tag{2.3.236}$$

Furthermore, one may consider (2.3.236) as a quadratic equation for u_t, whose solutions are

$$u_t = -\Omega u_\phi \pm \frac{1}{g^{tt}}\sqrt{\left(g^{t\phi}u_\phi\right)^2 - \left(g^{rr}\left(u_r\right)^2 + g^{\theta\theta}\left(u_\theta\right)^2 + g^{\phi\phi}\left(u_\phi\right)^2 - 1\right)}\,. \tag{2.3.237}$$

In the last equation we have used $\Omega = g^{t\phi}/g^{tt}$. Recall that the constants of motion in Kerr spacetime are $u_t = E$ and $u_\phi = L_z$. Interestingly the factor $(g^{tt})^{-1}$ in the formula (2.3.237) vanishes at the event horizon. Hence we may rewrite (2.3.237) as

$$E_\pm = -\Omega_H L_{z\pm}\,. \tag{2.3.238}$$

We understand the subscripts + and − in the equation above stand for the particle with positive and negative energy respectively.

It follows from the last equation that

$$E_\pm - \Omega L_{z\pm} \leq 0\,, \tag{2.3.239}$$

since outside the outer event horizon, $\Omega_H \geq \Omega$. Moreover, for a particle on the horizon with negative energy $E = -\left|E_-\right|$ and $L_z = \left|L_{z-}\right|$, (2.3.239) becomes

$$\left|E_-\right| \geq \Omega_H \left|L_{z-}\right|\,, \tag{2.3.240}$$

and accordingly

$$\left|\delta M\right| \geq \Omega_H \left|\delta J\right|.\,. \tag{2.3.241}$$

Plugging the last inequality into (2.3.234) leads us to a proof that the area of a classical Kerr black hole cannot decrease,

$$\delta A_{BH} = \frac{8\pi}{\kappa}\left(\delta M - \Omega_H \delta J\right) \geq 0\,. \tag{2.3.242}$$

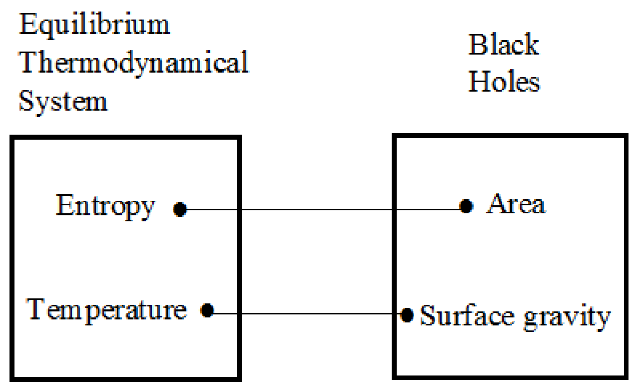

Figure 2.14: The relations between some variables of black holes and thermodynamical systems in equilibrium.

Figure 2.14 illustrates the connections between some physical aspects of Kerr black holes and some quantities in thermodynamics. The constancy of temperature for equilibrium thermodynamics systems is analogous to the constant surface gravity κ on the black hole horizon. The conservation of black hole energy is clearly related to the first law of thermodynamics. The relation becomes clear when we identify the exact relation between the surface gravity of black holes and the black hole temperature, and also the black hole area and entropy. Lastly, the non-decreasing of black hole area, which turns out later to become the non-decreasing of black hole entropy, is in agreement with the second law of thermodynamics.

To be more specific, the justification that a black hole should radiate at some temperatures comes from an analogy between the first law of black hole mechanics (2.3.232) and the first law of thermodynamics,

$$\delta E = T\delta S - P\delta V \,. \tag{2.3.243}$$

The work part PdV in (2.3.243) is clearly related to the angular momentum of black holes, thus the part $T\delta S$ must be related to $\kappa\delta A_{BH}/8\pi$. Consequently, the following relations must valid,

$$T_H = \eta\frac{\kappa}{8\pi} \quad , \quad S_{BH} = \frac{A_{BH}}{\eta} \,. \tag{2.3.244}$$

The constant η above is a free parameter to be determined.

Nevertheless, these nice analogies between black hole mechanics and thermodynamics still have one issue. We have learned that black holes are perfect absorbers, but they do not radiate. It is a problem, how can an object that does not radiate can have a temperature? The black body object in thermodynamics is a perfect absorber as well as a perfect emitter. Therefore, a black hole must radiate somehow. This is the motivation of Hawking back in 1970s [5, 6] to search a mechanism for black hole's radiation. Therefore, the concept of a black hole as an active thermodynamical object is complete. The next two subsections discuss the derivation of black hole's temperature by using some quantum mechanical techniques.

2.3.2 Unruh and Hawking temperatures

So far, the discussions about black holes have not included quantum mechanics yet. Hawking's works [5, 6] show that incorporating quantum mechanics in the study of black holes can give us a mechanism for black hole to radiate. Thus, black hole can behave like a black body; it absorbs and also emits. There are several quantum mechanical ways in showing the black hole's radiation, which convince us more that black holes do radiate. Treating black hole's radiation semiclassically, i.e. half classical and half quantum, leads one to derive an explicit temperature T_H of black holes as guessed in (2.3.244). This temperature is called as Hawking temperature, and the mechanism for black holes to radiate is called the Hawking process.

In this subsection, we review the derivation of Hawking temperature by obtaining the Unruh temperature first. Unruh temperature is the temperature measured by an accelerated observer in flat space. This temperature later can be related to the Hawking temperature by using the equivalence principle: an observer at rest in an environment with gravitation feels the same thing with an accelerated observer in a spacetime with no gravity. In the next subsection, the Hawking process from tunneling picture is given, to convince the reader that black holes do radiate. There are other several mechanisms for black holes radiations, interested reader can read [52].

Our starting equation is the massless relativistic scalar field equation in the $(1 + 1)$ Minkowski space,

$$\frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^2} \,. \qquad (2.3.245)$$

Here $(1 + 1)$ means one time and one spatial dimensions. Equation (2.3.245) can be solved by using the following separable functions

$$\Phi\left(t,x\right)=\begin{cases} f_{\omega}^{(-)}\left(t,x\right)\sim\exp\left(\pm i\omega\left(x-t\right)\right), \\ f_{\omega}^{(+)}\left(t,x\right)\sim\exp\left(\pm i\omega\left(x+t\right)\right), \end{cases} \tag{2.3.246}$$

where the frequency ω is always non-zero positive, i.e. $\omega > 0$. The function $\Phi\left(t,x\right)$ is a quantum mechanical wave function associated with a specific energy and momentum.

It follows from (2.3.246) that one can write

$$i\hbar\frac{\partial f_{\omega}^{(\pm)}}{\partial t}=\pm\hbar\omega f_{\omega}^{(\pm)}\,. \tag{2.3.247}$$

Consequently, the dependence on time in the wave function is

$$f_{\omega}^{(\pm)}\sim\exp\left(\mp i\omega t\right)\,. \tag{2.3.248}$$

Since we are discussing the vacuum for an inertial observer, $\Phi(t,x)$ represents a system where initially there is no any particle at all. Nevertheless, the uncertainty principle allows the field fluctuations to occur which yields a possibility for the vacuum to have the positive and negative energy particles excitation. The full expression for a wave function for the particle with negative energy is

$$f_{\omega}^{(-)}=\frac{\exp\left(i\omega\left(t-x\right)\right)}{2\sqrt{\pi\omega}}\,, \tag{2.3.249}$$

where the factor $2\sqrt{\pi\omega}$ is a normalization factor.

To get the wave equation for an accelerated observer, one can perform the coordinate transformation

$$t=\rho\sinh\left(a\tau\right)\quad,\quad x=\rho\cosh\left(a\tau\right) \tag{2.3.250}$$

to the equation (2.3.245). Consequently, the resulting metric after using the transformation (2.3.250) to the $(1+1)$ Minkowski space reads

$$ds^2=-a^2\rho^2d\tau^2+d\rho^2\,. \tag{2.3.251}$$

The spacetime (2.3.251) is known as the two dimensional Rindler spacetime. Writing the wave equation (2.3.245) in the two dimensional Rindler spacetime reads

$$\frac{1}{a^2\rho^2}\frac{\partial^2\Phi}{\partial\tau^2}=\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{\partial^2\Phi}{\partial\rho^2}\,. \tag{2.3.252}$$

The functions

$$F_{\omega}^{(\pm)} = \frac{\exp\left(\mp i\omega\tau\right)}{2\sqrt{\pi\omega}}\rho^{i\omega/a} \tag{2.3.253}$$

solve (2.3.252), where $F_{\omega}^{(+)}$ is the solution with positive energy and $F_{\omega}^{(-)}$ is associated with the negative energy.

The wave solution for negative energy (2.3.249) can be constructed from the superposition of the complete set of positive and negative energies wave functions (2.3.253),

$$f_{\omega}^{(-)}\left(t,x\right) = \int \left(a_{\omega'}\left(\omega\right)F_{\omega'}^{(+)}\left(\tau,\rho\right) + b_{\omega'}\left(\omega\right)F_{\omega'}^{(-)}\left(\tau,\rho\right)\right)d\omega'\,. \tag{2.3.254}$$

The probability amplitude of a particle with positive energy in the state $F_{\omega'}^{(+)}$ measured by an accelerated observer is the coefficient $a_{\omega'}$. Moreover, the number of particles with frequency ω' from all the vacuum modes in the inertial frame with frequency ω is given by

$$N\left(\omega'\right) = \int \left|a_{\omega'}\left(\omega\right)\right|^2 d\omega\,. \tag{2.3.255}$$

The coefficient $a_{\omega'}$ can be obtained by integrating

$$\int \left(F_{\omega''}^{(+)}\right)^* f_{\omega}^{(-)}\frac{d\rho}{a\rho} \tag{2.3.256}$$

and by employing the orthogonality between $\left(F_{\omega''}^{(+)}\right)^*$ and $F_{\omega'}^{(+)}$. The calculation is

$$\int \left(F_{\omega''}^{(+)}\right)^* f_{\omega}^{(-)}\frac{d\rho}{a\rho} = \int \left(a_{\omega'}F_{\omega'}^{(+)}\left(F_{\omega''}^{(+)}\right)^* + b_{\omega'}F_{\omega'}^{(-)}\left(F_{\omega''}^{(+)}\right)^*\right)d\omega'\frac{d\rho}{a\rho} = \frac{a_{\omega'}}{2\omega'}.$$

Hence we can have

$$a_{\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int \exp\left(i\omega'\left(\tau+t-x\right)\right)\rho^{-i\omega'/a}\frac{d\rho}{a\rho}\,. \tag{2.3.257}$$

The last formula is obtained after plugging the wave solution (2.3.249) in Minkowski spacetime $\{t,x\}$ and the wave solution with positive energy in (2.3.253) in Rindler spacetime $\{\tau,\rho\}$.

The Rindler spacetime itself is time-independent, thus allow us to choose any convenient time to get $a_{\omega'}$. In such consideration, we set $t=\tau=0$, then equation (2.3.257) becomes

$$a_{\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int \exp\left(-i\omega'\rho\right)\rho^{-1-i\omega'/a}\frac{d\rho}{a}\,. \tag{2.3.258}$$

In the formula above, we have replaced x to ρ in the exponential. Recall that at $\tau = 0$ from the relation $x = \rho \sinh(a\tau)$ we have $dx = d\rho$. To simplify (2.3.258), we can get some benefits by setting $z = i\omega\rho/a$ and recall that $-i = \exp(-i\pi/2)$. Finally the coefficient $a_{\omega'}$ is found to be

$$a_{\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\exp\left(\frac{-\pi\omega'}{2a}\right)A\left(\omega'\right) , \tag{2.3.259}$$

where

$$A\left(\omega'\right) = \int \exp\left(-za\right)\left(za\right)^{-1-i\omega'/a}\omega^{-1+i\omega'/a}dz\,. \tag{2.3.260}$$

Accordingly, the number of particle with the energy ω' is given by

$$N\left(\omega'\right) = \left|a_{\omega'}\left(\omega\right)\right|^2 = \frac{\exp\left(-\pi\omega'/a\right)}{4\pi^2}\left(\frac{\omega'}{\omega}\right)\left|A\left(\omega'\right)\right|^2 , \tag{2.3.261}$$

while for the negative energy $-\omega'$ we have

$$N\left(-\omega'\right) = \left|a_{-\omega'}\left(\omega\right)\right|^2 = -\frac{\exp\left(\pi\omega'/a\right)}{4\pi^2}\left(\frac{\omega'}{\omega}\right)\left|A\left(\omega'\right)\right|^2 . \tag{2.3.262}$$

It is clear that $\left|A\left(\omega'\right)\right|^2 = \left|A\left(-\omega'\right)\right|^2$, hence now one can establish a relation between $N\left(\omega'\right)$ and $N\left(-\omega'\right)$,

$$N\left(-\omega'\right) = -\exp\left(2\pi\omega'/a\right)N\left(\omega'\right) . \tag{2.3.263}$$

A function for $N\left(\omega'\right)$ which satisfies the last equation is

$$N\left(\omega'\right) = \frac{1}{\exp\left(2\pi\omega'/a\right) - 1} . \tag{2.3.264}$$

Consequently we can write

$$N\left(\omega'\right)d\omega' = \frac{d\omega'}{\exp\left(2\pi\omega'/a\right) - 1} \tag{2.3.265}$$

which has a very close form with the Plank distribution in one spatial dimension,

$$N\left(\nu\right)d\nu = \frac{2\pi d\nu}{\exp\left(h\nu/kT\right) - 1} . \tag{2.3.266}$$

By matching the last two equations, we arrive at a conclusion that the observers with acceleration in flat space feel a thermal distribution or thermal bath of particles at temperature

$$T_U = \frac{\hbar a}{2\pi kc} . \tag{2.3.267}$$

In the natural units, the Unruh temperature (2.3.267) reads $T_U = a/2\pi$. The superscript U in this temperature stands for Unruh, who is the first person to propose this temperature. Temperature (2.3.267) is known as the Unruh temperature.

We now use the equivalence principle to "connect" the Unruh temperature and the temperature measured by a static observer sitting in an environment with gravity [53]. Near the horizon of Kerr black hole, the Boyer-Lindquist metric for Kerr can be written as

$$ds_H^2 = \rho_H^2 \left(-\kappa^2 x^2 dt^2 + dx^2 + d\theta^2\right) + \frac{\Sigma_H^2}{\rho_H^2} \sin^2\theta d\tilde{\phi}^2 \,. \tag{2.3.268}$$

The metric (2.3.268) is obtained after employing the transformation

$$x^2 = \frac{r - r_+}{\kappa M r_+} \,. \tag{2.3.269}$$

In the metric (2.3.268) we have used

$$\Sigma_H^2 = 4M^2 r_+^2 \quad , \quad \rho_H^2 = r_+^2 + a^2\cos^2\theta \,, \tag{2.3.270}$$

and $d\tilde{\phi} = dt - \Omega_H d\phi$.

To match the discussion with Unruh temperature derivation, we need to keep only one spatial dimension in metric (2.3.268). Hence we choose $d\theta = d\phi = 0$ and $\theta = 0$ to avoid the dragging effect. Let us now consider the conformal transformation

$$\tilde{g}_{\mu\nu} = \rho_H^{-2} g_{\mu\nu} \,, \tag{2.3.271}$$

hence the metric (2.3.268) becomes

$$ds_H^2 = -\kappa^2 x^2 dt^2 + dx^2 \,. \tag{2.3.272}$$

One can see that the metric (2.3.272) is just the Rindler metric (2.3.251) after the identifications $a \to \kappa$, $\tau \to t$, and $\rho \to x$. Therefore, the temperature associated with the gravity near the black hole horizon can be read as

$$T_H = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{8\pi M r_+} \tag{2.3.273}$$

where we have replaced the acceleration a in Unruh temperature (2.3.267) with the surface gravity κ and use the result (2.2.203). Recall that the surface gravity is an acceleration at the horizon measured by an observer at infinity. From the equivalent principle, replacing a with κ is based on the fact that an accelerated observer in flat space whose acceleration is a will feel a thermal bath just like a static observer in gravitational environment with surface gravity κ. Hence, both of these observers will measure the same temperature. A more sophisticated calculation to derive the Hawking temperature can be done by some basic understandings in quantum mechanics and complex analysis, for example the original Hawking derivation [6].

After having an exact expression for the Hawking temperature, now we have the value of $\eta = 4$ in (2.3.244). Thus the black hole entropy is given by the formula

$$S_{BH} = \frac{A_{BH}}{4} \,. \tag{2.3.274}$$

Consequently, by using the are formula (2.2.163), the Bekenstein-Hawking entropy for Kerr black hole can be read as

$$S_{BH} = 2\pi M r_{+} \,. \tag{2.3.275}$$

2.3.3 Hawking radiation in the tunneling method

The derivation of Hawking temperature as performed in the previous subsection is one among several ways in deriving the Hawking temperature for black holes. There is a new method proposed quite recently which is easier to be digested from the physical and mathematical aspects, namely the tunneling method. In this method, Hawking radiation is described as a tunneling of quantum particle through a potential barrier at the horizon.

Let us start by writing the metric

$$ds^2 = -f\left(r\right)dt^2 + g\left(r\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,. \tag{2.3.276}$$

We then consider a massless particle [11] in the spacetime (2.3.276) whose dynamics are governed by the Klein-Gordon equation (2.1.27).

The general metric (2.3.276) is clearly spherically symmetric, and we consider the incoming and outgoing particles across the horizon would be in the radial direction only. Hence, we could simplify our discussion from $(3+1)$ gravity to $(1+1)$ case. This can be done by

[11] The method is also valid for massive case, shown earlier [54] that ultimately the final expressions match. Therefore for the sake of simplicity we consider the massless case only.

taking $d\theta$ and $d\phi$ to be zero, thus only dr and dt sectors of the metric (2.3.276) which are left. The appropriate semiclassical wave functions that correspond to this scheme is

$$\phi(r,t) = e^{-iS(r,t)}\,. \tag{2.3.277}$$

Expanding $S(r,t)$ in a powers of $\hbar$, we find

$$\begin{aligned} S(r,t) &= S_0(r,t) + \hbar S_1(r,t) + \hbar^2 S_2(r,t) + ... \\ &= S_0(r,t) + \sum_i \hbar^i S_i(r,t)\,. \end{aligned} \tag{2.3.278}$$

In this expansion the terms from $\mathcal{O}(\hbar)$ onwards are treated as quantum corrections over the semiclassical value S_0. The equation that we want to solve can be obtained by plugging (2.3.278) to the Klein-Gordon equation (2.1.27) where only g^{tt} and g^{rr} which matter.

We do not include the quantum correction in our discussion here, as done in [55], so we neglect all terms in the equation which couple to $\hbar$. Hence the equation that we need to solve is

$$\left(\frac{\partial S_0}{\partial t}\right)^2 = f\,(r)\,g\,(r)\left(\frac{\partial S_0}{\partial r}\right)^2\,. \tag{2.3.279}$$

Since the metric (2.3.276) is stationary, it has a timelike Killing vector. Hence we look for solutions of (2.3.279) which behave as

$$S_0 = \omega t + \tilde{S}_0(r)\,, \tag{2.3.280}$$

where ω is the energy of particle. The solution is

$$S(r,t) = \omega t \pm \omega \int_{r_{in}}^{r_{out}} \frac{dr}{\sqrt{f(r)g(r)}} \tag{2.3.281}$$

where $r_{in} = r_H - \varepsilon$ and $r_{out} = r_H + \varepsilon$. The $+(-)$ sign in front of the integral indicates the corresponding solution of ingoing (outgoing) particle. In (2.3.281), the integration over r is a complex one, where we perform an integration over a semicircle contour in the upper half complex plane where the pole is at horizon radius r_H. The nature of both t and r coordinates as complex variables can be understood from the exchange of metric coefficient's sign for dt^2 and dr^2 components when we go from outside to inside of a black hole's horizon. In fact, the authors of [56] show that one need to do some transformations which involve complex variables to relate the Kruskal-Szekeres coordinates in the region of exterior and interior of a Schwarzschild black hole.

Therefore the ingoing and outgoing solutions of the Klein-Gordon equation (2.1.27) under the background metric (2.3.276) is given by exploiting (2.3.277) and (2.3.281)

$$\phi_{\pm} = \exp\left(-i\omega\left(t \pm \int\limits_{r_{in}}^{r_{out}} \frac{dr}{\sqrt{f(r)\, g(r)}}\right)\right) , \tag{2.3.282}$$

where ϕ_+ is the ingoing wave function and ϕ_- is the outgoing one. Therefore the corresponding ingoing and outgoing probabilities of the particle, P_+ and P_- respectively, are given by[12]

$$P_{\pm} = |\phi_{\pm}|^2 = \exp\Big[2\omega\Big(\text{Im }t \pm \text{Im}\int_0^r \frac{dr}{\sqrt{f(r)g(r)}}\Big)\Big] . \tag{2.3.283}$$

Since in the classical limit, i.e. $\hbar \to 0$, everything falls into the black hole, the ingoing probability P_+ has to be unity. Thus (2.3.283) leads to

$$\text{Im } t = -\text{Im}\int_{r_{in}}^{r_{out}} \frac{dr}{\sqrt{f(r)g(r)}} . \tag{2.3.284}$$

From (2.3.284) one can easily find that Im $t = -2\pi M$ for the Schwarzschild spacetime which is precisely the imaginary part of the transformation $t \to t - 2i\pi M$ when one connects the outside and inside regions of a horizon as shown in [56]. Therefore the probability of the outgoing particle is

$$P_- = \exp\Big[-4\omega\text{Im}\int_{r_{in}}^{r_{out}} \frac{dr}{\sqrt{f(r)g(r)}}\Big] . \tag{2.3.285}$$

Now using the relation between the emission (outgoing) and absorption (ingoing) probabilities [54, 58, 59]

$$P_- = \exp\Big(-\frac{\omega}{T_H}\Big)P_+ = \exp\Big(-\frac{\omega}{T_H}\Big) \tag{2.3.286}$$

we obtain the temperature of the black hole as

$$T_H = \frac{\hbar}{4}\Big(\text{Im}\int_{r_{in}}^{r_{out}} \frac{dr}{\sqrt{f(r)g(r)}}\Big)^{-1} . \tag{2.3.287}$$

[12]This complex paths method had been discussed by Landau [57] to describe tunnelling processes.

Using this expression and knowing the metric coefficients $f(r)$ and $g(r)$, one can easily find out the temperature of the corresponding black hole.

In the Kerr black hole discussion, again to avoid the dragging effect we restrict the angle $\theta = 0$, thus we have

$$f(r) = g(r) = \Big(\frac{(r-r_+)(r-r_-)}{r^2+a^2}\Big) \,. \tag{2.3.288}$$

Plugging this to the formula (2.3.287) recovers the Hawking temperature for Kerr black holes (2.3.273).

2.4 Waves Scattering from Black Holes

2.4.1 Schwarzschild Case

Approaching the quantum theory of black holes can be done by studying the scattered waves by a black hole semiclassically. One can be start the discussion from the simplest case, i.e. the massless scalar $\Phi(t,r,\theta,\phi)$, whose dynamics are governed by the equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0 \,. \tag{2.4.289}$$

Plugging the corresponding $g^{\mu\nu}$ for Schwarzschild spacetime yields the last equation changes to

$$\begin{aligned}\frac{\partial\Phi}{\partial t^2} &- \frac{2}{r}\left(1-\frac{M}{r}\right)\left(1-\frac{2M}{r}\right)\frac{\partial\Phi}{\partial r} - \left(1-\frac{2M}{r}\right)^2\frac{\partial^2\Phi}{\partial r^2} \\ &- \left(1-\frac{2M}{r}\right)\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) - \left(1-\frac{2M}{r}\right)\frac{\partial^2\Phi}{\partial\phi^2} = 0 \,. \end{aligned} \tag{2.4.290}$$

The ansatz of separable solution

$$\Phi\left(t,r,\theta,\phi\right) = e^{-i\omega t}R\left(r\right)Y_{lm}\left(\theta,\phi\right) \tag{2.4.291}$$

solves equation (2.4.290), if the radial part of (2.4.291) satisfies

$$\left(1-\frac{2M}{r}\right)^2\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\left(1-\frac{M}{r}\right)\left(1-\frac{2M}{r}\right)\frac{\partial R}{\partial r} + \left(\omega^2 - \left(1-\frac{2M}{r}\right)\frac{l\left(l+1\right)}{r^2}\right)R = 0 \,. \tag{2.4.292}$$

We now introduce a new function

$$U(r) = rR(r) \,, \tag{2.4.293}$$

and a new coordinate which is called the "tortoise" coordinate

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right) . \tag{2.4.294}$$

Consequently, the radial equation (2.4.292) transforms to

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V_{eff}(r)\right) U = 0 \,. \tag{2.4.295}$$

The last equation is known as the Regge-Wheeler equation. The effective potential V_{eff} in the above equation is given by

$$V_{eff} = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) . \tag{2.4.296}$$

Related to this "tortoise coordinate" r_*, one can check that

$$\frac{d}{dr} = \frac{d}{dr_*} \,. \tag{2.4.297}$$

Moreover, one can verify that the new coordinate r_* covers the exterior of black holes only,

$$r_* \to +\infty \qquad \text{as} \qquad r \to +\infty \,, \tag{2.4.298}$$

$$r_* \to -\infty \qquad \text{as} \qquad r \to 2M \,. \tag{2.4.299}$$

Interestingly, the effective potential V_{eff} goes to zero at $r = +\infty$ and $r = 2M$. Therefore, at $r = +\infty$ and $r = 2M$, the solution of (2.4.295) is

$$U \sim \exp(\pm i\omega r_*) \,. \tag{2.4.300}$$

Now consider the ingoing mode of (2.4.300) at $r_* \to -\infty$ where there is no another waves emerge from the black hole, where the related wave function can be read as

$$U \sim \exp(-i\omega r_*) \,. \tag{2.4.301}$$

As in the case of one dimensional wave scattering, knowing the condition of the wave at $r_* \to -\infty$ allows us to tell the form of the wave solution at $r_* \to +\infty$ from the wave equation. A portion of the ingoing wave (2.4.301) will be absorbed by the black hole, whose amplitude is denoted by A_{in}, and another portion will be reflected back to infinity with an amplitude A_{out}

$$U \sim A_{in} e^{-i\omega r_*} + A_{out} e^{i\omega r_*} \quad \text{as} \quad r_* \to +\infty\,. \tag{2.4.302}$$

It is shown in appendix C that the ingoing and outgoing amplitudes satisfy

$$\left|A_{in}^2\right| = 1 + \left|A_{out}^2\right|\,. \tag{2.4.303}$$

Furthermore, we can define the coefficients of reflection and transmission as

$$R = \frac{A_{out}}{A_{in}} \quad \text{and} \quad T = \frac{1}{A_{in}} \tag{2.4.304}$$

respectively which allow us to rewrite the equation (2.4.303) as

$$1 = |T|^2 + |R|^2\,. \tag{2.4.305}$$

Numerical computations can give us the values of $|T|$ and $|R|$. In the special case when the frequency of Φ is very high, i.e. $M\omega \gg 1$, we can see that the ω^2 term in the Regge-Wheeler equation (2.4.295) will be the dominant term in the vicinity of black hole $r = 2M$. In this situation, the effective potential V_{eff} can be neglected, hence we can conclude that the appearance of black holes does not really affect the propagation of a wave with very high frequencies. In the other hand, when the wave has a very low frequency, i.e. $M\omega \ll 1$, this wave is almost scattered back by the black hole potential barrier entirely.

2.4.2 Kerr case

In this subsection, we generalize the analysis that we have performed previously on the wave scattering of Schwarzschild black holes. After plugging the Kerr tensor metric $g^{\mu\nu}$ into (2.4.289), we have

$$\begin{aligned}\left(\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right)\frac{\partial\Phi}{\partial t^2} \quad &- \quad \frac{\partial}{\partial r}\left(\Delta\frac{\partial\Phi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) \\ &+ \quad \frac{2Mar}{\Delta}\frac{\partial^2\Phi}{\partial t\partial\phi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\Phi}{\partial\phi^2} = 0\,. \end{aligned} \tag{2.4.306}$$

The ansatz in solving this equation is quite different compared to the one we have used in Schwarzschild case. We know that the Kerr geometry has an axial symmetry, rather than the spherical symmetry as in the Schwarzschild discussions. Consider that we fix the rotation about the z-axis. Hence, the eigenvalue of operator $L_z = -i\partial_z$ to the spherical harmonics eigenfunction $Y_{lm}(\theta, \phi)$ is m,

$$-i\frac{\partial}{\partial\phi}Y_{lm}(\theta,\phi) = mY_{lm}(\theta,\phi)\,. \tag{2.4.307}$$

A general solution for the last equation is

$$Y_{lm}(\theta,\phi) = e^{im\phi}S_{lm}(\theta)\,. \tag{2.4.308}$$

The purely θ dependent function S_{lm} will satisfy the angular equation derived from (2.4.306). This line of thought yields the proper ansatz for wave function Φ to solve (2.4.307) is

$$\Phi(t,r,\theta,\phi) = e^{-i\omega t+im\phi}R(r)S(\theta) \tag{2.4.309}$$

where the superscripts l and m for $S(\theta)$ have been suppressed since these two variables are not the spacetime coordinates. These indices simply mean that the solution $S(\theta)$ has the m and l components, and clearly this function will be a function of l and m. The "tortoise coordinate" for Kerr spacetime is expected to give[13]

$$\frac{d}{dr_*} = \frac{\Delta}{r^2+a^2}\frac{d}{dr}\,. \tag{2.4.310}$$

Up to an integration constant, the following transformation satisfies (2.4.310)

$$r_* = r + M\log\Delta + \frac{2M^2\arctan\left(\frac{(r-M)}{\sqrt{a^2-M^2}}\right)}{\sqrt{a^2-M^2}}\,. \tag{2.4.311}$$

Analogous to the new function (2.4.293), for Kerr spacetime we introduce

$$U(r) = \sqrt{r^2+a^2}R(r)\,. \tag{2.4.312}$$

It follows from (2.4.310) and (2.4.312) that the radial equation in Kerr spacetime can be written as

$$\begin{aligned}\frac{d^2U}{dr_*^2} - \frac{d}{dr_*}\left(\frac{r\Delta U}{\left(r^2+a^2\right)^2}\right) &- \left\{\frac{r^2\Delta^2}{\left(r^2+a^2\right)^4} + \frac{\Delta\left(l\left(l+1\right)+2am\omega-a^2\omega^2\right)}{\left(r^2+a^2\right)^2}\right\}U \\ &+ \frac{\left(\left(r^2+a^2\right)\omega-am\right)^2}{\left(r^2+a^2\right)^2}U = 0\,. \end{aligned} \tag{2.4.313}$$

[13]For a technical reason to simplify the equation of motion.

Taking the limit $r \to r_+$ in (2.4.311). Hence, at the vicinity of Kerr black holes, i.e. $r \to r_+$, we may write the equation (2.4.313) approximately

$$\frac{d^2U}{dr_*^2} + \left(\omega - \frac{am}{2Mr_+}\right) = 0\,. \tag{2.4.314}$$

Obtaining the last equation can be done simply by setting $\Delta \to 0$ approximation in the equation (2.4.313). Consequently, we have the solution

$$U \sim e^{i\pm(\omega - m\Omega_H)}r_* \quad \text{as} \quad r \to r_+ \tag{2.4.315}$$

where the angular velocity of the black hole is $\Omega_H = a/2Mr_+$. A novel feature obtained here is the relation between coefficients of reflection and transmission is given by

$$1 - |R|^2 = \frac{\omega - m\Omega_H}{\omega}\,|T|^2\;. \tag{2.4.316}$$

Equation (2.4.316), whose derivation is given in appendix C, reduces to the one we have in Schwarzschild black hole discussion (2.4.305) when $\Omega_H = 0$, i.e. the non-rotating case. An interesting outcome from the last formula can be stated as follows. Consider that the ingoing wave modes are $\omega < m\Omega_H$, hence $R > 1$. In the other words, the reflected outgoing amplitude exceeds that of the incoming one. This wave is amplified by the scattering from black holes, and it mines some energy from the scattering process. This amplification is called the superradiance effect. This effect is not an exclusive property of scalar wave only, it also applies to higher spins wave scattering from a Kerr black hole.

Chapter 3

Conformal Field Theory and AdS/CFT

In early of its development, conformal field theory (CFT) was used in attempts to describe the critical behavior of systems at second order phase transitions. These systems possess a unique behavior where near the critical points they are invariant under the scale transformation, $x \to \lambda x$. It was found in many aspects that these systems at critical points have one-to-one correspondence with a two dimensional Euclidean quantum field theory. Polyakov in his seminal work [60] realized the importance of conformal invariance in understanding the critical behavior. This idea was developed later in [62] where it is shown that in two dimensions, the conformal algebra contains an infinite number of generators.

Conformal field theory in two dimensions (CFT_2) has become one of the main topics in theoretical high energy physics due to the fact that it is closely related to the string theory; the most promising candidate for the theory of everything. The development of CFT_2 started much earlier before the rise of string theories, mostly related to the model constructions in statistical mechanics. CFT_2 is established in [62] where some main ideas were found by Polyakov in [60]. Since then, CFT_2 are continuing to develop and new discoveries on the connections between CFT_2 and other aspects of theoretical physics are constantly found. Related to our interest on gravity in this thesis, a very close relation between CFT_2 and gravity is found by Brown and Henneaux [63] and after several decades it becomes a specific branch of studies in quantum gravity [66].

In this section, we review some concepts in CFT, and concentrate on CFT_2. We derive the Cardy formula for entropy of a system described by CFT_2, which plays important role in chapter 4. The application of CFT is given in the context of holography, by showing the AdS/CFT calculations for scalars and vectors two point functions. The main references for this chapter are [64, 65].

3.1 Quantum Field Theory and Poincare Symmetry

3.1.1 Fields and Poincare symmetry

In particle physics, a particle is defined as a quantum excitation associated with a field. The field is an entity that is defined over all spacetime. Electromagnetic field is the most familiar example. It is created by an electric charge, extends all over the space-time, and the quanta related to this field are known as the photon. The electromagnetic field is static when the charge as a source of this field is not moving. When this charge is vibrating, it produces an electromagnetic wave. Historically, Planck proposed that the blackbody radiation is a collection of countable quanta of this electromagnetic fields, namely photons. This is the origin of quantum theory.

In 19th century physics, waves can only be transmitted by using such a vibrating medium. Before the advent of relativity, people believed that electromagnetic fields are transmitted trough a medium which was called the ether. The second postulate of special relativity ruled out the prediction of the ether, and people began to consider that the vacuum has the property of producing all kinds of fields, whose excitations are observed as quanta or particles. Studying the particles and fields in this direction is a branch of theoretical physics known as quantum field theory (QFT). Quantum Field Theory (QFT) is the mathematical and conceptual framework for contemporary elementary particle physics. In a rather informal sense, QFT is an extension of quantum mechanics combined with special relativity. QFT deals with particles and fields, i.e. systems with an infinite number of degrees of freedom.

An example of QFT, and the most successful one, is quantum electrodynamics (QED). QED is the quantum theory of electromagnetism. The final and complete form of QED was proposed by Feynman, Schwinger, and Tomonaga independently, and gave them the 1965 physics Nobel prize. QED explains how matters interact with electromagnetic fields from the relativistic point of view. One of the successes of QED, for example, is the accurate prediction of the g-factor[1] of muon [61],

$$\left(\frac{g-2}{2}\right)_{\mu} = (116584718.10 \pm 0.16) \times 10^{-11}\,. \tag{3.1.1}$$

This number is supported by the experimental verification [61],

$$\left(\frac{g-2}{2}\right)_{\mu} = (116592080 \pm 54) \times 10^{-11}\,. \tag{3.1.2}$$

This is one example, among many others, that QFT gives us a picture on how the universe works in a great accuracy.

Symmetry is an ingredient in constructing a QFT. Symmetry is the cornerstone of contemporary physics. It plays an important role in understanding the fundamental interactions. As a QFT, QED possesses the Poincare symmetry, i.e. the translation and Lorentz symmetries. A theory which consists of field $\phi(x)$ is said to have the translation symmetry

$$x^{\alpha\prime} = x^{\alpha} + a^{\alpha}\,, \tag{3.1.3}$$

which leads to a field transformation

$$\phi'(x) = U(a)\phi(x)\,, \tag{3.1.4}$$

leaves the theory to be invariant. The operator $U(a) = exp(ia^{\mu}p_{\mu})$ is an unitary operator for a finite translation, and the momentum operator $p_{\mu} = -i\partial_{\mu}$. To verify this operator, assume that the translation (3.1.3) is infinitesimal, i.e. $a \ll 1$, hence the translation operator $U(a)$ can be approximated up to the first order in a,

$$U(a) = 1 + a^{\mu}\partial_{\mu}\,. \tag{3.1.5}$$

Applying this operator to x^{ν} gives us

$$\tilde{U}\,(a)\,x^{\nu} = (1 + a^{\mu}\partial_{\mu})\,x^{\nu} = x^{\nu} + a^{\nu} = x^{\nu\prime}\,, \tag{3.1.6}$$

which is the translation operation.

[1]Without taking the relativistic effect into account, g-factor is given by the formula $g = 2m\omega_L/qB$, where ω_L is Larmor frequency, B is magnetic field, m is mass of particle, and q is particle's electric charge [61].

Lorentz transformation deals with an anti-symmetric tensor $\omega^{\mu\nu}$ as the transformation parameter,

$$x^\mu \to x^{\mu\prime} = x^\mu + \omega^\mu_\nu x^\nu \, . \tag{3.1.7}$$

The field transformation related to the coordinate change in (3.1.7) is

$$\exp\left(\frac{i}{2}\omega^{\mu\nu} J_{\mu\nu}\right) \tag{3.1.8}$$

where

$$J_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \, . \tag{3.1.9}$$

Hence we have

$$\delta x^\alpha = \frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}\right)^\alpha_\beta x^\beta \, . \tag{3.1.10}$$

Now we would like to see the generators associated with this kind of transformation. Up to the first order of δx, the variation of the fields can be read as

$$\delta\phi = \phi'\left(x' - \delta x\right) - \phi\left(x\right) = -\delta x^\alpha \partial_\alpha \phi\left(x\right) \, . \tag{3.1.11}$$

Plugging the "shift" of position (3.1.10) into the last equation, we have

$$\delta\phi = \frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}\right)^\alpha_\beta x^\beta \partial_\alpha \phi\left(x\right) \equiv \frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}\phi\left(x\right) \, , \tag{3.1.12}$$

where

$$L^{\mu\nu} = -\left(J^{\mu\nu}\right)^\alpha_\beta x^\beta \partial_\alpha = \left(x^\mu p^\nu - x^\nu p^\mu\right) \, . \tag{3.1.13}$$

The operator p^μ above is the momentum operator, $p^\mu = -i\partial^\mu$. One can check that the the operator (3.1.13) satisfies $so(3,1)$ Lie algebra,

$$\left[L^{\mu\nu}, L^{\rho\sigma}\right] = i\left(\eta^{\nu\rho}L^{\mu\sigma} - \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} + \eta^{\mu\sigma}L^{\nu\rho}\right) \, . \tag{3.1.14}$$

The numbers 3 and 1 in the parentheses of this group's name are related to the signature of the spacetime under consideration, $(-,+,+,+)$, i.e. 3 positives and 1 negative. If only we are working in the four dimensional Euclidean space, so instead of using the Minkowski metric $\eta^{\mu\nu}$ in (3.1.13) we have the delta Kronecker $\delta^{\mu\nu}$, then the group symmetry is $SO(4)$.

3.1.2 Squared Casimirs in Poincare group

We have understood that the Poincare group is built of momentum and Lorentz generators, p_μ and $L_{\mu\nu}$ respectively. In quantum mechanics we know that a quantity which is invariant under a transformation commutes with the generator of the corresponding transformation. For example the Heisenberg equation of motion,

$$i\frac{d\mathcal{O}}{dt} = [\mathcal{O}, H]$$

where H is Hamiltonian of a system, and $\mathcal{O}$ is a quantum mechanical operator. We understand that the Hamiltonian H is the generator of translation in time direction. The commutativity between $\mathcal{O}$ and H reflects that $\mathcal{O}$ is a conserved or invariant quantity in time evolution.

In Poincare group, the generators are responsible not only for the translation in time, but also the translations in space, and the rotational and boosts transformations as well. A quantity that commutes with all Poincare group generators is invariant under all of the transformations contained in the Poincare group.

A squared Casimir operator, or squared Casimir for short, of a group is constructed by the group's generators and commutes with all generators in the group. For Poincare group, there are two squared Casimirs. The first one is squared momentum, $p^2 = p^\mu p_\mu$. It is easy to check that $[p^2, p_\mu] = 0$. The commutation between p^2 and the generator of Lorentz transformation $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ can be computed as

$$\left[p^2, L_{\mu\nu}\right] = -i\left[\partial^\alpha\partial_\alpha, (x_\mu\partial_\nu - x_\nu\partial_\mu)\right] = -i\left[\partial^\alpha, (x_\mu\partial_\nu - x_\nu\partial_\mu)\right]\partial_\alpha - i\partial^\alpha\left[\partial_\alpha, (x_\mu\partial_\nu - x_\nu\partial_\mu)\right]$$

$$= -i\left[(\eta_{\alpha\mu}\partial_\nu - \eta_{\alpha\nu}\partial_\mu)\partial^\alpha + \partial^\alpha(\eta_{\alpha\mu}\partial_\nu - \eta_{\alpha\nu}\partial_\mu)\right] = 0\,.$$

For massive scalars, the invariance of p^2 under all Poincare transformations can be understood as the invariance of mass since the eigenvalue of p^2 for the scalar wave function $\phi(x)$ is m^2.

The second squared Casimir in Poincare group is the squared Pauli-Lubanski operator, $W^2 = W^\mu W_\mu$, where the Pauli-Lubanski operator is defined as

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}p_\nu L_{\rho\sigma}\,. \qquad (3.1.15)$$

For massive particle at rest, the eigenvalue of this squared Pauli-Lubanski operator is $-m^2 s(s+1)$. We notice from this operator, one can identify the mass and spin properties of a particle. Furthermore, these two properties are expected to be invariant under translations and Lorentz transformations. Algebraically it means this squared Pauli-Lubanski operator must commute with all generators in Poincare group.

3.2 CFT in D dimensions

3.2.1 Conformal Group and Algebra

Conformal field theory can be considered as a class of quantum field theory with conformal symmetry. The conformal symmetry preserves the form of the metric tensor to an arbitrary scale factor $\Lambda(x)$

$$g_{\mu\nu}(x) \rightarrow \Lambda^2(x)\, g_{\mu\nu}(x) \,, \tag{3.2.16}$$

where μ and ν indices run from 0 to $(D-1)$. We assign $\mu, \nu = 0$ to represent time coordinate. The set of transformations that preserves (3.2.16) in Minkowski spacetime[2] are

$$\text{Translation} \quad : \quad x^\mu \rightarrow x^\mu + a^\mu \,, \tag{3.2.17}$$

$$\text{Rotation} \quad : \quad x^\mu \rightarrow M^\mu_\nu x^\nu \,, \tag{3.2.18}$$

$$\text{Dilation} \quad : \quad x^\mu \rightarrow \lambda x^\mu \,, \tag{3.2.19}$$

$$\text{Special Conformal Transformation (SCT)} \quad : \quad x^\mu \rightarrow \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \,, \tag{3.2.20}$$

where a^μ and b^μ are constant vectors, λ is a scaling parameter, and M^μ_ν is the Lorentz transformation matrix. We notice that (3.2.17) together with (3.2.18) are just the Poincare transformation. We are already familiar with this transformation in the discussions of relativistic field theory. The expression (3.2.19) is the dilation transformation, which is also known as the scaling transformation. This kind of transformation is not too bizarre, since we can check that the free Maxwell equations[3]

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2} \tag{3.2.21}$$

[2]Minkowski spacetime is associated with metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.
[3]In the absence of electric charge and current.

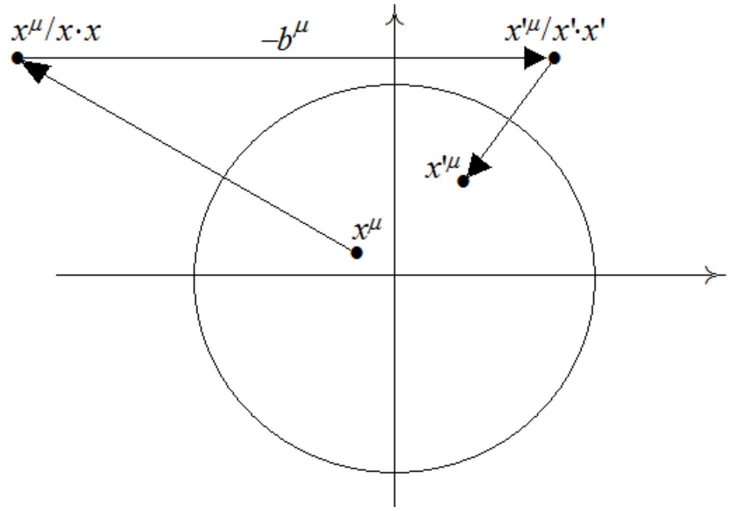

Figure 3.1: The illustration of a special conformal transformation (3.2.20).

are invariant under the dilation transformation (3.2.19). The vector fields $\vec{E}$ and $\vec{B}$ in the last equation are the electric and magnetic fields respectively. Perhaps, the most peculiar transformation that build the conformal symmetries is the equation (3.2.20). Figure 3.1 shows how this special conformal transformation works. Transformation (3.2.20) shifts the point x^μ to x'^μ where between these two points there are two inversions and one translation.

For each of the transformations in conformal symmetry, there is an associated generator. The followings are the generators of the symmetries in conformal field theory,

$$\text{Translation} \quad : \quad \mathbf{P}_\mu = -i\partial_\mu, \tag{3.2.22}$$

$$\text{Rotation} \quad : \quad \mathbf{L}_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), \tag{3.2.23}$$

$$\text{Dilation} \quad : \quad \mathbf{D} = -ix \cdot \partial, \tag{3.2.24}$$

$$\text{SCT} \quad : \quad \mathbf{K}_\mu = -i\left(2x_\mu x \cdot \partial - x^2\partial_\mu\right). \tag{3.2.25}$$

We have seen how the generators for translation and Lorentz transformation work in the previous section. Now we will verify the dilation and SCT generators. We set that the dilation parameter is α, hence the corresponding finite dilation transformation can be performed by using the operator

$$U_D = exp(i\alpha\mathbf{D}), \tag{3.2.26}$$

hence for the infinitesimal α we can expand the exponentiation above up to the first order,

$$U_D x^\sigma \approx (1 + \alpha x \cdot \partial) x^\sigma = (1 + \alpha) x^\sigma = \lambda x^\sigma \,. \tag{3.2.27}$$

In the last equation, the dilation parameter λ has been identified as $\alpha + 1$. Hence, we can see that $\mathbf{D}$ in (3.2.24) is really an operator that generates dilation. The generator for SCT can be verified by using similar way, where the finite SCT can be written as an exponentiation, $U_{SCT} = exp(ib \cdot \mathbf{K})$. Here b^μ is the SCT parameter. The infinitesimal SCT then can be read as

$$\begin{aligned} U_{SCT} x^\sigma \approx \left(1 + i b^\mu \mathbf{K}_\mu\right) x^\sigma &= x^\sigma + b^\mu \left(2 x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) x^\sigma \\ &= x^\sigma + 2 b^\mu x_\mu x^\sigma - x^2 b^\sigma \,. \end{aligned} \tag{3.2.28}$$

An algebra of a group is represented by the commutation relations between each of the group generators. The conformal group generators are given in (3.2.22) - (3.2.25), hence the algebra of conformal group basically will be constructed by all of these generators. The commutations between generators in (3.2.22) - (3.2.25) are

$$[\mathbf{D}, \mathbf{P}_\mu] = i \mathbf{P}_\mu \quad , \quad [\mathbf{D}, \mathbf{K}_\mu] = -i \mathbf{K}_\mu \,, \tag{3.2.29}$$

$$[\mathbf{K}_\mu, \mathbf{P}_\nu] = 2i \left(\eta_{\mu\nu} \mathbf{D} - \mathbf{L}_{\mu\nu}\right) , \tag{3.2.30}$$

$$[\mathbf{K}_\sigma, \mathbf{L}_{\mu\nu}] = 2i \left(\eta_{\sigma\mu} \mathbf{K}_\nu - \eta_{\sigma\nu} \mathbf{K}_\mu\right) \,, \tag{3.2.31}$$

$$[\mathbf{P}_\sigma, \mathbf{L}_{\mu\nu}] = 2i \left(\eta_{\sigma\mu} \mathbf{P}_\nu - \eta_{\sigma\nu} \mathbf{P}_\mu\right) \,, \tag{3.2.32}$$

$$[\mathbf{L}_{\mu\nu}, \mathbf{L}_{\rho\sigma}] = i \left(\eta_{\nu\rho} \mathbf{L}_{\mu\sigma} + \eta_{\mu\sigma} \mathbf{L}_{\nu\rho} - \eta_{\mu\rho} \mathbf{L}_{\nu\sigma} - \eta_{\nu\sigma} \mathbf{L}_{\mu\rho}\right) \,. \tag{3.2.33}$$

It is not easy to see what kind of group algebra associated with conformal group by looking at (3.2.29) - (3.2.33). However, in four dimensional Minskowski spacetime, we know that the equation (3.2.33) is just the $so(3,1)$ algebra.

Nevertheless, to see what kind of the algebra for conformal group in D dimension, we may define generators $\mathbf{J}_{AB}$ which is antisymmetric in A and B, i.e. $\mathbf{J}_{AB} = -\mathbf{J}_{BA}$. The indices A and B have two extra numbers compared to μ or ν, i.e. $A, B \;=\; -2, -1, 0, 1, ..., (D-1)$. The relations between J_{AB} with the generators in (3.2.22) - (3.2.25) can be written as

$$\mathbf{J}_{-2,-1} \equiv \mathbf{D} \quad , \quad \mathbf{J}_{-1,\mu} \equiv \frac{1}{2} \left(\mathbf{P}_\mu + \mathbf{K}_\mu\right) \quad , \quad \mathbf{J}_{-2,\mu} \equiv \left(\mathbf{P}_\mu - \mathbf{K}_\mu\right) \quad , \quad \mathbf{J}_{\mu\nu} \equiv \mathbf{L}_{\mu\nu} \,. \tag{3.2.34}$$

Thus from the algebra that we already have in (3.2.29) - (3.2.33), we can show the corresponding algebra for $\mathbf{J}_{AB}$ is

$$[\mathbf{J}_{AB}, \mathbf{J}_{CD}] = i\left(\eta_{AD}\mathbf{J}_{BC} + \eta_{BC}\mathbf{J}_{AD} - \eta_{AC}\mathbf{J}_{BD} - \eta_{BD}\mathbf{J}_{AC}\right) . \tag{3.2.35}$$

We observe that the equation (3.2.35) is very similar to (3.2.33). This signs that they are the same type of group algebra, but different in "size". We know that (3.2.33) is just the Lorentz group which in $D = 4$ is isomorphic to $SO(3,1)$. Therefore the algebra (3.2.35) is $so(D+1,1)$ for the metric tensor $\eta_{AB} = \text{diag}(-1,+1,...,+1)$.

After discussing the group algebra in conformal symmetry, we would like to see the infinitesimal coordinate transformation associated with this symmetry. Let us start with a local[4] infinitesimal transformation,

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x) . \tag{3.2.36}$$

We will see that the conformal transformation in D dimensions can constrain the general form of $\varepsilon^\mu(x)$. Under (3.2.36), a metric tensor $g_{\mu\nu}$ transforms as

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \partial_\mu\varepsilon_\nu(x) + \partial_\nu\varepsilon_\mu(x) . \tag{3.2.37}$$

Subsequently, from (3.2.16) we can write

$$\partial_\mu\varepsilon_\nu(x) + \partial_\nu\varepsilon_\mu(x) = \beta(x)\, g_{\mu\nu} , \tag{3.2.38}$$

which yields the relation $\Lambda^2(x) = 1+\beta(x)$. The equation (3.2.38) can be simplified by taking the trace on both side. This is done by multiplying both side with $g^{\mu\nu}$ which yields

$$\frac{2}{D}\partial\cdot\varepsilon = \beta(x) . \tag{3.2.39}$$

In the last equation we have used the relation $g_{\mu\nu}g^{\mu\nu} = D$ where D is the number of spacetime dimensions.

Consider the Euclidean spacetime flat spacetime in D dimensions, then the corresponding metric tensor has the form $g_{\mu\nu} = \text{diag}\underbrace{(1,...,1)}_{D-entries}$ which is simply the Kronecker delta $\delta_{\mu\nu}$. In this case we have

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = \frac{2}{D}\partial\cdot\varepsilon\delta_{\mu\nu} , \tag{3.2.40}$$

[4] Here local means coordinate dependent.

and furthermore after applying ∂^{μ} and ∂_{μ} successively to (3.2.40) and exchanging the indices μ and ν, we can obtain

$$\left(\left(D-2\right)\partial_{\mu}\partial_{\nu}+\delta_{\mu\nu}\Box\right)\partial\cdot\varepsilon=0\,. \tag{3.2.41}$$

The last expression is quite important since it tells us that $\epsilon^{\mu}\left(x\right)$ depends on x at most in quadratic term. The number of generators that build a conformal group in D dimension can be calculated as

$$N=\frac{\left(D+1\right)\left(D+2\right)}{2}\,. \tag{3.2.42}$$

3.2.2 Energy-Momentum Tensor in CFT

Related to the coordinate transformation (3.2.36), the tensor energy momentum $T^{\mu\nu}$ can be defined as

$$\delta S=\int d^{D}xT^{\mu\nu}\partial_{\mu}\varepsilon_{\nu}=\frac{1}{2}\int d^{D}xT^{\mu\nu}\left(\partial_{\mu}\varepsilon_{\nu}+\partial_{\nu}\varepsilon_{\mu}\right). \tag{3.2.43}$$

The symmetric behavior of energy-momentum tensor $T_{\mu\nu}$ under its index permutation allow us to perform the last step in (3.2.43) where the factor half appears. The integrand of (3.2.43) can be manipulated by using (3.2.40) to be

$$\frac{1}{2}T^{\mu\nu}\left(\partial_{\mu}\varepsilon_{\nu}+\partial_{\nu}\varepsilon_{\mu}\right)=\frac{1}{D}T^{\mu\nu}g_{\mu\nu}\left(\partial\cdot\varepsilon\right)=\frac{1}{D}T^{\mu}_{\mu}\left(\partial\cdot\varepsilon\right)\,.$$

The invariance of action, $\delta S=0$, for the non-trivial ε yields the tensor energy momentum $T^{\mu\nu}$ must be traceless. This property will play an important role in the next sections.

3.3 Conformal Field Theory in Two Dimesions (CFT$_2$)

3.3.1 Conformal group in 2 dimension

CFT in two dimensions are special. They have an infinite numbers of symmetry generators [62]. In this subsection we discuss in detail the coformal transformation for 2 dimensional spacetime. From (3.2.40) where the indices μ and ν run only from 0 to 1, we have

$$\partial_{0}\varepsilon_{0}=\partial_{1}\varepsilon_{1}\quad,\quad\partial_{0}\varepsilon_{1}=-\partial_{1}\varepsilon_{0}\,. \tag{3.3.44}$$

The two equations in (3.3.44) remind us on the Cauchy-Riemann equations in complex analysis. Hence we could introduce the following identifications

$$z = x^0 + ix^1 \quad , \quad \bar{z} = x^0 - ix^1, \tag{3.3.45}$$

together with the corresponding partial derivatives

$$\partial_1 = (\partial_z + \partial_{\bar{z}}) \quad , \quad \partial_2 = i\left(\partial_z - \partial_{\bar{z}}\right) . \tag{3.3.46}$$

In this complex coordinates z and $\bar{z}$, the conformal transformations are

$$z \to z' = f(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}) . \tag{3.3.47}$$

Furthermore one can check under transformation (3.3.47) the metric changes as

$$ds^2 = dzd\bar{z} \to \frac{df}{dz}\frac{d\bar{f}}{d\bar{z}}dzd\bar{z} = \left|\frac{df}{dz}\right|^2 dzd\bar{z}\,. \tag{3.3.48}$$

According to (3.2.42), for $D = 2$ there should be only six generators for a conformal group in 2 dimensions. Nevertheless, it turns out not to be that simple. We already know that in the original x coordinates, the expansion of ϵ in term of x can be at most in quadratic form. The existence of (3.3.44) in 2 dimensional conformal transformation allows us to perform the following expansion

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n \left(-z^{n+1}\right) \quad \text{and} \quad \bar{\varepsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right) \tag{3.3.49}$$

which is known as Taylor-Laurent expansion of $\varepsilon(z)$ at the origin. Then to the first order in ε, the difference of a scalar function $\phi(z, \bar{z})$ before and after transformation is

$$\phi(z', \bar{z}') - \phi(z, \bar{z}) = \left(\varepsilon(z)\,\partial + \bar{\varepsilon}(\bar{z})\,\bar{\partial}\right)\phi(z, \bar{z}) = \sum_{n=-\infty}^{\infty} \left(\varepsilon_n l_n + \bar{\varepsilon}_n \bar{l}_n\right)\phi(z, \bar{z}) \ . \tag{3.3.50}$$

In getting the last expression we have defined the generators

$$l_n = -z^{n+1}\partial \quad , \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}\,, \tag{3.3.51}$$

with $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. One can check that the generators in (3.3.51) obey the following commutation relations

$$[l_n, l_m] = (n - m)\, l_{n+m} \quad , \quad \left[\bar{l}_n, \bar{l}_m\right] = (n - m)\, \bar{l}_{n+m} \quad , \quad \left[l_n, \bar{l}_m\right] = 0\,. \tag{3.3.52}$$

The first commutation relation in (3.3.52) is one copy of the Witt algebra, where the second one is clearly another copy. At this point a question may arise, how can the conformal transformations in two dimensions are constructed by an infinite number of symmetry generators instead of six as it should be from the formula (3.2.42). Now let us just examine a copy of the Witt algebra generators $\{l_n\}$. It is easy to notice that there are some singular l_n's at $z = 0$. We can check that the collection of l_n's which are singular are those for $n \leq -1$. Furthermore by performing coordinate changing $z = -w^{-1}$, we have

$$l_n = -\left(-\frac{1}{w}\right)^{n-1} \partial_w \tag{3.3.53}$$

which is not singular for $n \leq +1$. Therefore, we conclude that the two dimensional conformal transformations are globally defined only for the symmetry generators $\{l_{-1}, l_0, l_{+1}\}$. The Witt algebra for these three generators is nothing but the $sl\,(2, \mathbb{C})$ algebra

$$[l_{\pm 1}, l_0] = \mp l_{\pm 1} \quad , \quad [l_+, l_-] = 2l_0\,. \tag{3.3.54}$$

That is why the presence of $SL\,(2, \mathbb{C})$ symmetry, or its subgroup $SL\,(2, \mathbb{R})$, hints the existence of two dimensional conformal symmetry.

3.3.2 Primary fields and correlation functions in CFT$_2$

A primary field $\phi\,(x)$ in 2 dimensional spacetime with planar spin s and conformal dimension Δ, under conformal transformations $z \to w\,(z)$ and $\bar{z} \to \bar{w}\,(\bar{z})$, transforms as

$$\phi\,(z, \bar{z}) \to \phi'\,(w, \bar{w}) = \left|\frac{\partial w}{\partial z}\right|^{-h} \left|\frac{\partial \bar{w}}{\partial \bar{z}}\right|^{-\bar{h}} \phi\,(z, \bar{z})\;. \tag{3.3.55}$$

In (3.3.55) we have defined the holomorphic (right) and antiholomorphic (left) conformal dimensions[5]

$$h = \frac{1}{2}\,(\Delta + s) \quad , \quad \bar{h} = \frac{1}{2}\,(\Delta - s)\;. \tag{3.3.56}$$

If the fields ϕ transform as in (3.3.55) only for $w\,(z) \in SL\,(2, \mathbb{C})/Z_2$, then we name this type of ϕ as quasi-primary fields. We may notice that all primary fields are quasi-primary ones, but not vice versa.

[5] In the next chapter, the holomorphic and antiholomorphic conformal dimensions are also called the left and right conformal dimensions respectively.

Now we will determine the 2-point and 3-point functions in 2 dimensional CFT solely driven by the conformal symmetry itself. From the scale invariance $z \to \lambda z$ we have[6]

$$\langle \phi_1(z_1)\,\phi_2(z_2)\rangle = \lambda^{h_1+h_2}\,\langle \phi_1(\lambda z_1)\,\phi_2(\lambda z_2)\rangle \ . \tag{3.3.57}$$

Dicatated by translation and rotating symmetries, the 2-point function must be in the form

$$\langle \phi_1(z_1)\,\phi_2(z_2)\rangle = f(|z_1 - z_2|) \ . \tag{3.3.58}$$

Furthermore by combining the conditions in (3.3.57) and (3.3.58), we could write the 2-point function as

$$\langle \phi_1(z_1)\,\phi_2(z_2)\rangle = \frac{C_{12}}{|z_1 - z_2|^{h_1+h_2}} \ . \tag{3.3.59}$$

This result for 2-point function can be extended to the D dimensional case,

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{h_1+h_2}} \ , \tag{3.3.60}$$

where x_1 and x_2 denote the two points in D dimensional spacetime.

