NUMERICAL SOLUTION
OF
TWO-DIMENSIONAL
NAVIER STOKES EQUATIONS

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MURLI MANOHAR GUPTA

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COMPUTATIONAL NOTE

The computations were carried out on an IBM 360/50 computer in the University of Saskatchewan Computation Center. The graphs were prepared on the CalComp Plotter attached with the SDS920 in the Linear Accelerator Laboratory, University of Saskatchewan. A linear interpolation was used for the plotting of data.

ABSTRACT

An attempt has been made to develop a convergent iterative method to obtain two-dimensional steady state solutions of the Navier-Stokes equations. In the last few years, various successful attempts have been made to apply numerical methods to obtain both the time dependent and steady state solutions of the Navier-Stokes equations for different types of boundary configurations. The non-linearity of the differential equations makes the convergence of the iterative method difficult. The limitations on the storage capacity of the available computing machines makes it impracticable to obtain solutions having any predetermined accuracy. In addition, the imposition of the natural boundary conditions which prescribe the velocities on the boundaries, also contributes to the complexity of the problem. A brief survey of some representative work done on the numerical solution of Navier-Stokes equations has been given here.

The Navier-Stokes equations have been reduced to a system of two second order coupled differential equations which are then solved numerically. Some finite difference schemes have recently been suggested which guarantee the convergence of the iterations for solving the two difference equations. However, these schemes require a certain number of parameters which have to be determined by numerical experimentation. Our aim here is to develop a procedure which would be economical and would require a minimum of numerical experimentation.

We have succeeded in showing that only one parameter besides the optimum over-relaxation factors is sufficient for convergence. Different criteria for convergence are studied and the one, which appears to be most efficient, has been used to obtain numerical solutions for the steady flow of a viscous fluid in a two-dimensional cavity flow. Another problem for which the numerical solutions have been obtained is the separating flow due to a thin plate of finite length placed at the point of symmetry of an otherwise two-dimensional stagnation point flow. A non-uniform mesh has been used for this problem and the numerical results for different values of Reynolds number have been obtained. These are given in the form of stream lines and equi-vorticity curves. We have also extended the difference schemes to problem of axisymmetric flows although no numerical results have been obtained.

The results obtained in this thesis may be considered as qualitatively satisfactory in the sense that they exhibit the characteristics of the fluid flow. These results can not be considered quantitatively reliable unless some insight is gained about the convergence of the numerical solution of the discretized problem to that of the differential equations. This can partly be done by repeating the calculations using smaller mesh sizes. However, this would require a considerable amount of computing time. It would be interesting to investigate the possibilities of improving the results using some higher order stable difference schemes or finer mesh sizes in the boundary layer only without unduly increasing the computing time.

PREFACE

The flow of an ideal fluid which is assumed to be frictionless and incompressible has been extensively investigated in classical hydrodynamics. The actual fluids are both viscous and compressible. In spite of this, the ideal fluid theory has succeeded in explaining certain characteristics of the flows such as wave motion, the lift and the induced drag of an airfoil, etc., due to the fact that for many common fluids such as water and air the viscosity is small. In these cases the effects of compressibility are also negligible at speeds which are small compared to the speed of sound. However, this theory fails to explain other phenomena such as no slippage on the surface of a solid body, boundary layer.

The fundamental equations governing the flow of a viscous fluid are the Navier-Stokes equations which are non-linear in character and have exact solutions only when the geometry of the flow is simple e.g. flow between parallel walls, in a circular pipe and the flow between two rotating cylinders. In these cases the non-linear terms drop out so that the resulting equations are linear and solvable. However, solutions for other geometries as well as for exterior flow over a given surface in an unbounded region is not known.

The problem of flow with finite viscosity has been handled by first considering two extreme cases, When the flow is very slow or when the Reynolds number is almost zero, the Navier-Stokes equations are approximated by a biharmonic equation, which is linear and can be solved for certain simple boundary configurations. The other extreme case occurs when the Reynolds number is very large and the terms due to viscosity can be neglected. However,this leads to the reduction of the order of differential

equations and the no-slip boundary condition can not be satisfied. For the flows with very large Reynolds numbers, the viscous effects are confined to a very thin boundary layer and in such cases certain terms in Navier-Stokes equations are neglected which leads to the Boundary Layer Equations. These equations are non-linear and give a good approximation for the flow in the boundary layer. However, the boundary layer theory fails to explain separation and the reverse flow which are often present in the actual flows. For intermediate values of Reynolds number any such simplification is not possible.

Various successful attempts have been made to apply numerical methods to obtain both the time-dependent and the steady state solutions of the Navier-Stokes equations for different types of boundary configurations. The non-linearity of the differential equations requires the use of certain iterative methods and the convergence of these iterative methods makes the problem difficult to handle. Furthermore, the limitations on the storage capacity of the available computing machines makes it impossible to obtain solutions having any predetermined accuracy. In addition, the imposition of the natural boundary conditions, which prescribes the velocities on the boundaries, also contributes to the complexity of the problem. A brief survey of some representative work done on Navier-Stokes equations in the last few years is given in Chapter 1.

In Chapter 2, we have studied the steady flow within a two-dimensional square which represents a model of two-dimensional cavity flow. A numerical method is developed with an aim to minimise the number of arbitrary parameters and to make the process more efficient and automatic. Different convergence criteria are studied and a fast and efficient criterion is proposed.

The numerical procedure developed in Chapter 2 is applied in Chapter 3 to study the separation due to a thin plate of finite length placed at the point of symmetry of a stagnation point flow. This problem was chosen because of various inherent difficulties present in solving it by numerical methods. The flow is physically interesting and the separation of the flow has been observed even at very small Reynolds number, therefore, the boundary layer approximations can not be made. We lose certain properties of symmetry in the discretised problem due to the use of non-uniform mesh. This creates some new problems not present in the cavity problem discussed in Chapter 2.

The numerical results are given in the form of equivorticity curves and stream line curves. In Chapter 4, we have outlined an extension of these methods to axially symmetric cases. The listing of a computer program for the two-dimensional modified stagnation point flow is also appended.

TABLE OF CONTENTS

1. INTRODUCTION

The two-dimensional flow of an incompressible viscous fluid is governed by the following differential equations:

$$u_t + uu_x + vu_y = -\frac{1}{\rho} p_x + \nu(u_{xx} + u_{yy}) \tag{1.1}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho} p_y + \nu(v_{xx} + v_{yy}) \tag{1.2}$$

$$u_x + v_y = 0 \quad , \tag{1.3}$$

where u, v are the components of the velocity, p the pressure, ρ the density and ν the kinematic viscosity. u, v depend upon the position (x,y) and time t while ρ and ν are assumed to remain constant.

The continuity equation (1.3) is satisfied by introducing a stream function Ψ and the vorticity ω such that

$$u = \Psi_y \quad , \quad v = -\Psi_x \quad ; \tag{1.4}$$

$$\omega = v_x - u_y = -(\Psi_{xx} + \Psi_{yy}) \equiv -\Delta\Psi \quad . \tag{1.5}$$

Eliminating pressure between (1.1) and (1.2) we get

$$\omega_t + \omega_x\Psi_y - \Psi_x\omega_y = \nu\Delta\omega \quad . \tag{1.6}$$

The equations (1.5) and (1.6) together with appropriate boundary conditions determine the time dependent viscous flow in terms of stream function and vorticity. These equations can be written in the non-dimensional form by introducing a characteristic length L and a characteristic velocity V_o; and using the same notation from now on for the non-dimensional quantities, we have

$$\omega = - \Delta\Psi \tag{1.7}$$

$$\omega_t + \omega_x \Psi_y - \Psi_x \omega_y = \frac{1}{R} \Delta\omega \ . \tag{1.8}$$

The flow now depends upon a single non-dimensional parameter

$$R = \frac{V_o L}{\nu} \ . \tag{1.9}$$

R is called the Reynolds number of the flow.

Considerable work has been done to obtain the numerical solutions of the above system of non-linear coupled equations for various boundary conditions.

Fromm[1] solved the problem of non-steady viscous fluid flow between parallel walls with periodic end-boundary conditions. The differential equations were discretized by using "Laplace's five point formula" for Δ-operator[2] and the first derivatives in Ψ and ω were replaced by central-differences. The stream function equation was solved by Liebmann's process , and the Dufort-Frankel Scheme[4] was used to compute vorticity at any particular step in time. For the stability of the numerical process the following condition was imposed:

$$\nu \delta t \leq \frac{h^2}{4} \ . \tag{1.10}$$

The flow around certain rectangular obstacles placed between parallel walls was studied using the above methods. The following procedure was adopted for solving the equations:

Starting from given values of Ψ and ω at any particular time t, one obtains ω at time $t + \delta t$ by using the Dufort-Frankel Scheme[4]. The vorticity is calculated on the boundaries using the periodicity condition and the values of the stream function and velocity prescribed

on the boundaries. The stream function is obtained everywhere by using the relaxation procedure. Now, the values of ω on the obstacles are obtained and the wall corrections are applied so that the ω-values satisfy equation (1.7). This completes one time step.

Hellums and Churchill[5] computed the two-dimensional flow and temperature fields for convection problems involving a viscous fluid. They obtained the steady state velocity and temperature fields for natural convection as the limiting transient solutions for large values of times. They used differential equations similar to (1.1 - 1.3) which were discretized using forward differences for the first derivatives and three point central differences for the second derivatives. This yielded explicit equations which gave the values of dependent variables at each time step.

Wilkes and Churchill[6] studied the natural convection of a fluid in a long horizontal enclosure of rectangular cross-section with one vertical side heated and the other cooled. The governing vorticity and energy transport equations were solved by an implicit alternating direction finite difference method . Each time step was divided into two half time steps and the equations used in the successive halves were implicit in x and y, respectively. The vorticity on the boundary points was calculated by using Ψ-values on the boundary and on the first two adjacent mesh points, and also the fact that the velocity was zero on the boundary. The stream function values were computed by using the finite difference form of equation (1.7). Successive over-relaxation (SOR) procedure[2] was used and the SOR parameter was found by numerical experimentation.

The results were numerically unstable for fluids with small

viscosity, i.e., for large Reynolds numbers. The reason given was the inconsistency of time steps between the interior and boundary vorticities which, it was suggested, could be overcome at the expense of iterating several times over each time step.

C.E. Pearson[8] calculated the time-dependent flow for the fluid injection problem in a rectangular cavity. The mesh points were divided into three groups: S_h consisting of the boundary mesh points; $R_{h,1}$ consisting of the first inner ring of points at one mesh distance from the boundary; and, $R_{h,2}$ consisting of all interior mesh points. The stream function and the velocities were prescribed on the boundaries and "this defined the stream function on S_h and $R_{h,1}$ for all times". The values of the vorticity on $R_{h,1}$ were thought of as being the "hinge" values that couple the problem together. The stream function was thus to be calculated on $R_{h,2}$ and the vorticity on $R_h (=R_{h,1} \cup R_{h,2})$. The numerical process at certain time step was started by solving the vorticity equation using an implicit process for diffusion calculations. An over-relaxation procedure was used. After a preset number of such ω-iterations, Ψ in $R_{h,2}$ was obtained by the optimal over-relaxation method and the vorticity on $R_{h,1}$ was obtained from the equation $\omega = - \Delta\Psi$. This completed one time-step.

For higher Reynolds numbers, the implicit difference equation scheme did not give the desired convergence and was replaced by the Peaceman-Rachford difference scheme[9]. The vorticity values on the "hinge" boundary $R_{h,1}$ were smoothed by the use of a weighting factor δ which was used to linearly combine the old and the new values of ω on $R_{h,1}$. With a higher amplitude factor in the fluid injection problem the process did not converge even when 95% of the old values of ω were

used by taking $\delta = 0.95$. At this stage another smoothing factor was introduced on the whole mesh for the vorticity and at each iteration step new vorticity values were obtained as a weighted average of the previous values and those which would have been obtained by an unmodified process.

R.E. Esch[10] attempted a similar problem but the boundary coupling was "hinged" on S_h. It was found that this coupling is more accurate and the errors were considerably lower. Some steady state solutions were also obtained by extending the time-dependent calculations to large times.

Greenspan et al.[11] solved the time-dependent viscous flow problem in a two-dimensional cavity. The first derivatives in the vorticity equation were replaced by central differences. The equations were hinged on the boundary S_h and computations were carried out using a procedure similar to the one used by Pearson[8]. The time derivative ω_t in (1.8) was replaced by

$$\frac{\omega(t + \Delta t) - \omega(t)}{\Delta t} , \tag{1.11}$$

and for stability the following condition was imposed:

$$\alpha = \frac{\Delta t}{R(\Delta x)^2} \leq \tfrac{1}{4} . \tag{1.12}$$

It was empirically found that the finite difference approximation of (1.8) yielded unstable schemes for $\alpha > \frac{1}{4}$. Moreover, for large R this finite difference replacement was expected to be unstable for all Δt. The mesh sizes used were $\frac{1}{4}$, 1/8, 1/16 and it was found that the convergence is slow and unsatisfactory as the mesh size is reduced to 1/16 for R = 64. The rate of convergence became slower as R was increased.

The time dependent problem for a viscous flow past a circular cylinder was attempted by Payne[12] who obtained the solutions at R = 40 and 100. He used the velocity components and the vorticity as the dependent variables. Kawaguti and Jain[13] attempted the same problem but used the stream function and the vorticity as dependent variables. At each time step Ψ was calculated using SOR procedures and ω was calculated using the non-linear equations. At R = 1, the time step had to be smaller than 0.002 and for R = 20, it had to be smaller than 0.05. The steady state solutions were obtained for R = 40 and the machine time required was two hours on CDC 3600. At higher Reynolds numbers, numerical instability was encountered. Various other attempts by Apelt[14], Harlow and Welch[15], Simuni[16] and others [see 14, 15] have also been made to solve these equations using variations of the schemes discussed above.

The time-dependent Navier-Stokes equations have generally been chosen in order to obtain the development of the flow with time and also with a hope that for large values of time, t, the solution would converge to the steady state solution. For obtaining steady state solutions one could interpret time as a parameter in the iteration process. However, this is not an efficient method if the interest is only to obtain the steady state solutions, since this introduces added complications of stability due to the time-derivatives. The time-step used has to be sufficiently small which again is restricted by the available computing machines. Thus, time-independent Navier-Stokes equations must be used if only steady state solutions are required. In this thesis we shall be concerned with steady state solutions only. The equations governing the steady state flow of a viscous incompressible fluid in their non-dimensional form are:

$$\Delta\Psi = -\omega \tag{1.13}$$

$$\Delta\omega + R(\Psi_x.\omega_y - \Psi_y.\omega_x) = 0 \ . \tag{1.14}$$

The problem of separated flow in a two-dimensional cavity has attracted considerable attention because of the simplicity of the model and boundedness of the region of integration. Kawaguti[17] obtained the solution of this model problem for various values of Reynolds numbers. The velocity components and the stream function had zero values on the three solid walls, whereas, on the moving wall, $\Psi = 0$, $\Psi_n = -1$. The ratio of the depth of the cavity to its length was β. The equations (1.13) and (1.14) were discretized and following difference equations were obtained:

$$4\Psi_o = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + h^2\omega_o \tag{1.15}$$

$$4\omega_o = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \frac{R}{4}\left[(\Psi_1-\Psi_3)(\omega_2-\omega_4) - (\Psi_2-\Psi_4)(\omega_1-\omega_3)\right] \ . \tag{1.16}$$

The subscripts 0, 1, 2, 3, 4 denote the functional values at the points (x,y), (x+h,y), (x,y+h), (x-h,y) and (x,y-h) respectively. The numerical procedure was started with certain initial values for Ψ, ω. The vorticity was calculated on the boundary using the values of Ψ, Ψ_n. Then Ψ, ω were corrected in the interior by using (1.15) and (1.16) which completed one cycle of iterations. For the square cavity (β=1) convergence was achieved for Reynolds numbers up to 64 with the mesh size 0.1. For R = 128, the process did not converge. In 1965, Kawaguti[18] attempted the solution of Navier-Stokes equations for the flow in a channel with a step. The numerical procedure was modified by introduction of a "relaxation factor" in all the difference equations used.

O.R. Burggraf[19] studied the structure of steady separated flows with special application to the cavity problem, both analytically

and numerically. The difference approximations were the same as those used by Kawaguti[17] except that a relaxation factor K was introduced in the difference analogue of the Ψ-, and ω-equations (1.13) and (1.14). It was claimed that higher order difference operators did not improve the results significantly at R = 0, and for large Reynolds numbers this increased the computational time considerably.

Since Kawaguti's work[17] showed that K = 1 did not give stable solutions, Burggraf decided that the system is not well-behaved and so "over relaxation" was not used. Hence $K < K^* < 1$ was prescribed for stability. The bounds on K^* were determined as functions of the Reynolds number by numerical experimentation. The iteration was terminated when the error between successive Ψ-iterates became less than a preassigned tolerance ε. As the Reynolds number was increased the number of iterations increased drastically. The maximum time allowed was 30 minutes which permitted accurate solutions to be obtained for $R \leq 200$ on IBM 7094. Oscillations in the solution were encountered for R = 400 and for higher Reynolds number, the numerical process diverged.

In 1963, D.M. Dix[20] solved the problem of the flow of a two-dimensional viscous incompressible magnetic fluid past a flat plate. The equations of motion were similar to the equations (1.13) and (1.14) except that extra terms due to magnetic effects were introduced. The numerical procedure was started with some initial values of Ψ and ω and equation (1.13) was solved (inner iterations for Ψ) to obtain the stream function. The vorticity values on the boundary were obtained by using the current value of Ψ and the value of Ψ_n. This vorticity was linearly combined with the previous ω-value to yield the ω-approximation on the boundary for the current over-all iteration step (outer iteration). The non-linear vorticity equation (1.14) was now solved

to obtain the new values of vorticity (inner iterations for ω). This completed one outer iteration. The process was repeated and was stopped when outer iterations had converged. The point relaxation was not convergent for the inner iterations so line relaxation was used. The weighting factor δ used to combine the vorticity values on the boundary was found by numerical experimentation and was such that more than 80% of the previous ω-values had to be used ($\delta = 0.8$) to achieve convergence.

In all these cases we have two types of convergence problems involved. The first requirement is that the inner iterations for Ψ, ω must be convergent. This in itself does not guarantee that the outer iterations will converge. In fact, for the problem of a biharmonic equation, J. Smith[21] has proved that the outer iterations do not converge if the mesh size is reduced beyond a certain fixed value. The same situation is true for equations (1.13) and (1.14). Therefore, in order to obtain convergence of outer iterations it is necessary to use some relaxation factors.

Greenspan[22] studied the problem of the flow in a two-dimensional model of a square cavity and developed a discretization technique for the vorticity equation which yields a diagonally dominant system of algebraic equations for any Reynolds number. This guarantees the convergence of inner iterations at least by using the Gauss-Seidel method[23]. The mesh was divided into three groups: S_h, $R_{h,1}$ and $R_{h,2}$. We have described this division earlier. The stream function and the velocities are prescribed on the boundaries and the Ψ-values on $R_{h,1}$ are extrapolated from the Ψ-values in $R_{h,2}$, using the given values of the normal derivatives for Ψ on S_h. The values of vorticity on the boundary S_h are computed from $\omega = - \Delta\Psi$ using the values of Ψ and the values of Ψ_n on S_h.

The numerical procedure was started with some initial values of Ψ, ω and the Ψ-system was iterated using a "modified" successive over-relaxation method. The modification is that after each sweep on $R_{h,2}$ a weighted average of the calculated values and the old values is taken as the new value of Ψ. After Ψ converges on $R_{h,2}$ the values on $R_{h,1}$ are computed by previously developed formulas. Next, the vorticity values on the boundary were computed using suitable finite difference formulas. Finally, a weighted average with the old values of ω was taken as the new boundary value of ω. The vorticity equation was then iterated on R_h using a similar "modified" SOR procedure. The weight δ used was the same as that used on the boundary. When ω converges to a preassigned tolerance in its inner iterations, one cycle of outer iterations is completed and the process is repeated till the outer iterations converge. A tolerance of 10^{-4} was prescribed for all inner and outer iterations. The results were obtained for various Reynolds numbers with the mesh size 0.05 using certain fixed relaxation parameters and weight factors. However, for R = 100 the convergence could not be achieved in 12 minutes on CDC 3600 when 40 outer iterations were completed.

