Fermionic Fields with Mass Dimension One as Supersymmetric Extension of the O'Raifeartaigh Model

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Abstract

The objective of this thesis is to derive a supersymmetric Lagrangian for fermionic fields with mass dimension one and to discuss their coupling to the O'Raifeartaigh model which is the simplest model permitting supersymmetry breaking. In addition it will be shown that eigenspinors of the charge conjugation operator (ELKO) exhibit a different transformation behaviour under discrete symmetries than previously assumed.

The calculations confirm that ELKO spinors are not eigenspinors of the parity operator and satisfy $(CPT)^2 = -\mathbb{I}$ which identifies them as representation of a nonstandard Wigner class. However, it is found that ELKO spinors transform symmetrically under parity instead of the previously assumed asymmetry. Furthermore, it is demonstrated that ELKO spinors transform asymmetrically under time reversal which is opposite to the previously reported symmetric behaviour. These changes affect the (anti)commutation relations that are satisfied by the operators acting on ELKO spinors. Therefore, ELKO spinors satisfy the same (anti)commutation relations as Dirac spinors, even though they belong to two different representations of the Lorentz group.

Afterwards, a supersymmetric model for fermionic fields with mass dimension one based on a general superfield with one spinor index is formulated. It includes the systematic derivation of all associated chiral and anti-chiral superfields up to third order in covariant derivatives. Starting from these fundamental superfields a supersymmetric on-shell Lagrangian that contains a kinetic term for the fermionic fields with mass dimension one is constructed. This on-shell Lagrangian is subsequently used to derive the on-shell supercurrent and to successfully formulate a consistent second quantisation for the component fields. In addition, the Hamiltonian in position space that corresponds to the supersymmetric Lagrangian is calculated. As the Lagrangian is by construction supersymmetric and the second quantisation of the component fields is consistent with their general supertranslations, the Hamiltonian is positive definite. This is confirmed by the results for the Hamiltonian in momentum space and the derivation of the creation and annihilation operators in momentum space. Based on these results, fermionic fields with mass dimension one represent an intriguing candidate for supersymmetric dark matter.

As an application the coupling of the fermionic fields with mass dimension one to the O'Raifeartaigh model is discussed. It turns out that the coupled model has two distinct

solutions. The first solution representing a local minimum of the superpotential spontaneously breaks supersymmetry in perfect analogy to the O'Raifeartaigh model. The second solution is more intriguing as it corresponds to a global minimum of the superpotential. In this case the coupling to the fermionic sector restores supersymmetry. This is, however, achieved at the cost of breaking Lorentz invariance. Finally, the mass matrices for the multiplets of the coupled model are presented. It turns out that it contains two bosonic triplets and one fermionic doublet which are mass multiplets. In addition it contains a massless fermionic doublet as well as one fermionic triplet which is not a mass multiplet but rather an interaction multiplet that contains component fields of different mass dimension.

These results show that the presented model for fermionic fields with mass dimension one is a viable candidate for supersymmetric dark matter that could be accessible to experiments in the near future.

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DM	dark matter
ELKO	Eigenspinors of the charge conjugation operator
LHC	Large Hadron Collider
MSSM	Minimal Supersymmetric Standard Model
SM	Standard Model

Chapter 1

Introduction

The recent proposal of matter fields with spin one-half and mass dimension one, "Eigenspinoren des Ladungskonjugationsoperators" (ELKO) – translating to eigenspinors of the charge conjugation operator – has opened a wide field of theoretical research previously thought to be unphysical (Ahluwalia-Khalilova and Grumiller, 2005a,b).

This belief was due to the nonlocality of the theory (Lee and Wick, 1966), even though the mathematical foundations were already formulated by Wigner (1939, 1964). From the present point of view, however, nonlocal degrees of freedom could become relevant close to the quantum gravity scale. It is further supported by the recent proposal of a set of local fermionic fields with mass dimension one (Ahluwalia et al., 2010).

From the theoretical point of view the study of fermionic fields with mass dimension one and thus ELKO spinors as a special case proves to be interesting. There are several reasons that make them a good candidate for dark matter (DM). First, ELKO spinors are of mass dimension one, even though they carry spin one-half (Ahluwalia-Khalilova and Grumiller, 2005a). Due to their mass dimension, they only interact very weakly with Standard Model (SM) spinors and gauge fields while the dominant contribution comes from unsuppressed quartic interactions with neutral scalar fields (da Rocha and Pereira, 2007). In the SM this is the Higgs field. Therefore, it is not surprising that ELKO spinors have not been detected yet and are first-principle DM candidates (Ahluwalia-Khalilova and Grumiller, 2005b). Second, they transform according to $(CPT)^2 = -\mathbb{I}$ and thus belong to a nonstandard Wigner class – class 5 in the Lounesto spinor field classification (Lounesto, 2002; da Rocha and Rodrigues Jr., 2006). Third, the simplest cosmological ELKO spinors (in absence of a preferred direction) satisfy scalar-like equations of motion and include a quartic self-interaction term (Ahluwalia-Khalilova and Grumiller, 2005b; Ahn and Shapiro, 2005). Finally, the model can be mapped to a scalar field theory and it is tempting to consider them as a source of inflation. This could have interesting consequences for cosmological models (Böhmer, 2007a,b; Böhmer et al., 2008).

ELKO spinors are also of special interest in light of recent work presented by Novello (2007) where it was shown that the spacetime metric can be interpreted as effective geometry without own dynamics. Instead the metric inherits its dynamics from two fundamental spinor fields. In this formalism the coupling of matter to spinor fields is kinematically identical to couplings in General Relativity. Within this spinor theory of gravity it is possible to construct a spatially homogeneous and isotropic universe as a special solution.

However, it can be shown that the Lagrangian presented by Ahluwalia-Khalilova and Grumiller (2005a,b) is not supersymmetric. Furthermore, there exists no mapping of the Lagrangian in field theoretical notation onto the superspace component field formalism that is commonly used to formulate models in particle theory. This makes it impossible to simply use the results from any of the previous publications on fermionic fields with mass dimension one to achieve a coupling or extension of one of the existing models in particle physics like the O'Raifeartaigh Model or the Minimal Supersymmetic Extension of the Standard Model.

This motivated the generalisation of ELKO spinors to the more fundamental concept of fermionic fields with mass dimension one in superspace. Therefore, all arguments presented above still apply. As mentioned above, for ELKO spinors there exists no mapping between fields in spacetime and component fields in superspace and it is necessary to formulate a model from ground up that is by construction supersymmetric and at the same time contains fermionic fields with mass dimension one. These component fields are no longer identified with ELKO spinors as introduced by Ahluwalia-Khalilova and Grumiller (2005a). Instead the fermionic component fields in superspace are solely constrained by their mass dimension as well as the requirement that it must be possible to formulate a supersymmetric Lagrangian describing dynamic fermionic fields with mass dimension one. Trivial solutions without possible kinetic terms are excluded. This Lagrangian builds the foundation for the formulation of a consistent model for fermionic fields with mass dimension one. Subsequently, this model is applied to extend the O'Raifeartaigh model which is the simplest model permitting spontaneous symmetry breaking.

In the following sections some of the fundamental concepts that are essential to achieve the objective of this thesis will be introduced. In Section 1.1 the general concept of units is presented and it is discussed when the term *mass dimension* becomes meaningful. Afterwards, in Section 1.2 discrete symmetries will be introduced. This includes a derivation of charge conjugation, parity, and time reversal that is consistent with the conventions used in this thesis. Then the concept of supersymmetry is presented in Section 1.3. Finally, in Section 1.4 the breaking of supersymmetry is explained.

1.1 The Mass Dimension

To understand how a mass dimension is assigned to a field or any other physical object or quantity in a meaningful way it is necessary to clarify the concept of mass dimensionality and more generally the concept of units.

In everyday life the units of interest are mass M, length L, and time T and are used to describe size and weight of physical objects as well as temporal separation between events. This rudimentary approach to units still holds in physics where any system of units, e. g., cgs, SI, or Planck units, can be boiled down to fundamental units for mass, length, and time. However, for a number of applications like general relativity or particle theory it proves useful to reduce this set of units using constraints that simplify the calculations and eliminate constants that appear repeatedly. The most common sets of constrained units are geometrised units, see, e. g., Hartle (2003), and natural units. In addition to these commonly used systems of units it is possible to conceive arbitrarily many other unit systems that may be advantageous for certain applications. Due to this ambiguity of the definition of the fundamental set of units, it is important to clearly specify the unit system used.

For the rest of this thesis natural units that are commonly used in particle theory and string theory will be adapted. In this convention the speed of light c and Planck's constant $\hbar$ are assumed to be dimensionless and equal to unity, $c = \hbar = 1$. These two constraints on the units then reduce the number of units such that L, M, and T can be expressed in terms of one single unit which is chosen to be the mass. Only then talking about the mass dimension of a physical object or in this specific case the mass dimension of component fields becomes meaningful.

The condition $c = 1$ leads to a relation between the units of length and time

$$1 = [c] = \frac{L}{T} \quad \Rightarrow \quad T = L\,. \tag{1.1}$$

Furthermore, setting $\hbar = 1$ and using the relation between length and time from the previous equation leads to a relation between length and mass

$$1 = [\hbar] = ML^2/T = ML \quad \Rightarrow \quad L = \frac{1}{M}\,. \tag{1.2}$$

If all units are expressed in terms of the mass, the new units for mass, length, and time are

$$M = M\,, \quad L = \frac{1}{M}\,, \quad T = \frac{1}{M}\,. \tag{1.3}$$

Obviously the mass still has the unit of a mass. However, in natural units, time and length have the units of an inverse mass.

Before proceeding to discuss mass dimensionality in the superfield formalism it is educational to review an example in classical field theory. In classical field theory it is assumed that bosonic fields satisfy the Klein-Gordon equation and obey commutation relations, while fermionic fields satisfy the Dirac equation in addition to the Klein-Gordon equation and obey anticommutation relations. It can then be shown that the simple Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi \tag{1.4}$$

containing the fermionic field ψ as well as its Dirac dual $\bar{\psi} = \psi^\dagger\gamma^0$ corresponds to the Hamiltonian

$$H = \int \mathrm{d}^3x\,\bar{\psi}\,(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi. \tag{1.5}$$

As the Hamiltonian represents the energy of the system, it has the following natural units

$$M = [H] = M^{-3}\left[\bar{\psi}\right] M\,[\psi] = M^{-2}\left[\bar{\psi}\right][\psi]\,. \tag{1.6}$$

The fermionic field and its Dirac dual that both appear in this equation have the same mass dimension. This implies that the fermionic field ψ in classical field theory has mass dimension

$$[\psi] = M^{3/2}\,. \tag{1.7}$$

The discussion of mass dimensionality in the superfield formalism follows in general the same principles that were used for the discussion in classical field theory. The only complication arises from the additional Grassmann valued superspace coordinates that have no corresponding counterparts in classical field theory. It can be shown that the Grassmann variables θ and $\bar{\theta}$ have mass dimension

$$\dim(\theta) = \dim\left(\bar{\theta}\right) = -\frac{1}{2}\,. \tag{1.8}$$

Using the expansion of the general superfield V in Grassmann variables which will be discussed in detail in Sections 3.1 and 3.4 together with the fact that the general superfield V has mass dimension 0, it is straightforward to calculate the mass dimension of the component fields, $\dim(C) = 0$, $\dim(\chi) = 1/2$, etc.

The transition from the general superfield V to the general superfield with one spinor index V_α introduces additional spinor indices. As the goal of this thesis is to formulate a model describing fermionic fields with mass dimension one, the mass dimension of the superfield V_α has to be chosen such that it contains at least one spinor field satisfying this property. Therefore, the superfield V_α must have an integer valued mass dimension. A mass dimension of less then 0 is excluded for obvious reasons, while a mass dimension of 1 or larger restricts the number of possible mass and kinetic terms significantly. Therefore, the general superfield with one spinor index must also have mass dimension 0. For the specific choice of V_α that will be presented in equation 3.17 this leads to mass dimensions for the component fields of

$$\dim(\kappa_\alpha) = 0\,, \tag{1.9}$$

$$\dim(M_{\beta\alpha}) = \dim(N_{\beta\alpha}) = \frac{1}{2}\,, \tag{1.10}$$

$$\dim(\psi_\alpha) = \dim(\chi_\alpha) = \dim(\omega_{\mu\alpha}) = 1\,, \tag{1.11}$$

$$\dim\left(R_{\dot{\beta}\alpha}\right) = \dim(S_{\beta\alpha}) = \frac{3}{2}\,, \tag{1.12}$$

$$\dim(\lambda_\alpha) = 2\,. \tag{1.13}$$

It can be seen that the general superfield with one spinor index contains component fields with mass dimension ranging from 0 to 2. More importantly it is found that it contains three possible candidates for a fermionic field with mass dimension one. Two of those are spinor fields, while the third one is a spinor-vector field.

1.2 Discrete Symmetries

A discussion of the discrete symmetry operators – namely charge conjugation, parity, and time reversal – can be found in most field theory textbooks, see, e. g., Peskin and Schroeder (1995, Sect. 3.6) or Landau and Lifshitz (1982, Sect. 26). Even though the book by Landau and Lifshitz (1982) does not contain an explicit derivation of the parity operator P, a general equation for P can be derived in analogy to the discussions for C and T.

In analogy to the conventions by Peskin and Schroeder (1995), the metric is chosen as

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)\,, \tag{1.14}$$

while the Dirac matrices in the Weyl representation is defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,, \tag{1.15}$$

where σ^μ are the Pauli matrices and $\bar{\sigma}^\mu = \left(\mathbb{I}, -\boldsymbol{\sigma}^i\right)$.

A complete summary of the conventions can be found in the Appendix. Appendix A.1 summarises the general conventions, while Appendix A.2 contains all ELKO specific definitions. The action of the operators on Dirac spinors can then be written as

$$C\psi(t,\mathbf{x}) = \eta_C \gamma^2 \psi^*(t,\mathbf{x})\,, \tag{1.16}$$

$$P\psi(t,\mathbf{x}) = \eta_P \gamma^0 \psi(t,-\mathbf{x})\,, \tag{1.17}$$

$$T\psi(t,\mathbf{x}) = \eta_T \gamma^1 \gamma^3 \psi^*(-t,\mathbf{x})\,, \tag{1.18}$$

where η_C, η_P, and η_T are arbitrary phase factors. Therefore, the derivation of the operators C, P, and T is reduced to the appropriate determination of the general phase factors which will depend on the conventions used. This will be the subject of the following subsections. It has to be emphasised that equations (1.16), (1.17), and (1.18) are only valid in the Weyl representation as well as for any other representation of γ^μ which is related to the Weyl representation by a real similarity transformation.

1.2.1 Derivation of the Charge Conjugation Operator

Equation (1.16) shows that the charge conjugation operator can be written as

$$C = \eta_C \gamma^2 K \,, \tag{1.19}$$

where K denotes the complex conjugation. A convenient choice for the phase factor should reproduce the charge conjugation operator as chosen by Ahluwalia-Khalilova and Grumiller (2005b),

$$C = \gamma^2 K \,. \tag{1.20}$$

Here it has been taken into account that the convention for γ^i used by Ahluwalia-Khalilova and Grumiller (2005b) differs by a factor of -1 from the one used by Peskin and Schroeder (1995). This corresponds to setting $\eta_C = 1$ and is convenient for the analysis of ELKO spinors as Ahluwalia-Khalilova and Grumiller (2005b) used this specific choice of C to normalise the ELKO spinors. Any other choice for η_C requires the modification of equations (A.11) to (A.14) by a phase factor to preserve the eigenvalues of ± 1. However, this choice for C makes it necessary to reanalyse the behaviour of the Dirac spinors under charge conjugation. It has to be emphasised that the changes only affect the prefactors – phase factors – and do not affect the symmetry properties like $(CPT)^2$ or commutation and anticommutation relations.

1.2.2 Derivation of the Parity Operator

On the left hand side of equation (1.17) the parity operator acts on the wavefunction while the right hand side is proportional to the wavefunction with inverted spatial coordinates. An explicit spatial inversion of coordinates has therefore to be included into the parity operator. The parity operator in its general form is then given by

$$P = \eta_P \gamma^0 \mathcal{R} \,, \tag{1.21}$$

where $\mathcal{R}$ is the space inversion operator as defined in equation (A.7). It is customary, if somewhat judicious, to choose the phase factor such that

$$Pu^{+}(\mathbf{p}) = u^{+}(\mathbf{p})\,. \tag{1.22}$$

It can be shown that this uniquely specifies the phase factor to $\eta_P = 1$. This choice of phase factor implies that $P^2 = 1$ for Dirac spinors, see, e. g., Peskin and Schroeder (1995). It deviates from Landau and Lifshitz (1982) where it is assumed that $P^2 = -1$ for Dirac spinors. The parity operator is then found to be

$$P = \gamma^0 \mathcal{R}\,. \tag{1.23}$$

1.2.3 Derivation of the Time Reversal Operator

The remaining operator is the time reversal operator that transforms a particle wavefunction according to equation (1.18). Here, the operator on the left hand side acts on the wavefunction while the right hand side is proportional to the complex conjugate wavefunction with inverted time coordinate. Both operations, complex conjugation as well as inversion of the time coordinate, have thus to be included into the general time reversal operator. The time reversal operator is then found to be

$$T = \eta_T \gamma^1 \gamma^3 K \mathcal{T}\,, \tag{1.24}$$

where K denotes complex conjugation and $\mathcal{T}$ encodes the reversal of the time coordinate.

Unlike the previous discussions for the charge conjugation and parity operators, the phase factor of the time reversal operator is no longer arbitrary. It is restricted by the requirement that $(CPT)^2 = \mathbb{I}$ for Dirac fields. Based on our previous derivations of the charge conjugation operator, equation (1.20), and the parity operator, equation (1.23), the action of CPT on a particle wavefunction is given by

$$CPT\psi(t, \mathbf{x}) = -\eta_{\mathrm{T}}^{*} \gamma_5 \psi(-t, -\mathbf{x})\,. \tag{1.25}$$

The behaviour under $(CPT)^2$ is then found to be

$$(CPT)^2\,\psi(t,\mathbf{x}) = -\,(\eta_{\mathrm{T}}^*)^2\,\psi(t,\mathbf{x})\,. \tag{1.26}$$

To satisfy the requirement $(CPT)^2\,\psi(t,\mathbf{x}) = \psi(t,\mathbf{x})$ for Dirac fields, the phase factor is restricted to $\eta_{\mathrm{T}}^* = \pm i$. This leaves two possible choices for the behaviour under CPT while the behaviour under $(CPT)^2$ is, of course, uniquely defined. In the following discussion the convention $\eta_{\mathrm{T}}^* = i$ and therefore,

$$T = -i\gamma^1\gamma^3 K\mathcal{T} \tag{1.27}$$

will be used. It implies that CPT acting on a particle wavefunction yields $CPT\psi(t,\mathbf{x}) = -i\gamma_5\psi(-t,-\mathbf{x})$.

1.3 Supersymmetry

Supersymmetry is by construction a symmetry between fermions and bosons, or in other words, a symmetry between the fields that constitute the matter in our universe and those that mediate the interactions. To distinguish particles from their superpartner the fermionic superpartner of a boson is denoted by the ending -ino, such as W-boson and Wino, while every fermion has a bosonic superpartner that is denoted by a prefix s-, e. g., electron and selectron. By construction a particle and its corresponding superpartner are members of the same supermultiplet and thus have the same mass as long as supersymmetry is not broken. Up to now no superpartners have been detected experimentally, which implies that any realistic supersymmetric theory must be broken – spontaneously or explicitly. The breaking of supersymmetry then eliminates the mass degeneracy among the different members of the multiplet without destroying the multiplet structure.