Since the special conformal transformation is simply an inversion that is followed by a translation, then the last constraint that can fix our 2-point function is the necessity to be invariant under an inversion $z \to -z^{-1}$. Finally the 2-point function for 2 dimensional conformal field theory can be shown to have the following form

$$\langle \phi_1(z_1)\,\phi_2(z_2)\rangle = \begin{cases} C_{12}\,|z_1 - z_2|^{-2h_1} & \text{for} \quad h_1 = h_2 \\ 0 & \text{for} \quad h_1 \neq h_2 \end{cases} \ . \tag{3.3.61}$$

In the case of ϕ depends on both z and $\bar{z}$, it would be straightforward to find the corresponding 2-point function as

$$\langle \phi_1(z_1,\bar{z}_1)\,\phi_2(z_2,\bar{z}_2)\rangle = C_{12}\,|z_1 - z_2|^{-2h}\,|\bar{z}_1 - \bar{z}_2|^{-2\bar{h}} \quad \text{if} \quad \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} \ . \tag{3.3.62}$$

Previously we have encountered the Witt algebra (3.3.52), which is the commutation relations between generators of conformal transformations in 2 dimensions. Furthermore this algebra can be extended by introducing the central charge quantity. This extended algebra

[6] Consider that we are dealing with holomorphic dependent field only. Nevertheless the discussion will be the same for the anti holomorphic dependence where in addition to h we also have to add $\bar{h}$.

is called the Virasoro algebra and will reduce to the Witt algebra for the vanishing central charge. There is a tricky way of deriving the Virasoro algebra, which is by looking at some constraints given by the generator's indices and the Jacobi identity between generators. The general form of the Virasoro algebra can be expressed in the following way:

$$[L_m, L_n] = (m-n)\, L_{m+n} + cp\,(m,n) \;, \tag{3.3.63}$$

where $p\,(m,n)$ is a function that depends on indices m and $n \in \mathbb{Z}$. We could notice from (3.3.63) that $p\,(m,n)$ must be antisymmetric in m and n exchange, i.e. $p\,(m,n) = -p\,(n,m)$. Then one can manage such that $p\,(1,-1) = 0$ and $p\,(n,0) = 0$. This can be seen by redefining

$$\tilde{L}_n \equiv L_n + \frac{cp\,(n,0)}{n} \;\; \text{for} \;\; n \neq 0 \;\; , \;\; \tilde{L}_0 \equiv L_0 + \frac{cp\,(1,-1)}{2} \,. \tag{3.3.64}$$

The next two steps to determine the exact form of $p\,(m,n)$ are checking the following Jacobi identities

$$[[L_m, L_n]\,, L_0] + [[L_n, L_0]\,, L_m] + [[L_0, L_m]\,, L_n] = 0\,, \tag{3.3.65}$$

$$[[L_{-n+1}, L_n]\,, L_{-1}] + [[L_n, L_{-1}]\,, L_{-n+1}] + [[L_{-1}, L_{-n+1}]\,, L_n] = 0\,. \tag{3.3.66}$$

From (3.3.65) we can see that only if $n \neq -m$ which provides a non-vanishing $p\,(n,m)$. Then by combining this fact with the previous ones that are supported by (3.3.64), we conclude that $p\,(m,-m)$ will be non-zero only for $|m| \geq 2$. Finally, by normalizing[7] $p\,(2,-2) = 1/2$, (3.3.66) gives us

$$p\,(m,-m) = \frac{1}{2}\begin{pmatrix} m+1 \\ 3 \end{pmatrix} = \frac{m\,(m^2-1)}{12}\,. \tag{3.3.67}$$

Finally the Virasoro algebra can be written explicitly as

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}\,. \tag{3.3.68}$$

3.3.3 Energy-momentum tensor for CFT$_2$

As a tensor, the covariant (lower indices) energy-momentum tensor is transformed as

$$T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}\,, \tag{3.3.69}$$

[7] This normalisation is taken to provide us the central charge of free boson to be unity.

for the coordinated change from x^μ to x'^μ. We focus on the two dimensional CFT, and switch from the two dimensional Euclidean coordinate x^0 and x^1 to a complex plane with the coordinates z and $\bar{z}$, where the relation between these coordinates are

$$x^0 = \frac{1}{2}(z+\bar{z}) \quad , \quad x^1 = -\frac{i}{2}(z-\bar{z}) \, . \tag{3.3.70}$$

Hence, the energy momentum tensor components in Euclidean and complex plane has relations

$$T_{zz} = \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial z}T_{00} + 2\frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial z}T_{01} + \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial z}T_{11} = \frac{1}{4}(T_{00} - 2iT_{01} - T_{11}) \, , \tag{3.3.71}$$

$$T_{\bar{z}\bar{z}} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial \bar{z}}T_{00} + 2\frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{11} = \frac{1}{4}(T_{00} + 2iT_{01} - T_{11}) \, , \tag{3.3.72}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial z}T_{00} + \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^1}{\partial z}T_{01} + \frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial x^1}{\partial z}T_{11} = \frac{1}{4}(T_{00} + T_{11}) = 0 \, . \tag{3.3.73}$$

The last equation is understood to be vanished due to the fact that the energy-momentum tensor in CFT has a vanishing trace. From this last equation we can also conclude that $T_{00} = -T_{11}$, thus from (3.3.71) and (3.3.72) we can obtain

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10}) \quad , \quad T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{00} + iT_{10}) \, . \tag{3.3.74}$$

The condition $\partial^\mu T_{\mu\nu} = 0$ obtained from translational invariance can be read in more detail

$$\frac{\partial T_{00}}{\partial x^0} = -\frac{\partial T_{10}}{\partial x^1} \quad , \quad \frac{\partial T_{01}}{\partial x^0} = -\frac{\partial T_{11}}{\partial x^1} \, . \tag{3.3.75}$$

Now we would like to see the dependence of $T_{zz}(z,\bar{z})$ and $T_{\bar{z}\bar{z}}(z,\bar{z})$ with respect to z and $\bar{z}$. At the moment, we consider that both functions depend on z and $\bar{z}$. One can verify that

$$\frac{\partial T_{zz}}{\partial z} \neq 0 \quad , \quad \frac{\partial T_{\bar{z}\bar{z}}}{\partial \bar{z}} \neq 0 \, . \tag{3.3.76}$$

However, it is interesting to find that

$$\begin{aligned} \frac{\partial T_{zz}}{\partial \bar{z}} &= \frac{1}{4}\left(\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial x^1}\right)(T_{00} - iT_{10}) = \frac{1}{4}\left(\frac{\partial T_{00}}{\partial x^0} + i\frac{\partial T_{00}}{\partial x^1} - i\frac{\partial T_{10}}{\partial x^0} + \frac{\partial T_{10}}{\partial x^1}\right) \\ &= \frac{1}{4}\left(\underbrace{\frac{\partial T_{00}}{\partial x^0} + \frac{\partial T_{01}}{\partial x^1}}_{\partial^\mu T_{\mu 0}=0} - i\left(\underbrace{\frac{\partial T_{10}}{\partial x^0} + \frac{\partial T_{11}}{\partial x^1}}_{\partial^\mu T_{\mu 1}=0}\right)\right) = 0 \, . \end{aligned} \tag{3.3.77}$$

$$\frac{\partial T_{\bar{z}\bar{z}}}{\partial z} = \frac{1}{4}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right)(T_{00} + iT_{10}) = \frac{1}{4}\left(\frac{\partial T_{00}}{\partial x^0} - i\frac{\partial T_{00}}{\partial x^1} + i\frac{\partial T_{10}}{\partial x^0} + \frac{\partial T_{10}}{\partial x^1}\right)$$

$$= \frac{1}{4}\left(\underbrace{\frac{\partial T_{00}}{\partial x^0} + \frac{\partial T_{01}}{\partial x^1}}_{\partial^\mu T_{\mu 0}=0} + i\left(\underbrace{\frac{\partial T_{10}}{\partial x^0} + \frac{\partial T_{11}}{\partial x^1}}_{\partial^\mu T_{\mu 1}=0}\right)\right) = 0\,. \tag{3.3.78}$$

Therefore, from the last two results (3.3.77) and (3.3.78) we can conclude that T_{zz} is holomorphic function, and $T_{\bar{z}\bar{z}}$ is anti-holomorphic function,

$$T_{zz}(z,\bar{z}) \equiv T(z) \quad , \quad T_{\bar{z}\bar{z}}(z,\bar{z}) \equiv \bar{T}(\bar{z})\,. \tag{3.3.79}$$

3.3.4 CFT on the torus and Partition Function

Before we discuss some properties of a CFT on the torus, we analyze some behaviors of a CFT on an infinite cylinder. Recall that we are working on the CFT_2 where the coordinates are x^0 (time) and x^1 (space). Let us now define w as the coordinate on a cylinder

$$w = x^0 + ix^1\,, \tag{3.3.80}$$

with identification

$$w \sim w + 2i\pi\,. \tag{3.3.81}$$

Let the time coordinate x^0 has an infinite range. The identification (3.3.81) can be viewed as a gluing of the spatial coordinate edges in our infinite strip as illustrated in figure 3.2.

The mapping between a cylinder with coordinate w and a complex plane with coordinate z is given by

$$z = e^w = e^{x^0 + ix^1}\,. \tag{3.3.82}$$

The initial time translation $x^0 \to x^0 + a$ in our new complex plane is identified as a complex dilation $z \to e^a z$, while the initial space translation $x^1 \to x^1 + b$ becomes a rotation $z \to (\cos b + i \sin b)z$. The illustration of this mapping is given in figure 3.3.

Under the mapping (3.3), a primary field transforms as

$$\phi_{\text{cyl.}}(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) = z^{-h}\bar{z}^{-\bar{h}}\phi(z,\bar{z})\,. \tag{3.3.83}$$

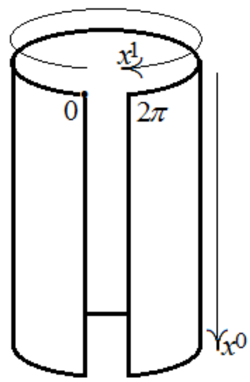

Figure 3.2: A cartoon illustration to construct a cylinder from a two dimensional strip. In the figure we have considered that the edges of our x^1 coordinate are $x^1 = 0$ and $x^1 = 2\pi$.

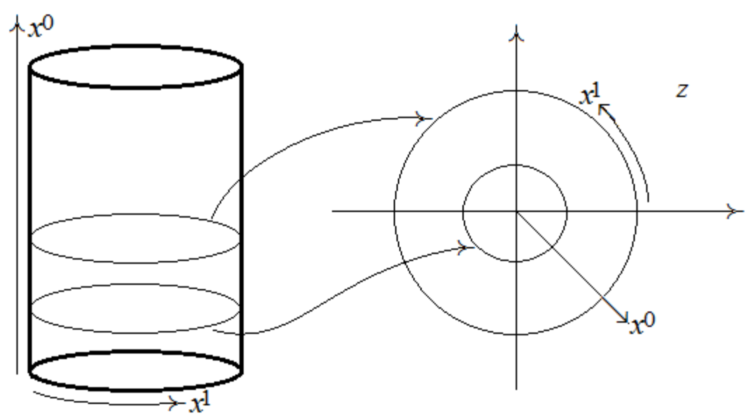

Figure 3.3: An ilustration for the cylinder to complex plane mapping.

Here we assign $\phi_{\text{cyl.}}(w, \bar{w})$ as the fields defined on the cylinder and $\phi(z, \bar{z})$ as the fields defined on the complex plane. To be simple, we can focus on the holomorphic part only, i.e. there is no dependence on $\bar{z}$, which gives us

$$\phi_{\text{cyl.}}(w) = \left(\frac{\partial w}{\partial z}\right)^h \phi(z) = z^h \sum_n \phi_n z^{-h-n} = \sum_n \phi_n z^{-n} = \sum_n \phi_n e^{-nw} . \tag{3.3.84}$$

Next we study the transformation of the energy-momentum tensor $T(z)$ dictated by the mapping (3.3.82). In general, the energy-momentum tensor is not a primary field. The transformation of the energy-momentum tensor under a general mapping $z \to f(z)$ is given by

$$T(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f) + \frac{c}{12} S(f, z) , \tag{3.3.85}$$

where

$$S(z, w) = \left(\frac{\partial z}{\partial w}\right)^{-2} \left(\frac{\partial z}{\partial w}\frac{\partial^3 z}{\partial w^3} - \frac{3}{2}\left(\frac{\partial^2 z}{\partial w^2}\right)^2\right) \tag{3.3.86}$$

is called as the Schwarzian derivative. In (3.3.85) one may observe that the energy-momentum tensor becomes a primary field with the conformal dimension $h = 2$ when the theory under consideration is centerless, i.e. $c = 0$. For $z = e^w$, the corresponding Schwarzian derivative is $S(z, w) = -1/2$. Hence the formula (3.3.85) gives us the energy-momentum tensor on the cylinder

$$T_{\text{cyl.}} = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12} S(z, w) = z^2 T(z) - \frac{c}{24} . \tag{3.3.87}$$

By plugging the Laurent expansion for this energy-momentum tensor

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \tag{3.3.88}$$

where

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \tag{3.3.89}$$

the reading of (3.3.87) becomes

$$T_{\text{cyl.}} = \sum_{n \in \mathbb{Z}} L_n z^{-n} - \frac{c}{24} = \sum_{n \in \mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) z^{-n} = \sum_{n \in \mathbb{Z}} (L_{\text{cyl.}})_n z^{-n} , \tag{3.3.90}$$

where

$$(L_{\text{cyl.}})_n = \left(L_n - \frac{c}{24}\delta_{n,0}\right) . \tag{3.3.91}$$

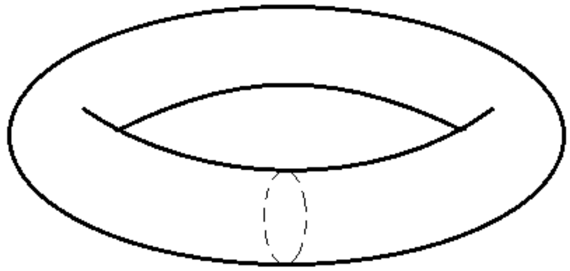

Figure 3.4: Cartoon description of a torus.

The zero mode of Laurent expansion coefficient above is shifted due to the presence of central charge,

$$\left(L_{\text{cyl.}}\right)_0 = L_0 - \frac{c}{24}\,. \tag{3.3.92}$$

After discussing some CFT properties on an infinite cylinder, now we perform an analysis for a CFT on a torus.

To make a torus from an infinite cylinder, we need to cut the cylinder and glue its edges together. In the complex plane, as depicted in figure 3.5, we can construct a torus by identifications of each points which differ by a linear combination of the two basic lattice vectors. Nevertheless, it is possible that our torus gets twisted before gluing, which is reflected by the presence of real part of modular parameter τ in the figure 3.5. Hence the modular parameter which is defined as

$$\tau = \tau_1 + i\tau_2 \tag{3.3.93}$$

describes the shape of the torus.

We now derive the proper partition function for CFT on the torus. The path integral over all paths $x(t)$ with an Euclidean action $S_E(x)$, subject to the boundary condition $x(0) = x(\beta)$, gives us a partition function reads

$$Z = \int \mathcal{D}x \exp\left(-S_E\left(x\right)\right) = \operatorname{Tr}\exp\left(-\beta H\right)\,. \tag{3.3.94}$$

As usual H in the last formula stands for the Hamiltonian of the system, and β is the inverse of temperature T, $\beta = 1/k_B T$.

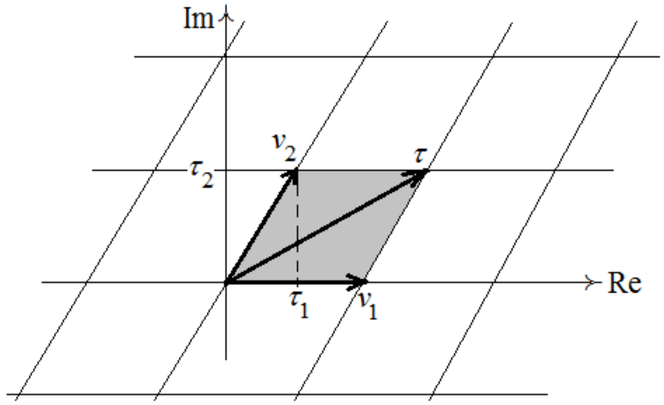

Figure 3.5: Lattice of a torus.

The Hamiltonian H in (3.3.94) is a generator of time translation. On the torus, the CFT evolves with respect to the complex parameter τ, whose real part is considered as spatial and the imaginary part is (Euclidean) time. Therefore, in the absence of translation in the real part of τ, then the partition function on the torus can be read as[8]

$$Z = \mathrm{Tr}\exp\left(-2\pi\tau_2 H\right) . \tag{3.3.95}$$

When the torus gets twisted, i.e. the real part of τ presents, the corresponding partition function now reads

$$Z = \mathrm{Tr}\exp\left(-2\pi(\tau_2 H - i\tau_1 P)\right) . \tag{3.3.96}$$

In the field theory, we already know that the T_{00} component of energy-momentum tensor is the energy density, and the T_{0k} is the momentum density related to the translation along spatial dimension x^k. In regard to the CFT on the torus, we can get the corresponding energy-momentum tensor obtained from the one defined on cylinder (3.3.91). Hence the Hamiltonian and momentum operator can be obtained as

$$H = \frac{1}{2\pi}\int dx^1 T_{00} = \frac{1}{2\pi}\oint \left(T_{\text{cyl.}}\left(z\right)dz + \bar{T}_{\text{cyl.}}\left(\bar{z}\right)d\bar{z}\right) , \tag{3.3.97}$$

$$P = \frac{1}{2\pi}\int dx^1 T_{01} = \frac{i}{2\pi}\oint \left(T_{\text{cyl.}}\left(z\right)dz - \bar{T}_{\text{cyl.}}\left(\bar{z}\right)d\bar{z}\right) , \tag{3.3.98}$$

[8]The 2π factor appearing in the partition function is the periodicity of the coordinate, and the factor "i" comes from the fact we are working in Euclidean time.

respectively. It follows that, on the torus, one can write the Hamiltonian operator as

$$H = L_0 - \frac{c}{24} + \bar{L}_0 - \frac{\bar{c}}{24}\,, \tag{3.3.99}$$

and the momentum operator as

$$P = L_0 - \frac{c}{24} - \bar{L}_0 + \frac{\bar{c}}{24}\,. \tag{3.3.100}$$

This yields the reading of the partition function on a torus becomes

$$Z\left(\tau,\bar{\tau}\right) = Trq^{\left(L_0 - \frac{c}{24}\right)\tau}\bar{q}^{\left(\bar{L}_0 - \frac{\bar{c}}{24}\right)\bar{\tau}}\,, \tag{3.3.101}$$

where $q = \exp\left(2i\pi\right)$ and $\bar{q} = \exp\left(-2i\pi\right)$.

3.3.5 Cardy Formula for Entropy in CFT$_2$

In a general treatment, the modular parameter τ can be shown to be transformed as

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{3.3.102}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL\left(2,\mathbb{Z}\right)/\mathbb{Z}_2\,. \tag{3.3.103}$$

This transformation is known as the modular transformation. In particular, a modular transformation that has a specific form

$$\tau \to -1/\tau\,, \tag{3.3.104}$$

is known as the modular S-transformation. Pictorially, we can depict this mapping as in figure 3.6. The shaded areas in the figures (a) and (b) describe the unit lattice of the same torus. The invariance of torus with respect to this S-modular transformation restricts the partition function for a CFT on the torus (3.3.101) to be invariant under the same transformation, i.e. $Z(\tau) = Z(-1/\tau)$. The partition function on the torus of modulus τ, in the absence of the central charge, reads

$$\tilde{Z}(\tau,\bar{\tau}) = \text{Tr}\; e^{2\pi i\tau L_0}e^{-2\pi i\bar{\tau}\bar{L}_0} = \sum_{h,\bar{h}}\rho(h,\bar{h})e^{2\pi ih\tau}e^{-2\pi i\bar{h}\bar{\tau}}\,. \tag{3.3.105}$$

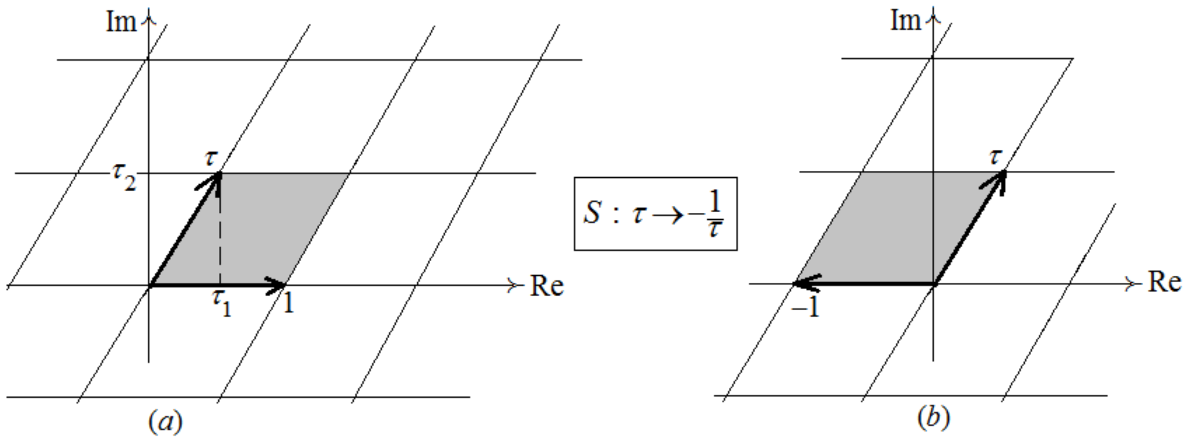

Figure 3.6: Lattice of a torus.

In writing the last formula, we have used the eigen equations

$$L_0\left|h,\bar{h}\right\rangle = h\left|h,\bar{h}\right\rangle \quad , \quad \bar{L}_0\left|h,\bar{h}\right\rangle = \bar{h}\left|h,\bar{h}\right\rangle \ , \tag{3.3.106}$$

and $\rho(h,\bar{h})$ is the density of states with eigenvalues h and $\bar{h}$.

To simplify the discussion, we treat τ and $\bar{\tau}$ as two independent complex variables. Hence, $q = e^{2\pi i\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$ are also independent each other. Computing the density of state $\rho(h,\bar{h})$ can be done by performing the integration,

$$\rho(h,\bar{h}) = \frac{1}{(2\pi i)^2}\int \frac{dq}{q^{h+1}}\frac{d\bar{q}}{\bar{q}^{\bar{h}+1}} Z(q,\bar{q})\,. \tag{3.3.107}$$

Another simplification can be performed by considering that the partition function that depends only on q for a while, and restore apply the result as if it depends on $\bar{q}$ also at the end. From (3.3.105) we can do the following trick,

$$\tilde{Z}(\tau) = e^{\frac{2\pi i c}{24}\tau} Z(\tau)\,. \tag{3.3.108}$$

Now we employ the S-modular invariance for the partition function,

$$\tilde{Z}(\tau) = e^{\frac{2\pi i c}{24}\tau} Z(-1/\tau) = e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} \tilde{Z}(-1/\tau)\,, \tag{3.3.109}$$

which gives us the integration to get the density matrix

$$\rho(h) = \int d\tau\, e^{-2\pi i h\tau} e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} Z(-1/\tau)\,. \tag{3.3.110}$$

In evaluating (3.3.110), we can apply the saddle point approximation, i.e. separating the integrand into a rapidly varying phase and a slowly varying prefactor. In the original proposal by Cardy where he discusses the CFT_2 on an infinite long strip, the lowest energy of the system depends linearly on the length of the strip, i.e. $E_0 \sim l$ where l is the strip's length. Accordingly, we are allowed to consider the case of asymptotic h, since we know in the absence of $\bar{L}_0$ in the theory, the Hamiltonian of the system is just L_0. Hence for large h, the extremum of exponent in (3.3.110) is obtained when

$$\tau \approx i\sqrt{c/24h}\,. \tag{3.3.111}$$

Substituting the result (3.3.111) back into the integral (3.3.110), we obtain

$$\rho(L_0) \approx \exp\left\{2\pi\sqrt{\frac{cL_0}{6}}\right\}\;, \tag{3.3.112}$$

which is known as the Cardy formula. Accordingly, by taking into account the contribution of anti holomorphic part in the theory, the Cardy formula for entropy can be read as

$$S_{Cardy} = 2\pi\left(\sqrt{\frac{c_L L_0}{6}} + \sqrt{\frac{c_R \bar{L}_0}{6}}\right)\;. \tag{3.3.113}$$

Sometime the holomorphic part is called as the left mover version of the theory, where the associated central charge is c_L, and the anti holomorphic part is called as the right mover one, where the corresponding central charge is c_R. Alternatively, since we know that the Eigenvalue of L_0 is energy E, hence the reading of this Cardy formula can also be

$$S_{Cardy} = 2\pi\left(\sqrt{\frac{c_L E_L}{6}} + \sqrt{\frac{c_R E_R}{6}}\right)\;. \tag{3.3.114}$$

In the last formula, quantities with subscript "L" which stands for "left" come from the holomorphic side of the theory. The subscript "R" which stands for "right" represents the anti holomorphic contribution. These "left" and "right" terminologies are analogous to the left and right moving of the traveling waves, $y = A\sin(kx + \omega t)$ and $y = A\sin(kx - \omega t)$ respectively. Writing $x^0 = kx$ and $x^1 = -i\omega t$ allows us to rewrite the left moving wave solution as a holomorphic function, $y = A\sin(z)$, and the right moving wave as an anti holomorphic one, $y = A\sin(\bar{z})$.

In fact, there is a relation between entropy, energy, and temperature,

$$\frac{\partial S}{\partial E} = \frac{1}{T}\,, \tag{3.3.115}$$

which allow us to rewrite the formula (3.3.114) as

$$S_{Cardy} = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right)\,. \tag{3.3.116}$$

The form of Cardy formula for the entropy of a two dimensional CFT as expressed in the last equation is the one which is widely used in the discussion of Kerr/CFT correspondence, which is the main topic of this thesis.

3.3.6 $\mathbf{CFT_2}$ scattering cross section

An amplitude responsible for the emission of a particle with frequency ω is written as [67]

$$\mathcal{M} \sim \int dt\, \langle f|\, \mathcal{O}\left(t\right) |i\rangle\, e^{-i\omega t}\,. \tag{3.3.117}$$

Using this amplitude, the rate of emission can be obtained by squaring and summing over all final states

$$\sum_f |M|^2 \sim \int dt dt'\, \langle i|\, \mathcal{O}^\dagger\left(t\right) \mathcal{O}\left(t'\right) |i\rangle\, e^{-i\omega(t-t')}\,, \tag{3.3.118}$$

where the sum over all final states has given us the identity matrix. We observe that this rate is proportional to the correlation function

$$\int dt\, \left\langle \mathcal{O}^\dagger\left(0\right) \mathcal{O}\left(t\right)\right\rangle e^{-i\omega t}\,, \tag{3.3.119}$$

where the initial state $|i\rangle$ has been taken to be a vacuum $|0\rangle$ and the intial time $t' = 0$.

Field theory on a cylinder has been used as a model in describing a system with finite temperature. Previously, we have discussed in details about the CFT on a cylinder. Here we will apply it to discuss a thermal system. The temperature is introduced as an inverse of the circumference of the cylinder. A specific mapping between an infinite flat plane with the coordinate z to a cylinder with the coordinate w is given by

$$w = \frac{L}{2\pi}\ln z \quad , \quad z = \exp\frac{2\pi w}{L}\,. \tag{3.3.120}$$

Hence the transformation of two point function $\langle\mathcal{O}(z)\mathcal{O}(z')\rangle$ where $\mathcal{O}(z)$ is a local operator with conformal dimension h can be read

$$\langle \mathcal{O}\left(w_1\right)\mathcal{O}\left(w_2\right)\rangle = \left(\frac{dz}{dw}\right)^h_{w=w_1}\left(\frac{dz}{dw}\right)^h_{w=w_2}\langle \mathcal{O}\left(z_1\right)\mathcal{O}\left(z_2\right)\rangle\ . \tag{3.3.121}$$

Using the transformation (3.3.120) and the two point function constrained by the conformal symmetry (3.3.59) gives us

$$\langle \mathcal{O}\left(w_1\right)\mathcal{O}\left(w_2\right)\rangle = \left(\frac{2\pi\exp\left(\pi\left(w_1-w_2\right)/L\right)}{\left(z_1-z_2\right)/L}\right)^{2h} \tag{3.3.122}$$

from the equation (3.3.121). As we are discussing the finite temperature case, inserting the temperature T to the formula can be done by considering that the cylinder circumference is proportional to the inverse of temperature T, i.e. $L\sim 1/T$. By using the formula of hyperbolic sine

$$\sinh(x) = \frac{\exp(x)-\exp(-x)}{2}\,, \tag{3.3.123}$$

we can perform some algebraic manipulations on (3.3.122) which gives us

$$\left\langle \mathcal{O}^\dagger\left(0\right)\mathcal{O}\left(t\right)\right\rangle \sim \left(\frac{\pi T}{\sinh\left(\pi T t\right)}\right)^{2h}\ . \tag{3.3.124}$$

Our next task is to compute (3.3.119) by using the last equation. Using some complex analysis, the authors of [67] obtain the result as

$$\int_0^{1/T} dt\left(\frac{\pi T}{\sinh\left(\pi T t\right)}\right)^{2h} e^{-i\omega(t-i\varepsilon)} \sim \left(T\right)^{2h-1} e^{-\omega/2T}\left|\Gamma\left(h+i\frac{\omega}{2\pi T}\right)\right|^2\ . \tag{3.3.125}$$

This rate is important later in establishing the microscopic dual calculation for absorption cross section.

3.4 AdS/CFT proposal

3.4.1 Introduction

In 1993, 't Hooft [10] proposed the holographic world idea, which was pushed further by Susskind with some more concrete examples [11]. An important finding in this holographic world studies is the conclusion where the number of microstates of matter (quantum theory) in two spatial dimensions (area) is equal with the number of microstates of black holes (gravitational theory) in three spatial dimensions (volume). Our best understanding of matter comes from quantum theory. In the other side, a black hole is an object in a gravitational theory. The gravitational theory could be the Einstein theory or beyond. This relation between the numbers of microstates in quantum and gravitational theories, in different dimensions of spacetime they are living, becomes a remarkable hint in the search of quantum gravity theories afterwards.

The AdS/CFT correspondence, or AdS/CFT duality, relates a quantum field theory (QFT) in $D-1$ dimensions and a quantum gravity theory in D dimensions. The AdS/CFT duality is a concrete realization of the holographic world proposal. This duality says, in some limits, that a string theory on the manifold $\mathrm{AdS}_{D+1} \times \mathcal{M}$ is equivalent to a specific D-dimensional conformal field theory living on the boundary of AdS_{D+1}. Here $\mathcal{M}$ is the compactification manifold. This proposal was given by Maldacena in his celebrated paper [9]. Yet, he did not specify how exactly these two distinguished theories are mapped each other. A detailed proposal on how the quantum field theory is mapped to the gravity theory in supporting Maldacena's idea was given by Gubser, Klebanov, and Polyakov [68] and by Witten [69].

The AdS/CFT duality is a discovery in the context of superstring theory. Superstring theory itself needs some extra dimensions. Consequently, it is quite natural to suspect in the context of holography, a quantum field theories that is holographically dual to the gravity according to superstring theory would live on the hypersurface embedded in a higher dimensional space where the superstring theory lives. In the development of this AdS/CFT duality, studies of this subject has been expanded to several domains. They include the studies of quantum field theories at the strong coupling, physics of black holes, relativistic fluid dynamics, and even some applications in condensed matter physics [70, 71, 72, 73, 74]. This reflects how remarkable this idea is.

Witten's prescription of the precise mapping between a gravity theory in the bulk and a conformal field theory on the boundary related to the AdS/CFT correspondence [69] has a close relation to the Kadanoff-Wilson renormalization group approach in the study of lattice systems. Let us consider a system where there is no gravity in a lattice with the lattice spacing a and the Hamiltonian

$$H = \sum_{x,i} J_i(x,a)\ \mathcal{O}^i(x)\ . \tag{3.4.126}$$

In the Hamiltonian (3.4.126), the variable x denotes the lattice sites and the operators $\mathcal{O}^i$ is coupled to the sources (could be coupling constants) $J_i(x,a)$ at the point x of the lattice. We include the argument a associated with J_i, to stress the point that J_i corresponds to a lattice spacing a. In the renormalization group approach, we coarse grain the lattice by expanding the lattice spacing and by replacing multiple sites by a single site. In this process, the Hamiltonian (3.4.126) does not change, but different operators are weighed differently. Accordingly, in each steps of coarse grain, the couplings $J_i(x,a)$ change. Suppose that we double the lattice spacing in each steps, it will affect the couplings J_i as

$$J_i(x,a) \Rightarrow J_i(x,2a) \Rightarrow J_i(x,4a) \Rightarrow J_i(x,8a) \Rightarrow \cdots\ . \tag{3.4.127}$$

In this process we observe the dependence of sources with respect to the scaling which allow us to write the sources as $J_i(x,u)$, where $u = (a, 2a, 4a, \cdots)$ is the length scale of a system under consideration. The evolution of the couplings with the scale is determined by flow equations

$$u\frac{\partial}{\partial u}\ J_i(x,u)\ =\ \beta_i\Big(J_j(x,u), u\Big)\ , \tag{3.4.128}$$

where β_i is the so-called β-function associated with source J_i. At weak coupling, the β_i-functions can be determined by using perturbation theory.

According to the AdS/CFT proposal, we consider u as an extra dimension. In this point of view, the succession of lattices at different sizes, i.e. different values of u, are considered as layers of a new higher-dimensional space. In addition, the sources $J_i(x,u)$ are regarded as fields in a space with the extra dimension u,

$$J_i(x,u) = \phi_i(x,u)\,. \tag{3.4.129}$$

The dynamical equations for the sources ϕ_i's will be dictated by some action. Specifically in the AdS/CFT duality, the dynamics of the ϕ_i's is coming from some gravity theories. Consequently, we can consider this AdS/CFT duality as a geometrization of the quantum

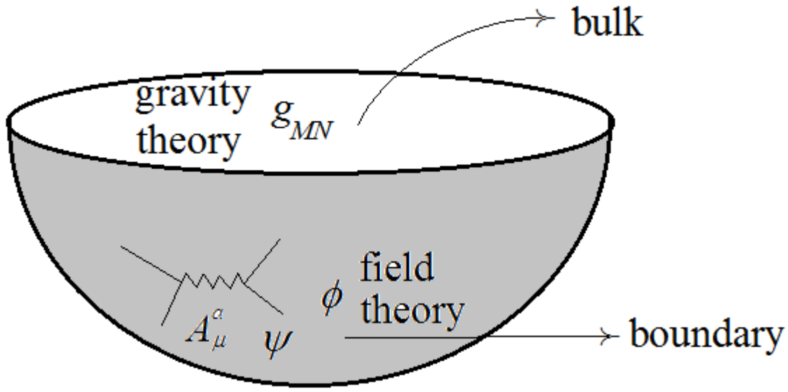

Figure 3.7: Ilustration of the gravity theory lives in the bulk and field theory lives on the boundary according to the AdS/CFT correspondence.

dynamics described by the renormalization group. The microscopic couplings or sources of the field theory can be identified to be the values of the bulk fields living at the boundary of the space with extra dimensions. That is why we mentioned that the field theory lives on the boundary of the higher-dimensional space, which we call as the bulk, where the gravity theory lives. Figure 3.7 illustrates this idea.

The same tensor structure must be shown by the associated field theory operator and quantities in the gavity theory. Therefore, a "bulk" scalar field will be dual to a scalar operator on the boundary, a "bulk" vector field A_μ will be dual to a current J^μ on the boundary, and a "bulk" graviton field $g_{\mu\nu}$ with spin-two will be dual to a symmetric second-rank tensor $T_{\mu\nu}$ on the boundary. This $T_{\mu\nu}$ will be naturally identified as the energy-momentum tensor $T_{\mu\nu}$ of the boundary field theory. In the next two subsections, we show how to verify this AdS/CFT correspondence by matching the two point functions on each sides of the duality, for scalar and vector fields.

3.4.2 AdS/CFT two point function for scalar fields

In this section, based on [69, 75], we will review the gravitational two point function for free massless scalars in AdS$_{D+1}$ background, and match this two point function from the prediction of CFT_D. Let us start by writing an action for scalar fields in AdS_{D+1},

$$S(\phi) = \frac{1}{2}\int d^{D+1}x\sqrt{|g|}\partial_\mu\phi\partial^\mu\phi\,. \tag{3.4.130}$$

In the action above, and in the rest of this AdS/CFT computation, we will use the measure $\sqrt{|g|}$ instead of $\sqrt{-g}$, due to the fact that one can use the Lorentzian or Euclidean version of the AdS spacetime. In the case of Euclidean, as the one that we will use in this and next subsection, all signatures in the spacetime will be positive, hence we do not need the negative sign inside of the square root. In the case of Lorentzian, as we use in the other sections which are related to the Schwarzschild and Kerr spacetimes, the negative sign is needed to make sure the quantity inside of the square root will be positive. The corresponding equation of motion for ϕ from this action is

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right) = 0\,. \tag{3.4.131}$$

The field $\phi(x)$ has a definite value $\phi_0(x')$ on the boundary, where $x \in \mathrm{AdS}_{D+1}$ and $x' \in E^D$. Here E^D is the D dimensional Euclidean flat spacetime. The spacetime metric for an AdS_{D+1} spacetime reads

$$ds^2 = \left(x^0\right)^{-2}\sum_{k=0}^{D}\left(dx^k\right)^2. \tag{3.4.132}$$

Now, we would like to find the Green function solution to the equation (3.4.131),

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu K\left(x,x'\right)\right) = 0\,. \tag{3.4.133}$$

In the AdS/CFT discussion, the Green function or propagator $K(x,x')$ is called as the bulk-to-boundary propagator. Once we get the solution for this propagator, the solution for scalar fields can be written as

$$\phi(x) = \int_{E^D} d^Dx' K\left(x,x'\right)\phi_0\left(x'\right). \tag{3.4.134}$$

Following Witten [69], the integrand in (3.4.134) is evaluated in Euclidean space (3.4.132). The coordinate x^k for $k = 1, ..., D$ are coordinates of the boundary, E^D, and the boundary is placed at $x^0 = 0$. The component of metric tensor for this spacetime then can be read as

$$g_{\mu\nu} = (x^0)^{-2}\delta_{\mu\nu}\quad,\quad \sqrt{|g|} = (x^0)^{-(D+1)}\quad,\quad g^{\mu\nu} = (x^0)^2\delta^{\mu\nu}\,. \tag{3.4.135}$$

The solution for $K(x,x')$ is given as [69]

$$K\left(x^0,\vec{x};\vec{x}'\right)=C\left(\frac{x^0}{\left(x^0\right)^2+|\vec{x}\ -\vec{x}\,'|^2}\right)^D\,, \tag{3.4.136}$$

where C is some constants. This propagator solution satisfies the Laplace equation (3.4.131) for $x^0\neq 0$ and $\vec{x}\neq\vec{x}'$.

Witten used a trick to derive the propagator solution (3.4.136). Initially, we could discuss the equation (3.4.131) at $x^0=\infty$. Therefore, only the coordinate x^0 which matters in this consideration, and the corresponding equation is

$$\partial_0\left(\sqrt{|g|}\partial^0 K\left(x^0\right)\right)=0\,. \tag{3.4.137}$$

Furthermore, since $\partial^0=g^{00}\partial_0$, and the propagator K is a function of x^0 only, we can rewrite the last equation as

$$\frac{d}{dx^0}\left(\left(x^0\right)^{-D+1}\frac{d}{dx^0}K\left(x^0\right)\right)=0\,. \tag{3.4.138}$$

We can try the ansatz $K\left(x^0\right)=C\left(x^0\right)^P$ to solve equation (3.4.138), where C is just some constants. We can find that this ansatz solves the equation (3.4.138) with P are the roots of

$$P\left(P-D\right)=0\,. \tag{3.4.139}$$

The solution $P=0$ cannot fulfill our boundary condition at $x^0=0$, where we are expecting $K(x,x')$ reduces to a delta function at the boundary $x^0=0$. Therefore, the accepted solution is

$$K\left(x^0,\vec{x};\infty\right)=C\left(x^0\right)^D\,. \tag{3.4.140}$$

We add the argument ∞ inside of the K function's dependence to show that this solution is obtained for the condition $x^0=\infty$.

However, the boundary where the field theory lives is taken at $x^0=0$. Therefore we need to map the solution (3.4.140) from $x^0=\infty$ to $x^0=0$. As can be seen from the appendix D, the appropriate mapping is

$$x^0\rightarrow\frac{x^0}{\left(x^0\right)^2+|\vec{x}\,|^2}\,. \tag{3.4.141}$$

Hence, under this mapping, the propagator solution (3.4.140) transforms to

$$K\left(x^0,\vec{x};\infty\right)\rightarrow K\left(x^0,\vec{x};0\right)=C\left(\frac{x^0}{\left(x^0\right)^2+|\vec{x}\,|^2}\right)^D\,. \tag{3.4.142}$$

The classical solution (3.4.134) now can be read in detail as

$$\phi\left(x^0,\vec{x}\right)=c\int d^Dx'\left(\frac{x^0}{\left(x^0\right)^2+\left|\vec{x}\,-\vec{x}\,'\right|^2}\right)^D\phi_0\left(\vec{x}'\right). \tag{3.4.143}$$

According to Witten [69], the precise statement of AdS/CFT is

$$\left\langle \exp\int_{E^D}\phi_0\mathcal{O}\right\rangle_{CFT}=Z_{grav}\left(\phi_0\right)\,, \tag{3.4.144}$$

where

$$Z_{grav}\left(\phi_0\right)=\exp\left(-S\left(\phi\right)\right)\,. \tag{3.4.145}$$

Inserting the solution (3.4.143) into the classical action (3.4.130), we have

$$S\left(\phi\right)=-\frac{CD}{2}\int d^Dxd^Dx'\frac{\phi_0\left(\vec{x}'\right)\phi_0\left(\vec{x}\right)}{\left|\vec{x}\,-\vec{x}\,'\right|^{2D}}\,. \tag{3.4.146}$$

Plugging the action (3.4.146) into the formula (3.4.144), and using the quantum field theoretic prescription to obtain the two point function from a generating functional yield

$$\left\langle\mathcal{O}\left(\vec{x}\right)\mathcal{O}\left(\vec{x}'\right)\right\rangle\sim\left.\frac{\delta^2Z_{grav}\left(\phi_0\right)}{\delta\phi_0\left(\vec{x}\right)\delta\phi_0\left(\vec{x}'\right)}\right|_{\phi_0=0}=\frac{\delta^2S\left(\phi\right)}{\delta\phi_0\left(\vec{x}\right)\delta\phi_0\left(\vec{x}'\right)} \tag{3.4.147}$$

we have

$$\left\langle\mathcal{O}\left(\vec{x}\right)\mathcal{O}\left(\vec{x}'\right)\right\rangle\sim\frac{1}{\left|\vec{x}\,-\vec{x}\,'\right|^{2D}}\,, \tag{3.4.148}$$

which is what we expect for a two point function in a CFT (3.3.60) with the conformal dimension D.

3.4.3 AdS/CFT two point function for gauge fields

We now turn our discussion to the gauge field case, where the mapping between the theories in the bulk and on the boundary is the "vectorial" version of (3.4.144)

$$\left\langle \exp\int_{E^D}A_0J\right\rangle_{CFT}=Z_{grav}\left(A_0\right)\,. \tag{3.4.149}$$

We restrict our problem to the massless Abelian gauge field A_μ case only. In general, it can be extended to the massive non-Abelian case, where the associated gauge fields are A_μ^a. In the formula (3.4.149), J is a vector operator in a CFT. The associated source free field equation for massless Abelian gauge fields in a curved spacetime is

$$\partial_\mu\left(\sqrt{|g|}F^{\mu\nu}\right)=0\,. \tag{3.4.150}$$

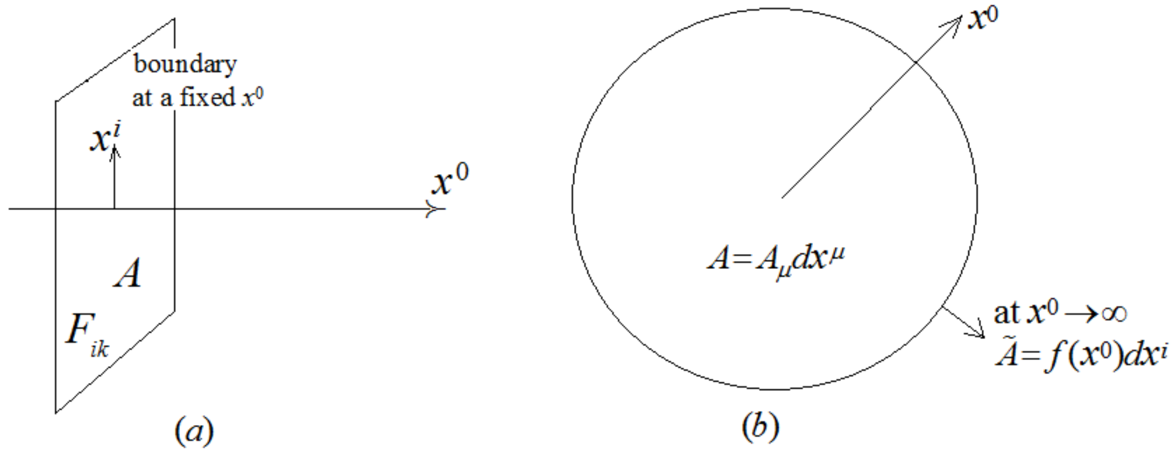

Figure 3.8: The illustration of the gauge fields on the boundary.

As usual, the field strength tensor $F_{\mu\nu}$ is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \,. \tag{3.4.151}$$

We continue to follow Witten [69] and Petersen [75] in discussing the gauge fields in AdS space. The equation (3.4.150) is known as the free Maxwell equation in curved space. In the same spirit with the previous discussion for scalar fields, the first step here is to seek a way in constructing a propagator which reduces to a delta function on the boundary, i.e. has the form $\delta\left(\vec{x}-\vec{x}'\right)$ at $x^0=0$. The gauge fields in the bulk can be written as a 1-form[9], we expect the propagator solution for equation (3.4.150) is also a 1-form. We are looking for a solution in the form

$$\mathbf{A}(x^0,\vec{x}\,) = a_i(\vec{x}\,)dx^i \,. \tag{3.4.152}$$

Here we also use the trick as we performed before, getting the propagator first when the boundary is placed at infinity, $x^0\to\infty$. In such case, the propagator will be $\vec{x}$ independent. Since the gauge field A_μ is expressed in a 1-form, the same will apply to the propagator solution that we are looking right now, i.e. the propagator is a 1-form also.

Hence, as the boundary is placed at infinity (the illustration is given in fugure 3.8 (b)), we are looking for the 1-form propagator solution $\tilde{\mathbf{A}} = \tilde{A}_i dx^i$ where $\tilde{A}_i = f(x^0)$ and $i\geq 1$.

[9]A brief discussions on forms is given in appendix E.

In general, the bulk 1-form gauge field is $\mathbf{A} = A_\mu dx^\mu$, where $\mu \geq 0$. However, since we are discussing the dynamics of the fields at $x^0 = \infty$, we have turned off the degree of freedom along $\vec{x}$. Consequently, the corresponding field strength tensor $\tilde{F}_{\mu\nu}$ associated with the propagator $\tilde{\mathbf{A}}$ on this boundary is

$$\begin{aligned} \tilde{F}_{0i} &= \frac{df(x^0)}{dx^0} = -\tilde{F}_{i0}\,, \\ \tilde{F}^{0i} &= (x^0)^4 \frac{df(x^0)}{dx^0} \\ \sqrt{|g|}\tilde{F}^{0i} &= (x^0)^{-n+3} \frac{df(x^0)}{dx^0}\,. \end{aligned} \tag{3.4.153}$$

Since $\tilde{A}_i$ is x^0 dependent only, then all components of $\tilde{F}_{\mu\nu}$ except $\tilde{F}_{0i}$ or $\tilde{F}_{i0}$ are zero. Then the equation of motion (3.4.150) gives us

$$\frac{d}{dx^0}(\sqrt{g}\tilde{F}^{0i}) = \frac{d}{dx^0}\left(\left(x^0\right)^{3-D} \frac{df\,(x^0)}{dx^0}\right) = 0\,. \tag{3.4.154}$$

The function $f(x^0)$ which solves this equation is

$$f(x^0) = \text{const.}\,(x^0)^{(D-2)} \tag{3.4.155}$$

which yields a possible solution for $\tilde{A}$ as

$$\tilde{\mathbf{A}} = \frac{D-1}{D-2}(x^0)^{(D-2)} dx^i\,. \tag{3.4.156}$$

We have set the constant in (3.4.155) to be $(D-1)/(D-2)$ for the latter convenient.

The solution $\tilde{\mathbf{A}}$ in (3.4.156) applies at $x^0 = \infty$. To map this solution to the boundary $x^0 = 0$, we can use the mapping (3.4.141), which gives

$$\tilde{\mathbf{A}} = \frac{D-1}{D-2}\left(\frac{x^0}{(x^0)^2 + |\vec{x}\,|^2}\right)^{D-2} d\left(\frac{x^i}{(x^0)^2 + |\vec{x}\,|^2}\right)\,. \tag{3.4.157}$$

It turns out that a nice result will be obtained if we remove the "pure gauge" part,

$$\left(\frac{D}{D-2}\right) d\left(\frac{(x^0)^{D-2}x^i}{((x^0)^2 + |\vec{x}\,|^2)^{D-1}}\right) \tag{3.4.158}$$

from the solution for A in (3.4.157). Recall that in the discussion of free Maxwell fields, we can perform a pure gauge transformation for the field solution

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + d\mathbf{a} \tag{3.4.159}$$

which yields $\mathbf{A}'$ is also a solution. The proof is very simple, since $\mathbf{F} = d\mathbf{A}$, then $\mathbf{F}' = d\mathbf{A}' = d\mathbf{A}$ which is clearly coming from the fact that $d(d\mathbf{a}) = 0$. Subtracting the "pure gauge" part from the solution (3.4.157) yields

$$\begin{aligned}
\tilde{\mathbf{A}} &= \frac{D-1}{D-2}\left(\frac{x^0}{(x^0)^2+|\vec{x}\,|^2}\right)^{D-2} d\left(\frac{x^i}{(x^0)^2+|\vec{x}\,|^2}\right) - \frac{1}{D-2} d\left(\frac{(x^0)^{D-2}x^i}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}\right) \\
&= \frac{1}{D-2}\left\{(D-1)(x^0)^{D-2}\frac{x^i}{((x^0)^2+|\vec{x}\,|^2)^{D-2}} d\left(\frac{1}{(x^0)^2+|\vec{x}\,|^2}\right)\right. \\
&\quad \left. + (D-1)\frac{(x^0)^{D-2}}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}dx^i - d\left(\frac{(x^0)^{D-2}x^i}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}\right)\right\} \\
&= \frac{1}{D-2}\left\{(x^0)^{D-2}x^i d\left(\frac{1}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}\right) + (D-1)\frac{(x^0)^{D-2}}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}dx^i\right. \\
&\quad \left. - d\left(\frac{(x^0)^{D-2}x^i}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}\right)\right\} \\
&= \frac{1}{D-2}\left\{-\frac{1}{((x^0)^2+|\vec{x}\,|^2)^{D-1}} d\left((x^0)^{D-2}x^i\right) + (D-1)\frac{(x^0)^{D-2}}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}dx^i\right\} \\
&= \frac{1}{((x^0)^2+|\vec{x}\,|^2)^{D-1}}\left\{-(x^0)^{D-3}dx^0 x^i + (x^0)^{D-2}dx^i\right\} . \qquad (3.4.160)
\end{aligned}$$

Therefore, the solution $\tilde{A}_\mu$ can be read from the last equation as

$$\tilde{A}_0 = -\frac{(x^0)^{D-3}\,x^i}{\left((x^0)^2+|\vec{x}\,|^2\right)^{D-1}} \quad , \quad \tilde{A}_i = \frac{(x^0)^{D-2}}{\left((x^0)^2+|\vec{x}\,|^2\right)^{D-1}} . \qquad (3.4.161)$$

After applying the the transformation $\vec{x} \to \vec{x} - \vec{x}'$, we now have

$$\tilde{A}_0 = -\frac{(x^0)^{D-3}\,(x-x')^i}{\left((x^0)^2+|\vec{x}\,-\vec{x}\,'|^2\right)^{D-1}} \quad , \quad \tilde{A}_i = \frac{(x^0)^{D-2}}{\left((x^0)^2+|\vec{x}\,-\vec{x}\,'|^2\right)^{D-1}} . \qquad (3.4.162)$$

Now we arrive at the key point in the AdS/CFT prescription. The bulk field $\mathbf{A}(x^0, \vec{x})$ can be written in terms of the boundary fields $a_i(\vec{x})$ by using the bulk-to-boundary propagator $\tilde{\mathbf{A}}(x^0, \vec{x}; \vec{x}')$ constructed from the solutions (3.4.162),

$$\begin{aligned}
\mathbf{A}(x^0,\vec{x}\,) &= \int d^Dx' \tilde{\mathbf{A}}(x^0,\vec{x}\,;\vec{x}\,')a_i(\vec{x}\,') \\
&= \int d^Dx' \left\{\frac{(x^0)^{D-2}}{((x^0)^2+|\vec{x}\,-\vec{x}\,'|^2)^{D-1}}a_i(\vec{x}\,')dx^i\right. \\
&\quad \left. -(x^0)^{D-3}dx^0\frac{(x-x')^i a_i(\vec{x}\,')}{((x^0)^2+|\vec{x}\,-\vec{x}\,'|^2)^{D-1}}\right\} . \qquad (3.4.163)
\end{aligned}$$

One way to express the delta function is [76]

$$\delta(x) = \lim_{q\to\infty}\frac{q}{\pi(1+q^2x^2)}. \qquad (3.4.164)$$

By changing $p \to 1/\varepsilon$, the delta function (3.4.164) transforms to

$$\delta\left(x\right) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi\left(\varepsilon^2 + x^2\right)}\,, \tag{3.4.165}$$

which can be generalized to

$$\delta\left(x\right) = \lim_{\varepsilon \to 0} \frac{\varepsilon^\beta}{(\varepsilon^2 + |\vec{x}\,|^2)^\alpha} \tag{3.4.166}$$

for $2\alpha - D = \beta > 0$. The last formula is singular at $\vec{x}^2 = 0$ and vanishing elsewhere. As we take the limit $x^0 \to 0$, the first term in the equation (3.4.163), i.e. the term that couples to dx^i, contains the delta function (3.4.166). Therefore, after the integrating over dx', we get the result $a_i(\vec{x})dx^i$. The second term in equation (3.4.163), the term that couples to dx^0, vanishes as $x^0 \to 0$. This is not too obvious, but we can see this by considering the contribution of $(\vec{x} - \vec{x}')$ term after a delta function $\delta(\vec{x} - \vec{x}')$ applies to it, and Taylor expand $a_i(\vec{x}')$ around $\vec{x}$. Hence, the field (3.4.163) will reduce to (3.4.152) as $x^0 \to 0$.

We are now instructed to evaluate the classical action for the classical solution (3.4.163). An action for the free Maxwell fields in the form language can be read as

$$S(\mathbf{A}) = \frac{1}{2}\int_{AdS_{D+1}} \mathbf{F} \wedge *\mathbf{F} \tag{3.4.167}$$

where $\mathbf{F} = d\mathbf{A}$ and the associated equation of motion is $d\mathbf{F} = 0$. By using the Stokes theorem, this action can be read as

$$S(\mathbf{A}) = \frac{1}{2}\int_{AdS_{D+1}} d\mathbf{A} \wedge *\mathbf{F} = \frac{1}{2}\int_{AdS_{D+1}} d(\mathbf{A} \wedge *\mathbf{F}) = \frac{1}{2}\int_{\partial AdS_{D+1}} \mathbf{A} \wedge *\mathbf{F}\,. \tag{3.4.168}$$

Hence, the action at the boundary of AdS_{D+1}, denoted by ∂AdS_{D+1}, can be read as

$$S(A) \sim \int d^D x \sqrt{h} A^\ell n^0 F_{0\ell}\,. \tag{3.4.169}$$

In the integrand of boundary action above, n^0 is the normal vector of this boundary and h is the determinant of boundary metric tensor

$$h_{ij} = \frac{1}{(x^0)^2}\delta_{ij}, \quad i, j = 1, ..., D\,. \tag{3.4.170}$$

We have the liberty to take

$$n_\mu = (-\frac{1}{x^0}, 0, ..., 0); \quad n^\mu = (-x^0, 0, ..., 0) \tag{3.4.171}$$

and $\sqrt{h} = (x^0)^{-D}$. Furthermore, the 2-form $\mathbf{F} = d\mathbf{A}$ can be computed from (3.4.163),

$$\begin{aligned}
\mathbf{F} &= (D-2)(x^0)^{D-3}dx^0 \wedge \int d^Dx' \frac{a_i(x')dx^i}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D-1}} \\
&\quad -2(D-1)(x^0)^{D-1}dx^0 \wedge \int d^Dx' \frac{a_i(x')dx^i}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D}} \\
&\quad +2(D-1)(x^0)^{D-3}dx^\ell \wedge dx^0 \int d^Dx' \frac{(x^\ell - (x')^\ell)a_i(x')(x^i - (x')^i)}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D}} \\
&\quad -(x^0)^{D-3}dx^i \wedge dx^0 \int d^Dx' \frac{a_i(x')}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D-1}} + \cdots
\end{aligned} \tag{3.4.172}$$

The dots in the last line of (3.4.172) represents the terms that are not needed in computing the action (3.4.169). In fact, by using the relation $dx^0 \wedge dx^i = -dx^i \wedge dx^0$, we can get

$$\begin{aligned}
\mathbf{F} &= (D-1)(x^0)^{D-3}dx^0 \wedge \int d^Dx' \frac{a_i(x')dx^i}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D-1}} \\
&\quad -2(D-1)(x^0)^{D-1}dx^0 \wedge \int d^Dx' \frac{a_i(x')dx^i}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D}} \\
&\quad -2(D-1)(x^0)^{D-3}dx^0 \wedge \int d^Dx' \frac{|\vec{x} - \vec{x}\,'| \cdot d\vec{x}\, a_i(x')(x^i - (x')^i)}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D}} + \cdots
\end{aligned} \tag{3.4.173}$$

The boundary action (3.4.169) can be rewritten as

$$S(\mathbf{A}) \sim \int d^Dx'(x^0)^{-D+3} A_i(x^0, \vec{x}\,') F_{0i}(x^0, \vec{x}\,') \,. \tag{3.4.174}$$

Therefore, the components of $F_{\mu\nu}$ that we need to compute the action above are F_{0i},

$$\begin{aligned}
F_{0i}(x^0, \vec{x}\,) &= (x^0)^{D-3} \Bigg\{ (D-1) \int d^Dx' \frac{a_i(x')}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D-1}} \\
&\quad -2(D-1) \int d^Dx'(x_i - x'_i) \frac{a_k(x')(x^k - (x')^k)}{((x^0)^2 + |\vec{x} - \vec{x}\,'|^2)^{D}} \Bigg\} \\
&\quad +\mathcal{O}((x^0)^{D-1}) \,,
\end{aligned} \tag{3.4.175}$$

which is the result in (3.4.173). Only the term with $(x^0)^{D-3}$ survives for $x^0 \to 0$, and we find

$$S(\mathbf{A}) = \int d^Dx d^Dx' a_i(\vec{x}\,) a_j(\vec{x}\,') \left(\frac{\delta^{ij}}{|\vec{x} - \vec{x}\,'|^{2D-2}} - \frac{2(x-x')^i(x-x')^j}{|\vec{x} - \vec{x}\,'|^{2D}} \right) . \tag{3.4.176}$$

The corresponding two point from CFT is

$$\langle J_i(\vec{x}\,) J_j(\vec{x}\,') \rangle \sim \frac{1}{|\vec{x} - \vec{x}\,'|^{2(D-1)}} \left\{ \delta_{ij} - \frac{2(x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}\,'|^2} \right\} , \tag{3.4.177}$$

which can be obtained by taking

$$\langle J_i(\vec{x}\,)J_j(\vec{x}\,')\rangle \sim \left.\frac{\delta^2 Z_{grav}\,(\mathbf{A})}{\delta a_i(\vec{x}\,)\delta a_j(\vec{x}\,')}\right|_{a_i=0} = \frac{\delta^2 S\,(\mathbf{A})}{\delta a_i(\vec{x}\,)\delta a_j(\vec{x}\,')}\,. \tag{3.4.178}$$

Hence, we arrive at a conclusion that the AdS/CFT conjecture works for free massless gauge fields also.

Chapter 4

Kerr/CFT Correspondence

In the previous section, we have discussed briefly the AdS/CFT correspondence. Apart from the fact that it has changed the direction of researches in theoretical high energy physics in the last seventeen years, it seems this duality proposal has no contact to the real black holes phenomenon yet. In 2008, the AdS/CFT correspondence idea was extended to the case of extremal rotating black holes, namely the Kerr/CFT correspondence [12]. This correspondence may explain the real world since astronomical data supports the existence of the near extremal rotating black holes [13].

In the same spirit with AdS/CFT, according to this Kerr/CFT correspondence, a lower dimensional field theory defined on the boundary of the near horizon of extremal Kerr (NHEK) can read "holographically" the semiclassical dynamics related to the Kerr black holes. In this section, we will review the Kerr/CFT correspondence. We will start with the extremal case, where we can obtain a central charge associated to the extremal Kerr black holes. This central charge is derived in the same fashion as Brown and Hannaeux did for AdS_3 space in 80's [63]. The derivation of central charge for extremal Kerr black holes in this chapter follows the method in [77]. It is natural to expect that the Kerr/CFT correspondence is not an exclusive property of extremal or near extremal Kerr black holes. It must be the property of generic or non-extremal Kerr black holes as well. However, one cannot show the conformal symmetries of the near horizon of generic Kerr black holes. It turns out that the conformal symmetry can be seen from the radial wave equation by using the low energy scalar probe in the near region of black holes, as we will see also in this chapter.

4.1 Extremal and near-Extremal Kerr/CFT

4.1.1 The NHEK geometry

The study of near horizon geometry [14] of extremal Kerr black holes, in many respects, shows that a "small portion" of the spacetime in this region is similar to $AdS_2 \times S^2$. The "AdS" structure that appears in this region of (extremal) Kerr spacetime is the signal for the possibility in showing the conformal property of the near horizon of extremal Kerr (NHEK). Furthermore, this conformal property motivates Guica et al [12] to calculate the corresponding central charge of NHEK, by using the similar procedure performed by Brown and Henneaux [63] in getting the central charge of AdS_3 spacetime.

In this section we will use the notations $(\hat{t}, \hat{r}, \theta, \hat{\phi})$ to represent the time, radius, azimuth, and altitude coordinates of Kerr (2.2.158), to distinguish with those that represent NHEK, (t, r, θ, ϕ). Hence the Kerr metric can be read as

$$ds^2 = -\frac{\Delta}{\rho}\left(d\hat{t} - a\sin^2\theta d\hat{\phi}\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(\hat{r}^2 + a^2\right)d\hat{\phi} - ad\hat{t}\right)^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \rho^2 d\theta^2\,, \qquad (4.1.1)$$

where now $\Delta \equiv \hat{r}^2 - 2M\hat{r} + a^2$ and $\rho^2 \equiv \hat{r}^2 + a^2\cos^2\theta$.