Weiss and Florsheim[24] obtained the solutions of slow viscous fluids past a cavity. They assumed that the dividing stream line is coincident with the mouth of the cavity of finite depth. Takematsu[25] used a cavity of infinite depth and showed that the dividing stream line is not coincident with the cavity mouth but it penetrates to a considerable depth into the cavity.

In 1967, Julius Smith[21] studied the convergence problem of a biharmonic equation which also corresponds to the Navier-Stokes equations for $R \to 0$. This equation was split into two second order Poisson equations

which were discretized and solved independently by convergent iterative methods. Smith proved that although the iterative method for solving each of the Poisson equations may be chosen to be convergent, the iterations for successively solving the overall system diverge as the mesh size is decreased. However, he also proved that the overall iterations converge when a "relaxation" parameter is introduced in the outer iterations. Some of the important results of this work are given below:

The biharmonic equation is

$$\Delta\Delta u = f \ . \tag{1.17}$$

The values of u, u_n are prescribed on the boundary of the domain. The equation (1.17) is split into two Poisson equations

$$\Delta u = v \ , \tag{1.18a}$$

$$\Delta v = f \ . \tag{1.18b}$$

These equations are discretized and the basic iteration is defined as follows:

$$LV_{m+1} + 2h^{-2}MU_m = h^2F + h^{-2}D_2 \tag{1.19a}$$

$$LU_{m+1} = h^2V_{m+1} + D_1 \ , \tag{1.19b}$$

which on eleminating V_{m+1} leads to

$$U_{m+1} = HU_m + L^{-2}D \quad , \quad H = -L^{-2}M \ . \tag{1.20}$$

U, V are discrete approximations of the functions u, v respectively. D, D_1, D_2 depend upon the boundary conditions and L, M are certain matrices associated with the block iteration of the discrete Poisson

equations. It is proved that

$$\tau_h = \rho(L^{-2}M) = (h\sigma_h)^{-1} , \tag{1.21}$$

where $\sigma_h \to \sigma$ as $h \to 0$; $0 < \sigma < \infty$. Thus

$$\rho(H) = 2(\sigma h)^{-1} \qquad \text{as } h \to 0 ,$$

and the basic iteration scheme (1.20) diverges as $h \to 0$. A relaxation factor δ is introduced in the basic iteration so that the modified iteration becomes:

$$LV_{m+1} + 2h^{-2}M\bar{U}_m = h^2F + h^{-2}D_2 \tag{1.22a}$$

$$LU_{m+1} = h^2V_{m+1} + D_1 \tag{1.22b}$$

$$\bar{U}_{m+1} = \delta \cdot U_{m+1} + (1 - \delta)\,\bar{U}_m , \tag{1.22c}$$

which gives

$$\bar{U}_{m+1} = \bar{H}\bar{U}_m + \omega L^{-2}D , \tag{1.23}$$

$$\bar{H} = \delta H + (1 - \delta)\, I . \tag{1.24}$$

It is proved that, for

$$\delta = (1 + \tau_h)^{-1} ,$$

$$\rho(\bar{H}) \leq \tau_h\,(1 + \tau_h)^{-1} \sim 1 - \sigma h \qquad \text{as } h \to 0 ,$$

which shows that the modified iteration converges.

Some particular cases of the inner iteration matrices L, M were also considered in this study and rates of convergence were derived for the cases when Jacobi or Richardson's method is used for inner iterations. A Chebyshev scheme for the overall (outer) iteration was also discussed.

Although the above work was limited to the solution of biharmonic equations, we expect similar results to hold for Navier-Stokes equations. This is evident from the difficulties encountered by Kawaguti, Burggraf and others which were attributed by them to high Reynolds number. Most of these studies have failed to recognize that this difficulty prevails even at small Reynolds number provided the mesh size is sufficiently reduced. So far most of the attempts were directed towards obtaining some qualitative information about the steady state solutions using a crude mesh size. The problem of convergence with the mesh size tending to zero was not considered.

Even with a crude mesh size, problems of convergence were encountered for large Reynolds number of the flow and it was natural to attribute this difficulty to the Reynolds number. Whenever the numerical procedure started diverging, it became customary to introduce some kind of relaxation parameter with the sole aim of obtaining convergence of the iterative process. Invariably, these parameters were chosen arbitrarily and this choice was based on the experience with the problem while the actual values of these parameters for a particular problem are obtained by a large number of time consuming numerical experiments. It is the aim of this study to use some convergent procedures for the inner iterations while the divergence of the outer iterations is controlled by a minimum number of parameters.

2. A MODEL OF TWO-DIMENSIONAL CAVITY FLOW

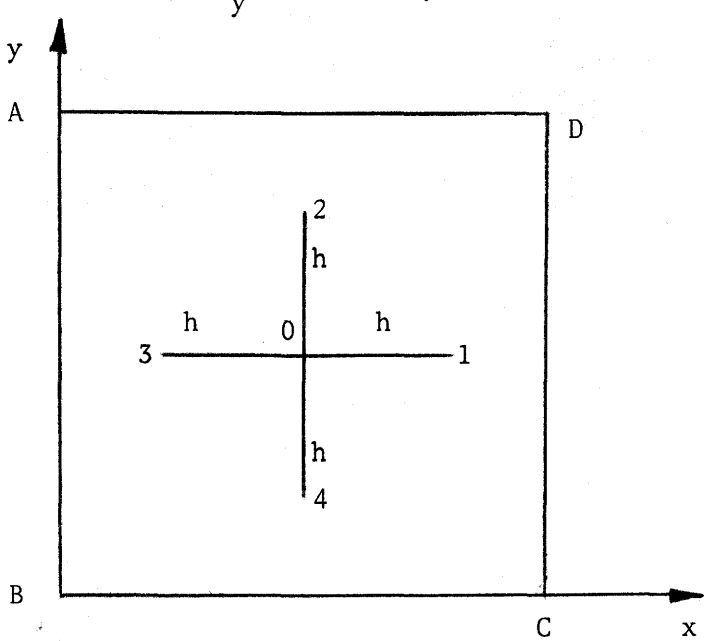
We consider the model of the steady flow of an incompressible viscous fluid within a square cavity. The equations governing the motion in non-dimensional form are:

$$\Delta\Psi = -\omega \tag{2.1}$$

$$\Delta\omega + R\,(\Psi_x\omega_y - \omega_x\,\Psi_y) = 0\ . \tag{2.2}$$

where $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, R is the Reynolds number. The velocities and the stream function are prescribed on the boundaries of the square ABCD (fig. 1). Taking the coordinate axes as shown, the boundary conditions are given by:

$$\begin{aligned} &\Psi = 0\ ,\ \Psi_x = 0 \quad \text{on } x = 0\ ,\ x = 1 \\ &\Psi = 0\ ,\ \Psi_y = 0 \qquad y = 0 \\ &\Psi = 0\ ,\ \Psi_y = -1 \qquad y = 1 \end{aligned} \tag{2.3}$$



(fig. 1)

The square ABCD is covered by a set of equidistant mesh points. The mesh distance is h. The mesh points are grouped into three sets as follows:

S_h - Consisting of all the mesh points on the boundary,

$R_{h,1}$ - Consisting of the first inner ring of mesh points at one mesh distance from the boundary,

$R_{h,2}$ - Consisting of the remaining interior mesh points.

We order the mesh points on each row from left to right, the rows being taken in order from the bottom to the top. We denote the value of any function f at the five points (x, y), (x + h,y), (x,y + h), (x - h,y), (x,y - h) by f_0, f_1, f_2, f_3 and f_4 respectively. The finite-difference analogue of the Poisson equation is given by the five point formula:

$$\Delta_h \Psi_0 \equiv h^{-2} (-4\Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4) = -\omega_0 . \qquad (2.4)$$

The error of this discretization is

$$\Delta\Psi - \Delta_h \Psi = \frac{h^2}{12} (\Psi_{xxxx} + \Psi_{yyyy}) + 0(h^4) .$$

The non-linear vorticity equation (2.2) is discretized according to the technique proposed by Greenspan[22]. The finite difference analogue of $\Delta\omega$ is

$$h^{-2} (-4\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4) .$$

The derivatives Ψ_x, Ψ_y are approximated by

$$\Psi_x = \frac{\alpha}{2h} \quad , \quad \alpha = \Psi_1 - \Psi_3 \; ; \qquad (2.5a)$$

$$\Psi_y = \frac{\beta}{2h} \quad , \quad \beta = \Psi_2 - \Psi_4 . \qquad (2.5b)$$

For the ω-derivatives we use

$$\omega_x = \begin{cases} \dfrac{\omega_2 - \omega_0}{h} & \alpha \geq 0 \\ \dfrac{\omega_0 - \omega_4}{h} & \alpha < 0 \end{cases} , \tag{2.6a}$$

$$\omega_y = \begin{cases} \dfrac{\omega_0 - \omega_3}{h} & \beta \geq 0 \\ \dfrac{\omega_1 - \omega_0}{h} & \beta < 0 \end{cases} . \tag{2.6b}$$

Substituting these finite differences in (2.2) we obtain a system of linear algebraic equations defined for each mesh point in R_h $(= R_{h,1} \cup R_{h,2})$. This system has a real coefficient matrix which is irreducible and diagonally dominant for any Reynolds number. The vorticity equation has a discretization error given by

$$\pm \frac{Rh}{2} (\Psi_x \omega_{yy} \pm \Psi_y \omega_{xx}) - \frac{Rh^2}{12} (\Psi_{xxx} \omega_y - \Psi_{yyy} \cdot \omega_x) - \frac{h^2}{12} (\omega_{xxxx} + \omega_{yyyy}) + 0 (h^3) .$$

We approximate the derivative Ψ_n at the point (x,y) on the boundary S_h by using Ψ-values at the three points (x, y), $(x + h, y)$, $(x + 2h, y)$ as follows:

$$(\Psi_x)\ (x, y) = \alpha_0 \Psi\ (x, y) + \alpha_1 \Psi\ (x + h, y) + \alpha_2 \Psi\ (x + 2h, y)$$

$$= \alpha_0 \Psi + \alpha_1 \left(\Psi + h\Psi_x + \frac{h^2}{2} \Psi_{xx} + \ldots\right)$$

$$+ \alpha_2 \left(\Psi + 2h\Psi_x + \frac{4h^2}{2} \Psi_{xx} + \ldots\right) .$$

On comparing coefficients, we obtain

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + 2\alpha_2 = h^{-1}$$

$$\alpha_1 + 4\alpha_2 = 0$$

which gives

$$\alpha_0 = -1.5\, h^{-1} \quad , \quad \alpha_1 = 2h^{-1} \quad , \quad \alpha_2 = -0.5h^{-1} \quad ,$$

so that

$$\Psi_x \cong \frac{1}{2h} [-3\Psi(x, y) + 4\Psi(x + h, y) - \Psi(x + 2h, y)] \quad .$$

The discretization error of this approximation is

$$-\frac{h^3}{3} \Psi_{xxx} + 0(h^4) \quad .$$

On the boundary AB of the cavity $\Psi = \Psi_x = 0$, and hence

$$\Psi(x + h, y) \cong \frac{1}{4} \Psi(x + 2h, y) \quad . \tag{2.7a}$$

Similarly, on the boundary BC, $\Psi = 0 = \Psi_y$ so that

$$\Psi(x, y + h) \cong \frac{1}{4} \Psi(x, y + 2h) \quad . \tag{2.7b}$$

On the boundary CD, $\Psi = \Psi_x = 0$ which yields

$$\Psi(x - h, y) \cong \frac{1}{4} \Psi(x - 2h, y) \quad . \tag{2.7c}$$

Finally, on the boundary DA, $\Psi = 0$; $\Psi_y = -1$, therefore

$$\Psi(x, y - h) \cong \frac{h}{2} + \frac{1}{4} \Psi(x, y - 2h) \quad . \tag{2.7d}$$

We obtain the vorticity approximations on the boundary S_h by using the values of Ψ, Ψ_n at the adjacent points. On AB ($x = 0$) we have $\Psi_x = \Psi = 0$ so that

$$(\Psi_x)_{x=0} \cong \frac{\Psi(h, y) - \Psi(-h, y)}{2h} = 0$$

$$\text{i.e.,} \qquad \Psi(-h, y) = \Psi(h, y) \quad .$$

From the equation (2.4) we have

$$\omega(0, y) = -(\Delta_h \Psi)_{x=0}$$

$$= -h^{-2}[-4\Psi(0, y) + \Psi(h, y) + \Psi(0, y + h) + \Psi(-h, y) + \Psi(0, y - h)]$$

$$= -2h^{-2}\Psi(h, y)$$

$$\text{i.e.} \qquad \omega(0, y) = -2h^{-2}\Psi(h, y) \quad . \tag{2.8a}$$

Similarly for the boundaries BC and CD we obtain

$$\omega(x, 0) = -2h^{-2}\Psi(x, h) \quad . \tag{2.8b}$$

$$\omega(1, y) = -2h^{-2}\Psi(1 - h, y) \quad . \tag{2.8c}$$

On the boundary DA ($y = 1$), $\Psi_y = -1$ which gives

$$\frac{\Psi(x, 1 + h) - \Psi(x, 1 - h)}{2h} = -1$$

$$\text{i.e.} \qquad \Psi(x, 1 + h) = \Psi(x, 1 - h) - 2h$$

which leads to,

$$\Psi(x, 1) = 2h^{-1} - 2h^{-2}\Psi(x, 1 - h) \quad . \tag{2.8d}$$

NUMERICAL PROCEDURE

The discretization of the Poisson equation (2.1) yields a linear algebraic system of equations which has a real, symmetric, diagonally dominant coefficient matrix. The non-linear vorticity equation yields a coefficient matrix which is, in general, non-symmetric and diagonally dominant.

A real n x n matrix $A = (a_{ij})$ is said to be symmetric if $a_{ij} = a_{ji}$. A is singular if the determinant of A is zero. A is said to be reducible if there exists a permutation matrix P such that

$$P A P^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} , \tag{2.9}$$

where A_{11} is a r x r submatrix and A_{22} is an (n-r) x (n-r) submatrix, $1 \leq r \leq n$. If no such permutation matrix exists then A is said to be irreducible. A is diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad , \quad 1 \leq i \leq n \quad . \tag{2.10}$$

A is irreducibly diagonally dominant if A is irreducible and diagonally dominant with strict inequality in (2.10) for at least one i. A is said to have property (A) if there exists a permutation matrix P such that $P A P^T$ is diagonally block-tridiagonal[2]. The spectral radius $\rho(A)$ of A is defined by

$$\rho(A) = \max_i |\lambda_i| \quad , \tag{2.11}$$

where λ_i are the eigenvalues of A.

We state the following well known results[2,3]:

Theorem: Let $A = (a_{ij})$ be an n x n irreducibly diagonally dominant matrix. Then the matrix A is non-singular. If all the diagonal entries of A are positive real numbers, then the eigenvalues of A satisfy

$$\mathrm{Re}\ \lambda_i > 0 \quad , \quad 1 \leq i \leq n \quad .$$

Consider the matrix equation

$$Ax = k \quad , \tag{2.12}$$

where k is a given column vector.

A can be expressed as the matrix sum

$$A = D - E - F \quad ,$$

where $D = \text{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\}$ and E, F are respectively strictly lower and upper triangular matrices. The matrix equation (2.12) can be solved using iterative procedures one of which is the point Jacobi method defined by

$$X^{(m+1)} = Bx^{(m)} + D^{-1}k \quad , \quad B = D^{-1}(E + F) \quad . \tag{2.13}$$

The matrix B is called the point Jacobi matrix. Another iterative procedure is point successive over-relaxation (SOR) which is defined by

$$X^{(m+1)} = T_{\omega}x^{(m)} + \omega (I - \omega L)^{-1} D^{-1}k \quad , \tag{2.14}$$

$$T_{\omega} = (I - \omega L)^{-1}\{(1 - \omega) I + \omega U\} \quad , \quad L \equiv D^{-1} E \quad , \quad U \equiv D^{-1} F \quad . \tag{2.15}$$

The point SOR matrix T_{ω} corresponds to the relaxation factor ω. The special case $\omega = 1$ is called the method of successive displacements (Gauss-Seidel method) and the G-S matrix is given by

$$M = T_1 = (I - L)^{-1} U \quad . \tag{2.16}$$

Theorem: Let $A = (a_{ij})$ be an irreducibly diagonally dominant n x n matrix. Then both the associated point Jacobi and the point Gauss-Seidel iterations for (2.12) are convergent for any initial vector approximations.

This theorem assures that successive over-relaxation procedures will be convergent for the algebraic systems obtained from discretization of (2.1) and (2.2) at least for $\omega = 1$. Moreover, from the following theorem

due to Kahan[3] this relaxation factor is restricted to the open interval (0,2).

Theorem: If $\rho\ (T_\omega)$ is the spectral radius of the SOR matrix T_ω, then for all real ω,

$$\rho\ (T_\omega) \geq |\omega - 1| \quad , \tag{2.17}$$

with equality only if all eigen values of T_ω are of modulas $|\omega - 1|$.

When A is symmetric, the spectral radius of T_ω can be minimized by using an optimal value of the relaxation factor ω. Since the ordering of mesh points used for the square region ABCD is a consistent order [26, page 107], this optimum value of ω can be obtained using the following results due to Young[7]:

Theorem: The relaxation parameter ω which gives the least spectral radius of T_ω for any consistent ordering is independent of the ordering and is given by

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \mu^2(B)}} \quad , \tag{2.18}$$

where $\mu(B)$ is the dominating eigenvalue of the point Jacobi matrix B.

When A is non-symmetric, (2.18) is not applicable and the optimum relaxation factor has to be estimated using numerical experimentation.

The numerical procedure is started with some initial values of stream function and vorticity. The stream function equations are iterated using a successive over-relaxation procedure defined by:

$$\Psi_0^{(j)} = (1 - r_\Psi)\ \Psi_0^{(j-1)} + \frac{r_\Psi}{4} (\Psi_1^{(j-1)} + \Psi_2^{(j-1)} + \Psi_3^{(j)} + \Psi_4^{(j)} + h^2\omega_0) \quad , \tag{2.19}$$

where r_Ψ is the over-relaxation factor, superscripts denote the number of the

iteration cycle and subscripts have their usual meaning. The sweeps are carried out in $R_{h,2}$ and these inner iterations are continued until the error between the successive Ψ-inner iterates becomes less than a certain tolerance ε_Ψ or the number of inner-iterations exceed a certain number N_Ψ. The values $(\varepsilon_\Psi, N_\Psi)$ need not be fixed beforehand.

Ψ is now computed on $R_{h,1}$ using the formulas (2.7a - 2.7d) developed earlier which defines Ψ on the whole mesh. The vorticity $\overline{\omega}$ is now computed on the boundary S_h using the formulas (2.8a - 2.8d) which are then smoothed by a linear combination with the old values of ω on S_h. The following weighted average formula is used:

$$\omega_{S_h} = \delta \cdot \omega_{old} + (1 - \delta)\ \overline{\omega}_{new} \quad , \tag{2.20}$$

where δ is to be found by numerical experimentation.

Now ω is computed on R_h using a SOR procedure defined by

$$\omega_0^{(j)} = (1 - r_\omega)\ \omega_0^{(j-1)} + \frac{r_\omega}{k_0}\left(k_1 \cdot \omega_1^{(j-1)} + k_2 \cdot \omega_2^{(j-1)} + k_3 \cdot \omega_3^{(j)} + k_4 \cdot \omega_4^{(j)}\right) \tag{2.21}$$

where r_ω is a proper over-relaxation factor and k_i are the coefficients of ω_i obtained by the substitution of (2.5), (2.6) into (2.2) which gives the following finite difference analogue of (2.2):

$$k_0\omega_0 = k_1\omega_1 + k_2\omega_2 + k_3\omega_3 + k_4\omega_4 \quad . \tag{2.22}$$

The inner iterations for ω are continued until either the error between successive ω-inner iterates becomes less than a certain tolerance ε_ω which is not pre-assigned, or until the number of ω-inner iterates exceed a certain number N_ω. The selection of these parameters is discussed later on.

Thus one cycle of overall (outer) iterations is completed. The outer iterations are continued until either the error between successive

outer iterates becomes less than ε or the number of the outer iterations exceed the prescribed maximum number N. ε is pre-assigned and depends upon the accuracy required.