In its modern formulation supersymmetry was introduced in the fundamental papers by Wess and Zumino (1974a,b,c) which is commonly referred to as the Wess-Zumino model. They succeeded to formulate a model that unifies spacetime symmetries and internal symmetries. To achieve this goal, they relaxed the requirement that supersymmetry generators form a Lie algebra, which had restricted the generators to have integer-valued spin, thus being bosonic and satisfying commutation relations. Furthermore, they admitted half-integer-valued fermionic supersymmetry generators that satisfy anticommutation relations. This resulted in an algebra that contained both commutation and anticommutation relations which identify the supersymmetry algebra as a graded Lie algebra. This approach conveniently circumvented the no-go theorem by Coleman and Mandula (1967) which excluded any such model based on bosonic symmetry generators. At the same time it implies that the generators of supersymmetry in any supersymmetric model must be fermionic operators, so that the Coleman-Mandula theorem would not apply. Subsequently, this result was generalised by Haag et al. (1975) who showed that supersymmetry is the only model that is able to unify spacetime symmetries and internal symmetries.

Shortly after its introduction, O'Raifeartaigh (1975) generalised the Wess-Zumino model by assuming distinguishable superfields instead of identical superfields. With a specific choice of structure constants, he was able to show that his model contains regions in the parameter space where at least one of the auxiliary fields acquires a nonvanishing expectation value and thus supersymmetry is spontaneously broken. This model is known as the O'Raifeartaigh model and is the simplest model that permits supersymmetry breaking, which will be discussed in more detail in the next section. Explicit calculations for the O'Raifeartaigh model will also be presented in Chapter 7, where the coupling of fermionic fields with mass dimension one to the O'Raifeartaigh model is discussed. Since then supersymmetry has been incorporated into more sophisticated models which culminated in the formulation of the Minimally Supersymmetric extension of the Standard Model (Drees et al., 2004; Dine, 2007), superstring theory (Green et al., 1987a,b; Dine, 2007), and supergravity (Wess and Bagger, 1982). It is possible to formulate supersymmetric theories with up to 11 dimensions. A very elegant proof can be given in the context of superstring theory utilising the commutator of quantum Lorentz charges and the commutation relations of the Virasoro operators. This fixes the dimension of spacetime for superstring theories to 10. Overall, there are five distinct superstring theories, which can then be unified to one 11-dimensional superstring theory, commonly referred to as M-theory.

The minimal supersymmetric model is generated by the four independent components of the two 2-spinor operators Q and $\bar{Q}$ and does not contain any central charges. They act on functions similar to a derivative operator and thus can be thought of as infinitesimal superspace transformations. Alternatively, the two 2-spinor operators can be arranged into one single 4-spinor operator, which is the namesake of $N = 1$ supersymmetry. Here, N

refers to the number of 4-spinor supersymmetry generators. In addition to the minimal model there exist extended supersymmetric models with $N = 2, 4, 8$. Any N-extended supersymmetric model will contain particles with spin of at least $N/4$. This results in stringent restrictions on the number of distinct supersymmetric theories and their ability to describe certain aspects of physics realistically. One of the fundamental cornerstones of field theoretical models is renormalisability. To formulate a renormalisable flat-space field theory it cannot contain particles with spin greater than or equal to 3/2, see, e. g., Sohnius (1985). Therefore, the maximally renormalisable supersymmetric theory is $N = 4$, since $N = 8$ contains at least one spin-2 particle that renders the model non-renormalizable. Similar arguments also apply for the successful inclusion of gravity which cannot be coupled consistently to fields of spin 5/2 or higher. This implies that any extended theory in excess of $N = 8$ is incapable to describe gravity. Therefore, the number of supersymmetric theories is restricted to the four distinct cases, $N = 1, 2, 4, 8$.

A discussion of the supersymmetry algebra can be found in any standard textbook on supersymmetry (Wess and Bagger, 1982; Dine, 2007; Misra, 1992) and thus the reader is referred to the reference literature for further information. As the calculations in this thesis are based on the minimal supersymmetric model which is the only supersymmetric model with chiral representations, the $N = 1$ supersymmetry algebra without central charges can be summarised to

$$[P_\mu, P_\nu] = 0\,, \tag{1.28}$$

$$[P_\mu, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\right)\,, \tag{1.29}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}\right)\,, \tag{1.30}$$

$$[B_r, B_s] = ic_{rs}{}^{t}B_t\,, \tag{1.31}$$

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0\,, \tag{1.32}$$

$$[Q_\alpha, P_\mu] = \left[\bar{Q}_{\dot\alpha}, P_\mu\right] = 0\,, \tag{1.33}$$

$$[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}\left(\sigma_{\mu\nu}\right)_\alpha{}^{\beta} Q_\beta\,, \tag{1.34}$$

$$\left[\bar{Q}_{\dot\alpha}, M_{\mu\nu}\right] = -\frac{1}{2}\bar{Q}_{\dot\beta}\left(\bar\sigma_{\mu\nu}\right)^{\dot\beta}{}_{\dot\alpha}\,, \tag{1.35}$$

$$[Q_\alpha, B_r] = b_r Q_\alpha\,, \tag{1.36}$$

$$\left[\bar{Q}_{\dot\alpha}, B_r\right] = -\bar{Q}_{\dot\alpha} b_r\,, \tag{1.37}$$

$$\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} P_\mu\,, \tag{1.38}$$

$$\left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0\,. \tag{1.39}$$

Here P_μ denotes the energy-momentum operator, $M_{\mu\nu}$ are the Lorentz generators, B_r are internal symmetry generators, and ${c_{rs}}^t$ and b_r are coefficients that satisfy certain symmetry properties. The infinitesimal transformations of a superfield with the generators of the supersymmetry algebra P_μ, $M_{\mu\nu}$, Q_α, and $\bar{Q}_{\dot\alpha}$ can then be used to construct an irreducible representation of the supersymmetry algebra. It is also worth mentioning that extracting the energy as the 0-component of the energy-momentum operator from the supersymmetry algebra always leads to a positive ground-state energy.

There are a number of arguments that suggest that supersymmetry is a good candidate for describing physics in the TeV range. First, a supersymmetry breaking scale of the order of a few TeV solves the hierarchy problem as low-energy physics is decoupled from the Planck scale. Second, in the Minimally Supersymmetric extension of the Standard Model, all couplings are unified at an energy of the order of 10^{16} GeV. Third, supersymmetric theories automatically provide a suitable candidate for dark matter in form of the lightest supersymmetric particle which is commonly assumed to be the neutralino. Being the lightest superpartner this particle is stable as its decay into Standard Model particles would otherwise be detectable.

The most promising approach to formulate a model that includes gravity and reduces to the Standard Model in the low-energy limit is to assume that physics at the Planck scale is described by supergravity. At intermediate energies between the Planck scale and the supersymmetry breaking scale a transition between supergravity and $N = 1$ supersymmetry takes place. Finally, another transition takes place at the supersymmetry breaking scale, and for energies below this scale any supersymmetric effect are strongly suppressed.

1.4 Supersymmetry Breaking

Supersymmetry is a very elegant and more importantly the only way to unify spacetime symmetries and internal symmetries (Haag et al., 1975). It implies that all members of the same supermultiplet which includes the particles as well as their superpartners must have the same mass if supersymmetry is preserved, but up to now no superpartners have been detected experimentally. This negative experimental result leaves two possible conclusions. First, there is no supersymmetry. This would be rather disappointing, considering the success of the Minimal Supersymmetric Standard Model as well as the promising work in superstring theory and supergravity. Alternatively, supersymmetry is broken to lift the mass degeneracy between the members of the supermultiplets and shifts the mass of the lightest supersymmetric particle to the supersymmetry breaking scale at a few TeV. Of course, supersymmetry breaking could theoretically shift the energies of the supersymmetric partners to even higher energies, e. g., up to the Planck scale, which again leads to the hierarchy problem that motivated the introduction of supersymmetry in the first place. Therefore, if supersymmetry is supposed to describe the real world, it must be broken either spontaneously or explicitly and the mass of the lightest supersymmetric particle should be of the order of a few TeV.

From the field theoretical point of view the breaking of supersymmetry only leads to minor inconveniences that can be dealt with. As long as a model exhibits supersymmetry, the diverging contributions from bosonic and fermionic fields cancel identically. The breaking of supersymmetry spoils this perfect cancellation of divergences. The resulting divergences are, however, only of logarithmic nature that can be accommodated by the introduction of a running coupling constant.

Supersymmetry breaking depends strongly on the underlying supersymmetric model. Models that exhibit global supersymmetry can be broken either spontaneously or explicitly, while models with local supersymmetry can only be broken spontaneously. If supersymmetry is not an accident, it must be a local symmetry which is realised in both superstring theory as well as supergravity.

To determine whether supersymmetry is broken within a specific model, it is sufficient to determine if certain conditions are satisfied. The main indicator for the occurrence of supersymmetry breaking is a nonzero expectation value of the superfield component with the highest mass dimension. It corresponds to the F-component of a chiral superfield or the D-component of a general superfield. The most general way to determine the expectation values and vacuum energy is to minimise the superpotential. Depending on the number of superfields involved, this can be a rather difficult task. It can be simplified by also utilising the equations of motion for the auxiliary fields. They usually do not provide a complete set of solutions but are significantly easier to solve. These results can then be used to reduce the initial set of equations that was derived in the minimisation of the superpotential. The reduced set of equations can then be solved for the remaining expectation values.

The simplest model exhibiting spontaneous symmetry breaking, which was also the first one proposed, is the O'Raifeartaigh model (O'Raifeartaigh, 1975). The discussion of this model which is outlined in Section 7.1 reveals that for a specific choice of coupling constants in the O'Raifeartaigh model one of the F-components, usually assumed to be F_3, acquires a nonzero expectation value. This leads to a nonzero expectation value of the component field A_3, which is undetermined in the classical calculation and is assumed to be proportional to the scale parameter μ. As quantum corrections create a potential for A_3, this problem is resolved at the one loop level. A similar effect arises for the coupling of the O'Raifeartaigh model to a fermionic sector as shown in Sections 7.3 and 7.4, where the expectation value becomes proportional to the mass scale of the fermionic sector as well as the coupling strength between the O'Raifeartaigh model and the fermionic sector.

In addition, there is a wide range of possible terms that can be introduced to achieve an explicit symmetry breaking. However, these symmetry breaking terms are far from arbitrary and are constrained by the properties of the model. Overall, explicit symmetry breaking terms can be narrowed down to three different types, scalar mass terms of the form $m_\Phi^2 \left|\Phi^2\right|$, gaugino mass terms $m_\lambda \lambda\lambda$, and trilinear scalar couplings $\Gamma\Phi\Phi\Phi$.

Supersymmetry can also be broken by the introduction of at least one additional set of fields. Furthermore, it is assumed that the interactions between the additional set of fields and the component fields of the visible sector are suppressed. Therefore, the additional set of fields is referred to as the hidden sector. The resulting small interaction between the visible and the hidden sector is then responsible for the breaking of supersymmetry in the visible sector. This is very interesting in light of the later discussion of the coupling of the O'Raifeartaigh model to the fermionic sector which is de facto a hidden sector model. However, in this case the visible sector described by the O'Raifeartaigh model already spontaneously breaks supersymmetry, while the hidden sector is supersymmetric and restores the supersymmetry of the coupled model at the cost of breaking Lorentz invariance.

Chapter 2

Transformation Properties and Symmetry Behaviour of ELKO Spinors

A review of the fundamental literature on ELKO spinors (Ahluwalia-Khalilova and Grumiller, 2005b) revealed inconsistencies in the behaviour of ELKO spinors under discrete symmetry operations. The authors of the paper refer to it as "*apparently paradoxical asymmetry*". A careful analysis of the transformation behaviour of ELKO spinors under discrete symmetry transformations using the operators derived in Section 1.2 reveals that this problem is resolved and that ELKO spinors exhibit a different transformation behaviour than previously assumed.

In this chapter, the transformation behaviour of ELKO spinors under charge conjugation, parity, and time reversal will be discussed. Furthermore, the symmetry structure of ELKO spinors in form of (anti)commutation relations will be analysed and compared to Dirac spinors.

In Section 2.1 the transformation behaviour of Dirac spinors under discrete symmetry operations which are derived in Section 1.2 are reviewed. In Sections 2.2 to 2.4 the transformation behaviour of ELKO spinors under charge conjugation, parity, and time reversal is discussed in detail. Afterwards, in Section 2.5 the behaviour under CPT and $(CPT)^2$ as well as the (anti)commutativity of the operators acting on ELKO spinors is analysed. Finally, the results are summarised in Section 2.7.

At this point it is important to mention that the results presented in this chapter differ slightly from those published in Wunderle and Dick (2009). This is due to the fact that the conventions used in this publication were chosen in agreement with those used by Ahluwalia-Khalilova and Grumiller (2005b). To achieve a consistent treatment of the field theoretical content as well as supersymmetry and superfield formalism, the previously published material was reworked to conform with the conventions by Peskin and Schroeder

(1995) that are used throughout this thesis.

2.1 Transformation and Symmetry Properties of Dirac Spinors

For later reference the behaviour of Dirac spinors under discrete symmetry transformations is breifly reviewed. The behaviour of Dirac spinors, equation (A.5), under charge conjugation, parity, and time reversal as defined in equations (1.20), (1.23), and (1.27) is summarised in the following equations:

$$Cu^{\pm}(\mathbf{p}) = iv^{\pm}(\mathbf{p}), \qquad Cv^{\pm}(\mathbf{p}) = iu^{\pm}(\mathbf{p}), \tag{2.1}$$

$$Pu^{\pm}(\mathbf{p}) = u^{\pm}(\mathbf{p}), \qquad Pv^{\pm}(\mathbf{p}) = -v^{\pm}(\mathbf{p}), \tag{2.2}$$

$$Tu^{\pm}(\mathbf{p}) = \pm iu^{\mp}(-\mathbf{p}), \qquad Tv^{\pm}(\mathbf{p}) = \pm iv^{\mp}(-\mathbf{p}). \tag{2.3}$$

These results can then be used to calculate the transformation of Dirac spinors under CPT and more importantly under $(CPT)^2$. For $u^+(\mathbf{p})$ it is found that

$$CPTu^{+}(\mathbf{p}) = v^{-}(-\mathbf{p}), \tag{2.4}$$

$$(CPT)^2 \, u^{+}(\mathbf{p}) = u^{+}(\mathbf{p}). \tag{2.5}$$

Similar calculations can be performed for the remaining Dirac spinors. Eq. (2.4) shows that a Dirac spinor transforms under CPT into an antiparticle with opposite spin and momentum. Furthermore, equation (2.5) confirms that $(CPT)^2 = \mathbb{I}$ for Dirac spinors. These results are as expectated.

Another important property of Dirac spinors is their symmetry behaviour which is encoded in the commutation and anticommutation relations between the three operators. By using equations (2.1), (2.2), and (2.3) it can be shown that

$$\{C, P\} \, u^{+}(\mathbf{p}) = 0, \tag{2.6}$$

$$[C, T] \, u^{+}(\mathbf{p}) = 0, \tag{2.7}$$

$$[P, T] \, u^{+}(\mathbf{p}) = 0. \tag{2.8}$$

Again, the discussion is restricted to $u^+(\mathbf{p})$, however, the same is valid for any Dirac spinor. Eq. (2.6) shows clearly that C and P anticommute while equations (2.7) and (2.8) reveal that C and T as well as P and T commute. This is the expected symmetry behaviour for Dirac spinors. The result also shows that the explicit choice of phase factors does not modify the symmetry behaviour of the theory as long as they are chosen such that $(CPT)^2 = \mathbb{I}$.

2.2 ELKO Spinors under Charge Conjugation

The behaviour of ELKO spinors under charge conjugation was previously presented in Ahluwalia-Khalilova and Grumiller (2005b). For convenience the derivation of this result will be briefly reviewed.

The analysis of the behaviour of $\lambda^S_{\{-,+\}}(\mathbf{p})$ under charge conjugationis based on the definitions of the charge conjugation operator, equation (1.20), and the boosted ELKO spinor, equation (A.19),

$$\begin{aligned} C\lambda^S_{\{-,+\}}(\mathbf{p}) &= i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix}\begin{pmatrix} -i\Theta\phi^+_{\rm L}(\mathbf{0}) \\ \left[\phi^+_{\rm L}(\mathbf{0})\right]^* \end{pmatrix} \\ &= -i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} -\Theta\left[\phi^+_{\rm L}(\mathbf{0})\right]^* \\ i\phi^+_{\rm L}(\mathbf{0}) \end{pmatrix} \\ &= \lambda^S_{\{-,+\}}(\mathbf{p}). \end{aligned} \tag{2.9}$$

Similarly, the behaviour of $\lambda^A_{\{-,+\}}(\mathbf{p})$ can be determined

$$\begin{aligned} C\lambda^A_{\{-,+\}}(\mathbf{p}) &= i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix}\begin{pmatrix} i\Theta\phi^+_{\rm L}(\mathbf{0}) \\ \left[\phi^+_{\rm L}(\mathbf{0})\right]^* \end{pmatrix} \\ &= -i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} -\Theta\left[\phi^+_{\rm L}(\mathbf{0})\right]^* \\ -i\phi^+_{\rm L}(\mathbf{0}) \end{pmatrix} \\ &= -\lambda^A_{\{-,+\}}(\mathbf{p})\,. \end{aligned} \tag{2.10}$$

These results confirm that ELKO spinors are eigenspinors of the charge conjugation operator and the expected eigenvalues are reproduced. The calculations for the remaining ELKO spinors can be performed in analogy and the behaviour of ELKO spinors under charge conjugation can thus be summarised to

$$C\lambda^S_{\{-,+\}}(\mathbf{p}) = \lambda^S_{\{-,+\}}(\mathbf{p})\,, \tag{2.11}$$

$$C\lambda^S_{\{+,-\}}(\mathbf{p}) = \lambda^S_{\{+,-\}}(\mathbf{p})\,, \tag{2.12}$$

$$C\lambda^A_{\{-,+\}}(\mathbf{p}) = -\lambda^A_{\{-,+\}}(\mathbf{p})\,, \tag{2.13}$$

$$C\lambda^A_{\{+,-\}}(\mathbf{p}) = -\lambda^A_{\{+,-\}}(\mathbf{p})\,. \tag{2.14}$$

These transformation properties under charge conjugation yield $C^2 = \mathbb{I}$.

2.3 ELKO Spinors under Parity Transformation

Using the definitions of the parity operator from equation (1.23) and the boosted ELKO spinors from equations (A.19) and (A.20), the behaviour of ELKO spinors under parity transformation is analysed. For reasons of brevity the transformations of $\lambda^{S/A}_{\{-,+\}}(\mathbf{p})$ under parity are discussed in detail while the results for $\lambda^{S/A}_{\{+,-\}}(\mathbf{p})$ are briefly summarised. They are calculated in analogy to the first case and the details differ only slightly based on their explicit form.