In getting the NHEK geometry, Bardeen and Horowitz introduce the transformation[14],

$$t = \frac{\lambda\hat{t}}{2M}\,, \qquad y = \frac{\lambda M}{\hat{r} - M}\,, \qquad \phi = \hat{\phi} - \frac{\hat{t}}{2M}\,, \qquad (4.1.2)$$

where $\lambda \to 0$ while (t, y, ϕ, θ) are kept to be fixed. The resulting metric after performing this transformation is

$$ds^2 = 2GJ\Omega^2\left(\frac{-dt^2 + dy^2}{y^2} + d\theta^2 + \Lambda^2\left(d\phi + \frac{dt}{y}\right)^2\right) \qquad (4.1.3)$$

which is known as the NHEK spacetime, where

$$\Omega^2 \equiv \frac{1 + \cos^2\theta}{2}\,, \qquad \Lambda \equiv \frac{2\sin\theta}{1 + \cos^2\theta}\,. \qquad (4.1.4)$$

This NHEK geometry is not asymptotically flat, i.e. taking $y \to \infty$ of (4.1.3) does not produce the Minkowski spacetime (2.1.5). One can see this easily from the fact that at the limit $\lambda \to 0$, the "time" coordinate is restricted only $t \to 0$. We expect that a well defined global time coordinate will span $-\infty \leq t \leq \infty$. To get a set of global coordinates (r, τ, φ), we can do the following transformations

$$\begin{aligned} y &= \left(\cos\tau\sqrt{1+r^2}+r\right)^{-1}, && (4.1.5)\\ t &= y\sin\tau\sqrt{1+r^2}\,, && (4.1.6)\\ \phi &= \varphi + \ln\left(\frac{\cos\tau + r\sin\tau}{1+\sin\tau\sqrt{1+r^2}}\right). && (4.1.7) \end{aligned}$$

This transformations yields the metric (4.1.3) becomes

$$d\bar{s}^2 = 2GJ\Omega^2\left(-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \Lambda^2(d\varphi + rd\tau)^2\right). \tag{4.1.8}$$

As it is mentioned in [14], the NHEK geometry has an enhanched symmetries compared to the original (extremal) Kerr spacetime. We know already that a Kerr spacetime, whether it is extremal or not, possesses a spacetime symmetry shown by two Killing vectors ξ_t and ξ_ϕ. The NHEK geometry, instead of having just two Killing vectors, it has four which turn out to be the representation of $SL(2,\mathbb{R})\times U(1)$ isometry group. The $U(1)$ symmetry is generated by

$$\zeta_0 = -\partial_\varphi\,. \tag{4.1.9}$$

and the $SL(2,\mathbb{R})$ isometry group is generated by the Killing vectors

$$\begin{aligned} \tilde{J}_0 &= 2\partial_\tau\,, && (4.1.10)\\ \tilde{J}_1 &= 2\sin\tau\frac{r}{\sqrt{1+r^2}}\partial_\tau - 2\cos\tau\sqrt{1+r^2}\partial_r + \frac{2\sin\tau}{\sqrt{1+r^2}}\partial_\varphi\,, && (4.1.11)\\ \tilde{J}_2 &= -2\cos\tau\frac{r}{\sqrt{1+r^2}}\partial_\tau - 2\sin\tau\sqrt{1+r^2}\partial_r - \frac{2\cos\tau}{\sqrt{1+r^2}}\partial_\varphi\,. && (4.1.12) \end{aligned}$$

4.1.2 The Asymptotic Symmetry Group and Diffeomorphism Generators

In [63], it is shown that by adopting an appropriate boundary conditions for AdS_3 spacetime at spatial infinity, we can get a central term in the Poisson bracket algebra of diffeomorphism charges. The corresponding central charge for AdS_3 spacetime is obtained after using the classical to quantum transition prescription $\{,\}_{PB}$ to $[,]$ for the Possion bracket algebra of

AdS_3 diffeomorphism charges at infinity where finally one can get the corresponding Virasoro algebra for AdS_3,

$$c = \frac{3l}{2G} \tag{4.1.13}$$

where the cosmological constant $\Lambda = l^{-2}$ and we have restored the Newton Gravitational constant G. One of the important steps in getting this AdS_3 central charge is setting an appropriate asymptotically boundary conditions for the metric components of AdS_3 spacetime. It is allowed to set some boundary conditions or fall-off conditions for such spacetime due to the fact that this spacetime is not flat at infinity.

The lacking of asymptotically flatness of NHEK allows us to use a set of fall-off conditions that is appropriate in giving us a non-trivial central charge after performing the Asymptotic Symmetry Group (ASG) method. The ASG method is defined as the set of allowed diffeomorphism modulo the set of trivial diffeomorphism,

$$\text{ASG} = \frac{\text{Allowed Symmetry Transformations}}{\text{Trivial Symmetry Transformations}}. \tag{4.1.14}$$

Here "allowed" means the transformation that is consistent with the specified boundary conditions, and "trivial" means the generator of transformations which vanishes after we have implemented the constraints and reduced it to a boundary integral.

To determine the allowed difeomorphisms, we need to specify a boundary condition by assigning the appropriate $p \in \mathbb{Z}$ in each components of NHEK deviation metric $h_{\mu\nu} = \mathcal{O}(r^p)$. We choose the boundary conditions

$$\begin{pmatrix} h_{\tau\tau} = \mathcal{O}(r^2) & h_{\tau\varphi} = \mathcal{O}(1) & h_{\tau\theta} = \mathcal{O}(\frac{1}{r}) & h_{\tau r} = \mathcal{O}(\frac{1}{r^2}) \\ h_{\varphi\tau} = h_{\tau\varphi} & h_{\varphi\varphi} = O(1) & h_{\varphi\theta} = \mathcal{O}(\frac{1}{r}) & h_{\varphi r} = \mathcal{O}(\frac{1}{r}) \\ h_{\theta\tau} = h_{\tau\theta} & h_{\theta\varphi} = h_{\varphi\theta} & h_{\theta\theta} = \mathcal{O}(\frac{1}{r}) & h_{\theta r} = \mathcal{O}(\frac{1}{r^2}) \\ h_{r\tau} = h_{\tau r} & h_{r\varphi} = h_{\varphi r} & h_{r\theta} = h_{\theta r} & h_{rr} = \mathcal{O}(\frac{1}{r^3}) \end{pmatrix}. \tag{4.1.15}$$

The diffeomorphisms which preserve the boundary conditions (4.1.15) are of the form

$$\zeta = [-r\epsilon'(\varphi) + O(1)]\partial_r + [C + O(\frac{1}{r^3})]\partial_\tau + [\epsilon(\varphi) + O(\frac{1}{r^2})]\partial_\varphi + O(\frac{1}{r})\partial_\theta \,, \tag{4.1.16}$$

where $\epsilon(\varphi)$ is an arbitrary smooth function, and C is an arbitrary constant. The subleading terms in (4.1.15) are considered as the trivial diffeomorphisms, and the leading terms

$$\zeta_\epsilon = \epsilon(\varphi)\partial_\varphi - r\frac{\partial\epsilon(\varphi)}{\partial\varphi}\partial_r \tag{4.1.17}$$

is then the ASG of NHEK. By the periodicity $\varphi \sim \varphi + 2\pi$, it is convenient to define $\epsilon_n(\varphi) = -e^{-in\varphi}$ and $\zeta_n = \zeta(\epsilon_n)$. Therefore the vector (4.1.17) can be read now as

$$\zeta_n = -e^{-in\phi}\partial_\phi - inre^{-in\phi}\partial_r\,. \tag{4.1.18}$$

It is clear that there is an infinite number of boundary condition that can be assigned, but using different conditions may lead to the different physics. Under the Lie brackets, the symmetry generators (4.1.18) obey the Virasoro algebra

$$i[\zeta_m, \zeta_n]_{L.B.} = (m-n)\zeta_{m+n}\,. \tag{4.1.19}$$

The diffeomorphism ζ is generated by a conserved charge $Q_\zeta[g]$ [12]. The Poisson bracket between this conserved charges, say $Q_\zeta[g]$ and $Q_\xi[g]$, is found to be the conserved charge of commutation between two isomorphisms, $Q_{[\zeta,\xi]}[g]$, plus a central term,

$$\{Q_\zeta, Q_\xi\}_{P.B.} = Q_{[\zeta,\xi]} + K\left[\zeta,\xi\right]\,, \tag{4.1.20}$$

where the central term $K\left[\zeta,\xi\right]$ is given by

$$K\left[\zeta,\xi\right] = \oint \mathbf{k}_\zeta\left(\bar{g}, \mathcal{L}_\xi \bar{g}\right). \tag{4.1.21}$$

The detail of 2-form $\mathbf{k}_\zeta$ can be read as

$$\mathbf{k}_\zeta\left(\bar{g}_{\mu\nu}, \mathcal{L}_\xi \bar{g}_{\mu\nu}\right) = \frac{k^{\mu\nu}}{32\pi}\varepsilon_{\mu\nu\alpha\beta}dx^\alpha \wedge dx^\beta\,, \tag{4.1.22}$$

where

$$\begin{aligned} k^{\mu\nu} \;=\; & \zeta^\nu\nabla^\mu h - \zeta^\nu\nabla_\rho h^{\mu\rho} + \frac{h}{2}\nabla^\nu\zeta^\mu - h^{\nu\rho}\nabla_\rho\zeta^\mu + \zeta_\rho\nabla^\nu h^{\mu\rho} \\ & -\left(\zeta^\mu\nabla^\nu h - \zeta^\mu\nabla_\rho h^{\nu\rho} + \frac{h}{2}\nabla^\mu\zeta^\nu - h^{\mu\rho}\nabla_\rho\zeta^\nu + \zeta_\rho\nabla^\mu h^{\nu\rho}\right)\,. \end{aligned} \tag{4.1.23}$$

In the last formula, the metric tensor $h_{\mu\nu}$ is the variation of $\bar{g}_{\mu\nu}$ with the deformation parameter ξ, i.e. $h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$. A detail discussion in getting this central term is given in appendix F.

4.1.3 Central Charge

As we have seen in the previous subsection, we will need the Lie derivative of NHEK metric tensor with the diffeomeorphism parameter is given in (4.1.15). To obtain a central charge from the central term (4.1.21), let us start by writing the non-vanishing metric elements of (4.1.8) are

$$\begin{aligned} \bar{g}_{\tau\tau} &= -2GJ\Omega^2((1+r^2)-\Lambda^2 r^2)\ ,\ \bar{g}_{\varphi t}=\bar{g}_{t\varphi}=2GJ\Omega^2\Lambda^2 r\,, \\ \bar{g}_{\varphi\varphi} &= 2GJ\Omega^2\Lambda^2\ ,\ \bar{g}_{\theta\theta}=2GJ\Omega^2\ ,\ \bar{g}_{rr}=\frac{2GJ\Omega^2}{1+r^2}\,. \end{aligned} \tag{4.1.24}$$

Accordingly, the contravariant version of these non-vanishing metric components can be read as

$$\begin{aligned} \bar{g}^{tt} &= -\frac{1}{2GJ\Omega^2(1+r^2)}\ ,\ \bar{g}^{\varphi t}=\bar{g}^{ta}=\frac{r}{2GJ\Omega^2(1+r^2)}\,, \\ \bar{g}^{\varphi\varphi} &= \frac{1}{2GJ\Omega^2\Lambda^2}-\frac{r^2}{2GJ\Omega^2(1+r^2)}\ ,\ \bar{g}^{\theta\theta}=\frac{1}{2GJ\Omega^2}\ ,\ \bar{g}^{rr}=\frac{1+r^2}{2GJ\Omega^2}\,. \end{aligned} \tag{4.1.25}$$

In performing the computation of central charge, we need the following Christoffel symbols

$$\begin{aligned} \Gamma^t_{r\varphi} &= -\frac{\Lambda^2}{2(1+r^2)}\ ,\ \Gamma^t_{rt}=\frac{r}{1+r^2}-\frac{\Lambda^2 r}{2(1+r^2)}\ ,\ \Gamma^r_{rr}=-\frac{r}{1+r^2}\,, \\ \Gamma^\varphi_{r\varphi} &= \frac{\Lambda^2 r}{2(1+r^2)}\ ,\ \Gamma^\theta_{r\theta}=0\ ,\ \Gamma^t_{rr}=0\ ,\ \Gamma^\varphi_{rt}=\frac{1-r^2}{2(1+r^2)}+\frac{\Lambda^2 r^2}{2(1+r^2)}\,. \end{aligned} \tag{4.1.26}$$

Using the diffeomorphism parameter (4.1.18), the Lie derivative of NHEK metric tensor

$$h_{\mu\nu}=\mathcal{L}_{\xi_n}\bar{g}_{\mu\nu}=\nabla_\mu\xi_{n\nu}+\nabla_\nu\xi_{n\mu} \tag{4.1.27}$$

which equivalently can be read as

$$\mathcal{L}_{\xi_n}\bar{g}_{\mu\nu}=\xi^\rho_n\partial_\rho\bar{g}_{\mu\nu}+\bar{g}_{\mu\rho}\partial_\nu\xi^\rho_n+\bar{g}_{\rho\nu}\partial_\mu\xi^\rho_n\,. \tag{4.1.28}$$

Therefore, the Lie derivative $=\mathcal{L}_{\xi_n}\bar{g}_{\mu\nu}=h_{\mu\nu}$ can be obtained as

$$\begin{aligned} h_{rr} &= \xi^r_n\partial_r\bar{g}_{rr}+2\bar{g}_{rr}\partial_r\xi^r_n=-\frac{4ine^{-in\varphi}GJ\Omega^2}{(1+r^2)^2}\,, \\ h_{r\varphi} &= \bar{g}_{rr}\partial_\varphi\xi^r_n=-\frac{2n^2re^{-in\varphi}GJ\Omega^2}{1+r^2}\,, \\ h_{\tau\tau} &= \xi^r_n\partial_r\bar{g}_{tt}=4inr^2e^{-in\varphi}GJ\Omega^2(1-\Lambda^2)\,, \\ h_{\varphi\varphi} &= 2\bar{g}_{\varphi\varphi}\partial_\varphi\xi^\varphi_n=-4ine^{-in\varphi}GJ\Omega^2\Lambda^2\,. \end{aligned} \tag{4.1.29}$$

The contravariant version of the tensor metric components above are

$$\begin{aligned}
h^{rr} &= \bar{g}^{rr}\bar{g}^{rr}h_{rr} = -\frac{2ine^{-in\varphi}}{2GJ\Omega^2}\,,\\
h^{r\varphi} &= \bar{g}^{rr}\bar{g}^{\varphi\varphi}h_{r\varphi} = -\frac{n^2re^{-in\varphi}}{2GJ\Omega^2}\Big(\frac{1}{\Lambda^2}-\frac{r^2}{1+r^2}\Big)\,,\\
h^{rt} &= \bar{g}^{rr}\bar{g}^{t\varphi}h_{r\varphi} = -\frac{n^2r^2e^{-in\varphi}}{2GJ\Omega^2(1+r^2)}\,,\\
h^{tt} &= \bar{g}^{tt}\bar{g}^{tt}h_{tt} + 2\bar{g}^{tt}\bar{g}^{t\varphi}h_{t\varphi} + \bar{g}^{t\varphi}\bar{g}^{t\varphi}h_{\varphi\varphi} = \frac{2inr^2e^{-in\varphi}}{2GJ\Omega^2(1+r^2)^2}\,,\\
h^{t\varphi} &= \bar{g}^{tt}\bar{g}^{\varphi t}h_{tt} + (\bar{g}^{tt}\bar{g}^{\varphi\varphi} + \bar{g}^{t\varphi}\bar{g}^{\varphi t})h_{t\varphi} + \bar{g}^{t\varphi}\bar{g}^{\varphi\varphi}h_{\varphi\varphi} = \frac{inre^{-in\varphi}}{GJ\Omega^2(1+r^2)^2}\,,\\
h^{\varphi\varphi} &= \bar{g}^{\varphi t}\bar{g}^{\varphi t}h_{tt} + 2\bar{g}^{\varphi t}\bar{g}^{\varphi\varphi}h_{t\varphi} + \bar{g}^{\varphi\varphi}\bar{g}^{\varphi\varphi}h_{\varphi\varphi} = \frac{ine^{-in\varphi}}{GJ\Omega^2}\left(\frac{1}{\Lambda^2}-\frac{r^2\,(2+r^2)}{1+r^2}\right).
\end{aligned}\tag{4.1.30}$$

The component of 2-form in the central term (4.1.21) that survives after performing the θ integration and taking asymptotic r is

$$\begin{aligned}
k^{rt} &= \xi^t_m\nabla^r h - \xi^t_m\nabla_\rho h^{r\rho} + \frac{h}{2}\nabla^t\xi^r_m - h^{t\rho}\nabla_\rho\xi^r_m + \xi_{m\rho}\nabla^t h^{r\rho}\\
&\quad -\xi^r_m\nabla^t h + \xi^r_m\nabla_\rho h^{t\rho} - \frac{h}{2}\nabla^r\xi^t_m + h^{r\rho}\nabla_\rho\xi^t_m - \xi_{m\rho}\nabla^r h^{t\rho}\,.
\end{aligned}\tag{4.1.31}$$

In the last equation, we have replaced the diffeomorphism parameter ζ to ξ_m in the tensor $k^{\mu\nu}$ which is given in (4.1.23). It is related to the case that we are dealing where rather than having two different diffeomorphism parameters ζ and ξ, we are discussing the central term that depends on the diffeomorphism parameters which differ in their modes, i.e. $K[\xi_m,\xi_n]$.

The fact that α is a free parameter in (4.1.38) allow us just to take the term that couple to m^3 only after setting $m+n=0$ in our k^{rt} calculation. The reason is the terms that coupled to m can be swept away by choosing an appropriate value of α. Hence each terms in k^{rt} above can be computed as follows

$$\begin{aligned}
\xi^r_m\nabla_\rho h^{t\rho} &= \xi^r_m(\partial_\rho h^{t\rho} + \Gamma^t_{\rho\sigma}h^{\sigma\rho} + \Gamma^\rho_{\rho\sigma}h^{t\sigma}) = \xi^r_m(\partial_\varphi h^{t\varphi} + \partial_r h^{tr} + 2\Gamma^t_{r\varphi}h^{r\varphi} + 2\Gamma^t_{rt}h^{rt} + \Gamma^\rho_{\rho r}h^{tr})\,,\\
&\sim \frac{imn^2r^2(r^2-1)e^{-i(m+n)\varphi}}{2GJ\Omega^2(1+r^2)^2},\\
-h^{t\rho}\nabla_\rho\xi^r_m &= -h^{t\rho}(\partial_\rho\xi^r_m + \Gamma^r_{\rho\sigma}\xi^\sigma_m) = -h^{t\varphi}\partial_\varphi\xi^r_m - h^{tr}(\partial_r\xi^r_m + \Gamma^r_{rr}\xi^r_m)\\
&\sim \frac{imn^2r^2e^{-i(m+n)\varphi}}{2GJ\Omega^2(1+r^2)^2}\Big(\frac{2m}{n}-1\Big)\,,
\end{aligned}$$

$$
\begin{aligned}
h^{r\rho}\nabla_\rho \xi^t_m &= h^{r\rho}(\partial_\rho \xi^t_m + \Gamma^t_{\rho\sigma}\xi^\sigma_m) = (h^{r\varphi}\Gamma^t_{\varphi r} + h^{rt}\Gamma^t_{tr})\xi^r_m \\
&\sim \frac{imn^2r^2(r^2-1)e^{-i(m+n)\varphi}}{4GJ\Omega^2(1+r^2)^2}, \\
\xi_{m\rho}\nabla^t h^{r\rho} &= \xi^\rho_m \bar{g}^{rr}(\bar{g}^{tt}\nabla_t h_{r\rho} + \bar{g}^{ta}\nabla_a h_{r\rho}) \\
&= \xi^\rho_m \bar{g}^{rr}\bar{g}^{tt}(\partial_t h_{r\rho} - \Gamma^\sigma_{tr}h_{\sigma\rho} - \Gamma^\sigma_{t\rho}h_{r\sigma}) + \xi^\rho_m \bar{g}^{rr}\bar{g}^{ta}(\partial_a h_{r\rho} - \Gamma^\sigma_{\varphi r}h_{\sigma\rho} - \Gamma^\sigma_{\varphi\rho}h_{r\sigma}) \\
&= \xi^r_m \bar{g}^{rr}\bar{g}^{tt}(-\Gamma^\varphi_{tr}h_{\varphi r} - \Gamma^\varphi_{tr}h_{r\varphi}) + \xi^\varphi_m \bar{g}^{rr}\bar{g}^{t\varphi}\partial_\varphi h_{r\varphi} + \xi^r_m \bar{g}^{rr}\bar{g}^{t\varphi}\partial_\varphi h_{rr} \\
&\quad + \xi^r_m \bar{g}^{rr}\bar{g}^{t\varphi}(-\Gamma^\varphi_{\varphi r}h_{\varphi r} - \Gamma^\varphi_{\varphi r}h_{r\varphi}) \\
&\sim \frac{imn^2r^2e^{-i(m+n)\varphi}}{r^2-1(1+r^2)}\Big(\frac{3-r^2}{1+r^2} - \frac{n}{m}\Big), \\
-\xi_{m\rho}\nabla^r h^{t\rho} &= -\xi_{m\rho}\bar{g}^{rr}(\partial_r h^{t\rho} + \Gamma^t_{r\sigma}h^{\sigma\rho} + \Gamma^\rho_{r\sigma}h^{t\sigma}) = -\xi_{mr}\bar{g}^{rr}(\partial_r h^{tr} + \Gamma^t_{rt}h^{tr} + \Gamma^t_{r\varphi}h^{\varphi r} + \Gamma^r_{rr}h^{tr}) \\
&\sim \frac{imn^2r^2e^{-i(m+n)\varphi}}{4GJ\Omega^2(1+r^2)}\left(1 - \frac{4}{1+r^2}\right),
\end{aligned} \tag{4.1.32}
$$

where at the final result of each terms above, we keep only those which give m^3 contributions. However, the condition $m=-n$ is not applied yet at the moment. As we take the limit of $r\to\infty$, many terms above vanish, where finally we have

$$
k^{rt} = \frac{i(m-n)n^2e^{-i(m+n)\varphi}}{2GJ\Omega^2}\,. \tag{4.1.33}
$$

From (F.0.40) and (F.0.52), we get

$$
\oint (d^2x)_{\mu\nu}k^{\mu\nu} = \oint 2(d^2x)_{rt}k^{rt}, \tag{4.1.34}
$$

and explicitly one can show

$$
(d^2x)_{rt} = 2(GJ\Omega^2)^2\Lambda^2 d\theta d\varphi\,. \tag{4.1.35}
$$

Consequently, the central term (4.1.21) is obtained as

$$
K[\xi_m,\xi_n] = -\frac{i(m-n)n^2}{16\pi}\oint 2JG\Omega^2\Lambda^2e^{-i(m+n)\varphi}d\theta d\varphi \tag{4.1.36}
$$

$$
= -\frac{i(m-n)n^2J}{2}\delta_{m+n}\,. \tag{4.1.37}
$$

The classical version of the charge Q_{ξ_m} associated to the diffeomorphism parameter ξ_m is defined in (F.0.36). Before proceeding to the classical to quantum transition prescription, i.e. $\{,\}_{P.B}\to i\,[,]$ transition, we redefine

$$
Q_{\xi_m} = L_m - \alpha\delta_m\,, \tag{4.1.38}
$$

where α is some constants. From equations (F.0.36) and (F.0.52), it is easy to see that if ξ_m is scaled by a factor, the right hand side of (4.1.38) also needs to be scaled by the same factor. Specifically, one has

$$Q_{[\xi_m,\xi_n]} = Q_{-i(m-n)\xi_{m+n}} = -i(m-n)\Big(L_{m+n} - \alpha\delta_{m+n}\Big)\,. \tag{4.1.39}$$

Note that we have used the relation (4.1.19) in writing the last equation. Accordingly, from equation (F.0.38), we can write

$$\begin{aligned}[L_m\,,\,L_n] &= i\Big\{Q_{\xi_m}\,,\,Q_{\xi_n}\Big\}_{P.B.} = i\Big(Q_{[\xi_m,\xi_n]} + K[\xi_m,\xi_n]\Big) \\ &= (m-n)L_{m+n} - 2m\alpha\delta_{m+n} + iK[\xi_m,\xi_n]\,. \end{aligned} \tag{4.1.40}$$

Comparing this result with the Virasoro algebra,

$$[L_m\,,\,L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,, \tag{4.1.41}$$

gives us

$$K[\xi_m,\xi_n] = -i\frac{c}{12}m\Big(m^2 - 1 + \frac{24\alpha}{c}\Big)\delta_{m+n}\,. \tag{4.1.42}$$

From the last equation, we understand why the value central charge c is controlled by the coefficient of m^3 in the central term $K[\xi_m,\xi_n]$. It is because we can choose an appropriate α to remove the contribution of linear m in (4.1.42). Using the result for $K[\xi_m,\xi_n]$ as obtained in (4.1.37), the central charge reads

$$c = 12J\,. \tag{4.1.43}$$

If we restore the units in the calculation, where from the beginning of this computation we have set that $\hbar = c = G = k_B = 1$, the central charge (4.1.43) would be

$$c = \frac{12J}{\hbar}\,. \tag{4.1.44}$$

Astronomical data strongly suggests that the astrophysical object GRS 1915+105 is a near-extremal rotating black hole. It has the rotational parameter bound $a \geq 0.98M$ [13], where an extremal black hole has $a = M$. Therefore, the associated central charge for this object would be $c = (2 \pm 1) \times 10^{79}$, which is a large number.

4.1.4 Temperature

The vacuum state near the Kerr black hole horizon is the Frolov-Throne vacuum [78], which is a generalization of the Hartle-Hawking vacuum for Schwarzschild black hole. The Frolov-Throne vacuum takes into account the rotation of black holes, from which a correction to the Hartle-Hawking vacuum comes. To construct the Frolov-Thorne vacuum for generic Kerr, starts by expanding the quantum fields in eigenmodes of the asymptotic energy ω and angular momentum m. As an example, we could write an expansion for scalar field Φ as

$$\Phi = \sum_{\omega,m,l} \phi_{\omega ml} e^{-i\omega\hat{t}+im\hat{\phi}} f_l(r,\theta)\,. \tag{4.1.45}$$

After we trace over the region inside the horizon, the vacuum is a diagonal density matrix in the energy-angular momentum eigenbasis with a Boltzmann weighting factor

$$e^{-\hbar\frac{\omega-\Omega_H m}{T_H}}\,. \tag{4.1.46}$$

In the non-rotating case, $\Omega_H = 0$, (4.1.46) reduces to the Hartle-Hawking vacuum.

A procedure to take the limit of the near horizon region and near extremal black hole [19] allows us to have

$$e^{-i\omega\hat{t}+im\hat{\phi}} = e^{-\frac{i}{\lambda}(2M\omega-m)t+im\phi} = e^{-in_R t+in_L\phi}, \tag{4.1.47}$$

where

$$n_L \equiv m, \qquad n_R \equiv \frac{1}{\lambda}(2M\omega - m) \tag{4.1.48}$$

are the left and right charges associated to ∂_ϕ and ∂_t in the near-horizon region. In terms of these variables the Boltzmann factor (4.1.46) is

$$e^{-\hbar\frac{\omega-\Omega_H m}{T_H}} = e^{-\frac{n_L}{T_L}-\frac{n_R}{T_R}}\,, \tag{4.1.49}$$

where the dimensionless left and right temperatures are

$$T_L = \frac{r_+ - M}{2\pi(r_+ - a)}\,, \qquad T_R = \frac{r_+ - M}{2\pi\lambda r_+}\,. \tag{4.1.50}$$

In the case of extremal limit $a \to M$, (4.1.50) reduce to

$$T_L = \frac{1}{2\pi}\,, \qquad T_R = 0\,. \tag{4.1.51}$$

4.1.5 Microscopic origin of the Bekenstein-Hawking-Kerr entropy

In the previous subsection, only one copy of the temperatures which is non-zero, i.e. T_L. Accordingly, the associate central charge c_L would be $12J$. To get the corresponding black hole entropy via CFT description, we employ the famous Cardy formula (3.3.116)

$$S_{CFT} = 2\pi\sqrt{\frac{c_L E}{6}}\,, \tag{4.1.52}$$

where E is the energy. Note that the formula (3.3.116) contains both left and right movers contributions, where in the last formula we have only the left sector. It is related to the fact that in the extremal case, there is only a single copy of conformal symmetry that we can read in the NHEK spacetime structure. This conformal structure which leads to the computation of central charge (4.1.43).

The first law of thermodynamics dictates that $dE = TdS$, so we could have

$$dS_{CFT} = 2\pi\sqrt{\frac{c_L}{6}}\frac{dE}{\sqrt{E}} = 2\pi\sqrt{\frac{c_L}{6}}\frac{T}{2\sqrt{E}}dS_{CFT} \tag{4.1.53}$$

which provides us

$$E = \frac{c_L}{6}\pi^2 T^2\,. \tag{4.1.54}$$

Hence we can write the alternative form of Cardy formula as

$$S_{CFT} = \frac{1}{3}\pi^2 c_L T_L\,, \tag{4.1.55}$$

after we plug the corresponding T_L rather than T in the last expression. Having in our hand $T_L = 1/2\pi$ then (4.1.55) gives us

$$S_{CFT} = 2\pi J \tag{4.1.56}$$

which is exactly what we have for Bekenstein-Hawking entropy of Kerr black holes (2.3.275) after setting $r_+ = M$.

4.1.6 The bulk-to-boundary propagator and the 2-point function

The bulk-to-boundary propagator has an important role in the AdS/CFT correspondence prescription to the holographic calculation of correlation functions. This prescription may be applied to the Kerr/CFT correspondence discussions. As we have briefly reviewed in section 3.4, the bulk-to-boundary propagator can be found by finding the Green function solutions to the field's equation of motion in the bulk with some requirements that must be satisfied on the boundary. Nevertheless, performing the same prescription in Kerr/CFT discussion seems to be very hard, due to the equation of motion that corresponds to a test particle in Kerr background, as well as NHEK or near-NHEK, is much more complicated compared to those in AdS. That is why, rather than deriving the bulk-to-boundary propagator in Kerr/CFT by using Green function technique, we could simply dictate the form of this propagator. Indeed, this way is less elegant compared to the case in AdS/CFT correspondence, but it is found that this method works properly.

In this subsection we will discuss the bulk to boundary propagator that can be used in Kerr/CFT correspondence. This propagator is proposed by Becker et al [18] where they discuss the near-NHEK case and employ the ingoing radial solution of the scalar wave equation in the far or asymptotically region of Kerr geometry. Before we show the corresponding propagator, let us discuss the near-NHEK geometry first, and get the the corresponding radial solutions around this region. The near-NHEK spacetime is just a slight modification of the NHEK geometry as we have discussed in the previous subsection. The modification is we consider a Kerr black hole which is rotating near to the extremal case. Therefore, in this case $T_H \to 0$ and $\hat{r} \to r_+$, but we keep the dimensionless near-horizon temperature

$$T_R \equiv \frac{2MT_H}{\lambda} = \frac{\tau_H}{4\pi\lambda} \tag{4.1.57}$$

fixed as $\lambda \to 0$. Accordingly, at this near extremality condition,

$$\begin{aligned} r_+ &= M + \lambda M 2\pi T_R + O(\lambda^2)\,, &(4.1.58)\\ a &= M - 2M(\lambda\pi T_R)^2 + O(\lambda^3)\,. &(4.1.59)\end{aligned}$$

In getting the near-NHEK (near horizon of the near extremal Kerr) geometry from the metric (4.1.1), instead of using the transformation (4.1.2), we employ

$$\begin{aligned} t &= \lambda\frac{\hat{t}}{2M}\,, &(4.1.60)\\ r &= \frac{\hat{r}-r_+}{\lambda r_+}\,, &(4.1.61)\\ \phi &= \hat{\phi} - \frac{\hat{t}}{2M}\,, &(4.1.62)\end{aligned}$$

which gives us the metric

$$ds^2 = 2J\Gamma\left(-r(r+4\pi T_R)dt^2 + \frac{dr^2}{r(r+4\pi T_R)} + d\theta^2 + \Lambda^2\left(d\phi + (r+2\pi T_R)dt\right)^2\right) . \quad (4.1.63)$$

The metric (4.1.63) is referred as the near-NHEK metric.

Now we consider a scalar field on near-NHEK Kerr whose modes expansion can be read as

$$\Phi = e^{-i\omega\hat{t}+im\hat{\phi}}R(\hat{r})S(\theta) . \quad (4.1.64)$$

We assign the "quantum numbers" n_L and n_R whose definitions are

$$n_L = m \ , \quad n_R = \frac{1}{\lambda}\left(2M\omega - m\right) . \quad (4.1.65)$$

These "quantum numbers" obey

$$e^{-in_R t+in_L\phi} = e^{-i\omega\hat{t}+im\hat{\phi}} . \quad (4.1.66)$$

The Boltzmann factor associated to the Frolov-Thorne vacuum state is $e^{-\frac{\omega-m\Omega_H}{T_H}}$. Identifying the "Boltzmann factor" that comes from the microscopic one (left and right movers CFT) and the macroscopic theory (semiclassical gravity), we may write

$$e^{-\frac{\omega-m\Omega_H}{T_H}} = e^{-\frac{n_L}{T_L}-\frac{n_R}{T_R}} . \quad (4.1.67)$$

The last equation gives us the definition of the temperatures in left and right movers CFT, or the left and right temperatures for short

$$T_L = \frac{r_+ - M}{2\pi(r_+ - a)} \ , \quad T_R = \frac{r_+ - M}{2\pi\lambda r_+} . \quad (4.1.68)$$

In this near extremal limit, T_R and n_R are kept to be fixed as $T_H \to 0$ and $\lambda \to 0$. From the definition of near horizon quantum number n_R in (4.1.65), as $\lambda \to 0$ we have

$$m \approx 2M\omega . \quad (4.1.69)$$

In the other hand, the near extremal condition also tells us the angular velocity at the horizon may be approximated as

$$\Omega_H \approx \frac{1}{2M} . \quad (4.1.70)$$

Therefore, we can see only the near superradiant wave modes $m - m\Omega_H \to 0$ which survive the near extremal limit.

The radial equation of scalar wave in Kerr geometry (4.1.1),

$$\Delta\partial_{\hat{r}}^2 R + 2(\hat{r} - M)\partial_{\hat{r}} R + \left(\frac{[(\hat{r}^2 + a^2)\omega - am]^2}{\Delta} + 2am\omega - K_\ell \right) R = 0\,. \tag{4.1.71}$$

By using the dimensionless redefinition

$$x = \frac{\hat{r} - r_+}{r_+}\,, \tag{4.1.72}$$

the radial equation (4.1.71) reduces to

$$x(x + \tau_H)R'' + (2x + \tau_H)R' + (V - K_\ell)\, R = 0\,. \tag{4.1.73}$$

In the equation above, the prime stands for ∂_x, and the "potential" V is given by

$$V = m^2 + \frac{[x(x+2)m + \tau_H(m + \frac{n_R}{2\pi T_R})]^2}{4x(x + \tau_H)}\,. \tag{4.1.74}$$

If we consider the far region, $x \gg \tau_H$, the radial wave equation (4.1.73) becomes

$$x^2 R'' + 2xR' + \left(\frac{1}{4} m^2 (2 + x)^2 + m^2 - K_\ell \right) R = 0\,. \tag{4.1.75}$$

The function

$$\begin{aligned} R_{far} &= N \Big[A e^{-\frac{1}{2}imx} x^{-\frac{1}{2}+\beta} \; {}_1F_1(\frac{1}{2} + \beta + im, 1 + 2\beta, imx) \\ &\quad + B e^{-\frac{1}{2}imx} x^{-\frac{1}{2}-\beta} \; {}_1F_1(\frac{1}{2} - \beta + im, 1 - 2\beta, imx) \Big] \end{aligned} \tag{4.1.76}$$

solves the equation (4.1.75), where A, B are constant coefficients, and N is the overall normalization to be determined later for convenience. The function ${}_1F_a(a; b; z)$ is known as the confluent hypergeometric function of the first kind whose integral representation can be read as

$${}_1F_1\left(a; b; z\right) = \frac{\Gamma\left(b\right)}{\Gamma\left(b-a\right)\Gamma\left(a\right)} \int\limits_0^1 e^{zt} t^{a-1} \left(1-t\right)^{b-a-1} dt\,. \tag{4.1.77}$$

The parameter β in solution (4.1.76) is defined as

$$\beta^2 = K_\ell - 2m^2 + \frac{1}{4}\,. \tag{4.1.78}$$

Next we consider the near region, given by $x \ll 1$. The corresponding wave equation obtained from (4.1.73) is

$$x(x+\tau_H)R'' + (2x+\tau_H)R' + \left(\frac{\left(\frac{\tau_H n_R}{2\pi T_R} + m(2x+\tau_H)\right)^2}{4x(x+\tau_H)} + m^2 - K_\ell\right) R = 0\,. \qquad (4.1.79)$$

The solutions to this near region equation are

$$\begin{aligned} R^{in}_{near} &= N x^{-\frac{i}{2}(m+\frac{n_R}{2\pi T_R})} \left(\frac{x}{\tau_H}+1\right)^{-\frac{i}{2}(m-\frac{n_R}{2\pi T_R})} \\ &\quad \times {}_2F_1\left(\frac{1}{2}+\beta-im, \frac{1}{2}-\beta-im; 1-i(m+\frac{n_R}{2\pi T_R}); -\frac{x}{\tau_H}\right)\,, \\ R^{out}_{near} &= N x^{\frac{i}{2}(m+\frac{n_R}{2\pi T_R})} \left(\frac{x}{\tau_H}+1\right)^{-\frac{i}{2}(m-\frac{n_R}{2\pi T_R})} \\ &\quad \times {}_2F_1\left(\frac{1}{2}+\beta+i\frac{n_R}{2\pi T_R}, \frac{1}{2}-\beta+i\frac{n_R}{2\pi T_R}; 1+i(m+\frac{n_R}{2\pi T_R}); -\frac{x}{\tau_H}\right)\,. \end{aligned} \qquad (4.1.80)$$

The integral representation of hypergeometric function that we used in the solutions above is given by

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt\,. \qquad (4.1.81)$$

These near and far regions coincide in an area, namely matching region. Therefore, the near and far solutions will also coincide. In this matching region, the far solution reduces to

$$R_{far} \to N\left(Ax^{-\frac{1}{2}+\beta} + Bx^{-\frac{1}{2}-\beta}\right) \qquad (4.1.82)$$

which is obtained after taking the limit $x \ll 1$ of the far solution (4.1.76). Taking this limit to the far solution can be considered as the approximation of the far solution to the outer part of the near region. In the other hand, taking the far limit of near solution can be done by setting $x \gg \tau_H$ of the solutions (4.1.80), which is

$$\begin{aligned} R^{in}_{near} \to \quad & N\tau_H^{\frac{1}{2}-\frac{i}{2}\left(m+\frac{n_R}{2\pi T_R}\right)} n \\ & \times \left(\frac{\Gamma(-2\beta)\Gamma\left(1-im-i\frac{n_R}{2\pi T_R}\right)\tau_H^{\beta}}{\Gamma\left(\frac{1}{2}-\beta-im\right)\Gamma\left(\frac{1}{2}-\beta-i\frac{n_R}{2\pi T_R}\right)} + \frac{\Gamma(2\beta)\Gamma\left(1-im-i\frac{n_R}{2\pi T_R}\right)\tau_H^{-\beta}}{\Gamma\left(\frac{1}{2}+\beta-im\right)\Gamma\left(\frac{1}{2}+\beta-i\frac{n_R}{2\pi T_R}\right)}\right)\,. \end{aligned} \qquad (4.1.83)$$

We only consider the ingoing one since we are discussing the wave that will penetrate to the near-NHEK region. To get the coefficients A and B in (4.1.82), we can match solutions

(4.1.82) and (4.1.83), which gives us

$$\begin{aligned} A &= \frac{\Gamma(2\beta)\Gamma(1-im-i\frac{n_R}{2\pi T_R})}{\Gamma(\frac{1}{2}+\beta-im)\Gamma(\frac{1}{2}+\beta-i\frac{n_R}{2\pi T_R})}\tau_H^{\frac{1}{2}-\beta-\frac{i}{2}(m+\frac{n_R}{2\pi T_R})}, \\ B &= \frac{\Gamma(-2\beta)\Gamma(1-im-i\frac{n_R}{2\pi T_R})}{\Gamma(\frac{1}{2}-\beta-im)\Gamma(\frac{1}{2}-\beta-i\frac{n_R}{2\pi T_R})}\tau_H^{\frac{1}{2}+\beta-\frac{i}{2}(m+\frac{n_R}{2\pi T_R})}. \end{aligned} \tag{4.1.84}$$

A form of the bulk-to-boundary propagator for scalar fields in Kerr/CFT correspondence is given [18]

$$K(r,t',\phi';t,\phi) = \int dm \int d\omega\; R^{in}_{\text{asymp.}}\, e^{-im(\phi-\phi')}e^{i\omega(t-t')}\,, \tag{4.1.85}$$

where $R^{in}_{\text{asymp.}}$ is the asymptotic expansion of the ingoing radial solution in near region (4.1.80)

$$R^{in}_{\text{asymp.}}(r,m,\omega) \;\sim\; N\Big[A\Big(r^{-\frac{1}{2}+\beta}+\mathcal{O}(r^{-3/2+\beta})\Big)+B\Big(r^{-\frac{1}{2}-\beta}+\mathcal{O}(r^{-3/2-\beta})\Big)\Big]\,. \tag{4.1.86}$$

Here, the normalization N is chosen to be A^{-1}. Consequently, the propagator can be read as

$$\begin{aligned} K(\phi,t,r;\phi',t') &= \int dm\, d\omega \left(r^{-\frac{1}{2}+\beta}+\frac{B(n_L,n_R)}{A(n_L,n_R)}\,r^{-\frac{1}{2}-\beta}\right)e^{-im(\phi-\phi')+i\omega(t-t')}+\dots \\ &\approx r^{-\frac{1}{2}+\beta}\,\delta(\phi-\phi')\,\delta(t-t')+r^{-\frac{1}{2}-\beta}\int dm\, d\omega\,\frac{B(n_L,n_R)}{A(n_L,n_R)}\,e^{-im(\phi-\phi')+i\omega(t-t')} \\ &\quad +\dots \end{aligned}$$

The leading behavior of the propagator

$$K(\phi,t,r;\phi',t') \to r^{-\frac{1}{2}+\beta}\,\delta(\phi-\phi')\,\delta(t-t') \tag{4.1.87}$$

is exactly what we expect from a bulk-to-boundary propagator, up to a scale factor to the scalar field. Inserting the bulk-to-boundary propagator into

$$\langle\mathcal{O}(t_1,\phi_1)\mathcal{O}(t_2,\phi_2)\rangle \;\sim\; \int d\phi\, dt\,\sqrt{-g}g^{rr}\,\bar{K}(r,t,\phi;t_1,\phi_1)\,\partial_r K(r,t,\phi;t_2\phi_2)\,\Big|_{r=r_{\text{B}}}, \tag{4.1.88}$$

with r_B is the radius of boundary, one can extract the two-point function behavior appropriate for the retarded Green's function:

$$\langle\mathcal{O}(t_1,\phi_1)\mathcal{O}(t_2,\phi_2)\rangle \sim \text{contact term} + \beta\int dm\; d\omega\;\frac{B(n_L,n_R)}{A(n_L,n_R)}e^{-im(\phi_2-\phi_1)+i\omega(t_2-t_1)}+\dots \tag{4.1.89}$$

Therefore, after performing the regularization, the retarded Green's function can be read-off directly from (4.1.89), and is

$$G_R \sim \frac{B(n_L,n_R)}{A(n_L,n_R)}\,, \tag{4.1.90}$$

in agreement with the prescription in AdS/CFT [79].

4.2 Non-Extremal Kerr/CFT

4.2.1 Wave equation and $SL(2,\mathbb{R})$ squared Casimir

We start by writing a general form of our scalar wave solution which can be decomposed as

$$\Phi(t,r,\theta,\phi) = e^{-i\omega t + im\phi} R(r) S(\theta)\,. \tag{4.2.91}$$

Plugging this ansatz into the massive Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) + \mu^2\Phi = 0 \tag{4.2.92}$$

gives us

$$\begin{aligned} &\frac{1}{S(\theta)\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta S(\theta)\right) - \frac{\left(\Delta - a^2\sin^2\theta\right)m^2}{\Delta\sin^2\theta} + \frac{2a\left(\Delta - (r^2+a^2)\right)m\omega}{\Delta} \\ + &\frac{\left(r^2+a^2-\Delta a^2\sin^2\theta\right)\omega^2}{\Delta} - \left(r^2+a^2\cos^2\theta\right)\mu^2 = \frac{1}{R(r)}\partial_r\left(\Delta\partial_r R(r)\right)\,. \end{aligned} \tag{4.2.93}$$

Initially, it was not expected that the scalar wave equation in Kerr spacetime is separable. It was Teukolsky [80] who first showed that the wave equation for field perturbations in the Kerr background are separable. Interestingly, this fact does not apply only for scalar perturbation, but also for spins 1/2, 1, and 2. For the massive scalar perturbation equation (4.2.93), the angular part of the equation is

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{m^2}{\sin^2\theta} + (\omega^2-\mu^2)a^2\cos^2\theta\right]S(\theta) = -K_l S(\theta)\,. \tag{4.2.94}$$

The corresponding radial one can be read as

$$\begin{aligned} \left[\partial_r(\Delta\partial_r) + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + (r^2+2M(r+2M))\omega^2 - \mu^2 r^2\right]R(r) \\ = K_l R(r)\,. \end{aligned} \tag{4.2.95}$$

The constants K_l are the corresponding eigen values on a sphere. Both of the two equations (4.2.94) and (4.2.95) can be solved by using the Heun functions.

However, we can restrict our discussion to the low frequencies only, i.e. $M\omega \ll 1$, hence the last term in the square bracket of (4.2.95) can be neglected. Then the spacetime in our discussion can be divided into 2 regions,

$$\text{"Near"}:\quad r \ll \frac{1}{\omega}\,, \tag{4.2.96}$$

$$\text{“Far”}: \quad r \gg M\,. \tag{4.2.97}$$

Expression (4.2.96) means that the wavelength of our test particle is very large compared to the radius of curvature, and (4.2.97) indicates the very far region from black hole horizon, i.e. a large number multiple of M. This two regions overlap in the matching region

$$M \ll r \ll \omega^{-1}\,. \tag{4.2.98}$$

The wave equation (4.2.95) can be solved both in the near and far regions by using some special functions. In addition, to get a full solution, one need to match the obtained solutions in the near and far regions along a surface in the matching region (4.2.98).

In the near region, the angular equation (4.2.94) for the low frequency of scalar field reduces to

$$\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) - \frac{m^2}{\sin^2\theta}\right] S\left(\theta\right) = -K_l S\left(\theta\right) \quad , \quad l = -m, ..., +m\,. \tag{4.2.99}$$

This is just the standard Laplacian on 2-sphere, where the separation constant is $K_l = l\left(l+1\right)$. Moreover, the radial one (4.2.95) becomes

$$\left[\partial_r(\Delta\partial_r) + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)}\right] R(r) = l(l+1)\, R(r)\,, \tag{4.2.100}$$

which can be solved analytically and the solutions are hypergeometric functions. We know that an equation that is solved by the hypergeometic functions possesses $SL\left(2,\mathbb{R}\right)$ symmetry. This hints us the hidden conformal symmetry that we will explore in the next subsection.

4.2.2 Hidden Conformal Symmetry

To show the $SL\left(2,\mathbb{R}\right)$ symmetry of the solution space for radial equation (4.2.100), first we introduce the following coordinates

$$\begin{aligned}
w^+ &= \sqrt{\frac{r-r_+}{r-r_-}}\, e^{2\pi T_R \phi}\,, \\
w^- &= \sqrt{\frac{r-r_+}{r-r_-}}\, e^{2\pi T_L \phi - t/2M}\,, \\
y &= \sqrt{\frac{r-r_+}{r-r_-}}\, e^{\pi(T_R+T_L)\phi - t/4M}\,,
\end{aligned} \tag{4.2.101}$$

where

$$T_L = \frac{r_+ + r_-}{4\pi a}, \quad T_R = \frac{r_+ - r_-}{4\pi a}\,. \tag{4.2.102}$$

Accordingly, we can define the vector fields

$$\begin{aligned} H_1 &= i\partial_+\,, \\ H_0 &= i\left(w^+\partial_+ + \frac{1}{2}y\partial_y\right)\,, \\ H_{-1} &= i\left(\left(w^+\right)^2\partial_+ + w^+y\partial_y - y^2\partial_-\right)\,, \end{aligned} \tag{4.2.103}$$

which obey the algebra of $SL(2,\mathbb{R})$ group

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1}, \quad [H_{-1}, H_1] = -2iH_0\,, \tag{4.2.104}$$

and similarly for $\bar{H}_0$ and $\bar{H}_{\pm 1}$. It turns out the following vectors

$$\begin{aligned} \bar{H}_1 &= i\partial_-\,, \\ \bar{H}_0 &= i\left(w^-\partial_- + \frac{1}{2}y\partial_y\right)\,, \\ \bar{H}_{-1} &= i\left(\left(w^-\right)^2\partial_- + w^-y\partial_y - y^2\partial_+\right)\,, \end{aligned} \tag{4.2.105}$$

are also the $SL(2,\mathbb{R})$ generators, where they satisfy the algebra

$$[\bar{H}_0, \bar{H}_{\pm 1}] = \mp i \bar{H}_{\pm 1}, \quad [\bar{H}_{-1}, \bar{H}_1] = -2i\bar{H}_0\,, \tag{4.2.106}$$

Writing the vectors above in $\{t, r, \theta, \phi\}$ coordinates, we can have

$$\begin{aligned} H_{\pm 1} &= ie^{\mp 2\pi T_R \phi}\left(\pm\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\left(\frac{r-M}{\Delta^{1/2}}\right)\partial_\phi + \frac{2T_L}{T_R}\left(\frac{Mr-a^2}{\Delta^{1/2}}\right)\partial_t\right)\,, \\ H_0 &= \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t\,, \end{aligned} \tag{4.2.107}$$

and

$$\begin{aligned} \bar{H}_{\pm 1} &= ie^{\mp 2\pi T_L \phi + \frac{t}{2M}}\left(\pm\Delta^{1/2}\partial_r - \frac{a}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right)\,, \\ \bar{H}_0 &= 2iM\partial_t\,. \end{aligned} \tag{4.2.108}$$

In group theory we learn about the squared Casimir, i.e. operator that commutes with all generators of the group. In subsection 3.1.1, we have seen that p^2 is a squared Casimir in the Poincare group. This squared Casimir, applied to the massive scalar wave, has the eigenvalue the squared mass of the scalar, m^2. This mass is invariant under all fields transformation in the Poincare group, which is the characteristic of squared Casimir eigenvalue.

The squared Casimir in the $sl(2,\mathbb{R})$ algebra, constructed by the vectors in (4.2.103) is

$$\begin{aligned}\mathcal{H}^2 &= -H_0^2 + (H_1 H_{-1} + H_{-1} H_1) \\ &= \frac{1}{4}\left(y^2\partial_y^2 - y\partial_y\right) + y^2\partial_+\partial_- \,. \end{aligned} \tag{4.2.109}$$

This squared Casimir commutes with all generators in (4.2.103),

$$\left[\mathcal{H}^2, H_+\right] = 0\ ,\ \ \left[\mathcal{H}^2, H_-\right] = 0\ ,\ \ \left[\mathcal{H}^2, H_0\right] = 0\,. \tag{4.2.110}$$

From the other copy of $SL(2,\mathbb{R})$ group, built from the generators (4.2.105), the corresponding squared Casimir can be found to be

$$\begin{aligned}\bar{\mathcal{H}}^2 &= -\bar{H}_0^2 + \left(\bar{H}_1 \bar{H}_{-1} + \bar{H}_{-1} \bar{H}_1\right) \\ &= \frac{1}{4}\left(y^2\partial_y^2 - y\partial_y\right) + y^2\partial_+\partial_- \,. \end{aligned} \tag{4.2.111}$$

As the squared Casimir of the group formed by the generators (4.2.105), $\bar{\mathcal{H}}^2$ will commute with those generators,

$$\left[\bar{\mathcal{H}}^2, \bar{H}_+\right] = 0\ ,\ \ \left[\bar{\mathcal{H}}^2, \bar{H}_-\right] = 0\ ,\ \ \left[\bar{\mathcal{H}}^2, \bar{H}_0\right] = 0\,. \tag{4.2.112}$$

As we can observe, the expressions of $\mathcal{H}^2$ and $\bar{\mathcal{H}}^2$ in terms of $\{y, w^+, w^-\}$ coordinates are just the same, which means these two squared Casimir will have the same eigenvalue when it applies to a corresponding eigen function. It turns out that writing these squared Casimir in (t, r, θ, ϕ) coordinates, we can get

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = \partial_r(\Delta\partial_r) + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)}\,, \tag{4.2.113}$$

which is just the operator that applies to $R(r)$ in the left hand side of equation (4.2.100). The eigenvalue of this operator is $l(l+1)$,

$$\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = l(l+1)\Phi\,, \tag{4.2.114}$$

which is clearly the conserved quantum number in this system. In this sense, we have shown the hidden conformal symmetry for the generic Kerr spacetime which is read by putting a low frequency scalar probe in the near region of this spacetime. It is hidden since we cannot directly see this conformal symmetry from the spacetime structure as the way we did for the extremal Kerr case, i.e. we study the near horizon of extremal Kerr that can be shown to have a conformal structure in subsection 4.1.1. To be more precise, unlike in the extremal case where we have only a copy of $SL(2,\mathbb{R})$ symmetry, in the non-extremal or generic case we have two copies of this symmetry, denoted by $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. These two copies of $SL(2,\mathbb{R})$ symmetry are generated by the vectors (4.2.103) and (4.2.105). The symmetry group $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ is acting on the solution space of the wave equation, not in the background geometry.

However, the vectors (4.2.107) and (4.2.108) is not periodic under the identification

$$\phi \sim \phi + 2\pi \ . \tag{4.2.115}$$

This can be interpreted as the spontaneously symmetry breaking of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ by (4.2.115). Hence, under (4.2.115) the conformal coordinates behave as

$$w^+ \sim e^{4\pi^2 T_R} w^+ \ \ , \quad w^- \sim e^{4\pi^2 T_L} w^- \ \ , \ \ y \sim e^{2\pi^2 (T_R + T_L)} y \, . \tag{4.2.116}$$

The identification (4.2.116) is generated by the element of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ group

$$g = e^{-4\pi^2 i T_R H_0 - 4\pi^2 i T_L \bar{H}_0} \tag{4.2.117}$$

which is the identification for a CFT partition function at finite temperature (T_L, T_R). So the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ symmetry is spontaneously broken into $U(1)_L \times U(1)_R$ subgroup generated by $(\bar{H}_0, H_0)$ for $\phi \sim \phi + 2\pi$.

4.2.3 CFT temperature and entropy

At a fixed radius, from (4.2.101) we can write the relation between the conformal coordinates $\{w^+, w^-\}$ and Boyer-Lindquist coordianates $\{\phi, t\}$ is

$$w^\pm = e^{\pm t^\pm} \, , \tag{4.2.118}$$

where

$$\begin{aligned} t^+ &= 2\pi T_R \phi \, , \\ t^- &= \frac{t}{2M} - 2\pi T_L \phi \, . \end{aligned} \tag{4.2.119}$$

This is precisely the relation between Minkowski ($w^{\pm}$) and Rindler ($t^{\pm}$) coordinates.

Under the periodic identification of $\phi \sim \phi + 2\pi$, the Rindler coordinates will have the identifications

$$t^{+} \sim t^{+} + 4\pi^2 T_R \quad , \quad t^{-} \sim t^{-} - 4\pi^2 T_L \, . \tag{4.2.120}$$

Observing from Minkowski vacuum by tracing over the quantum state, we will get a thermal density matrix at temperature (T_L, T_R). Hence Kerr black holes should be dual to a finite temperature (T_L, T_R) mixed state in the dual CFT.

From the extremal Kerr discussion, we have obtain the value of central charge is $12J$. Assuming this central charge can also be used in general Kerr black holes, so we have $c_L = c_R = 12J$. Thus by using Cardy formula

$$S_{CFT} = \frac{\pi^2}{3} \left(c_L T_L + c_R T_R \right) \, , \tag{4.2.121}$$

together with the CFT left and right temperatures (4.2.102), the CFT entropy is

$$S_{CFT} = 2\pi M r_+ = S_{BH} \, . \tag{4.2.122}$$

4.2.4 Scalar absorption

In the near region, i.e. $\omega r << 1$, the ingoing solution of equation (4.2.100) is

$$\begin{aligned} R_{in}(r) \;&=\; \left(\frac{r - r_+}{r - r_-} \right)^{-i\frac{(\omega - m\Omega_H)}{4\pi T_H}} (r - r_-)^{-1-\ell} \\ &\times F\left(1 + \ell - i\frac{4M^2 - 2Q^2}{r_+ - r_-}\omega + i\frac{m\Omega_H}{2\pi T_H}, 1 + \ell - i2M\omega; 1 - i\frac{(\omega - m\Omega_H)}{2\pi T_H}; \frac{r - r_-}{r - r_+} \right) \, , \end{aligned} \tag{4.2.123}$$

where the Hawking temperature for Kerr black hole is given by

$$T_H = \frac{1}{8\pi} \frac{r_+ - r_-}{M r_+} \, . \tag{4.2.124}$$

The outgoing one can be obtained by taking the complex conjugate of the above solution, $R_{out}(r) = R^{*}_{in}(r)$. The function $F(a, b; c; z)$ in (4.2.123) stands for the hypergeometric function and $\Omega_H = a(2Mr_+)^{-1}$ is the angular velocity at the horizon. In the matching region, a region where the far and near regions intersect, the ingoing wave behaves as

$$R_{in}(r \gg M) \sim A r^{\ell} \tag{4.2.125}$$

with

$$A = \frac{\Gamma(1 - i\frac{\omega - m\Omega_H}{2\pi T_H})\Gamma(1+2\ell)}{\Gamma(1+\ell - i2M\omega)\,\Gamma(1+\ell - i\frac{4M^2}{r_+ - r_-}\omega + \frac{i}{2\pi T_H} m\Omega_H)} \,. \qquad (4.2.126)$$

Accordingly, the absorption cross section can be written as[1]

$$\begin{aligned} \sigma_{abs} &\sim |A|^{-2} \\ &\sim \sinh\left(\frac{\omega - m\Omega}{2T_H}\right) |\Gamma\left(1+\ell - i2M\omega\right)|^2 \left|\Gamma\left(1+\ell - i\frac{4M^2}{r_+ - r_-}\omega + \frac{i}{2\pi T_H} m\Omega_H\right)\right|^2 . \end{aligned}$$

To compare the absorption cross section of a near-region scalar field in the Kerr black hole background with the finite-temperature absorption cross section for the corresponding 2 dimensional CFT, we need explore the first law of black hole thermodynamics

$$T_H \delta S = \delta M - \Omega_H \delta J \,, \qquad (4.2.127)$$

where

$$S = 2Mr_+ \,, \qquad (4.2.128)$$

is the entropy of generic Kerr black hole. We identify $\omega = \delta M$ and $m = \delta J$. Then we look for the conjugate charges δE_R and δE_L such that

$$\delta S = \frac{\delta E_L}{T_L} + \frac{\delta E_R}{T_R} \qquad (4.2.129)$$

which again the left and right temperatures are given in (4.2.102). The solutions are

$$\begin{aligned} \delta E_L &= \frac{2M^3}{J}\delta M \;, \\ \delta E_R &= \frac{2M^3}{J}\delta M - \delta J \,, \end{aligned} \qquad (4.2.130)$$

which allow us to identify the left and right moving frequencies

$$\begin{aligned} \omega_L \equiv \delta E_L &= \frac{2M^3}{J}\omega \,, \\ \omega_R \equiv \delta E_R &= \frac{2M^3}{J}\omega - m \,. \end{aligned} \qquad (4.2.131)$$

[1] The formula where $\sigma_{abs} \sim |A|^{-2}$ is obtained after taking the ratio between the absorption flux as a function of near region ingoing radial solution to the incoming flux as a function of far region ingoing radial solution, $\sigma_{abs} = \frac{\mathcal{F}_{abs}\left(R^{in}_{near}\right)}{\mathcal{F}_{in}\left(R^{in}_{far}\right)}$.

By using of the above formula, we can rewrite the absorption cross section as

$$\sigma_{abs} \sim T_L^{2h_L-1} T_R^{2h_R-1} \sinh\left(\frac{\omega_L}{2T_L} + \frac{\omega_R}{2T_R}\right) \left|\Gamma(h_L + i\frac{\omega_L}{2\pi T_L})\right|^2 \left|\Gamma(h_R + i\frac{\omega_R}{2\pi T_R})\right|^2 , \quad (4.2.132)$$

which is precisely the finite-temperature absorption cross section for a two dimensional CFT in (3.3.125). In the last formula, we have left and right sector since the generic Kerr black hole is conjectured to be dual to both left and right movers of CFT_2. That is why we have the product of these two CFT's in the absorption rate formula above.

Chapter 5

Hidden Conformal Symmetries of Charged Rotating Black Holes

The Kerr/CFT correspondence for generic Kerr black holes is established by showing the hidden conformal symmetry of the system first. We learned in section 4.2 that the hidden conformal symmetry appears in the radial part of scalar wave equation in the Kerr background. The scalar probe must be in the low energy condition and sitting in the near region. In such case, the radial wave equation can be rewritten as the $SL(2,\mathbb{R})$ squared Casimir eigen equation. Subsequently, the Kerr/CFT correspondence in non-extremal case is established by showing the matching of microscopic and macroscopic entropy, and also the agreement between the absorption cross section computations from microscopic and macroscopic point of views.

In this chapter, we extend the hidden conformal symmetry discussions to the charged black holes, namely the Kerr-Newman and Kerr-Sen black holes. We also apply the idea of deformed hidden conformal symmetry [36] to these charged black holes. For Kerr-Sen black holes, the discussion of hidden conformal symmetry is extended to the extremal case, but without the deformation consideration. It is because the deformation procedure works only in the non-extremal condition. We also show an alternative derivation for the central charge of Kerr-Sen black holes, where the result matches that of [16]. The materials in this chapter are based on the paper [90] and preprint [111].

5.1 Generalized Hidden Conformal Symmetry for Kerr-Sen Black Holes

5.1.1 Twisted Classical Solution in Heterotic String Theory

Black hole solutions are contained not only in the general relativity theory, but also in the low energy effective field theory describing string theories, for example the heterotic string theory. In this subsection, we derive the spacetime solution in the low energy effective field theory of heterotic string theory. The black hole solutions in the string theory framework may have some properties which are qualitatively different from the ones that appear in ordinary Einstein gravity. They may have more charges and fields, associated with the Yang-Mills fields or the antisymmetric tensor gauge field, and a non-trivial dilaton field as well. When all of these charges or extra fields vanish, the solutions reduce to the ones known in standard Einstein gravity.

In [35], Sen constructed an exact classical solution in the low energy effective field theory describing the heterotic string theory. His solution describes a black hole carrying finite amount of charge and angular momentum. He used the twisting procedure, which generates inequivalent classical solutions starting from a given classical solution of string theory. It is found that the electrically neutral rotating black hole solution in string theory matches the expression of Kerr solution. Therefore, the twisting procedure can be used to get the rotating charged black hole solution in the low energy effective field theory of heterotic string theory starting from the Kerr solution. In this subsection, we show as detail as possible on how Sen obtained this rotating charged black hole solution in the low energy limit of heterotic string theory.

Let us start by writing the low energy effective action of heterotic string theory,

$$S = \int d^4x\sqrt{-g}e^{-\Phi}\left(R - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} + g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{12}H_{\kappa\lambda\mu}H^{\kappa\lambda\mu}\right). \tag{5.1.1}$$

Here g is the determinant of metric tensor $g_{\mu\nu}$, R is the Ricci scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field strength tensor for the vector fields A_μ, the dilaton field is denoted by Φ, and the tensor field with three indices $H_{\kappa\mu\nu}$ is defined as

$$H_{\kappa\mu\nu} = \partial_\kappa B_{\mu\nu} + \partial_\nu B_{\kappa\mu} + \partial_\mu B_{\nu\kappa} - \frac{1}{4}\left(A_\kappa F_{\mu\nu} + A_\nu F_{\kappa\mu} + A_\mu F_{\nu\kappa}\right) , \tag{5.1.2}$$

where $B_{\mu\nu}$ is an antisymmetric second rank tensor field. The term in (5.1.2) that contains A_μ fields is called the gauge Chern-Simons term. In the other literature, the dilaton field may be rescaled $\Phi \to 2\Phi$, together with another field rescaling $A_\mu \to 2\sqrt{2}A_\mu$ and $B_{\mu\nu} \to 8B_{\mu\nu}$ to give the same equation of motion [81].