Thus we use r_ψ, ε_ψ, N_ψ for the ψ-inner iterations; r_ω, ε_ω, N_ω for the ω-inner iterations and δ, ε, N for the outer iterations. The value of r_ψ can be obtained from (2.18) where $\mu(B)$ is unknown. For the Laplace operator in a square region of unit length, we have [2, page 230]

$$\mu(B) = \cos \pi h \quad , \tag{2.23}$$

where h is the mesh size.

Alternatively μ can be estimated as follows:

We iterate (2.19) with a value $\tilde{\omega}$ of r_ψ smaller than the anticipated optimum value and find the dominant eigenvalue η of this iteration matrix using the Power method[26].

$$\eta = \frac{|| \psi_i^{(n+1)} - \psi_i^{(n)} ||}{|| \psi_i^{(n)} - \psi_i^{(n-1)} ||} \quad , \tag{2.24}$$

where the subscripts denote a component of the ψ-vector and the superscripts denote the order of the iterates. When these ratios become constant, $\mu(B)$ is computed from

$$\mu(B) = \frac{\eta + \tilde{\omega} - 1}{\tilde{\omega} \sqrt{\eta}} . \tag{2.25}$$

The relaxation factor for the non-symmetric diagonally dominant ω-system (2.22) is a function of the mesh size h and the Reynolds number R. Since the ordering is consistent and the matrix of this system has property (A), the dominating eigenvalue of this system is related to the corresponding Jacobi eigenvalue by equation (2.25); but the Jacobi eigenvalues may no longer remain real and hence (2.18) can not be used to obtain the optimum relaxation

factor r_ω. However, for small Reynolds numbers the symmetry of the iteration matrix is not completely destroyed and we iterate with $r_\omega = 1$ for some values of Ψ. The dominant eigenvalue η is obtained from (2.24), and (2.25), (2.18) are used to obtain the optimum relaxation factor $(r_\omega)_{opt}$. For large Reynolds numbers this analysis gives a value which is higher than the optimum relaxation factor. In such cases a better value is obtained by iterating (2.21) using different values of r_ω. Estimates of dominant eigenvalue are obtained for different values of r_ω and the factor which gives the smallest dominant eigenvalue is taken as an estimate of the optimum over-relaxation factor for ω. This value of r_ω may not remain optimum at later stages. However, when this happens we can obtain the new optimum value by a small reduction in the value of r_ω.

Thus the values of the over-relaxation parameters r_Ψ, r_ω can be calculated before starting the numerical procedure. Successive under-relaxation (i.e. $0 < r_\omega < 1$) was not used for the ω-iterations since it can be shown[29] that successive under-relaxation is very effective when all the Jacobi eigenvalues are pure imaginary, i.e., when the coefficient matrix of the ω-system is skew-symmetric. In such cases a formula for obtaining the optimal under-relaxation parameter can also be obtained.

There are five more parameters ε_Ψ, ε_ω, N_Ψ, N_ω, δ which are to be found by numerical experimentation. These experiments are costly and the numerical procedure should be such that a minimum number of these parameters are required to be calculated experimentally. The other parameters should be defined by the process itself. Some of the parameters viz. ε, N, N_Ψ, N_ω, ε_Ψ and ε_ω are not difficult to prescribe. In fact, they will depend upon the accuracy required. The only parameter whose correct value has to be obtained by a considerable amount of numerical experimentation is δ.

RESULTS

Numerical work was started with h = 0.125, R = 10* for which Greenspan[22] achieved convergence using $\Psi \equiv 0$, $\omega \equiv 0$ initially and $r_\Psi = 1.8$, $r_\omega = 1.0$, $\delta = 0.0$. The same data was used with $\varepsilon_\Psi = \varepsilon_\omega = 10^{-4}$ and arbitrarily large numbers prescribed for N_Ψ, N_ω. At the end of 10 outer iterations, convergence to 0.0021 was achieved when 595 inner iterations had been used. In the next experiment N_Ψ, N_ω were restricted to 30 and the outer iteration residual was 0.0017 at the 10th outer iteration, when 465 inner iterations were used. The outer convergence to 10^{-4} was obtained in 17 outer iterations when 670 inner iterations had been used. This showed that allowing Ψ, ω to converge to 10^{-4} in the inner iterations was more time-consuming and unnecessary. The optimum relaxation factors were determined as $r_\Psi = 1.45$, $r_\omega = 1.445$ and the above experiments repeated with $\varepsilon_\Psi = \varepsilon_\omega = 10^{-4}$. When N_Ψ, N_ω were reduced from 30 to 10, the convergence rate was increased considerably. However, when these numbers were further reduced to 1, oscillations were encountered which showed that these numbers should be greater than 1 and that some other type of convergence criterion was necessary. Some experiments were performed where inner iteration tolerances ε_Ψ, ε_ω depended upon the residue R_{max} in the current outer iterations. Some of the results are:

ε_Ψ	ε_ω	Total Inner Iter.	Total Outer Iter.
$R_{max}/10$	$R_{max}/10$	145	10
$R_{max}/10$	R_{max}	107	10
R_{max}	R_{max}	89	13

* Results have been obtained for Reynolds numbers lying between 10 - 100000, and it is assumed throughout that the flow remains laminar.

Some other results obtained with different values of δ are as follows:

$\varepsilon_\Psi = R_{max}/10$; $\varepsilon_\omega = R_{max}$

δ	total inner iter.
0.0	107
0.1	100
0.2	96
0.3	88
0.4	71 ;

$\varepsilon_\Psi = R_{max}$; $\varepsilon_\omega = R_{max}/10$

δ	total inner iter.
0.0	140
0.1	153
0.2	157
0.3	171
0.4	172

The above results confirm that it is neither necessary nor efficient to let the inner iterations for ω converge to the same order of accuracy as the inner iterations for Ψ. In fact, the rate of convergence of the outer iterations is retarded if we allow the ω-inner iterates to converge to a predetermined tolerance at each stage.

The optimum relaxation factors for Ψ were computed from (2.18) to be

$h = 0.1 \qquad r_\Psi = 1.48$

$h = 0.05 \qquad r_\Psi = 1.715$

$h = 0.025 \qquad r_\Psi = 1.85$.

The estimates of optimum relaxation factors for ω for various Reynolds numbers were:

h = 0.1	R =	10	r_ω = 1.54
		100	1.4
		1000	1.0
h = 0.05	R =	10	r_ω = 1.728
		100	1.3
		1000	1.0
h = 0.025	R =	10	r_ω = 1.82

For R = 100, h = 0.05 some experiments were performed with different tolerances ε_Ψ, ε_ω and numbers N_Ψ, N_ω. The results for outer convergence up to 10^{-4} are as follows:

	ε_Ψ	ε_ω	N_Ψ	N_ω	δ	Total inner iter.
(1)	10^{-4}	10^{-4}	*	*	0.6	473
(2)	10^{-4}	10^{-4}	20	20	0.6	316
(3)	10^{-4}	R_{max}	*	*	0.7	366

(* - These numbers prescribed were arbitrarily large so that the inner convergence criterion is satisfied.)

In the first case an excessive number of ω-iterations had to be performed while in the second case the parameters N_Ψ, N_ω had to be prescribed which can only be done experimentally. However, in the third case ω is required to converge to R_{max} which defines ε_ω during the computations automatically. The Ψ-system is allowed to converge to 10^{-4} and this convergence does not necessitate a large number of inner iterations. Thus the convergence criterion (3) is certainly better than the other two criteria.

Another convergence criterion is to use $\varepsilon_\psi = \varepsilon_\omega = R_{max}$ which worked well with R = 10 and the convergence was achieved with $\delta = 0.2$, 0.4. This criterion was also used to compute the solutions for R = 10 as the mesh size is reduced from 0.05 to 0.025. The solutions obtained with h = 0.05 were used as initial approximations and convergent solutions to 0.5×10^{-4} were obtained. The time taken on an IBM 360/50 computer was about 45 minutes when 40 outer iterations were completed. All the results reported so far are for the mesh sizes 0.1 or 0.05. The time taken for R = 10 with h = 0.05 was reported by Greenspan[22] to be 10 minutes on CDC 3600. We used h = 0.025 in order to estimate the time required and also to study the convergence to the exact solution as $h \to 0$.

For higher Reynolds numbers convergent solutions were obtained with h = 0.1 and $\delta > 0$. When the step size was reduced to 0.05 some convergence problems were encountered. However, these could be overcome by choosing a suitable value of δ to dampen the vorticity values on the boundary S_h. The convergence criterion used was $\varepsilon_\psi = 10^{-4}$, $\varepsilon_\omega = R_{max}$. The total number of inner iterations and the execution time taken for solutions to converge to 10^{-4} in outer iterations for various Reynolds numbers are given below:

R	Total Inner Iter.	Outer Iter.	Time Taken (Approx.)
10	366	10	4.0 Minutes
100	466	14	8.0
1000	816	20	8.5
10000	766	13	8.0

The experiments of Greenspan[22] were repeated for some Reynolds numbers to compare the computer time taken. He used a "modified" SOR technique

for inner iterations using 0.1, 0.7 as the weights. Fixed relaxation parameters $r_\Psi = 1.8$, $r_\omega = 1.0$, $\delta = 0.7$ were used and the inner iterations were required to converge to 10^{-4}. Moreover, convergence could not be achieved for R = 100 within 12 minutes when 40 outer iterations had been completed. The results of these repeated experiments are:

R	Total Inner Iter.	Outer Iter. Residual	Total Time Taken
10	1000 (366)	10^{-4} (10^{-4})	9.5 (4.0) min.
1000	1922 (816)	0.020 (10^{-4})	20 (8.5) min.

The corresponding figures obtained by using the method described here are given in the parentheses.

In the place of δ it is possible to use some other parameter to overcome the divergence of outer iterations as follows:

(1) A weight factor ξ could be used to dampen Ψ after it had converged in a particular inner iteration. If the old value of Ψ is Ψ_{old} and the value obtained after inner iteration convergence is Ψ_{new}, the final value is obtained by using:

$$\Psi = \xi \cdot \Psi_{old} + (1 - \xi) \cdot \Psi_{new} , \quad 0 \leq \xi \leq 1 \quad . \tag{2.26}$$

This could be used on the whole mesh R_h.

(2) ξ could be used only on $R_{h,1}$.

But the numerical experiments have shown that the use of δ defined as in equation (2.20) and used for points on the boundary alone proves to be the most efficient.

DISCUSSION

Greenspan[22] used a modified SOR method for the inner iterations. The modification consisted in the introduction of a weight factor and the iterations were defined by

$$\bar{\Psi}_0^{(j)} = (1 - r_\Psi)\, \Psi_0^{(j-1)} + \frac{r_\Psi}{4} (\Psi_1^{(j-1)} + \Psi_2^{(j-1)} + \bar{\Psi}_3^{(j)} + \bar{\Psi}_4^{(j)} + h^2\omega_0) \quad . \tag{2.27a}$$

After each sweep over the domain $R_{h,2}$, Ψ was defined by the weighted average

$$\Psi_0^{(j)} = \xi\Psi_0^{(j-1)} + (1 - \xi)\, \bar{\Psi}_0^{(j)} \quad , \quad 0 \leq \xi \leq 1 \quad . \tag{2.27b}$$

A fixed relaxation factor $r_\Psi = 1.8$ was used and "a suitable value of ξ was used whenever it was needed."

To examine the effect of this modification, let us assume that r_Ψ has its optimum value in (2.27a) when an unmodified SOR method is used to obtain $\bar{\Psi}_{new}$ from Ψ_{old} in one iteration. Now, the application of (2.27b) with $\xi > 0$ will result in replacing a proportionate amount of $\bar{\Psi}_{new}$ by Ψ_{old}. When (2.27a) is again applied in the next iteration, some extra work has to be done to recover the amount of $\bar{\Psi}_{new}$ which was not used. Hence the use of (2.27b) and $\xi > 0$ with the optimal SOR method will slow down the convergence of the optimal over-relaxation procedure.

For iterations of the ω-system, Greenspan used $r_\omega = 1.0$ as the relaxation factor and the weight factor $\delta = 0.7$ was used on R_h to modify the process. Although the relation between r_ω and δ is not simple, the value of ω used in the inner iterations slows down the convergence tremendously. In fact, for $R = 10$ and $h = 0.05$, 250 inner iterations were used for convergence up to 10^{-4} with $r_\omega = 1.0$, $\delta = 0.7$. However,

if instead of (2.27) the optimum value of r_ω (= 1.728) were used in accordance with (2.21), only 60 inner iterations would have been required to achieve the same convergence. It may be emphasized that Greenspan used the weight factor $\delta = 0.7$ both for S_h and R_h to obtain convergent solutions. On the other hand, we have found that use of δ over R_h slows down the rate of convergence.

Our aim was to use as few arbitrary parameters as possible. In case of [22], the four parameters r_ψ, r_ω, ξ, δ were taken as the arbitrary quantities. In fact the introduction of ξ and δ is artificial and unnecessary. The optimum factors can easily be obtained using the available theory and a minimum of experimentation. This enables us to eliminate four arbitrary parameters.

It was stated in [22] that the outer iteration convergence is greatly accelerated as δ is allowed to increase. We were unable to verify this and found that for a given h and R, the convergence is fastest when δ is within a particular range. As δ is taken outside this range, the convergence slows down and in some cases divergence is encountered. We obtained solutions for R = 100, h = 0.05 using $\delta = 0.2$ and for R = 10, h = 0.025 using $\delta = 0.4$. Greenspan[22] reported that for $\delta \leq 0.5$, h = 0.05 he could not obtain convergent solutions.

Recently, Greenspan[31] has reported some more results for the step size 0.025. He obtained the outer iteration convergence to 10^{-4} for R = 50 and the computer time taken was 60 minutes. For R = 10,000 the outer iteration convergence to 0.004 was obtained in 260 minutes on CDC 3600. The values of δ used for h = 0.025 were as high as 0.95.

All the results given above have been obtained starting with $\Psi \equiv 0$, $\omega \equiv 0$ initially. A more judicious choice of the initial

values could also be made. In fact, using the solutions obtained for $h = 0.1$ the convergence for $h = 0.05$ was achieved in a lesser amount of time than that with zero initial values.

In order to improve the accuracy of the solutions, it is necessary to reduce the mesh size. However, if we use an uniform mesh, such a reduction will mean a considerable increase in the computing time. The reduction in the mesh size is partly necessitated by the fact that the gradients have larger values near the walls, and hence a finer mesh should be used near the walls to improve the solutions. Moreover, the derivatives on the boundaries can be approximated by using finite difference approximations of higher order.

3. A MODIFIED STAGNATION POINT FLOW

Consider the two-dimensional flow near a stagnation point at a plane rigid surface. This point is created due to a fluid which approaches the rigid surface and divides into streams proceeding away from the point in question. In the steady state case, the vorticity generated at the boundary will be confined to a layer adjacent to the boundary and outside this boundary layer the vorticity will have a zero value. It is convenient first to determine the flow in the outer irrotational region and then to use this flow as an outer boundary condition for the flow in the layer of non-zero vorticity. Ignoring the layer of non-zero vorticity and thus ignoring the no-slip condition, the flow in the outer irrotational region is described by the stream function

$$\Psi = kxy \quad , \tag{3.1}$$

where x and y are rectilinear coordinates parallel and normal to the boundary. The corresponding velocity distribution is

$$u = kx \quad ; \quad v = - ky \quad , \tag{3.2}$$

u, v are the velocity components in the x- and y-directions respectively and k is a positive constant. In the case of the stagnation point on a body fixed in a stream, k must be proportional to the speed of the body and also depends upon the shape of the body as a whole.

The distribution of vorticity in the thin layer near the boundary is determined from the equation

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \quad , \tag{3.3}$$

with the boundary conditions u = 0, v = 0 at y = 0 and the flow tends to the form (3.2) at the outer edge of the layer. The existence of the no-slip condition in this thin layer will change the dependence of the velocity components on y and we try to build a solution such that u is proportional to x throughout the boundary layer. For such a solution we write

$$\Psi = xf(y) \quad . \tag{3.4a}$$

Correspondingly,

$$u = xf'(y) \quad , \quad v = - f(y) \quad ; \tag{3.4b}$$

$$\omega = v_x - u_y = - xf''(y) \quad , \tag{3.4c}$$

where f(y) is an unknown function and primes denote differentiation with respect to y. Substitution in (3.3) gives

$$- f'f'' + ff''' + \nu f^{iv} = 0 \quad , \tag{3.5a}$$

and the boundary conditions are

$$f = f' = 0 \quad \text{at} \quad y = 0 \quad , \tag{3.5b}$$

$$f \to ky \quad \text{as} \quad y \to \infty \quad . \tag{3.5c}$$

If a function f(y) satisfying (3.5) can be found, our guess (3.4) is successful. One integration of (3.5a) and use of outer boundary condition (3.5c) gives:

$$f'^2 - ff'' - \nu f''' = k^2 \quad . \tag{3.6}$$

The coefficients in this equation can be non-dimensionalized by the transformation

$$y = \left(\frac{\nu}{k}\right)^{\frac{1}{2}} \cdot \eta \quad ; \quad f(y) = (\nu k)^{\frac{1}{2}} \phi(\eta) \quad . \tag{3.7}$$

This transformation gives:

$$\phi'^2 - \phi\phi'' - \phi''' = 1 \quad , \tag{3.8a}$$

$$\phi = \phi' = 0 \quad ; \quad \eta = 0 \quad , \tag{3.8b}$$

$$\phi \to \eta \quad ; \quad \text{as} \quad \eta \to \infty \quad . \tag{3.8c}$$

Hiemenz (1911) obtained a numerical solution of (3.8) which was later improved by Howarth (1935). The thickness of the non-zero vorticity layer is found from the numerical solution to be

$$\delta = 2.4 \, (\nu / k)^{\frac{1}{2}} \quad . \tag{3.9}$$

This thickness is independent of the distance along the boundary and it approaches zero as the effect of convection (represented by k) becomes dominant over the effect of diffusion (represented by ν). As δ is independent of x, the effect of the vorticity layer on the irrotational flow is approximately to displace it in the y-direction in the same way as a simple shift of the boundary. This simple shift of the whole field of the irrotational flow does not affect the velocity distribution within that field.

In the steady flow of fluids past bodies, backflow in the rear is a typical feature and this becomes more pronounced as the Reynolds number increases[28]. The stream lines close to the forward face of the body leave the body at some point and pass round the large standing eddies. This breaking away of the stream lines is an example of boundary-layer separation. The breakaway of stream lines may occur even at Reynolds numbers near 10, but it takes on particular importance at large Reynolds numbers because the stream lines leaving the surface then carry vorticity of large magnitude away from the surface. The separation is due to the presence of

a solid boundary at which the no-slip condition is satisfied and vorticity is generated. This was experimentally observed by Föttinger[30] who introduced a thin rigid plate at the plane of symmetry of the stagnation point flow at a plane solid wall. The plate coincides with a stream line of the first flow so that in case of a completely inviscid fluid and slip being allowed at the plate, the flow would not show any change. Since the no-slip conditions hold, a boundary layer must form on the plate. The deceleration of fluid on the plane of symmetry of the original flow is too severe for a steady thin boundary layer on the plate to be possible; more and more vorticity accumulates in the ever thickening boundary layer and this finally produces separation on the plate so that the oncoming irrotational flow no longer passes close to the original wall. However, the effect of this plate is confined to a certain neighborhood of the original stagnation point while further downstream the flow would go over to the original stagnation point flow.