The action of the parity operator on $\lambda^S_{\{-,+\}}(\mathbf{p})$ is then given by

$$P\lambda^S_{\{-,+\}}(\mathbf{p}) = \gamma^0\sqrt{\frac{E+m}{2m}}\left(1 - \frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} \mathcal{R}i\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^* \\ \mathcal{R}\phi^+_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}. \tag{2.15}$$

At this point the behaviour of the ELKO spinor components under spatial inversion has to be analysed. With the help of equations (A.6), (A.7), (A.16), and (A.17) it can be shown that they transform as

$$\mathcal{R}\phi^+_{\mathrm{L}}(\mathbf{0}) = -i\phi^-_{\mathrm{L}}(\mathbf{0}) = i\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^*\,, \tag{2.16}$$

$$\mathcal{R}\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^* = i\Theta\left[\phi^-_{\mathrm{L}}(\mathbf{0})\right]^* = i\phi^+_{\mathrm{L}}(\mathbf{0}). \tag{2.17}$$

Obviously the states $-i\phi^-_{\mathrm{L}}(\mathbf{0})$ and $i\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^*$ in equation (2.16) have the same helicity. The same is valid for $i\Theta\left[\phi^-_{\mathrm{L}}(\mathbf{0})\right]^*$ and $i\phi^+_{\mathrm{L}}(\mathbf{0})$ in equation (2.17).

Rewriting the spatially inverted components in equation (2.15) using terms proportional to $\phi_{\rm L}^{-}(\mathbf{0})$ results in

$$P\lambda^S_{\{-,+\}}(\mathbf{p}) = -i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix}\phi_{\rm L}^{-}(\mathbf{0})\\ -i\Theta\left[\phi_{\rm L}^{-}(\mathbf{0})\right]^*\end{pmatrix} \tag{2.18}$$

$$\neq -i\sqrt{\frac{E+m}{2m}}\left(1+\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix}-i\Theta\left[\phi_{\rm L}^{-}(\mathbf{0})\right]^*\\ \phi_{\rm L}^{-}(\mathbf{0})\end{pmatrix}$$

$$= -i\lambda^A_{\{+,-\}}(\mathbf{p}). \tag{2.19}$$

This shows clearly that a transformation of the form $P\lambda^S_{\{-,+\}}(\mathbf{p}) \propto \lambda^{S/A}_{\{+,-\}}(\mathbf{p})$ as proposed in Ahluwalia-Khalilova and Grumiller (2005b) is only approximately satisfied. However, there are two obvious problems. First, the prefactor contains a different sign. This problem could be resolved by introducing a prefactor similar to the one that will arise in our discussion of the time reversal operator. Second, the ordering of the spinor components is opposite to what we expect. In this case no simple solution exists – other than introducing an additional γ^0, which means redefining P.

A different picture arises if the solutions proportional to $\phi_{\rm L}^{+}(\mathbf{0})$ in equations (2.16) and (2.17) are used to rewrite equation (2.15). The parity transformed ELKO spinor can then be written as

$$P\lambda^S_{\{-,+\}}(\mathbf{p}) = \gamma^0\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix}-\phi_{\rm L}^{+}(\mathbf{0})\\ i\Theta\left[\phi_{\rm L}^{+}(\mathbf{0})\right]^*\end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix}i\Theta\left[\phi_{\rm L}^{+}(\mathbf{0})\right]^*\\ -\phi_{\rm L}^{+}(\mathbf{0})\end{pmatrix}$$

$$= -\lambda^A_{\{-,+\}}(\mathbf{p}). \tag{2.20}$$

This result points to a symmetric behaviour of ELKO spinors under parity transformation. To confirm this finding, similar calculations have to be performed for $\lambda^A_{\{-,+\}}(\mathbf{p})$. With the help of the previously derived transformation properties of the ELKO spinor components under spatial inversion, equations (2.16) and (2.17), it can be shown that

$$\begin{aligned}
P\lambda^{A}_{\{-,+\}}(\mathbf{p}) &= \gamma^0 \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m}\right) \begin{pmatrix} -\mathcal{R} i\Theta \left[\phi^{+}_{\mathrm{L}}(\mathbf{0})\right]^{*} \\ \mathcal{R}\phi^{+}_{\mathrm{L}}(\mathbf{0}) \end{pmatrix} \\
&= \gamma^0 \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m}\right) \begin{pmatrix} \phi^{+}_{\mathrm{L}}(\mathbf{0}) \\ i\Theta \left[\phi^{+}_{\mathrm{L}}(\mathbf{0})\right]^{*} \end{pmatrix} \\
&= \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m}\right) \begin{pmatrix} i\Theta \left[\phi^{+}_{\mathrm{L}}(\mathbf{0})\right]^{*} \\ \phi^{+}_{\mathrm{L}}(\mathbf{0}) \end{pmatrix} \\
&= \lambda^{S}_{\{-,+\}}(\mathbf{p}). \qquad (2.21)
\end{aligned}$$

This verifies that ELKO spinors transform symmetrically under parity transformation.

Similar calculations can now be repeated for the remaining ELKO spinors. This involves the analysis of the behaviour of components containing the negative helicity eigenstate $\phi^{-}_{\mathrm{L}}(\mathbf{0})$ under space inversion. These relations are found in analogy to equations (2.16) and (2.17):

$$\mathcal{R}\phi^{-}_{\mathrm{L}}(\mathbf{0}) = -i\phi^{+}_{\mathrm{L}}(\mathbf{0}) = -i\Theta \left[\phi^{-}_{\mathrm{L}}(\mathbf{0})\right]^{*}, \qquad (2.22)$$

$$\mathcal{R}\Theta \left[\phi^{-}_{\mathrm{L}}(\mathbf{0})\right]^{*} = i\Theta \left[\phi^{+}_{\mathrm{L}}(\mathbf{0})\right]^{*} = -i\phi^{-}_{\mathrm{L}}(\mathbf{0}). \qquad (2.23)$$

The spatially inverted components can again be expressed in terms of $\phi^{+}_{\mathrm{L}}(\mathbf{0})$ or $\phi^{-}_{\mathrm{L}}(\mathbf{0})$. By using the appropriate relations the transformation behaviour of the remaining ELKO spinors can be derived. The results for all ELKO spinors are summarised in the following equations

$$P\lambda^{S}_{\{-,+\}}(\mathbf{p}) = -\lambda^{A}_{\{-,+\}}(\mathbf{p}), \qquad (2.24)$$

$$P\lambda^{S}_{\{+,-\}}(\mathbf{p}) = \lambda^{A}_{\{+,-\}}(\mathbf{p}), \qquad (2.25)$$

$$P\lambda^{A}_{\{-,+\}}(\mathbf{p}) = \lambda^{S}_{\{-,+\}}(\mathbf{p}), \qquad (2.26)$$

$$P\lambda^{A}_{\{+,-\}}(\mathbf{p}) = -\lambda^{S}_{\{+,-\}}(\mathbf{p}). \qquad (2.27)$$

These results for the properties under parity transformation directly imply $P^2 = -\mathbb{I}$.

2.4 ELKO Spinors under Time Reversal

Finally, the behaviour of ELKO spinors under time reversal is discussed. As we will see, ELKO spinors have asymmetric transformation properties under time reversal and therefore the discussion is restricted to $\lambda^S_{\{-,+\}}(\mathbf{p})$ and $\lambda^A_{\{+,-\}}(\mathbf{p})$.

For $\lambda^S_{\{-,+\}}(\mathbf{p})$ it is found that

$$\begin{aligned}
T\lambda^S_{\{-,+\}}(\mathbf{p}) &= i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} i\phi^+_{\mathrm{L}}(\mathbf{0}) \\ \Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^* \end{pmatrix} \\
&= i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} i\Theta\left[\phi^-_{\mathrm{L}}(\mathbf{0})\right]^* \\ -\phi^-_{\mathrm{L}}(\mathbf{0}) \end{pmatrix} \\
&= -i\sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\lambda^A_{\{+,-\}}(\mathbf{0}) \\
&= -i\frac{1-\frac{|\mathbf{p}|}{E+m}}{1+\frac{|\mathbf{p}|}{E+m}}\lambda^A_{\{+,-\}}(\mathbf{p}),
\end{aligned} \tag{2.28}$$

where the previously derived relations between the positive and negative helicity eigenstates from equations (2.16) and (2.17) were used. This result shows that a self-conjugate ELKO spinor is transformed into an anti-self-conjugate ELKO spinor with opposite helicity. Additionally, the time reversed spinor contains a prefactor. This result differs again from the predictions of Ahluwalia-Khalilova and Grumiller (2005b) – that predicts a transformation of the form $T\lambda^S_{\{-,+\}}(\mathbf{p}) \propto \lambda^A_{\{-,+\}}(\mathbf{p})$ – but does not come as a surprise. The change of the transformation behaviour under time reversal is a direct consequence of the change of behaviour under parity, since $(CPT)^2 = -\mathbb{I}$ for ELKO spinors should be preserved.

A similar calculation for $\lambda^A_{\{+,-\}}(\mathbf{p})$ shows that

$$\begin{aligned}
T\lambda^A_{\{+,-\}}(\mathbf{p}) &= i\sqrt{\frac{E+m}{2m}}\left(1+\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} -i\phi^-_{\mathrm{L}}(\mathbf{0}) \\ \Theta\left[\phi^-_{\mathrm{L}}(\mathbf{0})\right]^* \end{pmatrix} \\
&= i\sqrt{\frac{E+m}{2m}}\left(1+\frac{|\mathbf{p}|}{E+m}\right)\begin{pmatrix} i\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^* \\ \phi^+_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= i\sqrt{\frac{E+m}{2m}}\left(1+\frac{|\mathbf{p}|}{E+m}\right)\lambda^S_{\{-,+\}}(\mathbf{0}) \\
&= i\frac{1+\frac{|\mathbf{p}|}{E+m}}{1-\frac{|\mathbf{p}|}{E+m}}\lambda^S_{\{-,+\}}(\mathbf{p}).
\end{aligned} \tag{2.29}
$$

It can be seen that an anti-self-conjugate spinor is transformed into a self-conjugate spinor with opposite helicity. Furthermore, the prefactor that arises in the transformation of spinors with helicity index $\{-,+\}$ is the inverse of those arising in the transformation of spinors with helicity index $\{+,-\}$. Therefore, these prefactors cancel under repeated application of the time reversal operator and an ELKO spinor transforms into minus itself. This implies that $T^2 = -\mathbb{I}$.

Similar calculations can be performed for the remaining ELKO spinors and the results for the behaviour of all ELKO spinors under time reversal can be summarised to

$$
T\lambda^S_{\{-,+\}}(\mathbf{p}) = -i\frac{1-\frac{|\mathbf{p}|}{E+m}}{1+\frac{|\mathbf{p}|}{E+m}}\lambda^A_{\{+,-\}}(\mathbf{p}), \tag{2.30}
$$

$$
T\lambda^S_{\{+,-\}}(\mathbf{p}) = i\frac{1+\frac{|\mathbf{p}|}{E+m}}{1-\frac{|\mathbf{p}|}{E+m}}\lambda^A_{\{-,+\}}(\mathbf{p}), \tag{2.31}
$$

$$
T\lambda^A_{\{-,+\}}(\mathbf{p}) = -i\frac{1-\frac{|\mathbf{p}|}{E+m}}{1+\frac{|\mathbf{p}|}{E+m}}\lambda^S_{\{+,-\}}(\mathbf{p}), \tag{2.32}
$$

$$
T\lambda^A_{\{+,-\}}(\mathbf{p}) = i\frac{1+\frac{|\mathbf{p}|}{E+m}}{1-\frac{|\mathbf{p}|}{E+m}}\lambda^S_{\{-,+\}}(\mathbf{p}). \tag{2.33}
$$

2.5 ELKO Spinors under CPT and $(CPT)^2$

Now that the transformation properties of ELKO spinors under charge conjugation, parity, and time reversal are known, the calculation of the behaviour under CPT and $(CPT)^2$ is straightforward. As before, the calculations will only be shown for one ELKO spinor as the calculations for the remaining ELKO spinors do not yield any additional information. Under CPT the ELKO spinor $\lambda^S_{\{-,+\}}(\mathbf{p})$ transforms as follows

$$
CPT\lambda^S_{\{-,+\}}(\mathbf{p}) = -i\frac{1-\frac{|\mathbf{p}|}{E+m}}{1+\frac{|\mathbf{p}|}{E+m}}\lambda^S_{\{+,-\}}(\mathbf{p}). \tag{2.34}
$$

This result shows that the self-conjugate ELKO spinor is transformed into a self-conjugate ELKO spinor with opposite helicity times a prefactor. Thus, the interpretation is not as obvious as for Dirac spinors which does not come as a surprise. Now the behaviour under $(CPT)^2$ is derived by repeated operation of CPT

$$(CPT)^2\,\lambda^S_{\{-,+\}}(\mathbf{p}) = -\lambda^S_{\{-,+\}}(\mathbf{p})\,. \tag{2.35}$$

The result shows that ELKO spinors satisfy

$$(CPT)^2 = -\mathbb{I}\,, \tag{2.36}$$

which identifies them as a representation of a nonstandard Wigner class.

2.6 (Anti)commutation Relations for ELKO Spinors

The last remaining task is to analyse the (anti)commutation relations of the operators acting on ELKO spinors. With the help of the derived transformation properties under charge conjugation, parity, and time reversal this discussion is again clear-cut. For this set of operators there are three independent (anti)commutation relations that need to be evaluated. It can be shown that any ELKO spinor satisfies

$$\{C,P\}\,\lambda^S_{\{-,+\}}(\mathbf{p}) = 0\,, \tag{2.37}$$

$$[C,T]\,\lambda^S_{\{-,+\}}(\mathbf{p}) = 0\,, \tag{2.38}$$

$$[P,T]\,\lambda^S_{\{-,+\}}(\mathbf{p}) = 0\,. \tag{2.39}$$

These results show that for ELKO spinors C and P anticommute while C and T as well as P and T commute. This behaviour is in perfect analogy to the Dirac case which was outlined in Section 4. It deviates from previous proposals assuming that ELKO spinors satisfy $[C,P]=0$, $[C,T]=0$, and $\{P,T\}=0$.

2.7 Conclusions

The presented calculations confirm that ELKO spinors are not eigenspinors of the helicity operator. However, the observed transformation behaviour under parity and time reversal is different than previously thought. It has been shown that ELKO spinors transform symmetrically under parity transformation if the appropriate transformations of the helicity eigenstates under spatial inversion are chosen. The results also imply $P^2 = -1$. Furthermore, a change in the transformation behaviour under time reversal has been found. It has also been shown that ELKO spinors transform asymmetrically under time reversal and aquire an additional prefactor. This prefactor cancels through repeated application of the time reversal operator and it is found that $T^2 = -\mathbb{I}$. Even though the transformation behaviour under parity and time reversal changes, $(CPT)^2 = -\mathbb{I}$ is unchanged. This is expected as ELKO spinors are a representation of a nonstandard Wigner class while Dirac spinors satisfy $(CPT)^2 = \mathbb{I}$. Finally, it has been shown that ELKO spinors satisfy exactly the same (anti)commutation relations as Dirac spinors $\{C, P\} = 0$, $[C, T] = 0$, and $[P, T] = 0$. This result is especially intriguing as it shows that ELKO and Dirac spinors behave similarly under discrete symmetry transformations, even though they belong to two different representations.

Chapter 3

A Supersymmetric Lagrangian

3.1 Constructing a Theory Based on the General Scalar Superfield

The most straightforward approach to formulate a supersymmetric theory for fermionic fields with integer-valued mass dimension is to formulate a theory in analogy to the commonly used formalism where fermionic fields have half-integer-valued mass dimension. This is done by starting from the general scalar superfield

$$\begin{aligned} V = C - i\theta\chi + i\bar{\chi}'\bar{\theta} - \frac{i}{2}\theta^2\left(M - iN\right) + \frac{i}{2}\bar{\theta}^2\left(M + iN\right) - \theta\sigma^\mu\bar{\theta}A_\mu \\ + i\bar{\theta}^2\theta\left(\lambda - \frac{i}{2}\partial\!\!\!/\bar{\chi}'\right) - i\theta^2\bar{\theta}\left(\bar{\lambda}' - \frac{i}{2}\bar{\partial\!\!\!/}\chi\right) - \frac{1}{2}\theta^2\bar{\theta}^2\left(D + \frac{1}{2}\Box C\right), \end{aligned} \tag{3.1}$$

and redefining the mass dimensions of the component fields appropriately, e. g., $\dim(C) = 1/2$, $\dim(\chi) = 1$, etc. The chiral superfields X and W_α are then defined as

$$X = \frac{i}{2}\bar{D}^2 V\,, \tag{3.2}$$

$$W_\alpha = \frac{i}{4}\bar{D}^2 D_\alpha V\,, \tag{3.3}$$

where the covariant derivatives are given by

$$D_\alpha = \partial_\alpha - i\partial\!\!\!/_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\,, \tag{3.4}$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + i\theta^\beta\partial\!\!\!/_{\beta\dot{\alpha}}\,. \tag{3.5}$$

It has to be emphasised that this choice of conventions differs by a factor of $-i$ from the conventions used by Wess and Bagger (1982). It ensures that the covariant derivatives and supersymmetry generators are Hermitian while the ones used by Wess and Bagger (1982)

Contribution	Mass Dimension	Possible Contributions
VV	$\dim(VV) = 1$	$(mVV)_D$
XV	$\dim(XV) = 2$	$(XV)_D$
$DVDV$	$\dim(DVDV) = 2$	$(DVDV)_D$
VX	$\dim(VX) = 2$	$(VX)_D$
DWV	$\dim(DWV) = 3$	mass dimension too large for D-component
WDV	$\dim(WDV) = 3$	mass dimension too large for D-component
XX	$\dim(XX) = 3$	$(XX)_F$
DVW	$\dim(DVW) = 3$	mass dimension too large for D-component
VDW	$\dim(VDW) = 3$	mass dimension too large for D-component

Table 3.1: Contributions to the Lagrangian based on the general scalar superfield if χ is identified with the fermionic field of mass dimension one. In addition to the contributions built from products of unbarred superfields, the Hermitian conjugates are permitted as well.

are not Hermitian. This deviation, however, does not affect the results for the chiral and anti-chiral superfields as they are normalised such that the prefactor of the lowest superfield component is unity which absorbs any overall prefactor of the previously mentioned kind.

However, there are two fundamental problems that prevent a feasible theory using this approach. The first problem is that all possible contributions to the Lagrangian fail to produce a nonvanishing kinetic term for the fermionic fields. The second problem is encountered during second quantisation of the Lagrangian. It can be shown that already the simplest possible Lagrangian leads to negative energy solutions. In the following subsections these two problems are discussed in detail.

3.1.1 A Non-kinetic Supersymmetric Lagrangian

The general scalar superfield has two possible candidates for a fermionic field with mass dimension one, χ and λ. For simplicity, the discussion is restricted to the case for χ as fermionic field with mass dimension one. Similar calculations can be repeated for λ. Due to the shift in mass dimension of the component fields the maximum number of covariant derivatives necessary to be considered is then increased by two and the discussion becomes more involved.