In [82], Hassan and Sen show that a set of "new" fields $\left\{g'_{\mu\nu}, B'_{\mu\nu}, A'_\mu, \Phi'\right\}$ will satisfy the same equation of motions derived from the action (5.1.1) if the relation between the "new" and "old" fields are

$$\mathrm{M}' = \Omega \mathrm{M} \Omega^T \quad , \quad \Phi' = \Phi + \ln \frac{\det \mathrm{g}'}{\det \mathrm{g}} , \tag{5.1.3}$$

and each field contained in (5.1.3) is time independent. The matrices M and Ω are given by

$$\mathrm{M} = \begin{pmatrix} \left(\mathrm{K}^\mathrm{T}-\eta\right)\mathrm{g}^{-1}(\mathrm{K}-\eta) & \left(\mathrm{K}^\mathrm{T}-\eta\right)\mathrm{g}^{-1}(\mathrm{K}+\eta) & -\left(\mathrm{K}^\mathrm{T}-\eta\right)\mathrm{g}^{-1}\mathrm{A} \\ \left(\mathrm{K}^\mathrm{T}+\eta\right)\mathrm{g}^{-1}(\mathrm{K}-\eta) & \left(\mathrm{K}^\mathrm{T}+\eta\right)\mathrm{g}^{-1}(\mathrm{K}+\eta) & -\left(\mathrm{K}^\mathrm{T}+\eta\right)\mathrm{g}^{-1}\mathrm{A} \\ -\mathrm{A}^\mathrm{T}\mathrm{g}^{-1}(\mathrm{K}-\eta) & -\mathrm{A}^\mathrm{T}\mathrm{g}^{-1}(\mathrm{K}+\eta) & \mathrm{A}^\mathrm{T}\mathrm{g}^{-1}\mathrm{A} \end{pmatrix} , \tag{5.1.4}$$

and

$$\Omega = \begin{pmatrix} \mathrm{I}_{7\times7} & ... & ... \\ ... & \cosh\alpha & \sinh\alpha \\ ... & \sinh\alpha & \cosh\alpha \end{pmatrix} . \tag{5.1.5}$$

The dots in (5.1.5) represent zero components in the matrix, and $\mathrm{I}_{7\times7}$ is the 7×7 identity matrix. The superscript T in the formula above denotes the matrix transposition. Each of the matrices K, g^{-1}, and η are the matrix expressions of the tensors $K_{\mu\nu}$, $g^{\mu\nu}$, and $\eta_{\mu\nu}$ respectively. Sen shows the flat tensor metric $\eta_{\mu\nu}$ quite differently; the time component is put at the right bottom corner instead of at the left top as usual,

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{5.1.6}$$

It follows that, the matrix K in such convention will have the form

$$K = \begin{pmatrix} K_{rr} & K_{r\theta} & K_{r\phi} & K_{rt} \\ K_{\theta r} & K_{\theta\theta} & K_{\theta\phi} & K_{\theta t} \\ K_{\phi r} & K_{\phi\theta} & K_{\phi\phi} & K_{\phi t} \\ K_{tr} & K_{t\theta} & K_{t\phi} & K_{tt} \end{pmatrix} , \tag{5.1.7}$$

whose components are given by

$$K_{\mu\nu} = -B_{\mu\nu} - g_{\mu\nu} - \frac{1}{4} A_\mu A_\nu \,. \tag{5.1.8}$$

The associated column vector A, which contains the components of gauge fields A_μ, is written as

$$A = \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \\ A_t \end{pmatrix} . \tag{5.1.9}$$

The fields $\left\{g'_{\mu\nu}, B'_{\mu\nu}, A'_\mu, \Phi'\right\}$ and $\{g_{\mu\nu}, B_{\mu\nu}, A_\mu, \Phi\}$ that are obtained from (5.1.3) satisfy the same equation of motions derived from the action (5.1.1). One can simply understand the idea behind Sen's work [35] as follows. We know that the low energy effective action of heterotic string theory in four dimensions (5.1.1) reduces to the Einstein-Hilbert action (2.1.33) when all the non-gravitational fields are turned off. It means that the solution to the vacuum Einstein equations is also a solution to a set of equations of motion derived from (5.1.1). Hence, by using equation (5.1.3) we may get a set of new solution where the non-gravitational fields are not vanishing.

Since the Kerr metric (2.2.157) solves the vacuum Einstein equations, this metric is also a solution of equations of motion derived from the action (5.1.1). To avoid the confusion among the readers, in regard to our needs in this subsection, we rewrite the metric (2.2.157) where the black hole mass is denoted by "m" instead of M,

$$ds^2 = -dt^2 + \varrho^2 \left(d\theta^2 + \frac{dr^2}{\Delta} \right) + \left(r^2 + a^2\right) \sin^2\theta d\phi^2 + \frac{2mr}{\varrho^2} \left(dt - a \sin^2\theta d\phi\right)^2 , \tag{5.1.10}$$

where $\varrho = r^2 + \cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$. Sen showed from the Kerr solution (5.1.10), one can use the equation (5.1.3) to get a new solution which contains more fields that appear in the theory. In the followings, we will show in detail how this works.

Starting from a set of fields where all the non-gravitational fields are vanishing, and the gravitational one is described by the Kerr metric, the non-vanishing components of matrix M in (5.1.3) can be read as

$$
\begin{aligned}
&M_{11} = \frac{(g_{rr}+1)^2}{g_{rr}} \ , \ M_{15} = \frac{{g_{rr}}^2-1}{g_{rr}} \ , \ M_{22} = \frac{(g_{\theta\theta}+1)^2}{g_{\theta\theta}} \ , \ M_{26} = \frac{{g_{\theta\theta}}^2-1}{g_{\theta\theta}} \ , \\
&M_{33} = \frac{-{g_{\phi\phi}}^2 g_{tt} - 2\, g_{\phi\phi}\, g_{tt} - g_{tt} + {g_{t\phi}}^2 g_{\phi\phi} + 2\, {g_{t\phi}}^2}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M_{34} = \frac{g_{t\phi}\ (-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2 - 1)}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \\
&M_{37} = \frac{-{g_{\phi\phi}}^2 g_{tt} + g_{tt} + {g_{t\phi}}^2 g_{\phi\phi}}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M_{38} = M_{47} \frac{g_{t\phi}\ (-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2 + 1)}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \\
&M_{43} = M_{78} = \frac{g_{t\phi}\ (-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2 - 1)}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M_{44} = \frac{-2\, {g_{t\phi}}^2 + {g_{t\phi}}^2 g_{tt} - g_{\phi\phi}\, {g_{tt}}^2 + 2\, g_{\phi\phi}\, g_{tt} - g_{\phi\phi}}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \\
&M_{48} = \frac{{g_{t\phi}}^2 g_{tt} - g_{\phi\phi}\, {g_{tt}}^2 + g_{\phi\phi}}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M_{55} = \frac{(g_{rr}-1)^2}{g_{rr}} \ , \ M_{66} = \frac{(g_{\theta\theta}-1)^2}{g_{\theta\theta}} \ , \\
&M_{77} = \frac{(g_{\phi\phi}-1)^2 g_{tt} + {g_{t\phi}}^2 (g_{\phi\phi}-2)}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M_{88} = \frac{{g_{t\phi}}^2 (2+g_{tt}) - g_{\phi\phi}\, (g_{tt}+1)^2}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ .
\end{aligned}
\tag{5.1.11}
$$

We have used the notation M_{jk}, where the indices i and j run from 1 to 9, to denote the j-th row and k-th column component of matrix M. Since the matrix M depends only on the tensor metric for Kerr spacetime $g_{\mu\nu}$, and we know that the metric tensor is symmetric under its indices permutation, the tensorial notation M_{jk} of the matrix M is also symmetric under j and k permutation.

The non-vanishing components of the transformed matrix M' which obey the equation (5.1.3) are

$$
\begin{aligned}
&M'_{11} = M_{11} \ , \ M'_{15} = M_{15} \ , \ M'_{22} = M_{22} \ , \ M'_{26} = M_{26} \ , \\
&M'_{33} = M_{33} \ , \ M'_{34} = M_{34} \ , M'_{37} = M_{37} \ , \ M_{38} = \frac{g_{t\phi}\ (-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2 + 1)\cosh\alpha}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \\
&M'_{39} = \frac{g_{t\phi}\sinh\alpha\ (-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2 + 1)}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M'_{44} = M_{44} \ , \ M'_{47} = M_{47} \, , \\
&M'_{48} = \frac{({g_{t\phi}}^2 g_{tt} - g_{\phi\phi}\, {g_{tt}}^2 + g_{\phi\phi})\cosh\alpha}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \ M'_{49} = \frac{({g_{t\phi}}^2 g_{tt} - g_{\phi\phi}\, {g_{tt}}^2 + g_{\phi\phi})\cosh\alpha}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \\
&M'_{55} = M_{55} \ , \ M'_{66} = M_{66} \ , \ M'_{77} = M_{77} \ , \ M'_{78} = \frac{\cosh{(a)}\, g_{t\phi}\ (-1 + {g_{t\phi}}^2 - g_{\phi\phi}\, g_{tt})}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ , \\
&M'_{79} = \frac{\sinh{(a)}\, g_{t\phi}\ (-1 + {g_{t\phi}}^2 - g_{\phi\phi}\, g_{tt})}{-g_{\phi\phi}\, g_{tt} + {g_{t\phi}}^2} \ ,
\end{aligned}
$$

$$M'_{88} = \frac{\cosh^2\alpha\,(2\,g_{t\phi}{}^2 + g_{t\phi}{}^2 g_{tt} - g_{\phi\phi}\,g_{tt}{}^2 - 2\,g_{\phi\phi}\,g_{tt} - g_{\phi\phi})}{-g_{\phi\phi}\,g_{tt} + g_{t\phi}{}^2}\,,$$
$$M'_{89} = \frac{\cosh\alpha\,(2\,g_{t\phi}{}^2 + g_{t\phi}{}^2 g_{tt} - g_{\phi\phi}\,g_{tt}{}^2 - 2\,g_{\phi\phi}\,g_{tt} - g_{\phi\phi})\sinh\alpha}{-g_{\phi\phi}\,g_{tt} + g_{t\phi}{}^2}\,,$$
$$M'_{99} = \frac{\sinh^2\alpha\,(2\,g_{t\phi}{}^2 + g_{t\phi}{}^2 g_{tt} - g_{\phi\phi}\,g_{tt}{}^2 - 2\,g_{\phi\phi}\,g_{tt} - g_{\phi\phi})}{-g_{\phi\phi}\,g_{tt} + g_{t\phi}{}^2}\,, \qquad (5.1.12)$$

and we find that $M'_{jk} = M'_{kj}$. We now proceed to get the solutions of each fields $g'_{\mu\nu},\ A'_\mu,\ B'_{\mu\nu}$ and Φ' from the transformed matrix M' above. An explicit expression of M' in terms of $\mathrm{A},\ \mathrm{K},\ \mathrm{g}$ and η is

$$\mathrm{M}' = \begin{pmatrix} \left(\mathrm{K}'^{\mathrm{T}}-\eta\right)\mathrm{g}'^{-1}(\mathrm{K}'-\eta) & \left(\mathrm{K}'^{\mathrm{T}}-\eta\right)\mathrm{g}'^{-1}(\mathrm{K}'+\eta) & -\left(\mathrm{K}'^{\mathrm{T}}-\eta\right)\mathrm{g}'^{-1}\mathrm{A}' \\ \left(\mathrm{K}'^{\mathrm{T}}+\eta\right)\mathrm{g}'^{-1}(\mathrm{K}'-\eta) & \left(\mathrm{K}'^{\mathrm{T}}+\eta\right)\mathrm{g}'^{-1}(\mathrm{K}'+\eta) & -\left(\mathrm{K}'^{\mathrm{T}}+\eta\right)\mathrm{g}'^{-1}\mathrm{A}' \\ -\mathrm{A}^{\mathrm{T}}\mathrm{g}'^{-1}(\mathrm{K}'-\eta) & -\mathrm{A}^{\mathrm{T}}\mathrm{g}'^{-1}(\mathrm{K}'+\eta) & \mathrm{A}'^{\mathrm{T}}\mathrm{g}'^{-1}\mathrm{A}' \end{pmatrix}, \qquad (5.1.13)$$

where the matrices g' and A' are the matrix expressions of the new metric tensor $g'_{\mu\nu}$ and vector A'_μ respectively, and the matrix K' is the matrix expression of

$$K'_{\mu\nu} = -B'_{\mu\nu} - g'_{\mu\nu} - \frac{1}{4}A'_\mu A'_\nu\,. \qquad (5.1.14)$$

First, we would like to solve the tensor metric $g'_{\mu\nu}$. The only assumption that we need to make, as also mentioned in the original paper by Sen [35] where he is looking for an axially symmetric spacetime, is that the only non-vanishing off-diagonal metric component is $g'_{t\phi}$. Hence, we are looking for a new metric, provided by the formula (5.1.3), in the form

$$\mathrm{g}' = \begin{pmatrix} g'_{rr} & 0 & 0 & 0 \\ 0 & g'_{\theta\theta} & 0 & 0 \\ 0 & 0 & g'_{\phi\phi} & g'_{\phi t} \\ 0 & 0 & g'_{t\phi} & g'_{tt} \end{pmatrix}. \qquad (5.1.15)$$

It is quite tricky to get the solution for g'. Since in general the matrix M' contains not only $g'_{\mu\nu}$, but also A'_μ and $B'_{\mu\nu}$, we need to perform some operations to this matrix where finally we can get a set of equations which consists of graviton $g'_{\mu\nu}$ only. In this regard, it would be useful to show that the matrix M' is composed by several block matrices, i.e.

$$\mathrm{M}' = \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{C} \\ \mathcal{D} & \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} & \mathcal{I} \end{pmatrix}, \qquad (5.1.16)$$

where from (5.1.13) it is understood that

$$\begin{aligned}
&\mathcal{A} \equiv \left(\mathrm{K}'^{\mathrm{T}} - \eta\right) \mathrm{g}'^{-1} \left(\mathrm{K}' - \eta\right) \,, \; \mathcal{B} \equiv \left(\mathrm{K}'^{\mathrm{T}} - \eta\right) \mathrm{g}'^{-1} \left(\mathrm{K}' + \eta\right) \,, \; \mathcal{C} \equiv -\left(\mathrm{K}'^{\mathrm{T}} - \eta\right) \mathrm{g}'^{-1} \mathrm{A}' \,, \\
&\mathcal{D} \equiv \left(\mathrm{K}'^{\mathrm{T}} + \eta\right) \mathrm{g}'^{-1} \left(\mathrm{K}' - \eta\right) \,, \; \mathcal{E} \equiv \left(\mathrm{K}'^{\mathrm{T}} + \eta\right) \mathrm{g}'^{-1} \left(\mathrm{K}' + \eta\right) \,, \; \mathcal{F} \equiv -\left(\mathrm{K}'^{\mathrm{T}} + \eta\right) \mathrm{g}'^{-1} \mathrm{A}' \,, \\
&\mathcal{G} \equiv -\mathrm{A}'^{\mathrm{T}} \mathrm{g}'^{-1} \left(\mathrm{K}' - \eta\right) \,, \; \mathcal{H} \equiv -\mathrm{A}'^{\mathrm{T}} \mathrm{g}'^{-1} \left(\mathrm{K}' + \eta\right) \,, \; \mathcal{I} \equiv \mathrm{A}'^{\mathrm{T}} \mathrm{g}'^{-1} \mathrm{A}' \,.
\end{aligned} \tag{5.1.17}$$

The components of g' can be obtained using the following equation,

$$\mathcal{A} + \mathcal{E} - \mathcal{B} - \mathcal{D} = 4\eta \mathrm{g}'^{-1} \eta \,. \tag{5.1.18}$$

Explicitly, the right hand side of equation (5.1.18) can be expressed as

$$4\eta \mathrm{g}'^{-1} \eta = \begin{pmatrix} \frac{4}{g'_{rr}} & 0 & 0 & 0 \\ 0 & \frac{4}{g'_{\theta\theta}} & 0 & 0 \\ 0 & 0 & -\frac{4g'_{tt}}{g'_{\phi\phi} g'_{tt} + g'^2_{t\phi}} & -\frac{4g'_{t\phi}}{g'_{\phi\phi} g'_{tt} + g'^2_{t\phi}} \\ 0 & 0 & -\frac{4g'_{t\phi}}{g'_{\phi\phi} g'_{tt} + g'^2_{t\phi}} & -\frac{4g'_{\phi\phi}}{g'_{\phi\phi} g'_{tt} + g'^2_{t\phi}} \end{pmatrix} . \tag{5.1.19}$$

Plugging each components of M' from the results in (5.1.12) into the left hand side of equation (5.1.18), we get a set of equations for $g'_{\mu\nu}$, whose solutions can be written as

$$\begin{aligned}
g'_{t\phi} &= -\frac{2mra \cosh^2 \frac{\alpha}{2} \sin^2 \theta}{\rho^2 + 2mr \sinh^2 \frac{\alpha}{2}} \,, \; g'_{rr} = \frac{\rho^2 + 2mr \sinh^2 \frac{\alpha}{2}}{r^2 - 2mr + a^2} \,, \; g'_{\theta\theta} = \rho^2 + 2mr \sinh^2 \frac{\alpha}{2} \,, \\
g'_{\phi\phi} &= \left(\frac{(r^2 + a^2)\rho^2 + 2mra^2 \sin^2 \theta + 4mr (r^2 + a^2) \sinh^2 \frac{\alpha}{2} + 4m^2 r^2 \sinh^4 \frac{\alpha}{2}}{\rho^2 + 2mr \sinh^2 \frac{\alpha}{2}} \right) \sin^2 \theta \,, \\
g'_{tt} &= -\frac{\rho^2 - 2mr}{\rho^2 + 2mr \sinh^2 \frac{\alpha}{2}} \,.
\end{aligned} \tag{5.1.20}$$

After having all the components of $g'_{\mu\nu}$ in our hand, the next job is to get the gauge field A'_{μ}. It can be done by subtracting $\mathcal{G}$ and $\mathcal{H}$ which gives us a set of equations for A',

$$\mathcal{G} - \mathcal{H} = 2\mathrm{A}'^{\mathrm{T}} \mathrm{g}'^{-1} \eta \,. \tag{5.1.21}$$

Inserting the components of M′ in (5.1.12) into the last equation yields

$$\begin{aligned} \frac{2A'_r}{g'_{rr}} &= 0\,, \quad \frac{2A'_\theta}{g'_{\theta\theta}} \;=\; 0\,, \\ \frac{-A'_\phi g'_{tt} + A'_t g'_{t\phi}}{-g'_{\phi\phi} g'_{tt} + g'^2_{t\phi}} &= -\frac{mra\sinh\alpha}{\left(\rho^2\left(r^2+a^2\right) - 2r^3m - 2a^2mr\left(1+\sin^2\theta\right)\right)}\,, \\ \frac{-A'_\phi g'_{t\phi} + A'_t g'_{\phi\phi}}{-g'_{\phi\phi} g'_{tt} + g'^2_{t\phi}} &= \frac{m^2ra\sinh\alpha\left(\cosh\alpha\left(a^2r + r^3 - ra^2\sin^2\theta\right)\right)}{\rho^2\left(\rho^2\left(r^2+a^2\right) - 2r^3m - 2a^2mr\left(1+\sin^2\theta\right)\right)} \\ &\quad + \frac{m^2ra\sinh\alpha\left(ra^2\sin^2\theta + m^{-1}\left(\rho^2 - mr\right)\left(r^2+a^2\right)\right)}{\rho^2\left(\rho^2\left(r^2+a^2\right) - 2r^3m - 2a^2mr\left(1+\sin^2\theta\right)\right)}\,. \end{aligned} \tag{5.1.22}$$

The solutions to the last four equations for the gauge field A'_μ can be obtained as

$$\begin{bmatrix} A'_r \\ A'_\theta \\ A'_\phi \\ A'_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{2mra\sinh\alpha\sin^2\theta}{\rho^2+2mr\sinh^2\frac{\alpha}{2}} \\ \frac{2mr\sinh\alpha}{\rho^2+2mr\sinh^2\frac{\alpha}{2}} \end{bmatrix}. \tag{5.1.23}$$

Now the only unknown fields contained in M′ is the antisymetric tensor fields $B'_{\mu\nu}$. Obtaining the $B'_{\mu\nu}$ fields can be done by solving the following matrix equation,

$$\mathcal{D} + \mathcal{E} - \mathcal{A} - \mathcal{B} = 4\eta \mathrm{g}'^{-1}\mathrm{K}'\,. \tag{5.1.24}$$

It turns out that the only non-vanishing component of $B'_{\mu\nu}$ is

$$B'_{t\phi} = -B'_{\phi t} = \frac{2mra\sinh^2\frac{\alpha}{2}\sin^2\theta}{\rho^2 + 2mr\sinh^2\frac{\alpha}{2}}\,. \tag{5.1.25}$$

Finally, the dilaton field Φ' is given by the formula (5.1.3)

$$\Phi' = -\ln\frac{\rho^2 + 2mr\sinh^2\frac{\alpha}{2}}{\rho^2}\,. \tag{5.1.26}$$

It is a common believe that a theory describes the real world if it is written in the Einstein frame. The action (5.1.1) is still in the string frame, which is noticed from the coupling between Riemann tensor and the exponentiation of dilaton, i.e. $e^{-\Phi}R$. The Einstein frame version of (5.1.1) can be achieved by performing the Weyl rescaling of the tensor metric,

$$g'^E_{\mu\nu} \equiv e^{-\Phi'} g'_{\mu\nu}\,, \tag{5.1.27}$$

where ${g'}^{E}_{\mu\nu}$ is the metric in Einstein frame, and ${g'}_{\mu\nu}$ is the metric solution (5.1.20). Plugging the dilaton solution (5.1.26) into the Weyl rescaling (5.1.27), one can obtain the metric

$$
\begin{aligned}
{ds'}^2_E = &-\frac{\rho^2-2mr}{\rho^2+2mr\sinh^2\frac{\alpha}{2}}dt^2+\frac{\rho^2+2mr\sinh^2\frac{\alpha}{2}}{r^2+a^2-2mr}dr^2+\left(\rho^2+2mr\sinh^2\frac{\alpha}{2}\right)d\theta^2 \\
&-\frac{4mra\cosh^2\frac{\alpha}{2}\sin^2\theta}{\rho^2+2mr\sinh^2\frac{\alpha}{2}}dtd\phi+\left(\left(r^2+a^2\right)\rho^2+2mra^2\sin^2\theta+4mr\left(r^2+a^2\right)\sinh^2\frac{\alpha}{2}\right. \\
&\left.+4m^2r^2\sinh^4\frac{\alpha}{2}\right)\frac{\sin^2\theta}{\rho^2+2mr\sinh^2\frac{\alpha}{2}}d\phi^2\,, \qquad (5.1.28)
\end{aligned}
$$

where we have understood that ${ds'}^2_E={g'}^{E}_{\mu\nu}dx^\mu dx^\nu$. The metric (5.1.28) describes a black hole solution with mass M, electric charge Q, and angular momentum J after we redefine

$$
m=M-\frac{Q^2}{2M}\ ,\ e^\alpha=\frac{2M+\sqrt{2}Q}{2M-\sqrt{2}Q}\,. \qquad (5.1.29)
$$

The rotational parameter a definition as the ratio of black hole angular momentum J with respect to the black hole mass M is still unchanged. Hence, in terms of M and Q, the metric (5.1.28) can be rewritten as

$$
\begin{aligned}
{ds'}^2_E &= -(1-\frac{2Mr}{{\rho_{KS}}^2})dt^2+{\rho_{KS}}^2(\frac{dr^2}{\Delta_{KS}}+d\theta^2) \\
&- \frac{4Mra}{{\rho_{KS}}^2}\sin^2\theta dtd\phi+\{r(r+2b)+a^2+\frac{2Mra^2\sin^2\theta}{{\rho_{KS}}^2}\}\sin^2\theta d\phi^2, \qquad (5.1.30)
\end{aligned}
$$

where

$$
{\rho_{KS}}^2 = r(r+2b)+a^2\cos^2\theta\,, \qquad (5.1.31)
$$

$$
\Delta_{KS} = r(r+2b)-2Mr+a^2\,, \qquad (5.1.32)
$$

and $b=Q^2/2M$. Furthermore, the non-gravitational fields can be also rewritten as

$$
\Phi' = -\frac{1}{2}\ln\frac{r(r+2b)+a^2\cos^2\theta}{r^2+a^2\cos^2\theta}\,, \qquad (5.1.33)
$$

$$
{A'}_t = \frac{-rQ}{{\rho'}^2}\,, \qquad (5.1.34)
$$

$$
{A'}_\phi = \frac{rQa\sin^2\theta}{{\rho'}^2}\,, \qquad (5.1.35)
$$

$$
{B'}_{t\phi} = \frac{bra\sin^2\theta}{{\rho'}^2}\,. \qquad (5.1.36)
$$

The outer horizon of black hole is located at $r_+ = M - b + \sqrt{(M-b)^2 - a^2}$, while the Hawking temperature, angular velocity of horizon and electrostatic potential are given by

$$T_H = \frac{\sqrt{(2M^2-Q^2)^2-4J^2}}{4\pi M(2M^2-Q^2+\sqrt{(2M^2-Q^2)^2-4J^2})}\,, \tag{5.1.37}$$

$$\Omega_H = \frac{J}{M(2M^2-Q^2+\sqrt{(2M^2-Q^2)^2-4J^2})}\,, \tag{5.1.38}$$

$$V_H = Q/2M\,. \tag{5.1.39}$$

For $b = 0$, all non-gravitational fields (3.2.19)-(5.1.36) vanish and metric (5.1.30) changes simply to the metric of Kerr black hole. For generic non-zero b, the Kerr-Sen solution (5.1.30) (along with the non-gravitational fields (3.2.19)-(5.1.36)) is an interesting gravitational system in the context of string theory; quite different from Kerr solution in general relativity. The Kerr-Sen black hole (5.1.30) approaches to the metric of charged Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) black hole, which is a charged black hole in string theory [83, 84]. The metric that describes this GMGHS spacetime is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r\left(r-\frac{Q^2}{M}\right)\left(d\theta^2+\sin^2 d\phi^2\right)\,, \tag{5.1.40}$$

where we have turned off the dilaton field. The Kerr-Sen metric (5.1.28) reduces to the GMGHS metric (5.1.40) by setting the rotational parameter $a \to 0$ followed by a coordinate transformation $r \to r - Q^2/M$. It is clear that setting $Q = 0$ in this GMGHS metric, we obtain the Schwarzschild solution (2.1.50). Consequently, for the low energy limit of heterotic string theory, we can conclude that it contains a family of black holes which is tabulated in table 5.1.

Table 5.1: Black holes families in the low energy limit of heterotic string theory

	$J = 0$	$J \neq 0$
$Q = 0$	Schwarzschild	Kerr
$Q \neq 0$	GMGHS	Kerr-Sen

It is interesting to note that two out of four members of this family are just the same as the members of black holes in Einstein-Maxwell theory at the same physical conditions, static neutral and rotating neutral cases. This may be understood from the fact where in the absence of electric charge Q in this low energy limit of heterotic string theory, all non-gravitational fields are vanishing, which yields the theory is indistinguishable to the vacuum Einstein theory. The same fact applies to the Einstein-Maxwell theory, when $Q = 0$ this theory reduces to the vacuum Einstein's gravity.

5.1.2 Central charge for extremal Kerr-Sen from the stretched horizon technique

In this subsection, we derive the central charge associated with the extremal Kerr-Sen black holes via stretched horizon technique developed in [85, 24]. Quite recently the authors of [87] extend this stretched horizon technique to work better in non-extremal case. Our result here is in agreement with the work in [16]. Let us begin with the extremal Kerr-Sen metric in Boyer-Lindquist coordinates, written in ADM form as

$$\begin{aligned} ds^2 &= -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\ &= -N^2 dt^2 + \frac{\Sigma}{\Delta'_{KS}} dr^2 + \frac{\Xi \sin^2\theta}{\Sigma}\left(d\phi + N^\phi dt\right)^2 + \Sigma\, d\theta^2 \,, \end{aligned} \tag{5.1.41}$$

where q_{ij} denotes the spatial metric on a constant time slice, and

$$\begin{aligned} &\Xi = (r(r+2b) + {r_+}^2)^2 - (r - r_+)^2 {r_+}^2 \sin^2\theta \,, \\ &\Delta'_{KS} = (r - r_+)^2, \qquad \Sigma = r(r+2b) + {r_+}^2 \cos^2\theta \,, \\ &N = \sqrt{\frac{\Sigma \Delta'_{KS}}{\Xi}}, \qquad N^\phi = \frac{2Mrr_+}{\Xi}\,, \quad r_+ = 2(M-b)\,, \end{aligned} \tag{5.1.42}$$

where the outer horizon[1] is $r_+ = M - b$, M is black hole mass, and Q is black hole electric charge.

The only nonvanishing component of the canonical momentum is

$$\pi^{r\phi} = \frac{\sqrt{q}}{2N} q^{rr} \partial_r N^\phi \,, \tag{5.1.43}$$

$$\pi^{\theta\phi} = \frac{\sqrt{q}}{2N} q^{\theta\theta} \partial_\theta N^\phi \,. \tag{5.1.44}$$

[1]Note that the extremal limit of Kerr-Sen black holes is when $a = M - b$. The Outer horizon for generic case is $r_+ = M - b + \sqrt{(M-b)^2 - a^2}$.

Near the horizon, the shift vector N^ϕ can be expanded as $N^\phi \approx -\Omega_{_H} + \varepsilon$, where $\Omega_{_H}$ is the horizon angular velocity and the small parameter ε is given by

$$\varepsilon = (r - r_+)\partial_r N^\phi\big|_{r=r_+} = -\frac{(r-r_+)}{2r_+(r_+ + b)}\,. \tag{5.1.45}$$

Under a diffeomorphism generated by a vector field ξ^μ, the metric transforms as [85]

$$\begin{aligned}
\delta_\xi N\ &= \bar{\partial}_t \xi^\perp + \hat{\xi}^i \partial_i N\,,\\
\delta_\xi N^i &= \bar{\partial}_t \hat{\xi}^i - N q^{ij}\partial_j \xi^\perp + q^{ik}\partial_k N \xi^\perp + \hat{\xi}^j \partial_j N^i\,,\\
\delta_\xi q_{ij} &= q_{ik}\left(\partial_j \hat{\xi}^k - \frac{\partial_j N^k}{N}\xi^\perp\right) + q_{jk}\left(\partial_i \hat{\xi}^k - \frac{\partial_i N^k}{N}\xi^\perp\right) + \frac{1}{N}\xi^\perp \bar{\partial}_t q_{ij} + \hat{\xi}^k \partial_k q_{ij}\,,
\end{aligned} \tag{5.1.46}$$

where

$$\bar{\partial}_t = \partial_t - N^i \partial_i = \partial_t + \Omega_{_H}\partial_\phi - \varepsilon \partial_\phi \tag{5.1.47}$$

is a convective derivative, and the quantities $(\xi^\perp, \hat{\xi}^i)$ are the "surface deformation parameters". These surface deformation parameters are related to diffeomorphism parameters ξ^μ as

$$\xi^\perp = N\xi^t \quad , \quad \hat{\xi}^i = \xi^i + N^i \xi^t\,. \tag{5.1.48}$$

One can observe that q_{rr} is not well defined at the horizon $\mathcal{H}$, as well as the vanishing N there. As it is suggested in [85], we set a set of boundary conditions at the stretched horizon $\mathcal{H}_s$ first, i.e. a surface near the horizon, and then take the limit $\mathcal{H}_s \to \mathcal{H}$. Carlip [85] mentioned that there is not only one way to stretch the horizon. One way to do so is choosing the surface with constant angular velocity. This choice is the one that close to NHEK boundary condition applied in [12]. It can be understood from the fact that the only metric components that deviate at the boundary of NHEK (fall off conditions) after performing diffeomorphism in [12] are h_{rr} and $h_{\phi\phi}$. It needs $\delta_\xi N^\phi$ to be vanishing which in turn gives us

$$\bar{\partial}_t \hat{\xi}^\phi - N^2 q^{\phi\phi}\partial_\phi \xi^t + \hat{\xi}^r \partial_r N^\phi = 0 \tag{5.1.49}$$

where a possible solution can be written as

$$\hat{\xi}^r = (r - r_+)\partial_\phi \hat{\xi}^\phi, \qquad \xi^t = \mathcal{O}(r - r_+)\,. \tag{5.1.50}$$

The symmetries in canonical general relativity are generated by the quantity

$$H[\xi^{\perp},\hat{\xi}^{i}]=\int d^3x\left(\xi^{\perp}\mathcal{H}+\hat{\xi}^{i}\mathcal{H}_i\right) \tag{5.1.51}$$

where

$$\mathcal{H}=\frac{1}{\sqrt{q}}\left(\pi^{ij}\pi_{ij}-\pi^2\right)-\sqrt{q}\,{}^{(3)}R,\qquad \mathcal{H}^i=-2D_j\pi^{ij} \tag{5.1.52}$$

and q_{ij} is the spatial metric, π^{ij} is the canonical momentum, D_j is the spatial covariant derivative compatible with q_{ij}, ${}^{(3)}R$ is spatial curvature scalar related to the constant time slice spacetime denoted by metric tensor q_{ij}, $\mathcal{H}$ is the Hamiltonian, and $\mathcal{H}^i$ is the momentum constraints. After introducing a boundary term[2] $B[\xi]$,

$$\bar{H}[\xi]=H[\xi]+B[\xi], \tag{5.1.53}$$

we have a Poisson bracket between two generators as

$$\left\{\bar{H}[\xi],\bar{H}[\eta]\right\}=\bar{H}[\{\xi,\eta\}_{SD}]+K[\xi,\eta]\,. \tag{5.1.54}$$

The surface deformations $\{\xi,\eta\}_{SD}$ are [86]

$$\begin{aligned}\{\xi,\eta\}^{\perp}_{SD}&=\hat{\xi}^{i}D_i\eta^{\perp}-\hat{\eta}^{i}D_i\xi^{\perp}\\ \{\xi,\eta\}^{i}_{SD}&=\hat{\xi}^{k}D_k\hat{\eta}^{i}-\hat{\eta}^{k}D_k\hat{\xi}^{i}+q^{ik}\left(\xi^{\perp}D_k\eta^{\perp}-\eta^{\perp}D_k\xi^{\perp}\right)\,.\end{aligned} \tag{5.1.55}$$

Bringing expression (5.1.54) into a "Virasoro algebra" form gives us a the central charge $c=12K[\xi,\eta]$ where from the diffeomorphism (5.1.50) we can have the nonvanishing $K[\xi,\eta]$ as [85]

$$\begin{aligned}K[\xi,\eta]&=-\frac{1}{8\pi G}\int_{\partial\Sigma}d^2x\frac{\sqrt{\sigma}}{\sqrt{q}}\,n^k(\hat{\eta}_k\pi^{mn}D_m\hat{\xi}_n-\hat{\xi}_k\pi^{mn}D_m\hat{\eta}_n)\\ &=-\frac{1}{8\pi G}\int_{\mathcal{H}_s}d^2x\sqrt{\sigma}\left(n^r q_{rr}\hat{\xi}^r\left(\frac{1}{2N}q^{rr}\partial_r N^{\phi}\right)q_{rr}\partial_\phi\hat{\eta}^r\right.\\ &\quad\left.-\,n^r q_{rr}\hat{\eta}^r\left(\frac{1}{2N}q^{rr}\partial_r N^{\phi}\right)q_{rr}\partial_\phi\hat{\xi}^r\right)\\ &=-\frac{1}{16\pi G}\int_{\mathcal{H}_s}d^2x\sqrt{\sigma}\,\frac{n_r}{N}\partial_r N^{\phi}(r-r_+)^2\left(\partial_\phi\hat{\xi}^{\phi}\partial_\phi^2\hat{\eta}^{\phi}-\partial_\phi\hat{\eta}^{\phi}\partial_\phi^2\hat{\xi}^{\phi}\right)\,,\end{aligned} \tag{5.1.56}$$

[2] In the canonical general relativity discussions, a boundary term $B[\xi]$ must be introduced to cancel the boundary term coming from the variation of $H[\xi]$.

where σ is the determinant of metric tensor for the manifold $\partial\Sigma$. At the near horizon we have

$$\frac{n_r}{N} = \frac{\sqrt{\Xi}}{\Delta'_{KS}} \approx \frac{r(r+2b)+{r_+}^2}{(r-r_+)^2}, \qquad \partial_r N^\phi \approx -\frac{1}{2r_+(r_++b)}\,, \tag{5.1.57}$$

thus we can evaluate

$$\begin{aligned} K[\xi,\eta] &= \frac{1}{16\pi G}\int_{\mathcal{H}_s} d^2x\sqrt{\sigma}\,\left(\partial_\phi\hat{\xi}^\phi\partial_\phi^2\hat{\eta}^\phi - \partial_\phi\hat{\eta}^\phi\partial_\phi^2\hat{\xi}^\phi\right) \\ &= \frac{1}{16\pi G}\frac{A_H}{2\pi}\int d\phi\,\left(\partial_\phi\hat{\xi}^\phi\partial_\phi^2\hat{\eta}^\phi - \partial_\phi\hat{\eta}^\phi\partial_\phi^2\hat{\xi}^\phi\right)\,, \end{aligned} \tag{5.1.58}$$

where $A_H = \int d^2x\sqrt{\sigma} = 8\pi {r_+}^2$ is the horizon area.

Finally we have the central charge

$$c = 48\pi\frac{1}{16\pi G}\frac{A_H}{2\pi} = \frac{3A_H}{2\pi G} = 12J\,, \tag{5.1.59}$$

and by using the Cardy formula to obtain the entropy

$$S = \frac{\pi^2}{3}cT = 2\pi J = \frac{A_H}{4G}\,, \tag{5.1.60}$$

which matches the Bekenstein-Hawking entropy for Kerr-Sen black holes [16]. In (5.1.60) we have used Frolov-Thorne temperature for Kerr-Sen black holes

$$T = \frac{1}{2\pi}\,. \tag{5.1.61}$$

5.1.3 The Charged Scalar Field in Background of Kerr-Sen Spacetimes

We consider a massless scalar field Φ with charge e as a probe in background (5.1.30). The minimally coupled Klein-Gordon equation for the massless scalar field Φ is

$$\left(\nabla_\mu - ieA_\mu\right)\left(\nabla^\mu - ieA^\mu\right)\Phi = 0\,, \tag{5.1.62}$$

where A_μ is given by (5.1.34) and (5.1.35). We separate the coordinates in scalar wave function as

$$\Phi\left(r,t,\theta,\phi\right) = \exp\left(im\phi - i\omega t\right)S\left(\theta\right)R\left(r\right)\,, \tag{5.1.63}$$

where the radial and angular functions $R(r)$ and $S(\theta)$ are solutions to the radial equation

$$\partial_r \left(\Delta_{KS}\partial_r R\left(r\right)\right) + \left(\frac{\left(\gamma r - ma\right)^2}{\Delta_{KS}} + \omega^2 \Delta_{KS} + 2\delta r - \sigma\right) R\left(r\right) = 0\,, \tag{5.1.64}$$

and angular equation

$$\frac{1}{\sin\theta}\partial_\theta \left(\left(\sin\theta\right)\partial_\theta S\left(\theta\right)\right) + \left(\sigma - \frac{m^2}{\sin^2\theta} - \omega^2 a^2 \sin^2\theta\right) S\left(\theta\right) = 0\,, \tag{5.1.65}$$

respectively. In equation (5.1.64), $\gamma = 2M\omega - eQ$, $\delta = \gamma\omega$, σ is the separation constant, and Δ_{KS} is given in (5.1.32). We notice the radial equation (5.1.64) can be rearranged to

$$\left(\partial_r \left(\Delta_{KS}\partial_r\right) + \frac{\left(2M\omega r_+ - eQr_+ - ma\right)^2}{\left(r - r_+\right)\left(r_+ - r_-\right)} - \frac{\left(2M\omega r_- - eQr_- - ma\right)^2}{\left(r - r_-\right)\left(r_+ - r_-\right)}\right) R\left(r\right) = \left(\sigma - f\left(r\right)\right) R\left(r\right)\,, \tag{5.1.66}$$

where $f\left(r\right) = \left(\Delta_{KS} + 4M\left(M + r\right)\right)\omega^2 - \left(2M + r\right)2eQ\omega + e^2Q^2$, and the inner horizon of black hole is $r_- = M - b - \sqrt{\left(M - b\right)^2 - a^2}$. To simplify the equation of motion (5.1.66), we consider the low frequency scalar field $\omega << 1/M$ and so in the near region geometry defined by $r << 1/\omega$ and with assumption that electric charge of scalar field satisfies $eQ << 1$[3], we can neglect $f(r)$ in the right hand side of (5.1.66).

As we notice, the electric charge of scalar field e couples to the black hole charge Q in the radial equation (5.1.66). The existence of eQ term in radial equation is necessary to investigate the dual CFTs in general picture. The lack of eQ term in the radial equation of neutral scalar field (as in reference [29]) hinders the general picture of Kerr-Sen geometry.

In fact, the Kerr/CFT correspondence calculation in the general picture shows the electric charge of Kerr-Newman black hole as well as the angular momentum of black hole enters in the CFT quantities such as the central charges and the hidden conformal symmetry generators [34]. Indeed to realize the (hidden) conformal symmetry of charged rotating black holes in the general picture, one must consider a charged scalar field. An immediate result of CFT calculations in the general picture is that by looking at the dual CFT quantities, one can observe the presence of electric charge (hair) of the black hole. The authors in [34] proposed that each macroscopic hair of black holes, may be associated to a dual CFT.

[3] From the definition of γ, we can acknowledge that eQ has the same dimension as $M\omega$.

In the general picture, we consider the $SL(2,\mathbb{Z})$ transformation for the torus generated by ϕ and χ coordinates, given by [23] transformation

$$\begin{pmatrix} \phi' \\ \chi' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \eta & \tau \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} , \tag{5.1.67}$$

where the two $U(1)$ symmetries of black hole are associated with ϕ and χ coordinates. The first symmetry is simply the rotational symmetry of the black hole along ϕ direction. The second symmetry is associated to the rotational symmetry of the uplifted black hole into five-dimensions (the fifth coordinate is χ) and in fact this symmetry is equivalent to the original gauge symmetry of the four-dimensional charged rotating black hole. Such a transformation doesn't change the phase of the charged scalar field (6.1.19) with electric charge e; $e^{im\phi+ie\chi} = e^{im'\phi'+ie'\chi'}$ which yields $m = \alpha m' + \eta e', e = \beta m' + \tau e'$. Consequently, in ϕ' picture, the radial equation (5.1.66) for low frequency massless charged scalar field in the near region of Kerr-Sen spacetime can be rewritten as

$$\begin{aligned} \partial_r\left(\Delta_{KS}\partial_r R\left(r\right)\right) &+ \left(\frac{\left(2Mr_+\omega - \left(Qr_+\beta + a\alpha\right)m'\right)^2}{\left(r-r_+\right)\left(r_+-r_-\right)} - \frac{\left(2Mr_-\omega - \left(Qr_-\beta + a\alpha\right)m'\right)^2}{\left(r-r_-\right)\left(r_+-r_-\right)}\right)R\left(r\right) \\ &= l\left(l+1\right)R\left(r\right) , \end{aligned} \tag{5.1.68}$$

where we have chosen the separation constant $\sigma = l\left(l+1\right)$. To get the χ' picture, we should turn off the momentum along ϕ' coordinate. In this case, the radial equation (5.1.66) for low frequency massless charged scalar field in the near region of Kerr-Sen becomes the same as equation (5.1.68) by replacing α, β and m' to η, τ and e' respectively,

$$\begin{aligned} \partial_r\left(\Delta_{KS}\partial_r R\left(r\right)\right) &+ \left(\frac{\left(2Mr_+\omega - \left(Qr_+\tau + a\eta\right)e'\right)^2}{\left(r-r_+\right)\left(r_+-r_-\right)} - \frac{\left(2Mr_-\omega - \left(Qr_-\tau + a\eta\right)e'\right)^2}{\left(r-r_-\right)\left(r_+-r_-\right)}\right)R\left(r\right) \\ &= l\left(l+1\right)R\left(r\right) . \end{aligned} \tag{5.1.69}$$

5.1.4 Hidden Conformal Symmetry of Kerr-Sen Geometry in General Picture

In this section, we find the hidden conformal symmetry of the radial equation (5.1.68) for the massless charged scalar field in the near region of Kerr-Sen black hole in general picture.

We define ω^+, ω^- and y *as the conformal coordinates* in terms of coordinates t, r and ϕ' by

$$\omega^+ = \sqrt{\frac{r-r_+}{r-r_-}}\exp(2\pi T_R\phi' + 2n_R t), \tag{5.1.70}$$

$$\omega^- = \sqrt{\frac{r-r_+}{r-r_-}}\exp(2\pi T_L\phi' + 2n_L t), \tag{5.1.71}$$

$$y = \sqrt{\frac{r_+-r_-}{r-r_-}}\exp(\pi(T_R+T_L)\phi' + (n_R+n_L)t)\,. \tag{5.1.72}$$

In terms of conformal coordinates, we also define the right and left moving vector fields

$$H_1 = i\partial_+\,, \qquad H_0 = i(\omega^+\partial_+ + \frac{1}{2}y\partial_y)\,, \qquad H_{-1} = i((\omega^+)^2\partial_+ + \omega^+ y\partial_y - y^2\partial_-)\,, \tag{5.1.73}$$

and

$$\bar{H}_1 = i\partial_-\,, \qquad \bar{H}_0 = i(\omega^-\partial_- + \frac{1}{2}y\partial_y)\,, \qquad \bar{H}_{-1} = i((\omega^-)^2\partial_- + \omega^- y\partial_y - y^2\partial_+)\,, \tag{5.1.74}$$

respectively.

The vectors ∂_+, ∂_-, and ∂_y in terms of coordinates t, r and ϕ', can be written as

$$\partial_+ = e^{-(2\pi T_R\phi' + 2n_R t)}\left({\Delta_{KS}}^{1/2}\partial_r + Z_{\phi+}\partial_{\phi'} - Z_{t+}\partial_t\right)\,, \tag{5.1.75}$$

$$\partial_y = e^{-(\pi(T_L+T_R)\phi' + (n_R+n_L)t)}\left(Z_{ry}\partial_r + Z_{\phi y}\partial_{\phi'} - Z_{ty}\partial_t\right)\,, \tag{5.1.76}$$

$$\partial_- = e^{-(2\pi T_L\phi' + 2n_L t)}\left({\Delta_{KS}}^{1/2}\partial_r + Z_{\phi-}\partial_{\phi'} - Z_{t-}\partial_t\right)\,. \tag{5.1.77}$$

where

$$Z_{\phi+} = \frac{(n_R(r_+-r_-) - n_L(r_++r_-) + 2n_L r)}{4\pi{\Delta_{KS}}^{1/2}(n_L T_R - n_R T_L)}\,,$$

$$Z_{t+} = \frac{(T_R(r_+-r_-) - T_L(r_++r_-) + 2T_L r)}{4{\Delta_{KS}}^{1/2}(n_L T_R - n_R T_L)}\,,$$

$$Z_{ry} = -\frac{2\Delta_{KS}}{\sqrt{(r-r_-)(r_+-r_-)}}\,,$$

$$Z_{\phi y} = \sqrt{\frac{r_+-r_-}{r-r_-}}\frac{(n_L-n_R)}{2\pi(n_L T_R - n_R T_L)}\,,$$

$$Z_{ty} = \sqrt{\frac{r_+-r_-}{r-r_-}}\frac{(T_L-T_R)}{2(n_L T_R - n_R T_L)}\,,$$

$$Z_{\phi-} = \frac{(n_R(r_++r_-) - n_L(r_+-r_-) - 2n_L r)}{4\pi{\Delta_{KS}}^{1/2}(n_L T_R - n_R T_L)}\,,$$

and

$$Z_{t-} = \frac{(T_R\,(r_+ + r_-) - T_L\,(r_+ - r_-) - 2T_R r)}{4{\Delta_{KS}}^{1/2}\,(n_L T_R - n_R T_L)}\,.$$

The vector fields (5.1.73) satisfy the $sl(2,\mathbb{R})$ algebra

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1}\,, \qquad [H_{-1}, H_1] = -2iH_0\,, \tag{5.1.78}$$

and similarly for $\bar{H}_1, \bar{H}_0$ and $\bar{H}_{-1}$. The squared Casimir of $SL(2,\mathbb{R})_R$ and $SL(2,\mathbb{R})_L$ groups with generators $H_{\pm1}, H_0$ and $\bar{H}_{\pm1}, \bar{H}_0$ respectively, are equal and in conformal coordinates are given by

$$\begin{aligned}\mathcal{H}^2 &= -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) \\ &= \frac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2\partial_+\partial_-.\end{aligned} \tag{5.1.79}$$

In terms of r, t and ϕ' coordinates, the $\mathcal{H}^2$ can be written as

$$\mathcal{H}^2 \;=\; \partial_r(r - r_+)(r - r_-)\partial_r - \frac{(r_+ - r_-)[\pi(T_L + T_R)\partial_t - (n_L + n_R)\partial_\phi]^2}{16\pi^2(T_L n_R - T_R n_L)^2(r - r_+)} \tag{5.1.80}$$

$$+\frac{(r_+ - r_-)[\pi(T_L - T_R)\partial_t - (n_L - n_R)\partial_\phi]^2}{16\pi^2(T_L n_R - T_R n_L)^2(r - r_-)}. \tag{5.1.81}$$

The "bar" version of this squared Casimir is

$$\bar{\mathcal{H}}^2 = -\bar{H}_0^2 + \frac{1}{2}(\bar{H}_1 \bar{H}_{-1} + \bar{H}_{-1}\bar{H}_1) \tag{5.1.82}$$

which turns out to have the same expression $\mathcal{H}^2$ in terms of r, t and ϕ' coordinates.

The squared Casimirs (5.1.79) and (5.1.82) reduce simply to the radial equation (5.1.68) in ϕ' picture,

$$\mathcal{H}^2 R(r) = \bar{\mathcal{H}}^2 R(r) = l(l+1)R(r), \tag{5.1.83}$$

after choosing the right and left temperatures T_R and T_L

$$T_R = \frac{r_+ - r_-}{4\pi a\alpha} \quad , \quad T_L = \frac{r_+ + r_-}{4\pi a\alpha}\,, \tag{5.1.84}$$

and

$$n_R = -\frac{(r_+ - r_-)\,\beta Q}{8\alpha a M} \quad , \quad n_L = -\frac{(2a\alpha + (r_+ + r_-)\,\beta Q)}{8\alpha a M}\,, \tag{5.1.85}$$

where α and β are the parameters of $SL(2,\mathbb{Z})$ modular transformation (5.1.67).

As we notice, the temperatures of CFT dual to Kerr-Sen black hole in ϕ' picture depend only on α, while n_L and n_R depend on both α and β. The dependence of CFT temperatures on $SL(2,\mathbb{Z})$ parameters for Kerr-Sen is different than Kerr-Newman black hole. In the latter case, the CFT temperatures in ϕ' picture depend on both parameters α and β. The CFT temperatures (5.1.84), n_L and n_R (5.1.85) reduce to the results in J picture when $\alpha = 1$ and $\beta = 0$ [29].

In χ' picture, the radial equation is given by equation (5.1.68) where one replaces α, β and m' to η, τ and e' respectively. After changing to the conformal coordinates (5.1.70)-(5.1.72) (with replacing ϕ' to χ'), we find the squared Casimir of $SL(2,\mathbb{R})_R$ and $SL(2,\mathbb{R})_L$ reduce to the radial equation in χ' picture by choosing the right and left temperatures T_R and T_L as

$$T_R = \frac{r_+ - r_-}{4\pi a\eta} \quad , \quad T_L = \frac{r_+ + r_-}{4\pi a\eta} \, , \tag{5.1.86}$$

and

$$n_R = -\frac{(r_+ - r_-)\,\tau Q}{8\eta a M} \quad , \quad n_L = -\frac{(2a\eta + (r_+ + r_-)\,\tau Q)}{8\eta a M} \, , \tag{5.1.87}$$

where η and τ are the parameters of $SL(2,\mathbb{Z})$ modular transformation (5.1.67). We notice for unit element of $SL(2,\mathbb{Z})$ where $\eta = 0$ and $\tau = 1$, the temperatures are not finite that indicates the Q picture for the Kerr-Sen geometry is not well defined. The same type of calculation for Kerr-Newman black hole in χ' picture shows taking $\eta = 0$ and $\tau = 1$ leads to the well defined Q picture for the Kerr-Newman black hole [34]. The non-existent Q picture for the Kerr-Sen geometry hinders uplifting the black hole into five dimensional spacetime, in contrast to Kerr-Newman black hole.

We note that equation (5.1.83) signals the existence of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ hidden conformal symmetry in ϕ' picture for the Kerr-Sen black hole. We should emphasize that $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ is only a local hidden conformal symmetry for the solution space of massless charged scalar field in near region of Kerr-Sen geometry. The local symmetry is generated by the vector fields (5.1.73),(5.1.74). The reason is these vectors in ϕ' picture are not periodic under $\phi' \sim \phi' + 2\alpha\pi$ identification, so they can't be defined globally. We may conclude the existence of local $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ hidden conformal symmetry in ϕ' picture, suggests that we assume the dynamics of the near region can be described by a dual CFT. To verify this assumption, we try to find the microscopic entropy of the dual CFT which according to the Cardy formula, is given by

$$S_{CFT} = \frac{\pi^2}{3}(c_L T_L + c_R T_R)\,, \tag{5.1.88}$$

where T_R and T_L are the CFT temperatures in ϕ picture, given by (5.1.84). The central charges of dual CFT for extremal Kerr-Sen black holes were obtained in [16] based on analysis of the asymptotic symmetry group. For the case of non-extremal black hole, we assume that the conformal symmetry connects smoothly to that of the extremal case; so we consider the central charges given by

$$c_R = c_L = 12\alpha J\,. \tag{5.1.89}$$

We notice in the case of $\alpha = 1$, (5.1.89) reduces to $12J$ which is the central charge in the J picture. The central charges (5.1.89) and temperatures (5.1.84) yield the microscopic entropy of CFT (5.1.88) in ϕ' picture as

$$S_{CFT} = 2\pi M r_+\,, \tag{5.1.90}$$

which is in complete agreement with the macroscopic Bekenstein-Hawking entropy of Kerr-Sen spacetime. The macroscopic Bekenstein-Hawking entropy of Kerr-Sen black hole is given by [16, 88]

$$S = \pi\left(2M^2 - Q^2 + \sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2}\right)\,, \tag{5.1.91}$$

which is equal to S_{CFT} upon substitution $r_+ = M - b + \sqrt{(M-b)^2 - a^2}$, $J = aM$, and $b = Q^2/2M$.

5.1.5 Absorption Cross Section of Near Region Scalars in ϕ' Picture

In this section, to further show that the dynamics of the near region can be described by a dual CFT in ϕ' picture, we consider the absorption cross section of scalars in the near region of Kerr-Sen spacetime. We find that the absorption cross section could be reproduced correctly by dual CFT. In this regard, we introduce the new coordinate p, given by [67]

$$p = \frac{r - r_+}{r - r_-}\,. \tag{5.1.92}$$

By using the following relation that is obtained from (5.1.92),

$$\Delta_{KS}\partial_r = (r_+ - r_-)\, p\partial_p\,, \tag{5.1.93}$$

one can rewrite the radial part of Klein-Gordon equation (5.1.64) in terms of new coordinate p as

$$p\,(1-p)\,\partial_p^2 R\,(p) + (1-p)\,\partial_p R\,(p) + \left(\frac{C_1^2}{p} - C_2^2 - \frac{C_3}{1-p}\right) R\,(p) = 0\,, \tag{5.1.94}$$

where the constants C_1, C_2 and C_3 are

$$C_1 = \left(\frac{2Mr_+\omega - (Qr_+\beta + a\alpha)\, m'}{r_+ - r_-}\right), \tag{5.1.95}$$

$$C_2 = \left(\frac{2Mr_-\omega - (Qr_-\beta + a\alpha)\, m'}{r_+ - r_-}\right), \tag{5.1.96}$$

$$C_3 = l\,(l+1)\ . \tag{5.1.97}$$

The in-going solution for the equation (5.2.267) is

$$R_{in}\,(r) = Cp^{-iC_1}\,(p-1)^{-l}\,{}_2F_1\left(-l - i\,(C_1 - C_2)\,, -l - i\,(C_1 + C_2)\,; 1 - 2iC_1; p\right)\,, \tag{5.1.98}$$

where C is a constant of integration and ${}_2F_1$ is the hypergeometric function. The in-going solution (5.2.269) on the outer boundary of the matching region where $r \gg M$ behaves as,

$$R_{in} \sim Ar^l\,, \tag{5.1.99}$$

where $A = {}_2F_1\left(-l - i\,(C_1 - C_2)\,, -l - i\,(C_1 + C_2)\,; 1 - 2iC_1; 1\right)$. We should mention in finding the in-going solution, we consider the low frequency condition, $\omega \ll 1/M$ in near region, $r \ll 1/\omega$, along with the assumption of small probe $eQ \ll 1$. Using the Gauss' theorem for Gamma functions, we can rewrite the factor A in equation (5.1.99) as

$$A = \frac{\Gamma\,(1 - 2iC_1)\,\Gamma\,(2l+1)}{\Gamma\left(l + 1 - i\frac{\left(2M\omega - m_\beta Q(1-\beta)\right)(r_+ + r_-) - 2m_\beta a\beta}{r_+ - r_-}\right)\Gamma\left(l + 1 + i\left(2M\omega - m_\beta Q\,(1-\beta)\right)\right)}\,. \tag{5.1.100}$$

Hence, we find the absorption cross section, given by

$$P_{abs} \sim |A|^{-2} = \sinh\,(2\pi C_1)\,\frac{|\Gamma\,(l + 1 - iB_1)|^2\,|\Gamma\,(l + 1 - iB_2)|^2}{2\pi C_1\,(\Gamma\,(2l+1))^2}\,, \tag{5.1.101}$$

where

$$B_1 = \frac{(2M\omega - m'Q\beta)(r_+ + r_-) - 2m'a\alpha}{r_+ - r_-}, \quad (5.1.102)$$

$$B_2 = (2M\omega - m'Q\beta) , \quad (5.1.103)$$

and C_1 and C_2 are given by (5.1.95) and (5.1.95), respectively. To find the possible agreement between macroscopic cross section P_{abs} and the microscopic cross section of dual CFT, we need to identify some parameters of the theories. In this regard, we consider the first law of thermodynamics for the charged rotating black holes which can be written as

$$T_H \delta S_{BH} = \delta M - \Omega_H \delta J - V_H \delta Q \,, \quad (5.1.104)$$

where T_H and Ω are given by (5.1.37) and (5.1.38), and V_H is the electrostatic potential. In the case of neutral rotating black holes, δJ can be identified as m and δM as ω [28]. In addition to these identifications, for charged black holes we identify δQ as e.

To find the conjugate charges, we calculate the variation of entropy from gravitational point of view, δS_{BH} as well as the variation of entropy from CFT, δS_{CFT}. These two variations should be equal, so we find

$$\frac{\delta M - \Omega_H \delta J - V_H \delta Q}{T_H} = \frac{\delta E_L}{T_L} + \frac{\delta E_R}{T_R} \,, \quad (5.1.105)$$

where Ω_H and V_H are given by (5.1.38) and (5.1.39) respectively. The absorption cross section (5.1.101) can be written as a thermal CFT absorption cross section if we identify $\delta E_L = \tilde{\omega}_L$ and $\delta E_R = \tilde{\omega}_R$ where

$$\tilde{\omega}_L = \frac{(2M\omega - m'Q\beta)(r_+ + r_-)}{2a\alpha} \,, \quad (5.1.106)$$

and

$$\tilde{\omega}_R = \frac{(2M\omega - m'Q\beta)(r_+ + r_-)}{2a\alpha} - m' \,. \quad (5.1.107)$$

The variables $\tilde{\omega}_R$ and $\tilde{\omega}_L$ are introduced somehow to accommodate three sets of CFT parameters: the frequencies $\omega_{L,R}$, the charges $q_{L,R}$, and the chemical potentials[4] $\mu_{L,R}$. The relations between these variables are written as

$$\tilde{\omega}_{L,R} = \omega_{L,R} - q_{L,R}\mu_{L,R} \,, \quad (5.1.108)$$

[4]This chemical potential is just an analogy with the similar terminology that we have in thermodynamics. A brief introduction to the chemical potential in thermodynamics context is given in the appendix G.

where

$$\omega_L = \frac{2M\omega\left(r_+ + r_-\right)}{2a\alpha} \quad , \quad \omega_R = \omega_L - m' \, , \tag{5.1.109}$$

$$\mu_L = \mu_R = \frac{Q\beta\left(r_+ + r_-\right)}{2a\alpha} \, , \tag{5.1.110}$$

and $q_L = q_R = m'$. We also notice that for $\beta = 0$, i.e. the absence of left and right chemical potential, and $\alpha = 1$, the left and right frequencies (5.1.106) and (5.1.107) reduce to standard left and right frequencies for Kerr-Sen geometry with an electrically neutral test field [29]. In fact, by equation (5.1.107), (5.1.106), and (5.1.84), the macroscopic cross section (5.1.101) can be expressed as

$$P_{abs} \sim T_L^{2h_L-1} T_R^{2h_R-1} \sinh\left(\frac{\omega_L}{2T_L} + \frac{\omega_R}{2T_R}\right) \left|\Gamma\left(h_L + i\frac{\omega_L}{2\pi T_L}\right)\right|^2 \left|\Gamma\left(h_R + i\frac{\omega_R}{2\pi T_R}\right)\right|^2 , \tag{5.1.111}$$

where we set $h_L = h_R = l + 1$. Equation (5.1.111) is the well known finite temperature absorption cross section for a $2D$ CFT [28].