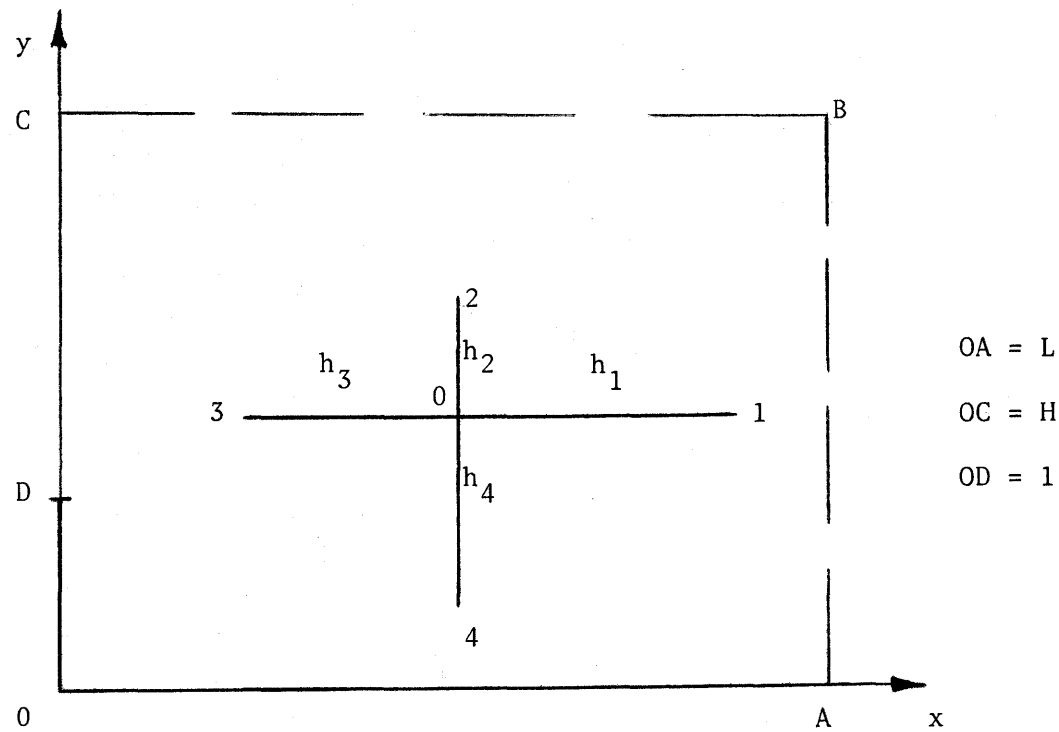
We shall consider this particular example because of the possibility of separation at low Reynolds numbers. The length of the plate is used to non-dimensionalize the equations of motion and also to form the Reynolds number of the flow. This problem is difficult to handle using the techniques and simplifications of boundary layer theory because of the separation and back flow in the corner.

The equations of steady flow of a viscous incompressible fluid in non-dimensional form are given by:

$$\Delta\Psi = -\omega \tag{3.10}$$

$$\Delta\omega + R\,(\Psi_x\omega_y - \omega_x\Psi_y) = 0 \quad . \tag{3.11}$$

R is the reynolds number of the flow, $R = VL/\nu$, where V is the characteristic velocity, L the length of the small plate taken as the characteristic length and ν is the kinematic viscosity.



(fig. 2)

For the purpose of numerical calculations we consider a rectangular region OABCD (fig. 2) where OA is the plane rigid surface and OD is the small plate of non-dimensional unit length. To start with, two of the boundaries, viz. AB and BC, are undetermined. It is assumed that the flow assumes its asymptotic values on these boundaries. Thus, on these boundaries the stream function is given by

$$\Psi = xf(y) = x(\nu k)^{\frac{1}{2}} \phi(\eta) \quad , \quad \eta = \sqrt{k/\nu} \cdot y \quad .$$

Introducing non-dimensional quantities, we get

$$\Psi = \frac{x}{\sqrt{R}} \phi \ (\eta) \quad , \quad \eta = \sqrt{R} \ . \ y \quad . \tag{3.12a}$$

Similarly, the vorticity is given by

$$\omega = - \ x \ \sqrt{R} \ \phi'' \ (\eta) \quad . \tag{3.12b}$$

Thus, the boundary conditions on the boundaries OABCD are given by:

$$u = 0 \quad , \ v_x = 0 \quad \text{on} \quad x = 0 \quad , \quad y > 1 \quad \text{(by symmetry)} \quad ,$$

which yields

$$\Psi = 0 \quad , \quad \Psi_{xx} = 0 \quad , \quad \omega = 0 \quad ; \quad x = 0 \quad , \quad y > 1 \tag{3.13a}$$

$$\Psi = 0 \quad , \quad \Psi_x = 0 \qquad ; \quad x = 0 \quad , \quad 0 \leq y \leq 1 \tag{3.13b}$$

$$\Psi = 0 \quad , \quad \Psi_y = 0 \qquad ; \quad y = 0 \tag{3.13c}$$

$$\left.\begin{aligned} \Psi &= \frac{x}{\sqrt{R}} \phi \ (\eta) \\ \omega &= - \ x\sqrt{R} \ \phi'' \ (\eta) \end{aligned}\right\} \quad \begin{aligned} x &= L \\ &\\ y &= H \quad , \end{aligned} \tag{3.13d}$$

where x = L and y = H are the estimated free boundaries such that OA = L, OC = H.

FINITE DIFFERENCES

Since the velocity gradients are large near the boundaries OA and OD (refer Fig. 2), we require a finer mesh at least near these boundaries. In fact, the boundary layer thickness far away from the small plate (i.e., where the original stagnation point flow (3.4) prevails) is found to be $\simeq 5.0/\sqrt{R}$ from the numerical solution of (3.8). At this distance from OA,

the velocity component u attains 99.99% of the corresponding value in the case of potential flow given by (3,2). Thus at R = 9, the boundary layer thickness is $\simeq$ 1.7, at R = 100 it is $\simeq$ 0.5 and at R = 2500 it is $\simeq$ 0.1. Thus for regions far away from OD, we consider only a layer of this thickness outside which the vorticity and stream function take their asymptotic values given by (3.1). In order to describe the flow in this layer satisfactorily, we are required to take a reasonable number of mesh points across the layer. This puts some sort of upper limit on the mesh size in this region. If it is decided to choose a uniform mesh, the number of mesh points becomes unmanageably large and hence requires an enormous amount of computing time. We, therefore, consider a non-uniform mesh for this problem. A coarser mesh could be used as we go away from the boundaries OA and OD.

We derive the finite difference formulas using variable mesh sizes. The region OABCD is covered by a rectangular mesh. If the mesh distance in the four directions from any point (x, y) is h_1, h_2, h_3, h_4, we denote the values of a function f at the five points (x, y), $(x + h_1, y)$, $(x, y + h_2)$, $(x - h_3, y)$ and $(x , y - h_4)$ by f_0, f_1, f_2, f_3 and f_4 respectively. With this notation, the finite difference analogue $\Delta_h u$ of Δu is given by [2]:

$$\Delta_h u_0 \equiv - c_0 u_0 + c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 \quad , \qquad (3.14a)$$

where,

$$\begin{aligned} c_0 &= \frac{2}{h_1 h_3} + \frac{2}{h_2 h_4} \\ c_1 &= \frac{2}{h_1 (h_1 + h_3)} \\ c_2 &= \frac{2}{h_2 (h_2 + h_4)} \\ c_3 &= \frac{2}{h_3 (h_1 + h_3)} \\ c_4 &= \frac{2}{h_4 (h_2 + h_4)} \end{aligned} \qquad (3.14b)$$

The discretization error is given by

$$\Delta u - \Delta_h u \equiv \frac{h_1 - h_3}{3} u_{xxx} + \frac{h_2 - h_4}{3} u_{yyy} + \theta(h^2) \quad .$$

The finite difference analogue of the Poisson equation (3.10) in Ψ is given by

$$\Delta_h \Psi_0 = - \omega_0 \quad . \tag{3.15}$$

The non-linear equation (3.11) in ω is discretized using a technique similar to the one used for equal mesh size (2.5, 2.6). The finite difference analogue of $\Delta\omega$ is $\Delta_h\omega$ with Δ_h defined by (3.14). The derivatives Ψ_x, Ψ_y are approximated by

$$\Psi_x = \frac{\Psi_1 - \Psi_3}{h_1 + h_3} = \frac{\alpha}{h_1 + h_3} \quad , \tag{3.16a}$$

$$\Psi_y = \frac{\Psi_2 - \Psi_4}{h_2 + h_4} = \frac{\beta}{h_2 + h_4} \quad , \tag{3.16b}$$

where $\alpha = \Psi_1 - \Psi_3$, $\beta = \Psi_2 - \Psi_4$.

The following approximations are used for ω-derivatives:

$$\omega_x = \begin{cases} \dfrac{\omega_2 - \omega_0}{h_2} & \alpha \geq 0 \qquad (3.17a) \\ \dfrac{\omega_0 - \omega_4}{h_4} & \alpha < 0 \quad , \qquad (3.17b) \end{cases}$$

$$\omega_y = \begin{cases} \dfrac{\omega_o - \omega_3}{h_3} & \beta \geq 0 \qquad (3.17c) \\ \dfrac{\omega_1 - \omega_0}{h_1} & \beta < 0 \quad . \qquad (3.17d) \end{cases}$$

Substituting these approximations in (3.11) we obtain a system of finite difference equations which, when written in a matrix form, has a diagonally dominant coefficient matrix. In fact, the finite difference analogue of (3.11) is

$$- k_0\omega_0 + k_1\omega_1 + k_2\omega_2 + k_3\omega_3 + k_4\omega_4 = 0 \quad . \tag{3.18}$$

The coefficients k_i are defined as follows:

(1) $\alpha \geq 0$, $\beta \geq 0$

$$\begin{aligned} k_0 &= c_0 + \frac{a}{h_2} + \frac{b}{h_3} \\ k_1 &= c_1 \\ k_2 &= c_2 + \frac{a}{h_2} \\ k_3 &= c_3 + \frac{b}{h_3} \\ k_4 &= c_4 \quad ; \end{aligned} \tag{3.19a}$$

(2) $\alpha \geq 0$, $\beta < 0$

$$\begin{aligned} k_0 &= c_0 + \frac{a}{h_2} - \frac{b}{h_1} \\ k_1 &= c_1 - \frac{b}{h_1} \\ k_2 &= c_2 + \frac{a}{h_2} \\ k_3 &= c_3 \\ k_4 &= c_4 \quad ; \end{aligned} \tag{3.19b}$$

(3) $\alpha < 0$, $\beta \geq 0$

$$\begin{aligned} k_0 &= c_0 - \frac{a}{h_4} + \frac{b}{h_3} \\ k_1 &= c_1 \\ k_2 &= c_2 \\ k_3 &= c_3 + \frac{b}{h_3} \quad , \quad k_4 = c_4 - \frac{a}{h_4} \quad ; \end{aligned} \tag{3.19c}$$

(4) $\alpha < 0$, $\beta < 0$

$$k_0 = c_0 - \frac{a}{h_4} - \frac{b}{h_1}$$

$$k_1 = c_1 - \frac{b}{h_1}$$

$$k_2 = c_2 \qquad (3.19d)$$

$$k_3 = c_3$$

$$k_4 = c_4 - \frac{a}{h_4} \quad .$$

Here $a = \frac{R\alpha}{h_1 + h_3}$, $b = \frac{R\beta}{h_2 + h_4}$ and the quantities c_i are defined in (3.14b).

The discretization error is given by

$$(h_1 - h_3)\ (\tfrac{1}{3}\,\omega_{xxx} - \tfrac{R}{2}\,\Psi_{xx}\omega_y) + (h_2 - h_4)\ (\tfrac{1}{3}\,\omega_{yyy} + \tfrac{R}{2}\,\Psi_{yy}\omega_x)$$

$$+ \frac{Rh}{2}\,(\Psi_x\omega_{yy} \pm \Psi_y\omega_{xx}) + \theta\,(h^2) \ ,$$

where $h = \max_i h_i$.

The derivative Ψ_y at the point (x, y) is approximated by using the values of Ψ at (x, y), (x, y + h), (x, y + 2h), (x, y + 3h) numbered 0, 1, 2, 3 respectively as follows:

$$(\Psi_y)_0 = \alpha_0\Psi_0 + \alpha_1\Psi_1 + \alpha_2\Psi_2 + \alpha_3\Psi_3$$

$$= \alpha_0\Psi(x,y) + \alpha_1\Psi(x,y+h) + \alpha_2\Psi(x,y+2h) + \alpha_3\Psi(x,y+3h)$$

$$= \alpha_0\Psi + \alpha_1(\Psi + h\Psi_y + \frac{h^2}{2}\,\Psi_{yy} + \frac{h^3}{6}\,\Psi_{yyy} + \ldots)$$

$$+ \alpha_2(\Psi + 2h\Psi_y + \frac{4h^2}{2}\,\Psi_{yy} + \frac{8h^3}{6}\,\Psi_{yyy} + \ldots)$$

$$+ \alpha_3(\Psi + 3h\Psi_y + \frac{9h^2}{2}\,\Psi_{yy} + \frac{27h^3}{6}\,\Psi_{yyy} + \ldots) \ ,$$

which on comparing coefficients gives:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = h^{-1}$$

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 0$$

$$\alpha_1 + 8\alpha_2 + 27\alpha_3 = 0 \quad .$$

This has the solution

$$\alpha_0 = -\frac{11}{6h} \quad , \quad \alpha_1 = \frac{3}{h} \quad , \quad \alpha_2 = -\frac{9}{6h} \quad , \quad \alpha_3 = \frac{1}{3h} \quad .$$

Hence we have the following approximation:

$$(\Psi_y)_0 = \frac{1}{6h}(-11\Psi_0 + 18\Psi_1 - 9\Psi_2 + 2\Psi_3) \quad . \qquad (3.20)$$

Similarly (Ψ_x) at (x, y) is approximated using values of Ψ at (x, y), (x +h, y), (x + 2h, y), (x + 3h, y) numbered 0, 1, 2, 3 respectively by

$$(\Psi_x)_0 = \frac{1}{6h}(-11\Psi_0 + 18\Psi_1 - 9\Psi_2 + 2\Psi_3) \quad . \qquad (3.21)$$

The error in these approximations is given by:

$$\frac{h^3}{4}\Psi^{(4)}(x, y) + \theta(h^4) \quad .$$

The derivative Ψ_{xx} at (x, y) is approximated by the formula

$$(\Psi_{xx})_0 = \alpha_0\Psi_0 + \alpha_1\Psi_1 + \alpha_2\Psi_2 + \alpha_3\Psi_3 \quad ,$$

where the subscripts 0, 1, 2, 3 denote the functional values at the four points (x, y), (x + h, y), (x + 2h, y) and (x + 3h, y) respectively. Expanding in Taylor's series and comparing coefficients we obtain the following

equations:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$$

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 2h^{-2}$$

$$\alpha_1 + 8\alpha_2 + 27\alpha_3 = 0 \quad .$$

These equations have the solution

$$\alpha_0 = 2h^{-2} \quad , \quad \alpha_1 = -5h^{-2} \quad , \quad \alpha_2 = 4h^{-2} \quad , \quad \alpha_3 = -h^{-2}$$

which yields,

$$(\Psi_{xx})_0 = h^{-2} (2\Psi_0 - 5\Psi_1 + 4\Psi_2 - \Psi_3) \quad . \qquad (3.22)$$

The error of this approximation is given by

$$-\frac{11}{12} h^2 \Psi_{xxxx} + 0 (h^3) \quad .$$

The accuracy of this approximation can be increased by using one more mesh point (x + 4h, y) numbered 4. This formula is given by

$$(\Psi_{xx})_0 = \frac{1}{12h^2} (35\Psi_0 - 104\Psi_1 + 114\Psi_2 - 56\Psi_3 + 11\Psi_4) \quad , \qquad (3.23)$$

which has the error given by

$$\frac{5}{6} h^3 \Psi_{xxxxx} + 0 (h^4) \quad .$$

On the plane rigid surface OA (y = 0) we have $\Psi = \Psi_y = 0$, therefore, using (3.20) we obtain an approximation for Ψ (x, h) as follows:

$$\Psi(x, h) \cong \frac{1}{18}\left(9\Psi(x, 2h) - 2\Psi(x, 3h)\right) \quad . \tag{3.24}$$

On the plate OD ($x = 0$, $0 \leq y \leq 1$) we have $\Psi = \Psi_x = 0$, therefore, from (3.21) we obtain an approximation for $\Psi(h, y)$ as

$$\Psi(h, y) \cong \frac{1}{18}\left(9\Psi(2h, y) - 2\Psi(3h, y)\right) \quad . \tag{3.25}$$

On the boundary DC ($x = 0$, $y > 1$) $\Psi = \Psi_{xx} = 0$, therefore, $\Psi(h, y)$ can be approximated by using (3.22) as follows:

$$\Psi(h, y) \cong \frac{1}{5}\left(4\Psi(2h, y) - \Psi(3h, y)\right) \quad . \tag{3.26}$$

The vorticity is prescribed by (3.13) on the free boundaries AB, BC and on CD ($x = 0$, $y > 1$). On the rigid boundary OA ($y = 0$) normal derivative $\Psi_y = 0$, so that

$$\Psi_y \cong \frac{\Psi(x, h) - \Psi(x, -h)}{2h} = 0$$

or

$$\Psi(x, h) = \Psi(x, -h) \quad .$$

This defines the vorticity on OA by

$$\begin{aligned} \omega_{OA} &= -\Delta\Psi|_{OA} \\ &\cong -\frac{2}{h_2^2}\Psi(x, h_2) \quad , \end{aligned} \tag{3.27}$$

since $\Psi = 0$ on OA .

Similarly for OD, one has

$$\omega_{OD} \cong -\frac{2}{h_1^2}\Psi(h_1, y) \quad . \tag{3.28}$$

NUMERICAL PRODECURE

The same numerical procedure as was used for the problem discussed in Chapter 2 can be used here with some modifications. For the problem under consideration the boundary values of Ψ and ω are known on the boundaries AB and BC. Since maximum information is available at the top right corner B, the sweeps were performed on the points from the right to the left on each row, the rows being taken in order from the top to the bottom. It is expected that the process will converge faster using this ordering.

The mesh is divided into three groups as follows:

S_h : consisting of all the mesh points on the boundary.

$R_{h,1}$: consisting of the mesh points of the type (h_1, y) or (x, h_2), i.e., the points one mesh distance away from the boundaries where the gradients of Ψ are prescribed.

$R_{h,2}$: consisting of the remaining interior points.

We denote $R_h = R_{h,1} \cup R_{h,2}$.

The discretization of the equations (3.10) and (3.11) leads to systems of linear algebraic equations which have real diagonally dominant coefficient matrices. However, due to the unequal mesh sizes these matrices are non-symmetric. The numerical procedure is started with some initial values of Ψ and ω and the system (3.15) is iterated using a successive over-relaxation method defined by

$$\Psi_0^{(j)} = (1 - r_\Psi)\, \Psi_0^{(j-1)} + \frac{r_\Psi}{c_0} [c_1 \Psi_1^{(j)} + c_2 \Psi_2^{(j)} + c_3 \Psi_3^{(j-1)} + c_4 \Psi_4^{(j-1)} + \omega_0^{(j-1)}] , \tag{3.29}$$

where c_i are defined by (3.14b), r_Ψ is the overrelaxation factor and subscripts and superscripts have their usual meaning. This inner iteration for Ψ is continued in the region $R_{h,2}$ until either the error between successive

Ψ-inner iterates becomes less than a certain tolerance ε_Ψ or the number of inner-iterations exceed a certain number N_Ψ. These quantities need not be fixed beforehand. After Ψ converges, Ψ is computed on $R_{h,1}$ using the formulas (3.24 - 3.26) which define Ψ on R_h for the current overall (outer) iteration. The vorticity values $\bar{\omega}$ are now computed on the boundaries OA and OD using (3.27, 3.28) and a weighting factor δ is used to smooth them on OA and OD. The formula used is:

$$\omega_{OA,OD} = \delta \cdot \omega_{old} + (1 - \delta)\, \bar{\omega}_{new} \quad . \tag{3.30}$$

The inner iterations for ω are now performed in the region R_h using the SOR method defined by:

$$\omega_0^{(j)} = (1 - r_\omega)\, \omega_0^{(j-1)} + \frac{r_\omega}{k_0} (k_1 \cdot \omega_1^{(j)} + k_2 \cdot \omega_2^{(j)} + k_3 \cdot \omega_3^{(j-1)} + k_4 \cdot \omega_4^{(j-1)}) \tag{3.31}$$

r_ω is the relaxation factor and the k_i are defined by (3.19). These inner iterations are continued until the convergence criterion defined by $(\varepsilon_\omega , N_\omega)$ is satisfied. This completes one outer iteration. The iterations are continued until outer iteration convergence to (ε, N) is achieved.