If χ is identified with the fermionic field with mass dimension one, it can be shown that the mass dimensions of the general superfield V, covariant derivative D_α, and chiral superfields X and W_α are

$$\dim(V) = \frac{1}{2}\,, \quad \dim(D_\alpha) = \frac{1}{2}\,, \quad \dim(X) = \frac{3}{2}\,, \quad \dim(W_\alpha) = 2\,. \tag{3.6}$$

These results for the building blocks of the Lagrangian can be utilised to work out all possible contributions to the Lagrangian. The contributions have to satisfy three basic requirements. First, all contributions to the Lagrangian have to be Lorentz scalars and thus cannot contain any uncontracted indices. Second, all structure constants must have positive mass dimension for the theory to be renormalizable. Third, the contributions must have the appropriate mass dimension to contribute either via the F-component or the D-component. For contributions via the D-component no further restrictions on the symmetry properties of the fields apply. This is no longer the case for contributions via the F-component. In this case the fields that are multiplied together must either be all chiral or all anti-chiral.

For χ the list of contributions is rather short and all possible terms are summarised in Table 3.1. It groups the contributions into three groups depending on the mass dimension of the superfield product without structure constants, which is related to the number of covariant derivatives needed to derive the products using solely the general superfield and covariant derivatives. It is possible to conceive terms with higher mass dimension; however, those terms cannot contribute to the Lagrangian and are therefore irrelevant for the following discussion. For simplicity the discussion is restricted to the unbarred fields while the Hermitian conjugated components have to be considered for the Lagrangian as well.

The first group of terms with mass dimension one consists of a single term which is the product of two general superfields. As the general superfield is neither chiral nor antichiral the only possible contribution to the Lagrangian is a mass term via the D-component.

The second group containing all terms with mass dimension 2 then encompasses all terms that can be constructed using two general superfields and two covariant derivatives. This results in three possible contributions to the kinetic term via the D-component. There can be no contributions to the mass term via the F-component as neither V nor DV are chiral or anti-chiral.

Finally, the third group summarises all terms with mass dimension three which contain two general superfields as well as four covariant derivatives. Due to the mass dimension, only contributions via the F-component are possible. The only term that satisfies the necessary symmetry requirements is XX which contributes to the kinetic term.

This means that there is one contribution to the mass term as well as four contributions to the kinetic term. On the first glance this seems to ensure the existence of a viable theory. However, explicit calculations reveal that neither one of the four kinetic terms in question nor any combination of them is able to produce a kinetic term for χ which was originally identified with the fermionic field with mass dimension one. A similar discussion can be repeated for the case where λ is identified with the fermionic field with mass dimension one. Although the discussion for λ produces an even larger number of potential contributions to the kinetic term, neither of these terms results in a nonvanishing kinetic term. Therefore, it can be concluded that it is impossible to construct a viable theory – other than the trivial solution for a constant background spinor field – based on the general scalar superfield that is able to describe fermionic fields with mass dimension one.

3.1.2 Problems with Second Quantisation

The second major problem arises from the second quantisation of the component fields. A simple way to demonstrate this is to start with the simplest possible Lagrangian for a fermionic field with mass dimension one

$$\mathcal{L} = \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi} \psi \,. \tag{3.7}$$

The corresponding Hamiltonian is then found to be

$$H = \vec{\nabla} \bar{\psi} \vec{\nabla} \psi + m^2 \bar{\psi} \psi \,. \tag{3.8}$$

Inserting the quantised Dirac field

$$\psi = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_s \left(a_\mathbf{p}^s u^s(\mathbf{p}) e^{-ip\cdot x} + b_\mathbf{p}^{s\,\dagger} v^s(\mathbf{p}) e^{ip\cdot x} \right) \,, \tag{3.9}$$

$$\bar{\psi} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_s \left(b_\mathbf{p}^s \bar{v}^s(\mathbf{p}) e^{-ip\cdot x} + a_\mathbf{p}^{s\,\dagger} \bar{u}^s(\mathbf{p}) e^{ip\cdot x} \right) \,, \tag{3.10}$$

into the Hamiltonian yields

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} m E_{\mathbf{p}} \sum_s \left(a_{\mathbf{p}}^{s\,\dagger} a_{\mathbf{p}}^{s} + b_{\mathbf{p}}^{s} b_{\mathbf{p}}^{s\,\dagger} \right) , \tag{3.11}$$

where the creation and annihilation operators obey the well known anti-commutation relations

$$\left\{ a_{\mathbf{p}}^{r}, a_{\mathbf{q}}^{s\dagger} \right\} = \left\{ b_{\mathbf{p}}^{r}, b_{\mathbf{q}}^{s\dagger} \right\} = (2\pi)^3 \, \delta(\mathbf{p} - \mathbf{q}) \delta_{rs} \, . \tag{3.12}$$

After removing the zero-point energy from the Hamiltonian, it is given by

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} m E_{\mathbf{p}} \sum_s \left(a_{\mathbf{p}}^{s\,\dagger} a_{\mathbf{p}}^{s} - b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^{s} \right) . \tag{3.13}$$

This reveals immediately that the constructed theory is not viable. This is due to the fact that the creation operator $b^\dagger$ can be used to lower the energy arbitrarily and obtain negative energy solutions. In retrospect this outcome could have been expected since all bosonic fields were replaced with fermionic fields and all commutation relations with anti-commutation relations; see, e. g., Peskin and Schroeder (1995, Sect. 3.5).

3.2 The General Superfield with one Spinor Index

In the previous section it was shown that a theory based on the general scalar superfield cannot be viable. This motivated an ansatz based on the general superfield with one spinor index. So far only few references to the general spinor superfield exist in the literature. One exception being the article by Gates Jr. (1977) that contains an expansion of a spinor superfield in Grassmann variables. In addition, an expansion of the chiral spinor superfield was given by Siegel (1979). Their results are also included in the book by Gates Jr. et al. (2001). As our notation differs from previous publications and is based on spinor superfields with different mass dimension the spinor superfield is introduced in detail and all chiral and anti-chiral superfields up to third order in covariant derivatives are derived.

In analogy to the general scalar superfield, the general superfield with one spinor index can immediately be written down as an expansion in θ and $\bar{\theta}$

$$V_\alpha = \kappa_\alpha + \theta^\beta M_{\beta\alpha} + \bar{\theta}^{\dot{\beta}} N_{\dot{\beta}\alpha} + \theta^\beta\theta^\gamma\psi_{\alpha\beta\gamma} + \bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}}\chi_{\alpha\dot{\beta}\dot{\gamma}} + \theta^\beta\bar{\theta}^{\dot{\gamma}}\omega_{\alpha\beta\dot{\gamma}}$$
$$+ \theta^\beta\theta^\gamma\bar{\theta}^{\dot{\delta}} R_{\dot{\delta}\alpha\beta\gamma} + \theta^\beta\bar{\theta}^{\dot{\gamma}}\bar{\theta}^{\dot{\delta}} S_{\alpha\beta\dot{\gamma}\dot{\delta}} + \theta^\beta\theta^\gamma\bar{\theta}^{\dot{\delta}}\bar{\theta}^{\dot{\epsilon}}\lambda_{\alpha\beta\gamma\dot{\delta}\dot{\epsilon}} \,. \tag{3.14}$$

To bring this ansatz into a more convenient form the Grassmann variables need to be contracted over the respective indices. After absorbing some of the prefactors into the component fields, the general superfield is found to be

$$V_\alpha = \kappa_\alpha + \theta^\beta M_{\beta\alpha} - \bar{\theta}^{\dot{\beta}} N_{\dot{\beta}\alpha} + \theta^2\psi_\alpha + \bar{\theta}^2\chi_\alpha + \theta\sigma^\mu\bar{\theta}\left(\sigma_\mu\right)^{\beta\dot{\gamma}}\omega_{\alpha\beta\dot{\gamma}}$$
$$- \theta^2\bar{\theta}^{\dot{\delta}} R_{\dot{\delta}\alpha} + \bar{\theta}^2\theta^\beta S_{\beta\alpha} + \theta^2\bar{\theta}^2\lambda_\alpha \,, \tag{3.15}$$

where κ, ψ, χ, and λ are Majorana spinors, M, N, R, and S are complex second rank spinors, and ω is a 3rd rank spinor. The four complex second rank spinors contain 32 bosonic degrees of freedom while the four majorana spinors contain 16 fermionic degrees of freedom. As the number of bosonic and fermionic degrees of freedom must be the same for a supersymmetric theory, the 3rd rank spinor must also have 16 fermionic degrees of freedom. It is then tempting to rewrite the 3rd rank spinor as a vector of Majorana spinors

$$\left(\sigma_\mu\right)^{\beta\dot{\gamma}}\omega_{\alpha\beta\dot{\gamma}} = \left(\sigma_\mu\right)^{\beta\dot{\gamma}}\left(\sigma^\nu\right)_{\beta\dot{\gamma}}\omega_{\nu\alpha} = 2\omega_{\mu\alpha} \,, \tag{3.16}$$

which has 16 degrees of freedom as well. After appropriate rescaling of this term the most general superfield with one spinor index is given by

$$V_\alpha = \kappa_\alpha + \theta^\beta M_{\beta\alpha} - \bar{\theta}^{\dot{\beta}} N_{\dot{\beta}\alpha} + \theta^2\psi_\alpha + \bar{\theta}^2\chi_\alpha + \theta\sigma^\mu\bar{\theta}\omega_{\mu\alpha}$$
$$- \theta^2\bar{\theta}^{\dot{\beta}} R_{\dot{\beta}\alpha} + \bar{\theta}^2\theta^\beta S_{\beta\alpha} + \theta^2\bar{\theta}^2\lambda_\alpha \,. \tag{3.17}$$

3.2.1 The Chiral Superfields

For the general scalar superfield the chiral and antichiral fields are derived by repeated operation of the covariant derivatives D and $\bar{D}$. By definition the chiral and antichiral superfields satisfy the following relations

$$\bar{D}_{\dot{\alpha}} X = 0\,, \tag{3.18}$$

$$D_\alpha Y = 0\,, \tag{3.19}$$

where it is assumed that X is a chiral superfield and Y is an antichiral superfield which can have an arbitrary number of spinor indices. The chiral and anti-chiral superfields up to third order in covariant derivatives are then found by calculating $\bar{D}^2 V$ and $D^2 V$, as well as $\bar{D}^2 D V$ and $D^2 \bar{D} V$. For the general scalar superfield the anticommutation relations for the covariant derivatives imply that the spinor indices of the covariant derivatives that appear twice have to be contracted. This results in one chiral and one anti-chiral scalar superfield, as well as one chiral and one anti-chiral spinor superfield.

The derivation of the chiral and anti-chiral superfields based on the general superfield with one spinor index can then be performed in perfect analogy. The only difference is that the resulting superfields acquire additional spinor indices as well – one chiral and one anti-chiral spinor field, as well as one chiral and one anti-chiral second rank spinor field. It is interesting to note that the chiral second rank spinor field contains a special case where the second rank spinor field is reduced to a chiral scalar field. This is not possible for the anti-chiral second rank spinor field due to the index structure which contains an odd number of dotted and undotted indices.

The chiral spinor field is found by repeated operation of the covariant derivative $\bar{D}$ onto the general superfield

$$\begin{aligned} X_\alpha &= -\frac{1}{4}\bar{D}^2 V_\alpha \\ &= \chi_\alpha + \theta^\beta \left(S_{\beta\alpha} + \frac{i}{2}\partial\!\!\!/_\beta{}^{\dot{\delta}} N_{\dot{\delta}\alpha} \right) + \theta^2 \left(\lambda_\alpha + \frac{i}{2}\partial^\mu \omega_{\mu\alpha} - \frac{1}{4}\Box\kappa_\alpha \right) - i\theta\partial\!\!\!/\bar{\theta}\chi_\alpha \\ &\quad + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}}{}^\beta \left(S_{\beta\alpha} + \frac{i}{2}\partial\!\!\!/_\beta{}^{\dot{\delta}} N_{\dot{\delta}\alpha} \right) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\chi_\alpha\,. \end{aligned} \tag{3.20}$$

Comparison with the general expansion of a chiral field with one spinor index

$$X_\alpha = \chi_\alpha + \theta^\beta M_{\beta\alpha} + \theta^2\lambda_\alpha - i\theta\partial\!\!\!/\bar{\theta}\chi_\alpha + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}}{}^\beta M_{\beta\alpha} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\chi_\alpha \tag{3.21}$$

enables us to rewrite the chiral spinor field in a very elegant way

$$X_\alpha = \exp\left(-i\theta\partial\!\!\!/\bar{\theta}\right)\left(\chi_\alpha + \theta^\beta\left(S_{\beta\alpha} + \frac{i}{2}\partial\!\!\!/_\beta{}^{\dot{\delta}} N_{\dot{\delta}\alpha}\right) + \theta^2\left(\lambda_\alpha + \frac{i}{2}\partial^\mu\omega_{\mu\alpha} - \frac{1}{4}\Box\kappa_\alpha\right)\right)\,. \quad (3.22)$$

A similar notation is possible for the remaining chiral and anti-chiral superfields that will be introduced shortly. As this notation is not used in the further discussion an explicit notation in exponential form is not given for Y, Z, Z_0, and Z' but can be derived as well.

The calculations for the anti-chiral spinor field can be performed in a similar way where the repeated operation of the covariant derivatives D on the general superfield replaces the repeated operation of $\bar{D}$

$$\begin{aligned} Y_\alpha &= -\frac{1}{4}D^2 V_\alpha \\ &= \psi_\alpha - \bar{\theta}^{\dot{\beta}}\left(R_{\dot{\beta}\alpha} + \frac{i}{2}\bar{\partial\!\!\!/}_{\dot{\beta}}{}^\gamma M_{\gamma\alpha}\right) + \bar{\theta}^2\left(\lambda_\alpha + \frac{i}{2}\partial^\mu\omega_{\mu\alpha} - \frac{1}{4}\Box\kappa_\alpha\right) - i\theta\partial\!\!\!/\bar{\theta}\psi_\alpha \\ &\quad - \frac{i}{2}\theta^\gamma\bar{\theta}^2\partial\!\!\!/_\gamma{}^{\dot{\beta}}\left(R_{\dot{\beta}\alpha} + \frac{i}{2}\bar{\partial\!\!\!/}_{\dot{\beta}}{}^\delta M_{\delta\alpha}\right) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\psi_\alpha\,. \end{aligned} \quad (3.23)$$

At this point it is instructive to have a brief look at the component fields of the chiral and anti-chiral superfields. It can be seen that the component fields χ, S and N only appear in the chiral spinor field, while ψ, R, and M only appear in the anti-chiral spinor field. Furthermore, the component fields λ, ω and κ appear in the θ^2 and $\bar{\theta}^2$ components of the chiral and anti-chiral fields respectively. It turns out that the component fields λ, ω, and κ that appear in both superfields are auxiliary fields and don't contribute to the on-shell Lagrangian.

To third order in covariant derivatives there is again one chiral and one anti-chiral superfield which are now spinor fields of second rank. The chiral second rank spinor field is found to be

$$
\begin{aligned}
Z_{\gamma\alpha} &= -\frac{1}{4}\bar{D}^2 D_\gamma V_\alpha \\
&= \left(S_{\gamma\alpha} - \frac{i}{2}\partial\!\!\!/_{\gamma}{}^{\dot\beta} N_{\dot\beta\alpha}\right) + \theta^\beta\left(2\epsilon_{\beta\gamma}\lambda_\alpha + (\sigma^{\nu\mu})_{\beta\gamma}\,\partial_\nu\omega_{\mu\alpha} + \frac{1}{2}\epsilon_{\beta\gamma}\Box\kappa_\alpha\right) \\
&\quad - i\theta^2\left(\partial\!\!\!/_{\gamma}{}^{\dot\beta} R_{\dot\beta\alpha} - \frac{i}{2}\Box M_{\gamma\alpha}\right) - i\theta\partial\!\!\!/\bar\theta\left(S_{\gamma\alpha} - \frac{i}{2}\partial\!\!\!/_{\gamma}{}^{\dot\beta} N_{\dot\beta\alpha}\right) \\
&\quad + \frac{i}{2}\theta^2\bar\theta^{\dot\delta}\partial\!\!\!/^{\beta}{}_{\dot\delta}\left(2\epsilon_{\beta\gamma}\lambda_\alpha + (\sigma^{\nu\mu})_{\beta\gamma}\,\partial_\nu\omega_{\mu\alpha} + \frac{1}{2}\epsilon_{\beta\gamma}\Box\kappa_\alpha\right) \\
&\quad - \frac{1}{4}\theta^2\bar\theta^2\Box\left(S_{\gamma\alpha} - \frac{i}{2}\partial\!\!\!/_{\gamma}{}^{\dot\beta} N_{\dot\beta\alpha}\right). \qquad (3.24)
\end{aligned}
$$

A special case arises if the two undotted indices of the second rank spinor field are contracted. It is then reduced to a scalar superfield

$$
\begin{aligned}
Z_0 &= \frac{1}{4}\bar{D}^2 D V \\
&= \mathrm{Tr}\left(S + \frac{i}{2}\partial\!\!\!/ N\right) - \theta^\beta\left(2\lambda_\beta + \sigma^{\nu\mu}\partial_\nu\omega_\mu + \frac{1}{2}\Box\kappa_\beta\right) + i\theta^2\,\mathrm{Tr}\left(\partial\!\!\!/ R + \frac{i}{2}\Box M\right) \\
&\quad - i\theta\partial\!\!\!/\bar\theta\,\mathrm{Tr}\left(S + \frac{i}{2}\partial\!\!\!/ N\right) - \frac{i}{2}\theta^2\bar\theta^{\dot\delta}\partial\!\!\!/^{\beta}{}_{\dot\delta}\left(2\lambda_\beta + \sigma^{\nu\mu}\partial_\nu\omega_\mu + \frac{1}{2}\Box\kappa_\beta\right) \\
&\quad - \frac{1}{4}\theta^2\bar\theta^2\Box\,\mathrm{Tr}\left(S + \frac{i}{2}\partial\!\!\!/ N\right). \qquad (3.25)
\end{aligned}
$$

The calculations for the anti-chiral second rank spinor field are nearly identical and it is found to be

$$
\begin{aligned}
Z' &= -\frac{1}{4}D^2\bar{D}_{\dot\gamma} V_\alpha \\
&= \left(R_{\dot\gamma\alpha} + \frac{i}{2}\partial\!\!\!/^{\beta}{}_{\dot\gamma} M_{\beta\alpha}\right) - \bar\theta^{\dot\beta}\left(2\epsilon_{\dot\beta\dot\gamma}\lambda_\alpha - (\bar\sigma^{\nu\mu})_{\dot\beta\dot\gamma}\,\partial_\nu\omega_{\mu\alpha} + \frac{1}{2}\epsilon_{\dot\beta\dot\gamma}\Box\kappa_\alpha\right) \\
&\quad + \bar\theta^2\left(i\bar{\partial\!\!\!/}_{\dot\gamma}{}^{\beta} S_{\beta\alpha} - \frac{1}{2}\Box N_{\dot\gamma\alpha}\right) + i\theta\partial\!\!\!/\bar\theta\left(R_{\dot\gamma\alpha} + \frac{i}{2}\partial\!\!\!/^{\beta}{}_{\dot\gamma} M_{\beta\alpha}\right) \\
&\quad + \frac{i}{2}\theta^\delta\bar\theta^2\partial\!\!\!/_{\delta}{}^{\dot\beta}\left(2\epsilon_{\dot\beta\dot\gamma}\lambda_\alpha - (\bar\sigma^{\nu\mu})_{\dot\beta\dot\gamma}\,\partial_\nu\omega_{\mu\alpha} + \frac{1}{2}\epsilon_{\dot\beta\dot\gamma}\Box\kappa_\alpha\right) \\
&\quad - \frac{1}{4}\theta^2\bar\theta^2\Box\left(R_{\dot\gamma\alpha} + \frac{i}{2}\partial\!\!\!/^{\beta}{}_{\dot\gamma} M_{\beta\alpha}\right). \qquad (3.26)
\end{aligned}
$$

Unlike for the chiral second rank spinor field, no special case exists for the anti-chiral second rank spinor field. This is due to the odd number of dotted and undotted indices which makes it impossible to contract the indices to achieve a scalar superfield.