5.1.6 Generalized Hidden Conformal Symmetry with Deformation Parameter

In sections (5.1.4) and (5.1.5), we considered the propagation of a scalar field in the background of a generic non-extremal Kerr-Sen black hole and found evidences for a hidden conformal field theory in ϕ' picture. The metric function of Kerr-Sen black hole has two roots r_+ and r_- where the scalar wave equation (5.1.66) have poles in both locations. We may deform the wave equation (5.1.66) near the inner horizon r_- since for the non-extremal Kerr-Sen black hole we can consider r to be far enough from r_- such that the linear and quadratic terms in frequency which are coming from the expansion near the inner horizon can be dropped [36]. In this regard we consider the deformation of radial equation (5.1.68) for the massless scalar filed by deformation parameter κ as

$$\left[\partial_r\left(\Delta_{KS}\partial_r\right) + \frac{\left(2Mr_+\omega - a_1 m'\right)^2}{\left(r - r_+\right)\left(r_+ - r_-\right)} - \frac{\left(2M\kappa r_+\omega - a_2 m'\right)^2}{\left(r - r_-\right)\left(r_+ - r_-\right)}\right] R\left(r\right) = l\left(l+1\right) R\left(r\right) \, , \tag{5.1.112}$$

where $a_1 = Qr_+\beta + a\alpha$ and $a_2 = Q\kappa r_+\beta + a\alpha$. The deformation parameter κ and $r - r_-$ should satisfy $\kappa M^2 a_2 m'\omega \ll 2\sqrt{(M-b)^2 - a^2}(r - r_-)$ as well as $\kappa^2 M^4 \omega^2 \ll 2\sqrt{(M-b)^2 - a^2}(r -$

r_-) to drop the linear and quadratic terms in frequency from the expansion near the inner horizon pole while we still keep the near region geometry and low frequency scalar field as an electrically charged probe. We look now to a new set of vector fields that make $sl(2,\mathbb{R})$ algebra in such a way that the squared Casimir of the algebra represents the deformed radial equation (5.1.112) of the scalar field. We consider the set of vector fields $L_\pm$ and L_0 given by

$$L_\pm = e^{\pm\rho t \pm \sigma\phi}\left(\mp\sqrt{\Delta_{KS}}\partial_r + \frac{C_2 - \Delta_{KS} r}{\sqrt{\Delta_{KS}}}\partial_{\phi'} + \frac{C_1 - \gamma r}{\sqrt{\Delta_{KS}}}\partial_t\right), \tag{5.1.113}$$

$$L_0 = \gamma\partial_t + \delta\partial_{\phi'}, \tag{5.1.114}$$

which should satisy $[L_+, L_-] = 2L_0\,, [L_\pm, L_0] = \pm L_0$ as well as making the squared Casimir

$$L_0^2 - \frac{1}{2}\left(L_+L_- + L_-L_+\right) = \partial_r\left(\Delta_{KS}\partial_r\right) + \frac{\left(2Mr_+\omega - a_1 m'\right)^2}{\left(r - r_+\right)\left(r_+ - r_-\right)} - \frac{\left(2M\kappa r_+\omega - a_2 m'\right)^2}{\left(r - r_-\right)\left(r_+ - r_-\right)}. \tag{5.1.115}$$

The coefficients of ∂_r and ∂_r^2 in (5.2.231) gives two equations

$$\rho C_1 + \sigma C_2 + M = 0\,, \tag{5.1.116}$$

and

$$1 + \rho\gamma + \sigma\delta = 0\,. \tag{5.1.117}$$

In addition, the coefficient of $\partial_{\phi'}^2$ and ∂_t^2 yield

$$-\delta^2\left(r - r_+\right)\left(r - r_-\right) + C_2^2 - 2C_2\delta r + \delta^2 r^2 = a_1^2\frac{r - r_-}{r_+ - r_-} - a_2^2\frac{r - r_+}{r_+ - r_-}, \tag{5.1.118}$$

and

$$C_1^2 - \gamma^2\left(r - r_+\right)\left(r - r_-\right) - 2C_1\gamma r + \gamma^2 r^2 = \frac{4M^2 r_+^2}{\left(r_+ - r_-\right)}\left(\left(r - r_-\right) - \kappa^2\left(r - r_+\right)\right). \tag{5.1.119}$$

The last possible term in (5.2.231) that is proportional to the mixed derivative $\partial_\phi\partial_t$ is

$$\begin{aligned} -C_2C_1 + \delta r C_1 - \delta r^2\gamma \quad &+ \quad \gamma\left(r - r_+\right)\left(r - r_-\right)\delta + C_2\gamma r \\ &= -\frac{2Mr_+}{\left(r_+ - r_-\right)}\left(a_1\left(r - r_-\right) - \kappa a_2\left(r - r_+\right)\right). \end{aligned} \tag{5.1.120}$$

The two different classes of solutions to equation (5.1.155) (that we show by subscripts a and b are,

$$\delta_a = \frac{a_1 + a_2}{r_+ - r_-}\,, \qquad C_{2a} = \frac{a_1 r_- + a_2 r_+}{r_+ - r_-}\,, \tag{5.1.121}$$

$$\delta_b = \frac{a_2 - a_1}{r_+ - r_-}\,, \qquad C_{2b} = \frac{a_2 r_+ - a_1 r_-}{r_+ - r_-}\,. \tag{5.1.122}$$

These solutions substituted into equations (5.1.119) and (5.2.237) give the corresponding C_1 and γ, that are given by

$$\gamma_a = \frac{2Mr_+ (\kappa + 1)}{r_+ - r_-}\,, \qquad C_{1a} = \frac{2Mr_+ (\kappa r_+ + r_-)}{r_+ - r_-}\,, \tag{5.1.123}$$

$$\gamma_b = \frac{2Mr_+ (\kappa - 1)}{r_+ - r_-}\,, \qquad C_{1b} = \frac{2Mr_+ (\kappa r_+ - r_-)}{r_+ - r_-}\,. \tag{5.1.124}$$

So, the generators of $SL(2,\mathbb{R})$ for $a-$solutions are

$$L_{\pm a} = e^{\pm \rho_1 t \mp \left(\frac{b(1+\kappa)}{a\alpha(1-\kappa)} + 2\pi T_R\right)\phi} \left[\mp\sqrt{\Delta_{KS}}\partial_r + \left(Q\beta \frac{r_+ (r_- + \kappa r_+ - r(1+\kappa))}{r_+ - r_-} \right.\right. \tag{5.1.125}$$

$$\left.\left. - \frac{\alpha\Omega_H}{2\pi T_H}(r - (M - b)) \right) \frac{\partial_{\phi'}}{\sqrt{\Delta_{KS}}} + \left(\frac{r - r_+}{2\pi\Omega_H \alpha (T_L + T_R)} - \frac{r - (M-b)}{2\pi T_H} \right) \frac{\partial_t}{\sqrt{\Delta_{KS}}} \right]\,,$$

and

$$L_{0a} = \left(\frac{1}{2\pi T_H} - \frac{1}{2\pi \Omega_H \alpha (T_L + T_R)} \right) \partial_t + \left(\frac{Q\beta (1+\kappa)}{8\pi M T_H} + \frac{\Omega_H \alpha}{2\pi T_H} \right) \partial_{\phi'}\,, \tag{5.1.126}$$

where

$$\rho_1 \equiv \frac{b}{Mr_+(1-\kappa)} + \frac{Q\beta}{2Ma\alpha(1-\kappa)} (M(1+\kappa) - \kappa r_+ - r_-)\,.$$

For the second class of solutions, we find

$$L_{\pm b} = e^{\pm \rho_2 t \mp \left(\frac{b}{a\alpha} + 2\pi T_L\right)\phi} \left[\mp\sqrt{\Delta_{KS}}\partial_r + \left(2Mr_+ \Omega_H \alpha + \frac{Q\beta r_+ (\kappa r_+ - r_- + r - \kappa r)}{r_+ - r_-} \right) \frac{\partial_{\phi'}}{\sqrt{\Delta_{KS}}} \right.$$

$$\left. + \left(2Mr_+ + \frac{(r - r_+)}{2\pi\alpha\Omega_H (T_L + T_R)} \right) \frac{\partial_t}{\sqrt{\Delta_{KS}}} \right]\,, \tag{5.1.127}$$

and

$$L_{0b} = \left(\frac{-1}{2\pi\alpha\Omega_H (T_L + T_R)} \right) \partial_t + \left(\frac{Q\beta(\kappa - 1)}{8\pi T_H M} \right) \partial_{\phi'}\,, \tag{5.1.128}$$

where

$$\rho_2 \equiv \frac{Q\beta\left(r_+\left(r_- - \kappa r_+\right) + Mr_+\left(\kappa - 1\right)\right)}{2Mr_+a\alpha\left(\kappa - 1\right)} + 2\pi\alpha\Omega_H\left(T_R + T_L\right) .$$

As we notice, the generators (5.1.125), (5.1.126), (5.1.127) and (5.1.128) of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ reduce exactly to the generators of generalized hidden conformal symmetry of Kerr black hole [36], in the limit where $\alpha = 1$ and $b = 0$. The left and right temperatures are given by $T_L = T_R\frac{1+\kappa}{1-\kappa}$ and $T_R = \frac{r_+ - r_-}{4\pi\alpha a}$ respectively. This means the right temperature of generalized hidden CFT doesn't get any contribution from the deformation parameter κ and so is the same as the right temperature of hidden CFT while the left temperature is affected by the deformation parameter κ. Demanding the agreement of microscopic entropy of CFT given by (5.1.88) to the Bekenstein-Hawking entropy of Kerr-Sen black hole (5.1.91) requires the central charges are given by

$$c_L = c_R = \frac{6\left(1-\kappa\right)a\alpha Mr_+}{\sqrt{\left(M-b\right)^2 - a^2}} . \tag{5.1.129}$$

These central charges reduce to central charges of generalized hidden CFT of Kerr black hole where $\alpha = 1$ and $b = 0$. As we mentioned earlier, the charged Gibbons-Maeda-Garfinkle-Horowitz-Strominger black hole is a special case of Kerr-Sen black hole when the rotational parameter is zero. In this limit, one can show the solutions to equations (5.1.116) and (5.1.117) exist only for special values of parameter κ. In the b-branch, the solutions are $\sigma = 0$ along with we get

$$\rho = \frac{M-2b}{4M\left(M-b\right)}, \kappa = -\frac{M}{M-2b} . \tag{5.1.130}$$

Consequently, the generators of $SL(2,\mathbb{R})$ for b-solutions (5.1.127),(5.1.128) reduce to

$$L_{\pm b} = e^{\pm\left(\frac{M-2b}{4M(M-b)}\right)}\left(\mp\sqrt{\Delta_{GM}}\partial_r - 2\frac{Q\beta\left(M-r\right)\left(M-b\right)}{\left(M-2b\right)\sqrt{\Delta_{GM}}}\partial_\phi - 4\frac{M\left(M-r\right)\left(M-b\right)}{\left(M-2b\right)\sqrt{\Delta_{GM}}}\partial_t\right) , \tag{5.1.131}$$

$$L_{0b} = -4\frac{M\left(M-b\right)}{M-2b}\partial_t - 2\frac{\beta Q\left(M-b\right)}{M-2b}\partial_\phi , \tag{5.1.132}$$

where $\Delta_{GM} = r(r-2(M-b))$. So, these are the generators of $SL(2,\mathbb{R})$ for the Gibbons-Maeda-Garfinkle-Horowitz-Strominger black hole. The generators (5.1.125) and (5.1.126) of $SL(2,\mathbb{R})$ for a-solutions with $\kappa = M\left(M-2b\right)^{-1}$ give the same copy of generators as in (5.1.131) and (5.1.132) with renaming the generators by $L_\pm \to -L_\mp, L_0 \to -L_0$. We also note that generators (5.1.131) and (5.1.132) in the special case of $Q = 0$ reduce to the generators of $SL(2,\mathbb{R})_{Sch}$ for Schwarzschild black hole [89].

5.1.7 Hidden conformal symmetry for extremal Kerr-Sen

In this subsection we show the hidden conformal symmetry of extremal Kerr-Sen black holes, following the method developed in ref. [31]. In the previous subsection, we find that the non-extremal Kerr-Sen black holes do not posses the Q picture. In the extremal case, the same outcome is obtained, where there is also no Q picture for the hidden symmetry of Kerr-Sen black holes.

Let us start by taking the extremal limit of radial equation (5.1.64), which reads

$$\begin{aligned} &\partial_r\left(\Delta'_{KS}\partial_r R\left(r\right)\right) + \left(\lambda - f\left(r\right)\right)R\left(r\right) \\ = -&\left(\frac{\left(2M\omega r_+ - eQr_+ - ma\right)^2}{\left(r-r_+\right)^2} + \frac{2\left(2M\omega - eQ\right)\left(2M\omega r_+ - eQr_+ - ma\right)}{r-r_+}\right)R\left(r\right)\,, \end{aligned} \tag{5.1.133}$$

where

$$f\left(r\right) = \left(\Delta'_{KS} + 4M\left(M+r\right)\right)\omega^2 - \left(2M+r\right)2eQ\omega + e^2Q^2\,, \tag{5.1.134}$$

and $\Delta'_{KS} = (r-r_+)^2$. Again, the near region, low frequency, and weakly couple limits

$$r\omega \ll 1\,, \qquad \omega M \ll 1\,, \qquad eQ \ll 1\,, \tag{5.1.135}$$

are used here, hence we are able to neglect the function $f(r)$ in (5.1.134). Hence, for example when we consider the J picture, i.e. by setting $e = 0$, (5.1.134) reduces to

$$\left(\partial_r\left(\Delta'_{KS}\partial_r\right) + \frac{\left(2M\omega r_+ - ma\right)^2}{\left(r-r_+\right)^2} + \frac{2\left(2M\omega - eQ\right)\left(2M\omega r_+ - ma\right)}{r-r_+}\right)R\left(r\right) = \lambda R\left(r\right)\,. \tag{5.1.136}$$

The corresponding black hole entropy, angular velocity at the horizon, as well as the electrostatic potential can be written as

$$S_{BH} = 2\pi Ma\,, \tag{5.1.137}$$

$$\Omega_H = \frac{1}{2M}\,, \tag{5.1.138}$$

$$\Phi_H = \frac{Q}{2M}\,, \tag{5.1.139}$$

respectively. Rewriting (5.1.134) in the ϕ' picture gives us

$$\left(\partial_r\left(\Delta'_{KS}\partial_r\right)+\frac{\left(2M\omega r_+-m'(\alpha a+\beta Q r_+)\right)^2}{\left(r-r_+\right)^2}\right.$$
$$\left.+\frac{2\left(2M\omega-m'\beta Q\right)\left(2M\omega r_+-m'(\alpha a+\beta Q r_+)\right)}{r-r_+}\right)R\left(r\right)=\lambda R\left(r\right)\ . \quad (5.1.140)$$

We observe that the set of "conformal coordinates" (5.1.72) doesn't work properly in extremal case, i.e. when $r_+=r_-$. In regard to this problem, the authors of [31] introduce a set of "conformal coordinates", $\omega^\pm$ and y which suits the extremal case discussion. Adopting the "coformal coordinates" in [31] to the ϕ' picture provides

$$\begin{aligned}
\omega^+ &= \left(\frac{\phi'}{a}-\frac{1}{r-r_+}\right)\ , \\
\omega^- &= \frac{1}{2}\left(e^{2\pi T_L\phi'+2n_L t}-\frac{2}{\gamma_1}\right)\ , && (5.1.141)\\
y &= \sqrt{\frac{\gamma_1}{2(r-r_+)}}e^{\pi T_L\phi'+n_L t}\ . && (5.1.142)
\end{aligned}$$

The vectors (5.1.73) and (5.1.74) for the "conformal coordinates" (5.1.141) in terms of t, r and ϕ' coordinates can be read as

$$\begin{aligned}
H_+ &= i\frac{2}{\mathcal{Q}}(2\pi T_L\partial_t-2n_L\partial_{\phi'})\ , \\
H_0 &= i\left(-(r-r_+)\partial_r-\frac{\phi'}{2n_L}(2\pi T_L\partial_t-2n_L\partial_{\phi'})\right)\ , \\
H_- &= i\left\{-\beta_1\phi'(r-r_+)\partial_r-\frac{\partial_t}{(r-r_+)n_L}\right. \\
&\quad\left.+\left((\beta_1\phi')^2+\frac{4}{(r-r_+)^2}\right)\frac{1}{2\mathcal{Q}}(2\pi T_L\partial_t-2n_L\partial_{\phi'})\right\}\ , && (5.1.143)
\end{aligned}$$

and

$$\begin{aligned}
\bar{H}_+ &= 2ie^{-2\pi T_L\phi'-2n_L t}\left((r-r_+)\partial_r+\frac{\partial_t}{2n_L}-\frac{4(\pi T_L\partial_t-n_L\partial_{\phi'})}{(r-r_+)\mathcal{Q}}\right)\ , \\
\bar{H}_0 &= i\left(-e^{-2\pi T_L\phi'-2n_L t}(r-r_+)\partial_r+(1-e^{-2\pi T_L\phi'-2n_L t})\frac{\partial_t}{2n_L}\right. \\
&\quad\left.+\frac{2e^{-2\pi T_L\phi'-2n_L t}}{(r-r_+)\mathcal{Q}}(2\pi T_L\partial_t-2n_L\partial_{\phi'})\right)\ ,
\end{aligned}$$
$$(5.1.144)$$

$$
\begin{aligned}
\bar{H}_{-} &= i\left\{ -\frac{1}{2}\left(e^{2\pi T_L\phi'+2n_L t} - e^{-2\pi T_L\phi'-2n_L t}\right)(r-r_+)\partial_r \right. \\
&\quad + \left(e^{2\pi T_L\phi'+2n_L t} - 2 + e^{-2\pi T_L\phi'-2n_L t}\right)\frac{\partial_t}{4n_L} \\
&\quad \left. + \left(e^{2\pi T_L\phi'+2n_L t} + e^{-2\pi T_L\phi'-2n_L t}\right)\frac{(2\pi T_L\partial_t - 2n_L\partial_{\phi'})}{(r-r_+)\mathcal{Q}} \right\} . \qquad (5.1.145)
\end{aligned}
$$

where $\mathcal{Q} = -2n_L\beta_1$. The squared Casimir

$$
\mathcal{H}^2 = \bar{\mathcal{H}}^2 = -H_0^2 + \frac{1}{2}(H_1H_{-1} + H_{-1}H_1) = \frac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2\partial_+\partial_- \qquad (5.1.146)
$$

constructed from the vectors in (5.1.143) and (5.1.145) reads

$$
\mathcal{H}^2 = \partial_r(\Delta'_{KS}\partial_r) - \left(\frac{(4\pi T_L\partial_t - 4n_L\partial_{\phi'})}{(r-r_+)\mathcal{Q}}\right)^2 - \frac{(8\pi T_L\partial_t - 8n_L\partial_{\phi'})\beta_1\partial_t}{(r-r_+)\mathcal{Q}^2} . \qquad (5.1.147)
$$

Equation (5.1.147) matches (5.1.140) if we identify

$$
n_L = -\frac{\alpha a + \beta Q r_+}{4a\alpha M} \quad , \quad \beta_1 = \frac{2}{\alpha a} \quad , \quad T_L = \frac{1}{2\pi} . \qquad (5.1.148)
$$

The identification of T_L and n_L are in agreement with the previous results in (5.1.86) and (5.1.87) for the J picture setting and after taking the extremal limit. Note that in the extremal limit T_R and n_R in (5.1.86) and (5.1.87) become zero, which reflects the lacking of right mover dual CFT_2 for the system. The absence of T_R and n_R in this hidden conformal symmetry for Kerr-Sen black holes is in agreement with the extremal Kerr/CFT correspondence proposal, which is reviewed in section 4.1, where only the left mover of CFT_2 which is dual to the extremal Kerr black holes. The fact that we have two copies of $SL(2,\mathbb{R})$ does not mean that we have left and right movers CFT that dual to the extremal Kerr-Sen black holes, but rather that they are just the different representations of $SL(2,\mathbb{R})$ symmetry that are belong to the same CFT_2, i.e. the left mover one.

Finding that (5.1.147) matches (5.1.140) reflects that we we can also find the hidden conformal symmetry for extremal Kerr-Sen black holes. In other words, the hidden conformal symmetry is not an exclusive property of the non-extremal Kerr-Sen black holes only, though we had such feeling before when we notice that the mapping (5.1.72) is not well behaved when we take the extremal limit.

In [113], the authors show the hidden conformal symmetry of extremal Kerr-Sen in J picture only. In contrast to the work we performed in this thesis, the authors of [113] discuss the neutral scalar test particle in the Kerr-Sen background. By doing so, there is no basis to judge the lack of Q picture for the hidden conformal symmetry of Kerr-Sen black holes. From (5.1.148), aiming to obtain Q picture results by setting $\alpha = 0$ doesn't work, from which we can state that the extremal Kerr-Sen black hole doesn't have Q picture hidden conformal symmetry. The situation for extremal Kerr-Newman black holes is different. The authors of [23] show that the extremal Kerr-Newman black holes possess both the J and Q picture hidden conformal symmetries.

Setting $\beta = 0$ and $\alpha = 1$ of the results in (5.1.148) gives the hidden conformal symmetry generators in J picture, which from (5.1.143) and (5.1.145) one can derive the corresponding generators as the followings

$$\begin{aligned}
H_+ &= i2Ma\left(\partial_t + \frac{1}{2M}\partial_\phi\right) ,\\
H_0 &= i\left(-\left(r-r_+\right)\partial_r + 2M\phi\left(\partial_t + \frac{1}{2M}\partial_\phi\right)\right) ,\\
H_- &= i\left(-\frac{2}{a}\phi\left(r-r_+\right)\partial_r + \frac{4M}{\left(r-r_+\right)}\partial_t + 2Ma\left(\left(\frac{\phi}{a}\right)^2 + \left(\frac{1}{r-r_+}\right)^2\right)\left(\partial_t + \frac{1}{2M}\partial_\phi\right)\right) ,
\end{aligned} \tag{5.1.149}$$

and

$$\begin{aligned}
\bar{H}_+ &= 2ie^{-\phi+\frac{t}{2M}}\left(\left(r-r_+\right)\partial_r - 2M\partial_t - \frac{2Ma}{\left(r-r_+\right)}\left(\partial_t + \frac{1}{2M}\partial_\phi\right)\right) ,\\
\bar{H}_0 &= i\left(-e^{-\phi+\frac{t}{2M}}\left(r-r_+\right)\partial_r - 2M\left(1-e^{-\phi+\frac{t}{2M}}\right)\partial_t + \frac{2Mae^{-\phi+\frac{t}{2M}}}{\left(r-r_+\right)}\left(\partial_t + \frac{1}{2M}\partial_\phi\right)\right) ,\\
\bar{H}_- &= i\left(-\frac{1}{2}\left(e^{\phi-\frac{t}{2M}} - e^{-\phi+\frac{t}{2M}}\right)\left(r-r_+\right)\partial_r - 2M\left(e^{\phi-\frac{t}{2M}} - 2 + 2e^{-\phi+\frac{t}{2M}}\right)\partial_t\right.\\
&\quad \left. - \frac{Ma}{\left(r-r_+\right)}\left(e^{\phi-\frac{t}{2M}} + e^{-\phi+\frac{t}{2M}}\right)\left(\partial_t + \frac{1}{2M}\partial_\phi\right)\right) .
\end{aligned} \tag{5.1.150}$$

The squared Casimir (5.1.146) constructed from the vectors in (5.1.149) and (5.1.150) is just the left hand side of equation (5.1.136). It means that these vectors are the generators of hidden conformal symmetry of the extremal Kerr-Sen black holes in J picture.

5.1.8 An alternative to construct the hidden symmetry

In subsection 5.1.6, we have applied the method by Lowe et al [36] in constructing the deformed hidden conformal symmetry generators for the non-extremal Kerr-Sen black holes.

Here, we would like to show that the method proposed by Lowe et al [36] can be used to get a set of hidden conformal symmetry generators for the extremal Kerr-Sen black holes. Note that we do not deform the equation of motion, since the deformation technique works only for the non-extremal case, i.e. we are allowed to consider the case of back holes which are far from the extremal condition.

As we have seen before, the first step is defining a set of general generators

$$\begin{aligned} L_\pm &= e^{\pm\rho t \pm \sigma\phi}\left(\mp\sqrt{\Delta'_{KS}}\partial_r + \frac{C_1-\gamma r}{\sqrt{\Delta'_{KS}}}\partial_t + \frac{C_2-\delta r}{\sqrt{\Delta'_{KS}}}\partial_\phi\right) \\ L_0 &= \gamma\partial_t + \delta\partial_\phi\,, \end{aligned} \tag{5.1.151}$$

where the $sl(2,\mathbb{R})$ algebra $[L_\pm, L_0] = \pm L_\pm$ and $[L_+, L_-] = 2L_0$ are satisfied. We now focus on the J picture only, where the equation of motion that we are considering is (5.1.134) with $e = 0$. In this consideration, the squared Casimir of generators (5.1.151) should match the left hand side of equation (5.1.136), i.e.

$$L_0^2 - \frac{1}{2}\left(L_+L_- + L_-L_+\right) = \partial_r\left(\Delta'_{KS}\partial_r\right)R\left(r\right) + \left(\frac{\left(2M\omega r_+ - ma\right)^2}{\left(r-r_+\right)^2} + \frac{2\left(2M\omega\right)\left(2M\omega r_+ - ma\right)}{r-r_+}\right). \tag{5.1.152}$$

Matching the coefficients of ∂_r and ∂_r^2 from the left and right hand sides of equation (5.1.152) gives

$$\rho C_1 + \sigma C_2 + M = 0\,, \tag{5.1.153}$$

and

$$1 + \rho\gamma + \sigma\delta = 0\,. \tag{5.1.154}$$

Furthermore, from the coefficients of ∂_ϕ^2 and ∂_t^2 in (5.1.152) we can have

$$2C_2\delta r + 2\delta^2 r r_+ - \delta^2 r_+^2 + C_2^2 = a^2\,, \tag{5.1.155}$$

and

$$-2C_1\gamma r + 2\gamma^2 r r_+ - \gamma^2 r_+^2 + C_1^2 = 4M^2 r_+\left(2r - r_+\right)\,. \tag{5.1.156}$$

Finally, the mixed derivative $\partial_\phi\partial_t$ in (5.1.152) gives us

$$2C_2C_1 + 4\gamma\delta r r_+ - 2\gamma\delta r_+^2 - 2\delta r C_1 - 2C_2\gamma r = 4Mar\,. \tag{5.1.157}$$

By using (5.1.153), (5.1.154), (5.1.155), (5.1.156), and (5.1.157), we obtain two set of solutions of $C_1, C\ C_2,\ \delta,\ \gamma,\ \sigma$ and ρ as tabulated in table 5.2.

Table 5.2: Solutions in J picture

	$(+)$	$(-)$
δ	0	0
C_2	a	$-a$
C_1	0	0
γ	$-2M$	$2M$
ρ	$\frac{1}{2M}$	$-\frac{1}{2M}$
σ	$-\frac{M}{a}$	$\frac{M}{a}$

Consider the generators (5.1.151) constructed from the $(+)$ family coefficients as the $L^+_\pm$ and L^+_0, and the $(-)$ family coefficients as the $L^-_\pm$ and L^-_0. We can show that there is a mapping between the two set of generators,

$$L^+_\pm = -L^-_\mp \quad , \quad L^+_0 = -L^-_0 \, . \tag{5.1.158}$$

It is just a reflection of the invariance of squared Casimir

$$L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) \tag{5.1.159}$$

by the transformation

$$L_0 \to -L_0 \quad , \quad L_\pm \to -L\mp \, . \tag{5.1.160}$$

Interestingly, in extremal case the generators (5.1.151) can reveal only a single copy of $SL(2,\mathbb{R})$ symmetry of the system. It agrees the previous conclusion that there is only a single dual CFT for extremal Kerr-Sen black holes. Explicitly, these generators (5.1.151) for extremal Kerr-Sen can be read as

$$\begin{aligned} L^+_\pm &= e^{\pm\frac{t}{2M}\mp\frac{M\phi}{a}}\left(\mp\sqrt{\Delta'_{KS}}\partial_r + \frac{2Mr}{\sqrt{\Delta'_{KS}}}\partial_t + \frac{a}{\sqrt{\Delta'_{KS}}}\partial_\phi\right), \\ L^+_0 &= -2M\partial_t \, . \end{aligned} \tag{5.1.161}$$

It is not easy to see how the generators (5.1.161) can be mapped to those in (5.1.149) or (5.1.150). Nevertheless, the mapping should exist since the generators in (5.1.161), (5.1.149), and (5.1.150) describes a single copy of the conformal symmetry that is hidden in the equation (5.1.136).

5.1.9 Macroscopic absorption cross section

The radial equation can be written as

$$\partial_r \Delta'_{KS} \partial_r R(r) + \frac{K_1^2}{(r-r_+)^2} R(r) + \frac{K_2}{r-r_+} R(r) = \lambda R(r)\,, \tag{5.1.162}$$

where

$$\begin{aligned} K_1 &= 2M\omega r_+ - eQr_+ - ma\,, \\ K_2 &= 2\left(2M\omega - eQ\right)\left(2M\omega r_+ - eQr_+ - ma\right)\,. \end{aligned} \tag{5.1.163}$$

Introducing $z = \frac{-2iK_1}{r-r_+}$, we get the Whittaker equation

$$R''(z) + (-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2})R(z) = 0\,, \tag{5.1.164}$$

where

$$k = i(2M\omega - eQ), \quad m^2 = \frac{1}{4} + \lambda\,. \tag{5.1.165}$$

This equation has the solution

$$R(z) = C_1 R_+(z) + C_2 R_-(z)\,, \tag{5.1.166}$$

where

$$R_\pm(z) = e^{-\frac{z}{2}} z^{\frac{1}{2} \pm m} F(\frac{1}{2} \pm m - k, 1 \pm 2m, z) \tag{5.1.167}$$

are two linearly independent solution.

At the near horizon where we have a very large z, the Kummer function $F(\alpha, \gamma, z)$ in (5.1.167) could be expanded asymptotically as

$$F(\alpha, \gamma, z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} e^{-i\alpha\pi} z^{-\alpha} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha - \gamma}\,. \tag{5.1.168}$$

To cancel the outgoing mode in the solutions, we need to have

$$C_1 = -\frac{\Gamma(1-2m)}{\Gamma(\frac{1}{2} - m - k)} C, \qquad C_2 = \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2} + m - k)} C \tag{5.1.169}$$

to thus the ingoing modes only that left.

When r goes asymptotically to infinity, $z \to 0$, $F(\alpha, \gamma, z) \to 1$, the solution has asymptotic behavior

$$R \sim C_1 r^{-h} + C_2 r^{1-h}\,, \tag{5.1.170}$$

where h is the conformal weight of the scalar

$$h = \frac{1}{2} + m = \frac{1}{2} + \sqrt{\frac{1}{4} + \lambda}\,. \tag{5.1.171}$$

The retarded Green's function could be read directly [79]

$$G_R \sim \frac{C_1}{C_2} \propto \frac{\Gamma(1-2h)\Gamma(h-k)}{\Gamma(2h-1)\Gamma(1-h-k)}\,. \tag{5.1.172}$$

5.1.10 Microscopic point of view

As it is mentioned in the original extremal Kerr/CFT proposal [12], only one copy of CFT_2 that dual to the black hole at the near horizon. It is surprising that by using prescription by authors of [31], that we still can uncover the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ hidden conformal symmetry for extremal black holes which was discussed in section 5.1.8. The discussions in section 5.1.9 seems to be more natural in describing extremal black holes, because it is consistent with what we expect that only a copy of $SL(2,\mathbb{R})$ exists. However, at the end we still have one CFT temperature left, i.e. T_L. Now by using the Cardy formula (3.3.116) we can obtain the entropy of extremal Kerr-Sen black holes

$$S = \frac{\pi^2}{3} c_L T_L = 2\pi J\,, \tag{5.1.173}$$

which is in agreement with the macroscopic Bekenstein-Hawking entropy.

The first law of thermodynamics tells us

$$\delta S = \frac{\delta M - \Omega_H \delta J - \Phi \delta Q}{T_H} \tag{5.1.174}$$

which in [28] can be used to match the microscopic and macroscopic parameters. At extremal condition, we know that

$$T_H \to 0$$

as well as

$$\delta M - \Omega_H \delta J - \Phi \delta Q \to 0.$$

A little bit work on this equation gives us a separation in δS,

$$\delta S = 2\pi(2M\delta M - Q\delta Q) + 4\pi \frac{(2M^2 - Q^2)\delta M - a\delta J - QM\delta Q}{2\sqrt{M^2 - a^2 - Q^2}}\,, \tag{5.1.175}$$

where the second term in (5.1.175) vanishes when $M - b = a$, which also means $2M\delta M - Q\delta Q = \delta J$ that would be useful to show

$$\delta S = 2\pi(2M\delta M - Q\delta Q)\,. \tag{5.1.176}$$

Then we find that the identifications

$$\delta Q = e, \quad \delta M = \omega, \quad \delta E_L = \omega_L - q_L\mu_L\,, \tag{5.1.177}$$

with

$$\omega_L = 2M\omega, \quad \mu_L = Q, \quad q_L = e\,, \tag{5.1.178}$$

yield the equation (5.1.137) can be rewritten as

$$\delta S = \frac{\delta E_L}{T_L} = \frac{\omega_L - q_L\mu_L}{T_L}\,. \tag{5.1.179}$$

The retarded charged scalar Green's function in the extremal Kerr-Sen black hole could be rewritten as

$$G_R \sim \frac{\Gamma(1-2h)\Gamma(h - i(2M\omega - eQ))}{\Gamma(2h-1)\Gamma(1-h-i(2M\omega - eQ))} \tag{5.1.180}$$

$$= \frac{\Gamma(1-2h)}{\Gamma(2h-1)}\frac{\Gamma\left(h - i\frac{\omega_L - q_L\mu_L}{2\pi T_L}\right)}{\Gamma\left(1 - h - i\frac{\omega_L - q_L\mu_L}{2\pi T_L}\right)}\,, \tag{5.1.181}$$

where the conformal wight is given in (5.1.171).

In a two-dimensional conformal field theory, the two-point functions of the primary operators are determined by the conformal invariance. The retarded correlator $G_R(\omega_L, \omega_R)$ is analytic on the upper half complex $\omega_{L,R}$ plane and its value along the positive imaginary $\omega_{L,R}$ axis gives the Euclidean correlator:

$$G_E(\omega_{L,E}, \omega_{R,E}) = G_R(i\omega_{L,E}, i\omega_{R,E}), \quad \omega_{L,E}, \omega_{R,E} > 0\,. \tag{5.1.182}$$

At finite temperature, $\omega_{L,E}$ and $\omega_{R,E}$ take discrete values of the Matsubara frequencies

$$\omega_{L,E} = 2\pi m_L T_L, \quad \omega_{R,E} = 2\pi m_R T_R\,, \tag{5.1.183}$$

where m_L, m_R are integers for bosonic modes and are half integers for fermionic modes. For an operator of dimensions (h_L, h_R), charges (q_L, q_R) at temperatures (T_L, T_R) and chemical potentials (μ_L, μ_R), the momentum space Euclidean correlator is given by[67]

$$G_E \sim T_L^{2h_L-1} e^{i\tilde{\omega}_{L,E}/2T_L} \Gamma\left(h_L - \frac{\tilde{\omega}_{L,E}}{2\pi T_L}\right) \Gamma\left(h_L + \frac{\tilde{\omega}_{L,E}}{2\pi T_L}\right) , \quad (5.1.184)$$

with the Euclidean frequency

$$\tilde{\omega}_{L,E} = \omega_{L,E} - iq_L\mu_L, \quad \omega_{L,E} = i\omega_L \,. \quad (5.1.185)$$

The absorption cross section can be read from the retarded Green's function

$$\sigma \sim \mathrm{Im} G_R \propto \sinh\left(\frac{\tilde{\omega}_L}{2T_L}\right) \left|\Gamma(h_L + i\frac{\tilde{\omega}_L}{2\pi T_L})\right|^2 , \quad (5.1.186)$$

which agrees with the finite temperature absorption cross section for a 2D chiral CFT

$$\sigma \sim T_L^{2h_L-1} \sinh\left(\frac{\tilde{\omega}_L}{2T_L}\right) \left|\Gamma(h_L + i\frac{\tilde{\omega}_L}{2\pi T_L})\right|^2 . \quad (5.1.187)$$

Unlike the absorption cross section in the non extremal case, we have only a copy of CFT represented in the last result. The reason is, as it appears also in the computing the central charge for extremal black holes, one of $SL(2,\mathbb{R})$ symmetry breaks when the system in extreme case.

5.2 Deformed Hidden Conformal Symmetry for Kerr-Newman Black Holes

5.2.1 Einstein-Maxwell Theory

We discussed the vacuum Einstein gravitational system in section 2.1. In such system, a mass curves the spacetime where the spacetime outside of the mass is literally empty, i.e. not even electromagnetic radiations exist. Nevertheless, we know that the gravitational interaction is not the only long range interaction in the universe. Since our discussion is still classical, the interactions that we need to consider are the long range type, i.e. the gravitational and electromagnetic interactions. Taking into account the electromagnetism in the Einstein gravitational theory framework can be done by adding a term in the Einstein-Hilbert action (2.1.33) which represents the electromagnetic contribution, which can be written as

$$S_{EM} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - \frac{1}{4}F^2\right). \tag{5.2.188}$$

The superscript "EM" in action above stands for Einstein-Maxwell, not to be confused with Electro-Magnetic. A theory that is described by (5.2.188) is called the Einstein-Maxwell theory.

We observe the presence of F^2 term in (5.2.188),

$$F^2 = F_{\mu\nu}F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\,, \tag{5.2.189}$$

that represents the contribution of the electric field $\vec{E}$ and magnetic field $\vec{B}$ in the theory,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{5.2.190}$$

In terms of the gauge fields A_μ, the field strength tensor $F_{\mu\nu}$ in a curved spacetime is given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\,. \tag{5.2.191}$$

However, since Christoffel symbol of the second kind $\Gamma^\sigma_{\mu\nu}$ contained in the covariant derivative ∇_μ,

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu}A_\sigma\,, \tag{5.2.192}$$

is symmetric in its lower indices permutation, the reading of the field strength tensor $F_{\mu\nu}$ in a curved spacetime would be just like the one in the flat spacetime,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.2.193}$$

The field strength tensor $F_{\mu\nu}$ is invariant under the gauge transformation

$$A'_\mu = A_\mu + \partial_\mu\Lambda\,, \tag{5.2.194}$$

where scalar function Λ is called the gauge parameter.

In the form language, one can reformulate the Maxwell theory in such a more elegant way. The gauge fields can be expressed as a 1-form,

$$\mathbf{A} = A_\mu dx^\mu \,. \tag{5.2.195}$$

Accordingly, the gauge transformation (5.2.194) can be rewritten as

$$\mathbf{A}' = \mathbf{A} + d\Lambda \,, \tag{5.2.196}$$

and the corresponding field strength tensor $F_{\mu\nu}$ is replaced by a 2-form $\mathbf{F}$ which is defined by

$$\mathbf{F} = d\mathbf{A} \,. \tag{5.2.197}$$

Clearly the last formula reminds us a relation in the form language where one can obtain a 2-form by performing an exterior derivative "d" to a 1-form. Furthermore, by using the form language, the action for Maxwell fields in curved spacetime can be read as

$$S_M = \frac{1}{2}\int \mathbf{F} \wedge *\mathbf{F} \,. \tag{5.2.198}$$

As one can see from the action (5.2.188), there are two kinds of fields in the Einstein-Maxwell theory, i.e. the graviton $g_{\mu\nu}$ and the gauge field A_μ. Performing the variations of the Einstein-Maxwell action with respect to each of these fields yield a set of equations of motion which rule the dynamics of $g_{\mu\nu}$ and A_μ. Using the identities,

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} \quad \text{and} \quad \delta g = -g g_{\mu\nu}\delta g^{\mu\nu} \,, \tag{5.2.199}$$

the variation of Einstein-Maxwell action (5.2.188) with respect to $g^{\mu\nu}$ can be written as

$$\begin{aligned} \frac{\delta S_{EM}}{\delta g^{\mu\nu}} &= \frac{1}{16\pi}\int d^4x \left\{ \left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right)\left(R - \frac{1}{4}F^2\right) + \sqrt{-g}\frac{\delta\left(R - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\right)}{\delta g^{\mu\nu}} \right\} \\ &= \frac{1}{16\pi}\int d^4x \left\{ -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(R - \frac{1}{4}F^2\right) + \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}F_{\mu\alpha}F^{\alpha}_{\nu}\right)\right\} . \end{aligned} \tag{5.2.200}$$

The principle of least action requires that the integrand in the last formula vanishes, hence we have an equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}T_{\mu\nu} \tag{5.2.201}$$

where the energy momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F^2 \,. \tag{5.2.202}$$

The nonzero of right hand side in equation (5.2.201) signals the non vacuum property of the spacetime in Einstein-Maxwell theory. The energy momentum tensor $T_{\mu\nu}$ does not appear in the vacuum Einstein gravitational system.

The variation of the Einstein-Maxwell action (5.2.188) with respect to the gauge field A_μ gives another equation of motion in the theory. Since the gauge field appears in the action in its first order derivative, it would more convenient to perform the Euler-Lagrange equation in deriving the equation of motion for A_μ. The associated Lagrangian density is

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi}\left(R - \frac{1}{4}F^2\right) \,. \tag{5.2.203}$$

It follows from the Lagrangian density (5.2.203) that

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \,. \tag{5.2.204}$$

Therefore, the Euler-Lagrange equation with the Lagrangian density (5.2.203) becomes

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu A_\nu\right)} = 0 \,. \tag{5.2.205}$$

Using the equation (5.2.191), from (5.2.205) one can show that

$$\nabla_\mu \left(\sqrt{-g}F^{\mu\nu}\right) = 0 \tag{5.2.206}$$

which is known as the free Maxwell equation in the curved spacetime.

At this point we have obtained two equations, i.e. (5.2.201) and (5.2.206), which describe the behavior of graviton and gauge field in Einstein-Maxwell system. In subsection 2.2.1 we have encountered a quite tedious derivation to rediscover the Kerr solution. Indeed, we will face some more complexities in deriving the tensor metric together with the corresponding gauge fields which obey the equations (5.2.201) and (5.2.206). Therefore, we just write down the solution, which was first found by Ezra Newman [91, 92]. The solution for $g_{\mu\nu}$ in the Einstein-Maxwell theory is

$$ds^2 = -\frac{\Delta_{KN} - a^2\sin^2\theta}{\varrho}\left[dt + \frac{(2Mr - Q^2)a\sin^2\theta}{\Delta_{KN} - a^2\sin^2\theta}d\phi\right]^2 + \varrho\frac{dr^2}{\Delta_{KN}} + \varrho d\theta^2 + \frac{\varrho\Delta_{KN}\sin^2\theta}{\Delta_{KN} - a^2\sin^2\theta}d\phi^2 \,, \tag{5.2.207}$$

where

$$\varrho = r^2 + a^2 \cos^2\theta \,, \tag{5.2.208}$$

$$\Delta_{KN} = r^2 - 2Mr + a^2 + Q^2 \,. \tag{5.2.209}$$

The metric (5.2.207) is known as the Kerr-Newman solution, which describes the spacetime outside of an electrically charged rotating massive object. The gauge field associated to the solution (5.2.207) is

$$\mathbf{A} = -\frac{Qr}{\varrho}\left(dt - a\sin^2\theta d\phi\right) \,. \tag{5.2.210}$$

In the limit of $a = 0$, the Kerr-Newman metric (5.2.207) reduces to the Reissner-Nordstrom solution,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{5.2.211}$$

which describes the spacetime outside of a static and electrically charged massive object. The associated gauge field to the Reissner-Nordstrom solution is

$$\mathbf{A} = -\frac{Qr}{\varrho}dt \,, \tag{5.2.212}$$

which is the $a \to 0$ limit of (5.2.210). Furthermore, taking $Q = 0$ from the equation (5.2.211), we recover the Schwarzschild spacetime (2.1.50). It is clear that in the limit of $Q = 0$, the gauge field A_μ does not present.

In the Einstein-Maxwell theory, we have seen some solutions which are connected each other, i.e. the Kerr-Newman, Reissner-Nordstrom , and Schwarzschild solutions, and all of them may describe black holes. It shows the existence of a family of black holes in the Einstein-Maxwell theory, which is tabulated in the table 5.3.

Table 5.3: Black holes families in Einstein-Maxwell theory

	$J = 0$	$J \neq 0$
$Q = 0$	Schwarzschild	Kerr
$Q \neq 0$	Reissner-Nordstrom	Kerr-Newman

5.2.2 The hidden conformal symmetries for Kerr-Newman black holes

The inner and outer horizons, r_- and r_+ respectively, can be expressed as

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}\,. \tag{5.2.213}$$

For the extremal Kerr-Newman black holes, $M^2 = a^2 + Q^2$ which provides $r_+ = r_- = M$. The Bekenstein-Hawking entropy, Hawking temperature, angular velocity and the electric potential at the horizon of the black hole (5.2.207) can be read as

$$S_{BH} = \pi(r_+^2 + a^2)\,, \tag{5.2.214}$$

$$T_H = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}\,, \tag{5.2.215}$$

$$\Omega_H = \frac{a}{r_+^2 + a^2}\,, \tag{5.2.216}$$

$$\Phi_H = \frac{Q r_+}{r_+^2 + a^2}\,, \tag{5.2.217}$$

respectively.

We consider a massless charged test scalar field in the background of the Kerr-Newman black hole. The minimally coupled equation of motion for the scalar field is

$$(\nabla_\alpha - ieA_\alpha)(\nabla^\alpha - ieA^\alpha)\Phi = 0\,, \tag{5.2.218}$$

where e is the electric charge of scalar field. There are two Killing vectors ∂_t and ∂_ϕ for the Kerr-Newman black holes (5.2.207). We separate the coordinates in the solutions to equation (5.2.218) as

$$\Phi(t, r, \theta, \phi) = \mathrm{e}^{-i\omega t + im\phi} R(r) S(\theta)\,. \tag{5.2.219}$$

Using (5.2.219) in equation (5.2.218) leads to two differential equations for angular function $S(\theta)$ and the radial function $R(r)$,

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta S(\theta)) - \left[a^2\omega^2\sin^2\theta + \frac{m^2}{\sin^2\theta} - K_l\right] S(\theta) = 0\,, \tag{5.2.220}$$

$$\partial_r(\Delta_{KN}\partial_r R(r)) + \left[\frac{[(r^2 + a^2)\omega - eQr - ma]^2}{\Delta_{KN}} + 2ma\omega - K_l\right] R(r) = 0\,, \tag{5.2.221}$$

where K_l is the separation constant. Furthermore, the radial equation (5.2.221) can be rewritten as

$$\partial_r(\Delta_{KN}\partial_r R(r))+\left[\frac{\left[(r_+^2+a^2)\omega-am-Qr_+q\right]^2}{(r-r_+)(r_+-r_-)}-\frac{\left[(r_-^2+a^2)\omega-am-Qr_-q\right]^2}{(r-r_-)(r_+-r_-)}\right]R(r)$$
$$+f(r)R(r)=K_lR(r)\,, \tag{5.2.222}$$

where $f(r)=\omega^2r^2+2(\omega M-eQ)\omega r+\omega^2a^2-\omega^2Q^2+(2\omega M-eQ)^2$. To simplify the radial equation (5.2.222) and find the hidden conformal symmetry, we consider the low frequency scalar field $\omega M\ll 1$ where the non-extremal condition guarantees $\omega a\ll 1$ and $\omega Q\ll 1$. Moreover, we assume small electric charge for the scalar field $eQ\ll 1$. These conditions in the near region geometry $\omega r\ll 1$, lead to neglect the function $f(r)$ in the radial equation (5.2.222). So, we find

$$\partial_r(\Delta_{KN}\partial_r R(r))+\left[\frac{\left[(2Mr_+-Q^2)\omega-am-Qr_+e\right]^2}{(r-r_+)(r_+-r_-)}-\frac{\left[(2Mr_--Q^2)\omega-am-Qr_-e\right]^2}{(r-r_-)(r_+-r_-)}\right]R(r)$$
$$=l(l+1)R(r)\,, \tag{5.2.223}$$

where we set the separation constant $K_l=l(l+1)$.

Considering a charged probe in the background of a rotating charge black hole in Kerr/CFT correspondence leads to new features that are quite distinct for rotating charged black holes [34, 90] . As the first example, there are two different individual CFT_2 that are holographically dual to the Kerr-Newman black hole. The twofold hidden conformal symmetries are in J picture where the charge of probe is neglected and in Q picture where the probe co-rotates with the horizon. In J picture the electric charge of probe is set to be zero while in Q picture the scalar wave expansion is restricted to be in the $m=0$ mode. Each of the two pictures provides the hidden conformal symmetry and so establishes the correspondence to the CFT_2 [34].

Therefore, one may expect the twofold hidden conformal symmetries must exist for other four-dimensional rotating charged black holes. However, the Kerr-Sen black hole which is a rotating charged black hole in four-dimensions doesn't possess the twofold hidden conformal symmetries. More specifically, the four-dimensional Kerr-Sen black hole as the solutions to the low energy limit of heterotic string theory don't have the hidden conformal symmetry in a well defined Q picture [90]. One may consider the absence of Q picture for the Kerr-Sen black hole as a counterexample to the "microscopic hair conjecture" that only exists in Einstein-Maxwell theory [34].

In revealing the twofold picture of hidden conformal symmetries for the Kerr-Newman black holes, the scalar wave function can be expanded as

$$\Phi = e^{-\omega t + im\phi + ie\chi} R(r) S(\theta) \,, \tag{5.2.224}$$

where the internal dimension χ has the same $U(1)$ symmetry as the coordinate ϕ. The existence of two coordinates with $U(1)$ symmetry leads the twofold hidden symmetries for the Kerr-Newman black holes. We note that the twofold hidden conformal symmetries of the Kerr-Newman suggest the unique central charge in each picture. In J picture, the central charge depends only on the angular momentum J while in Q picture, the central charge depends only on the black holes charge Q. In both pictures, all the results for microscopic entropy, absorption cross section, and real time correlators are in favor of Kerr/CFT correspondence. In the next following sections, we confirm that the deformed hidden conformal symmetry for Kerr-Newman black hole exists in both J and Q pictures, as well as finally can be collected in a single picture namely general picture [23].

5.2.3 Deformed hidden conformal symmetry in J picture

The radial equation (5.2.223) has two poles on outer horizon r_+ and inner horizon r_- where the Kerr-Newman metric function Δ_{KN} (5.2.209) vanishes. For the Kerr-Newman black holes far from the extremality, we note that r is far enough from r_-. As a result of this, we can drop the linear and quadratic terms in frequency [90, 36]. These terms are coming from the expansion near the inner horizon. So we deform the radial equation (5.2.223) near the inner horizon r_- by deformation parameter κ as

$$\left[\frac{\left[(2Mr_+ - Q^2)\omega - am - Qr_+e\right]^2}{(r-r_+)(r_+-r_-)} - \frac{\left[(2M\kappa r_+ - Q^2)\omega - am - Q\kappa r_+e\right]^2}{(r-r_-)(r_+-r_-)}\right] R(r)$$
$$= -\partial_r(\Delta_{KN}\partial_r R(r)) + l(l+1)R(r) \,. \tag{5.2.225}$$

where κ satisfies

$$(2M^2\kappa - Q^2)am\omega \ll 2\sqrt{M^2 - a^2 - Q^2}(r - r_-) \tag{5.2.226}$$

as well as

$$(2M^2\kappa - Q^2)^2\omega^2 \ll 2\sqrt{M^2 - a^2 - Q^2}(r - r_-)\,. \tag{5.2.227}$$

These two constraints on κ guarantee that equation (5.2.225) is still in low frequency limit and so one can neglect the linear and quadratic terms in frequency that come from the expansion near the inner horizon. Moreover, the constraints do not change drastically the near region geometry of the black hole. We note that physical justification for the deformation in the radial equation near the inner horizon is related to the fact that the solutions to the exact radial equation (5.2.222) (before going to the near region and considering the low frequency limit and small electric charge for the scalar field) are singular at the inner horizon. However, it is shown that the back-reaction of the field on the internal geometry of black hole replaces the inner Cauchy horizon by a null curvature spacelike singularity that covers the inner horizon of the black hole [93, 94, 95]. As a result, the region behind the null spacelike singularity that includes the inner horizon is not the physical region of interest in the solutions of the radial equation (5.2.223). In other words, one can consider the deformation of the radial equation near the inner horizon only, given by (5.2.225), as the radial equation describes the dynamics of the test field outside of the null spacelike singularity. The other interesting feature of deformation of the inner horizon is that it doesn't change the location of other singularities of the radial equation (5.2.223) that are located on the outer horizon and far infinity.

Let us consider first the deformed equation (5.2.225) in the J picture that can be written as

$$\partial_r(\Delta_{KN}\partial_r R(r)) + \left[\frac{\left[(2Mr_+ - Q^2)\omega - am\right]^2}{(r - r_+)(r_+ - r_-)} - \frac{\left[(2M\kappa r_+ - Q^2)\omega - am\right]^2}{(r - r_-)(r_+ - r_-)}\right] R(r) = l(l+1)R(r)\,. \tag{5.2.228}$$

We consider the following vector fields

$$\begin{aligned} L_\pm &= e^{\pm\rho t \pm\sigma\phi}\left(\mp\sqrt{\Delta_{KN}}\partial_r + \frac{C_1 - \gamma r}{\sqrt{\Delta_{KN}}}\partial_t + \frac{C_2 - \delta r}{\sqrt{\Delta_{KN}}}\partial_\phi\right), & (5.2.229)\\ L_0 &= \gamma\partial_t + \delta\partial_\phi\,, & (5.2.230)\end{aligned}$$

that make the $sl(2,\mathbb{R})$ algebra given by $[L_\pm, L_0] = \pm L_\pm$ and $[L_+, L_-] = 2L_0$ [90, 36]. Furthermore, we require the squared Casimir of $SL(2,\mathbb{R})$ represents the deformed radial equation

(5.2.228). Hence we find

$$L_0^2 - \frac{1}{2}\left(L_+L_- + L_-L_+\right)$$

$$= \partial_r\left(\Delta_{KN}\partial_r\right) + \frac{\left((2Mr_+ - Q^2)\omega - am\right)^2}{(r-r_+)\,(r_+-r_-)} - \frac{\left((2M\kappa r_+ - Q^2)\omega - am\right)^2}{(r-r_-)\,(r_+-r_-)}\,. \qquad (5.2.231)$$

We notice that the following automorphism for the generators $L_\pm$ and L_0,

$$L_\pm \to -L_\pm \quad , \quad L_0 \to L_0\,, \qquad (5.2.232)$$

does not change the $sl(2,\mathbb{R})$ algebra and so the squared Casimir is invariant.

We get the following two equations for the coefficients of ∂_r and ∂_r^2 in (5.2.231)

$$\rho C_1 + \sigma C_2 + M = 0\,, \qquad (5.2.233)$$

and

$$1 + \rho\gamma + \sigma\delta = 0\,. \qquad (5.2.234)$$

In addition, the coefficients of ∂_ϕ^2 and ∂_t^2 in (5.2.231) give two other equations as

$$-\delta^2\,(r-r_+)\,(r-r_-) + C_2^2 - 2C_2\delta r + \delta^2 r^2 = a^2\,, \qquad (5.2.235)$$

and

$$C_1^2 - \gamma^2\,(r-r_+)\,(r-r_-) - 2C_1\gamma r + \gamma^2 r^2 = \frac{(2Mr_+ - Q^2)^2}{(r_+-r_-)}\left((r-r_-) - \kappa^2\,(r-r_+)\right) -$$

$$-\frac{4MQ^2 r_+}{(r_+-r_-)}\left(r - r_- - \kappa\,(r-r_+)\right) + Q^4\,. \qquad (5.2.236)$$

Finally, we get the following equation which is the coefficient of $\partial_\phi\partial_t$ in (5.2.231)

$$-C_2C_1 + \delta r C_1 - \delta r^2\gamma + \gamma\,(r-r_+)\,(r-r_-)\,\delta + C_2\gamma r =$$

$$= -\frac{2Mr_+a}{(r_+-r_-)}\left((r-r_-) - \kappa\,(r-r_+)\right) + 2aQ^2\,. \qquad (5.2.237)$$

From equation (5.2.235), we find two classes of solutions,

$$\delta_a^J = \frac{2a}{r_+-r_-}\,, \qquad C_{2a}^J = \frac{a(r_++r_-)}{r_+-r_-}\,, \qquad (5.2.238)$$

$$\delta_b^J = 0\,, \qquad C_{2b}^J = a\,. \qquad (5.2.239)$$

Substituting (5.2.238) and (5.2.239) into equations (5.2.236) and (5.2.237), we find C_1 and γ that are given by

$$\gamma_a^J = \frac{2Mr_+(\kappa+1) - 2Q^2}{r_+ - r_-}\,, \qquad C_{1a}^J = \frac{2Mr_+(\kappa r_+ + r_-)}{r_+ - r_-} - Q^2\left(\frac{r_+ + r_-}{r_+ - r_-}\right), \tag{5.2.240}$$

$$\gamma_b^J = \frac{2Mr_+(\kappa-1)}{r_+ - r_-}\,, \qquad C_{1b}^J = \frac{2Mr_+(\kappa r_+ - r_-)}{r_+ - r_-} - Q^2\,. \tag{5.2.241}$$

Solving (5.2.233) and (5.2.234) for σ and ρ gives all the conformal generators (5.2.229) and (5.2.230) where all the constants are given in table 5.4.

Table 5.4: Solutions for deformed conformal generators in J picture

	branch a	branch b
δ	$\frac{2a}{r_+-r_-}$	0
γ	$\frac{2Mr_+(\kappa+1)-2Q^2}{r_+-r_-}$	$\frac{2Mr_+(\kappa-1)}{r_+-r_-}$
C_1	$\frac{2Mr_+(\kappa r_++r_-)}{r_+-r_-} - Q^2\left(\frac{r_++r_-}{r_+-r_-}\right)$	$\frac{2Mr_+(\kappa r_+-r_-)}{r_+-r_-} - Q^2$
C_2	$\frac{a(r_++r_-)}{r_+-r_-}$	a
ρ	0	$-\frac{r_+-r_-}{2(\kappa-1)Mr_+}$
σ	$\frac{(r_--r_+)}{2a}$	$\frac{2Mr_+(\kappa r_+-r_--M(\kappa-1))-Q^2(r_+-r_-)}{2aMr_+(\kappa-1)}$

We note that multiplying all the coefficients in table 5.4 with -1 also are solutions to equations (5.2.233), (5.2.234), (5.2.235), (5.2.236) and (5.2.237). However, these solutions correspond to the invariance of the squared Casmir $L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+)$ by renaming the vector fields as

$$L_0 \to -L_0 \quad , \quad L_\pm \to -L\mp\,. \tag{5.2.242}$$

We also note that in the limit of $Q = 0$, the vector fields in the J picture reduce correctly to the generators of deformed conformal symmetry for the Kerr black holes [36].

Furnished by the explicit expressions for the deformed conformal generators in branch a

$$\begin{aligned} L_\pm^a &= e^{\mp 2\pi T_R \phi}\left[\mp\sqrt{\Delta_{KN}}\partial_r - \tfrac{1}{2\pi T_H}\tfrac{r-M}{\sqrt{\Delta_{KN}}}(\Omega_H\partial_\phi + \partial_t) + \tfrac{1}{2\pi\Omega_H(T_L+T_R)}\tfrac{r-r_+}{\sqrt{\Delta_{KN}}}\partial_t\right], \\ L_0^a &= \frac{1}{2\pi T_H}(\Omega_H\partial_\phi + \partial_t) - \frac{1}{2\pi\Omega_H(T_L+T_R)}\partial_t\,, \end{aligned} \tag{5.2.243}$$

and branch b

$$\begin{aligned} L_\pm^b &= e^{\pm 2\pi\Omega(T_L+T_R)t \mp 2\pi T_L\phi}\left[\mp\sqrt{\Delta_{KN}}\partial_r + \tfrac{2Mr_+-Q^2}{\sqrt{\Delta_{KN}}}(\Omega\partial_\phi + \partial_t) + \tfrac{1}{2\pi\Omega_H(T_L+T_R)}\tfrac{r-r_+}{\sqrt{\Delta_{KN}}}\partial_t\right], \\ L_0^b &= -\frac{1}{2\pi\Omega_H(T_L+T_R)}\partial_t\,, \end{aligned} \tag{5.2.244}$$

where T_H and Ω_H are defined in (5.2.215) and (5.2.216) and the left and right moving CFT temperatures are given by

$$T_R = \frac{r_+ - r_-}{4\pi a}, \quad T_L = \frac{T_R(1+\kappa)}{1-\kappa} - \frac{Q^2 T_R}{M r_+ (1-\kappa)}\,. \tag{5.2.245}$$

One can verify that taking the left and right central charges

$$c_R = c_L = \frac{6aMr_+(1-\kappa)}{\sqrt{M^2 - a^2 - Q^2}}\,, \tag{5.2.246}$$

leads to the exact Bekenstein-Hawking entropy for Kerr-Newman black holes (5.2.214), if we use the Cardy formula

$$S_{Cardy} = \frac{\pi^2}{3}(c_R T_R + c_L T_L)\,. \tag{5.2.247}$$

We notice that for the special case of deformation parameter given by $\kappa = r_+/r_-$, we find the generators of hidden conformal symmetry for the Kerr-Newman black holes [34]. In fact, for $\kappa = r_+/r_-$, the deformed generators $L^a_\pm$ and L^a_0 (up to automorphisms (5.2.232)) reduce to conformal generators $H_\pm$ and H_0 in [34] according to

$$L^a_k = -iH_k\,, \tag{5.2.248}$$

where $k = +, -, 0$. The generators in branch b for $\kappa = r_+/r_-$ reduce to the other copy of conformal generators $\bar{H}_k$ in [34] by the mapping

$$L^b_k = i\bar{H}_k\,. \tag{5.2.249}$$

The left and right temperatures (5.2.245) as well as central charge (5.2.246) reduce to the corresponding results in [34] after setting $\kappa = r_-/r_+$. An interesting open question is to derive the deformed central charges by using either ASG or stretched horizon techniques.

5.2.4 Deformed hidden conformal symmetry in Q picture

The deformed radial equation (5.2.225) in the Q picture is

$$\partial_r(\Delta_{KN}\partial_r R(r)) + \left[\frac{[(2Mr_+ - Q^2)\omega - Qr_+e]^2}{(r-r_+)(r_+-r_-)} - \frac{[(2M\kappa r_+ - Q^2)\omega - Q\kappa r_+ e]^2}{(r-r_-)(r_+-r_-)}\right]R(r) = l(l+1)R(r)\,. \tag{5.2.250}$$

Matching the squared Casimir of $sl(2,\mathbb{R})$ algebra to Laplacian in equation (5.2.250) gives the same equations (5.2.233) and (5.2.234) for the coefficients of ∂_r and ∂_r^2 in J picture. However, the other equations are different and their solutions again provide two branches. The solutions are represented in table 5.5.

Table 5.5: Solutions for conformal generators in Q picture

	branch a	branch b
δ	$\frac{Qr_+(1+\kappa)}{r_+-r_-}$	$\frac{Qr_+(\kappa-1)}{r_+-r_-}$
γ	$\frac{2Mr_+(\kappa+1)-2Q^2}{r_+-r_-}$	$\frac{2Mr_+(\kappa-1)}{r_+-r_-}$
C_1	$\frac{2Mr_+(\kappa r_++r_-)}{r_+-r_-} - Q^2\left(\frac{r_++r_-}{r_+-r_-}\right)$	$\frac{2Mr_+(\kappa r_+-r_-)}{r_+-r_-} - Q^2$
C_2	$\frac{Qr_+(\kappa r_++r_-)}{r_+-r_-}$	$\frac{Qr_+(\kappa r_+-r_-)}{r_+-r_-}$
ρ	$\frac{r_+-r_-}{2Q^2}$	$\frac{M(\kappa-1)-\kappa r_++r_-}{Q^2(\kappa-1)}$
σ	$-\frac{M(r_+-r_-)}{Q^3}$	$\frac{(r_+-r_-)(Mr_+(\kappa+1)-Q^2)}{r_+(\kappa-1)Q^3}$

In fact, we note that multiplying all the solutions in table 5.5 by -1 also satisfy the full set of equations. It is similar to what happens in the J picture case, these solutions correspond to invariance of the squared Casimir under renaming (5.2.242). Considering the right and left temperatures to be proportional to σ in branches a and b as

$$T_R = \frac{M(r_+-r_-)}{2\pi Q^3} \quad , \quad T_L = T_R\frac{(1+\kappa)}{(1-\kappa)} - \frac{T_RQ^2}{Mr_+(1-\kappa)}\,, \tag{5.2.251}$$

one can produce the correct Bekenstein-Hawking entropy of the Kerr-Newman black holes using the Cardy formula by the central charges

$$c_L = c_R = \frac{3Q^3r_+(1-\kappa)}{\sqrt{M^2-a^2-Q^2}}\,. \tag{5.2.252}$$

We note that the dependence of T_L in (5.2.251) to T_R has exactly the same form as the left temperature in J picture (5.2.245). We notice that for special value of $\kappa = r_-/r_+$, the mappings (5.2.248) and (5.2.249) show that deformed conformal generators correctly reduce to conformal generators of the Kerr-Newman black hole. The temperatures (5.2.251) and the central charges (5.2.252) reduce to the left and right temperatures and the central charge of CFT dual to Kerr-Newman black hole [34]. We note that in Q picture in which the coefficients of conformal generators are given in table 5.5, the deformed hidden conformal symmetry generators are (5.2.229) and (5.2.230), replacing the coordinate ϕ with the internal coordinate χ.