The boundaries AB and BC are determined before using the above mentioned iterative process. These boundaries are assumed to lie in the region where the effect of the obstacle is negligible. They are determined by numerical experimentation in the following manner: At first we calculate the solutions using a crude mesh size and the boundaries are taken at some distance away from the obstacle. These distances depend upon the Reynolds number and physical considerations. The boundaries are then moved away and the calculations repeated. This process is repeated until one arrives at a point when any further change does not significantly affect the solutions near the obstacle (which is the region of interest to us). This determines

the boundaries for a particular Reynolds number.

The coefficient matrices for the algebraic systems are non-symmetric and it becomes necessary to estimate the optimal relaxation factors (r_ψ, r_ω) by actually computing the eigenvalue with the largest modulus for different values of the parameter and then choosing a proper relaxation factor. This procedure has been described in Chapter 2.

RESULTS

The problem was solved for three values of the Reynolds numbers viz. 9, 100, and 400 . First of all the outer boundaries were estimated. The crude mesh size used was 0.1. The values of the outer boundary lengths found sufficient for different Reynolds numbers are:

$R = 9 , L = 4.0 , H = 3.0$

$R = 100 , L = 5.5 , H = 1.6$

$R = 400 , L = 6.0 , H = 1.3 .$

The relaxation factors were found to be:

$R = 9 , r_\psi = 1.82 , r_\omega = 1.6$

$R = 100 , r_\psi = 1.8 , r_\omega = 1.15$

$R = 400 , r_\psi = 1.8 , r_\omega = 1.09$

The smallest mesh size α chosen near OA and OC and the largest mesh size β chosen far from OA and OC were:

$R = 9 , \alpha = 0.05 , \beta = 0.2$

$R = 100 , \alpha = 0.05 , \beta = 0.1$

$R = 400 , \alpha = 0.03 , \beta = 0.2 .$

The solutions were obtained for R = 9 using two convergence criterions as discussed in Chapter 2. With $N_\psi = N_\omega = 40$, $\delta = 0.7$ and $\varepsilon_\psi = \varepsilon_\omega = \varepsilon = 10^{-4}$ the time taken was 16 minutes on IBM 360/50 when 10 outer iterations were completed on this grid consisting of 1485 mesh points. Convergent solutions were also obtained using $\delta = 0.6$ and the criterion developed in Chapter 2 viz. $\varepsilon = 10^{-4}$ and $\varepsilon_\omega = R_{max}$. The time taken for this convergence was 15 minutes when 499 inner and 13 outer iterations had been completed on the same grid. The separation and back flow in the corner O were observed and the point of separation was at $h \simeq 0.26$ (h is the non-dimensional distance on OD from O).

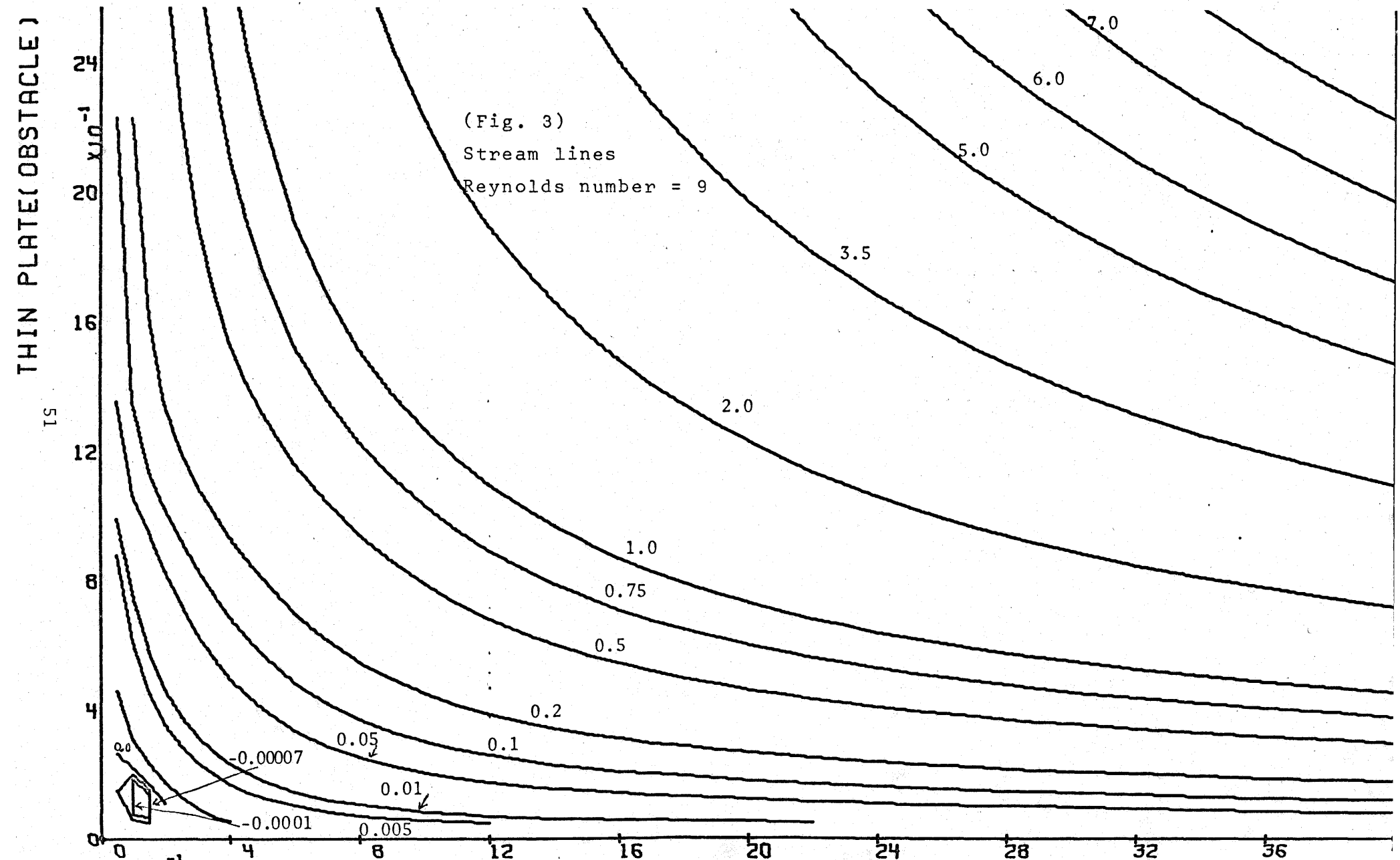
In the case of R = 100, convergent solutions were obtained with $\delta = 0.7$ using fixed numbers $N_\psi = N_\omega = 30$. A tolerance of 10^{-5} was taken for the inner iterations of ψ ($\varepsilon_\psi = 10^{-5}$) and $\varepsilon_\omega = 10^{-4}$. Monotonic convergence to 10^{-5} was achieved in about 23 minutes when 18 outer and 780 inner iterations had been completed on the grid consisting of 1408 mesh points. Convergent solutions were also obtained using the second convergence criterion ($\varepsilon_\psi = 10^{-4}$, $\varepsilon_\omega = R_{max}$) and the time taken was 19 minutes when 509 inner iterations had been used and convergence to 10^{-4} had resulted in 20 outer interations. The point of separation was at $h \simeq 0.45$.

With R = 400, the smallest mesh size used near the boundaries OA and OD (fig. 2) was 0.03. The convergence of outer iterations up to 10^{-4} was achieved in about 60 minutes on the mesh consisting of 2176 points. The relaxation parameters used were $r_\psi = 1.8$, $r_\omega = 1.09$ and $\delta = 0.9$. The point of separation on the plate OD was at $h \simeq 0.66$.

The separation was observed in all of these examples for R = 9, 10 and 400. There was a region of reverse flow and this region enlarged as Reynolds number was increased. These results are given (fig. 3-8) in the

form of stream line curves and the equivorticity curves.

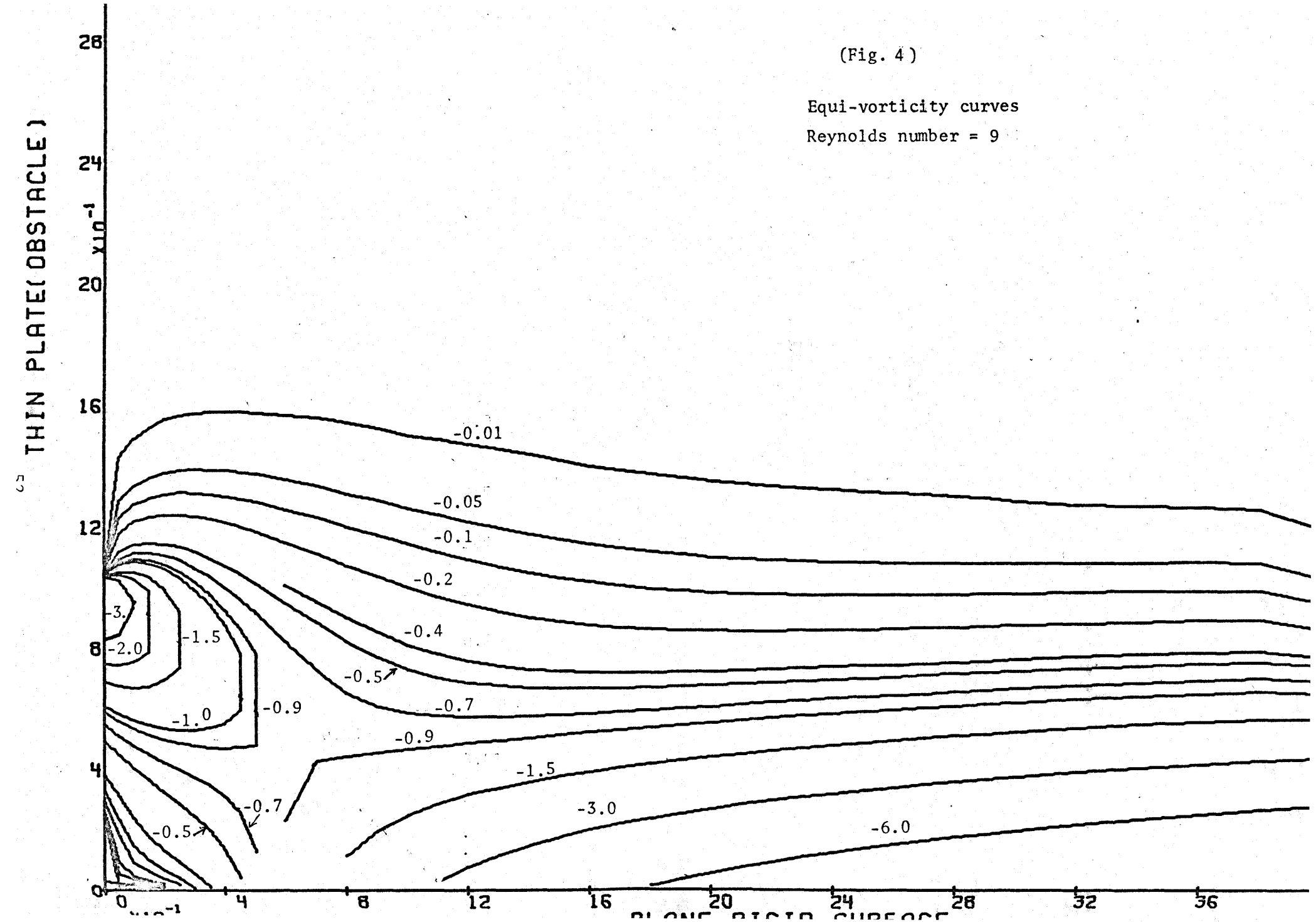
For higher Reynolds numbers we have to take a smaller mesh size near the boundary OD and also one needs to take OA large enough. This increases the number of mesh points considerably. The value of δ needed for convergence also increases with R. The time required for obtaining a solution for higher Reynolds number also increases considerably.

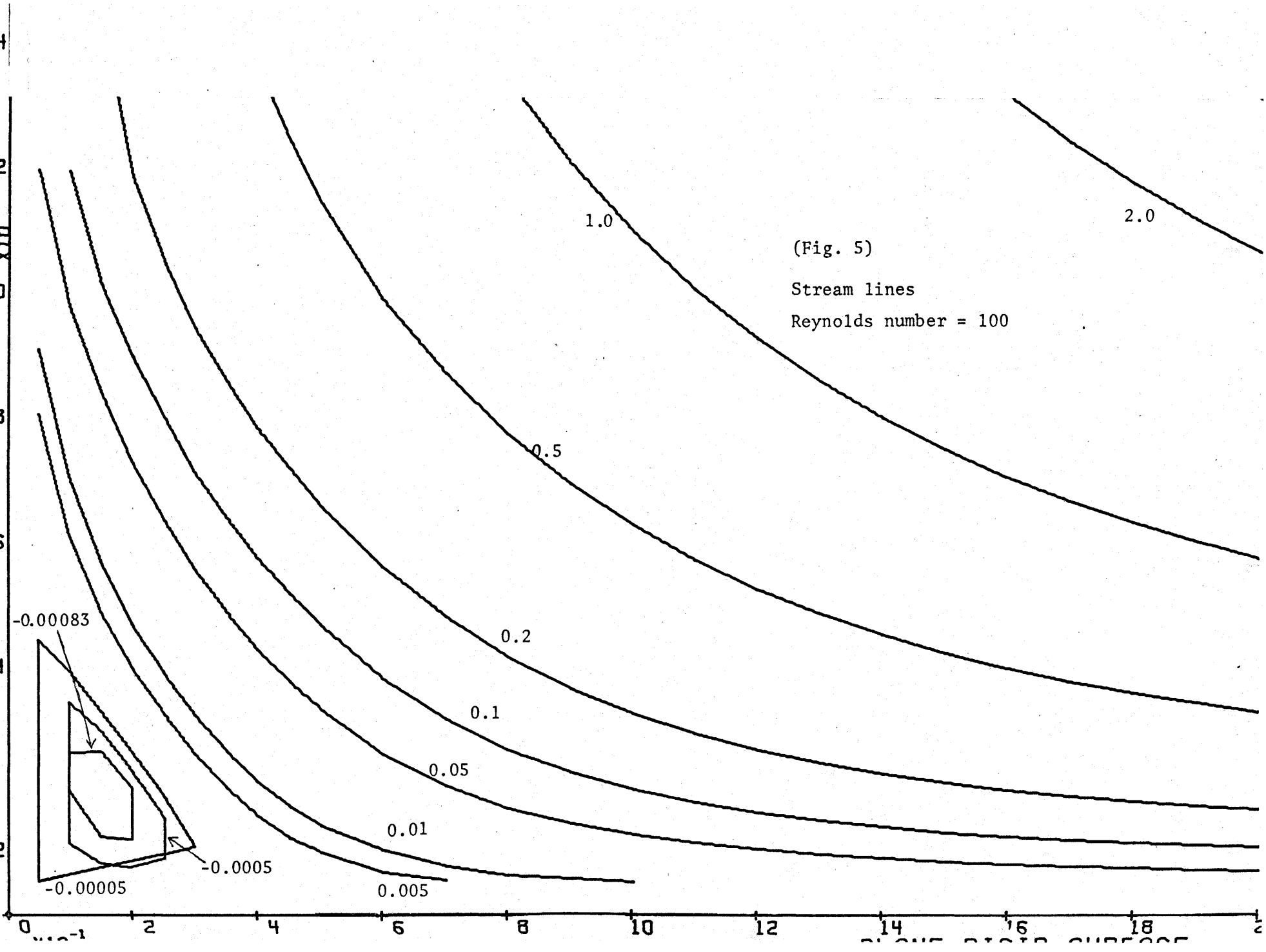

THIN PLATE(OBSTACLE)
(Fig. 3)
Stream lines
Reynolds number = 9
7.0
6.0
5.0
3.5
2.0
1.0
0.75
0.5
0.2
0.1
0.05
0.01
0.005
-0.00007
-0.0001
0
4
8
12
16
20
24
28
32
36

(Fig. 4)