3.2.2 Unitary Supertranslations

For the later discussion of the supercurrent and the derivation of the commutators and anticommutators for the second quantisation of the component fields it is necessary to derive the superfield variation of the general superfield with one spinor index. This also specifies the superfield variations of the component fields and thus the superfield variation of the on-shell component fields. The following derivation follows closely the discussion for the general scalar superfield in Dick (2009) and was adapted accordingly to compensate for the additional spinor index.

The starting point for the derivation of the behaviour of a superfield under unitary supertranslations is the definition of a superspace eigenstate

$$\left|x_0, \theta_0, \bar{\theta}_0\right\rangle \ , \tag{3.27}$$

which has the eigenvalues

$$x^\mu \left|x_0, \theta_0, \bar{\theta}_0\right\rangle = x_0 \left|x_0, \theta_0, \bar{\theta}_0\right\rangle \ , \tag{3.28}$$

$$\theta_\alpha \left|x_0, \theta_0, \bar{\theta}_0\right\rangle = \theta_{0\alpha} \left|x_0, \theta_0, \bar{\theta}_0\right\rangle \ , \tag{3.29}$$

$$\bar{\theta}_{\dot{\alpha}} \left|x_0, \theta_0, \bar{\theta}_0\right\rangle = \bar{\theta}_{0\dot{\alpha}} \left|x_0, \theta_0, \bar{\theta}_0\right\rangle \ . \tag{3.30}$$

Here θ_α, $\bar{\theta}_{\dot{\alpha}}$, and x^μ are operators acting on the superspace eigenstate while the eigenvalues are denoted by a subscript 0 for the original eigenstate and with a prime for the translated state. This differs somewhat from the usual convention to denote operators with a hat to achieve a consistent notation throughout this thesis. Therefore, a state that is shifted under unitary supertranslations can be written as

$$\left|x', \theta', \bar{\theta}'\right\rangle = \exp\left(iay \cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right) \left|x_0, \theta_0, \bar{\theta}_0\right\rangle \ , \tag{3.31}$$

where the prefactors a, b, and c still need to be determined in agreement with the previous choice for the position space representation of the components of the supersymmetry algebra. An arbitrary operator $\mathcal{O}$ acting on the shifted state can also be expressend as

$$\begin{aligned}\mathcal{O}\left|x',\theta',\bar{\theta}'\right\rangle &= \mathcal{O}\exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right)\left|x_0,\theta_0,\bar{\theta}_0\right\rangle \\ &= \exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right)\exp\left(-iay\cdot P - ib\zeta Q - ic\bar{Q}\bar{\zeta}\right) \\ &\quad\times\mathcal{O}\exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right)\left|x_0,\theta_0,\bar{\theta}_0\right\rangle\,. \end{aligned} \tag{3.32}$$

Using the Campbell-Baker-Hausdorff formula

$$e^{-iG\lambda}Ae^{iG\lambda} = \sum_j (-i\lambda)^j \overset{j}{[G,A]} \tag{3.33}$$

this product of operators can be decomposed into an infinite sum of commutators

$$\begin{aligned}\mathcal{O}\left|x',\theta',\bar{\theta}'\right\rangle &= \exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right) \\ &\quad\times\sum_j (-i\lambda)^j \overset{j}{\left[ay\cdot P + b\zeta Q + c\bar{Q}\bar{\zeta},\mathcal{O}\right]}\left|x_0,\theta_0,\bar{\theta}_0\right\rangle\,. \end{aligned} \tag{3.34}$$

To evaluate the commutators in the previous equation it proves useful to utilise the following three commutators and anticommutators.

$$\{\partial_\beta,\theta_\alpha\} = \epsilon_{\beta\alpha}\,, \tag{3.35}$$

$$\left\{\partial_{\dot{\beta}},\bar{\theta}_{\dot{\alpha}}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}}\,, \tag{3.36}$$

$$[P_\nu, x_\mu] = i\eta_{\nu\mu}\,. \tag{3.37}$$

For the operator θ it is found that

$$\begin{aligned}\theta_\alpha\left|x',\theta',\bar{\theta}'\right\rangle &= \exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right) \\ &\quad\times\sum_j (-i\lambda)^j \overset{j}{\left[ay\cdot P + b\zeta Q + c\bar{Q}\bar{\zeta},\theta\right]}\left|x,\theta,\bar{\theta}\right\rangle\,, \end{aligned} \tag{3.38}$$

where the n-th commutator has to be derived recursively. Conveniently, the first commutator is given by

$$\left[ay^\mu P_\mu + b\zeta^\beta Q_\beta + c\bar{Q}_{\dot{\beta}}\bar{\zeta}^{\dot{\beta}},\theta_\alpha\right] = ib\zeta_\alpha\,. \tag{3.39}$$

This implies that the second commutator already vanishes

$$\overset{2}{[}\ ay^\mu P_\mu + b\zeta^\beta Q_\beta + c\bar{Q}_{\dot\beta}\bar{\zeta}^{\dot\beta}, \theta_\alpha \Big] = 0\,. \tag{3.40}$$

Therefore, all higher order contributions to the infinite sum must vanish identically as well. The eigenvalue of the shifted state is then found to be

$$\begin{aligned}\theta'_\alpha \left|x', \theta', \bar\theta'\right\rangle &= \exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar\zeta\right)(\theta_\alpha + b\zeta_\alpha)\left|x_0, \theta_0, \bar\theta_0\right\rangle \\ &= (\theta_{0\alpha} + b\zeta_\alpha)\left|x', \theta', \bar\theta'\right\rangle\,,\end{aligned} \tag{3.41}$$

which corresponds to

$$\theta'_\alpha = \theta_{0\alpha} + b\zeta_\alpha\,. \tag{3.42}$$

A similar calculation can be repeated for the operator $\bar\theta$. It is found that

$$\begin{aligned}\bar\theta_{\dot\alpha} \left|x', \theta', \bar\theta'\right\rangle &= \exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar\zeta\right) \\ &\quad \times \sum_j (-i\lambda)^j \overset{j}{[}ay\cdot P + b\zeta Q + c\bar{Q}\bar\zeta, \bar\theta_{\dot\alpha}]\left|x_0, \theta_0, \bar\theta_0\right\rangle\,.\end{aligned} \tag{3.43}$$

Again the the n-th commutator must be calculated recursively starting with the first order commutator

$$\left[ay\cdot P + b\zeta^\beta Q_\beta + c\bar{Q}_{\dot\beta}\bar\zeta^{\dot\beta}, \bar\theta_{\dot\alpha}\right] = ic\bar\zeta_{\dot\alpha}\,. \tag{3.44}$$

Like in the previous case this result implies that the second commutator already vanishes identically

$$\overset{2}{[}\ ay\cdot P + b\zeta^\beta Q_\beta + c\bar{Q}_{\dot\beta}\bar\zeta^{\dot\beta}, \bar\theta_{\dot\alpha}\Big] = 0\,. \tag{3.45}$$

The eigenvalue of the shifted state is then found to be

$$\begin{aligned}\bar\theta'_{\dot\alpha} \left|x', \theta', \bar\theta'\right\rangle &= \exp\left(iay\cdot P + ib\zeta Q + ic\bar{Q}\bar\zeta\right)\left(\bar\theta_{\dot\alpha} + c\bar\zeta_{\dot\alpha}\right)\left|x_0, \theta_0, \bar\theta_0\right\rangle \\ &= \left(\bar\theta_{0\dot\alpha} + c\bar\zeta_{\dot\alpha}\right)\left|x', \theta', \bar\theta'\right\rangle\,,\end{aligned} \tag{3.46}$$

which alternatively can be written as

$$\bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{0\dot{\alpha}} + c\bar{\zeta}_{\dot{\alpha}} \,. \tag{3.47}$$

Finally, the behaviour of the eigenvalue of the operator x^μ is analysed

$$\begin{aligned} x^\mu \left|x', \theta', \bar{\theta}'\right\rangle &= \exp\left(iay \cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right) \\ &\times \sum_j (-i\lambda)^j \overset{j}{\left[ay \cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}, x^\mu\right]} \left|x_0, \theta_0, \bar{\theta}_0\right\rangle \,. \end{aligned} \tag{3.48}$$

As for the previous discussion of θ and $\bar{\theta}$ it is necessary to determine the n-th commutator recursively. The first commutator is found to be

$$\left[ay^\nu P_\nu + b\zeta^\alpha Q_\alpha + c\bar{Q}_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}, x^\mu\right] = iay^\mu - b\zeta\sigma^\mu\bar{\theta} + c\theta\sigma^\mu\bar{\zeta} \,. \tag{3.49}$$

At first glance it seems as if the series expansion doesn't terminate after the first commutator. However, the explicit calculation of the second commutator reveals that it vanishes identically

$$\overset{2}{\left[ay^\nu P_\nu + b\zeta^\alpha Q_\alpha + c\bar{Q}_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}, x^\mu\right]} = ibc\zeta^\gamma \left(\sigma^\mu\right)_\gamma{}^{\dot{\alpha}}\bar{\zeta}_{\dot{\alpha}} - ibc\zeta_\alpha \left(\sigma^\mu\right)^\alpha{}_{\dot{\gamma}}\bar{\zeta}^{\dot{\gamma}} = 0 \,. \tag{3.50}$$

For this calculation it proves convenient to reuse the results from the previous calculations for θ and $\bar{\theta}$. The identically vanishing second commutator then terminates the infinite series and the eigenvalue for the operator x^μ acting on the translated state is found to be

$$\begin{aligned} x^\mu \left|x', \theta', \bar{\theta}'\right\rangle &= \exp\left(iay \cdot P + ib\zeta Q + ic\bar{Q}\bar{\zeta}\right)\left(x^\mu + ay^\mu + i\left(b\zeta\sigma^\mu\bar{\theta} - c\theta\sigma^\mu\bar{\zeta}\right)\right)\left|x_0, \theta_0, \bar{\theta}_0\right\rangle \\ &= \left(x_0^\mu + ay^\mu + i\left(b\zeta\sigma^\mu\bar{\theta}_0 - c\theta_0\sigma^\mu\bar{\zeta}\right)\right)\left|x', \theta', \bar{\theta}'\right\rangle \,. \end{aligned} \tag{3.51}$$

Therefore, the eigenvalue of the translated state under the operation of the operator x is given by

$$x'^\mu = x_0^\mu + ay^\mu + i\left(b\zeta\sigma^\mu\bar{\theta}_0 - c\theta_0\sigma^\mu\bar{\zeta}\right) \,. \tag{3.52}$$

Combining the results for the operators θ, $\bar{\theta}$, and x^μ yields a translated superspace eigenstate of

$$\left|x', \theta', \bar{\theta}'\right\rangle = \left|x_0 + a y_0 + i\left(b\zeta\sigma\bar{\theta}_0 - c\theta_0\sigma\bar{\zeta}\right), \theta_0 + b\zeta, \bar{\theta}_0 + c\bar{\zeta}\right\rangle \,, \tag{3.53}$$

where the prefactors a, b, and c are still arbitrary. As a convention it is assumed that the discussion is restricted to pure superspace translations for which the spatial translation vanishes and thus $ay_0 = 0$. Furthermore, the translations of the superspace coordinates θ and $\bar{\theta}$ are chosen to be positive which results in $b = c = 1$. Alternatively it would be possible to adopt a negative translation of the superspace coordinates which was realized in Dick (2009). This results in a relation between the original and shifted state of the following form

$$\left|x', \theta', \bar{\theta}'\right\rangle = \left|x + i\left(\zeta\sigma\bar{\theta} - \theta\sigma\bar{\zeta}\right), \theta + \zeta, \bar{\theta} + \bar{\zeta}\right\rangle = \exp\left(i\zeta Q + i\bar{Q}\bar{\zeta}\right)\left|x, \theta, \bar{\theta}\right\rangle \,, \tag{3.54}$$

where the subscript 0 was dropped as it is no longer necessary to distinguish between operators and eigenvalues. It expresses the eigenstate at the shifted superspace coordinates in terms of the superspace coordinates of the original superspace eigenstate. It can be seen that a superspace translation, unlike a translation of normal fields, not only induces a spatial translation, but also results in a shift of the superspace coordinates θ and $\bar{\theta}$. This is similar to the consecutive application of two Lorentz boosts with velocities that are not colinear which induces a Lorentz boost as well as a rotation.

Now that the behaviour of a superspace eigenstate under unitary supertranslation is known, the calculation of the translated general superfield with one spinor index is straightforward

$$\begin{aligned} V'\left(x, \theta, \bar{\theta}\right) &= \left\langle x, \theta, \bar{\theta}\right| \exp\left(i\zeta Q + i\bar{Q}\bar{\zeta}\right)\left|V\right\rangle \\ &= \left\langle x - i\left(\zeta\sigma\bar{\theta} - \theta\sigma\bar{\zeta}\right), \theta - \zeta, \bar{\theta} - \bar{\zeta}\right| V\rangle \\ &= V\left(x - i\left(\zeta\sigma\bar{\theta} - \theta\sigma\bar{\zeta}\right), \theta - \zeta, \bar{\theta} - \bar{\zeta}\right). \end{aligned} \tag{3.55}$$

As for the superspace eigenstate, a unitary supertranslation acting on a general superfield induces a spatial translation as well as a shift of superspace coordinates. In terms of the component fields the translated superfield can then be written as

$$
\begin{aligned}
{V'}_\alpha(x,\theta,\bar\theta) &= {V'}_\alpha\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right), \theta-\zeta, \bar\theta-\bar\zeta\right) \\
&= \kappa_\alpha\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) + \left(\theta^\beta - \zeta^\beta\right) M_{\beta\alpha}\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \\
&\quad - \left(\bar\theta^{\dot\beta} - \bar\zeta^{\dot\beta}\right) N_{\dot\beta\alpha}\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) + \left(\theta-\zeta\right)^2 \psi_\alpha\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \\
&\quad + \left(\bar\theta - \bar\zeta\right)^2 \chi_\alpha\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \\
&\quad + \left(\theta-\zeta\right)\sigma^\mu\left(\bar\theta-\bar\zeta\right)\omega_{\mu\alpha}\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \\
&\quad - \left(\theta-\zeta\right)^2\left(\bar\theta^{\dot\beta} - \bar\zeta^{\dot\beta}\right) R_{\dot\beta\alpha}\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \\
&\quad + \left(\bar\theta-\bar\zeta\right)^2\left(\theta^\beta - \zeta^\beta\right) S_{\beta\alpha}\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \\
&\quad + \left(\theta-\zeta\right)^2\left(\bar\theta-\bar\zeta\right)^2 \lambda_\alpha\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right). \qquad (3.56)
\end{aligned}
$$

To express the translated component fields in terms of the component fields at the original superspace coordinates a Taylor expansion of the component fields can be used. In Dick (2009) the transformation to all orders is presented. For our purposes an expansion up to first order in the transformation parameters ζ and $\bar\zeta$ of the form

$$
\kappa_\alpha\left(x - i\left(\zeta\sigma\bar\theta - \theta\sigma\bar\zeta\right)\right) \approx \kappa_\alpha(x) - i\left(\zeta\sigma^\nu\bar\theta - \theta\sigma^\nu\bar\zeta\right)\partial_\nu\kappa_\alpha(x) \qquad (3.57)
$$

is sufficient. After appropriately rewriting equation (3.56), neglecting all terms of second or higher order in the transformation parameters ζ and $\bar\zeta$, and collecting the terms with corresponding orders in the Grassmann variables θ and $\bar\theta$ the shifted superfield is given by

$$
\begin{aligned}
V'_\alpha(x,\theta,\bar\theta) = {}& \kappa_\alpha(x) - \zeta^\beta M_{\beta\alpha}(x) + \bar\zeta^{\dot\beta} N_{\dot\beta\alpha}(x) \\
& + \theta^\beta\left(M_{\beta\alpha}(x) + i\left(\sigma^\mu\right)_{\beta\dot\gamma}\bar\zeta^{\dot\gamma}\partial_\mu\kappa_\alpha(x) - 2\zeta_\beta\psi_\alpha(x) - \left(\sigma^\mu\right)_{\beta\dot\gamma}\bar\zeta^{\dot\gamma}\omega_{\mu\alpha}(x)\right) \\
& - \bar\theta^{\dot\beta}\left(N_{\dot\beta\alpha}(x) - i\left(\bar\sigma^\mu\right)_{\dot\beta\gamma}\zeta^\gamma\partial_\mu\kappa_\alpha(x) - 2\bar\zeta_{\dot\beta}\chi_\alpha(x) - \left(\bar\sigma^\mu\right)_{\dot\beta\gamma}\zeta^\gamma\omega_{\mu\alpha}(x)\right) \\
& + \theta^2\left(\psi_\alpha(x) - \frac{i}{2}\bar\zeta^{\dot\delta}\left(\bar\sigma^\mu\right)_{\dot\delta}{}^\beta\partial_\mu M_{\beta\alpha}(x) + \bar\zeta^{\dot\beta}R_{\dot\beta\alpha}(x)\right) \\
& + \bar\theta^2\left(\chi_\alpha(x) - \frac{i}{2}\zeta^\delta\left(\sigma^\mu\right)_\delta{}^{\dot\beta}\partial_\mu N_{\dot\beta\alpha}(x) - \zeta^\beta S_{\beta\alpha}(x)\right) \\
& + \theta\sigma^\mu\bar\theta\left(\omega_{\mu\alpha}(x) + \frac{i}{2}\zeta^\delta\left(\sigma^\nu\bar\sigma_\mu\right)_\delta{}^\beta\partial_\nu M_{\beta\alpha}(x) - \frac{i}{2}\bar\zeta^{\dot\delta}\left(\bar\sigma^\nu\sigma_\mu\right)_{\dot\delta}{}^{\dot\beta}\partial_\nu N_{\dot\beta\alpha}(x)\right. \\
& \left. -\zeta_\gamma\left(\sigma_\mu\right)^{\gamma\dot\beta}R_{\dot\beta\alpha}(x) + \bar\zeta_{\dot\gamma}\left(\bar\sigma_\mu\right)^{\dot\gamma\beta}S_{\beta\alpha}(x)\right)
\end{aligned}
$$

$$- \theta^2\bar{\theta}^{\dot{\beta}}\left(R_{\dot{\beta}\alpha}(x) - i\left(\bar{\sigma}^\mu\right)_{\dot{\beta}\gamma}\zeta^\gamma\partial_\mu\psi_\alpha(x) + \frac{i}{2}\left(\bar{\sigma}^\mu\sigma^\nu\right)_{\dot{\beta}\dot{\epsilon}}\bar{\zeta}^{\dot{\epsilon}}\partial_\nu\omega_{\mu\alpha}(x) - 2\bar{\zeta}_{\dot{\beta}}\lambda_\alpha(x)\right)$$
$$+ \bar{\theta}^2\theta^\beta\left(S_{\beta\alpha}(x) + i\left(\sigma^\mu\right)_{\beta\dot{\gamma}}\bar{\zeta}^{\dot{\gamma}}\partial_\mu\chi_\alpha(x) + \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_{\beta\delta}\zeta^\delta\partial_\nu\omega_{\mu\alpha}(x) - 2\zeta_\beta\lambda_\alpha(x)\right)$$
$$+ \theta^2\bar{\theta}^2\left(\lambda_\alpha(x) - \frac{i}{2}\zeta^\gamma\left(\sigma^\mu\right)_\gamma{}^{\dot{\beta}}\partial_\mu R_{\dot{\beta}\alpha}(x) - \frac{i}{2}\bar{\zeta}^{\dot{\delta}}\left(\bar{\sigma}^\mu\right)_{\dot{\delta}}{}^\beta\partial_\mu S_{\beta\alpha}(x)\right). \quad (3.58)$$