5.2.5 Deformed hidden conformal symmetry in general picture

As it was mentioned in introduction, the Kerr-Newman black holes have two conformal pictures as ϕ' and χ' pictures. These pictures correspond respectively to two separated $U(1)$ symmetries with respect to coordinates ϕ and χ. The third conformal picture (general picture) can be obtained by using the modular group $SL(2,\mathbb{Z})$ of the torus (ϕ,χ). In this picture, the $SL(2,\mathbb{Z})$ transformation for the torus is given by [24, 23]

$$\begin{pmatrix} \phi' \\ \chi' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \eta & \tau \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{5.2.253}$$

where $\begin{pmatrix} \alpha & \beta \\ \eta & \tau \end{pmatrix}$ is any $SL(2,\mathbb{Z})$ group element. Under transformation (5.2.253), the phase factor of the charged scalar field (5.2.219) with the electric charge e is invariant; $e^{im\phi+ie\chi} = e^{im'\phi'+ie'\chi'}$ which yields

$$m = \alpha m' + \eta e' \quad , \quad e = \beta m' + \tau e' \,. \tag{5.2.254}$$

In ϕ' picture, we set $e' = 0$, hence the deformed radial equation is

$$\begin{aligned} &\partial_r \left(\Delta_{KN} \partial_r R(r)\right) \\ &+ \left(\frac{\left((2Mr_+ - Q^2)\omega - (Qr_+\beta + a\alpha)\, m'\right)^2}{(r-r_+)(r_+-r_-)} - \frac{\left((2M\kappa r_+ - Q^2)\omega - (Q\kappa r_+\beta + a\alpha)\, m'\right)^2}{(r-r_-)(r_+-r_-)} \right) R(r) \\ &= l(l+1) R(r) \,, \end{aligned} \tag{5.2.255}$$

Similar to J and Q pictures, we match the squared Casimir of $SL(2,\mathbb{R})$ to the left hand side of equation (5.2.255) and solve for the coefficients of the vector fields (5.2.229) and (5.2.230). We find there are two classes of solutions for δ, γ, C_1, and C_2 that are given by

$$\delta^G_a = \frac{a_1 + a_2}{r_+ - r_-}\,, \qquad C^G_{2a} = \frac{a_1 r_- + a_2 r_+}{r_+ - r_-}\,, \tag{5.2.256}$$

$$\delta^G_b = \frac{a_2 - a_1}{r_+ - r_-}\,, \qquad C^G_{2b} = \frac{a_2 r_+ - a_1 r_-}{r_+ - r_-}\,, \tag{5.2.257}$$

$$\gamma_a^G = \frac{2Mr_+(\kappa+1) - 2Q^2}{r_+ - r_-}\,, \qquad C_{1a}^G = \frac{2Mr_+(\kappa r_+ + r_-)}{r_+ - r_-} - Q^2\left(\frac{r_+ + r_-}{r_+ - r_-}\right), \tag{5.2.258}$$

$$\gamma_b^G = \frac{2Mr_+(\kappa-1)}{r_+ - r_-}\,, \qquad C_{1b}^G = \frac{2Mr_+(\kappa r_+ - r_-)}{r_+ - r_-} - Q^2\,, \tag{5.2.259}$$

where

$$a_1 = Qr_+\beta + a\alpha \quad , \quad a_2 = Q\kappa r_+\beta + a\alpha\,. \tag{5.2.260}$$

Table 5.6 shows the full set of solutions for branch a and b.

Table 5.6: Solutions for conformal generators in general picture

	branch a	branch b
δ	$\frac{2\alpha a+(\kappa+1)\beta Q r_+}{r_+-r_-}$	$\frac{(\kappa-1)\beta Q r_+}{r_+-r_-}$
γ	$\frac{2Mr_+(\kappa+1)-2Q^2}{r_+-r_-}$	$\frac{2Mr_+(\kappa-1)}{r_+-r_-}$
C_1	$\frac{2Mr_+(\kappa r_++r_-)}{r_+-r_-} - Q^2\left(\frac{r_++r_-}{r_+-r_-}\right)$	$\frac{2Mr_+(\kappa r_+-r_-)}{r_+-r_-} - Q^2$
C_2	$\frac{Q\beta r_+(\kappa r_++r_-)+a\alpha(r_++r_-)}{r_+-r_-}$	$\frac{Q\beta r_+(\kappa r_+-r_-)+a\alpha(r_+-r_-)}{r_+-r_-}$
ρ	$\frac{Q\beta(r_+-r_-)}{2(2Ma\alpha+Q^3\beta)}$	$-\frac{\alpha a(r_+-r_-)-Q\beta r_+(M(\kappa-1)-\kappa r_++r_-)}{(\kappa-1)r_+(2Ma\alpha+Q^3\beta)}$
σ	$-\frac{M(r_+-r_-)}{(2Ma\alpha+Q^3\beta)}$	$\frac{(r_+-r_-)((\kappa+1)Mr_+-Q^2)}{(\kappa-1)r_+(2Ma\alpha+Q^3\beta)}$

In this picture, the left and right CFT temperatures are given by

$$T_R = \frac{M(r_+ - r_-)}{2\pi(2Ma\alpha + Q^3\beta)} \quad , \quad T_L = T_R\frac{(1+\kappa)}{(1-\kappa)} - \frac{T_R Q^2}{Mr_+(1-\kappa)}\,. \tag{5.2.261}$$

The agreement between microscopic CFT entropy and the Hawking-Bekenstein entropy requires that the central charges are

$$c_L = c_R = \frac{3(1-\kappa)r_+(2Ma\alpha + Q\beta)}{\sqrt{M^2 - a^2 - Q^2}}\,. \tag{5.2.262}$$

We note that the right temperature of generalized CFT is independent of deformation parameter κ. However, the left temperature non-trivially depends on the deformation parameter κ. Moreover, we should note that the deformed hidden conformal symmetry generators in general ϕ' picture are given by (5.2.229) and (5.2.230), replacing the coordinate ϕ by ϕ'. The solutions for coefficients (tabulated in table 5.6) reduce to the corresponding coefficients in table 5.4 in J picture where we set $\alpha = 1, \beta = 0$, and reduce to the coefficients in table 5.5 in Q picture where $\alpha = 0, \beta = 1$. As a result the generators in the ϕ' picture reduce to the corresponding generators in J and Q pictures respectively.

5.2.6 Scattering of charged scalars in the Kerr-Newman background based on deformed radial equation

In this section, we consider the absorption cross section of the scalar fields in the background of Kerr-Newman black holes in different pictures.

5.2.7 J picture

We rewrite the deformed equation in J picture (5.2.225) as

$$\partial_r\left(\Delta_{KN}\partial_r R(r)\right)+\left(\frac{(g_+^J)^2(r_+-r_-)}{(r-r_+)}-\frac{(g_-^J)^2(r_+-r_-)}{(r-r_-)}-K_l\right)R(r)=0 \tag{5.2.263}$$

where

$$g_+^J = \frac{(2Mr_+-Q^2)\,\omega-am}{r_+-r_-}, \tag{5.2.264}$$

$$g_-^J = \frac{(2M\kappa r_+-Q^2)\,\omega-am}{r_+-r_-}. \tag{5.2.265}$$

We define the new coordinate [67]

$$p=\frac{r-r_+}{r-r_-}, \tag{5.2.266}$$

and so the deformed equation (5.2.263) becomes

$$p(1-p)\,\partial_p^2 R(p)+(1-p)\,\partial_p R(p)+\left(\frac{g_+^J}{p}-g_-^J-\frac{K_l}{1-p}\right)R(p)=0, \tag{5.2.267}$$

where we used the following identity

$$\Delta_{KN}\partial_r=(r_+-r_-)\,p\partial_p. \tag{5.2.268}$$

The in-going solution for the equation (5.2.267) is

$$R_{in}(r)=Const.p^{-ig_+^J}(p-1)^{-l}\,{}_2F_1\left(-l-i\left(g_+^J-g_-^J\right),-l-i\left(g_+^J+g_-^J\right);1-2ig_+^J;p\right), \tag{5.2.269}$$

where ${}_2F_1$ is the hypergeometric function. The in-going solution (5.2.269) on the outer boundary of the matching region where $r>>M$ behaves as,

$$R_{in}\sim Ar^l, \tag{5.2.270}$$

where $A = {}_2F_1\left(-l - i\left(g_+^J - g_-^J\right), -l - i\left(g_+^J + g_-^J\right); 1 - 2ig_+^J; 1\right)$. We should mention in finding the in-going solution, we consider the low frequency condition, $\omega << 1/M$ in near region, $r << 1/\omega$. Using the Gauss' theorem for hypergeometric functions, we can rewrite the factor A in equation (5.1.99) as

$$A = \frac{\Gamma\left(1 - 2ig_+^J\right)\Gamma(2l+1)}{\Gamma\left(l + 1 - 2i\left(\frac{(Mr_+(1+\kappa) - Q^2)\omega - am}{r_+ - r_-}\right)\right)\Gamma\left(l + 1 - 2i\left(\frac{Mr_+(1-\kappa)\omega}{r_+ - r_-}\right)\right)}. \tag{5.2.271}$$

Hence, we find the absorption cross section, given by

$$P_{abs} \sim |A|^{-2} = \sinh\left(2\pi g_+^J\right)\frac{|\Gamma(l+1-iB_1)|^2\,|\Gamma(l+1-iB_2)|^2}{2\pi g_+^J\left(\Gamma(2l+1)\right)^2}, \tag{5.2.272}$$

where

$$B_1 = 2\left(\frac{(Mr_+(1+\kappa) - Q^2)\omega - am}{r_+ - r_-}\right), \tag{5.2.273}$$

$$B_2 = 2\left(\frac{Mr_+(1-\kappa)\omega}{r_+ - r_-}\right). \tag{5.2.274}$$

In supporting the Kerr/CFT duality in this scattering process, we need to associate the absorption cross section (5.2.272) with the results from 2D CFT. In other words, we want to match the absorption cross section (5.2.272) computed from gravitational side to the corresponding cross section in the dual 2D CFT in J picture,

$$P_{abs} \sim T_L^{J\,2h_L-1} T_R^{J\,2h_R-1} \sinh\left(\frac{\omega_L^J}{2T_L^J} + \frac{\omega_R^J}{2T_R^J}\right)\left|\Gamma\left(h_L + i\frac{\omega_L^J}{2\pi T_L^J}\right)\right|^2\left|\Gamma\left(h_R + i\frac{\omega_R^J}{2\pi T_R^J}\right)\right|^2 \tag{5.2.275}$$

which is known as the finite temperature absorption cross section in a 2D CFT [67]. To match and find the possible agreement between (5.2.275) and (5.2.272), we consider the first law of thermodynamics for the charged rotating black holes

$$T_H \delta S_{BH} = \delta M - \Omega_H \delta J - \Phi_H \delta Q. \tag{5.2.276}$$

where T_H,Ω_H and Φ_H are given by (5.2.215), (5.2.216) and (5.2.217). For a 2D CFT with the Cardy entropy [96]

$$S_{CFT} = 2\pi\left(\sqrt{\frac{c_L E_L}{6}} + \sqrt{\frac{c_R E_R}{6}}\right), \tag{5.2.277}$$

the variation of entropy can be read as

$$\delta S_{CFT} = \frac{\delta E_L}{T_L} + \frac{\delta E_R}{T_R}\,. \tag{5.2.278}$$

Matching the variations of entropy (5.2.276) and CFT entropy (5.2.278) gives

$$\frac{\delta M - \Omega_H \delta J - \Phi_H \delta Q}{T_H} = \frac{\delta E_L^J}{T_L^J} + \frac{\delta E_R^J}{T_R^J}\,. \tag{5.2.279}$$

In the last equation and also in (5.2.275), the superscripts J show the corresponding quantities in the J picture. We can identify δM as ω, δJ as m, δQ as e, $\delta E_{R,L}^J = \omega_{R,L}^J$ in (5.2.279). Therefore a set of left and right frequencies that satisfy the equation (5.2.279) are

$$\omega_L^J = \frac{\omega\left(Mr_+ (\kappa+1) - Q^2\right)}{a} \quad , \quad \omega_R^J = \omega_L^J - m\,. \tag{5.2.280}$$

For $\kappa = r_-/r_+$, these left and right frequencies definitely reduce to the left and right frequencies in J picture for the Kerr-Newman black hole [34]. The fact which supports the existence of dual 2D CFT for the deformed Kerr-Newman/CFT correspondence is the agreement between (5.2.275) with (5.2.272) if the $\omega_{L,R}^J$ are as in (5.2.280). In the formula (5.2.275), the left and right conformal weights $h_{L,R}$ are equal to $l+1$. We notice that these conformal weights are the same in the other Q and general pictures that we discuss in next two subsections.

5.2.8 Q picture

In Q picture, the charged test particle is co-rotating with the black hole horizon, thus we can turn off the rotational parameter a. The absorption cross section and the deformed radial equation are given by (5.2.272) and (5.2.263) with replacing g^J to g^Q where

$$g_+^Q = \frac{(2Mr_+ - Q^2)\,\omega - Qr_+ e}{r_+ - r_-}\,, \tag{5.2.281}$$

$$g_-^Q = \frac{(2M\kappa r_+ - Q^2)\,\omega - Q\kappa r_+ e}{r_+ - r_-}\,. \tag{5.2.282}$$

Thus we find the corresponding absorption cross section, given by

$$P_{abs} \sim |A|^{-2} = \sinh\left(2\pi g_+^J\right) \frac{\left|\Gamma\left(l+1-iB_1^Q\right)\right|^2 \left|\Gamma\left(l+1-iB_2^Q\right)\right|^2}{2\pi g_+^J \left(\Gamma\left(2l+1\right)\right)^2}\,, \tag{5.2.283}$$

where

$$B_1^Q = \left(\frac{(2Mr_+ (1+\kappa) - 2Q^2)\,\omega - Q(1+\kappa) r_+ e}{r_+ - r_-} \right) , \tag{5.2.284}$$

$$B_2^Q = \left(\frac{2Mr_+ (1-\kappa)\,\omega - Q(1-\kappa) r_+ e}{r_+ - r_-} \right) . \tag{5.2.285}$$

In this picture, to match and find the possible agreement between the cross section (5.2.283) and the finite temperature absorption cross section of CFT, we again consider the first law of thermodynamics for the charged rotating black holes. The matching of microscopic and macroscopic entropy variations in Q picture now can be read as

$$\frac{\delta M - \Omega_H \delta J - \Phi_H \delta Q}{T_H} = \frac{\delta E_L^Q}{T_L^Q} + \frac{\delta E_R^Q}{T_R^Q} . \tag{5.2.286}$$

We identify $\delta E_{L,R}$ with the left and right frequencies $\tilde{\omega}_{L,R}^Q$ that are related to three quantities; the left and right frequencies $\omega_{L,R}^Q$, charges $q_{L,R}$, and chemical potentials $\mu_{L,R}$ where

$$\omega_R^Q = \omega_L^Q = \frac{2M\omega}{Q^3} \left(Mr_+ (1+\kappa) - Q^2 \right) , \tag{5.2.287}$$

$$\mu_R = \mu_L + 1 = \frac{Mr_+ (1+\kappa)}{Q^2} . \tag{5.2.288}$$

The frequencies $\tilde{\omega}_{L,R}$ are given by

$$\tilde{\omega}_{L,R}^Q = \omega_{L,R}^Q - q_{L,R}\mu_{L,R} , \tag{5.2.289}$$

where the charges $q_{L,R} = e$. Substituting equations (5.2.287), (5.2.288) and (5.2.289) into the 2D CFT absorption cross section

$$P_{abs} \sim T_L^{Q^{2h_L - 1}} T_R^{Q^{2h_R - 1}} \sinh \left(\frac{\tilde{\omega}_L^Q}{2T_L} + \frac{\tilde{\omega}_R^Q}{2T_R^Q} \right) \left| \Gamma \left(h_L + i \frac{\tilde{\omega}_L^Q}{2\pi T_L^Q} \right) \right|^2 \left| \Gamma \left(h_R + i \frac{\tilde{\omega}_R^Q}{2\pi T_R^Q} \right) \right|^2 , \tag{5.2.290}$$

shows that the CFT_2 cross section agrees with the absorption cross section which is derived from gravitational point of view in Q picture (5.2.283). Also as it is expected, when $\kappa = r_-/r_+$, the left and right frequencies for the deformed CFT_2 as well as the chemical potential reduce to the corresponding quantities in [34].

5.2.9 General picture

The $SL(2,\mathbb{Z})$ transformation between (ϕ,χ) and (ϕ',χ') yields the relations (5.2.254). The ϕ' picture under consideration is given by setting $e'=0$. The deformed radial equation (5.2.255) in ϕ' picture can be rewritten as,

$$\partial_r\left(\Delta_{KN}\partial_r R(r)\right)+\left(\frac{(g_+^G)^2(r_+-r_-)}{(r-r_+)}-\frac{(g_-^G)^2(r_+-r_-)}{(r-r_-)}-K_l\right)R(r)=0\,, \tag{5.2.291}$$

where g_+^G and g_-^G are

$$g_+^G = \frac{(2Mr_+-Q^2)\,\omega-(Qr_+\beta+a\alpha)m'}{r_+-r_-}\,, \tag{5.2.292}$$

$$g_-^G = \frac{(2M\kappa r_+-Q^2)\,\omega-(Q\kappa r_+\beta+a\alpha)m'}{r_+-r_-}\,. \tag{5.2.293}$$

The absorption cross section is given by

$$P_{abs}\sim|A|^{-2}=\sinh\left(2\pi g_+^G\right)\frac{\left|\Gamma\left(l+1-iB_1^G\right)\right|^2\left|\Gamma\left(l+1-iB_2^G\right)\right|^2}{2\pi g_+^G\left(\Gamma(2l+1)\right)^2}\,, \tag{5.2.294}$$

where

$$B_1^G = \left(\frac{(2Mr_+(1+\kappa)-2Q^2)\,\omega-(Q(1+\kappa)r_+\beta+2a\alpha)m'}{r_+-r_-}\right)\,, \tag{5.2.295}$$

$$B_2^G = \left(\frac{2Mr_+(1-\kappa)\,\omega-Q(1-\kappa)r_+\beta m'}{r_+-r_-}\right)\,. \tag{5.2.296}$$

Matching the macroscopic and microscopic entropy requires that we introduce the generalized frequencies $\tilde{\omega}_{L,R}^G$ in terms of three quantities; frequencies $\omega_{L,R}^G$, charges $q_{L,R}^G$, and chemical potentials $\mu_{L,R}^G$,

$$\tilde{\omega}_{L,R}^G=\omega_{L,R}^G-q_{L,R}^G\mu_{L,R}^G\,. \tag{5.2.297}$$

In (5.2.297),

$$\omega_{L,R}^G=\frac{2M\omega\left(Mr_+(1+\kappa)-Q^2\right)}{2\alpha Ma+\beta Q^3}\,, \tag{5.2.298}$$

$$\mu_R^G=\frac{M\left(2\alpha a+\beta Qr_+(1+\kappa)\right)}{2\alpha Ma+\beta Q^3}\quad,\quad \mu_L^G=\frac{\beta Q\left(Mr_+(1+\kappa)-Q^2\right)}{2\alpha Ma+\beta Q^3}\,, \tag{5.2.299}$$

and $q_{L,R}^G=m'$. Substituting the relevant quantities in the CFT absorption cross section

$$P_{abs}\sim T_L^{G^{2h_L-1}}T_R^{G^{2h_R-1}}\sinh\left(\frac{\tilde{\omega}_L^G}{2T_L}+\frac{\tilde{\omega}_R^G}{2T_R^G}\right)\left|\Gamma\left(h_L+i\frac{\tilde{\omega}_L^G}{2\pi T_L^G}\right)\right|^2\left|\Gamma\left(h_R+i\frac{\tilde{\omega}_R^G}{2\pi T_R^G}\right)\right|^2 \tag{5.2.300}$$

shows exact agreement between the CFT absorption cross section (5.2.300) and the corresponding cross section from gravitational side (5.2.294).

5.2.10 The (deformed) hidden conformal symmetries of Kerr and Reissner-Nordstrom

We notice that the deformed conformal generators $L_n^{a,b}$, $n = +, -, 0$ in (5.2.243) and (5.2.244) reduce to the deformed conformal generators L_n, $n = +, -, 0$ for the Kerr black holes when we set $Q = 0$ [36]. Furthermore, setting the rotation parameter $a = 0$ with special value of deformation parameter κ, we find the conformal generators of Schwarzschild black holes in agreement with [36] and [89].

Plugging the results for $C_1, C_2, \delta, \gamma, \rho$ and σ for branch a from table 5.5 in Q picture along with $a = 0$, we find the deformed hidden conformal generators for Reissner-Nordstrom black holes as

$$\begin{aligned} L_{\pm}^a &= e^{\pm\left(\frac{\pi Q T_R}{M}\right)t \mp (2\pi T_R)\chi}\left(+\frac{r_+}{Q^2 2\pi T_R}\left(\kappa r_+ - r_- - r\left(1+\kappa\right)\right)\partial_\chi \right. \\ &\quad \left. \mp\sqrt{\Delta_{RN}}\partial_r + \left(\frac{M}{2\pi r_+}\left(\kappa r_+ + r_- - r\left(1+\kappa\right)\right) + \frac{Q^2}{4\pi r_+^2}\left(2r - r_+ - r_-\right)\right)\partial_t\right), \end{aligned} \tag{5.2.301}$$

$$L_0^a = \frac{2M}{2\pi Q^3 T_R}\left(M r_+\left(\kappa+1\right) - Q^2\right)\partial_t + \frac{M r_+\left(\kappa+1\right)}{2\pi Q^2 T_R}\partial_\chi \,, \tag{5.2.302}$$

where $\Delta_{RN} = r^2 - 2Mr + Q^2$. The right temperature T_R is given by

$$T_R = \frac{\left(r_+ - r_-\right)M}{2\pi Q^3} \,. \tag{5.2.303}$$

Solving $\Delta_{RN} = 0$ gives the outer and inner Reissner-Nordstrom black holes horizons are $r_+ = M + \sqrt{M^2 - Q^2}$ and $r_- = M - \sqrt{M^2 - Q^2}$ respectively. This right temperature T_R in (5.2.303) matches the right temperature in [97] after considering the unit length for the uplifted extra dimension.

The second copy of deformed hidden conformal symmetry generators for the Reissner-Nordstrom black hole can be obtained from the branch b generators for the Kerr-Newman black hole in appropriate limit of $a = 0$. We notice that $\sigma = -2\pi T_L$ and so the deformed hidden conformal generators for the Reissner-Nordstrom read as,

$$\begin{aligned} L_{\pm}^b &= e^{\pm\left(\frac{(1+\kappa)\pi Q T_R}{(1-\kappa)M}\right)t \mp (2\pi T_L)\chi}\left(+\frac{M r_+}{Q^2 2\pi T_R}\left(\kappa r_+ - r_- - r\left(\kappa-1\right)\right)\partial_\chi \right. \\ &\quad \left. \mp\sqrt{\Delta_{RN}}\partial_r + \left(\frac{M}{2\pi r_+}\left(\kappa r_+ - r_- + r\left(1-\kappa\right)\right) - \frac{Q^2}{4\pi r_+^2}\left(r_+ - r_-\right)\right)\partial_t\right), \end{aligned} \tag{5.2.304}$$

$$L_0^b = \frac{M^2 r_+\left(\kappa-1\right)}{\pi Q^3 T_R}\partial_t + \frac{M r_+\left(\kappa-1\right)}{2\pi Q^2 T_R}\partial_\chi \,. \tag{5.2.305}$$

We also note that for the special value of deformation parameter $\kappa = r_-/r_+$, the previously obtained generators (5.2.301), (5.2.302), (5.2.304) and (5.2.305) reduce to the hidden conformal generators for the Reissner-Nordstrom black hole [97] after setting the unit length for the uplifted extra dimension.

Nevertheless, we notice that the hidden conformal generators (5.2.301), (5.2.302), (5.2.304) and (5.2.305) with $\kappa = r_+/r_-$ for the Reissner-Nordstrom black holes do not simply reduce to the conformal generators for Schwarzschild black holes [89] by setting $Q = 0$. In this limit, as it is clear from table 5.5 , the coefficients ρ and σ do not have any finite values, though the other four coefficients are well-defined. The situation is similar in reduction of hidden conformal generators of Kerr-Sen black holes to hidden conformal generators of Gibbons-Maeda-Garfinkle-Horowitz-Strominger black holes [90] or reduction of conformal generators for Kerr black holes [28] to Schwarzschild black holes by setting $a = 0$. To overcome this problem, as it was noticed in [90, 36], we set $\sigma = 0$ for the neutral black holes and so the equations (5.2.233) and (5.2.234) become

$$\rho C_1 + M = 0 \quad , \quad 1 + \rho\gamma = 0 \,. \tag{5.2.306}$$

We note that ρ, C_1, and γ contain the free deformation parameter κ. In branch b, we choose $\kappa = -1$ and our deformed conformal generators reduce exactly to those derived in [89] by the mapping

$$L_0^b = -iH_0 \quad , \quad L_\pm^b = iH_\pm \,. \tag{5.2.307}$$

Chapter 6

Vector Fields in Kerr/CFT

The materials in this chapter are based on our paper [112]. In this chapter, the discussions are based on the wave equation for spin-1 objects in Kerr background given in [46] together with their solutions by Chandrasekhar [47]. After reviewing the equations and solutions of vector fields in Kerr spacetime, we will show the appropriate boundary action for the Maxwell fields in this spacetime. From this boundary action, we derive the two point function for vector fields by borrowing the AdS/CFT prescription (3.4.144). We will show that the non gauge dependent part of the two point function computed in gravitational theory and CFT sides agree each other, which support the Kerr/CFT correspondence proposal.

6.1 Spin-1 fields in the background of Kerr black holes

6.1.1 Construction of solutions in Newman-Penrose formalism

In this section, we briefly review the derivation of solutions to Maxwell equations in the background of Kerr black hole [47] and fix the notation in the article [1]. In Boyer-Lindquist coordinate, the Kerr metric read as

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(adt - \left(r^2+a^2\right)d\phi\right)^2 , \qquad (6.1.1)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$. For later convenience, the corresponding contravariant components of the metric tensor for (6.1.1) are given by

$$g^{rr} = \frac{\Delta}{\rho^2}\,, \quad g^{\theta\theta} = \frac{1}{\rho^2}\,, \quad g^{tt} = \frac{\left(\Delta a^2\sin^2\theta - (r^2+a^2)^2\right)}{\Delta\rho^2}\,, \qquad (6.1.2)$$

[1] There is a slight difference on some notations in constructing the solutions to Maxwell's equations in the background of Kerr black hole in literature such as [98] and [47]. In this chapter, we mainly follow [47].

$$g^{t\phi} = \frac{-2Mra}{\Delta\rho^2}\,, \quad g^{\phi\phi} = \frac{\left(\Delta - a^2\sin^2\theta\right)}{\Delta\rho^2\sin^2\theta}\,. \tag{6.1.3}$$

Stationary and axisymmetric properties of the Kerr black hole suggest that the solution to Maxwell equations in this spacetime can be written as a superposition of waves with different frequencies ω and different periods $2m\pi$, $m = 0, 1, 2, ...$ for coordinate ϕ. Thus, the existence of Killing vectors ∂_t and ∂_ϕ for Kerr spacetime (6.1.1) enable us to write down the dependence of spin-1 field solutions to t and ϕ coordinates as $e^{-i\omega t+im\phi}$.

In his seminal work [46], Teukolsky showed that the equations of motions for the fields (with different spin weights) in Kerr background are separable in radial and angular directions. In Newman-Penrose (NP) formalism, the real null-vectors l^μ and n^μ and the complex null-vector m^μ for Kerr spacetime (6.1.1) are given by [47]

$$l^\mu = \Delta^{-1}\left(r^2 + a^2, \Delta, 0, a\right)\,, \tag{6.1.4}$$

$$n^\mu = \frac{1}{2\rho^2}\left(r^2 + a^2, -\Delta, 0, a\right)\,, \tag{6.1.5}$$

$$m^\mu = \frac{1}{\bar{\rho}\sqrt{2}}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right)\,, \tag{6.1.6}$$

in (t, r, θ, ϕ) coordinate system where $\bar{\rho} = r + ia\cos\theta$ and $\bar{\rho}* = r - ia\cos\theta$. Contracting the vectors l^μ, n^μ and m^μ by ∂_μ, we get the following differential operators

$$l = \mathcal{D}_0 \quad , \quad n = -\frac{\Delta}{2\rho^2}\mathcal{D}_0^\dagger\,, \tag{6.1.7}$$

$$m = \frac{1}{\bar{\rho}\sqrt{2}}\mathcal{L}_0^\dagger\,. \tag{6.1.8}$$

We also consider the operator

$$\bar{m} = \frac{1}{\bar{\rho}^*\sqrt{2}}\mathcal{L}_0\,. \tag{6.1.9}$$

The differential operators (6.1.7), (6.1.8) and (6.1.9) act on any function that its dependence on coordinates t and ϕ is given by $e^{-i\omega t+im\phi}$. The operators $\mathcal{D}_0$, $\mathcal{D}_0^\dagger$, $\mathcal{L}_0$ and $\mathcal{L}_0^\dagger$ are special cases of

$$\mathcal{D}_n = \frac{\partial}{\partial r} + \frac{iK}{\Delta} + 2n\left(\frac{r-M}{\Delta}\right) \quad , \quad \mathcal{D}_n^\dagger = \frac{\partial}{\partial r} - \frac{iK}{\Delta} + 2n\left(\frac{r-M}{\Delta}\right)\,, \tag{6.1.10}$$

$$\mathcal{L}_n = \frac{\partial}{\partial\theta} + Q + n\cot\theta \quad , \quad \mathcal{L}_n^\dagger = \frac{\partial}{\partial\theta} - Q + n\cot\theta\,, \tag{6.1.11}$$

where K and Q are given by

$$K = -\left(r^2 + a^2\right)\omega + am\,, \tag{6.1.12}$$

and

$$Q = -a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\,, \tag{6.1.13}$$

and $n \in \mathbb{Z}$. As we notice, the operators $\mathcal{D}_n$ and $\mathcal{D}_n^\dagger$ are purely radial dependent operators, whereas $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ are purely angular dependent operators.

Contracting the field-strength tensor $F_{\mu\nu}$ with the basis vectors (6.1.4) - (6.1.6) yield three complex scalars Φ_0, Φ_1 and Φ_2 which can be read as

$$\Phi_0 = F_{\mu\nu}l^\mu m^\nu\,, \tag{6.1.14}$$

$$\Phi_1 = \frac{\bar{\rho}^*}{\sqrt{2}}F_{\mu\nu}\left(l^\mu n^\nu + \bar{m}^\mu m^\nu\right)\,, \tag{6.1.15}$$

and

$$\Phi_2 = 2\left(\bar{\rho}^*\right)^2 F_{\mu\nu}\bar{m}^\mu n^\nu\,. \tag{6.1.16}$$

The Maxwell's equations in the background (6.1.1) are given by

$$g^{\mu\nu}\nabla_\nu F_{\mu\rho} = 0\,, \tag{6.1.17}$$

along with the Bianchi identity

$$\nabla_\mu F_{\nu\rho} + \nabla_\rho F_{\mu\nu} + \nabla_\nu F_{\rho\mu} = 0\,. \tag{6.1.18}$$

Equation (6.1.18) indicates that there is no source for Maxwell fields in the gravitational background (6.1.1). Inserting all the spin coefficients and directional derivatives into Maxwell's equations gives a set of four equations in NP formalism

$$\left(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_0 = \left(\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\right)\Phi_1\,, \tag{6.1.19}$$

$$\left(\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_1 = \left(\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right)\Phi_2\,, \tag{6.1.20}$$

$$\left(\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_2 = -\Delta\left(\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}^*}\right)\Phi_1\,, \tag{6.1.21}$$

and

$$\left(\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_1 = -\Delta\left(\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}^*}\right)\Phi_0\,. \tag{6.1.22}$$

The equations (6.1.19) - (5.1.65) can be decoupled to two differential equations for Φ_0 and Φ_2 by noticing that two operators

$$\begin{aligned} Y_m &= \mathcal{D} + m\left(\bar{\rho}^*\right)^{-1}\,, & (6.1.23)\\ Z_m &= \mathcal{L} + ima\sin\theta\left(\bar{\rho}^*\right)^{-1}\,, & (6.1.24)\end{aligned}$$

commute, i.e. $[Y_m, Z_n] = 0$. In (6.1.23) and (6.1.24), $\mathcal{D}$ can be either $\mathcal{D}_n$ or $\mathcal{D}_n^\dagger$ and $\mathcal{L}$ can be either $\mathcal{L}_n$ or $\mathcal{L}_n^\dagger$ respectively. The two decoupled differential equations for Φ_0 and Φ_2 are

$$\left[\left(\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\left(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right) + \Delta\left(\mathcal{D}_1 + \frac{1}{\bar{\rho}^*}\right)\left(\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}^*}\right)\right]\Phi_0 = 0\,, \tag{6.1.25}$$

and

$$\left[\left(\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\left(\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}^*}\right) + \Delta\left(\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\right)\left(\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right)\right]\Phi_2 = 0\,. \tag{6.1.26}$$

We notice that to obtain equation (6.1.25), we have used the identity $\mathcal{D}_0\Delta = \Delta\mathcal{D}_1$. Using the identities,

$$\begin{aligned} \Delta\left(\mathcal{D}_1 + \frac{1}{\bar{\rho}^*}\right)\left(\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}^*}\right) &= \Delta\mathcal{D}_1\mathcal{D}_1^\dagger - \frac{2iK}{\bar{\rho}^*}\,, & (6.1.27)\\ \left(\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\left(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right) &= \mathcal{L}_0^\dagger\mathcal{L}_1 + \frac{2iQa\sin\theta}{\bar{\rho}^*}\,, & (6.1.28)\end{aligned}$$

and

$$\begin{aligned} \Delta\left(\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}^*}\right)\left(\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right) &= \Delta\mathcal{D}_0^\dagger\mathcal{D}_0 + \frac{2iK}{\bar{\rho}^*}\,, & (6.1.29)\\ \left(\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\left(\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}^*}\right) &= \mathcal{L}_0\mathcal{L}_1^\dagger - \frac{2iQa\sin\theta}{\bar{\rho}^*}\,, & (6.1.30)\end{aligned}$$

where K and Q are given by (6.1.12) and (6.1.13) respectively, we can simplify equations (6.1.25) and (6.1.26) to

$$\left(\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + \mathcal{L}_0^\dagger\mathcal{L}_1 + 2i\omega\left(r + ia\cos\theta\right)\right)\Phi_0 = 0\,, \tag{6.1.31}$$

and

$$\left(\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 + \mathcal{L}_0\mathcal{L}_1^\dagger - 2i\omega\left(r + ia\cos\theta\right)\right)\Phi_2 = 0\,. \tag{6.1.32}$$

The equations (6.1.31) and (6.1.32) are clearly separable in r and θ and called the Teukolsky equations for the massless particles with spin weight 1. For convenience, we set

$$\Phi_0 = \Psi_+, \qquad \Phi_2 = \Psi_-\,, \tag{6.1.33}$$

where $\Psi_\pm \equiv S_\pm(\theta)\,R_\pm(r)$ and $R_\pm(r)$ and $S_\pm(\theta)$ are functions of r and θ only, respectively. The functions $\Psi_\pm$ contain the r and θ dependence of Teukolsky wave functions $\tilde{\Psi}_\pm$ for Maxwell field perturbation with spin weights ± 1

$$\tilde{\Psi}_\pm = e^{-i\omega t + im\phi}R_\pm(r)\,S_\pm(\theta) \equiv e^{-i\omega t + im\phi}\Psi_\pm\,. \tag{6.1.34}$$

Plugging (6.1.33) into equations (6.1.31) and (6.1.32) we obtain a set of equations

$$\left(\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + 2i\omega r\right)R_+ = \lambda R_+\,, \tag{6.1.35}$$

$$\left(\mathcal{L}_0^\dagger\mathcal{L}_1 - 2a\omega\cos\theta\right)S_+ = -\lambda S_+\,, \tag{6.1.36}$$

and

$$\left(\Delta\mathcal{D}_0\mathcal{D}_0^\dagger - 2i\omega r\right)R_- = \lambda R_-\,, \tag{6.1.37}$$

$$\left(\mathcal{L}_0\mathcal{L}_1^\dagger + 2a\omega\cos\theta\right)S_- = -\lambda S_-\,, \tag{6.1.38}$$

for the radial $R_\pm$ and angular $S_\pm$ functions where λ is the separation constant.

The radial solutions to Teukolsky equations have been found in reference [99]. The radial solutions also have been obtained for near horizon near extremal Kerr in reference [37] by taking near and far region limits of a generic Teukolsky equation [46]. Applying the operator $(\mathcal{L}_0 + ia\bar{\rho}^{*-1}\sin\theta)$ to (6.1.19) and $(\mathcal{D}_0 + \bar{\rho}^{*-1})$ to (5.1.63) and adding them up, we find

$$\left(\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\right)\left(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right)\Phi_0 = \left(\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\right)\left(\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right)\Phi_2\,. \tag{6.1.39}$$

Furthermore equation (6.1.39) simplifies to

$$\mathcal{L}_0\mathcal{L}_1\Phi_0 = \mathcal{D}_0^2\Phi_2\,. \tag{6.1.40}$$

As we notice from (6.1.33) and (6.1.34), the complex scalars Φ_0 and Φ_2 are separable functions in terms of coordinates r and θ. Plugging the identifications (6.1.33) into equation (6.1.40), we get the equation

$$\frac{\mathcal{L}_0\mathcal{L}_1 S_+}{S_-} = \frac{\Delta\mathcal{D}_0^2 R_-}{\Delta R_+}\,, \tag{6.1.41}$$

that leads to

$$\mathcal{D}_0^2 R_- = C R_+\,, \tag{6.1.42}$$

which is one of the Teukolsky - Starobinsky identities [47].

The proof of this identity can be given in the following way. We apply the operator $\mathcal{D}_0^2$ to the left hand side of equation (6.1.37),

$$\begin{aligned}
\mathcal{D}_0^2\left(\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 R_-\right) - 2i\omega\mathcal{D}_0^2\left(rR_-\right) &= \mathcal{D}_0^2\left(\Delta\mathcal{D}_0 - 2iK\right)\mathcal{D}_0 R_- - 2i\omega r\mathcal{D}_0^2 R_- - 4i\omega\mathcal{D}_0 R_- \\
&= \mathcal{D}_0\Delta\mathcal{D}_1\mathcal{D}_0^2 R_- - 2i\mathcal{D}_0\left(K\mathcal{D}_0^2 R_- - 2\omega r\mathcal{D}_0 R_-\right) \\
&\quad -2i\omega r\mathcal{D}_0^2 R_- - 4i\omega\mathcal{D}_0 R_- \\
&= \mathcal{D}_0\left(\Delta\mathcal{D}_1^\dagger + 2iK\right)\mathcal{D}_0^2 R_- - 2i\mathcal{D}_0\left(K\mathcal{D}_0^2 R_-\right) + 2i\omega r\mathcal{D}_0^2 R_- \\
&= \left(\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + 2i\omega r\right)\mathcal{D}_0^2 R_-\,.
\end{aligned} \tag{6.1.43}$$

In the first line of equation above, we have used the relation between $\mathcal{D}_0$ and $\mathcal{D}_0^\dagger$, derived from the definitions of both operators in (6.1.10),

$$\Delta\mathcal{D}_n^\dagger = \Delta\mathcal{D}_n - 2iK\,, \tag{6.1.44}$$

and

$$\Delta\mathcal{D}_{n+1} = \mathcal{D}_n\Delta\,. \tag{6.1.45}$$

The same operator $\mathcal{D}_0^2$ must be applied also to the right hand side of equation (6.1.37), which finally gives us an equation

$$\left(\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + 2i\omega r\right)\mathcal{D}_0^2 R_- = \lambda\mathcal{D}_0^2 R_-\,. \tag{6.1.46}$$

Interestingly, from the equation (6.1.35), we observe that $\mathcal{D}_0^2 R_-$ obeys the same eigen equation as R_+. This allows us to write equation (6.1.42), i.e. $\mathcal{D}_0^2 R_-$ is a constant multiple of R_+. By performing this analysis, we have obtained an identity,

$$\mathcal{D}_0^2 \left(\Delta \mathcal{D}_0^\dagger \mathcal{D}_0 - 2i\omega r\right) = \left(\Delta \mathcal{D}_1 \mathcal{D}_1^\dagger + 2i\omega r\right) \mathcal{D}_0^2 \,. \tag{6.1.47}$$

It is interesting to note that ΔR_+ and R_- are a complex conjugate pair. This can be seen by adding a Δ on both sides of equation (6.1.35) and apply the identity (6.1.45), where we can get

$$\left(\Delta \mathcal{D}_0 \mathcal{D}_0^\dagger + 2i\omega r\right) \Delta R_+ = \lambda \Delta R_+ \,. \tag{6.1.48}$$

Hence, by using this fact, the complex conjugate version of (6.1.42) can be shown. This can be pursued by applying the operator $\Delta \left(\mathcal{D}_0^\dagger\right)^2$ to the both side of equation (6.1.48), i.e.

$$\begin{aligned}
\lambda \Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ &= \Delta \left(\mathcal{D}_0^\dagger\right)^2 \left(\Delta \mathcal{D}_0 \mathcal{D}_0^\dagger \Delta R_+\right) + 2i\omega \Delta \left(\mathcal{D}_0^\dagger\right)^2 (r \Delta R_+) \\
&= \Delta \left(\mathcal{D}_0^\dagger\right)^2 \left(\Delta \mathcal{D}_0^\dagger + 2iK\right) \mathcal{D}_0^\dagger \Delta R_+ + 2i\omega r \Delta \left(\mathcal{D}_0^\dagger\right)^2 R_- + 4i\omega \Delta \mathcal{D}_0^\dagger \Delta R_+ \\
&= \Delta \mathcal{D}_0^\dagger \Delta \mathcal{D}_1^\dagger \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ + 2i \Delta \mathcal{D}_0^\dagger \left(K \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ - 2\omega r \mathcal{D}_0^\dagger \Delta R_+\right) \\
&\quad + 2i\omega r \Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ + 4i\omega \Delta \mathcal{D}_0^\dagger \Delta R_+ \\
&= \Delta \mathcal{D}_0^\dagger \left(\Delta \mathcal{D}_1 - 2iK\right) \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ \\
&\quad + 2i \Delta \mathcal{D}_0^\dagger \left(K \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+\right) - 2i\omega r \Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ \\
&= \Delta \left(\Delta \mathcal{D}_1^\dagger \mathcal{D}_1 - 2i\omega r\right) \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ \\
&= \left(\mathcal{D}_0^\dagger \mathcal{D}_0 - 2i\omega r\right) \Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ \,.
\end{aligned} \tag{6.1.49}$$

From the last computation, we find that $\Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+$ satisfies the same eigen equation as R_-, hence we can say that $\Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+$ is a constant multiple of R_-, i.e.

$$\Delta \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ = C^* R_- \,. \tag{6.1.50}$$

From equation (6.1.49), it is easy to show that

$$\lambda \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ = \left(\Delta \mathcal{D}_1^\dagger \mathcal{D}_1 - 2i\omega r\right) \left(\mathcal{D}_0^\dagger\right)^2 \Delta R_+ \,, \tag{6.1.51}$$

from which, by matching this equation to (6.1.48), we can have an identity

$$\left(\mathcal{D}_0^\dagger\right)^2\left(\Delta\mathcal{D}_0\mathcal{D}_0^\dagger+2i\omega r\right)=\left(\Delta\mathcal{D}_1^\dagger\mathcal{D}_1-2i\omega r\right)\left(\mathcal{D}_0^\dagger\right)^2 . \tag{6.1.52}$$

It turns out that the last identity is just the complex conjugate of (6.1.47).

In (6.1.42), C is the Starobinsky constant which in general can be complex valued,

$$|C|^2=\lambda^2-4\alpha^2\omega^2, \tag{6.1.53}$$

and α is defined as

$$\alpha^2=a^2-\frac{am}{\omega} . \tag{6.1.54}$$

The relation (6.1.53) can be proven by using the results (6.1.42) and (6.1.50). First let us apply the operator $\Delta(\mathcal{D}_0^\dagger)^2$ to the equation (6.1.42),

$$\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta\mathcal{D}_0^2R_-=C\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta R_+ . \tag{6.1.55}$$

Replacing the term $\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta R_+$ in the last equation by using the relation (6.1.50) we can have

$$\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta\mathcal{D}_0^2R_-=|C|^2R_- . \tag{6.1.56}$$

Therefore, related to R_-, we have a new identity

$$\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta\mathcal{D}_0^2=|C|^2 , \tag{6.1.57}$$

as long as the following equation is satisfied

$$\Delta\mathcal{D}_0^\dagger\mathcal{D}_0=2i\omega r+\lambda . \tag{6.1.58}$$

In fact, one can verify that

$$\begin{aligned}\mathcal{D}_0^\dagger\Delta\mathcal{D}_0 &= \left(\partial_r-\frac{iK}{\Delta}\right)\Delta\left(\partial_r-\frac{iK}{\Delta}\right)\\ &= \left(\partial_r+\frac{iK}{\Delta}\right)\Delta\left(\partial_r-\frac{iK}{\Delta}\right)-2iK\left(\partial_r+\frac{iK}{\Delta}\right)+\left(\partial_r+\frac{iK}{\Delta}\right)2iK\\ &= \mathcal{D}_0\Delta\mathcal{D}_0^\dagger-2iK\partial_r+2i\partial_rK\\ &= \mathcal{D}_0\Delta\mathcal{D}_0^\dagger-4i\omega r .\end{aligned} \tag{6.1.59}$$

Let us perform some algebraic manipulations on the left hand side of equation (6.1.57) by sing the results in (6.1.57) and (6.1.59).

$$\begin{aligned}\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta\mathcal{D}_0^2 &= \Delta\mathcal{D}_0^\dagger\left(\mathcal{D}_0\Delta\mathcal{D}_0^\dagger-4i\omega r\right)\mathcal{D}_0\\ &= \Delta\mathcal{D}_0^\dagger\mathcal{D}_0\left(\Delta\mathcal{D}_0^\dagger\mathcal{D}_0\right)-4i\omega\Delta\mathcal{D}_0^\dagger\mathcal{D}_0-4i\omega\Delta\mathcal{D}_0\\ &= \Delta\mathcal{D}_0^\dagger\mathcal{D}_0\left(\lambda+2i\omega r\right)-4i\omega\Delta\mathcal{D}_0^\dagger\mathcal{D}_0-4i\omega\Delta\mathcal{D}_0\,. \end{aligned} \tag{6.1.60}$$

It is easy to see that

$$\begin{aligned}\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 r &= \Delta\mathcal{D}_0^\dagger\left(r\mathcal{D}_0+1\right)\\ &= r\Delta\mathcal{D}_0^\dagger\mathcal{D}_0+\Delta\left(\mathcal{D}_0+\mathcal{D}_0^\dagger\right)\\ &= r\Delta\mathcal{D}_0^\dagger\mathcal{D}_0+2\Delta\mathcal{D}_0-2iK\,, \end{aligned} \tag{6.1.61}$$

hence the result in (6.1.60) can be simplified to be

$$\begin{aligned}\Delta\left(\mathcal{D}_0^\dagger\right)^2\Delta\mathcal{D}_0^2 &= \left(\lambda-2i\omega r\right)\Delta\mathcal{D}_0^\dagger\mathcal{D}_0+4\omega K\\ &= \left(\lambda-2i\omega r\right)\left(\lambda+2i\omega r\right)+4\omega\left(am-\left(r^2+a^2\right)\omega\right)\\ &= \lambda^2-4a^2\omega^2+4am\omega=\left|C\right|^2\,, \end{aligned} \tag{6.1.62}$$

which proves equation (6.1.53). For later convenience, we consider the angular functions S_+ and S_- normalized to unity

$$\int_0^\pi S_+^2\sin\theta d\theta=\int_0^\pi S_-^2\sin\theta d\theta=1\,. \tag{6.1.63}$$

6.1.2 Chandrasekhar's solutions for Maxwell fields in Kerr background

In this section, we derive in detail the solutions to Maxwell's equations in Kerr background by using the three complex scalars (6.1.14), (6.1.15) and (6.1.16) that are related to Maxwell field strength tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. We consider the gauge field A_μ as $(A_t, A_r, A_\theta, A_\phi)$ in spherical coordinates. The complex scalars Φ_0 and Φ_2 given by (6.1.14) and (6.1.16), can be written as

$$\Phi_0 = \frac{1}{\bar{\rho}\sqrt{2}}\left(\mathcal{L}_0^\dagger\left(\frac{r^2+a^2}{\Delta}A_t + A_r + \frac{a}{\Delta}A_\phi\right) - \mathcal{D}_0\left(iaA_t\sin\theta + A_\theta + \frac{iA_\phi}{\sin\theta}\right)\right)\ , \tag{6.1.64}$$

$$\Phi_2 = -\frac{1}{\bar{\rho}\sqrt{2}}\left(\Delta\mathcal{D}_0^\dagger\left(-iaA_t\sin\theta + A_\theta - \frac{iaA_\phi}{\sin\theta}\right) + \mathcal{L}_0\left(-\Delta A_r + \left(r^2+a^2\right)A_t + aA_\phi\right)\right)\ . \tag{6.1.65}$$

To simplify some expressions that will be handled hereafter, we define the following functions

$$\Delta F_+ = \left(r^2+a^2\right)A_t + \Delta A_r + aA_\phi \quad , \quad \Delta F_- = \left(r^2+a^2\right)A_t - \Delta A_r + aA_\phi\,, \tag{6.1.66}$$

$$G_+ = iaA_t\sin\theta + A_\theta + i\frac{A_\phi}{\sin\theta} \quad , \quad G_- = -iaA_t\sin\theta + A_\theta - i\frac{A_\phi}{\sin\theta}\,. \tag{6.1.67}$$

In addition, the following definitions would also be helpful [47]

$$\xi_+(r) = C^{-1}\left(r\mathcal{D}_0 - 1\right)R_- \quad , \quad \xi_-(r) = C^{-1}\left(r\mathcal{D}_0^\dagger - 1\right)\left(\Delta R_+\right)\ , \tag{6.1.68}$$

$$\zeta_+(\theta) = C^{-1}\left(\cos\theta\mathcal{L}_1^\dagger + \sin\theta\right)S_- \quad , \quad \zeta_-(\theta) = C^{-1}\left(\cos\theta\mathcal{L}_1 + \sin\theta\right)S_+\,, \tag{6.1.69}$$

where C is the Starobinksy constant (6.1.53).

One can easily verify that the r-dependent functions $\xi_\pm$ and θ-dependent functions $\zeta_\pm$ satisfy the following differential equations

$$\mathcal{D}_0\xi_+ = rR_+, \qquad \Delta\mathcal{D}_0^\dagger\xi_- = rR_-\,, \tag{6.1.70}$$

$$\mathcal{L}_0^\dagger\zeta_+ = S_+\cos\theta, \qquad \mathcal{L}_0\zeta_- = \cos\theta S_-\,. \tag{6.1.71}$$

The differential equation (6.1.31) combined with (6.1.64) yields the following equation

$$\Delta\mathcal{L}_0^\dagger F_+ - \Delta\mathcal{D}_0 G_+ = \sqrt{2}\left(ia\Delta R_+\mathcal{L}_0^\dagger\zeta_+ + S_+\Delta\mathcal{D}_0\xi_+\right)\ , \tag{6.1.72}$$

where we have used the definitions in (6.1.66), (6.1.67), (6.1.68) and (6.1.69). In a similar way, the differential equation (6.1.32) along with equation (6.1.65) yields the following relation

$$\Delta\mathcal{D}_0^\dagger G_- + \mathcal{L}_0\Delta F_- = -\sqrt{2}\left(\Delta\mathcal{D}_0^\dagger S_-\xi_- + ia\mathcal{L}_0 R_-\zeta_-\right)\ . \tag{6.1.73}$$

We can solve (6.1.72) and (6.1.73) to find $F_\pm$ and $G_\pm$ in terms of $R_\pm$, $S_\pm$, $\zeta_\pm$ and $\xi_\pm$. The solutions are given by

$$F_+ = \sqrt{2}\left(iaR_+\zeta_+ + \mathcal{D}_0 H_+\right)\,, \tag{6.1.74}$$

$$F_- = \sqrt{2}\left(-iaR_-\zeta_- + \mathcal{D}_0^\dagger H_-\right)\,, \tag{6.1.75}$$

$$G_+ = \sqrt{2}\left(-\xi_+ S_+ + \mathcal{L}_0^\dagger H_+\right)\,, \tag{6.1.76}$$

and

$$G_- = \sqrt{2}\left(-\xi_- S_- + \mathcal{L}_0 H_-\right)\,, \tag{6.1.77}$$

where $H_\pm$ are any two arbitrary functions that depend on both r and θ coordinates. The presence of arbitrary functions $H_\pm$ in the solutions (6.1.74)-(6.1.77) is the result of identity $[\mathcal{D}_0, \mathcal{L}_0^\dagger] = 0$. These functions show the freedom of Maxwell fields A_μ in the Kerr background. Plugging (6.1.74) - (6.1.77) back to (6.1.66) and (6.1.67) provides us the general set of explicit solutions for A_μ which includes the arbitrary functions $H_\pm$. As we notice to find the solutions for A_μ, we have used only the equations (6.1.14) and (6.1.16) for the complex scalars Φ_0 and Φ_2. The gauge condition is the remaining equation (6.1.15) for Φ_1. Using equations (6.1.19) - (5.1.65), one can find the following equation

$$\begin{aligned} &\left(l^\mu n^\nu + \bar{m}^\mu m^\nu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ &= -\frac{\sqrt{2}}{\left(\bar{\rho}*\right)^2}\left[\left(\zeta_+\mathcal{L}_1 S_+ - \zeta_-\mathcal{L}_1^\dagger S_-\right) - ia\left(\xi_-\mathcal{D}_0 R_- - \xi_+\mathcal{D}_0^\dagger(\Delta R_+)\right)\right]\,. \end{aligned} \tag{6.1.78}$$

Plugging the known results for A_μ in equation (6.1.15) (that we call it as the Chandrasekhar gauge) and comparing the result with equation (6.1.78) yields the following equation that the arbitrary functions $H_\pm$ must satisfy

$$\mathcal{D}_0^\dagger\frac{\Delta\mathcal{D}_0 H_+}{\left(\bar{\rho}*\right)^2} + \mathcal{L}_1\frac{\mathcal{L}_0^\dagger H_+}{\left(\bar{\rho}*\right)^2} - \mathcal{D}_0\frac{\Delta\mathcal{D}_0^\dagger H_-}{\left(\bar{\rho}*\right)^2} - \mathcal{L}_1^\dagger\frac{\mathcal{L}_0 H_-}{\left(\bar{\rho}*\right)^2} = 0\,. \tag{6.1.79}$$

The equation (6.1.79) imposes a constraint on the choices for the arbitrary functions $H_\pm$. We very roughly can compare the Chandrasekhar gauge with the well known Lorentz gauge for the Maxwell fields in Minkowski spacetime. The Maxwell's equations in Minkowski spacetime are invariant under the gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$ where $\Lambda(x)$ is an arbitrary function. The Lorentz gauge $\partial_\mu A^\mu = 0$ restricts the arbitrary function $\Lambda(x)$ to a function that satisfies the wave equation $\Box\Lambda(x) = 0$. The Chandrasekhar gauge (6.1.15) resembles the Lorentz gauge. The constraint equation (6.1.79) for $H_\pm$ resembles to the wave equation $\Box\Lambda(x) = 0$ for $\Lambda(x)$, where the arbitrary functions $H_\pm$ play the role of $\Lambda(x)$. The full solutions for Maxwell fields in Kerr spacetime that include the gauge functions $H_\pm$, are given by [47],

$$\begin{aligned} A_t &= \frac{ia}{\sqrt{2}\rho^2}\left(\left(\Delta R_+\zeta_+ - R_-\zeta_-\right) - \left(\xi_+ S_+ - \xi_- S_-\right)\sin\theta\right) \\ &+ \frac{1}{\sqrt{2}\rho^2}\left(\Delta\left(\mathcal{D}_0 H_+ - \mathcal{D}_0^\dagger H_-\right) + ia\left(\mathcal{L}_0^\dagger H_+ - \mathcal{L}_0 H_-\right)\sin\theta\right) , \end{aligned} \tag{6.1.80}$$

$$A_r = \frac{ia}{\sqrt{2}\Delta}\left(\Delta R_+\zeta_+ + R_-\zeta_-\right) + \frac{\Delta}{\sqrt{2}}\left(\mathcal{D}_0 H_+ + \mathcal{D}_0^\dagger H_-\right) , \tag{6.1.81}$$

$$A_\theta = -\frac{1}{\sqrt{2}}\left(\xi_+ S_+ + \xi_- S_-\right) + \frac{1}{\sqrt{2}}\left(\mathcal{L}_0^\dagger H_+ + \mathcal{L}_0 H_-\right) , \tag{6.1.82}$$

and

$$\begin{aligned} A_\phi &= -\frac{i}{\sqrt{2}\rho^2}\left(a^2\left(\Delta R_+\zeta_+ - R_-\zeta_-\right)\sin^2\theta - \left(r^2+a^2\right)\left(\xi_+ S_+ - \xi_- S_-\right)\sin\theta\right) \\ &- \frac{1}{\sqrt{2}}\left(a\Delta\left(\mathcal{D}_0 H_+ - \mathcal{D}_0^\dagger H_-\right)\sin^2\theta + i\left(r^2+a^2\right)\left(\mathcal{L}_0^\dagger H_+ - \mathcal{L}_0 H_-\right)\sin\theta\right) \end{aligned} \tag{6.1.83}$$

Choosing both $H_\pm$ to be zero gives the simplest solutions to the second order differential equation (6.1.79). Using this choice and comparing equations (6.1.74), (6.1.75), (6.1.76) and (6.1.77) with equations (6.1.66) and (6.1.67), we get the Maxwell fields as

$$A_t = \frac{ia}{\rho^2\sqrt{2}}\left(\Delta R_+\zeta_+ - R_-\zeta_- - \sin\theta\left(\xi_+ S_+ - \xi_- S_-\right)\right) , \tag{6.1.84}$$

$$A_r = \frac{ia}{\sqrt{2}}\left(R_+\zeta_+ + \frac{R_-\zeta_-}{\Delta}\right) , \tag{6.1.85}$$

$$A_\theta = -\frac{1}{\sqrt{2}}\left(\xi_+ S_+ + \xi_- S_-\right) , \tag{6.1.86}$$

and

$$A_\phi = \frac{-i}{\rho^2\sqrt{2}}\left(a^2\sin^2\theta\left(\Delta R_+\zeta_+ - R_-\zeta_-\right) - \sin\theta\left(r^2+a^2\right)\left(\xi_+ S_+ - \xi_- S_-\right)\right) , \tag{6.1.87}$$

where the functions $\zeta_\pm$ and $\xi_\pm$, given by (6.1.68) and (6.1.69) can be rewritten as

$$\xi_+ = \frac{1}{2CK}\left(\left(ir\lambda + 2\alpha^2\omega\right) R_- - irC\Delta R_+\right) , \tag{6.1.88}$$

$$\xi_- = \frac{1}{2CK}\left(-\left(ir\lambda - 2\alpha^2\omega\right) \Delta R_+ + irCR_-\right) , \tag{6.1.89}$$

$$\zeta_+ = \frac{1}{2CQ}\left(\left(-\lambda\cos\theta - \frac{2\alpha^2\omega}{a}\right) S_- - CS_+\cos\theta\right) , \tag{6.1.90}$$

$$\zeta_- = \frac{1}{2CQ}\left(\left(\lambda\cos\theta - \frac{2\alpha^2\omega}{a}\right) S_+ + CS_-\cos\theta\right) . \tag{6.1.91}$$

The functions K, Q, and α are given in (6.1.12), (6.1.13), and (6.1.54) respectively. As a consistency check, we substitute the equations (6.1.84)-(6.1.87) for the different components of Maxwell fields into equation (6.1.64) and find $\Phi_0 = R_+(r)S_+(\theta)$ in perfect agreement with what was considered in equation (6.1.33) for Φ_0. A similar calculation shows substituting the equations (6.1.84)-(6.1.87) into equation (6.1.65) yields $\Phi_2 = R_-(r)S_-(\theta)$ that is again in agreement with equation (6.1.33) for Φ_2

6.2 Boundary action for Maxwell fields in the background of Kerr black hole

The action for the Maxwell fields in gravitational background $g_{\mu\nu}$ with no current is

$$S = \frac{1}{4}\int d^4x\sqrt{-g}\mathbf{F}^*{}_{\mu\nu}\mathbf{F}^{\mu\nu} + c.c. \tag{6.2.92}$$

that leads to the Maxwell's equations (6.1.18) and (6.1.17). The *c.c.* term should be added in (6.2.92) to ensure that the action is real valued as we notice that the Chandrasekhar solutions for the Maxwell fields in Kerr spacetime (6.1.84) - (6.1.87) are basically complex quantities. The existence of ∂_t and ∂_ϕ Killing vectors in Kerr geometry leads to write down the dependence of Maxwell fields $\mathbf{A}$ on coordinates t and ϕ as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_t \\ \mathbf{A}_r \\ \mathbf{A}_\theta \\ \mathbf{A}_\phi \end{pmatrix} = e^{-i\omega t + im\phi}\begin{pmatrix} A_t \\ A_r \\ A_\theta \\ A_\phi \end{pmatrix} , \tag{6.2.93}$$

where A_μ's are given by (6.1.84) - (6.1.87). We note that in (6.2.92), $\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu$ and so we can write $S = 2S_0$ where

$$S_0 = \frac{1}{4}\int d^4x\sqrt{-g}\left(\partial_\mu \mathbf{A}^*_\nu\right)\mathbf{F}^{\mu\nu} + c.c.\,. \tag{6.2.94}$$

The integrand of S_0 can be written as the difference of a total derivative term and one other term which is, in fact, proportional to the Maxwell's equations (6.1.17). Taking a spherical boundary with radius r_B that is the boundary of near-NHEK geometry of Kerr black hole, we can convert the total derivative term to a boundary term, given by

$$S_B = \frac{1}{2}\int d^3x\,\sqrt{-g}\mathbf{A}^*_\nu \mathbf{F}^{r\nu}\big|_{r=r_B} + c.c.\,, \tag{6.2.95}$$

where d^3x stands for $dtd\phi d\theta$. In getting (6.2.95) from (6.2.94), we have done the following steps. First recall that the contraction of the field strength tensor $\mathbf{F}_{\mu\nu}$ can be written as

$$\begin{aligned}
\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} &= \left(\nabla_\mu \mathbf{A}^*_\nu - \nabla_\nu \mathbf{A}^*_\mu\right)\mathbf{F}^{\mu\nu} \\
&= 2\left(\nabla_\mu \mathbf{A}^*_\nu\right)\mathbf{F}^{\mu\nu} \qquad (6.2.96)\\
&= 2\left(\partial_\mu \mathbf{A}^*_\nu - \Gamma^\alpha_{\mu\nu}\mathbf{A}^*_\alpha\right)\mathbf{F}^{\mu\nu} \\
&= 2\left\{\left(\partial_\mu \mathbf{A}^*_\nu\right)\mathbf{F}^{\mu\nu} - \Gamma^\alpha_{\mu\nu}\mathbf{A}^*_\alpha \mathbf{F}^{\mu\nu}\right\} \\
&= 2\left\{\partial_\mu\left(\mathbf{A}^*_\nu \mathbf{F}^{\mu\nu}\right) - \mathbf{A}^*_\nu \partial_\mu \mathbf{F}^{\mu\nu} - \Gamma^\alpha_{\mu\nu}\mathbf{A}^*_\alpha \mathbf{F}^{\mu\nu}\right\} \\
&= 2\left\{\nabla_\mu\left(\mathbf{A}^*_\nu \mathbf{F}^{\mu\nu}\right) - \Gamma^\mu_{\mu\alpha}\mathbf{A}^*_\nu \mathbf{F}^{\alpha\nu} - \mathbf{A}^*_\nu \partial_\mu \mathbf{F}^{\mu\nu} - \Gamma^\alpha_{\mu\nu}\mathbf{A}^*_\alpha \mathbf{F}^{\mu\nu}\right\} \\
&= 2\nabla_\mu\left(\mathbf{A}^*_\nu \mathbf{F}^{\mu\nu}\right) - 2\mathbf{A}^*_\mu\left[\partial_\nu \mathbf{F}^{\nu\mu} + \Gamma^\nu_{\nu\alpha}\mathbf{F}^{\alpha\mu} + \Gamma^\mu_{\alpha\nu}\mathbf{F}^{\alpha\nu}\right] \\
&= 2\nabla_\mu\left(\mathbf{A}^*_\nu \mathbf{F}^{\mu\nu}\right) - 2\mathbf{A}^*_\mu \nabla_\nu \mathbf{F}^{\nu\mu}\,. \qquad (6.2.97)
\end{aligned}$$

Note that the last term inside of the square bracket in the last second line of equation (6.2.96) does not give the exact definition of covariant derivative of $\mathbf{F}^{\mu\nu}$, i.e. it should be $\Gamma^\mu_{\alpha\nu}\mathbf{F}^{\nu\alpha}$ instead of $\Gamma^\mu_{\alpha\nu}\mathbf{F}^{\alpha\nu}$. However, both of these "right" and "wrong" terms are vanished, due to the fact where contraction between symmetric and antisymmetric tensors is zero. Therefore, we can get the last line of equation (6.2.96). Furthermore, since we are discussing the case of free Maxwell fields in curved space, the last equation reduces to

$$\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} = 2\nabla_\mu\left(\mathbf{A}^*_\nu \mathbf{F}^{\mu\nu}\right)\,, \tag{6.2.98}$$

since

$$\nabla_\nu \mathbf{F}^{\nu\mu} = 0\,. \tag{6.2.99}$$

By using the Gauss' theorem[2] we can get the expression (6.2.95) from (6.2.94) where the quantity K^r in the appendix H is replaced by $\mathbf{A}^*_\nu \mathbf{F}^{r\nu}$.