Equi-vorticity curves

Reynolds number = 9

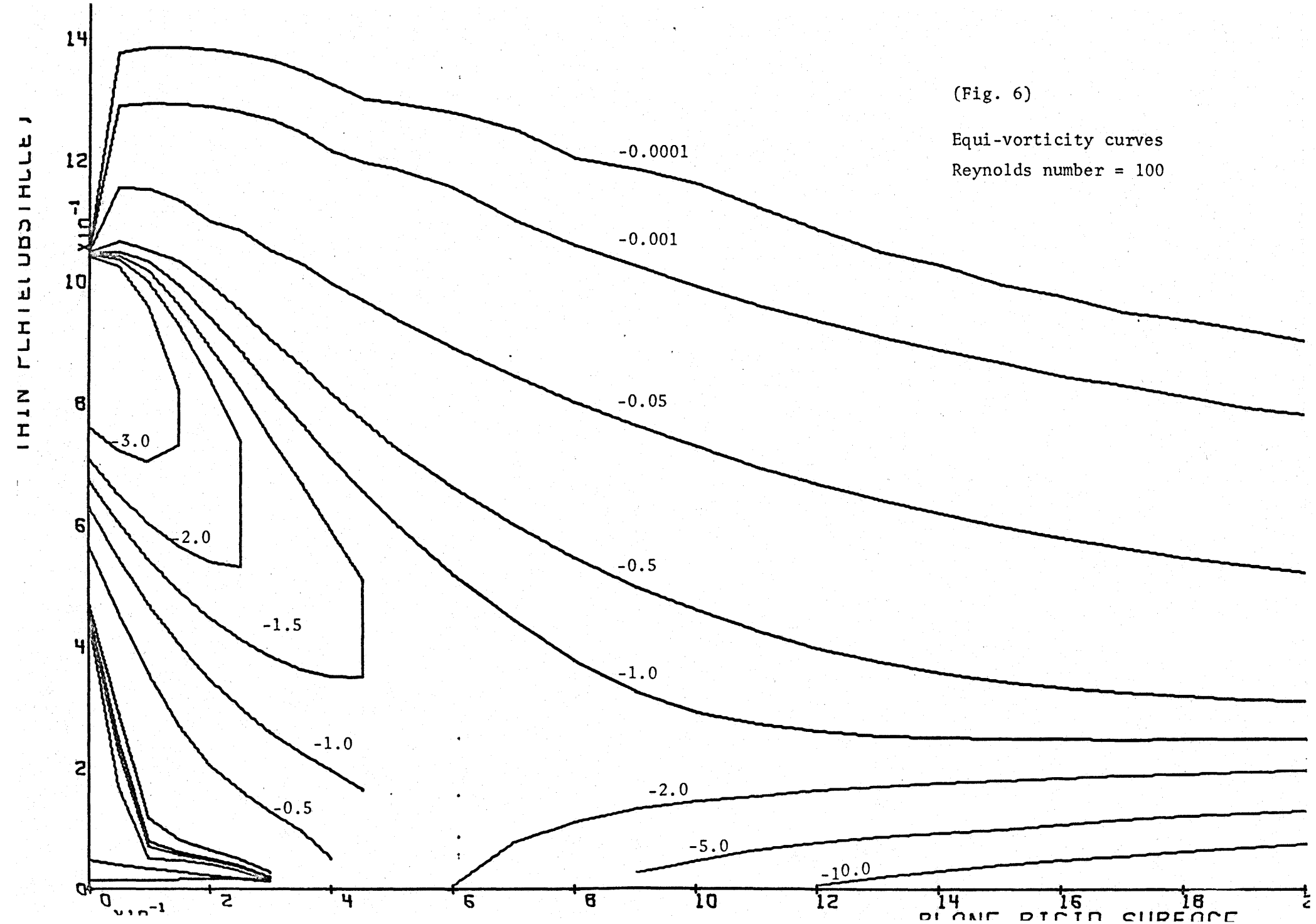


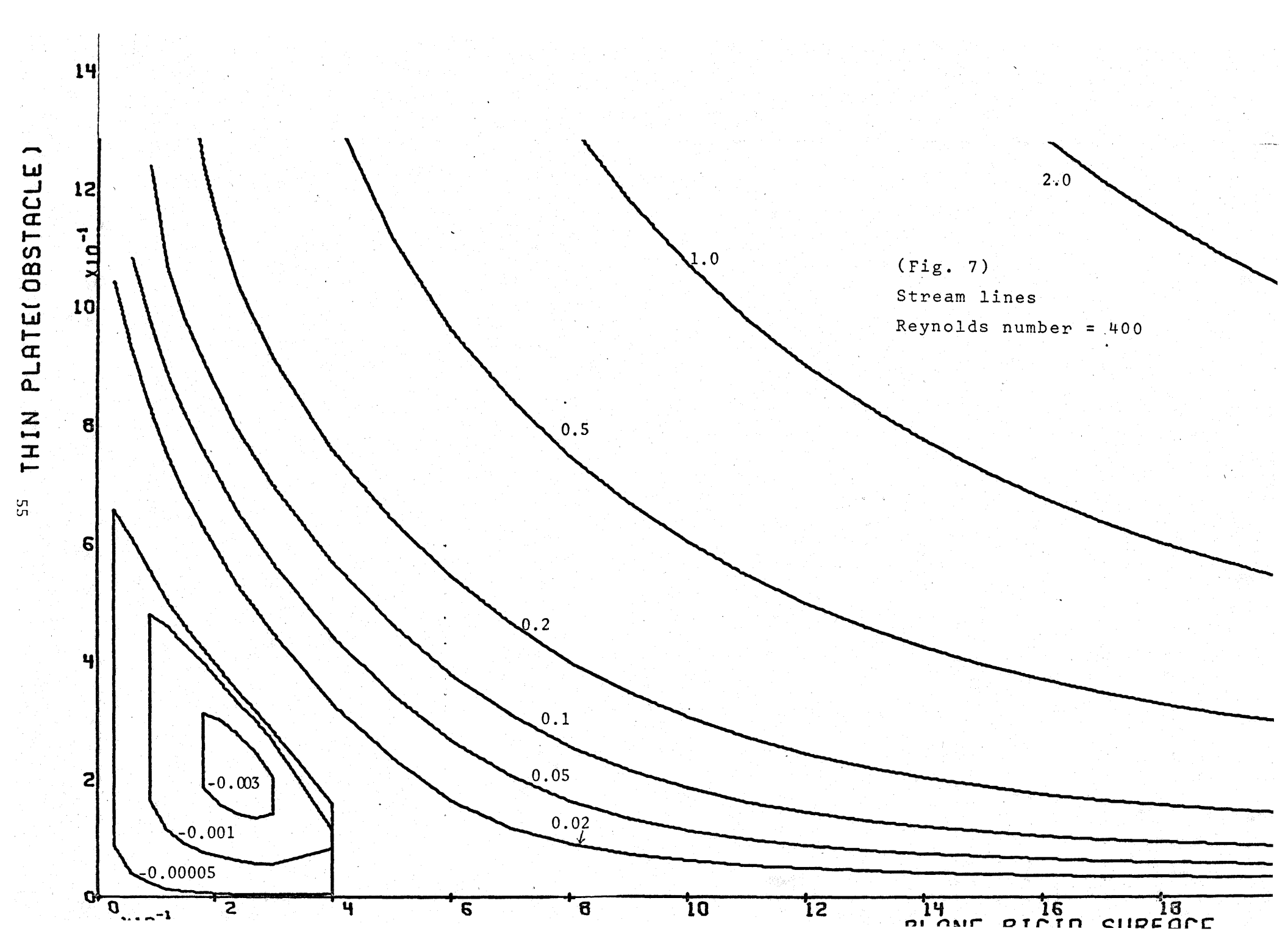


(Fig. 5)

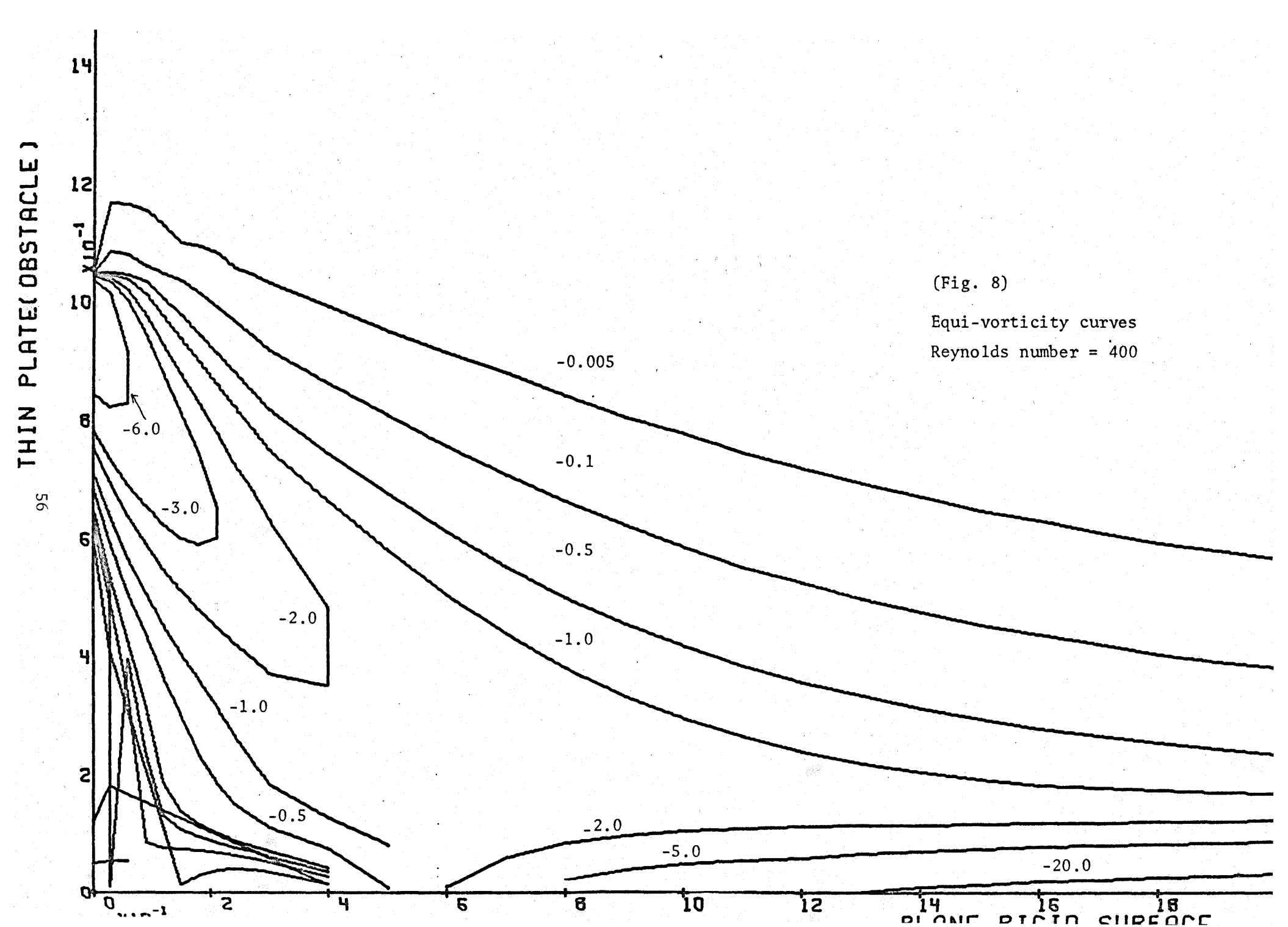
Stream lines

Reynolds number = 100


(Fig. 6)
Equi-vorticity curves
Reynolds number = 100
-0.0001
-0.001
-0.05
-0.5
-1.0
-2.0
-5.0
-10.0
-3.0
-2.0
-1.5
-1.0
-0.5
14
12
10
8
6
4
2
0
2
4
6
8
10
12
14
16
18



(Fig. 7)
Stream lines
Reynolds number = 400



(Fig. 8)

Equi-vorticity curves

Reynolds number = 400

4. AXI SYMMETRIC FLOW

In this chapter we shall consider a possible generalization of the numerical method developed earlier to the three-dimensional case. It may be noted that in the two-dimensional case, equation of continuity could be satisfied by introduction of stream function thereby reducing the number of equations of motion to two. In three-dimensional case such a simplification is not possible and we are required to solve four simultaneous equations. However, in case of axisymmetric flows which are essentially three-dimensional, it is still possible to introduce a stream function. There will now be three differential equations which can be solved using numerical procedure which is a generalization of the one used in Chapters 2 and 3. Except for the additional storage requirement and computer time, we do not anticipate any difficulty so far as the numerical procedure is concerned.

The axisymmetric flow of a viscous incompressible steady fluid is defined by the following equations:

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (4.1a)$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (4.1b)$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) \quad (4.1c)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad , \quad (4.1d)$$

where (r, θ, z) are cylinderical polar coordinates, (u, v, w) are velocity components and ν is kinematic viscosity.

The mass conservation equation (4.1d) is satisfied by introducing a stream function Ψ such that

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial z} \quad , \quad w = -\frac{1}{r}\frac{\partial \Psi}{\partial r} \quad . \tag{4.2}$$

The components of vorticity (ζ_r , ζ_θ, ζ_z) are given by:

$$\zeta_r = -\frac{\partial v}{\partial z} \quad , \quad \zeta_\theta = u_z - w_r \quad , \quad \zeta_z = \frac{1}{r}\frac{\partial}{\partial r}(rv) \quad . \tag{4.3}$$

We introduce Ω and W such that

$$\Omega = rv \quad , \quad W = r\zeta_\theta \quad ; \tag{4.4}$$

and from (4.2) and (4.3) we get

$$W = D^2\Psi \quad , \tag{4.5a}$$

$$D^2 \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) \quad .$$

From equation (4.1b) it follows:

$$\nu D^2\Omega = \frac{1}{r}\left(-\Psi_r\Omega_z + \Psi_z\Omega_r\right) \quad . \tag{4.5b}$$

Eliminating pressure between (4.1a) and (4.1c), and using (4.2 - 4.4) we get

$$\nu D^2 W = -\frac{1}{r}\left(\Psi_r W_z - W_r\Psi_z\right) - \frac{2\Omega\Omega_z}{r^2} - \frac{2W}{r^2}\Psi_z \quad . \tag{4.5c}$$

Introducing non-dimensional quantities, we get

$$D^2\Psi = W \tag{4.6a}$$

$$D^2\Omega + \frac{R}{r}\left(\Psi_r\Omega_z - \Psi_z\Omega_r\right) = 0 \tag{4.6b}$$

$$D^2 W + \frac{R}{r}\left(\Psi_r W_z - \Psi_z W_r\right) = -\frac{2R}{r^2}\left(\Omega\Omega_z + W\Psi_z\right) \quad , \tag{4.6c}$$

where $R = u_o L/\nu$ is the Reynolds number; u_o, L being some reference values of velocity and length respectively.

The equations (4.6) define the axisymmetric fluid flow which now depends only on one parameter R. Various boundary conditions can be associated with (4.6). As an example, consider the case of a cylinder with its top rotating with an angular velocity ω. If the radius and height of the cylinder are a and b respectively, we can use $L = a$, $u_o = a\omega$ to non-dimensionalize (4.5). The boundary conditions are:

$$z = 0 \quad , \quad u = v = w = 0 \tag{4.7a}$$

$$z = b/a \quad , \quad u = w = 0 \quad ; \quad v = r \tag{4.7b}$$

$$r = 1 \quad , \quad u = v = w = 0 \tag{4.7c}$$

$$r = 0 \quad , \quad u = v = 0 \quad ; \quad w_r = 0 \quad , \tag{4.7d}$$

(4.7d) is obtained using the fact that the flow is symmetric about r = 0.

In terms of Ψ, W, Ω, these conditions become:

$$z = 0 \quad , \quad \Psi = \Psi_z = 0 \quad ; \quad \Omega = 0 \tag{4.8a}$$

$$z = b/a \quad , \quad \Psi = \Psi_z = 0 \quad ; \quad \Omega = r^2 \tag{4.8b}$$

$$r = 0 \quad , \quad \Psi = \Omega = W = 0 \tag{4.8c}$$

$$r = 1 \quad , \quad \Psi = \Psi_r = 0 \quad ; \quad \Omega = 0 \quad . \tag{4.8d}$$

The equations (4.6) can be discretized using the technique described in the preceding chapters. If the mesh size is uniform, the finite difference analogue of $D^2 f$ is given by

$$D_h^2 f \equiv h^{-2} \left[-4f_0 + \left(1 - \frac{h}{2r}\right) f_1 + f_2 + \left(1 + \frac{h}{2r}\right) f_3 + f_4 \right] \quad , \tag{4.9}$$

where f_0, f_1, f_2, f_3, f_4 denote the functional values at the points (r, z), $(r + h, z)$, $(r, z + h)$, $(r - h, z)$ and $(r, z - h)$ respectively.

The finite difference analogue of (4.6a) is

$$D_h^2 \Psi_0 = W_0 \quad . \tag{4.10}$$

For (4.6b), the first derivatives are approximated as follows

$$\Psi_r = A/2h \quad , \quad A = \Psi_1 - \Psi_3 \quad ; \tag{4.11a}$$

$$\Psi_z = B/2h \quad , \quad B = \Psi_2 - \Psi_4 \quad . \tag{4.11b}$$

$$\Omega_r = \begin{cases} \dfrac{\Omega_0 - \Omega_3}{h} & B \geq 0 \quad , \\ \dfrac{\Omega_1 - \Omega_0}{h} & B < 0 \quad ; \end{cases} \tag{4.12a}$$

$$\Omega_z = \begin{cases} \dfrac{\Omega_2 - \Omega_0}{h} & A \geq 0 \quad , \\ \dfrac{\Omega_0 - \Omega_4}{h} & A < 0 \quad . \end{cases} \tag{4.12b}$$

This substitution yields a system of linear algebraic equations which has a real, non-symmetric and diagonally dominant coefficient matrix.

The following approximations can be used to discretize (4.6c):

$$\Psi_r = A/2h \quad ; \quad \Psi_z = B/2h \quad ; \quad \Omega_z = C/2h \quad , \tag{4.13}$$

where

$$A = \Psi_1 - \Psi_3 \quad ; \quad B = \Psi_2 - \Psi_4 \quad ; \quad C = \Omega_2 - \Omega_4 \quad .$$

$$W_z = \begin{cases} \dfrac{W_2 - W_0}{h} & A \geq 0 \quad , \\ \dfrac{W_0 - W_4}{h} & A < 0 \quad ; \end{cases} \tag{4.14a}$$

$$W_r = \begin{cases} \dfrac{W_0 - W_3}{h} & B \geq 0 \quad , \\ \dfrac{W_1 - W_0}{h} & B < 0 \quad . \end{cases} \tag{4.14b}$$

The terms on the right hand side of (4.6c) can be replaced by

$$- \frac{R}{hr^2} (\Omega_0 C + \overline{W}_0 B) \quad . \tag{4.15}$$

Here W is not replaced by W_0 as this will destroy the property of diagonal dominance of the resulting coefficient matrix. Instead, the previous approximation $\overline{W}_0$ can be used for W_0.

Thus the equations (4.6) can be discretized in such a way that the coefficient matrices are diagonally dominant in character. This ensures the convergence of the inner iterations by using the appropriate over relaxation procedures. The overall numerical process described in the preceding chapters can be easily extended to solve these equations.

CONCLUSIONS

In this study we have made an attempt to develop a convergent iterative method to obtain two-dimensional steady-state solutions of the Navier-Stokes equations. The finite difference scheme given by Greenspan[22] guarantees the convergence of inner iterations but requires at least two unknown parameters which have to be obtained by numerical experimentation. Even then the use of these parameters do not necessarily increase the rate of convergence. Our aim was to develop an efficient procedure which would be economical and would also require a minimum of numerical experimentation. We have succeeded in showing that only one parameter, namely δ, besides the optimal over-relaxation factors is sufficient for convergence. We have also developed difference schemes using unequal mesh sizes. Different criteria for convergence have been studied and the one, which appears to be most efficient, has been used to obtain numerical solutions for two different boundary configurations.

The numerical solutions of the discretized form of Navier-Stokes equations have been obtained by using a consistent finite-difference scheme. This scheme is convergent in the sense that the solution of the discretized problem converges to the solution of the boundary value problem as h (the mesh size) approaches zero, provided the scheme is stable and the rounding-off error is negligible. The results obtained here are only qualitatively satisfactory and in order to improve their accuracy, it is desirable to consider step sizes much smaller than those used here or elsewhere in the literature. However, any such reduction increases the computing time considerably. This has also been observed for the prototype cavity flow problem by Greenspan[31]. In order to overcome this difficulty, the following

alternatives can be investigated:

(1) Use of finer mesh and better approximations in the boundary layer;

(2) Use of higher order stable difference schemes and block iterative procedures.

In Chapter 4, we have extended the difference schemes to problems of axially symmetric flows which are of considerable physical interest. No numerical example of such a flow has been calculated.

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APPENDIX

COMPUTER PROGRAM FOR THE TWO-DIMENSIONAL MODIFIED STAGNATION POINT FLOW

PROGRAMMING LANGUAGE: FORTRAN IV

COMPUTING MACHINE: IBM 360, model 50

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A PROGRAM TO SOLVE TWO DIMENSIONAL NAVIER STOKES EQUATIONS,IN PARTICULAR,
FOR A MODIFIED STAGNATION POINT FLOW IN A RECTANGULAR REGION. THE BOUNDARY
CONDITIONS ARE PRESCRIBED AND OBTAINED BY USING SOLVE SUBROUTINE.
THE DISCRETE REGION IS BROKEN INTO BLOCKS SO THAT UNEQUAL MESH SIZES CAN
BE USED.THE NOTATION IS AS FOLLOWS.............
NO. OF BLOCKS= MX-HORIZONTAL,NX-VERTICAL
MESH SIZES= ALPHA(NX), BETA(NX)
IJ(I)-NO. OF HORZ. BLOCKS IN THE I-TH VERTICAL BLOCK FROM THE TOP
NO. OF MESH SIZES IN EACH BLOCK-M(MX+1),N(NX+1)..,M(1)=N(1)=3.
L=NO. OF MESHES FOR THE LENGTH OF THE OBSTACLE
ITERMX=MAXIMUM NO. OF OUTER ITERATIONS
ITRMXP=MAXIMUM NO. OF INNER ITERATIONS FOR PSI
ITRMXW=MAXIMUM NO. OF INNER ITERATIONS FOR W
TOLTST=TOLERANCE TEST CYCLE FOR OUTER ITERATIONS
TOLTSP=TOLERANCE TEST CYCLE FOR INNER ITERATIONS FOR PSI
TOLTSW=TOLERANCE TEST CYCLE FOR INNER ITERATIONS FOR W
IOUTER=NO. OF OUTER ITERATIONS AFTER WHICH OUTER DIVERGENCE IS CHECKED
IPUNCH=0 WILL PUNCH INITIAL DATA
IPRINT=0 WILL PRINT INITIAL DATA
ISOLVE=0 WILL READT INITIAL DATA FROM TAPE
IINERP=0 WILL CONTINUE OUTER ITERATIONS EVEN IF PSI DOES NOT CONVERGE.
IINERW=0 WILL CONTINUE OUTER ITERATIONS EVEN IF W DOES NOT CONVERGE.
IFINAL=0 WILL WILL PUNCH FINAL DATA
IRH1=0  WILL LINEARLY COMBINE PSI ONLY ON R-H.I
ITOLPP=0 WILL FIND INNER TOLERANCES FROM TOLSUB SUBROUTINE
ITOLWW=0 WILL WRITE INNER RESIDUALS AT EVERY FIFTH ITERATION
RELAXP,RELAXW  -WHEN TO START COMPUTING SPECTRAL RADIUUS
EIGENP,EIGENW  -WHEN TO START COMPUTING EIGENVALUES FROM EIGENW
JOUTPN,JOUTTP  -WHEN TO PUNCH OR TAPE THE OUTPUT DATA
TOLP,TOLW  - INNER TOLERANCES
OMEGAP,OMEGAW  - SOR PARAMETERS
XI,DELTA  - WEIGHT FACTORS
TOL = OUTER ITERATION TOLERANCE.

 PROGRAMMER--MURLI M. GUPTA
            UNIVERSITY OF SASKATCHEWAN,SASKATOON,CANADA.
       1968-69.
.....................................................................