The variation of the general superfield with one spinor index is then defined as the difference between the translated superfield and the superfield at the original superspace coordinates

$$\delta V\left(x,\theta,\bar{\theta}\right) = V'\left(x,\theta,\bar{\theta}\right) - V\left(x,\theta,\bar{\theta}\right). \quad (3.59)$$

Therefore, the variation of the component fields can be extracted immediately from equation (3.58)

$$\delta\kappa_\alpha = -\zeta^\beta M_{\beta\alpha}(x) + \bar{\zeta}^{\dot{\beta}}N_{\dot{\beta}\alpha}(x)\,, \quad (3.60)$$
$$\delta M_{\beta\alpha} = -2\zeta_\beta\psi_\alpha(x) + i\bar{\zeta}^{\dot{\gamma}}\left(\bar{\sigma}^\mu\right)_{\dot{\gamma}\beta}\partial_\mu\kappa_\alpha(x) - \bar{\zeta}^{\dot{\gamma}}\left(\bar{\sigma}^\mu\right)_{\dot{\gamma}\beta}\omega_{\mu\alpha}(x)\,, \quad (3.61)$$
$$\delta N_{\dot{\beta}\alpha} = -2\bar{\zeta}_{\dot{\beta}}\chi_\alpha(x) - i\zeta^\gamma\left(\sigma^\mu\right)_{\gamma\dot{\beta}}\partial_\mu\kappa_\alpha(x) - \zeta^\gamma\left(\sigma^\mu\right)_{\gamma\dot{\beta}}\omega_{\mu\alpha}(x)\,, \quad (3.62)$$
$$\delta\psi_\alpha = \bar{\zeta}^{\dot{\beta}}R_{\dot{\beta}\alpha}(x) - \frac{i}{2}\bar{\zeta}^{\dot{\beta}}\left(\bar{\sigma}^\mu\right)_{\dot{\beta}}{}^\gamma\partial_\mu M_{\gamma\alpha}(x)\,, \quad (3.63)$$
$$\delta\chi_\alpha = -\zeta^\beta S_{\beta\alpha}(x) - \frac{i}{2}\zeta^\beta\left(\sigma^\mu\right)_\beta{}^{\dot{\gamma}}\partial_\mu N_{\dot{\gamma}\alpha}(x)\,, \quad (3.64)$$
$$\delta\omega_{\mu\alpha} = \zeta^\beta\left(\sigma_\mu\right)_\beta{}^{\dot{\gamma}}R_{\dot{\gamma}\alpha}(x) + \frac{i}{2}\zeta^\beta\left(\sigma^\nu\bar{\sigma}_\mu\right)_\beta{}^\gamma\partial_\nu M_{\gamma\alpha}(x)$$
$$- \bar{\zeta}^{\dot{\beta}}\left(\bar{\sigma}_\mu\right)_{\dot{\beta}}{}^\gamma S_{\gamma\alpha}(x) - \frac{i}{2}\bar{\zeta}^{\dot{\beta}}\left(\bar{\sigma}^\nu\sigma_\mu\right)_{\dot{\beta}}{}^{\dot{\gamma}}\partial_\nu N_{\dot{\gamma}\alpha}(x)\,, \quad (3.65)$$
$$\delta R_{\dot{\beta}\alpha} = -2\bar{\zeta}_{\dot{\beta}}\lambda_\alpha(x) - i\zeta^\gamma\left(\sigma^\mu\right)_{\gamma\dot{\beta}}\partial_\mu\psi_\alpha(x) - \frac{i}{2}\bar{\zeta}^{\dot{\gamma}}\left(\bar{\sigma}^\nu\sigma^\mu\right)_{\dot{\gamma}\dot{\beta}}\partial_\nu\omega_{\mu\alpha}(x)\,, \quad (3.66)$$
$$\delta S_{\beta\alpha} = -2\zeta_\beta\lambda_\alpha(x) + i\bar{\zeta}^{\dot{\gamma}}\left(\bar{\sigma}^\mu\right)_{\dot{\gamma}\beta}\partial_\mu\chi_\alpha(x) - \frac{i}{2}\zeta^\gamma\left(\sigma^\nu\bar{\sigma}^\mu\right)_{\gamma\beta}\partial_\nu\omega_{\mu\alpha}(x)\,, \quad (3.67)$$
$$\delta\lambda_\alpha = -\frac{i}{2}\zeta^\beta\left(\sigma^\mu\right)_\beta{}^{\dot{\gamma}}\partial_\mu R_{\dot{\gamma}\alpha}(x) - \frac{i}{2}\bar{\zeta}^{\dot{\beta}}\left(\bar{\sigma}^\mu\right)_{\dot{\beta}}{}^\gamma\partial_\mu S_{\gamma\alpha}(x)\,. \quad (3.68)$$

These results then imply the variation of the on-shell component fields. After eliminating the auxiliary fields and using the definition of the component fields $\tilde{R}$ and $\tilde{S}$ from equations (3.75) and (3.76) the variation of the component field of the on-shell Lagrangian are found to be

$$\delta\psi_\alpha = \bar{\zeta}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha}\,, \quad (3.69)$$
$$\delta\chi_\alpha = -\zeta^\beta\tilde{S}_{\beta\alpha}\,, \quad (3.70)$$

Product	Mass Dimension	Contributions
VV	$\dim(VV) = 0$	$\big(m^2 VV\big)_D$
XV	$\dim(XV) = 1$	$(mXV)_D$, $(mYV)_D$
$DVDV$	$\dim(DVDV) = 1$	$(mDVDV)_D$, $\big(m\bar{D}V\bar{D}V\big)_D$
VX	$\dim(VX) = 1$	$(mVX)_D$, $(mVY)_D$
DZV	$\dim(DZV) = 2$	$(DZV)_D$, $\big(\bar{D}Z'V\big)_D$
ZDV	$\dim(ZDV) = 2$	$(ZDV)_D$, $\big(Z'\bar{D}V\big)_D$
XX	$\dim(XX) = 2$	$(mXX)_F$, $(mYY)_F$, $(XY)_D$, $(YX)_D$
DVZ	$\dim(DVZ) = 2$	$(DVZ)_D$, $\big(\bar{D}VZ'\big)_D$
VDZ	$\dim(VDZ) = 2$	$(VDZ)_D$, $\big(V\bar{D}Z'\big)_D$
DZX	$\dim(DZX) = 3$	mass dimension too large for D-component
ZZ	$\dim(ZZ) = 3$	$(ZZ)_F$, $(Z'Z')_F$
XDZ	$\dim(XDZ) = 3$	mass dimension too large for D-component

Table 3.2: Possible contributions to the Lagrangian for χ as fermionic field with mass dimension one based on the general superfield with one spinor index. The first two columns specify the product and mass dimensionality using the general superfield and chiral superfields only. The third column then summarises all possible contributions corresponding to the product outlined in the first column including the contributions that arise from the antichiral superfields.

$$\delta\tilde{R}_{\dot{\beta}\alpha} = m\bar{\zeta}_{\dot{\beta}}\chi_\alpha - 2i\zeta^\gamma \partial\!\!\!/_{\gamma\dot{\beta}}\psi_\alpha\,, \tag{3.71}$$

$$\delta\tilde{S}_{\beta\alpha} = m\zeta_\beta\psi_\alpha + 2i\bar{\zeta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}\beta}\chi_\alpha\,. \tag{3.72}$$

3.3 The Lagrangian for χ as Fermionic Field with Mass Dimension One

If χ is identified with the fermionic field with mass dimension one it can be shown that

$$\dim(V_\alpha) = 0\,, \quad \dim(D_\alpha) = \frac{1}{2}\,, \quad \dim(X_\alpha) = \dim(Y_\alpha) = 1\,, \quad \dim(Z_{\gamma\alpha}) = \dim\big(Z'_{\dot{\gamma}\alpha}\big) = \frac{3}{2}\,. \tag{3.73}$$

With these results for the mass dimensions of the building blocks of the Lagrangian all possible terms can be worked out. It is interesting to note that for χ as fermionic field with mass dimension one the mass dimension of the general superfield with one spinor index is 1/2 lower than for the general scalar superfield. This indicates that the structure of the theory based on the general superfield with one spinor index is richer as there are more allowed contributions to the Lagrangian. For convenience the discussion is restricted to the unbarred superfields while the Hermitian conjugates contribute to the Lagrangian as well.

The contributions to the Lagrangian have to satisfy the same basic requirements as outlined in the discussion of the general scalar superfield in Section 3.1.1 – no uncontracted spinor indices, positive mass dimension for coupling constants, and appropriate mass dimension for contribution via D- or F-component. All conceivable terms that are in agreement with these conditions are then summarised in Table 3.2.

As mentioned earlier, Table 3.2 contains significantly more possible contributions to the Lagrangian which are divided into four groups. The additional group is due to the lower mass dimensionality of the general superfield with one spinor index which now allows a spectrum for the mass dimensions ranging from 0 and 3.

The first group which contains only one term, the product of two general superfields with one spinor index without additional covariant derivatives, has mass dimension 0. For symmetry reasons the only possible contribution to the Lagrangian is a mass term via the D-component.

The group containing all terms with mass dimension 1 has then 6 possible terms. As V and DV are neither chiral nor anti-chiral all six terms are contributions to the mass term via the D-component.

In the third group all terms with mass dimension 2 are grouped together. It contains 12 terms of which 10 are contributions to the kinetic term via the D-component while 2 are contributions to the mass term via the F-component. It is worth mentioning that this is the only group that contains contributions to the kinetic term as well as contributions to the mass term. It is even more intriguing to notice that one specific type of superfield product of the form X_1X_2 where X_1 and X_2 can be either chiral or antichiral is able to produce both kind of contributions.

Finally, the fourth group which contains all terms with mass dimension 3 has two entries. Due to the mass dimension only contributions via the F-component are possible which means that both terms can only contribute to the kinetic term.

It is interesting to note that some of the terms contained in table 3.2, namely $DVDV$ and XV were previously considered by Gates and Siegel Gates Jr. and Siegel (1980, 1981). However, in these articles the authors assume the commonly used mass dimensions for fermionic and bosonic fields. This has two consequences. First, all kinetic terms in Gates Jr. and Siegel (1980, 1981) become mass terms in the present scenario due to the change of mass dimensionality. Second, all contributions summarised in groups three and four of table 3.2, and therefore the products of chiral superfields XX and YY do not exist without redefinition of mass dimensions to accommodate fermionic fields with mass dimension one and thus were not considered before.

3.4 The On-shell Lagrangian

A supersymmetric Lagrangian can be constructed by combining contributions that were found in the dimensional analysis of the previous section. It was mentioned earlier that the first two groups of Table 3.2 with mass dimension 0 and 1 respectively contain only contributions to the mass term while the group with mass dimension 3 only produces contributions to the kinetic term. Therefore, the following discussion for the construction of a supersymmetric Lagrangian will be resticted to the third group which is the only one containing kinetic as well as mass terms. This limits the number of superfield products that need to be calculated to 12. Explicit calculations reveal that this number can be narrowed down even further. It can be shown that the terms $(DZV)_D$, $(ZDV)_D$, $(XY)_D$, $(DVZ)_D$, $(VDZ)_D$ are identical up to a prefactor. Therefore, only the D-component of the terms XY and YX will be considered for the kinetic term. The Lagrangian can then be written in a very compact form

$$\mathcal{L} = (XY)_D + (YX)_D + \frac{m}{2}(XX)_F + \frac{m}{2}(YY)_F + h.c.\,. \tag{3.74}$$

From the previous derivation of the chiral superfield X in equation (3.21) and the anti-chiral superfield Y in equation (3.23) it can be seen that the component fields N, M, S, R, λ, and κ are not independent. Therefore, it is convenient to introduce the new component fields

$$\tilde{S}_{\beta\alpha} = S_{\beta\alpha} + \frac{i}{2}\partial\!\!\!/_{\beta}{}^{\dot{\gamma}} N_{\dot{\gamma}\alpha}\,, \tag{3.75}$$

$$\tilde{R}_{\dot{\beta}\alpha} = R_{\dot{\beta}\alpha} - \frac{i}{2}\bar{\partial\!\!\!/}_{\dot{\beta}}{}^{\tau} M_{\tau\alpha}\,, \tag{3.76}$$

$$\tilde{\lambda}_{\alpha} = \lambda_{\alpha} - \frac{1}{4}\Box\kappa_{\alpha}\,. \tag{3.77}$$

Furthermore, it can be seen that the spinor vector field ω^μ_α is always contracted with a four derivative and therefore it is convenient to introduce

$$\tilde{\omega}_\alpha = \partial^\mu \omega_{\mu\alpha}\,, \tag{3.78}$$

to simplify the equations. The chiral and anti-chiral superfields can then be written as

$$X_\alpha = \chi_\alpha + \theta^\beta \tilde{S}_{\beta\alpha} + \theta^2 \left(\tilde{\lambda}_\alpha + \frac{i}{2}\tilde{\omega}_\alpha\right) - i\theta\partial\!\!\!/\bar{\theta}\chi_\alpha + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}}{}^\beta \tilde{S}_{\beta\alpha} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\chi_\alpha\,, \tag{3.79}$$

$$Y_\alpha = \psi_\alpha - \bar{\theta}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha} + \bar{\theta}^2 \left(\tilde{\lambda}_\alpha - \frac{i}{2}\tilde{\omega}_\alpha\right) + i\theta\partial\!\!\!/\bar{\theta}\psi_\alpha + \frac{i}{2}\theta^\gamma\bar{\theta}^2\partial\!\!\!/_\gamma{}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\psi_\alpha\,. \tag{3.80}$$

This can be used to calculate the four contributions to the Lagrangian outlined in equation (3.74)

$$(X^\alpha X_\alpha)_F = \chi\tilde{\lambda} + \frac{i}{2}\chi\tilde{\omega} - \frac{1}{2}\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) + \tilde{\lambda}\chi + \frac{i}{2}\tilde{\omega}\chi\,, \tag{3.81}$$

$$(Y^\alpha Y_\alpha)_F = \psi\tilde{\lambda} - \frac{i}{2}\psi\tilde{\omega} - \frac{1}{2}\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) + \tilde{\lambda}\psi - \frac{i}{2}\tilde{\omega}\psi\,, \tag{3.82}$$

$$(X^\alpha Y_\alpha)_D = \partial_\mu\chi\partial^\mu\psi + \tilde{\lambda}\tilde{\lambda} - \frac{i}{2}\tilde{\lambda}\tilde{\omega} + \frac{i}{2}\tilde{\omega}\tilde{\lambda} + \frac{1}{4}\tilde{\omega}\tilde{\omega} + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T\partial\!\!\!/\tilde{R}\right), \tag{3.83}$$

$$(Y^\alpha X_\alpha)_D = \partial_\mu\psi\partial^\mu\Box\chi + \tilde{\lambda}\tilde{\lambda} + \frac{i}{2}\tilde{\lambda}\tilde{\omega} - \frac{i}{2}\tilde{\omega}\tilde{\lambda} + \frac{1}{4}\tilde{\omega}\tilde{\omega} + \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T\bar{\partial\!\!\!/}\tilde{S}\right). \tag{3.84}$$

Therefore, the Lagrangian is given by

$$\begin{aligned}\mathcal{L} = \partial_\mu\chi\partial^\mu\psi + \partial_\mu\psi\partial^\mu\chi + 2\tilde{\lambda}\tilde{\lambda} + \frac{1}{2}\tilde{\omega}\tilde{\omega} + \frac{m}{2}\chi\tilde{\lambda} + \frac{im}{4}\chi\tilde{\omega} + \frac{m}{2}\tilde{\lambda}\chi + \frac{im}{4}\tilde{\omega}\chi + \frac{m}{2}\psi\tilde{\lambda} - \frac{im}{4}\psi\tilde{\omega} \\ + \frac{m}{2}\tilde{\lambda}\psi - \frac{im}{4}\tilde{\omega}\psi + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T\partial\!\!\!/\tilde{R}\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T\bar{\partial\!\!\!/}\tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right).\end{aligned} \tag{3.85}$$

It can be seen that this Lagrangian still contains the auxiliary fields $\tilde{\lambda}$ and $\tilde{\omega}$. They can be eliminated from the Lagrangian using their equations of motion

$$\tilde{\omega}_\tau = -\frac{im}{2}\left(\chi_\tau - \psi_\tau\right)\,, \tag{3.86}$$

$$\tilde{\lambda}_\tau = -\frac{m}{4}\left(\chi_\tau + \psi_\tau\right)\,. \tag{3.87}$$

This process is also referred to as going "on-shell". The resulting on-shell Lagrangian is then found to be

$$\mathcal{L} = \partial_\mu \chi \partial^\mu \psi + \partial_\mu \psi \partial^\mu \chi - \frac{m^2}{4}\psi\chi - \frac{m^2}{4}\chi\psi \\ + \frac{i}{2}\mathrm{Tr}\Big(\tilde{S}^T \partial\!\!\!/ \tilde{R}\Big) + \frac{i}{2}\mathrm{Tr}\Big(\tilde{R}^T \bar{\partial\!\!\!/} \tilde{S}\Big) - \frac{m}{4}\mathrm{Tr}\Big(\tilde{S}^T \tilde{S}\Big) - \frac{m}{4}\mathrm{Tr}\Big(\tilde{R}^T \tilde{R}\Big). \tag{3.88}$$

It is solely dependent on the on-shell component fields χ, ψ, $\tilde{S}$, and $\tilde{R}$. On the first glance it seems that there are twice as many bosonic degrees of freedom as fermionic ones, because each of the second rank spinor fields has in general 8 degrees of freedom, while each of the complex spinor fields only encompasses four degrees of freedom. However, on-shell, the bosonic second rank spinor fields satisfy a Weyl type equation which reduces the number of bosonic on-shell degrees of freedom by a factor of 2. This means that the Lagrangian indeed has 8 fermionic and 8 bosonic degrees of freedom.

Chapter 4

The Supercurrent

In classical field theory the Noether theorem describes the connection between symmetry transformations that leave the Lagrangian invariant and the corresponding conserved quantities. It states that every symmetry results in a conserved current which can alternatively be expressed as a conserved charge. Even though supersymmetry is not a symmetry in the classical sense – as it represents a symmetry between fermionic and bosonic fields – the Lagrangian is invariant under the variation of the component fields as defined in equations (3.69) to (3.72). Therefore, according to Noether's theorem, a conserved supercurrent exists.