The field strength tensor components

$$\mathbf{F}^{r\nu} = g^{rr} g^{\nu\beta} \mathbf{F}_{r\beta} = g^{rr} g^{\nu\beta} \left(\partial_r \mathbf{A}_\beta - \partial_\beta \mathbf{A}_\mathbf{r}\right)\,, \tag{6.2.100}$$

can be written simply as $\mathbf{F} = \mathbf{\Xi A}$ where

$$\mathbf{\Xi} = g^{rr} \begin{pmatrix} g^{tt}\partial_r & -\left(g^{tt}\partial_t + g^{t\phi}\partial_\phi\right) & 0 & g^{t\phi}\partial_r \\ 0 & 0 & 0 & 0 \\ 0 & -g^{\theta\theta}\partial_\theta & g^{\theta\theta}\partial_r & 0 \\ g^{\phi t}\partial_r & -\left(g^{\phi t}\partial_t + g^{\phi\phi}\partial_\phi\right) & 0 & g^{\phi\phi}\partial_r \end{pmatrix}\,. \tag{6.2.101}$$

Using the above expressions, we can rewrite the boundary action (6.2.95) accordingly as

$$S_B = \frac{1}{2}\int d^3x\,\sqrt{-g}\mathbf{A}^\dagger \mathbf{\Xi A}\Big|_{r=r_B} + c.c., \tag{6.2.102}$$

where

$$\mathbf{A}^\dagger \mathbf{\Xi A} = g^{rr}\Big(g^{tt}\left(A^*_t\partial_r A_t - i\omega A^*_t A_r\right) + g^{t\phi}\left(A^*_t\partial_r A_\phi + imA^*_t A_r\right) - g^{\theta\theta}\left(A^*_\theta\partial_\theta A_r - A^*_\theta\partial_r A_\theta\right)$$

$$+g^{\phi t}\left(A^*_\phi\partial_r A_t - i\omega A^*_\phi A_r\right) + g^{\phi\phi}\left(A^*_\phi\partial_r A_\phi + imA^*_\phi A_r\right)\Big)\,. \tag{6.2.103}$$

6.3 Approximations for Maxwell fields in the near horizon limit

The solutions for Maxwell fields in Kerr background that are given in (6.1.84) - (6.1.87) contain the radial Teukolsky functions $R_\pm(r)$. In [37], the authors find the exact solutions to the radial Teukolsky equations for spin weight ± 1 in the corresponding "near" region $x \ll 1$ and "far" region $x \gg \tau_H$ where

$$x = \frac{r - r_+}{r_+}\,, \tag{6.3.104}$$

[2]See appendix H for a brief review.

where $\tau_H = \frac{r_+ - r_-}{r_+}$ is the dimensionless Hawking temperature [37, 41] which is related to the Hawking temperature T_H of the Kerr black holes by

$$T_H = \frac{\tau_H}{8\pi M}\,. \tag{6.3.105}$$

As we are discussing the near extremal rotating black holes, thus the dimensionless Hawking temperature τ_H would be very small number. This fact would play an important role later in getting the dominant terms of the action that describe the Maxwell fields in near horizon of near extremal Kerr black holes. To get the solutions everywhere, the incomplete solutions from "near" and "far" regions should match in the "matching" region. We elaborate this prescription as follows. The $R_\pm(r)$ and $S_\pm(\theta)$ in Teukolsky wave function (6.1.34) satisfy the equations,

$$\frac{\partial_r\left(\Delta^{\pm1+1}\partial_r R_\pm\right)}{\Delta^{\pm1}}\left(\frac{H^2 \mp 2i\left(r-M\right)H}{\Delta} \pm 4i\omega r + 2am\omega + 1 \pm 1 - K_l\right)R_\pm = 0\,, \tag{6.3.106}$$

which is known as the radial Teukolsky equation, and

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta S_\pm\left(\theta\right)\right) - \left(\frac{m\left(m \pm 2\cos\theta\right)+1}{\sin^2\theta} + a^2\omega^2\sin^2\theta \pm 2a\omega\cos\theta - K_l\right)S_\pm\left(\theta\right) = 0\,, \tag{6.3.107}$$

which is the corresponding angular one for spin ±1. K_l is the separation constant and $H = \omega(r^2+a^2) - am$. Following [37], for $x = (r - r_+)/r_+$, the radial equation for spin ±1 can be written as

$$x\left(x+\tau_H\right)\partial_r\left(\partial_r R_\pm\right) + \left(1\pm1\right)\left(2x+\tau_H\right)\partial_r R_\pm + V_\pm R_\pm = 0\,, \tag{6.3.108}$$

where

$$V_\pm = \frac{\left(r_+\omega x^2 + 2r_+\omega x + \frac{1}{2}n\tau_H\right)^2 \mp i\left(2x+\tau_H\right)\left(r_+\omega x^2 + 2r_+\omega x + \frac{1}{2}n\tau_H\right)}{x\left(x+\tau_H\right)}$$
$$\pm 4ir_+\omega\left(1+x\right) + 2am\omega + 1 \pm 1 - K_l\,. \tag{6.3.109}$$

The solutions for "near" region are given in [37] as

$$R_\pm^{near} = \left(\frac{x}{\tau_H}+1\right)^{i\left(\frac{n}{2}-m\right)\mp1} x^{-\frac{in}{2}\mp1}{}_2F_1\left(\frac{1}{2}+\beta\mp1-im, \frac{1}{2}-\beta\mp1-im, 1\mp1-in, -\frac{x}{\tau_H}\right)\,, \tag{6.3.110}$$

where β is given in (6.3.119). Considering only real and positive valued β in (6.3.119) plays a role in deriving the corresponding two-point function from the variation of the boundary action (6.4.194).

In the "far" region, i.e. $x \gg \tau_H$, the approximation to the radial equation is

$$x^2 R''_\pm + (1 \pm 1) 2x R'_\pm + V^{far}_\pm R_\pm = 0 \tag{6.3.111}$$

with

$$V^{far}_\pm = -K_\ell + m^2 + \frac{m^2}{4}(x+2)^2 \pm imx + 1 \pm 1 \,. \tag{6.3.112}$$

The solution is

$$\begin{aligned} R^{far}_\pm \;=\; & \mathcal{A}_\pm x^{-\frac{1}{2}+\beta\mp 1} e^{-imx/2} \;{}_1F_1\left(\frac{1}{2}+\beta \mp 1 + im, 1+2\beta, imx\right) \\ & + \mathcal{B}_\pm x^{-\frac{1}{2}-\beta\mp 1} e^{-imx/2} \;{}_1F_1\left(\frac{1}{2}-\beta \mp 1 + im, 1-2\beta, imx\right) . \end{aligned} \tag{6.3.113}$$

In the matching region $\tau_H \ll x \ll 1$, the far and near solutions must be coincide, hence the coefficient $\mathcal{A}_\pm$ and $\mathcal{B}_\pm$ can be fixed

$$R^{match}_\pm \to \mathcal{A}_\pm x^{-\frac{1}{2}+\beta\mp 1} \tau_H^{\frac{1}{2}-i\frac{n}{2}-\beta} + \mathcal{B}_\pm x^{-\frac{1}{2}-\beta\mp 1} \tau_H^{\frac{1}{2}-i\frac{n}{2}+\beta} \,, \tag{6.3.114}$$

where

$$\mathcal{A}_\pm = \frac{\Gamma(2\beta)\Gamma(1 \mp 1 - in)}{\Gamma(\frac{1}{2}+\beta - i(n-m))\Gamma(\frac{1}{2}+\beta \mp 1 - im)} \,, \tag{6.3.115}$$

$$\mathcal{B}_\pm = \frac{\Gamma(-2\beta)\Gamma(1 \mp 1 - in)}{\Gamma(\frac{1}{2}-\beta - i(n-m))\Gamma(\frac{1}{2}-\beta \mp 1 - im)} \,. \tag{6.3.116}$$

Considering the smallness of τ_H (asymptotic solutions), the solutions to Teukolsky radial equations (6.1.35) and (6.1.37) in the matching region can be read as [37]

$$R_+ = N_+ \tau_H^{-1-in/2} \left(\mathcal{A}_+ \left(\frac{r}{\tau_H}\right)^{\beta-3/2} + \mathcal{B}_+ \left(\frac{r}{\tau_H}\right)^{-\beta-3/2} \right) + \ldots \,, \tag{6.3.117}$$

$$R_- = N_- \tau_H^{1-in/2} \left(\mathcal{A}_- \left(\frac{r}{\tau_H}\right)^{\beta+1/2} + \mathcal{B}_- \left(\frac{r}{\tau_H}\right)^{-\beta+1/2} \right) + \ldots \,, \tag{6.3.118}$$

where β is given by

$$\beta^2 = \frac{1}{4} + K_l - 2m^2 \,. \tag{6.3.119}$$

The parameter K_l is related to λ in equations (6.1.35)-(6.1.38) by $K_l = \lambda + 2am\omega$ and we consider $K_l \geq 2m^2 - 1/4$ and so β is a real number. The "quantum number" n contained in the solutions above is given by

$$n = \frac{\omega - m\Omega_H}{2\pi T_H}, \tag{6.3.120}$$

and $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the horizon. We notice that since τ_H is a very small number and n is a finite number, so the equation (6.3.120) implies $\omega \sim m\Omega_H$. This means we consider only the Maxwell fields with frequency that is around the superradiant bound. The coefficients N_+ and N_- are the normalization constants that their ratio can be fixed (by using equation (6.1.42)) to

$$\frac{N_-}{N_+} = -\frac{\mathcal{K}_l r_+^2}{n\,(n+i)}\,. \tag{6.3.121}$$

where $\mathcal{K}_l = \sqrt{K_l^2 + m^2(m^2 + 1 - 2K_l)}$. In deriving the ratio (6.3.121), we considered the near horizon limit $r \to r_+$. As we notice from (6.2.103), we need to find the derivative of the gauge fields with respect to the coordinate r. From equations (6.3.117) and (6.3.118), we find the following equations

$$\partial_r R_+ = \left(\frac{\beta - 3/2}{r}\right) R_+ - Q_+, \quad \partial_r R_- = \left(\frac{\beta + 1/2}{r}\right) R_- - Q_-\,, \tag{6.3.122}$$

where

$$Q_+ \equiv \frac{2\beta\mathcal{B}_+}{r}\tau_H^{-1-in/2}\left(\frac{r}{\tau_H}\right)^{-\beta-3/2}, \quad Q_- \equiv \frac{2\beta\mathcal{B}_-}{r}\tau_H^{1-in/2}\left(\frac{r}{\tau_H}\right)^{-\beta+1/2}. \tag{6.3.123}$$

As we notice from expressions (6.1.84) and (6.1.87), the gauge field components A_t and A_ϕ depend on coordinates r and θ in a non-separable way, due to the presence of function $\rho = \sqrt{r^2 + a^2\cos^2\theta}$. As a result, performing the integration over the boundary in (6.2.95) becomes almost impossible. We can separate the dependence of gauge fields (6.1.84) and (6.1.87) on r and θ by making an approximation. The approximation is to set the coordinate r equal to the boundary radius r_B in all ρ and Δ that appear in (6.1.84) - (6.1.87). In this approximation, $\Delta_B = \Delta(r = r_B) = (r_B - r_+)(r_B - r_-)$ approaches to zero as the boundary radius $r_B \to r_+$. So, we can write the Maxwell fields on the boundary as

$$\mathbf{A}_t = e^{-i\omega t+im\phi}\left(\left(f_1 S_- + f_2 S_+\right) R_+ \Delta_B + \left(f_3 S_- + f_4 S_+\right) R_-\right), \tag{6.3.124}$$

$$\mathbf{A}_r = e^{-i\omega t+im\phi}\left(\left(f_5 S_- + f_6 S_+\right) R_+ + \left(f_7 S_- + f_8 S_+\right) R_- \Delta_B^{-1}\right), \tag{6.3.125}$$

$$\mathbf{A}_\theta = e^{-i\omega t+im\phi}\left(\left(f_9 S_- + f_{10} S_+\right) R_+ \Delta_B + \left(f_{11} S_- + f_{12} S_+\right) R_-\right), \tag{6.3.126}$$

$$\mathbf{A}_\phi = e^{-i\omega t+im\phi}\left(\left(f_{13} S_- + f_{14} S_+\right) R_+ \Delta_B + \left(f_{15} S_- + f_{16} S_+\right) R_-\right), \tag{6.3.127}$$

where f_i's $(i = 1, ..., 16)$ depend only on θ and are given by

$$f_1 = \frac{ia\sqrt{2}(-\lambda\cos\theta - 2\alpha^2\omega a^{-1})}{4\rho_B^2 C\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} - \frac{ia\sqrt{2}\sin\theta(ir_B\lambda - 2\alpha^2\omega)}{4C\left(am - \omega\left(r_B^2 + a^2\right)\right)\rho_B^2}, \tag{6.3.128}$$

$$f_2 = -\frac{ia\sqrt{2}\cos\theta}{4\rho_B^2\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} - \frac{ar_B\sqrt{2}\sin\theta}{4\left(am - \omega\left(r_B^2 + a^2\right)\right)\rho_B^2}, \tag{6.3.129}$$

$$f_3 = -\frac{ia\sqrt{2}\cos\theta}{4\rho_B^2\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} - \frac{ar_B\sqrt{2}\sin\theta}{4\left(am - \omega\left(r_B^2 + a^2\right)\right)\rho_B^2}, \tag{6.3.130}$$

$$f_4 = -\frac{ia\sqrt{2}\left(\lambda\cos\theta - 2\alpha^2\omega a^{-1}\right)}{4\rho_B^2 C\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} - \frac{ia\sqrt{2}\sin\theta\left(ir_B\lambda + 2\alpha^2\omega\right)}{4C\left(am - \omega\left(r_B^2 + a^2\right)\right)\rho_B^2}, \tag{6.3.131}$$

$$f_5 = \frac{ia\sqrt{2}\left(-\lambda\cos\theta - 2\alpha^2\omega a^{-1}\right)}{4C\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)}, \tag{6.3.132}$$

$$f_6 = -\frac{ia\sqrt{2}\cos\theta}{4\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)}, \tag{6.3.133}$$

$$f_7 = \frac{ia\sqrt{2}\cos\theta}{4\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)}, \tag{6.3.134}$$

$$f_8 = \frac{ia\sqrt{2}\left(\lambda\cos\theta - 2\alpha^2\omega a^{-1}\right)}{4C\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)}, \tag{6.3.135}$$

$$f_9 = \frac{\sqrt{2}\left(ir_B\lambda - 2\alpha^2\omega\right)}{4C\left(am - \omega\left(r_B^2 + a^2\right)\right)}, \tag{6.3.136}$$

$$f_{10} = \frac{ir_B\sqrt{2}}{4\left(am - \omega\left(r_B^2 + a^2\right)\right)}, \tag{6.3.137}$$

$$f_{11} = -\frac{ir_B\sqrt{2}}{4\left(am - \omega\left(r_B^2 + a^2\right)\right)}, \tag{6.3.138}$$

$$f_{12} = -\frac{\sqrt{2}\left(ir_B\lambda - 2\alpha^2\omega\right)}{4C\left(am - \omega\left(r_B^2 + a^2\right)\right)}, \tag{6.3.139}$$

$$f_{13} = \frac{ia^2\sqrt{2}\left(\lambda\cos\theta + 2\alpha^2\omega a^{-1}\right)\sin^2\theta}{4\rho_B^2 C\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} + \frac{i\left(r_B^2+a^2\right)\sqrt{2}\sin\theta\left(ir_B\lambda - 2\alpha^2\omega\right)}{4C\left(am - \omega\left(r_B^2+a^2\right)\right)\rho_B^2}, \tag{6.3.140}$$

$$f_{14} = \frac{ia^2\sqrt{2}\cos\theta\sin^2\theta}{4\rho_B^2\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} + \frac{r_B\left(r_B^2+a^2\right)\sqrt{2}\sin\theta}{4\left(am - \omega\left(r_B^2+a^2\right)\right)\rho_B^2}, \tag{6.3.141}$$

$$f_{15} = \frac{ia^2\sqrt{2}\cos\theta\sin^2\theta}{4\rho_B^2\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} + \frac{r_B\left(r_B^2+a^2\right)\sqrt{2}\sin\theta}{4\left(am - \omega\left(r_B^2+a^2\right)\right)\rho_B^2}, \tag{6.3.142}$$

$$f_{16} = \frac{ia^2\sqrt{2}\left(\lambda\cos\theta - 2\alpha^2\omega a^{-1}\right)\sin^2\theta}{4\rho_B^2 C\left(-a\omega\sin\theta + m\left(\sin\theta\right)^{-1}\right)} + \frac{i\left(r_B^2+a^2\right)\sqrt{2}\sin\theta\left(ir_B\lambda - 2\alpha^2\omega\right)}{4C\left(am - \omega\left(r_B^2+a^2\right)\right)\rho_B^2}. \tag{6.3.143}$$

In (6.3.128) - (6.3.143),

$$\rho_B = r_B^2 + a^2\cos^2\theta. \tag{6.3.144}$$

We note that for the near extremal Kerr black holes $\Omega_H \simeq \frac{1}{2a}$ and so $\omega \sim \frac{m}{2a}$. As a result for $r_B \to r_+$, all f_i's (except $f_5, \cdots, f_8$ that appear in radial component (6.3.125) of Maxwell field) become very large. In fact, due to the smallness of Δ_B in the near horizon limit of near extremal black hole, all components of Maxwell fields in the near horizon of near extremal Kerr background become very large. We consider the difference between am and $\omega(r_B^2 + a^2)$ in near horizon near extremal Kerr black hole to be the same order of Δ_B.

6.4 Two-point function of vector fields

In this section we explicitly calculate the boundary action (6.2.95) and find the two-point function of the vector fields. We rewrite the components of the Maxwell field (6.3.124) - (6.3.127) in a matrix form as

$$\mathbf{A} = e^{-i\omega t + im\phi}(R_+\mathbf{K}\mathbf{v}_+ + R_-\mathbf{L}\mathbf{v}_-), \tag{6.4.145}$$

where the matrices $\mathbf{K}$ and $\mathbf{L}$ are

$$\mathbf{K} = \begin{pmatrix} \Delta_B f_1 & 0 & 0 & \Delta_B f_2 \\ f_5 & \kappa_1 & 0 & f_6 \\ \Delta_B f_9 & 0 & \kappa_2 & \Delta_B f_{10} \\ \Delta_B f_{13} & 0 & 0 & \Delta_B f_{14} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \kappa_3 & f_3 & f_4 & 0 \\ 0 & f_7\Delta_B^{-1} & f_8\Delta_B^{-1} & 0 \\ 0 & f_{11} & f_{12} & 0 \\ 0 & f_{15} & f_{16} & \kappa_4 \end{pmatrix}, \tag{6.4.146}$$

and

$$\mathbf{v}_+ = \begin{pmatrix} S_- & 0 & 0 & S_+ \end{pmatrix}^T, \quad \mathbf{v}_- = \begin{pmatrix} 0 & S_- & S_+ & 0 \end{pmatrix}^T. \tag{6.4.147}$$

For later convenience, we show the first and the second term of (6.4.145) by $\mathbf{A}_+$ and $\mathbf{A}_-$, respectively. We notice the matrices $\mathbf{K}$ and $\mathbf{L}$ as well as vectors $\mathbf{v}_\pm$ depend only on angular coordinate θ, according to Teukolsky equations (6.1.36), (6.1.38) and equations (6.3.128) - (6.3.143) for f_i. The arbitrary constants κ_i, $i = 1, 2, 3, 4$ in (6.4.146) are introduced to provide the invertibility for matrices $\mathbf{K}$ and $\mathbf{L}$. We notice that $\kappa_i \to 0$ to reduce the Maxwell fields (6.4.145) to the solutions (6.3.124) - (6.3.127) and we perform this limit at the end of calculation wherever κ_i's appear. We may find that due to the gauge choice (6.1.15), there is a relation between the vectors $\mathbf{v}_+$ and $\mathbf{v}_-$ as $\mathbf{v}_- = \chi \mathbf{v}_+$ where the matrix χ is given by

$$\chi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{6.4.148}$$

Denoting the Maxwell fields and the Teukolsky functions on the boundary by $\mathbf{A}^B_\pm$ and $R^B_\pm$ respectively, we get

$$\frac{\mathbf{A}^B_+}{R^B_+} = e^{-i\omega t + im\phi} \mathbf{K} \mathbf{v}_+ \,, \tag{6.4.149}$$

and

$$\frac{\mathbf{A}^B_-}{R^B_-} = e^{-i\omega t + im\phi} \mathbf{L} \mathbf{v}_- \,. \tag{6.4.150}$$

As we notice, equations (6.4.149) and (6.4.150) enable us to consider the non-radial dependent parts of Maxwell fields as the ratio of boundary Maxwell fields to the boundary Teukolsky radial solutions. Using the relation between $\mathbf{v}_+$ and $\mathbf{v}_-$, we have

$$\mathbf{A} = \left(R_+ \mathbf{K} + R_- \mathbf{L} \chi\right) e^{-i\omega t + im\phi} \mathbf{v}_+ = \left(R_+ + R_- \mathbf{L} \chi \mathbf{K^{-1}}\right) \frac{\left(\mathbf{A}^B_+\right)}{R^B_+} \,, \tag{6.4.151}$$

or

$$\mathbf{A}^\dagger = \frac{\left(\mathbf{A}^B_+\right)^\dagger}{\left(R^B_+\right)^*} \left(R^*_+ + R^*_- \left(\mathbf{L} \chi \mathbf{K^{-1}}\right)^\dagger\right) . \tag{6.4.152}$$

We calculate the integrand $\mathbf{A}^\dagger \mathbf{\Xi} \mathbf{A}$ of the boundary action (6.2.95) now. We notice that though the matrix $\mathbf{\Xi}$ in (6.2.101) has real entries, but after acting on $\mathbf{A}$, the result is a complex-valued vector. We decompose the operator $\mathbf{\Xi}$ as

$$\mathbf{\Xi} = g^{rr} \left(\mathbf{\Pi} + \mathbf{\Theta}\right) \,, \tag{6.4.153}$$

where

$$\mathbf{\Pi} = \begin{pmatrix} g^{tt}\partial_r & 0 & 0 & g^{t\phi}\partial_r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g^{\theta\theta}\partial_r & 0 \\ g^{\phi t}\partial_r & 0 & 0 & g^{\phi\phi}\partial_r \end{pmatrix}, \tag{6.4.154}$$

contains only the derivatives with respect to r and

$$\mathbf{\Theta} = \begin{pmatrix} 0 & -\left(g^{tt}\partial_t + g^{t\phi}\partial_\phi\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -g^{\theta\theta}\partial_\theta & 0 & 0 \\ 0 & -\left(g^{\phi t}\partial_t + g^{\phi\phi}\partial_\phi\right) & 0 & 0 \end{pmatrix}, \tag{6.4.155}$$

contains the derivatives with respect to t and ϕ. The reason for performing this decomposition is due to the fact that the radial dependence of the Maxwell fields (6.3.124) - (6.3.127) are in terms of functions $R_\pm$, while the non-radial dependence are in $\frac{\mathbf{A}_+^B}{R_+^B}$ or $\frac{\mathbf{A}_-^B}{R_-^B}$ (according to (6.4.149) and (6.4.150)). We find $\mathbf{A}^\dagger\mathbf{\Xi}\mathbf{A}$ is given by

$$\begin{aligned} \mathbf{A}^\dagger\mathbf{\Xi}\mathbf{A} &= g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}\left(R_+ + R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger(\mathbf{\Pi}+\mathbf{\Theta})\left(R_+ + R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)\frac{\mathbf{A}_+^B}{R_+^B} \\ &= g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}R_+^*\left(\mathbf{\Pi}R_+ + R_+\mathbf{\Theta}\right)\frac{\mathbf{A}_+^B}{R_+^B} + g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}R_+^*\left(\mathbf{\Pi}R_- + R_-\mathbf{\Theta}\right)\mathbf{L}\chi\mathbf{K}^{-1}\frac{\mathbf{A}_+^B}{R_+^B} \\ &+ g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}\left(R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger\left(\mathbf{\Pi}R_+ + R_+\mathbf{\Theta}\right)\frac{\mathbf{A}_+^B}{R_+^B} \\ &+ g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}\left(R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger\left(\mathbf{\Pi}R_- + R_-\mathbf{\Theta}\right)\mathbf{L}\chi\mathbf{K}^{-1}\frac{\mathbf{A}_+^B}{R_+^B}\,. \end{aligned} \tag{6.4.156}$$

As the matrix $\mathbf{\Pi}$ contains the differential operator ∂_r, it would be helpful to split $\mathbf{\Pi}R_+$,

$$\mathbf{\Pi}R_+ = \begin{pmatrix} g^{tt}\partial_r R_+ & 0 & 0 & g^{t\phi}\partial_r R_+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g^{\theta\theta}\partial_r R_+ & 0 \\ g^{\phi t}\partial_r R_+ & 0 & 0 & g^{\phi\phi}\partial_r R_+ \end{pmatrix}$$

$$
= \begin{pmatrix} g^{tt}\left(\left(\frac{\beta-3/2}{r}\right)R_+ - Q_+\right) & 0 & 0 & g^{t\phi}\left(\left(\frac{\beta-3/2}{r}\right)R_+ - Q_+\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g^{\theta\theta}\left(\left(\frac{\beta-3/2}{r}\right)R_+ - Q_+\right) & 0 \\ g^{\phi t}\left(\left(\frac{\beta-3/2}{r}\right)R_+ - Q_+\right) & 0 & 0 & g^{\phi\phi}\left(\left(\frac{\beta-3/2}{r}\right)R_+ - Q_+\right), \end{pmatrix}, \tag{6.4.157}
$$

to two terms, given by

$$
\mathbf{\Pi} R_+ = R_+ \mathbf{\Pi}_1 - Q_+ \mathbf{\Pi}_2 \,. \tag{6.4.158}
$$

In (6.4.158), the matrix $\mathbf{\Pi}_2$ is given by

$$
\mathbf{\Pi}_2 = \begin{pmatrix} g^{tt} & 0 & 0 & g^{t\phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g^{\theta\theta} & 0 \\ g^{\phi t} & 0 & 0 & g^{\phi\phi} \end{pmatrix}, \tag{6.4.159}
$$

and $\mathbf{\Pi}_1 = \frac{\beta-3/2}{r}\mathbf{\Pi}_2$. A similar calculation shows that we can split $\mathbf{\Pi} R_-$ to two terms, as

$$
\mathbf{\Pi} R_- = R_- \mathbf{\Pi}_3 - Q_- \mathbf{\Pi}_4 \,, \tag{6.4.160}
$$

where $\mathbf{\Pi}_3 = \frac{\beta+1/2}{r}\mathbf{\Pi}_2$ and $\mathbf{\Pi}_4 = \mathbf{\Pi}_2$. So, in terms of $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$, $\mathbf{\Pi}_3$ and $\mathbf{\Pi}_4$, we get the following expression for (6.4.156)

$$
\begin{aligned}
\mathbf{A}^\dagger \mathbf{\Xi} \mathbf{A} \;=\;& g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_+(\mathbf{\Pi}_1 + \mathbf{\Theta}) - Q_+\mathbf{\Pi}_2\right)\frac{\mathbf{A}_+^B}{R_+^B} \\
+\;& g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_-(\mathbf{\Pi}_3 + \mathbf{\Theta}) - Q_-\mathbf{\Pi}_4\right)\mathbf{L}\chi\mathbf{K}^{-1}\frac{\mathbf{A}_+^B}{R_+^B} \\
+\;& g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} \left(R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger \left(R_+(\mathbf{\Pi}_1 + \mathbf{\Theta}) - Q_+\mathbf{\Pi}_2\right)\frac{\mathbf{A}_+^B}{R_+^B} \\
+\;& g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} \left(R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger \left(R_-(\mathbf{\Pi}_3 + \mathbf{\Theta}) - Q_-\mathbf{\Pi}_4\right)\mathbf{L}\chi\mathbf{K}^{-1}\frac{\mathbf{A}_+^B}{R_+^B} \,.
\end{aligned} \tag{6.4.161}
$$

Comparing the functions $Q_\pm$ in (6.3.123) to the leading terms of $R_\pm$ in (6.3.117) and (6.3.118), we find $Q_\pm \sim \tau_H^{2\beta} R_\pm$. So, we can neglect the terms that are proportional to $Q_\pm$ compared to the terms that are proportional to $R_\pm$ in (6.4.161). This yields the equation (6.4.162) transforms to

$$
\begin{aligned}
\mathbf{A}^\dagger \mathbf{\Xi} \mathbf{A} &= g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_+ (\mathbf{\Pi}_1 + \mathbf{\Theta})\right) \frac{\mathbf{A}_+^B}{R_+^B} \\
&+ g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_- (\mathbf{\Pi}_3 + \mathbf{\Theta})\right) \mathbf{L} \chi \mathbf{K}^{-1} \frac{\mathbf{A}_+^B}{R_+^B} \\
&+ g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} \left(R_- \mathbf{L} \chi \mathbf{K}^{-1}\right)^\dagger \left(R_+ (\mathbf{\Pi}_1 + \mathbf{\Theta})\right) \frac{\mathbf{A}_+^B}{R_+^B} \\
&+ g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} \left(R_- \mathbf{L} \chi \mathbf{K}^{-1}\right)^\dagger \left(R_- (\mathbf{\Pi}_3 + \mathbf{\Theta})\right) \mathbf{L} \chi \mathbf{K}^{-1} \frac{\mathbf{A}_+^B}{R_+^B} \,, \qquad (6.4.162)
\end{aligned}
$$

which is obtained after dropping the terms proportional to $Q_\pm$. One can show that the components of $\mathbf{A_+^B}$ are

$$
\begin{aligned}
A_{t+}^B &= e^{-i\omega t + im\phi} \left(f_1 S_- + f_2 S_+\right) R_+^B \Delta_B \,, \\
A_{r+}^B &= e^{-i\omega t + im\phi} \left(f_5 S_- + f_6 S_+\right) R_+^B \,, \\
A_{\theta+}^B &= e^{-i\omega t + im\phi} \left(f_9 S_- + f_{10} S_+\right) R_+^B \Delta_B \,, \\
A_{\phi+}^B &= e^{-i\omega t + im\phi} \left(f_{13} S_- + f_{14} S_+\right) R_+^B \Delta_B \,. \qquad (6.4.163)
\end{aligned}
$$

Therefore, each term of (6.4.162) can be rewritten as

$$
g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_+ \mathbf{\Pi}_1\right) \frac{\mathbf{A}_+^B}{R_+^B} = g^{rr} \frac{R_+^* R_+}{\left(R_+^B\right)^* R_+^B} \pi_1^{ij} A_{i+}^{B*} A_{j+}^B \,, \qquad (6.4.164)
$$

$$
g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_+ \mathbf{\Theta}\right) \frac{\mathbf{A}_+^B}{R_+^B} = g^{rr} \frac{R_+^* R_+}{\left(R_+^B\right)^* R_+^B} \theta_1^{ij} A_{i+}^{B*} A_{j+}^B \,, \qquad (6.4.165)
$$

$$
g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_- \mathbf{\Pi}_3\right) \mathbf{L} \chi \mathbf{K}^{-1} \frac{\mathbf{A}_+^B}{R_+^B} = g^{rr} \frac{R_+^* R_-}{\left(R_+^B\right)^* R_+^B} \pi_2^{ij} A_{i+}^{B*} A_{j+}^B \,, \qquad (6.4.166)
$$

$$
g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} R_+^* \left(R_- \mathbf{\Theta}\right) \mathbf{L} \chi \mathbf{K}^{-1} \frac{\mathbf{A}_+^B}{R_+^B} = g^{rr} \frac{R_+^* R_-}{\left(R_+^B\right)^* R_+^B} \theta_2^{ij} A_{i+}^{B*} A_{j+}^B \,, \qquad (6.4.167)
$$

$$
g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} \left(R_- \mathbf{L} \chi \mathbf{K}^{-1}\right)^\dagger \left(R_+ \mathbf{\Pi}_1\right) \frac{\mathbf{A}_+^B}{R_+^B} = g^{rr} \frac{R_-^* R_+}{\left(R_+^B\right)^* R_+^B} \pi_3^{ij} A_{i+}^{B*} A_{j+}^B \,, \qquad (6.4.168)
$$

$$
g^{rr} \frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*} \left(R_- \mathbf{L} \chi \mathbf{K}^{-1}\right)^\dagger \left(R_+ \mathbf{\Theta}\right) \frac{\mathbf{A}_+^B}{R_+^B} = g^{rr} \frac{R_-^* R_+}{\left(R_+^B\right)^* R_+^B} \theta_3^{ij} A_{i+}^{B*} A_{j+}^B \,, \qquad (6.4.169)
$$

$$g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}\left(R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger\left(R_-\mathbf{\Pi}_3\right)\mathbf{L}\chi\mathbf{K}^{-1}\frac{\mathbf{A}_+^B}{R_+^B}=g^{rr}\frac{R_-^*R_-}{\left(R_+^B\right)^*R_+^B}\pi_4^{ij}A_{i+}^{B*}A_{j+}^B\,, \tag{6.4.170}$$

$$g^{rr}\frac{\left(\mathbf{A}_+^B\right)^\dagger}{\left(R_+^B\right)^*}\left(R_-\mathbf{L}\chi\mathbf{K}^{-1}\right)^\dagger\left(R_-\mathbf{\Theta}\right)\mathbf{L}\chi\mathbf{K}^{-1}\frac{\mathbf{A}_+^B}{R_+^B}=g^{rr}\frac{R_-^*R_-}{\left(R_+^B\right)^*R_+^B}\theta_4^{ij}A_{i+}^{B*}A_{j+}^B\,, \tag{6.4.171}$$

where

$$\pi_1^{ij}A_{i+}^{B*}A_{j+}^B=\left(g^{tt}A_{t+}^{B*}+g^{t\phi}A_{\phi+}^{B*}\right)A_t^B+\left(g^{t\phi}A_{t+}^{B*}+g^{\phi\phi}A_{\phi+}^{B*}\right)A_\phi^B\,, \tag{6.4.172}$$

$$\theta_1^{ij}A_{i+}^{B*}A_{j+}^B=-\left(\left(-i\omega g^{\phi t}+img^{\phi\phi}\right)A_{\phi+}^{B*}+\left(-i\omega g^{tt}+img^{t\phi}\right)A_{t+}^{B*}\right)A_r^B\,, \tag{6.4.173}$$

$$\begin{aligned}\pi_2^{ij}A_{i+}^{B*}A_{j+}^B &= \frac{-1}{\Delta_B\left(f_{13}f_2-f_1f_{14}\right)}\left(A_{t+}^BA_{t+}^{B*}\left(g^{tt}\left(f_4f_{13}-f_3f_{14}\right)+g^{t\phi}\left(f_{16}f_{13}-f_{15}f_{14}\right)\right)\right.\\ &+ A_{t+}^BA_{\phi+}^{B*}\left(g^{t\phi}\left(f_4f_{13}-f_3f_{14}\right)+g^{\phi\phi}\left(f_{16}f_{13}-f_{15}f_{14}\right)\right)\\ &+ A_{\phi+}^BA_{t+}^{B*}\left(g^{tt}\left(f_3f_2-f_4f_1\right)+g^{t\phi}\left(f_{15}f_2-f_{16}f_1\right)\right)\\ &+ \left.A_{\phi+}^BA_{\phi+}^{B*}\left(g^{t\phi}\left(f_3f_2-f_4f_1\right)+g^{\phi\phi}\left(f_{15}f_2-f_{16}f_1\right)\right)\right)\,,\end{aligned} \tag{6.4.174}$$

$$\begin{aligned}\theta_2^{ij}A_{i+}^{B*}A_{j+}^B &= \frac{\left(A_{t+}^B\left(f_8f_{13}-f_7f_{14}\right)+A_{\phi+}^B\left(f_7f_2-f_8f_1\right)\right)}{\Delta_B^2\left(f_1f_{14}-f_{13}f_2\right)}\\ &\times \left(A_{t+}^{B*}\left(-i\omega g^{tt}+img^{t\phi}\right)+A_{\phi+}^{B*}\left(-i\omega g^{t\phi}+img^{\phi\phi}\right)\right)\,,\end{aligned} \tag{6.4.175}$$

$$\begin{aligned}\pi_3^{ij}A_{i+}^{B*}A_{j+}^B &= \frac{-(\beta-3/2)}{\Delta_Br_B\left(f_1^*f_{14}^*-f_{13}f_2\right)}\left(A_{t+}^BA_{t+}^{B*}\left(g^{tt}\left(f_3^*f_{14}^*-f_4^*f_{13}^*\right)+g^{t\phi}\left(f_5^*f_{14}^*-f_6^*f_{13}^*\right)\right)\right.\\ &+ A_{t+}^BA_{\phi+}^{B*}\left(g^{tt}\left(f_4^*f_1^*-f_3^*f_2^*\right)+g^{t\phi}\left(f_{16}^*f_1^*-f_{15}^*f_2^*\right)\right)\\ &+ A_{\phi+}^BA_{t+}^{B*}\left(g^{t\phi}\left(f_3^*f_{14}^*-f_4^*f_{13}^*\right)+g^{t\phi}\left(f_{15}^*f_{14}^*-f_{16}^*f_{13}^*\right)\right)\\ &+ \left.A_{\phi+}^BA_{\phi+}^{B*}\left(g^{tt}\left(f_4^*f_1^*-f_3^*f_2^*\right)+g^{t\phi}\left(f_{16}^*f_1^*-f_{15}^*f_2^*\right)\right)\right)\,,\end{aligned} \tag{6.4.176}$$

$$\begin{aligned}\theta_3^{ij}A_{i+}^{B*}A_{j+}^B &= \frac{-1}{\Delta_B\left(f_1^*f_{14}^*-f_{13}f_2\right)}\left(A_{r+}^BA_{t+}^{B*}\left(-i\omega g^{tt}\left(f_4^*f_{13}^*-f_3^*f_{14}^*\right)+img^{\phi\phi}\left(f_{16}^*f_3^*-f_{15}^*f_{14}^*\right)\right.\right.\\ &+ \left.g^{t\phi}\left(-i\omega\left(f_{16}^*f_3^*-f_{15}^*f_{14}^*\right)+im\left(f_4^*f_{13}^*-f_3^*f_{14}^*\right)\right)\right)\\ &+ A_{r+}^BA_{\phi+}^{B*}\left(-i\omega g^{tt}\left(f_3^*f_2^*-f_4^*f_1^*\right)+img^{\phi\phi}\left(f_{15}^*f_2^*-f_6^*f_1^*\right)\right.\\ &+ \left.\left.g^{t\phi}\left(-i\omega\left(f_{15}^*f_2^*-f_{16}^*f_1^*\right)+im\left(f_3^*f_2^*-f_4^*f_1^*\right)\right)\right)\right)\,,\end{aligned} \tag{6.4.177}$$

$$
\begin{aligned}
\pi_4^{ij} A_{i+}^{B*} A_{j+}^{B} &= \frac{-(\beta+1/2)}{\Delta_B^2 r_B \left|f_1 f_{14} - f_{13} f_2\right|^2} \\
&\times \left(A_{\phi+}^{B} A_{\phi+}^{B*} \left(g^{tt}\left(f_3^* f_2^* f_4 f_1 - f_3^* f_2^* f_3 f_2 - f_4^* f_1^* f_4 f_1 + f_4^* f_1^* f_3 f_2\right)\right.\right. \\
&+ g^{\phi\phi}\left(f_{15}^* f_2^* f_{16} f_1 + f_{16}^* f_1^* f_{15} f_2 - f_{15}^* f_2^* f_{15} f_2 - f_{16}^* f_1^* f_{16} f_1\right) \\
&+ g^{t\phi}\left(f_{16}^* f_1^* f_3 f_2 - f_4^* f_1^* f_{16} f_1 + f_3^* f_2^* f_{16} f_1 - f_{16}^* f_1^* f_4 f_1\right. \\
&- \left. f_3^* f_2^* f_{15} f_2 - f_{15}^* f_2^* f_3 f_2 + f_4^* f_1^* f_{15} f_2 + f_{15}^* f_2^* f_4 f_1\right) \\
&+ A_{\phi+}^{B} A_{t+}^{B*} \left(g^{tt}\left(f_4^* f_{13}^* f_4 f_1 + f_3^* f_{14}^* f_3 f_2 - f_3^* f_{14}^* f_4 f_1 - f_4^* f_{13}^* f_3 f_2\right)\right. \\
&+ g^{\phi\phi}\left(f_{15}^* f_{14}^* f_{15} f_2 + f_{16}^* f_{13}^* f_{16} f_1 - f_{15}^* f_{14}^* f_{16} f_1 - f_{16}^* f_{13}^* f_{15} f_2\right) \\
&+ g^{t\phi}\left(f_{15}^* f_{14}^* f_3 f_2 - f_{15}^* f_{14}^* f_4 f_1 - f_4^* f_{13}^* f_{15} f_2 + f_{16}^* f_{13}^* f_4 f_1\right. \\
&+ \left. f_3^* f_{14}^* f_{15} f_2 - f_{16}^* f_{13}^* f_3 f_2 - f_3^* f_{14}^* f_{16} f_1 + f_4^* f_{13}^* f_{16} f_1\right) \\
&+ A_{t+}^{B} A_{t+}^{B*} \left(g^{tt}\left(f_3^* f_{14}^* f_4 f_{13} + f_4^* f_{13}^* f_3 f_{14} - f_3^* f_{14}^* f_3 f_{14} - f_4^* f_{13}^* f_4 f_{13}\right)\right. \\
&+ g^{\phi\phi}\left(f_{16}^* f_{13}^* f_{15} f_{14} + f_{15}^* f_{14}^* f_{16} f_{13} - f_{16}^* f_{13}^* f_{16} f_{13} - f_{15}^* f_{14}^* f_{15} f_{14}\right) \\
&+ g^{t\phi}\left(f_{15}^* f_{14}^* f_4 f_{13} - f_3^* f_{14}^* f_{15} f_{14} - f_{15}^* f_{14}^* f_3 f_{14} + f_4^* f_{13}^* f_{15} f_{14}\right. \\
&+ \left. f_{16}^* f_{13}^* f_3 f_{14} - f_{16}^* f_{13}^* f_4 f_{13} + f_3^* f_{14}^* f_{16} f_{13} - f_4^* f_{13}^* f_{16} f_{13}\right) \\
&+ A_{t+}^{B} A_{\phi+}^{B*} \left(g^{tt}\left(f_3^* f_2^* f_3 f_{14} + f_4^* f_1^* f_4 f_{13} - f_4^* f_1^* f_3 f_{14} - f_3^* f_2^* f_4 f_{13}\right)\right. \\
&+ g^{\phi\phi}\left(f_{15}^* f_2^* f_{15} f_{14} - f_{15}^* f_2^* f_{16} f_{13} + f_{16}^* f_1^* f_{16} f_{13} - f_{16}^* f_1^* f_{15} f_{14}\right) \\
&+ g^{t\phi}\left(f_{15}^* f_2^* f_3 f_{14} - f_{16}^* f_1^* f_3 f_{14} - f_{15}^* f_2^* f_4 f_{13} - f_4^* f_1^* f_{15} f_{14}\right. \\
&+ \left.\left.\left. f_4^* f_1^* f_{16} f_{13} + f_{16}^* f_1^* f_4 f_{13} - f_3^* f_2^* f_{16} f_{13} + f_3^* f_2^* f_{15} f_{14}\right)\right)\right), \qquad (6.4.178)
\end{aligned}
$$

$$
\begin{aligned}
\theta_4^{ij} A_{i+}^{B*} A_{j+}^{B} &= \frac{1}{\Delta_B^3 \left|f_1 f_{14} - f_{13} f_2\right|^2} \\
&\times \left(A_{t+}^{B*} A_{t+}^{B} \left(f_7 f_{14} - f_8 f_{13}\right)\left(im\left(\left(f_{16}^* f_{13}^* - f_{15}^* f_{14}^*\right) g^{\phi\phi} + \left(f_4^* f_{13}^* - f_3^* f_{14}^*\right) g^{t\phi}\right)\right.\right. \\
&- \left. i\omega\left(\left(f_{16}^* f_{13}^* - f_{15}^* f_{14}^*\right) g^{\phi t} - \left(f_4^* f_{13}^* - f_3^* f_{14}^*\right) g^{tt}\right)\right) + A_{t+}^{B*} A_{\phi+}^{B} \left(f_7 f_{14} - f_8 f_{13}\right) \\
&\times \left(im\left(\left(f_{15}^* f_2^* - f_{16}^* f_1^*\right) g^{\phi\phi} + \left(f_3^* f_2^* - f_4^* f_1^*\right) g^{t\phi}\right)\right. \\
&- \left. i\omega\left(\left(f_{15}^* f_2^* - f_{16}^* f_1^*\right) g^{\phi t} - \left(f_4^* f_1^* - f_3^* f_2^*\right) g^{tt}\right)\right) \\
&+ A_{\phi+}^{B*} A_{t+}^{B} \left(f_8 f_1 - f_2 f_7\right)\left(im\left(\left(f_{16}^* f_{13}^* - f_{15}^* f_{14}^*\right) g^{\phi\phi} + \left(f_4^* f_{13}^* - f_3^* f_{14}^*\right) g^{t\phi}\right)\right. \\
&- \left. i\omega\left(\left(f_{16}^* f_{13}^* - f_{15}^* f_{14}^*\right) g^{\phi t} - \left(f_4^* f_{13}^* - f_3^* f_{14}^*\right) g^{tt}\right)\right) + A_{\phi+}^{B*} A_{\phi+}^{B} \\
&\times \left(f_8 f_1 - f_2 f_7\right)\left(im\left(\left(f_{15}^* f_2^* - f_{16}^* f_1^*\right) g^{\phi\phi} + \left(f_3^* f_2^* - f_4^* f_1^*\right) g^{t\phi}\right)\right. \\
&- \left.\left. i\omega\left(\left(f_{15}^* f_2^* - f_{16}^* f_1^*\right) g^{\phi t} - \left(f_4^* f_1^* - f_3^* f_2^*\right) g^{tt}\right)\right)\right). \qquad (6.4.179)
\end{aligned}
$$

To obtain (6.4.172) - (6.4.179), we consider only the terms that couple to g^{tt}, $g^{t\phi}$ and $g^{\phi\phi}$ because they are the leading order terms compared to the terms that couple to g^{rr} or $g^{\theta\theta}$. This fact can be seen from equations (6.1.2) and (6.1.3) where $\Delta = \Delta_B$ is a very small number. A simple analysis of eight equations (6.4.172) - (6.4.179) shows that π_4^{ij} and θ_4^{ij} are the dominant terms in (6.4.162). Both terms are in order of Δ_B^{-3} compared to π_2, θ_2, π_3 and θ_3 that are in order of Δ_B^{-2} and π_1 and θ_1 are in order of Δ_B^{-1}. We show the summation of the dominant terms π_4^{ij} and θ_4^{ij} by

$$\tilde{\theta}_4^{ij} = \pi_4^{ij} + \theta_4^{ij} \,. \tag{6.4.180}$$

We calculate explicitly and present all the terms of (6.4.162) in the near region limit where $\Delta_B \ll 1$. We find that the leading terms in (6.4.161) (or (6.4.162)) are the terms that contain $\left(\mathbf{L}\chi\mathbf{K}^{-1}\right)^{\dagger}\mathbf{\Pi_3}\mathbf{L}\chi\mathbf{K}^{-1}$ and $\left(\mathbf{L}\chi\mathbf{K}^{-1}\right)^{\dagger}\mathbf{\Theta}\mathbf{L}\chi\mathbf{K}^{-1}$ respectively. Both terms are in the order of Δ_B^{-3} as the contravariant components of the metric tensor, g^{tt} , $g^{t\phi}$ and $g^{\phi\phi}$, are in order of Δ_B^{-1}. Using the results in (6.4.172) - (6.4.179), the boundary action (6.2.102) turns out to be

$$S_B = \frac{1}{2}\int dt d\theta d\phi \sqrt{-g_B}\, g^{rr}(r_B)\frac{(R_-^B)^* R_-^B}{(R_+^B)^* R_+^B}\tilde{\theta}_4^{ij}(\theta) A_{i+}^{B*}(t,\theta,\phi) A_{j+}^{B}(t,\theta,\phi) + c.c.\,. \tag{6.4.181}$$

where $\tilde{\theta}_4^{ij}$ is given by (6.4.180). We show the (t,ϕ)-dependence of the boundary gauge fields A_{i+}^B by a_{i+}^B according to

$$A_{i+}^B = a_{i+}^B(t,\phi)\tilde{\theta}_i(\theta)\,, \tag{6.4.182}$$

where

$$a_{i+}^B = e^{-i\omega t + im\phi} R_+^B \Delta_B\,, \tag{6.4.183}$$

for $i = t, \theta, \phi$ and

$$a_{r+}^B = e^{-i\omega t + im\phi} R_+^B\,. \tag{6.4.184}$$

The θ-dependent functions $\tilde{\theta}_i(\theta)$ are given by

$$\begin{aligned}
\tilde{\theta}_t &= f_1 S_- + f_2 S_+\,,\\
\tilde{\theta}_r &= f_5 S_- + f_6 S_+\,,\\
\tilde{\theta}_\theta &= f_9 S_- + f_{10} S_+\,,\\
\tilde{\theta}_\phi &= f_{13} S_- + f_{14} S_+\,.
\end{aligned} \tag{6.4.185}$$

Now we will borrow the prescription in AdS/CFT, where the connection between gravity and quantum field theories is given by

$$Z_{CFT} = Z_{grav.}\,, \tag{6.4.186}$$

where in our case the gravitational theory partition function is given by

$$Z_{grav} = \exp(-S_B(a^B_{i+}))\,. \tag{6.4.187}$$

In this discussion, the field a^B_{i+} or a rescaled of it plays a role as the source on the boundary. It is clear that the vanishing of this field yields the boundary action to be vanished as well. Therefore, the following formula is valid,

$$\frac{\delta Z}{\delta a^B_{i+}}\bigg|_{a^B_{i+}=0} = -\frac{\delta S_B(a^B_{i+})}{\delta a^B_{i+}} \exp\left(-S_B(a^B_{i+})\right)\bigg|_{a^B_{i+}=0} = -\frac{\delta S_B(a^B_{i+})}{\delta a^B_{i+}}\,. \tag{6.4.188}$$

For the later convenient, instead of using a^B_{i+} as the source, we will used the real part of the rescaled one,

$$\mathcal{A}^B_{i+} = \frac{\Re(a^B_{i+})}{r_B^{\beta-2}\Delta_B^{3/2}}\,. \tag{6.4.189}$$

Taking the functional derivative of (6.4.181) twice with respect to $\mathcal{A}^B_{i+}$, we can get the two-point function as [3]

$$\frac{\delta^2 S_B}{\delta\mathcal{A}^B_{i+}\delta\mathcal{A}^B_{j+}} = r_B^{2\beta-4}\frac{\left(R^B_-\right)^* R^B_-}{\left(R^B_+\right)^* R^B_+}\mathcal{Z}^{ij}\,, \tag{6.4.190}$$

where

$$\mathcal{Z}^{ij} = \int_0^\pi d\theta \sin(\theta)\{\tilde{\theta}_4^{ij}\tilde{\theta}_i\tilde{\theta}_j^* + \tilde{\theta}_4^{*ij}\tilde{\theta}_i^*\tilde{\theta}_j\}_{n.s.}\,. \tag{6.4.191}$$

In (6.4.191), n.s. means there is no summation over indices i and j. Moreover, we note from the results in (6.4.172) - (6.4.179) that the leading terms in (6.4.190) correspond to indices i and j to be t and ϕ only.

This is an interesting result that confirms the dual CFT to four-dimensional Kerr black hole is a two-dimensional theory, in contrast to AdS/CFT correspondence that the dimension of dual CFT always is one dimension less than the dimension of the bulk theory. Although it looks very unlikely to perform the integration in (6.4.191) and find an exact analytical

[3] The rescaling of the boundary gauge fields is quite similar to the rescaling of the boundary gauge fields in the context of AdS/CFT correspondence [42].

expression for $\mathcal{Z}^{ij}$, however we can find the retarded Green's function for the spin-1 fields from the factor $\frac{(R_-^B)^* R_-^B}{(R_+^B)^* R_+^B}$ in (6.4.190). The term $\frac{(R_-^B)^* R_-^B}{(R_+^B)^* R_+^B}$ can be calculated explicitly by using the equations (6.3.117) and (6.3.118) as

$$\frac{(R_-^B)^* R_-^B}{(R_+^B)^* R_+^B} = \left|\frac{N_-}{N_+}\right|^2 \frac{r_B^4}{\mathcal{A}_+\mathcal{A}_+^*} \left(\mathcal{A}_-\mathcal{A}_-^* + \left(\frac{\tau_H}{r_B}\right)^{2\beta} \left(\mathcal{A}_-\mathcal{B}_-^* + \mathcal{B}_-\mathcal{A}_-^*\right) + \left(\frac{\tau_H}{r_B}\right)^{4\beta} \mathcal{B}_-\mathcal{B}_-^*\right), \tag{6.4.192}$$

and so the two-point function (6.4.190) becomes

$$\begin{aligned}\frac{\delta^2 S_B}{\delta\mathcal{A}_{i+}^B\delta\mathcal{A}_{j+}^B} &= \mathcal{Z}^{ij}\frac{N_-N_-^* r_B^{2\beta}}{N_+N_+^*}\left(\left|\frac{\mathcal{A}_-}{\mathcal{A}_+}\right|^2 + \left(\frac{\tau_H}{r_B}\right)^{2\beta}\left(\frac{\mathcal{A}_-\mathcal{B}_-^*}{\mathcal{A}_+\mathcal{A}_+^*} + \frac{\mathcal{B}_-\mathcal{A}_-^*}{\mathcal{A}_+\mathcal{A}_+^*}\right) + \left(\frac{\tau_H}{r_B}\right)^{4\beta}\mathcal{B}_-\mathcal{B}_-^*\right) \\ &= \mathcal{Z}^{ij} r_B^{2\beta}\left(M^4 + \frac{N_-N_-^*}{N_+N_+^*}(\frac{\tau_H}{r_B})^{2\beta}\frac{\mathcal{A}_-\mathcal{B}_-^*}{\mathcal{A}_+\mathcal{A}_+^*}\right. \\ &\qquad \left. + \frac{N_-N_-^*}{N_+N_+^*}(\frac{\tau_H}{r_B})^{2\beta}\frac{\mathcal{B}_-\mathcal{A}_-^*}{\mathcal{A}_+\mathcal{A}_+^*} + \frac{N_-N_-^*}{N_+N_+^*}(\frac{\tau_H}{r_B})^{4\beta}\mathcal{B}_-\mathcal{B}_-^*\right). \end{aligned} \tag{6.4.193}$$

We have used equations (6.3.115), (6.3.116), and (6.3.121) to simplify the first term of (6.4.193). Plugging for the ratios N_-^*/N_+^* and $\mathcal{B}_-^*/\mathcal{A}_+^*$ that appear in the second term of (6.4.193) as well as the ratios N_-/N_+ and $\mathcal{B}_-/\mathcal{A}_+$ in the third term from equations (6.3.115), (6.3.116) and (6.3.121), we find

$$\begin{aligned}\frac{\delta^2 S_B}{\delta\mathcal{A}_{i+}^B\delta\mathcal{A}_{j+}^B} &= \mathcal{Z}^{ij} r_B^{2\beta}\left(M^4 + \left(\frac{\mathcal{K}_l M^2}{n\,(n-i)}\right) r_B^{-2\beta}\frac{N_-\mathcal{A}_-}{N_+\mathcal{A}_+}G_R^*\right. \\ &\qquad \left. + \left(\frac{\mathcal{K}_l M^2}{n\,(n+i)}\right) r_B^{-2\beta}\frac{N_-^*\mathcal{A}_-^*}{N_+^*\mathcal{A}_+^*}G_R + \frac{N_-N_-^*}{N_+N_+^*}\frac{\tau_H^{4\beta}}{r_B^{4\beta}}\mathcal{B}_-\mathcal{B}_-^*\right). \end{aligned} \tag{6.4.194}$$

In (6.4.194), G_R stands for

$$G_R(n_L, n_R) = -n\,(i+n)\,T_R^{2\beta}\frac{\Gamma(-2\beta)}{\Gamma(2\beta)}\frac{\Gamma\left(\beta+\frac{1}{2}-in_R\right)\Gamma\left(\beta-\frac{1}{2}-in_L\right)}{\Gamma\left(\frac{1}{2}-\beta-in_R\right)\Gamma\left(\frac{3}{2}-\beta-in_L\right)}, \tag{6.4.195}$$

where n_L and n_R are related to m and ω by $m = n_L$, and $n = n_L + n_R$ and we have considered the normalization (6.3.121) as well as the relation between dimensionless Hawking temperature τ_H with the right temperature T_R

$$T_R = \frac{\tau_H}{4M\lambda}, \tag{6.4.196}$$

where $\lambda \to 0$. The first term in bracket in (6.4.194) clearly is a constant term compared to the other terms. The second term in (6.4.194) is the complex conjugate of the third term. In fact, we can ignore the fourth term of (6.4.194) compared to the other terms, as this term is proportional to $\tau_H^{4\beta}$. Dropping the complex conjugate term in (6.4.194) according to [41, 79] , we find that the field theoretical two-point function (6.4.190) is equal to $G_R \mathcal{Z}^{ij}$ up to a multiplicative factor that depends on momentum and is not a part of the retarded Green's function. The existence of multiplicative factor has also been found for the field theoretical two-point function of spinors [41]. We note that $G_R(n_L, n_R)$ (given in (6.4.195)) is in exact agreement with the proposed retarded Green's function for the spin-1 fields in reference [21]. Using the optical theorem for the obtained retarded Green's function (6.4.195), we get exactly the absorption cross section of spin-1 fields scattered off of the Kerr black hole [19]. Interestingly enough, as we mentioned before, the boundary vector field components that contribute to the leading term of two-point function are only $\mathcal{A}^B_{t+}$ and $\mathcal{A}^B_{\phi+}$. This fact is in agreement with the statement of Kerr/CFT correspondence that the dual boundary theory is a two-dimensional CFT. The two-point function (6.4.190) is a function of ω and m which are the conjugate momenta in t and ϕ directions, respectively.

6.5 Two-point function of the vector fields in CFT$_2$

According to Kerr/CFT correspondence [12, 28], the four-dimensional physics of rotating black holes is holographic to two-dimensional CFT. The Green's functions for field perturbations with different spins have been proposed in [37, 21]. In [41], the authors found that the spin-1/2 Green's function can be obtained from the field theoretical technique by varying the boundary action with respect to the spinor fields. They also found that the field theoretical result is in agreement with what is expected from CFT calculation. The correlation function for the spinor operators in CFT is widely known from AdS/CFT correspondence [42]. The correlation function of conformal vector fields in Lorentz gauge has been obtained in the context of AdS$_{d+1}$/CFT$_d$ correspondence in [42, 100] and in covariant gauge in [101]. One crucial point is that the correlator vanishes for $d = 2$ in Lorentz gauge [42, 100, 101]. However, we expect that the correlator of conformal vector operators must not vanish in Chandrasekhar gauge (6.1.78). The reason is that we know the semiclassical absorption cross section of spin-1 fields in Kerr background is not zero [37, 21]. Moreover, the correlation functions definitely depend on the gauge condition [102]. In fact, the general form for the correlator of conformal vector operators $\mathcal{O}_i$ (with conformal weight Δ) read as

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\mathcal{C}}{|x-y|^{2\Delta}}\left(\eta_{ij} + f_{ij}\left(x,y\right)\right) , \tag{6.5.197}$$

where $\mathcal{C}$ is a constant that depends on the number of dimensions of spacetime and the functions f_{ij} depend on the gauge condition. Although the explicit form of functions f_{ij} is known in Lorentz gauge [42] or covariant gauge [101], however it is very unlikely to find f_{ij}'s in Chandrasekhar gauge (6.1.78) due to the complicated structure of (6.1.78). Nevertheless, inspired by the fact that the two-point function (6.4.190) factorizes as $G_R\mathcal{Z}^{ij}$ to two factors (G_R which is not sensitive to vector indices and $\mathcal{Z}^{ij}$ which depends on vector indices), we may associate the former factor to $\frac{\mathcal{C}}{|x-y|^{2\Delta}}$ and the latter to $\eta_{ij} + f_{ij}\left(x,y\right)$. In this regard, we consider the finite temperature correlation function on a torus with circumferences $1/T_L$ and $1/T_R$ [41]

$$\langle \mathcal{O}\mathcal{O}\rangle \sim \left(\frac{\pi T_R}{\sinh\left(\pi T_R t_{12}^{+}\right)}\right)^{2h_R}\left(\frac{\pi T_L}{\sinh\left(\pi T_L t_{12}^{-}\right)}\right)^{2h_L} . \tag{6.5.198}$$

We note that one can obtain the two-point function of scalars [37, 21] and spin-1/2 fermions [41] just by plugging the suitable left and right conformal weights $h_R = h_L = \beta + 1/2$ and $h_R = \beta + 1/2, h_L = \beta$ in (6.5.198), respectively.