 DIMENSION AUX(61,61)
 DIMENSION PHI(200),PHI2T(200)
 DIMENSION FT(200) ,FD2T(200)
 DIMENSION ANRW(10),ADRW(10),EGNW(10)
 DIMENSION ANRP(10),ADRP(10),EGNP(10)
 INTEGER TOLTST,TOLTSP,TOLTSW,IOUTER
 INTEGER TTK,TTL,TTLW
 INTEGER RELAXP,RELAXW,EGNTSP,EGNTSW
 COMMON NNN,MMM,R,HHH,IJM
 COMMON X(61),Y(61)
 COMMON PSI(61,61),PSISAV(61,61),W(61,61),WSAV(61,61)
 COMMON OMEGAP,CWOO,OMEGAW,CP1,MX,NX
 COMMON ALPHA(10),BETA(10),IJ(10),M(11),N(11)
```

```
   FORMAT(10X,'OMEGAw =',F10.4,5X,'DELTA=',F10.4)
2  FORMAT(3I5)
1  FORMAT(I5)
2  FORMAT(F10.0)
   FORMAT(10X,2F15.8)
4  FORMAT(/,25X,'*** EIGENVALUE FOR PSI MATRIX=',F10.6,/)
5  FORMAT(///)
6  FORMAT(/,25X,'*** EIGENVALUE FOR w MATRIX=',F10.6,/)
7  FORMAT('****STEP SIZE =',F9.3,' IN HORIZONTAL DIRECTION UPTO THE G
  1RID NUMBERED ..' ,I4)
8  FORMAT('****STEP SIZE =',F9.3,' IN VERTICAL DIRECTION UPTO THE GRI
  1D NUMBERED .. ' ,I4)
9  FORMAT(/,5X,' NEW RELAXATION FACTOR NOW BEING USED=',F10.4,/)
0  FORMAT( 2 I 5)
1  FORMAT(F10.0,I5)
2  FORMAT(10X , 2F 15.0)
3  FORMAT (1H1)
4  FORMAT( 1H1,/,'  **** INITIAL VALUES OF PSI AND W' ,//,20X 'STREAM
  1 FUNCTION ,PSI ' )
5  FORMAT(8F5.0,F10.0,I1)
6  FORMAT(7I5)
7  FORMAT(2F10.0)
8  FORMAT(5X ,'RELAXATION FACTOR=',F10.4)
9  FORMAT(20X,12HRANGE OF X =,F6.2,3H TO,F6.2,/,
  120X,12HRANGE OF Y =,F6.2,3H TO,F6.2,/,
  2/,20X,39HRELAXATION FACTOR FOR STREAM FUNCTION =,F6.2,/,
  320X 33HRELAXATION FACTOR FOR VORTICITY =,F6.2,/,
  420X 'TOLERANCE FOR PSI=',E10.2,/,
  520X 'TOLERANCE FOR w=',E10.2,/,
  620X 'TOLERANCE FOR OUTER ITERATIONS =',E10.2,/,
  720X 22HTOLERANCE TEST CYCLE =,I5,/
  820X 30HMAXIMUM NUMBER OF ITERATIONS =,I6,/,
  920X,17HREYNOLDS NUMBER =,F10.2)
1  FORMAT(20X,46HMAXIMUM ITERATIONS FOR PSI-ONLY CALCULATIONS =,I6,/,
  120X,48HTOLERANCE TEST CYCLE FOR PSI-ONLY ITERATIONS =  ,I6,/
  220X 44HMAXIMUM ITERATIONS FOR W-ONLY CALCULATIONS =,I6 /
  320X 46HTOLERANCE TEST CYCLE FOR W-ONLY CALCULATIONS =I6,/
  420X 21HWEIGHT (XI) FOR PSI =,F6.2 /
  520X 22HWEIGHT (DELTA) FOR W =,F6.2,/
  620X 'NO. OF OUTER ITERATIONS BEFORE TESTING FOR THE-',/
  720X  'DIVERGENCE OF OUTER RESIDUAL =' , I6 ,// )
3  FORMAT(//56H ***** MAXIMUM NUMBER OF ITERATIONS USED. MAX RESIDUAL
  1 =,F12.4,8H AT ITER, I6)
5  FORMAT(/  20H ***** AT ITERATION ,I6,10H,TOLERANCE,E10.2,34H SATIS
  1FIED WITH MAXIMUM RESIDUAL =,E15.6, 9H FOR PSI(,I3,1H),'AT POINT (
  2',I2,1H,,I2,1H))
7  FORMAT(//44H DIVERGENCE. RUN STOPPED WITH MAX RESIDUAL = ,E12.4,8H
  1 AT ITER,I6)
9  FORMAT(1H1,// 4X,'Y',15X,'****STREAM FUNCTION ,PSI***',//)
0  FORMAT(1H1,// 4X ,'Y',15X,'***VORTICITY , W ***',//)
3  FORMAT(// 81H ***** MAXIMUM NUMBER OF ITERATIONS USED FOR PSI-ONLY
  1 ITERATIONS. MAX RESIDUAL = ,E12.4,8H AT ITER ,I6)
7  FORMAT(// 74H****** DIVERGENCE IN PSI-ONLY ITERATIONS. PROBLEM ABA
  1NDONED.MAX RESIDUAL =,E12.4,8H AT ITER ,I6)
2  FORMAT(/ 19H ***** AT ITERATION,I6,20H  MAXIMUM RESIDUAL =,E15.6,
  119H FOR LARGE PROBLEM.,'AT THE POINT (',I2,1H,,I2,1H))
7  FORMAT(/ 20H *****  AT ITERATION,I6,10H TOLERANCE,E10.2,34H SATISF
```

```
     1IED WITH MAXIMUM RESIDUAL = ,E15.6,8H  FOR W(,I3,1H),'AT POINT (',
     2I2,1H,,I2,1H))
    0 FORMAT(/ 19H ***** AT ITERATION,I6,10H TOLERANCE,E10.2,34H SATISFI
     1ED WITH MAXIMUM RESIDUAL =,E15.6, / )
    1 FORMAT(//,20X '****** FINAL RESULTS ARE ----******',//,20X,
     1 'STREAM FUNCTION,PSI' ,//)
    0 FORMAT(/ 19H ***** AT ITERATION,I6,20H  MAXIMUM RESIDUAL =,E15.6,
     119H FOR LARGE PROBLEM.,'AT THE POINT (',I2,1H,,I2,1H),/,
     2 ' ****** OUTER RESIDUAL INCREASING WITH OUTER ITERATIONS.'
     3 /, 5X  '****** PROBLEM ABANDONED ********' )
    5 FORMAT(// 54H ***** DIVERGENCE IN W-ONLY ITERATIONS. MAX RESIDUAL
     1=,E12.4,8H AT ITER,I6)
    7 FORMAT(// 79H *****  MAXIMUM NUMBER OF ITERATIONS USED FOR W-ONLY
     1ITERATIONS. MAX RESIDUAL = ,E12.4,8H AT ITER, I6)
    9 FORMAT(5F16.8)

      NPROB=0

      READ(5,906) ITERMX,TOLTST,ITRMXP,TOLTSP,ITRMXW,TOLTSW,TOUTER
      READ(5,901) R,IJM
      READ(5,90)MX,NX,L
      MX11=MX+1
      NX11=NX+1
      READ(5,91) (IJ(I),I=1,NX)
      READ(5,92) (ALPHA(I),I=1,MX)
      READ(5,92) (BETA(I),I=1,NX)
      READ(5,91) (M(I),I=1,MX11)
      READ(5,91) (N(I),I=1,NX11)
      MMAX=M(MX11)
      NMAX=N(NX11)
      READ(5,90) IPUNCH,IPRINT,ISOLVE
      READ(5,90 ) IINERP,IINERW,IFINAL
      READ(5,90) IRH1,ITOLPP,ITOLWW
      READ(5,900) RELAXP,RELAXW
      READ(5,900) EGNTSP,EGNTSW
      READ(5,901) DECRW,IDECRW
      READ(5,900) JOUTPR,JOUTTP
      READ(5,907) TOLP,TOLW
    1 READ(5,905) XMIN,XMAX,YMIN,YMAX,OMEGAP,OMEGAW,XI,DELTA ,TOL,ISOLVE

      IF(ISOLVE.NE.0) GO TO 2
      CALL INITLVL(PSI,W,NPROB,NMAX,MMAX)
    2 CONTINUE
      NPROB=NPROB+1

      PRINT PROBLEM DESCRIPTION AND PARAMETERS.
      WRITE(6,903)
      WRITE(6,97) (ALPHA(I),M(I+1),I=1,MX)
      WRITE(6,95)
      WRITE(6,98) (BETA(I),N(I+1),I=1,NX)
      WRITE(6,95)
      WRITE(6,909)XMIN,XMAX,YMIN,YMAX,OMEGAP,OMEGAW,TOLP,TOLW,TOL,TOLTST
     1,ITERMX,R
      WRITE(6,9091) ITRMXP,TOLTSP,ITRMXW,TOLTSW,XI,DELTA,TOUTER

    OMPUTE ADDITIONAL PARAMETERS
      M1=MMAX-1
```

```
      N1=NMAX-1
      CP1=1.0-OMEGAP
      XI1=1.0-XI
      R2=0.5*R
      R1=SQRT(R)
      R3=1.0/R1
      L11=L+1
      CW5=2.0/ALPHA(1)**2
      CW6=2.0/BETA(1)**2
      TTK=EGNTSP+1
      TTL=EGNTSW+1
      ITER=0
      ITOL=0
C     COORDINATES  COMPUTED
      X(1)=0.
      Y(1)=0.
      X(2)=ALPHA(1)
      X(3)=X(2)+ALPHA(1)
      Y(2)=BETA(1)
      Y(3)=Y(2)+BETA(1)
      DO 1500 II=1,MX
      H1=ALPHA(II)
      MII1=M(II)+1
      MII2=M(II+1)
      DO 1500 I=MII1,MII2
1500  X(I)=X(I-1)+H1
C     COMPUTING YCOORDINATES
      DO 1600 JJ=1,NX
      H2=BETA(JJ)
      NJJ1=N(JJ)+1
      NJJ2=N(JJ+1)
      DO 1600 J=NJJ1,NJJ2
1600  Y(J)=Y(J-1)+H2
      IF(ISOLVE.EQ.0) GO TO 2030
C     ZERO CODE READS INITIAL VALUES FROM PUNCHED CARDS.
C     ANY OTHER CODE WILL CALCULATE THE INITIAL VALUES FROM SOLVE.
1     MMM=M(2)
      NNN=N(2)
      HHH=BETA(1)
      CALL SOLVE(PHI,PHI2T)
C     FINDING BNDRYVALUES BY SOLVE  AND LINEARLY CALCULATING THE REST
      NOPT=N(2)
      DO 2000 I=1,NOPT
      FT(I)=PHI(I)
1000  FD2T(I)=PHI2T(I)
      DO 2010 I=NOPT,N1
      FT(I+1)=FT(I)+(Y(I+1)-Y(I))*R1
2010  FD2T(I+1)=0.0
C     ALL  INITIAL VALUES  CALCULATED
      DO 2020 I=1,MMAX
      DO 2020 J=1,NMAX
      PSI(I,J)=X(I)*R3*FT(J)
2020  W(I,J)=-X(I)*R1*FD2T(J)
2030  IF(IPUNCH.NE.0) GO TO 2040
C     ZERO CODE WILL PUNCH INITIAL DATA.
      REWIND 9
      WRITE(9) PSI
```

```
   WRITE(9) W
   REWIND 9
0 NNN=NMAX
   MMM=MMAX
RINT INITIAL DATA
   IF(IPRINT.NE.0) GO TO 10
ZERO CODE WILL PRINT INITIAL DATA.
   WRITE(6,904)
   CALL PRTMAT(PSI)
   WRITE(6,9050)
   CALL PRTMAT(W)
EGIN MAIN LOOP FOR OUTER ITERATIONS.


0 ITER=ITER+1
   ITOL=ITOL+1
   JINERP=5
   JINERW=5

   IF(ITOL-TOLTST) 105,102,102
AVE STREAM FUNCTION
2 DO 1021 J=2,N1
   DO 1021 I=2,M1
1 AUX(I,J)=PSI(I,J)
5 ITERP=0
6 ITOLP=0
2 ITERP=ITERP+1
   ITOLP=ITOLP+1
OMPUTE ONE SWEEP OF INNER REGION FOR STREAM FUNCTION
VE THE STREAM FUNCTION
   DO 1051 J=1,NMAX
   DO 1051 I=1,MMAX
1 PSISAV(I,J)=PSI(I,J)

   CALL PSINER

   IF(ITERP.LT.EGNTSP) GO TO 21
   CALL EIGEN(PSI,OMEGAP,PSISAV,ITERP,ANRP,ADRP,EGNP,TTK)
1 IF(ITOLP-TOLTSP) 24,29,29

4 CONTINUE
NTHE OBSTACLE SIDE OF  RH-1, BOUNDARYVALUES ARE:
   DO 23 J=3,L
3 PSI(2,J)=(9.0*PSI(3,J)-2.0*PSI(4,J))/18.0
NOT ON THE OBSTACLE
   DO 26 J=L11,N1
6 PSI(2,J)=(4.0*PSI(3,J)-PSI(4,J))/5.0
N BOTTOM BOUNDARY
   DO 25 I=2,M1
5 PSI(I,2)=(9.0*PSI(I,3)-2.0*PSI(I,4))/18.0
   GO TO 12

9 RMAX1P=0.0

   DO 207 J=3,N1
   DO 207 I=3,M1
   RES=ABS(PSI(I,J)-PSISAV(I,J))
```

```
   IF(RES-RMAX1P) 207,207,206
6 RMAX1P=RES
   IAUXP=I
   JAUXP=J
7 CONTINUE
   DO 261 J=L11,N1
   PSIOLD=PSI(2,J)
   PSINEW=(4.0*PSI(3,J)-PSI(4,J))/5.0
   PSI(2,J)=PSINEW
   RES=ABS(PSIOLD-PSINEW)
   IF(RES-RMAX1P)261,261,260
0 RMAX1P=RES
   IAUXP=I
   JAUXP=J
1 CONTINUE
   DO 271 J=3,L
   PSIOLD=PSI(2,J)
   PSINEW=(9.0*PSI(3,J)-2.0*PSI(4,J))/18.0
   PSI(2,J)=PSINEW
   RES=ABS(PSIOLD-PSINEW)
   IF(RES-RMAX1P)271,271,270
0 RMAX1P=RES
   IAUXP=I
   JAUXP=J
1 CONTINUE
   DO 281I=2,M1
   PSIOLD=PSI(I,2)
   PSINEW=(9.0*PSI(I,3)-2.0*PSI(I,4))/18.0
   PSI(I,2)=PSINEW
   RES=ABS(PSINEW-PSIOLD)
   IF(RES-RMAX1P) 281,281,280
0 RMAX1P=RES
   IAUXP=I
   JAUXP=J
1 CONTINUE
   IF(ITERP.LT.RELAXP) GO TO 31
   ATOLP=TOLTSP
   AAUXP=1.0/ATOLP
   PSIMU=(ABS(RMAX1P/RMAXP))**AAUXP
   WRITE(6,94) PSIMU
1 RMAXP=RMAX1P
MAXP IS THE MAXIMUM RESIDUAL FOR ALL GRID POINTS OF R-H.
   IF(ITOLww.NE.0) GO TO 331
A ZERO CODE WRITES INTERMEDIATE ERRORS AT THE STEPS OF FIVE ITERATIONS.
   IF(ITERP.NE.JINERP) GO TO 331
   WRITE(6,915) ITERP,TOLP,RMAXP,ITER,IAUXP,JAUXP
   JINERP=JINERP+5
1 CONTINUE
EST MAX RESIDUAL FOR DIVERGENCE.
   IF(RMAXP-1.E+3)35,35,32
2 WRITE(6,9017)RMAXP,ITERP
   WRITE(6,9009)
   CALL PRTMAT(PSI)
   WRITE(6,9050)
   CALL PRTMAT(W)
   GO TO 70
5 IF(RMAXP-TOLP) 40,45,45
```

```
0 WRITE(6,915)ITERP,TOLP,RMAXP,ITER,IAUXP,JAUXP
  IF(ITOL-TOLTST)50,405,405
5 RMAX1=0.0
ETMAXIMUM OUTER RESIDUAL FOR STREAM FUNCTION
  DO 42 J=2,M1
  DO 42 I=2,M1
  RES=ABS(PSI(I,J)-AUX(I,J))
  IF(RES -RMAX1) 42,42,41
1 RMAX1=RES
  IAUX=I
  JAUX=J
2 CONTINUE
1 IF(ITER.LT.TOUTER) GO TO 425
TEST FOR SUCCESSIVE OUTER DIVERGENCE.
  IF (RMAX1.GT.RMAX)  GO TO 470
5 RMAX=RMAX1
  ITOL=0
  WRITE(6,9042)ITER,RMAX,IAUX,JAUX
EST OUTER RESIDUAL FOR DIVERGENCE
  IF(RMAX-1.E+3) 435,435,432
2 WRITE(6,917)RMAX,ITER
OUTER DIVERGENCE FOR PSI.
  WRITE(6,9009)
  CALL PRTMAT(PSI)
  WRITE(6,9050)
  CALL PRTMAT(W)
  GO TO 70
EST THE OUTER RESIDUAL FOR CONVERGENCE
5 IF(RMAX-TOL) 440,440,445
0 WRITE(6,9440)ITER,TOL,RMAX
FINAL RESULTS.
   WRITE(6,9441)
  CALL PRTMAT(PSI)
  WRITE(6,9050)
  CALL PRTMAT(W)
  GO TO 70
5 IF(ITER-ITERMX) 50,947,947
7 WRITE(6,913)RMAX,ITER
OUTER ITERATIONS EXCEEDED AND NO CONVERGENCE.

  WRITE(6,9009)
  CALL PRTMAT(PSI)
  WRITE(6,9050)
  CALL PRTMAT(W)
  GO TO 70
EST IF  MAX NO. OF INNER ITERATIONS EXCEEDED FOR PSI.
5 IF(ITERP-ITRMXP) 106,47,47
7 WRITE(6,9013) RMAXP,ITERP
 INNER ITERATIONS FOR PSI EXCEEDED .
  IF(IINERP.EQ.0) GO TO 40
ZERO CODE TRANSFERS THE CONTROL TO 50 WHEN INNER ITERATIONS FOR
 PSI HAVE EQUALLED THE MAXIMUM LIMIT.
  WRITE(6,9009)
  CALL PRTMAT(PSI)
  WRITE(6,9050)
  CALL PRTMAT(W)
  GO TO 70
```

```
0 WRITE(6,9470)  ITER,RMAX1,IAUX,JAUX
TEST FOR SUCCESSIVE OUTER DIVERGENCE.      THE RESIDUAL INCREASING.
  WRITE(6,9009)
  CALL PRTMAT(PSI)
  WRITE(6,9050)
  CALL PRTMAT(W)
  GO TO 70

C RMAXW=1.E+6
ECALCULATE STREAM FUNCTION IN INNER REGION USING WEIGHTING.
  DO 241 J=2,N1
1 PSI(2,J)=XI*AUX(2,J)+XI1*PSI(2,J)
  DO 242 I=3,M1
2 PSI(I,2)=XI*AUX(I,2)+XI1*PSI(I,2)

  IF(IRH1.EQ.0) GO TO 51
ZERO CODE ONLY LINEARLY COMBINE ON R-H.1

  DO 243 J=3,N1
  DO 243 I=3,M1
3 PSI(I,J)=XI*AUX(I,J)+XI1*PSI(I,J)
1 CONTINUE
  IF(ITOLPP.NE.0) GO TO 511
  CALL TOLSUB(TOLP,TOLW,RMAX,NPROB)
1 CONTINUE
EGIN INNER  ITERATIONS FOR VORTICITY.