The general equation for the supercurrent is given by

$$J_{\mu\kappa} = \frac{\partial}{\partial \zeta^\kappa} \left(\sum_\phi \delta\phi \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} - \mathcal{K}_\mu \right) , \tag{4.1}$$

where the summation runs over all component fields of the Lagrangian. It has to be emphasised that this compact general equation for the full general supercurrent suppresses any indices of the component fields and also includes all Hermitian conjugate component fields as well. Furthermore, it is necessary to adjust the signs in agreement with the definition of the derivative with respect to the Grassmann variables ζ and $\bar{\zeta}$ as well as the insertion rules for the variation of the component fields.

The term $\mathcal{K}_\mu$ is related to the variation of the Lagrangian by

$$\partial^\mu \mathcal{K}_\mu = \delta\mathcal{L} \, , \tag{4.2}$$

which indicates that the variation of the Lagrangian is a four-divergence. This plays an important role in the explicit calculation of the variation as for most purposes it is assumed that the boundary terms vanish. For the variation of the Lagrangian and thus the calculation of $\mathcal{K}_\mu$, however, these are exactly the terms that are of interest.

As the full supercurrent J_μ is Hermitian there are two possible ways to derive it. First, it is possible to derive the full supercurrent using the complete on-shell Lagrangian including the Hermitian conjugate part. Alternatively it is possible to restrict the discussion to the on-shell Lagrangian without the Hermitian conjugate part and to calculate both $J_\mu^{1/2}$ as well as $\bar{J}_\mu^{1/2}$. Both ways are equivalent as the result of the first approach calculated from the complete Lagrangian can be constructed from the two contributions $J_\mu^{1/2}$ and $\bar{J}_\mu^{1/2}$ derived from the Lagrangian without Hermitian conjugate part alone. For simplicity the derivation using only the Lagrangian without Hermitian conjugate part will be presented first. This calculation was also performed using the other approach to demonstrate that the two methods are actually equivalent.

If the discussion is restricted to the Lagrangian without Hermitian conjugate part the general equation for $J_\mu^{1/2}$ can be written as

$$J_{\mu\kappa}^{1/2} = \frac{\partial}{\partial\zeta^\kappa}\left(\delta\chi^\tau \frac{\partial\mathcal{L}}{\partial\partial^\mu\chi^\tau} + \delta\psi^\tau \frac{\partial\mathcal{L}}{\partial\partial^\mu\psi^\tau} + \delta\tilde{S}^{\tau\omega}\frac{\partial\mathcal{L}}{\partial\partial^\mu\tilde{S}^{\tau\omega}} + \delta\tilde{R}^{\dot{\tau}\omega}\frac{\partial\mathcal{L}}{\partial\partial^\mu\tilde{R}^{\dot{\tau}\omega}} - \mathcal{K}_\mu\right). \tag{4.3}$$

Inserting the on-shell Lagrangian from equation (3.88) into the equation for the supercurrent yields

$$J_{\mu\kappa} = -3\tilde{S}_\kappa{}^\alpha\partial_\mu\psi_\alpha - \frac{im}{2}\psi^\alpha\left(\sigma_\mu\right)_\kappa{}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha} - i\partial_\nu\psi^\alpha\left(\sigma^\nu{}_\mu\right)_\kappa{}^\beta\tilde{S}_{\beta\alpha} - \frac{\partial}{\partial\zeta^\kappa}\mathcal{K}_\mu\,. \tag{4.4}$$

The remaining task is now to derive the explicit form of $\mathcal{K}_\mu$ by calculating the variation of the Lagrangian without Hermitian conjugate part

$$\begin{aligned}\delta\mathcal{L} = {} & \partial_\mu\delta\chi\partial^\mu\psi + \partial_\mu\chi\partial^\mu\delta\psi + \partial_\mu\delta\psi\partial^\mu\chi + \partial_\mu\psi\partial^\mu\delta\chi \\ & - \frac{m^2}{4}\delta\psi\chi - \frac{m^2}{4}\psi\delta\chi - \frac{m^2}{4}\delta\chi\psi - \frac{m^2}{4}\chi\psi\delta \\ & + \frac{i}{2}\mathrm{Tr}\left(\delta\tilde{S}^T\partial\!\!\!/\tilde{R}\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T\partial\!\!\!/\delta\tilde{R}\right) + \frac{i}{2}\mathrm{Tr}\left(\delta\tilde{R}^T\bar{\partial\!\!\!/}\tilde{S}\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T\bar{\partial\!\!\!/}\delta\tilde{S}\right) \\ & - \frac{m}{4}\mathrm{Tr}\left(\delta\tilde{S}^T\tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{S}^T\delta\tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\delta\tilde{R}^T\tilde{R}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{R}^T\delta\tilde{R}\right).\end{aligned} \tag{4.5}$$

It can be shown that the variation of the Lagrangian is a four divergence as expected which implies that

$$\mathcal{K}_\mu = \zeta^\beta \tilde{S}_{\beta\alpha}\partial_\mu\psi^\alpha - \bar{\zeta}^{\dot\beta}\tilde{R}_{\dot\beta\alpha}\partial_\mu\chi^\alpha + \frac{im}{2}\left(\bar{\sigma}_\mu\right)_{\dot\beta}{}^{\gamma}\bar{\zeta}^{\dot\beta}\chi^\alpha\tilde{S}_{\gamma\alpha} + \frac{im}{2}\left(\sigma_\mu\right)_{\beta}{}^{\dot\gamma}\zeta^\beta\psi^\alpha\tilde{R}_{\dot\gamma\alpha}$$
$$+ i\bar{\zeta}^{\dot\delta}\left(\bar{\sigma}_\mu{}^\nu\right)_{\dot\delta}{}^{\dot\gamma}\chi^\alpha\partial_\nu\tilde{R}_{\dot\gamma\alpha} + i\zeta^\delta\left(\sigma_\mu{}^\nu\right)_\delta{}^\gamma\psi^\alpha\partial_\nu\tilde{S}_{\gamma\alpha}\,. \tag{4.6}$$

This result can then be inserted into the equation for the supercurrent. After differentiating with respect to the transformation parameter ζ the supercurrent is found to be

$$J^{1/2}_{\mu\kappa} = -im\left(\sigma_\mu\right)_\kappa{}^{\dot\beta}\tilde{R}_{\dot\beta\alpha}\psi^\alpha + 2\left(\sigma_\mu\right)^{\beta\dot\gamma}\bar{\partial\!\!\!/}_{\dot\gamma\kappa}\psi^\alpha\tilde{S}_{\beta\alpha}\,. \tag{4.7}$$

The contribution to the full supercurrent $\bar{J}^{1/2}_\mu$ is defined in perfect analogy to $J^{1/2}_\mu$ by replacing the derivative with respect to the Grassmann variable ζ with a derivative with respect to $\bar{\zeta}$. As mentioned earlier the behaviour of the Grassmann derivative is rather subtle and depends on the conventions chosen. Therefore it is not immediately clear whether the transition between unbarred and barred Grassmann derivatives also implies a transition from right to left derivatives. In the present scenario where by convention all derivatives are written as right derivatives the change from left to right derivative introduces an additional overall minus sign

$$\bar{J}^{1/2}_{\mu\dot\kappa} = -\frac{\partial}{\partial\bar{\zeta}^{\dot\kappa}}\left(\delta\chi^\tau\frac{\partial\mathcal{L}}{\partial\partial^\mu\chi^\tau} + \delta\psi^\tau\frac{\partial\mathcal{L}}{\partial\partial^\mu\psi^\tau} + \delta\tilde{S}^{\tau\omega}\frac{\partial\mathcal{L}}{\partial\partial^\mu\tilde{S}^{\tau\omega}} + \delta\tilde{R}^{\dot\tau\omega}\frac{\partial\mathcal{L}}{\partial\partial^\mu\tilde{R}^{\dot\tau\omega}} - \mathcal{K}_\mu\right)\,. \tag{4.8}$$

Later on in the calculation it will become clear that this overall sign is justified as this choice for $\bar{J}^{1/2}_\mu$ in connection with the previous result for $J^{1/2}_\mu$ produces a consistent second quantization of the component fields. No second quantization exists if $\bar{J}^{1/2}_\mu$ is defined with a different prefactor. The supercurrent $\bar{J}^{1/2}_\mu$ for the Lagrangian without the complex conjugate part is then given by

$$\bar{J}^{1/2}_{\mu\dot\kappa} = 3\tilde{R}_{\dot\kappa\alpha}\partial_\mu\chi^\alpha + i\left(\sigma^\nu{}_\mu\right)_{\dot\kappa}{}^{\dot\beta}\partial_\nu\chi^\alpha\tilde{R}_{\dot\beta\alpha} + \frac{im}{2}\left(\bar{\sigma}_\mu\right)_{\dot\kappa}{}^{\beta}\tilde{S}_{\beta\alpha}\chi^\alpha + \frac{\partial}{\partial\bar{\zeta}^{\dot\kappa}}\mathcal{K}_\mu\,, \tag{4.9}$$

where the term $\mathcal{K}_\mu$ is already known from the discussion of $J^{1/2}_\mu$. After differentiation with respect to the Grassmann variable the final result for the contribution to the Hermitian conjugate supercurrent is

$$\bar{J}^{1/2}_{\mu\dot{\kappa}} = im\,(\bar{\sigma}_\mu)_{\dot{\kappa}}{}^{\beta}\,\tilde{S}_{\beta\alpha}\chi^\alpha + 2\,(\bar{\sigma}_\mu)^{\dot{\beta}\gamma}\,\partial\!\!\!/_{\gamma\dot{\kappa}}\chi^\alpha\tilde{R}_{\dot{\beta}\alpha}\,. \tag{4.10}$$

Together with the previous result for $J^{1/2}_\mu$ from equation (4.7) the construction of the full supercurrent is straightforward

$$\begin{aligned} J_{\mu\kappa} = &-im\,(\sigma_\mu)_\kappa{}^{\dot{\beta}}\,\tilde{R}_{\dot{\beta}\alpha}\psi^\alpha + 2\,(\sigma_\mu)^{\beta\dot{\gamma}}\,\bar{\partial\!\!\!/}_{\dot{\gamma}\kappa}\psi^\alpha\tilde{S}_{\beta\alpha} \\ &- im\,(\sigma_\mu)_\kappa{}^{\dot{\beta}}\,\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} + 2\,(\sigma_\mu)^{\beta\dot{\gamma}}\,\bar{\partial\!\!\!/}_{\dot{\gamma}\kappa}\bar{\chi}^{\dot{\alpha}}\bar{\tilde{R}}_{\beta\dot{\alpha}}\,. \end{aligned} \tag{4.11}$$

The supercurrent $\bar{J}_\mu$ is then simply found by Hermitian conjugation

$$\begin{aligned} \bar{J}_{\mu\kappa} = &\; im\,(\bar{\sigma}_\mu)_{\dot{\kappa}}{}^{\beta}\,\tilde{S}_{\beta\alpha}\chi^\alpha + 2\,(\bar{\sigma}_\mu)^{\dot{\beta}\gamma}\,\partial\!\!\!/_{\gamma\dot{\kappa}}\chi^\alpha\tilde{R}_{\dot{\beta}\alpha} \\ &+ im\,(\bar{\sigma}_\mu)_{\dot{\kappa}}{}^{\beta}\,\bar{\tilde{R}}_{\beta\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + 2\,(\bar{\sigma}_\mu)^{\dot{\beta}\gamma}\,\partial\!\!\!/_{\gamma\dot{\kappa}}\bar{\psi}^{\dot{\alpha}}\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}}\,. \end{aligned} \tag{4.12}$$

4.1 The Contribution of $\mathcal{L}_{h.c.}$ to the On-shell Supercurrent

Alternatively it is possible to derive the supercurrent using the the full Lagrangian. As the contribution from the unconjugated part of the Lagrangian was previously calculated the discussion can be restricted to the Hermitian conjugate part of the Lagrangian. Starting from the Lagrangian in equation (3.88) the Hermitian conjugate part is found to be

$$\begin{aligned} \mathcal{L}_{h.c.} = &\;\partial_\mu\bar{\chi}\partial^\mu\bar{\psi} + \partial_\mu\bar{\psi}\partial^\mu\bar{\chi} - \frac{m^2}{4}\bar{\psi}\bar{\chi} - \frac{m^2}{4}\bar{\chi}\bar{\psi} \\ &+ \frac{i}{2}\mathrm{Tr}\Big(\bar{\tilde{S}}^T\bar{\partial\!\!\!/}\bar{\tilde{R}}\Big) + \frac{i}{2}\mathrm{Tr}\Big(\bar{\tilde{R}}^T\partial\!\!\!/\bar{\tilde{S}}\Big) - \frac{m}{4}\mathrm{Tr}\Big(\bar{\tilde{S}}^T\bar{\tilde{S}}\Big) - \frac{m}{4}\mathrm{Tr}\Big(\bar{\tilde{R}}^T\bar{\tilde{R}}\Big)\,. \end{aligned} \tag{4.13}$$

Furthermore, the variations for the Hermitian conjugate on-shell component fields have to be determined. They can be found by Hermitian conjugation from the variation of the on-shell component fields from equations (3.69) to (3.72)

$$\delta\bar{\psi}_{\dot{\alpha}} = \zeta^\beta\bar{\tilde{R}}_{\beta\dot{\alpha}}\,, \tag{4.14}$$

$$\delta\bar{\chi}_{\dot{\alpha}} = -\bar{\zeta}^{\dot{\beta}}\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}}\,, \tag{4.15}$$

$$\delta\bar{\tilde{R}}_{\beta\dot{\alpha}} = -m\zeta_\beta\bar{\chi}_{\dot{\alpha}} - 2i\bar{\zeta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}\beta}\bar{\psi}_{\dot{\alpha}}\,, \tag{4.16}$$

$$\delta\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}} = -m\bar{\zeta}_{\dot{\beta}}\bar{\psi}_{\dot{\alpha}} + 2i\zeta^\gamma\partial\!\!\!/_{\gamma\dot{\beta}}\bar{\chi}_{\dot{\alpha}}\,. \tag{4.17}$$

The general equation for the Hermitian conjugate component of the supercurrent is given by

$$\begin{aligned}\left(J^{1/2}_{h.c.}\right)_{\mu\kappa} &= -\frac{\partial}{\partial\zeta^\kappa}\left(-\delta\bar{\chi}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\chi}} - \delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\psi}} + \delta\bar{\tilde{S}}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\tilde{S}}} + \delta\bar{\tilde{R}}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\tilde{R}}} - \bar{\mathcal{K}}_\mu\right)\\ &= -\frac{\partial}{\partial\zeta^\kappa}\left(\delta\bar{\chi}^{\dot{\tau}}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\chi}^{\dot{\tau}}} + \delta\bar{\psi}^{\dot{\tau}}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\psi}^{\dot{\tau}}} + \delta\bar{\tilde{S}}^{\dot{\tau}\dot{\omega}}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\tilde{S}}^{\dot{\tau}\dot{\omega}}} + \delta\bar{\tilde{R}}^{\tau\dot{\omega}}\frac{\partial\mathcal{L}}{\partial\partial^\mu\bar{\tilde{R}}^{\tau\dot{\omega}}} - \bar{\mathcal{K}}_\mu\right)\end{aligned} \tag{4.18}$$

where

$$\partial^\mu\bar{\mathcal{K}}_\mu = \delta\mathcal{L}_{h.c.}\,. \tag{4.19}$$

On the first glance the signs in the equation for $J^{1/2}_\mu$ seem inappropriate. This is caused by interplay between the definition of the derivatives with respect to Grassmann variables ζ and $\bar{\zeta}$ and the convention for the insertion of the variation of the component fields that was mentioned before. A detailed analysis of the five terms reveals that this choice for the prefactors is the only one that leads to a consistent second quantisation of the component fields. It also produces a structure in agreement with the result from $J^{1/2}_\mu$ containing two terms which is not necessarily the case for a different choice of signs. The Hermitian conjugate component of the supercurrent as outlined in equation (4.18) can then be calculated to

$$\left(J^{1/2}_{h.c.}\right)_{\mu\kappa} = -2\bar{\tilde{R}}_{\kappa\dot{\alpha}}\partial_\mu\bar{\chi}^{\dot{\alpha}} - \left(\sigma^\nu\bar{\sigma}_\mu\right)_\kappa{}^\beta\partial_\nu\bar{\chi}^{\dot{\alpha}}\bar{\tilde{R}}_{\beta\dot{\alpha}} + \frac{im}{2}\bar{\chi}^{\dot{\alpha}}\left(\sigma_\mu\right)_\kappa{}^{\dot{\beta}}\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}} + \frac{\partial}{\partial\zeta^\kappa}\bar{\mathcal{K}}_\mu\,. \tag{4.20}$$

Now we only have to determine $\bar{\mathcal{K}}_\mu$ from the variation of the Hermitian conjugate part of the Lagrangian $\delta\mathcal{L}_{h.c.}$ which is defined by

$$\begin{aligned}\delta\mathcal{L} = {}&\partial_\mu\delta\bar{\chi}\partial^\mu\bar{\psi} + \partial_\mu\bar{\chi}\partial^\mu\delta\bar{\psi} + \partial_\mu\delta\bar{\psi}\partial^\mu\bar{\chi} + \partial_\mu\bar{\psi}\partial^\mu\delta\bar{\chi}\\ &-\frac{m^2}{4}\delta\bar{\psi}\bar{\chi} - \frac{m^2}{4}\bar{\psi}\delta\bar{\chi} - \frac{m^2}{4}\delta\bar{\chi}\bar{\psi} - \frac{m^2}{4}\bar{\chi}\delta\bar{\psi}\\ &+\frac{i}{2}\mathrm{Tr}\left(\delta\bar{\tilde{S}}^T\bar{\partial\!\!\!/}\bar{\tilde{R}}\right) + \frac{i}{2}\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\partial\!\!\!/}\delta\bar{\tilde{R}}\right) + \frac{i}{2}\mathrm{Tr}\left(\delta\bar{\tilde{R}}^T\bar{\partial\!\!\!/}\bar{\tilde{S}}\right) + \frac{i}{2}\mathrm{Tr}\left(\bar{\tilde{R}}^T\bar{\partial\!\!\!/}\delta\bar{\tilde{S}}\right)\\ &-\frac{m}{4}\mathrm{Tr}\left(\delta\bar{\tilde{S}}^T\bar{\tilde{S}}\right) - \frac{m}{4}\mathrm{Tr}\left(\bar{\tilde{S}}^T\delta\bar{\tilde{S}}\right) - \frac{m}{4}\mathrm{Tr}\left(\delta\bar{\tilde{R}}^T\bar{\tilde{R}}\right) - \frac{m}{4}\mathrm{Tr}\left(\bar{\tilde{R}}^T\delta\bar{\tilde{R}}\right).\end{aligned} \tag{4.21}$$