Analytic continuing t to it and assuming the integer frequencies $\omega = 2\pi kT$ [41], the Fourier transform of two-point function (6.5.198) becomes

$$\begin{aligned}\widetilde{\langle \mathcal{O}\mathcal{O}\rangle} \quad &\sim \int_0^{1/T_R} dt_{12}^{+}e^{i\omega_R t_{12}^{+}}\left(\frac{\pi T_R}{\sin\left(\pi T_R t_{12}^{+}\right)}\right)^{2h_R}\int_0^{1/T_L} dt_{12}^{-}e^{i\omega_L t_{12}^{-}}\left(\frac{\pi T_L}{\sin\left(\pi T_L t_{12}^{-}\right)}\right)^{2h_L} \\ &\sim {T_R}^{2\beta}\frac{\Gamma\left(1-2h_R\right)\Gamma\left(1-2h_L\right)}{\Gamma\left(1-h_R+\frac{\omega_R}{2\pi T_R}\right)\Gamma\left(1-h_R-\frac{\omega_R}{2\pi T_R}\right)\Gamma\left(1-h_L+\frac{\omega_L}{2\pi T_L}\right)\Gamma\left(1-h_R-\frac{\omega_L}{2\pi T_L}\right)} .\end{aligned} \tag{6.5.199}$$

In computing the Fourier transform (6.5.200), we have used the formula

$$\int_0^{1/T} e^{i\omega t}\left(\frac{\pi T}{\sin\left(\pi T t\right)}\right)^{2h} dt = \frac{\left(\pi T\right)^{2h-1}2^{2h}e^{i\omega/2T}\Gamma\left(1-2h\right)}{\Gamma\left(1-h+\frac{\omega}{2\pi T}\right)\Gamma\left(1-h-\frac{\omega}{2\pi T}\right)} . \tag{6.5.200}$$

Identifying the frequencies as

$$\frac{\omega_R}{2\pi T_R} = -in_R,\ \frac{\omega_L}{2\pi T_L} = -in_L , \tag{6.5.201}$$

and the conformal weights as

$$h_R = \beta + 1/2,\ \ h_L = \beta - 1/2\,, \tag{6.5.202}$$

and plugging into (6.5.200) yields the two-point function

$$\widetilde{\langle \mathcal{O}\mathcal{O} \rangle} \sim T_R^{2\beta} \frac{\Gamma\left(-2\beta\right)\Gamma\left(\beta+\frac{1}{2}-in_R\right)\Gamma\left(\beta-\frac{1}{2}-in_L\right)}{\sin\left(2\pi\beta\right)\Gamma\left(2\beta\right)\Gamma\left(\frac{3}{2}-\beta-in_L\right)\Gamma\left(\frac{1}{2}-\beta-in_R\right)}\,, \tag{6.5.203}$$

which is in agreement with (6.4.195) that was obtained by using the variational method. We note that to get (6.5.203), we absorb some terms of (6.5.200) in the other part of two-point function that is associated to $\eta_{ij} + f_{ij}\left(x, y\right)$.

Chapter 7

Summary

The first three chapters of this thesis are the reviews on black holes in Einstein gravity, CFT together with the AdS/CFT correspondence, and the Kerr/CFT correspondence. The review on black holes is started with the introductions to some basic concepts in Einstein's general relativity. In the review, we restrict the discussion to the vacuum case only, where the corresponding action is known as the Einstein-Hilbert action (2.1.33) from which one can derive the vacuum Einstein gravitational equation (2.1.29). However, in chapter 5, the system under consideration is not vacuum anymore. In the vacuum case, there are two famous solutions to the Einstein equation. They are the Schwarzschild and Kerr spacetime solutions. The Schwarzschild solution describes the vacuum spacetime outside of a static mass. When the mass gets rotated, we use the Kerr solution to describe the spacetime outside of this mass. Both Schwarzschild and Kerr spacetimes contain the black hole solution. The black hole is formed when the mass, either in the static case or with some rotation, is contracted by the gravity and reaches the final state as a singularity covered by the event horizon. The blackness of a black hole comes from the fact that nothing can escape from the inside of the black hole's event horizon, not even light rays.

Discussing black holes only by using the Einstein's general relativity leads to an image that black holes are "dead" thermodynamical objects, i.e. they don't have the entropy. Nevertheless, Hawking's work in the 1970s show that incorporating quantum mechanics in understanding some physical aspects of black holes allows the radiation process to happen for black holes. Therefore, a black hole would behave like a normal object in thermodynamics where it absorbs and emits, thus it has entropy. It turns out that the entropy of a black hole is a large quantity, and yet there is still no clear picture of the general theory of relativity on how a black hole may have such large number of degrees of freedom. In fact, the entropy of a black hole is found to be a function of the black hole's area, instead of the volume, which is somehow against normal intuition. It is the deep insights of Bekenstein and Hawking in showing the dependence of a black hole's entropy on it's area, which lays the foundations of the holographic principle proposed by 't Hooft and Susskind [10, 11].

The holographic principle asserts that the number of possible states of a spacetime region is the same as that of a system with binary degrees of freedom, such as up and down spins, distributed on the boundary of the region. The number of such degrees of freedom is bounded by the number of Planck area which fits the region, hence it is not infinitely large. A realization of this principle, which is to date still the famous one, is the AdS/CFT correspondence proposed by Maldacena in 1997 [9]. In [9], he shows the equivalence between a gravitational theory in $D+1$ dimensions and a non-gravitational theory in D dimensions. The relation between these two distinguished theories which live in the different spacetimes, one in the bulk and the other in the boundary illustrated in figure 3.7, is very much like a "hologram". To be more specific, in his famous paper [9] Maldacena proposed that the type IIB superstring theory in $\mathrm{AdS}_5 \times S^5$ manifold describing gravity is equivalent to the four dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory living on the boundary of AdS_5 manifold. Note that the $\mathcal{N}=4$ supersymmetric Yang-Mills theory is a conformal field theory (CFT), from which the acronym of AdS/CFT comes.

Before the birth of AdS/CFT correspondence, conformal field theories have been used quite extensively at least in the three different areas of theoretical physics, i.e. statistical mechanics, interacting quantum field theory, and string theory. The discovery of AdS/CFT even attracts more attention on the researches in conformal field theories. Even though the AdS/CFT correspondence has been studied in many directions, it seems the real application of this idea is still quite far from satisfaction for at least two reasons. The first one is that the AdS background, where the gravitational system is defined in the AdS/CFT correspondence, is not the type of the spacetime on which we are living nowadays. The second reason is if one considers a four dimensional CFT, which somehow we can connect to real particle phenomena, as an "image" of a gravitational theory living in the higher dimensions, the concept of our universe which consists of more than four dimensions is still under question.

However, the ideas and deep insights in AdS/CFT correspondence are found to be very fruitful. Noticing that the near horizon geometry of extremal Kerr black holes (NHEK) has the AdS-like structure, the authors of [12] proposed the Kerr/CFT correspondence which claims that the extremal Kerr black holes have a holographic relation to the conformal field theory in two dimensions. Extremal Kerr black holes, or at least the near extremal ones, are very likely to exist in the universe. The Bekenstein-Hawking entropy for extremal Kerr black holes, $S_{BH} = 2\pi J$, can be recovered by using the Cardy formula known in CFT_2, $S_{Cardy} = \pi^2 cT/3$. The central charge c is computed for the NHEK spacetime, i.e. a slice of the spacetime near the extremal Kerr black holes, while the temperature T is the Frolov-Thorne temperature. Quite astonishing that the formula in CFT_2 can retrieve the result from gravitational theory, but at the same time it is a quite convincing evidence that Kerr/CFT correspondence may work in nature. Another clue for the Kerr/CFT correspondence comes from the agreement between the absorption cross sections computed from the gravitational theory and the CFT_2. As the Kerr/CFT correspondence is still a growing subject, the last two chapters are devoted to add more evidence of its existence.

A preliminary hint of the possibility of a system which may be holographic to a CFT_2 is the conformal symmetry possessed by the system. In the extremal case, the conformal symmetry can be seen directly in the spacetime structure of the near horizon of extremal Kerr black holes. In fact, when the black holes are not in the extremal case, the conformal structure can't be seen in the near horizon geometry, but it is hidden in the scalar wave equation for a low energy test particle in the near region. As it is reviewed in section 4.2, after showing the hidden conformal symmetry of Kerr black holes, one can establish the Kerr/CFT correspondence in the non-extremal condition. Inspired by the study of Kerr/CFT correspondence in non-extremal case [28], where the gravitational object is the electrically neutral rotating black hole, we broaden the discussion to the electrically charged rotating black holes. We study two black hole solutions, the Kerr-Sen black hole solution obtained in heterotic string theory, and Kerr-Newman black hole solution which is the solution in Eintein-Maxwell theory. Both of these black hole solutions are quite similar in their physical properties: they both have an electric charge, rotation, and mass. However, they are different, and the results in this thesis somehow can be used to distinguish these two black hole solutions.

In chapter 5, we find an extended family of the hidden conformal symmetry for Kerr-

Newman and Kerr-Sen black holes, which are characterized by a deformation parameter κ. We start by reviewing the derivation of Kerr-Sen black hole solution using the formula (5.1.3). A detail computation in obtaining the Kerr-Sen metric, which is the spacetime solution in the low energy limit of field theory describing heterotic string in four dimensions, is given starting from the Kerr solution. The Kerr-Sen solution contains black holes descriptions, which occur when the mass which is also carrying the electric charge collapses into a singularity covered by the critical radius $r_+ = M - b + \sqrt{(M^2 - b^2) - a^2}$. The dynamics of scalar probes outside of a Kerr-Sen black hole can be studied from the Klein-Gordon equation defined in the Kerr-Sen spacetime with $r > r_+$. Therefore, the study of hidden conformal symmetry as reviewed in section 4.2 can also be performed in the near region of Kerr-Sen black holes. In addition to the "near region" and "low frequency" limits applied to the scalar probe in studying the hidden conformal symmetry for Kerr black holes, for Kerr-Sen black hole case we also need to consider the "weakly interacting" condition, i.e. $eQ \ll 1$, where e and Q are the electric charges of probe and black hole respectively.

The hidden conformal symmetries for Kerr-Sen black holes are worked out in both extremal and non-extremal cases. The results obtained in these works point to the conjecture of Kerr-Sen/CFT correspondence, i.e. a duality between CFT_2 and Kerr-Sen black holes. Moreover, it supports the duality between a CFT_2 and rotating charged black holes, where now we have one more example in addition to the Kerr-Newman/CFT correspondence [23, 34]. The authors of [34] show that the hidden conformal symmetry for non-extremal Kerr-Newman black holes have two pictures, namely J and Q pictures. In [23], the discussion for Kerr-Newman/CFT correspondence is pushed further, where it is shown that the Kerr-Newman black holes in extremal and non-extremal conditions have the hidden conformal symmetry, and the J and Q pictures hidden conformal symmetries can be combined by using a $SL(2,\mathbb{Z})$ modular transformation. The $SL(2,\mathbb{Z})$ modular transformation performed in the hidden conformal symmetry discussion produces the equal ϕ' and χ' pictures, which are also called as the general picture, depending on which parameters in the corresponding $SL(2,\mathbb{Z})$ matrix that are set to be zero.

Both Kerr-Newman and Kerr-Sen black holes are characterized by the angular momentum J, electric charge Q, and mass M. Therefore, when we consider the charged scalar field in the background of Kerr-Sen spacetime, we expect to find the hidden conformal symmetries in J picture and Q picture as a Kerr-Neman black hole possesses. However, we are unable to show the hidden conformal symmetry for Kerr-Sen black holes in Q picture, i.e. only in J picture where we can observe the hidden conformal symmetry. Nevertheless, we still can construct the hidden conformal symmetry in general picture for Kerr-Sen black holes, though the lacking of Q picture is still contained. It can be noticed from the failure of obtaining the Q picture results from the hidden conformal symmetry generators of Kerr-Sen black holes in general picture.

The absence of Q picture hidden conformal symmetry for a Kerr-Sen black hole can be considered as one of the distinctions between Kerr-Sen and Kerr-Newman black holes. This may be understood from the fact that the Kerr-Sen geometries are not obtained from Einstein-Maxwell theory. Therefore, the "microscopic no hair conjecture" proposed in [34], which states that each of the macroscopic hair parameters besides the mass of black holes is associated to a holographic CFT_2 dual description, applies exclusively only in the Einstein-Maxwell theory. Moreover, the equation of motion for the dilaton in Kerr-Sen geometry is different from the equation of motion for the Klein-Gordon field and this renders the possibility of writing the equation in terms of squared Casimir (5.1.79) of $SL(2,\mathbb{R})_L$ and $SL(2,\mathbb{R})_R$. This observation is in agreement with the fact that the non-gravitational fields don't contribute to the central charge of conformal field theory [16].

As we know, there is only a single copy of conformal symmetry for an extremal Kerr-Sen black hole that can be read in the near horizon of this black hole [16]. To be explicit, the isometry group for the near horizon of extremal Kerr-Sen black hole is $SL(2,\mathbb{R})\times U(1)$, from which the single conformal symmetry is obvious. In fact, in the near region[1] of extremal Kerr-Sen black holes, we can obtain several sets of generators for the conformal symmetry of the system that are "hidden" in the low frequencies scalar wave equation. It resembles the situation in the non-extremal case where the symmetry is $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$, i.e. the generators are $\{H_\pm\,, H_0\}$ and $\left\{\bar{H}_\pm, \bar{H}_0\right\}$. However, the existence of several sets of hidden conformal symmetry generators for extremal Kerr-Sen black holes should not be associated to the multiple copies of $SL(2,\mathbb{R})$ symmetry of the system. These generators represent a

[1]One can verify that the near horizon consideration satisfies the near region condition.

single copy of the spacetime conformal symmetry at the near horizon. Finding the exact mapping between the sets of hidden conformal symmetry generators for Kerr or Kerr-Sen black holes would be an interesting future work.

In chapter 5, we also extend the discussion of hidden conformal symmetry for Kerr-Newman black holes in the case with deformation parameter κ. An advantage that we can get from the deformation case, which cannot be obtained in the case with no deformation [12, 23], is an ability in approaching the Schwarzschild limit from the results for Kerr black holes [36], or the Schwarzschild limit from the results for Reissner-Nordstrom black holes as shown in this thesis. The deformed hidden conformal generators for Kerr-Newman black holes are constructed explicitly in several different pictures, namely the ϕ', J, and Q pictures. The obtained deformed hidden conformal symmetry generators in the ϕ' picture reduces to the hidden conformal symmetry of the Kerr-Newman black holes in J and Q pictures [34], where the deformation parameter κ is set r_-/r_+. Setting $Q = 0$ in the J picture generators of the deformed hidden conformal symmetry for Kerr-Newman black hole provides the deformed hidden conformal symmetry generators for Kerr black hole as given in [36]. Obtaining the deformed hidden conformal symmetry generators for Reissner-Nordstrom black holes by setting $a = 0$ in the Q picture deformed hidden conformal symmetry generators of Kerr-Newman black hole [111] is a novel work presented in this thesis. The resulting deformed hidden conformal symmetry generators for Reissner-Nordstrom black holes agree with those obtained in [97] after setting $\kappa = r_-/r_+$. Furthermore, we can get the generators for Schwarzschild black holes [89] from the deformed hidden conformal symmetry generators of Reissner-Nordstrom black holes by setting $Q = 0$ after using a particular prescription, which can't be performed without the deformation scheme.

We also support the deformed Kerr-Newman/CFT correspondence by finding the absorption cross section of charged scalars in the Kerr-Newman background. We find a perfect agreement between the gravitational absorption cross section and CFT_2 cross section in three different conformal pictures for the Kerr-Newman black holes. Kerr and Reissner-Nordstrom black holes are members of the black hole family in Einstein-Maxwell theory. These black holes can be obtained by setting the physical parameters $Q = 0$ and $a = 0$ respectively from the Kerr-Newman black hole solution. Accordingly, the deformed Kerr/CFT and Reissner-Nordstrom /CFT correspondences observed by matching the absorption cross section formulas from gravitational and CFT_2 sides are the special cases of such analysis in the deformed Kerr-Newman/CFT correspondence, where the limits $Q = 0$ and $a = 0$ are taken respectively.

In chapter 6, we obtain the two-point function for the vector fields on the near horizon of near extremal Kerr black holes [112]. We borrow the famous AdS/CFT correspondence formula (3.4.144) where the gravitational action is an action for the vector fields in Kerr spacetime. We then consider the appropriate boundary action for the vector fields with respect to the boundary vector fields. One can verify the Kerr/CFT correspondence using the vector fields by following the prescription in the AdS/CFT correspondence, i.e. deriving the two point function for vector fields from the gravitational partition function and comparing the result with the vectorial two point function in CFT_2.

An interesting result that emerges from the explicit calculation of the boundary action is that the degrees of freedom of boundary vector fields (which is two) supports the original idea of Kerr/CFT correspondence that the dual theory to the four-dimensional Kerr black hole is a two-dimensional CFT. This is in contrast to the well known AdS_{d+1}/CFT_d result that the dimension of bulk theory is exactly one more than the dimension of dual CFT. In fact, the two-point function for the vector fields factorizes into two terms. The first term is not sensitive to the vector indices while the second term depends on vector indices as well as the gauge condition. The structure of the two-point function is exactly in agreement with the correlator of vector operators in a CFT. In deriving the two-point function of the vector fields, we have used some approximations and considered the leading terms of the boundary action. It is interesting to investigate the subleading terms of the boundary action to find their contributions to the two-point function and to their dual quantities in CFT. Moreover, deriving the correlator of conformal vector operators in Chandrasekhar gauge is another interesting task. The dependence of Kerr/CFT correspondence on the gauge condition is the other open question. We address these issues in future works.

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Appendix A

Ricci Tensor for Lewis metric

The component of Ricci tensor for the metric (2.2.73) are

$$\begin{aligned} R_{tt} &= e^{-Y}\left(\partial_1^2\Xi+(\partial_1\Xi)\,\partial_1\left(\Xi+\frac{1}{2}\ln X\right)+\frac{1}{2}(\partial_1\Xi)\,\partial_1(Z-Y)\right)\\ &+e^{-Z}\left(\partial_2^2\Xi+(\partial_2\Xi)\,\partial_2\left(\Xi+\frac{1}{2}\ln X\right)+\frac{1}{2}(\partial_2\Xi)\,\partial_2(Z-Y)\right)\\ &-\frac{1}{2}\left(\frac{V+W^2}{X^2}\right)\left(e^{-Y}(W\partial_1X-X\partial_1W)^2+e^{-Z}(W\partial_2X-X\partial_2W)^2\right)\ , \end{aligned} \tag{A.0.1}$$

$$\begin{aligned} R_{\phi\phi} &= -\left(\frac{e^{-Y}}{2X^2}\right)\left(\partial_1^2-\partial_1X^2+(\partial_1X)\,X\partial_1\left(\frac{1}{2}\ln X+\Xi+\frac{1}{2}(Y-Z)\right)\right)\\ &-\left(\frac{e^{-Z}}{2X^2}\right)\left(\partial_2^2-\partial_2X^2+(\partial_2X)\,X\partial_2\left(\frac{1}{2}\ln X+\Xi+\frac{1}{2}(Y-Z)\right)\right)\\ &-\left(\frac{V+W^2}{X^2}\right)\left(e^{-Y}(W\partial_1X-X\partial_1W)^2+e^{-Z}(W\partial_2X-X\partial_2W)^2\right)\ , \end{aligned} \tag{A.0.2}$$

$$\begin{aligned} R_{\phi t} &= \frac{e^{-(Z+Y)/2}}{X}\left(\partial_1\left(X^{-1/2}e^{-(Z+Y)/2}\left(V+\frac{W^2}{X}\right)^{-1/2}(W\partial_1X-X\partial_1W)\right)\right.\\ &\left.+\partial_2\left(X^{-1/2}e^{-(Z+Y)/2}\left(V+\frac{W^2}{X}\right)^{-1/2}(W\partial_2X-X\partial_2W)\right)\right)\ , \end{aligned} \tag{A.0.3}$$

$$\begin{aligned} R_{12} &= -e^{-(Y+Z)/2}\left(\partial_2^2\Xi+(\partial_1\Xi)(\partial_2\Xi)+\frac{\partial_1\partial_2X}{2X}-\frac{(\partial_1X)(\partial_2X)}{4X^2}\right.\\ &\left.-\frac{1}{2}(\partial_1Z)\,\partial_2\left(\frac{1}{2}\ln X+Y\right)-\frac{1}{2}(\partial_2Y)\,\partial_1\left(\frac{1}{2}\ln X+Y\right)\right)\\ &+\frac{1}{X^5}e^{-(Y+Z)/2}(W\partial_1X-X\partial_1W)(W\partial_2X-X\partial_2W)\ , \end{aligned} \tag{A.0.4}$$

$$\begin{aligned} R_{11} &= e^{-Y}\left((\partial_1\Xi)\,\partial_1\left(\frac{1}{2}\ln X+\frac{1}{2}Z\right)+\frac{1}{4}\frac{\partial_1X}{X}\partial_1Z+\frac{1}{2}e^{2Y}R\right)\\ &+e^{-Z}\left(\left(\partial_2^2\Xi\right)+(\partial_1\Xi)\,\partial_2\left(\Xi-\frac{1}{2}Z\right)+\frac{\left(\left(\partial_2^2X\right)X-\partial_2\left(X^2\right)\right)}{2X^2}\right.\\ &\left.+\frac{1}{2}\left(\frac{\partial_2X}{X}\right)\partial_2\left(\frac{\ln X}{2}+\Xi-\frac{Z}{2}\right)\right)\\ &+\frac{e^{-Y}(X\partial_1W-W\partial_1X)^2-e^{-Z}(X\partial_2W-W\partial_2X)^2}{4X^2\left(XV+W^2\right)}\ , \end{aligned} \tag{A.0.5}$$

$$
\begin{aligned}
R_{22} &= e^{-Z}\left(\left(\partial_2\Xi\right)\partial_2\left(\frac{\ln X}{2}+\frac{Y}{2}\right)+\frac{1}{4}\frac{\partial_2 X}{X}\partial_1 Y+\frac{1}{2}e^{2Z}R\right)\\
&\quad +e^{-Y}\left(\left(\partial_1^2\Xi\right)+\left(\partial_2\Xi\right)\partial_1\left(\Xi-\frac{1}{2}Y\right)+\frac{\left(\left(\partial_1^2 X\right)X-\partial_1\left(X^2\right)\right)}{2X^2}\right.\\
&\quad \left.+\frac{1}{2}\left(\frac{\partial_1 X}{X}\right)\partial_1\left(\frac{\ln X}{2}+\Xi-\frac{Y}{2}\right)\right)\\
&\quad +\frac{e^{-Y}\left(X\partial_2 W-W\partial_2 X\right)^2-e^{-Z}\left(X\partial_1 W-W\partial_1 X\right)^2}{4X^2\left(XV+W^2\right)}\,, \qquad \text{(A.0.6)}
\end{aligned}
$$

$$
R_{t1}=R_{\phi 1}=R_{t2}=R_{\phi 2}=0\,, \qquad \text{(A.0.7)}
$$

where the Ricci scalar R is

$$
\begin{aligned}
R &= -\frac{e^{-Y}}{X}\left(\frac{\left(\left(\partial_1^2 X\right)X-\partial_1\left(X^2\right)\right)}{X^2}+\frac{\partial_1 X}{X}\partial_1\left(\frac{1}{2}\ln X+\Xi+\frac{Z}{2}-\frac{Y}{2}\right)+2\partial_1^2\Xi\right.\\
&\quad \left.+\left(\partial_1\Xi\right)\partial_1\left(2\Xi+Z-Y\right)+\partial_1^2 Z+\frac{1}{2}\left(\partial_1 Z\right)\partial_1\left(Z-Y\right)\right)\\
&\quad -\frac{e^{-Z}}{X}\left(\frac{\left(\left(\partial_2^2 X\right)X-\partial_2\left(X^2\right)\right)}{X^2}+\frac{\partial_2 X}{X}\partial_2\left(\frac{1}{2}\ln X+\Xi+\frac{Z}{2}-\frac{Y}{2}\right)+2\partial_2^2\Xi\right.\\
&\quad \left.+\left(\partial_2\Xi\right)\partial_2\left(2\Xi+Y-Z\right)+\partial_2^2 Y+\frac{1}{2}\left(\partial_2 Y\right)\partial_2\left(Y-Z\right)\right)\\
&\quad +\frac{e^{-Y}\left(X\partial_1 W-W\partial_1 X\right)^2-e^{-Z}\left(X\partial_2 W-W\partial_2 X\right)^2}{2X^2\left(XV+W^2\right)}\,, \qquad \text{(A.0.8)}
\end{aligned}
$$

and the function Ξ is defined as

$$
\Xi=\frac{\ln(V-W^2/X)}{2}\,. \qquad \text{(A.0.9)}
$$

Appendix B

Riemann Tensor for Kerr Spacetime

To compute the Kretschmann scalar for Kerr spacetime, we need to find all the components of the corresponding Riemann tensor. For the metric (2.2.157), the corresponding nonzero covariant Riemann tensor (2.1.53) components are

$$
\begin{aligned}
R_{r\theta r\theta} &= -\frac{Mr\sigma}{\Delta\varrho^2}\,,\ R_{r\theta\phi t} = \frac{\tilde{\sigma}Ma\sin\theta\cos\theta}{\varrho^4}\,,\ R_{\phi t\phi t} = \frac{Mr\Delta\sin^2\theta\sigma}{\varrho^6}\,,\\
R_{r\phi r\phi} &= -\frac{\sin^2\theta Mr\sigma\left(r^4+2r^2a^2\left(1+\sin^2\theta\right)-4Mra^2\sin^2\theta+a^4\left(1+2\sin^2\theta\right)\right)}{\Delta\varrho^6}\,,\\
R_{r\phi rt} &= \frac{Ma\sigma r\left(3\,r^2-4\,Mr+3\,a^2\right)\sin^2\theta}{\Delta\varrho^6}\,,\ R_{r\phi\theta\phi} = \frac{3\,a^2\tilde{\sigma}\left(r^2+a^2\right)M\cos\theta\sin^3\theta}{\varrho^6}\,,\\
R_{r\phi\theta t} &= -\frac{aM\sin\theta\cos\theta\left(r^2+a^2\left(1+2\sin^2\theta\right)\right)\tilde{\sigma}}{\varrho^6}\,,\ R_{rtrt} = -\frac{Mr\left(2\Delta+a^2\sin^2\theta\right)\sigma}{\Delta\varrho^6}\,,\\
R_{rt\theta\phi} &= -\frac{aM\sin\theta\cos\theta\left(2r^2+2a^2+a^2\sin^2\theta\right)\tilde{\sigma}}{\varrho^6}\,,\ R_{rt\theta t} = \frac{3\,a^2M\tilde{\sigma}\cos\theta\sin\theta}{\varrho^6}\,,\\
R_{\theta\phi\theta\phi} &= \frac{Mr\sigma\left(-a^2\Delta\cos^2\theta+2\,r^4+5\,a^2r^2-2\,ra^2M+3\,a^4\right)\sin^2\theta}{\varrho^6}\,,\\
R_{\theta\phi\theta t} &= -\frac{raM\sigma\left(3\,r^2-2\,Mr+3\,a^2\right)\sin^2\theta}{\varrho^6}\,,\ R_{\theta t\theta t} = -\frac{Mr\sigma\left(\Delta+2\,a^2\sin^2\theta\right)}{\varrho^6}\,.
\end{aligned}
\tag{B.0.1}
$$

In the results above, we have used the notations $\Delta = r^2 - 2Mr + a^2$, $\varrho^2 = r^2 + a^2\cos^2\theta$, $\sigma = r^2 - 3a^2\cos^2\theta$, and $\tilde{\sigma} = 3r^2 - a^2\cos^2\theta$. In addition, the nonzero contravariant Riemann tensor for Kerr spacetime are

$$
\begin{aligned}
R^{r\theta r\theta} &= \frac{Mr\Delta\sigma}{\varrho^{10}}\,,\ R^{r\phi\theta\phi} = \frac{3a^2\cos\theta M\tilde{\sigma}}{\varrho^{10}\sin\theta}\,,\ R^{r\theta\phi t} = -\frac{Ma\cos\theta\tilde{\sigma}}{\varrho^8\sin\theta}\,,\\
R^{r\phi r\phi} &= \frac{-Mr\sigma\left(\Delta+2a^2\sin^2\theta\right)}{\varrho^{10}\sin^2\theta}\,,\ R^{r\phi rt} = -\frac{Mar\sigma\left(3\Delta+4Mr\right)}{\varrho^{10}}\,,\\
R^{r\phi\theta t} &= \frac{Ma\cos\theta\left(r^2+3a^2-2a^2\cos^2\theta\right)\tilde{\sigma}}{\varrho^{10}\sin\theta}\,,\ R^{rtr\phi} = -\frac{Mar\sigma\left(3\Delta+4Mr\right)}{\varrho^{10}}\,,\\
R^{rtrt} &= -\frac{Mr\sigma\left(2\,r^4+5\,a^2r^2-r^2a^2\cos^2\theta+2\,ra^2\cos^2\theta\,M-2\,ra^2M-a^4\cos^2\theta+3\,a^4\right)}{\varrho^{10}}\,,\\
R^{rt\theta\phi} &= \frac{Ma\cos\theta\left(2r^2+2a^2+a^2\sin^2\theta\right)\tilde{\sigma}}{\sin\theta\varrho^{10}}\,,\ R^{rt\theta t} = \frac{3Ma^2\left(r^2+a^2\right)\tilde{\sigma}\cos\theta\sin\theta}{\varrho^{10}}\,,
\end{aligned}
$$

$$R^{\theta\phi\theta\phi} = \frac{Mr\,(\varrho)\left(2\Delta + a^2\sin^2\theta\right)}{\sin^2\theta\varrho^{10}\Delta}\,,\ R^{\theta\phi\theta t} = \frac{aMr\sigma\,(3\Delta + 2Mr)}{\varrho^{10}\Delta}\,,$$

$$R^{\theta t\theta t} = \frac{Mr\sigma\,(r^4 - 2\,r^2a^2\cos^2\theta + 4\,a^2r^2 - 4\,ra^2M + 4\,ra^2\cos^2\theta\,M - 2\,a^4\cos^2\theta + 3\,a^4)}{\Delta\varrho^{10}}\,,$$

$$R^{\phi t\phi t} = \frac{Mr\sigma}{\varrho^6\Delta\sin^2\theta}\,. \tag{B.0.2}$$

Appendix C

Flux and Scattering Amplitudes

One can check that the function

$$\mathcal{F} = \frac{dU}{dr_*}U^* - \frac{dU^*}{dr_*}U \tag{C.0.1}$$

is conserved, i.e. $d\mathcal{F}/dr_* = 0$, where U is the solution of Regge-Wheeler equation (2.4.295). Hence, this function is called the probability flux or flux for short related to the Regge-Wheeler equation. The conservation of this flux can be used to show the relation between the ingoing and outgoing amplitudes contained in the wave solutions.

First let us discuss the Schwarzschild case, where the solutions for the near horizon and asymptotically flat regions are given in (2.4.301) and (2.4.302) respectively. From this solutions, we can compute

$$\mathcal{F}(r_* \to -\infty) = -2i\omega \tag{C.0.2}$$

which is the flux near the horizon, and

$$\begin{aligned} \mathcal{F}(r_* \to +\infty) &= \left(A_{out} i\omega e^{i\omega r_*} - A_{in} i\omega e^{-i\omega r_*}\right)\left(A^*_{out} e^{-i\omega r_*} + A^*_{in} e^{i\omega r_*}\right) \\ &+ \left(A^*_{out} i\omega e^{-i\omega r_*} - A^*_{in} i\omega e^{i\omega r_*}\right)\left(A_{out} e^{i\omega r_*} + A_{in} e^{-i\omega r_*}\right) \\ &= 2i\omega\left(|A_{out}|^2 - |A_{in}|^2\right) \end{aligned} \tag{C.0.3}$$

which is the flux at spatial infinity. The conservation of flux requires the last two equations to be equal, hence

$$|A_{in}|^2 = |A_{out}|^2 + 1\,. \tag{C.0.4}$$

Now we extend our discussion to the Kerr black hole. At spatial infinity, the solution U for Kerr spacetime behaves just like in the Schwarzschild spacetime. Therefore, the flux $\mathcal{F}$ at infinity for Kerr spacetime is also

$$\mathcal{F}(r \to -\infty) = -2i\omega\left(|A_{out}|^2 - |A_{in}|^2\right)\,. \tag{C.0.5}$$

The ingoing solution for U at the near horizon of Kerr black holes is

$$U \sim e^{(\omega - m\Omega_H)r_*}\,. \tag{C.0.6}$$

hence the associated flux in this region is

$$\mathcal{F}(r_* \to -\infty) = -2i\left(\omega - m\Omega_H\right)\,. \tag{C.0.7}$$

Making the equations (C.0.5) and (C.0.7) due to the flux conservation gives us

$$|A_{in}|^2 - |A_{out}|^2 = \frac{\omega - m\Omega_H}{\omega}\,. \tag{C.0.8}$$

By plugging $T = 1/A_{in}$ and $R = A_{out}/A_{in}$ in the last equation, we can get the formula (2.4.316) which hints the superradiant effect of Kerr black holes.

Appendix D

Isometry transformation in AdS_{D+1}

Isometry transformation is defined as a transformation that preserves the form of the metric, $ds^2 = ds'^2$. Related to the metric we are using to compute the wave equations in AdS_{D+1} spacetime in section 3.4, which can be read as

$$ds^2 = dx_\mu dx^\mu = \left(x^0\right)^{-2} \sum_{\alpha=0}^{D} \left(dx^\alpha\right)^2 , \tag{D.0.1}$$

we will see that the mapping

$$x^\mu \to y^\mu = \frac{x^\mu}{\left(x^0\right)^2 + \left(\vec{x}\right)^2} \tag{D.0.2}$$

is an isometry mapping, i.e.

$$\frac{dy^2}{\left(y^0\right)^2} = \frac{dx^2}{\left(x^0\right)^2} . \tag{D.0.3}$$

From the mapping (D.0.2), we have

$$\begin{aligned} dy^\mu &= \frac{dx^\mu \left(\left(x^0\right)^2 + \left(\vec{x}\right)^2\right) - x^\mu \left(2x^0 dx^0 + 2\vec{x} d\vec{x}\right)}{\left(\left(x^0\right)^2 + \left(\vec{x}\right)^2\right)^2} \\ &= \frac{x^2 dx^\mu - 2x^\mu x \cdot dx}{\left(x^2\right)^2} . \end{aligned} \tag{D.0.4}$$

Here we have used the dot "$\cdot$" notation in showing the contraction between two vectors. Recall that in this AdS_{D+1} spacetime denoted by the metric (D.0.1), the contravariant and covariant vectors are just the same, $x_\mu = x^\mu$. Therefore, the similarity between the contravariant and covariant vectors also applies to the transformed coordianates y^μ, i.e. $dy^\mu = dy_\mu$. It yields the reading of the metric in terms of the transformed coordinates as

$$ds'^2 = dy_\mu dy^\mu = \frac{dx^2}{x^4} . \tag{D.0.5}$$

From the last equation, it is easy to check that

$$\frac{dy^2}{\left(y^0\right)^2} = \frac{dx^2}{\left(x^0\right)^2} , \tag{D.0.6}$$

which is just (D.0.3). In the last equation we have used

$$y^0 = \frac{x^0}{\left(x^0\right)^2 + \left(\vec{x}\right)^2} \tag{D.0.7}$$

from the mapping (D.0.2).

Appendix E

Forms

This appendix contains a brief review of forms. An n-form $\mathbf{F}$ in terms of its components can be written as

$$\mathbf{F} = \frac{1}{n!} F_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{E.0.1}$$

Its Hodge-$*$ dual is defined by (note $|\epsilon_{\dots}| = \sqrt{|g|}$)

$$*\mathbf{F} = F^{\mu_1 \cdots \mu_p} \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{n-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \,. \tag{E.0.2}$$

One can also write it as

$$*\mathbf{F} \;=\; (d^{n-p}x)_{\mu_1 \cdots \mu_p} F^{\mu_1 \cdots \mu_p} \,, \tag{E.0.3}$$

$$(d^{n-p}x)_{\mu_1 \cdots \mu_p} \;=\; \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{n-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \,. \tag{E.0.4}$$

With this, Stokes's theorem $\int_\Sigma d * \mathbf{F} = \oint_{\partial\Sigma} *\mathbf{F}$ can be written as

$$\int_\Sigma (d^{n-p+1}x)_{\mu_2 \cdots \mu_p} \nabla_{\mu_1} F^{\mu_1 \mu_2 \cdots \mu_p} = \oint_{\partial\Sigma} (d^{n-p}x)_{\mu_2 \cdots \mu_p \mu_1} F^{\mu_1 \mu_2 \cdots \mu_p} \,. \tag{E.0.5}$$

A self-dual tensor satisfies the relations

$$*\mathbf{F} = \mathbf{F} \,,$$

and consequently

$$\mathbf{F} = * * \mathbf{F} \,.$$

Appendix F

Nother Charges and Central Term in Einstein Gravity

In [103], Iyer and Wald show that a diffeomorphism invariant 4-form Lagrangian[1] $\mathbf{L}$ can be written "manifestly covariant" form. Associated to the infinitesimal diffeomorphism transformation of this Lagrangian, we can find the corresponding 3-form Noether current $\mathbf{J}$, and the 2-from Noether charge as well $\mathbf{Q}$. Iyer and Wald use this analysis to prove the first law of black hole mechanics for arbitrary perturbations of a stationary black hole.

It turns out the method proposed by Iyer and Wald [103], known as the covariant phase space method, provides us a convenient way in studying the asymptotic symmetries of a spacetime. Asymptotic symmetries are defined as the transformations that yield the metric to be invariant up to what is allowed by the given boundary conditions. Carlip was the first one to calculate the central charge associated to the conformal symmetries related to a black hole horizon in [104]. Since then, there appear a lot of further developments, for example in [104, 106, 107, 108, 109, 110]. The materials in this appendix are based on [77].

Discussing the covariant phase method in Einstein theory of gravity is quite complicated. Therefore, to motivate the reader what is the essential of this method, we could start with the classical mechanics case. Consider the Lagrangian $L = L(q, \dot{q})$, where $q = q(t)$ describes the classical trajectory of a particle. The variation of this type Lagrangian with respect to the small variation of the path is

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \\ &= \Big(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\Big)\delta q + \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\delta q\Big) \,. \end{aligned} \tag{F.0.1}$$

where $\dot{q} \equiv \frac{dq}{dt}$. The corresponding equation of motion is

$$G = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \,. \tag{F.0.2}$$

In this case, the equation of motion G is expressed as a scalar. In a more general case, it could be any arbitrary n-rank tensor depending on the tensor rank of the dynamical fields and conjugate momenta under consideration. The variation of this equation of motion G can

[1] Since we are discussing the four dimensional theory, then we limit our discussion to 4-form Lagrangian only. In general, as it is shown in [103], we can extend the discussions to the n-dimensional case.

be read as

$$\begin{aligned}\delta G &= \delta\left(\frac{\partial L}{\partial q}\right) - \delta\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\\ &= \frac{\partial^2 L}{\partial q^2}\delta q + \frac{\partial^2 L}{\partial \dot{q}\partial q}\delta\dot{q} - \delta\dot{p}\\ &= 0\,, \end{aligned}\tag{F.0.3}$$

where the momentum can be written as

$$p = \frac{\partial L}{\partial \dot{q}}.\tag{F.0.4}$$

The second term in (F.0.1) can be rewritten as

$$\frac{d\Theta(q,\delta)}{dt}\tag{F.0.5}$$

where $\Theta(q,\delta) \equiv p\delta q$. Furthermore, we can define

$$\begin{aligned}\Omega(q;\delta_1,\delta_2) &= \delta_1\Theta(q,\delta_2) - \delta_2\Theta(q,\delta_1)\\ &= \delta_1 p\delta_2 q - \delta_2 p\delta_1 q\,,\end{aligned}\tag{F.0.6}$$

where δ_1 and δ_2 are the variations with respect to independent variables, say q_1 and q_2, where they do not depend each other, hence δ_1 and δ_2 are two independent variations. The time independence of $\Omega(q;\delta_1,\delta_2)$ is guaranteed if both $\delta_1 q$ and $\delta_2 q$ fulfill (F.0.3),

$$\frac{d\Omega(q;\delta_1,\delta_2)}{dt} = \delta_1\dot{p}\delta_2 q + \delta_1 p\delta_2\dot{q} - \delta_2\dot{p}\delta_1 q - \delta_2 p\delta_1\dot{q} = 0\,.\tag{F.0.7}$$

A Hamiltonian associated to the Lagrangian $L(q,\dot{q})$ is

$$H = p\dot{q} - L\tag{F.0.8}$$

whose variation can be read as

$$\delta H = \Omega\Big(q;\delta,\frac{d}{dt}\Big) = \delta\Theta\Big(q,\frac{d}{dt}\Big) - \frac{d}{dt}\Theta(q,\delta) = \delta p\dot{q} - \dot{p}\delta q\,.\tag{F.0.9}$$

The operator δ has been generalized to any possible operators including d/dt. Later, when we discuss the dynamics in Einstein theory of gravity, we might use the Lie derivative $\mathcal{L}_\xi$ as an explicit form of operator δ. From equation (F.0.9), we can get the Hamilton equations

$$\dot{q} = \frac{\partial H}{\partial p}\,,\quad \dot{p} = -\frac{\partial H}{\partial q}\,.\tag{F.0.10}$$

Now we use the generalized coordinates, $\phi^a = \{q,p\}$, $a = 1,2$, hence we can write

$$\Omega(\phi^a;\delta_1,\delta_2) = \Omega_{ab}\delta_1\phi^a\delta_2\phi^b\,,\tag{F.0.11}$$

where the corresponding matrix Ω_{ab} is

$$\Omega_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{F.0.12}$$

and its inverse

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{F.0.13}$$

By using this Ω_{ab} matrix, we can write the Poisson bracket of any two functions $f(q,p)$ and $g(q,p)$as

$$\Big\{ f \, , \, g \Big\}_{P.B.} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \Omega^{ab}\partial_a f \partial_b g \, . \tag{F.0.14}$$

We know that the Hamiltonian is the generator for time translation. For $f = f(q,p)$, the evolution of this function f with respect to time is

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} = \Big\{ f \, , \, H \Big\}_{P.B.} . \tag{F.0.15}$$

Furthermore, the last equation tells us that the conserved quantity, or charge, related to the time translational symmetry is the energy or Hamiltonian of the system.

We can generalize the discussion when there appear more coordinates to be handled. The corresponding Ω^{ab} to be dealt would be more complicated, and in analogy to (F.0.9), one can try to construct a charge Q_ξ related to any symmetric transformation δ_ξ,

$$\delta Q_\xi = \Omega(\phi^a; \delta, \delta_\xi) = \Omega_{ab}\delta\phi^a \delta_\xi \phi^b \, . \tag{F.0.16}$$

The Poisson bracket between two charges as defined in (F.0.16) is

$$\Big\{ Q_\xi \, , \, Q_\zeta \Big\}_{P.B.} = \Omega^{ab}\frac{\delta Q_\xi}{\delta \phi^a}\frac{\delta Q_\zeta}{\delta \phi^b} = \Omega(\phi^a; \delta_\zeta, \delta_\xi) \, . \tag{F.0.17}$$

In the followings, we will generalize the last formula by using a more complex treatment, i.e. in the form language.

Now consider a system with an action

$$S = \int_{\mathcal{M}} \mathbf{L} \, , \tag{F.0.18}$$

where the 4-form Lagrangian

$$\mathbf{L} = \mathcal{L} * \mathbf{1} \, . \tag{F.0.19}$$

The corresponding Lagrangian density in general is $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a, \partial_\mu \partial_\nu \phi^a, \cdots)$ and $*\mathbf{1} = \sqrt{|g|} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. We then consider a transformation

$$\delta_\epsilon \mathbf{L} = d\mathbf{M}_\epsilon \, , \tag{F.0.20}$$

that leaves the Lagrangian $\mathbf{L}$ to be invariant or up to a total derivative which vanishes after integration,

$$\delta S = \int_{\mathcal{M}} d\mathbf{M}_\epsilon = \oint_{\partial \mathcal{M}} \mathbf{M}_\epsilon = 0 \, . \tag{F.0.21}$$

Knowing $\mathbf{L}$ is a 4-form, we understand that $\mathbf{M}_\epsilon$ would be a 3-form. On the other hand, if we vary the dynamical fields ϕ^a, the variation of the Lagrangian can always be shown to have the form [103]

$$\delta_\epsilon \mathbf{L} = G_a \delta_\epsilon \phi^a * \mathbf{1} + d\mathbf{\Theta}(\phi^a, \delta_\epsilon)\,. \tag{F.0.22}$$

We notice that $G_a = 0$ is the equation of motion that is satisfied by ϕ^a. From the equations (F.0.21) and (F.0.22), a Noether current can be defined as

$$\mathbf{J}_\epsilon = \mathbf{\Theta}(\phi^a, \delta_\epsilon) - \mathbf{M}_\epsilon\,. \tag{F.0.23}$$

This current becomes a closed form when the equations of motion are fulfilled, $d\mathbf{J}_\epsilon = -G_a \cdot \delta_\epsilon \phi^a * \mathbf{1}$. Furthermore, when the Euler-Lagrange equation $G_a = 0$ is satisfied, we should have a 2-form $\mathbf{Q}_\epsilon$ whose relation to the 3-form current is $\mathbf{J}_\epsilon = d\mathbf{Q}_\epsilon$. This leads us to a definition of a conserved charge

$$Q_\epsilon = \int_{\mathcal{V}} d\mathbf{Q}_\epsilon = \oint_{\partial\mathcal{V}} \mathbf{Q}_\epsilon\,, \tag{F.0.24}$$

where $\mathcal{V}$ is a space-like slice of the spacetime manifold $\mathcal{M}$ and some appropriate boundary conditions have been applied.

No we consider a transformation that is generated by the Lie derivative $\delta_\xi \phi^a = \mathcal{L}_\xi \phi^a$,

$$\begin{aligned} \delta_\xi \mathbf{L} &= G_a \cdot \mathcal{L}_\xi \phi^a * \mathbf{1} + d\mathbf{\Theta}(\phi^a, \mathcal{L}_\xi) \\ &= \mathcal{L}_\xi \mathbf{L} = d(\xi \cdot \mathbf{L})\,. \end{aligned} \tag{F.0.25}$$

where $\xi \cdot \mathbf{L}$ means a contraction between ξ and $\mathbf{L}$, i.e. we perform the Einstein summation between the index of ξ and the first tensorial index of $\mathbf{L}$. Accordingly, the Noether current (F.0.23) is [103]

$$\mathbf{J}_\xi = \mathbf{\Theta}(\phi^a, \mathcal{L}_\xi) - \xi \cdot \mathbf{L}\,. \tag{F.0.26}$$

At this point, by an analogy to (F.0.6), we can define

$$\Omega(\phi^a; \delta_1, \delta_2) = -\int_{\mathcal{V}} \mathbf{w}(\phi^a; \delta_1, \delta_2)\,, \tag{F.0.27}$$

$$\mathbf{w}(\phi^a; \delta_1, \delta_2) = -\delta_1 \mathbf{\Theta}(\phi^a, \delta_2) + \delta_2 \mathbf{\Theta}(\phi^a, \delta_1)\,. \tag{F.0.28}$$

The conservation of $\Omega(\phi^a; \delta_1, \delta_2)$ is guaranteed if

$$d\mathbf{w}(\phi^a; \delta_1, \delta_2) = 0\,, \tag{F.0.29}$$

which yields

$$\oint_{\partial\mathcal{M}} \mathbf{w} = \int_{\mathcal{M}} d\mathbf{w} = 0. \tag{F.0.30}$$

Again, by an analogy to (F.0.9), we can construct a charge that corresponds to the transformation $\delta_\xi = \mathcal{L}_\xi$

$$\delta Q_\xi = \Omega(\phi^a; \delta, \mathcal{L}_\xi) = -\int_V \mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi)\,. \tag{F.0.31}$$

In the other hand, the variation of the Noether current (F.0.26) can be read as

$$\begin{aligned} \delta \mathbf{J}_\xi &= \delta \mathbf{\Theta}(\phi^a, \mathcal{L}_\xi) - \xi \cdot \delta \mathbf{L} \\ &= \delta \mathbf{\Theta}(\phi^a, \mathcal{L}_\xi) - \mathcal{L}_\xi \mathbf{\Theta}(\phi^a, \delta) + d\Big[\xi \cdot \mathbf{\Theta}(\phi^a, \delta)\Big]\,. \end{aligned} \tag{F.0.32}$$

The second line of the last equation is obtained when the equation of motion $G_a = 0$ is satisfied. Furthermore we can have

$$\mathbf{w}(\phi^a;\delta,\mathcal{L}_\xi) = \delta\mathbf{\Theta}(\phi^a,\mathcal{L}_\xi) - \mathcal{L}_\xi\mathbf{\Theta}(\phi^a,\delta) = d\mathbf{k}_\xi(\phi^a,\delta)\,, \tag{F.0.33}$$

and consequently

$$\delta Q_\xi = -\oint_{\partial\mathcal{V}} \mathbf{k}_\xi(\phi^a,\delta)\,. \tag{F.0.34}$$

The 2-form $\mathbf{k}_\xi$ above is defined as

$$\mathbf{k}_\xi(\phi^a,\delta) = \delta\mathbf{Q}_\xi - \xi\cdot\mathbf{\Theta}(\phi^a,\delta)\,. \tag{F.0.35}$$

From equation (F.0.34) we can define the charge as

$$Q_\xi(\phi) = \int_{\bar{\phi}}^{\phi} \delta Q_\xi + Q_\xi(\bar{\phi}) = -\int_{\bar{\phi}}^{\phi}\oint_{\partial\mathcal{V}} \mathbf{k}_\xi(\phi^a,\delta) + Q_\xi(\bar{\phi})\,. \tag{F.0.36}$$

To become a physically accepted charge, $Q_\xi(\phi)$ which is given in the integration (F.0.50) must be finite. The charge $Q_\xi(\bar{\phi})$ above is the value of charge in a specific background. Hence, by an analogy to (F.0.17), the Poisson bracket between two charges Q_ξ and Q_ζ is

$$\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.} \;=\; \Omega(\phi^a;\mathcal{L}_\zeta,\mathcal{L}_\xi) = -\oint_{\partial\mathcal{V}} \mathbf{k}_\xi(\phi^a,\mathcal{L}_\zeta)\,. \tag{F.0.37}$$

The works by Brown and Hanneaux [63] show that, with some appropriate boundary conditions, the Poisson bracket between Q_ξ and Q_ζ can be written as

$$\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.} = Q_{[\xi,\zeta]} + K[\xi,\zeta]\,. \tag{F.0.38}$$

It is $K[\xi,\zeta]$ which will play an important role in getting the central charge for NHEK for specific boundary conditions. This term is called as the central term. Moreover, Brown and Hanneaux [63] show that a constant shift in the charges,

$$Q_\xi \to Q_\xi + \alpha \tag{F.0.39}$$

will not change the expression of the central term $K[\xi,\zeta]$. A benefit obtained from this fact is we are allowed to shift the charges and choose the background which finally leaves only the central term in the Poisson bracket (F.0.38),

$$K[\xi,\zeta] = \Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.} = -\oint_{\partial\mathcal{V}} \mathbf{k}_\xi(\bar{\phi}^a,\mathcal{L}_\zeta)\,. \tag{F.0.40}$$

We now discuss the application of this covariant phase method for Einstein theory of gravity in vacuum. The 4-form Lagrangian can be read as

$$\mathbf{L} = \frac{R}{16\pi} * \mathbf{1}\,. \tag{F.0.41}$$

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \, , \tag{F.0.42}$$

and the equation of motion in the vacuum case is denoted by the vanishing of this tensor. When the variation of the metric is denoted by $h^{\mu\nu}$, i.e. $\delta g^{\mu\nu} = h^{\mu\nu}$, and the raising as well as lowering indices are performed by using the metric tensor $g_{\mu\nu}$, the variation of Einstein tensor (F.0.42) is

$$\begin{aligned} \delta G_{\mu\nu} &= \frac{1}{2}\Big[\nabla^\rho(\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\mu\rho}) - \partial^\rho\partial_\rho h_{\mu\nu} - \nabla_\mu\nabla_\nu h\Big] \\ &\quad -\frac{1}{2}\Big[\nabla_\mu\nabla_\nu h^{\mu\nu} - \partial^\rho\partial_\rho h - R^{\rho\sigma} h_{\rho\sigma}\Big] g_{\mu\nu} - \frac{R}{2} h_{\mu\nu} \, . \end{aligned} \tag{F.0.43}$$

The 3-form $\mathbf{\Theta}$ in (F.0.25) related to the variation of Lagrangian in (F.0.41) is

$$\mathbf{\Theta}(g_{\mu\nu}, \delta) = \frac{1}{16\pi}(d^3x)_\mu \Big[\nabla_\nu h^{\mu\nu} - \nabla^\mu h\Big] \, , \tag{F.0.44}$$

and consequently we have

$$\xi \cdot \mathbf{\Theta}(g_{\mu\nu}, \delta) = -\frac{1}{16\pi}(d^2x)_{\mu\nu}(I^{\mu\nu}_{\Theta_\xi}) \, , \tag{F.0.45}$$

with

$$I^{\mu\nu}_{\Theta_\xi} = \xi^\mu \nabla_\rho h^{\nu\rho} - \xi^\nu \nabla_\rho h^{\mu\rho} + \xi^\nu \nabla^\mu h - \xi^\mu \nabla^\nu h \, . \tag{F.0.46}$$

Therefore, the Noether current given in (F.0.26) associated with this $\mathbf{\Theta}(g_{\mu\nu}, \delta)$ transformations is

$$\begin{aligned} \mathbf{J}_\xi &= \frac{1}{16\pi}(d^3x)_\mu \Big[\nabla^\nu\nabla^\mu \xi_\nu + \partial^\rho\partial_\rho \xi^\mu - 2\nabla^\mu\nabla^\nu \xi_\nu - R\xi^\mu\Big] \\ &= -\frac{1}{16\pi}(d^3x)_\mu \nabla_\nu \Big[\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu\Big] \, . \end{aligned} \tag{F.0.47}$$

The 2-form charge that can give the above current is

$$\mathbf{Q}_\xi = -\frac{1}{16\pi}(d^2x)_{\mu\nu}(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \, . \tag{F.0.48}$$

In getting the charge in the last equation, we has used the fact that the Einstein tensor is vanished. The variation of the charge in (F.0.48) can be obtained as

$$\delta \mathbf{Q}_\xi = \frac{1}{16\pi}(d^2x)_{\mu\nu} I^{\mu\nu}_{Q_\xi} \, , \tag{F.0.49}$$

where

$$I^{\mu\nu}_{Q_\xi} = -\frac{h}{2}(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) + h^{\mu\rho}\nabla_\rho \xi^\nu - h^{\nu\rho}\nabla_\rho \xi^\mu - (\nabla^\mu h^{\nu\rho} - \nabla^\nu h^{\mu\rho})\xi_\rho \, . \tag{F.0.50}$$

Then finally, from equation (F.0.35), we can get the 2-form

$$\mathbf{k}_\xi(g_{\mu\nu}, \delta) = \frac{1}{16\pi}(d^2x)_{\mu\nu} k^{\mu\nu} \, , \tag{F.0.51}$$

with

$$k^{\mu\nu} = I^{\mu\nu}_{Q_\xi} + I^{\mu\nu}_{\Theta_\xi} = \xi^\nu \nabla^\mu h - \xi^\nu \nabla_\rho h^{\mu\rho} + \frac{h}{2} \nabla^\nu \xi^\mu - h^{\nu\rho} \nabla_\rho \xi^\mu + \xi_\rho \nabla^\nu h^{\mu\rho} - \left(\xi^\mu \nabla^\nu h - \xi^\mu \nabla_\rho h^{\nu\rho} + \frac{h}{2} \nabla^\mu \xi^\nu - h^{\mu\rho} \nabla_\rho \xi^\nu + \xi_\rho \nabla^\mu h^{\nu\rho} \right), \tag{F.0.52}$$

and

$$(d^{n-p}x)_{\mu_1 \cdots \mu_p} = \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{n-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} . \tag{F.0.53}$$

Appendix G

Chemical Potential

This appendix is a very brief introduction to the chemical potential that can be found in the discussion of a system with variable mass. Suppose that the number of moles N of substance in our system increases by dN at constant temperature T and pressure P. Let the internal energy, entropy, and and volume of our N moles of substance are U, S and V respectively. The corresponding quantities referred to one mole would be u, s and v, i.e.

$$u = \frac{U}{N} \quad , \quad s = \frac{S}{N} \quad , \quad v = \frac{V}{N} \,. \tag{G.0.1}$$

Changes in u, s and v where the mass is kept fixed can be read as

$$du = Tds - Pdv \,. \tag{G.0.2}$$

However, equation (G.0.1) tells us

$$\begin{aligned} du &= \frac{dU}{N} - \frac{UdN}{N^2} \,, \\ ds &= \frac{dS}{N} - \frac{SdN}{N^2} \,, \\ dv &= \frac{dV}{N} - \frac{VdN}{N^2} \,. \end{aligned} \tag{G.0.3}$$

From (G.0.2) and (G.0.3) we can get

$$dU = TdS - PdV + (U - TS + PdV)\,\frac{dN}{N} \,. \tag{G.0.4}$$

The coefficient of dN in equation above is the Gibbs free energy per mole, which for a one component system is called the chemical potential μ. Hence we can rewrite equation (G.0.4) as

$$dU = TdS - PdV + \mu dN \,. \tag{G.0.5}$$

When we have more than one substance, i.e. each substances has N_i moles, then the last formula can be generalized to

$$dU = TdS - PdV + \mu_i dN_i \,. \tag{G.0.6}$$

Then μ_i is the free Gibbs energy per mole for each components we have. Consequently, we can write

$$\mu_i = -T \left(\frac{\partial S}{\partial N_i} \right) \Bigg|_{U,V} \,. \tag{G.0.7}$$

Appendix H

Gauss' Theorem in Curved Space

In vector calculus, we are familiar with Gauss' theorem

$$\int_V \left(\nabla \cdot \vec{K}\right) dV = \oint_S \left(\vec{n} \cdot \vec{K}\right) dS \tag{H.0.1}$$

which says the integration over a volume V of a divergence of a vector $\vec{F}$ is equal to an integration over a closed surface of a scalar product between the vector $\vec{F}$ and a unit vector $\vec{n}$ that normal to the closed surface under consideration. See figure H.1 for an illustration. The curved spacetime version of equation (H.0.1) is

$$\int_V \sqrt{-g}\nabla_\mu K^\mu d^4x = \oint_{\partial V} K^\mu dS_\mu . \tag{H.0.2}$$

To prove the equation (H.0.2), first let us construct the spacetime we are discussing. The spacetime V is four dimensional manifold with metric tensor $g_{\mu\nu}$ in spherical coordinates t, r, θ and ϕ. The boundary ∂V is taken by setting the radius r to be fixed, say the boundary radius r_B, and described by three coordinates t, θ and ϕ. The center of volume V here is denoted by a dot in the middle of the left side picture in the figure H.2. Each of layers represented by the dashed closed curve has the same and fixed radius, and this layer grows radially outward from the center and finally reach the boundary with radius r_B. The points where the layer starts to grow and stop will be used later as the range of integration.

Now we can do some algebraic manipulation on (H.0.2),

$$\begin{aligned}
\int_V \sqrt{-g}\nabla_\mu K^\mu d^4x &= \int_V \partial_\mu \left(\sqrt{-g}K^\mu\right) d^4x \\
&= \int dr \oint \partial_r \left(\sqrt{-g}K^r\right) d^3x + \int dr \oint \partial_k \left(\sqrt{-g}K^k\right) d^3x \\
&= \int dr \frac{\partial}{\partial r} \oint \sqrt{-g}K^r d^3x \\
&= \oint \sqrt{-g}K^r d^3x \Big|_{r=0}^{r=r_B} \\
&= \oint \left(\sqrt{-g}K^r\right)\Big|_{r=r_B} d^3y \, .
\end{aligned} \tag{H.0.3}$$

The index k in the second line of equation above represents component of boundary coordinates, which are time angular coordinates θ and ϕ. The boundary ∂V is described by the coordinates y, and it is assumed the value of integrand in the last line of equation above vanishes at $r = 0$.

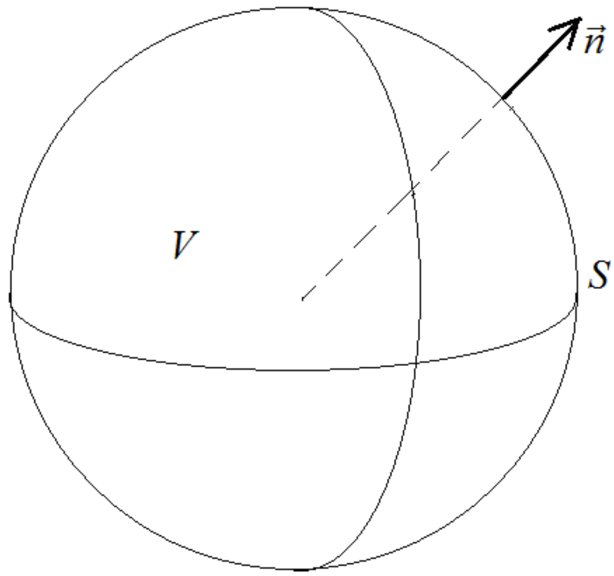

Figure H.1: An illustration of the volume and surface related to the Gauss' theorem.

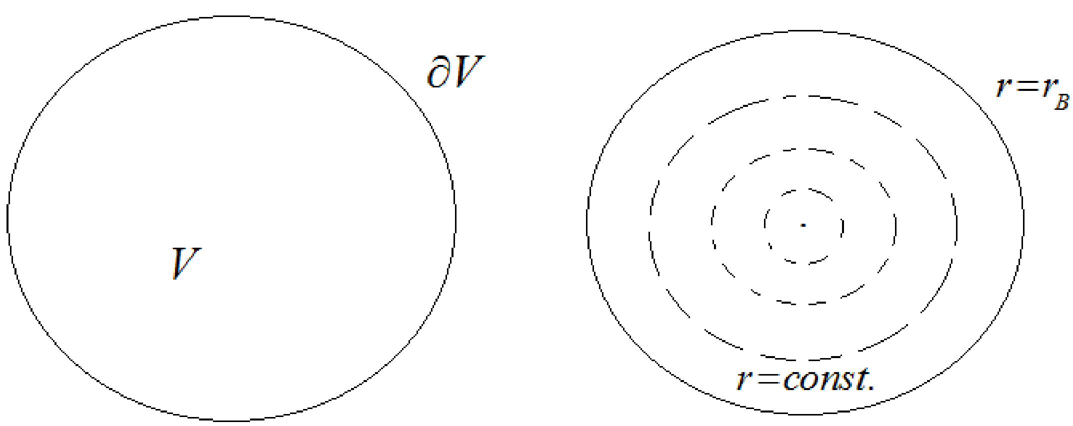

Figure H.2: A sketch of Gauss' theorem proof.