  DELTA1=1.0-DELTA
  CW00=1.0-OMEGAW
ORTICITY HAS ALREADY BEEN CALCULATED FOR BC,CD&DE.
ORTICITY ON THE OBSTACLE
  DO 5074 J=2,L
4 W(1,J)=DELTA*W(1,J)-DELTA1*CW5*PSI(2,J)
ORTICITY ON BOTTOM BOUNDARY.
  DO 5075 I=1,M1
5 W(I,1)=DELTA*W(I,1)-DELTA1*CW6*PSI(I,2)
TART COMPUTATION FOR W IN R-H.
  ITERW=0
5 ITOLW=0
OMPUTE ONE SWEEP OF VORTICITY IN THE INTERIOR
6 ITERW=ITERW+1
  ITOLW=ITOLW+1
  DO 1151 J=1,NMAX
  DO 1151 I=1,MMAX
1 WSAV(I,J)=W(I,J)


  CALL WINNER

  IF(ITERW.LT.EGNTSW) GO TO 625
  CALL EIGEN(W,OMEGAW,WSAV, ITERW,ANRW,ADRW,EGNW,TTL)
5 IF(ITOLW-TOLTSW) 506,63,63
3 RMAX1W=0.0
  DO 65 J=2,N1
  DO 65 I=2,M1
  RES=ABS(WSAV(I,J)-W(I,J))
```

```
   IF(RES-RMAX1W) 65,65,64
4  RMAX1W=RES
   IAUXW=I
   JAUXW=J
5  CONTINUE
   IF(ITERW.LT.RELAXW) GO TO 61
   ATOLW=TOLTSW
   AAUXW=1.0/ATOLW
   WMU=(ABS(RMAX1W/RMAXW))**AAUXW
   WRITE(6,96) WMU
1  IF(ITER.LT.IDECRW) GO TO 66
   IF(RMAX1W.LT.RMAXW) GO TO 66
   IF(OMEGAW.LE.1.0) GO TO 66
REDUCE OMEGAW BY THE AMOUNT DECRW
   OMEGAW=OMEGAW-DECRW
   WRITE(6,9) OMEGAW,DELTA
   CWOO=1.0-OMEGAW
   DELTA1=1.0-DELTA
6  RMAXW=RMAX1W
   IF(ITOLWW.NE.0) GO TO 631
A ZERO CODE WRITES INTERMEDIATE ERRORS AT THE STEPS OF FIVE ITERATIONS.
   IF(ITERW.NE.JINERW) GO TO 631
   WRITE(6,9067) ITERW,TOLW,RMAXW,ITER,IAUXW,JAUXW
   JINERW=JINERW+5
1  CONTINUE
EST VORTICITY FOR INNER-ITERATION DIVERGENCE
7  IF(RMAXW-1.E+3) 666,666,665
5  WRITE(6,9665) RMAXW,ITERW
DIVERGENCE IN INNER ITERATIONS FOR W.
   WRITE(6,9009)
   CALL PRTMAT(PSI)
   WRITE(6,9050)
   CALL PRTMAT(W)
   GO TO 70

6  IF(RMAXW-TOLW) 67,67,675
7  WRITE(6,9067)ITERW,TOLW,RMAXW,ITER,IAUXW,JAUXW
     GO TO 700
EST IF MAXIMUM NUMBER OF INNER ITERATIONS EXCEEDED FOR VORTICITY.
5  IF(ITERW-ITRMXW) 505,677,677
7  WRITE(6,9677) RMAXW,ITERW
THE NO. OF ITERATIONS FOR INNER W EXCEEDED THE LIMIT PRESCRIBED.
   IF(IINERW.EQ.0) GO TO 700
  ZERO CODE TRANSFERS THE CONTROL TO 10 WHEN INNER ITERATIONS FOR W
 HAVE EQUALLED THE MAXIMUM PRESCRIBED.
   WRITE(6,9009)
   CALL PRTMAT(PSI)
   WRITE(6,9050)
   CALL PRTMAT(W)
   GO TO 70
2  IF(ITER.LT.JOUTTP) GO TO 701
   REWIND 9
   WRITE(9) PSI
   WRITE(9) W
   REWIND 9
1  IF(ITER.NE.JOUTPN) GO TO 702
   CALL PNCHMT(PSI,NMAX,MMAX)
```

```
      CALL PNCHMT(W,NMAX,MMAX)
    2 IF(ITOLPP.NE.0) GO TO 10
      CALL TOLSUB(TOLP,TOLW,RMAX,NPROB)
      GO TO 10
    0 CONTINUE
      IF(IFINAL.NE.0) GO TO 1
     ZERO CODE PUNCHES FINAL RESULTS FOR THE PROBLEM.
      CALL PNCHMT(PSI,NMAX,MMAX)
      CALL PNCHMT(W,NMAX,MMAX)
      GO TO 1
      END
```

```
      SUBROUTINE WINNER
      COMMON NNN,MMM,R,HHH,IJM
      COMMON X(61),Y(61)
      COMMON PSI(61,61),PSISAV(61,61),W(61,61),WSAV(61,61)
      COMMON OMEGAP,CW00,OMEGAW,CP1,MX,NX
      COMMON ALPHA(10),BETA(10),IJ(10),M(11),N(11)
     TO TAKE CARE OF THE REGION R.H-1  WHICH HAS NO SIGNIFICANCE
      FOR CALCULATIONS FOR THE VORTICITY THE CALCULATIONS ARE TO BE DONE
     RIGHT UPTO THE FIRST GRID.  BUT THE PROGRAM DEFINES N(1) & M(1) FOR PSI.
    SO THE NO. OF THESE QUANTITIES IS DECREASED BY 1.
      M(1)=M(1)-1
      N(1)=N(1)-1

      DO 1150JJ=1,NX
      JJL=NX+1-JJ
      H2=BETA(JJL)
      NJJL=N(JJL+1)-N(JJL)
      IJAX=IJ(JJ)
      DO 1150 II=1,IJAX
      IIL=IJ(JJ)+1-II
      H1=ALPHA(IIL)
      MIIL=M(IIL+1)-M(IIL)

      DO 1150 J=1,NJJL
      IF(JJL.EQ.1) GO TO 1173
      IF(J.EQ.NJJL) GO TO 1171
3     H4=H2
      GO TO 1172
1     H4=BETA(JJL-1)
2     DO 1150 I=1,MIIL
      IF(IIL.EQ.1)GO TO 1183
      IF(I.EQ.MIIL)GO TO 1181
3     H3=H1
      GO TO 1182
1     H3=ALPHA(IIL-1)
2     IV=M(IIL+1)
      JV=N(JJL+1)
      K1=IV+1
      K2=IV-1
      K3=JV+1
      K4=JV-1
      CK0=2./(H1*H3) +2./(H2*H4)
      CK1=2./(H1*(H1+H3))
      CK2=2./(H2*(H2+H4))
      CK3=2./(H3*(H1+H3))
      CK4=2./(H4*(H2+H4))
      A=PSI(K1-I,JV-J)-PSI(K2-I,JV-J)
      B=PSI(IV-I,K3-J)-PSI(IV-I,K4-J)
      RA=R*A/(H1+H3)
      RB=R*B/(H2+H4)
      IF(A) 55,51,51
1     IF(B)53,52,52
2     CW0=CK0+RA/H2+RB/H3
      CW1=CK1
      CW2=CK2+RA/H2
      CW3=CK3+RB/H3
      CW4=CK4
```

```
   GO TO 60
3  CW0=CK0+RA/H2-RB/H1
   CW1=CK1-RB/H1
   CW2=CK2+RA/H2
   CW3=CK3
   CW4=CK4
   GO TO 60
5  IF(B)57,56,56
6  CW0=CK0-RA/H4+RB/H3
   CW1=CK1
   CW2=CK2
   CW3=CK3+RB/H3
   CW4=CK4-RA/H4
   GO TO 60
7  CW0=CK0-RA/H4-RB/H1
   CW1=CK1-RB/H1
   CW2=CK2
   CW3=CK3
   CW4=CK4-RA/H4

9  CW0=OMEGAw/CW0
   W(IV-I,JV-J)=CW00*W(IV-I,JV-J)+CW0*(CW1*W(K1-I,JV-J)+CW2*W(IV-I,K3
  1-J)+CW3*W(K2-I,JV-J)+CW4*W(IV-I,K4-J))
60 CONTINUE

  TO RESTORE THE ORIGINAL SITUATION THE NUMBERS N(1) & M(1) ARE
  BEING GIVEN ORIGINAL VALUES BY ADDING 1.
   M(1)=M(1)+1
   N(1)=N(1)+1
   RETURN
   END
```

```
      SUBROUTINE PSINER
      COMMON NNN,MMM,R,HHH,IJM
      COMMON X(61),Y(61)
      COMMON PSI(61,61),PSISAV(61,61),W(61,61),WSAV(61,61)
      COMMON OMEGAP,CW00,OMEGAW,CP1,MX,NX
      COMMON ALPHA(10),BETA(10),IJ(10),M(11),N(11)
      DO 1050 JJ=1,NX
      JJL=NX+1-JJ
      H2=BETA(JJL)
      NJJL=N(JJL+1)-N(JJL)
      IJAX=IJ(JJ)
      DO 1050 II=1,IJAX
      IIL=IJ(JJ)+1-II
      H1=ALPHA(IIL)
      MIIL=M(IIL+1)-M(IIL)
      DO 1050 J=1,NJJL
      IF(JJL.EQ.1) GO TO 1073
      IF(J.EQ.NJJL)GO TO 1071
3     H4=H2
      GO TO 1072
1     H4=BETA(JJL-1)
2     DO 1050 I=1,MIIL
      IF( IIL.EQ.1) GO TO 1083
      IF(I.EQ.MIIL) GO TO 1081
3     H3=H1
      GO TO 1082
1     H3=ALPHA(IIL-1)
2     IV=M(IIL+1)
      JV=N(JJL+1)
      K1=IV+1
      K2=IV-1
      K3=JV+1
      K4=JV-1
      CKC=2./(H1*H3)+2./(H2*H4)
      CK1=2./(H1*(H1+H3))
      CK2=2./(H2*(H2+H4))
      CK3=2./(H3*(H1+H3))
      CK4=2./(H4*(H2+H4))
      PSIAUX          =CK1*PSI(K1-I,JV-J)+CK2*PSI(IV-I,K3-J)+CK3*PSI(K2-I,
     1JV-J)+CK4*PSI(IV-I,K4-J)+W(IV-I,JV-J)
      PSI(IV-I,JV-J)=CP1*PSI(IV-I,JV-J)+OMEGAP/CKC*PSIAUX
0     CONTINUE
      RETURN
      END
```

```
      SUBROUTINE EIGEN(Z,OMEGA,ZSAV,ITER,ANR,ADR,EGN,ITT)
      DIMENSION  Z(61,61),ZSAV(61,61)
      DIMENSION ANR(10),ADR(10),EGN(10)
      DIMENSION A(10),AS(10)
      COMMON NNN,MMM,R,HHH,IJM
      COMMON X(61),Y(61)
    9 FORMAT(5F20.8)
    6 FORMAT( /,5X,'EIGENVALUES AT DIFFERENT POINTS',/)
      A(1)=Z(4,7)
      AS(1)=ZSAV(4,7)
      A(2)=Z(9,14)
      AS(2)=ZSAV(9,14)
      A(3)=Z(13,12)
      AS(3)=ZSAV(13,12)
      A(4)=Z(7,3)
      AS(4)=ZSAV(7,3)
      A(5)=Z(19,6)
      AS(5)=ZSAV(19,6)
      INK=5
      WRITE(6,96)
      DO 5 I=1,INK
      EGN(I)=0.0
      ANR(I)=A(I)-AS(I)
      IF(ITER.LT.ITT) GO TO 5
      EGN(I)=ANR(I)/ADR(I)
    5 ADR(I)=ANR(I)
      WRITE(6,9) (EGN(I),I=1,INK)
      RETURN
      END
```

```
      SUBROUTINE PRTMAT(AX)
      DIMENSION AX(61,61)
      COMMON N,M,AZ1,AZ2,IZ1
      COMMON X(61),Y(61)
      I=1
      IM=M
      N11=N+1
    1 IF(IM.GT.11) GO TO 10002
      DO 10005  J=1,N
      N11J=N11-J
    5 WRITE(6,10000) Y(N11J),(AX(K,N11J), K=I,M)
      WRITE(6,10007) (X(K),K=I,M)
      WRITE(6,10009)
      RETURN
    0 FORMAT(2X,F5.3,1X,11F10.5)
    7 FORMAT(//,3X, 2HX=,3X,11(5X,F5.3))
    9 FORMAT(///)
    2 IM=IM-11
      I9=I+10
      DO 10006 J=1,N
      N11J=N11-J
    6 WRITE(6,10000) Y(N11J),(AX(K,N11J),K=I,I9)
      WRITE(6,10007)(X(K),K=I,I9)
      WRITE(6,10009)
      IF(IM.EQ.0)  RETURN
      I=I+11
      GO TO 10001
      END
```

```
  SUBROUTINE SOLVE(PHI,PHI2T)
  STAGNATION POINT FLOW
PHI  AND  PHI2T ARE THE SOLUTION ARRAYS TO BE FINALLY USED IN
THE MAIN PROGRAM.
 X  IS THE STEP  SIZE USED IN THE MAIN PROGRAM.
IN IS THE NO. OF GRID POINTS IN THE VERTICAL DIRECTION .
  DIMENSION PHI(200),PHI2T(200)
  DOUBLE PRECISION FT(1,1500),FDT(1,1500),FD2T(1,1500)
  DOUBLE PRECISION FD3T(1,1500)
  DOUBLE PRECISION  H,H1,H2,EPS,AM
  DOUBLE PRECISION AB,BA,ERR,A
  DOUBLE PRECISION R
  DOUBLE PRECISION X,AMM
  COMMON IN,IJK,RAY,AXX,M
 IJK  IS A DUMMY VARIABLE WHICH HAS NO USE IN THIS SUBPROGRAM.
4 FORMAT(//,41H NO. OF GRIDS CALCULATED FROM SUBROUTINE=, I3)
6 FORMAT(23H P-C NOT CONVERGING  I=  ,I3)
  IT=0
  AM=1.0
  FT(1,1)=0.
  FDT(1,1)=0.
  FD2T(1,1)=1.232588000
  J=1
  N=1500
  R=RAY
  X=AXX
  AMM=X*DSQRT(R)/H
  H=0.01
  H1=H*H
  H2=H1*H
  I=1
  EPS=0.000001
  FD3T(J,1)=(FDT(J,I)*FDT(J,I)-AM*FT(J,I)*FD2T(J,I)-1.D0)
  FT(J,I&1)=FT(J,I)&H*FDT(J,I)&H1*FD2T(J,I)/2.D0&H2*FD3T(J,I)/6.D0
  FDT(J,I&1)=FDT(J,I)&H*FD2T(J,I)&H1*FD3T(J,I)/2.D0
  FD2T(J,I&1)=FD2T(J,I)&H*FD3T(J,I)
  FD3T(J,I&1)=(FDT(J,I&1)*FDT(J,I&1)-AM*FT(J,I&1)*FD2T(J,I&1)-1.D0)
  FT(J,I&1)=FT(J,I)&H*FDT(J,I)&H1*FD2T(J,I)/2.D0
  FT(J,I&1)=FT(J,I&1)&H2*(3.D0*FD3T(J,I)&FD3T(J,I&1))/24.D0
  FDT(J,I&1)=FDT(J,I)&H*FD2T(J,I)&H1*(2.D0*FD3T(J,I)&FD3T(J,I&1))/6.
 1D0
  AB=FDT(J,I&1)
  FD2T(J,I&1)=FD2T(J,I)&H*(FD3T(J,I)&FD3T(J,I&1))/2.D0
  FD3T(J,I&1)=(AB*AB-AM*FT(J,I&1)*FD2T(J,I&1)-1.D0)
  FT(J,I&2)=4.D0*FT(J,I&1)-3.D0*FT(J,I)-2.D0*H*FDT(J,I)
  FT(J,I&2)=FT(J,I&2)&H2*(FD3T(J,I)&3.D0*FD3T(J,I&1))/6. D0
  FDT(J,I&2)=2.D0*FDT(J,I&1)-FDT(J,I)&H1*FD3T(J,I&1)
  BA=FDT(J,I&2)
  FD2T(J,I&2)=FD2T(J,I)&2.D0*H*FD3T(J,I&1)
  FD3T(J,I&2)=(BA*BA-AM*FT(J,I&2)*FD2T(J,I&2)-1.D0)
4 A=H2*(27.D0*FD3T(J,I)&16.D0*FD3T(J,I&1)-3.D0*FD3T(J,I&2))/240.D0
  FT(J,I&1)=FT(J,I)&H*FDT(J,I)&H1*FD2T(J,I)/2.D0&A
  A=H1*(7.D0*FD3T(J,I)&6.D0*FD3T(J,I&1)-FD3T(J,I&2))/24.D0
  FDT(J,I&1)=FDT(J,I)&H*FD2T(J,I)&A
  A=H*(5.D0*FD3T(J,I)&8.D0*FD3T(J,I&1)-FD3T(J,I&2))/12. D0
  FD2T(J,I&1)=FD2T(J,I)&A
  A=H2*(FD3T(J,I)&18.D0*FD3T(J,I&1)&FD3T(J,I&2))/60.D0
```

```
  FT%J,I&2<#FT%J,I<&2.DO*H*FDT%J,I&1<&A
  A#H1*%FD3T%J,I<&10.DO*FD3T%J,I&1<&FD3T%J,I&2<</12.DO
  FDT%J,I&2<#FDT%J,I&1<*2.DO-FDT%J,I<&A
  A#H*%FD3T%J,I<&4.DO*FD3T%J,I&1<&FD3T%J,I&2<</3. DO
  FD2T%J,I&2<#FD2T%J,I<&A
  FD3T%J,I&1<#%FDT%J,I&1<*FDT%J,I&1<-AM*FT%J,I&1<*FD2T%J,I&1<-1.DO<
  FD3T%J,I&2<#%FDT%J,I&2<*FDT%J,I&2<-AM*FT%J,I&2<*FD2T%J,I&2<-1.DO<
  ERR#DABS%AB-FDT%J,I&1<<
  IF%ERR.GT.EPS< GO TO 15
  ERR#DABS%BA-FDT%J,I&2<<
  IF%ERR.GT.EPS< GO TO 15
  GO TO 16
5 IF%IT.GT.100< GO TO 41
  AB#FDT%J,I&1<
  BA#FDT%J,I&2<
  IT#IT&1
  GO TO 14
1 K#I&2
   WRITE%6,6< K
  GO TO 21
  CONTINUE
  I#I&2
  IT#0
  FD3T%J,I&1<#2.DO*FD3T%J,I<-FD3T%J,I-1<
8 A#H2*%18.DO*FD3T%J,I<&FD3T%J,I-1<&FD3T%J,I&1<</60.DO
  FT%J,I&1<#FT%J,I-1<&2.DO*H*FDT%J,I<&A
  A#H1*%10.DO*FD3T%J,I<&FD3T%J,I-1<&FD3T%J,I&1<</12. DO
  FDT%J,I&1<#2.DO*FDT%J,I<-FDT%J,I-1<&A
  A#H*%FD3T%J,I-1<&4.DO*FD3T%J,I<&FD3T%J,I&1<</3. DO
  FD2T%J,I&1<#FD2T%J,I-1<&A
  FD3T%J,I&1<#%FDT%J,I&1<*FDT%J,I&1<-AM*FT%J,I&1<*FD2T%J,I&1<-1.DO<
  IF%IT.GT.1< GO TO 40
9 IT#IT&1
  AB#FDT%J,I&1<
  GO TO 18
3 ERR#DABS%AB-FDT%J,I&1<<
  IF%ERR.LT.EPS< GO TO 20
  IF %IT.LT.100< GO TO 19
   WRITE%6,6< I
    GO TO 21
3 CONTINUE
  I#I&1
  IF%I.EQ.700< GO TO 211
  GO TO 17
1 DO 1001 IJ#1,800
  I#IJ&700
  AIJ#IJ
  FT%1,I<#FT%1,700<&0.01*AIJ
1 FD2T%1,I<#0.0
1 E#0.0
  L#1
  DO 1002 K#1,I ,M
  PHI%L<#FT%1,K<
  PHI2T%L<#FD2T%1,K<
  L#L&1
2 E#E&AMM*H
  IL#L-1
```

```
      WRITE(6,4) IL
      DO 1003 K=L,IN
      AAMM=AMM*0.01
      PHI2T(K)=0.0
      PHI(K)=PHI(K-1)+AAMM
 3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE INTLVL(PSI,W,NPROB,NMAX,MMAX)
      DIMENSION PSI(61,61),W(61,61)
      REWIND 9
      READ(9) PSI
      READ(9) W
      REWIND 9
      RETURN
      END


      SUBROUTINE TOLSUB(TOLP,TOLW,RMAX,NPROB)
      TOLW=RMAX
      RETURN
      END


      SUBROUTINE PNCHMT(X,N,M)
      DIMENSION  X(61,61)
    9 FORMAT(5F16.8)
      WRITE(7,99) ((X(I,J),J=1,N),I=1,M)
      RETURN
      END
```