The resulting variation of the Hermitian conjugate part of the Lagrangian is found to be a four-divergence as well

$$\bar{\mathcal{K}}_\mu = -\bar{\zeta}^{\dot{\beta}}\tilde{\bar{S}}_{\dot{\beta}\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}} + \zeta^\beta\tilde{\bar{R}}_{\beta\dot{\alpha}}\partial_\mu\bar{\chi}^{\dot{\alpha}} + \frac{im}{2}\bar{\zeta}^{\dot{\beta}}\left(\bar{\sigma}_\mu\right)_{\dot{\beta}}{}^{\gamma}\bar{\psi}^{\dot{\alpha}}\tilde{\bar{R}}_{\gamma\dot{\alpha}} + \frac{im}{2}\zeta^\beta\left(\sigma_\mu\right)_\beta{}^{\dot{\gamma}}\bar{\chi}^{\dot{\alpha}}\tilde{\bar{S}}_{\dot{\gamma}\dot{\alpha}}$$
$$+ i\bar{\zeta}^{\dot{\delta}}\left(\bar{\sigma}^\nu{}_\mu\right)_{\dot{\delta}}{}^{\dot{\gamma}}\partial_\nu\bar{\psi}^{\dot{\alpha}}\tilde{\bar{S}}_{\dot{\gamma}\dot{\alpha}} + i\zeta^\delta\left(\sigma^\nu{}_\mu\right)_\delta{}^{\gamma}\partial_\nu\bar{\chi}^{\dot{\alpha}}\tilde{\bar{R}}_{\gamma\dot{\alpha}}\,. \tag{4.22}$$

At this point it is possible to perform a simple consistency check involving the variaton of the Lagrangian. Based on the previous result for $\bar{\mathcal{K}}_\mu$ it can be shown that its Hermitian conjugate is

$$\left(\bar{\mathcal{K}}_\mu\right)^\dagger = \zeta^\beta\tilde{S}_{\beta\alpha}\partial_\mu\psi^\alpha - \bar{\zeta}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha}\partial_\mu\chi^\alpha + \frac{im}{2}\zeta^\beta\left(\sigma_\mu\right)_\beta{}^{\dot{\gamma}}\psi^\alpha\tilde{R}_{\dot{\gamma}\alpha} + \frac{im}{2}\bar{\zeta}^{\dot{\beta}}\left(\bar{\sigma}_\mu\right)_{\dot{\beta}}{}^{\gamma}\chi^\alpha\tilde{\bar{S}}_{\dot{\gamma}\dot{\alpha}}$$
$$+ i\zeta^\delta\left(\sigma^\nu{}_\mu\right)_\delta{}^{\gamma}\partial_\nu\psi^\alpha\tilde{S}_{\gamma\alpha} + i\bar{\zeta}^{\dot{\delta}}\left(\bar{\sigma}^\nu{}_\mu\right)_{\dot{\delta}}{}^{\dot{\gamma}}\partial_\nu\chi^\alpha\tilde{R}_{\dot{\gamma}\alpha}\,. \tag{4.23}$$

Comparing this result to the previously derived variation of the non-conjugate part of the Lagrangian

$$\mathcal{K}_\mu = \zeta^\beta\tilde{S}_{\beta\alpha}\partial_\mu\psi^\alpha - \bar{\zeta}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha}\partial_\mu\chi^\alpha + \frac{im}{2}\left(\bar{\sigma}_\mu\right)_{\dot{\beta}}{}^{\gamma}\bar{\zeta}^{\dot{\beta}}\chi^\alpha\tilde{S}_{\gamma\alpha} + \frac{im}{2}\left(\sigma_\mu\right)_\beta{}^{\dot{\gamma}}\zeta^\beta\psi^\alpha\tilde{R}_{\dot{\gamma}\alpha}$$
$$+ i\bar{\zeta}^{\dot{\delta}}\left(\bar{\sigma}_\mu{}^\nu\right)_{\dot{\delta}}{}^{\dot{\gamma}}\chi^\alpha\partial_\nu\tilde{R}_{\dot{\gamma}\alpha} + i\zeta^\delta\left(\sigma_\mu{}^\nu\right)_\delta{}^{\gamma}\psi^\alpha\partial_\nu\tilde{S}_{\gamma\alpha} \tag{4.24}$$

reveals immediately that $\bar{\mathcal{K}}_\mu$ is the Hermitian conjugate of $\mathcal{K}_\mu$. This is not surprising, as the Lagrangian is by construction symmetric between barred and unbarred component fields. The remaining calculation solely requires inserting the result for $\bar{\mathcal{K}}_\mu$ from equation (4.22) into the intermediate result for $\bar{J}^{1/2}_\mu$ from equation (4.20). The supercurrent is then given by

$$J^{1/2}_{\mu\kappa h.c.} = -im\left(\sigma_\mu\right)_\kappa{}^{\dot{\beta}}\tilde{\bar{S}}_{\dot{\beta}\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} + 2\left(\sigma^\nu\bar{\sigma}_\mu\right)_\kappa{}^{\beta}\tilde{\bar{R}}_{\beta\dot{\alpha}}\partial_\nu\bar{\chi}^{\dot{\alpha}}\,. \tag{4.25}$$

To verify that this result is in agreement with the one derived in the previous section it is important to recall that the full supercurrent is Hermitian

$$\left(J_\mu\right)^\dagger = \bar{J}_\mu\,, \tag{4.26}$$

where J_μ and $\bar{J}_\mu$ can be written in terms of their contributions from the barred and unbarred componentn fields

$$J_\mu = J_\mu^{1/2} + \left(J_{h.c.}^{1/2}\right)_\mu \,, \tag{4.27}$$

$$\bar{J}_\mu = \bar{J}_\mu^{1/2} + \left(\bar{J}_{h.c.}^{1/2}\right)_\mu \,. \tag{4.28}$$

This implies that the contributions to the supercurrent satisfy the following relations

$$J_\mu^{1/2} = \left(\bar{J}_{h.c.}^{1/2}\right)_\mu \,, \tag{4.29}$$

$$\bar{J}_\mu^{1/2} = \left(J_{h.c.}^{1/2}\right)_\mu \,. \tag{4.30}$$

To verify that these relations are satisfied it is necessary to calculate

$$\left(J_{\mu\kappa h.c.}^{1/2}\right)^\dagger = im\left(\bar{\sigma}_\mu\right)_{\dot{\kappa}}{}^{\beta}\tilde{S}_{\beta\alpha}\chi^\alpha - 2\left(\bar{\sigma}^\nu\sigma_\mu\right)_{\dot{\kappa}}{}^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha}\partial_\nu\chi^\alpha \,. \tag{4.31}$$

It is found that $\bar{J}_\mu^{1/2}$ in equation (4.10) as derived from the unbarred part of the Lagrangian is exactly the Hermitian conjugate of $\left(J_{h.c.}^{1/2}\right)_\mu$ that was calculated form the Hermitian conjugate part of the Lagrangian. Thus it can be concluded that either method can be used to derive the on-shell supercurrent. For simplicity the approach involving the Lagrangian without Hermitian conjugate part is preferred as it neither requires the calculation of $\bar{\mathcal{K}}_\mu$ nor the derivation of the variation of the barred component fields.

Chapter 5

The Hamiltonian in Position Space

The Hamiltonian in position space is usually derived from the Lagrangian by Legendre transformation. However, it is not immediately clear whether this approach is still valid for the present scenario that is based on a general superfield with one spinor index instead of a scalar superfield. Due to this uncertainty a more conservative approach based on the supersymmetry algebra was chosen.

This approach utilises the anticommutation relation between the barred and unbarred supersymmetry generator of the $N = 1$ supersymmetry algebra

$$2\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}} P_{\mu} = \left\{ Q_{\alpha}, \bar{Q}_{\dot{\beta}} \right\}, \tag{5.1}$$

which is proportional to the momentum operator. The 0-th component of the momentum operator then encompasses the Hamiltonian while the momentum of the system makes up the remaining three components.

At this point it can already be seen that a successful derivation of the Hamiltonian from the supersymmetry algebra requires the knowledge of the commutation and anticommutation relations of the component fields in position space. Therefore, the second quantisation of the component fields in position space will be discussed in Section 5.1. Afterwards in Section 5.2, these results will be used to derive an expression for the Hamiltonian in position space which is founded in the supersymmetry algebra and thus should be positive definite by construction. Finally, in Section 5.3 it will be shown that Legendre transformation of the Lagrangian yields the same Hamiltonian in position space as the approach based on the supersymmetry algebra. This establishes the equivalence between the two approaches.

5.1 Second Quantisation in Position Space

A viable supersymmetric theory of fermionic fields with mass dimension one requires a second quantisation that is in agreement with the superfield transformations of the component fields as derived in Section 3.2.2. This can be achieved by calculating the commutator between the component fields and the generators of the superspace translations

$$\delta\phi = -i\left[\phi, \zeta^{\alpha} Q_{\alpha} + \zeta_{\dot{\alpha}} Q^{\dot{\alpha}}\right] . \tag{5.2}$$

To generalise the notation the spinor indices of the field ϕ are suppressed and it can represent a scalar field as well as first, second, or higher rank spinor fields. Subsequently, the commutation and anticommutation relations of the barred component fields are derived from the results for the unbarred component fields by Hermitian conjugation.

The supersymmetry generators that appear in this equation are proportional to the supercurrent

$$Q_{\alpha} = \int \mathrm{d}\mathbf{x} J_{0\alpha} \, , \tag{5.3}$$

$$\bar{Q}_{\dot{\alpha}} = \int \mathrm{d}\mathbf{x} \bar{J}_{0\dot{\alpha}} \, . \tag{5.4}$$

In general the supersymmetry generators must contain the full supercurrent. However, the previous results for the superfield translations, equations (3.69) to (3.72), imply that no mixing between barred and unbarred component fields occurs. Therefore, it is sufficient to restrict the discussion in this section to the supercurrent arising from the Lagrangian without Hermitian conjugate contribution, as any cross terms vanish identically and define the constrained generators

$$Q^{1/2}_{\alpha} = \int \mathrm{d}\mathbf{x} J^{1/2}_{0\alpha} \, , \tag{5.5}$$

$$\bar{Q}^{1/2}_{\dot{\alpha}} = \int \mathrm{d}\mathbf{x} \bar{J}^{1/2}_{0\dot{\alpha}} \, . \tag{5.6}$$

To distinguish the constrained generators from the full generators as outlined in equations (5.3) and (5.4) an additional superscript 1/2 was incorporated into the notation in analogy to the notation for the supercurrent in Chapter 4. Inserting the results for the supercurrent from equations (4.7) and (4.10) then yields the following expression for the constrained supersymmetry generators

$$Q_{\alpha}^{1/2} = \int \mathrm{d}\mathbf{x} \left(-im \left(\sigma_{\mu}\right)_{\alpha}{}^{\dot{\gamma}} \tilde{R}_{\dot{\gamma}\beta}(x)\psi^{\beta}(x) + 2 \left(\sigma_{\mu}\right)^{\gamma\dot{\delta}} \tilde{S}_{\gamma\beta}(x)\bar{\partial\!\!\!/}_{\dot{\delta}\alpha}\psi^{\beta}(x) \right) , \tag{5.7}$$

$$\bar{Q}_{\dot{\alpha}}^{1/2} = \int \mathrm{d}\mathbf{x} \left(im \left(\bar{\sigma}_{\mu}\right)_{\dot{\alpha}}{}^{\gamma} \tilde{S}_{\gamma\beta}(x)\chi^{\beta}(x) + 2 \left(\bar{\sigma}_{\mu}\right)^{\dot{\gamma}\delta} \tilde{R}_{\dot{\gamma}\beta}(x)\partial\!\!\!/_{\delta\dot{\alpha}}\chi^{\beta}(x) \right) . \tag{5.8}$$

5.1.1 Superfield Transformation of χ

Inserting the constrained supersymmetry generators as defined in equations (5.7) and (5.8) into equation (5.2) for the commutator between component field χ and the generators of superspace translations yields a variation of χ of

$$\begin{aligned} \delta\chi_{\alpha}(x) = \int \mathrm{d}\mathbf{x}' \Big(& m\zeta^{\beta} \left(\sigma_0\right)_{\beta}{}^{\dot{\gamma}} \left\{ \chi_{\alpha}(x), \tilde{R}_{\dot{\gamma}\delta}(x')\psi^{\delta}(x') \right\} \\ & + 2i\zeta^{\beta} \left(\sigma_0\right)^{\gamma\dot{\delta}} \left\{ \chi_{\alpha}(x), \tilde{S}_{\gamma\epsilon}(x')\bar{\partial\!\!\!/}'_{\dot{\delta}\beta}\psi^{\epsilon}(x') \right\} \\ & - m\bar{\zeta}_{\dot{\beta}} \left(\bar{\sigma}_0\right)^{\dot{\beta}\gamma} \left\{ \chi_{\alpha}(x), \tilde{S}_{\gamma\delta}(x')\chi^{\delta}(x') \right\} \\ & + 2i\bar{\zeta}_{\dot{\beta}} \left(\bar{\sigma}_0\right)^{\dot{\gamma}\delta} \left\{ \chi_{\alpha}(x), \tilde{R}_{\dot{\gamma}\epsilon}(x')\partial\!\!\!/'_{\delta}{}^{\dot{\beta}}\chi^{\epsilon}(x') \right\} \Big) . \end{aligned} \tag{5.9}$$

Each of the contributions to the variation of the component field χ contains an anticommutator involving two fermionic fields and one bosonic field. They can be rewritten using the anticommutator relation

$$\{F_1, B_2F_2\} = B_2 \{F_1, F_2\} , \tag{5.10}$$

which results in

$$\begin{aligned} \delta\chi_{\alpha}(x) = \int \mathrm{d}\mathbf{x}' \Big(& m\zeta^{\beta} \left(\sigma_0\right)_{\beta}{}^{\dot{\gamma}} \tilde{R}_{\dot{\gamma}\delta}(x') \left\{ \chi_{\alpha}(x), \psi^{\delta}(x') \right\} \\ & + 2i\zeta^{\beta} \left(\sigma_0\right)^{\gamma\dot{\delta}} \tilde{S}_{\gamma\epsilon}(x') \left\{ \chi_{\alpha}(x), \bar{\partial\!\!\!/}'_{\dot{\delta}\beta}\psi^{\epsilon}(x') \right\} \\ & - m\bar{\zeta}_{\dot{\beta}} \left(\bar{\sigma}_0\right)^{\dot{\beta}\gamma} \tilde{S}_{\gamma\delta}(x') \left\{ \chi_{\alpha}(x), \chi^{\delta}(x') \right\} \\ & + 2i\bar{\zeta}_{\dot{\beta}} \left(\bar{\sigma}_0\right)^{\dot{\gamma}\delta} \tilde{R}_{\dot{\gamma}\epsilon}(x') \left\{ \chi_{\alpha}(x), \partial\!\!\!/'_{\delta}{}^{\dot{\beta}}\chi^{\epsilon}(x') \right\} \Big) . \end{aligned} \tag{5.11}$$

It can be seen that the second and fourth terms contain a four derivative $\partial\!\!\!/$ acting on one of the component fields in the anticommutator. These terms can be rewritten by splitting the four derivative into its time and spatial components and partially integrating over the spatial components. They are then each replaced by two terms – one containing a time derivative acting on one of the fields in the commutator and one simply containing the commutator of component fields. Furthermore, the boundary terms from the partial integration which are 3-divergences vanish identically and were ignored. This results in

$$\begin{aligned}\delta\chi_\alpha(x) = \int \mathrm{d}\mathbf{x}' \Big(& m\zeta^\beta \left(\sigma_0\right)_\beta{}^{\dot\gamma} \tilde{R}_{\dot\gamma\delta}(x') \left\{\chi_\alpha(x), \psi^\delta(x')\right\} + 2i\zeta^\beta \tilde{S}_{\beta\epsilon}(x') \left\{\chi_\alpha(x), \dot\psi^\epsilon(x')\right\} \\ & + 2i\zeta^\beta \left(\sigma_0\right)^{\gamma\dot\delta} \boldsymbol{\sigma}_{\dot\delta\beta} \cdot \boldsymbol{\nabla}' \tilde{S}_{\gamma\epsilon}(x') \left\{\chi_\alpha(x), \psi^\epsilon(x')\right\} \\ & - m\bar\zeta_{\dot\beta} \left(\bar\sigma_0\right)^{\dot\beta\gamma} \tilde{S}_{\gamma\delta}(x') \left\{\chi_\alpha(x), \chi^\delta(x')\right\} - 2i\bar\zeta^{\dot\beta} \tilde{R}_{\dot\beta\epsilon}(x') \left\{\chi_\alpha(x), \dot\chi^\epsilon(x')\right\} \\ & + 2i\bar\zeta_{\dot\beta} \left(\bar\sigma_0\right)^{\dot\gamma\delta} \boldsymbol{\sigma}_\delta{}^{\dot\beta} \cdot \boldsymbol{\nabla}' \tilde{R}_{\dot\gamma\epsilon}(x') \left\{\chi_\alpha(x), \chi^\epsilon(x')\right\}\Big) \,. \end{aligned} \tag{5.12}$$

By comparison with the previously derived superspace translation of χ in equation (3.70) it can be seen that the only nonvanishing contribution comes from the term proportional to $\zeta\tilde{S}$ while all other contributions have to vanish identically. This implies that three of the anticommutators vanish identically

$$\left\{\chi_\alpha(x), \psi_\beta(x')\right\} = 0\,, \tag{5.13}$$

$$\left\{\chi_\alpha(x), \dot\chi_\beta(x')\right\} = 0\,, \tag{5.14}$$

$$\left\{\chi_\alpha(x), \chi_\beta(x')\right\} = 0\,. \tag{5.15}$$

Furthermore, the only nonvanishing anticommutator satisfies

$$-\zeta^\beta \tilde{S}_{\beta\alpha}(x) = -2i\zeta^\beta \int \mathrm{d}\mathbf{x}' \tilde{S}_\beta{}^\gamma(x') \left\{\chi_\alpha(x), \dot\psi_\gamma(x')\right\}\,, \tag{5.16}$$

which has a solution of the form

$$\left\{\chi_\alpha(x), \dot\psi_\gamma(x')\right\} = a\epsilon_{\alpha\gamma}\delta(x-x')\,. \tag{5.17}$$

The prefactor a can then be determined by inserting this ansatz into equation (5.16)

$$\begin{aligned}\zeta^\beta \tilde{S}_{\beta\alpha}(x) &= 2i\zeta^\beta \int \mathrm{d}\mathbf{x}' \tilde{S}_\beta{}^\gamma(x') a\delta(x-x')\epsilon_{\alpha\gamma} \\ &= -2ia\zeta^\beta \tilde{S}_{\beta\alpha}(x)\,, \end{aligned} \tag{5.18}$$

which uniquely specifies the prefactor to $a = i/2$. The nonvanishing anticommutation relation involving χ is therefore given by

$$\left\{\chi_\alpha(x), \dot{\psi}_\gamma(x')\right\} = \frac{i}{2}\epsilon_{\alpha\gamma}\delta(x - x') \,. \tag{5.19}$$

5.1.2 Superfield Transformation of ψ

As the Lagrangian is symmetric with respect to the exchange of χ and ψ there is no difference between the calculation of $\delta\chi$ and $\delta\psi$. This means that the intermediate result from equation (5.12) can be used to simplify the calculation of $\delta\psi$ if it is adapted appropriately. It is then found that

$$\begin{aligned}\delta\psi_\alpha(x) = \int \mathrm{d}\mathbf{x}' \Big(& m\zeta^\beta (\sigma_0)_\beta{}^{\dot\gamma} \tilde{R}_{\dot\gamma\delta}(x') \left\{\psi_\alpha(x), \psi^\delta(x')\right\} + 2i\zeta^\beta \tilde{S}_{\beta\epsilon}(x') \left\{\psi_\alpha(x), \dot{\psi}^\epsilon(x')\right\} \\ & + 2i\zeta^\beta (\sigma_0)^{\gamma\dot\delta} \boldsymbol{\sigma}_{\dot\delta\beta} \cdot \boldsymbol{\nabla}' \tilde{S}_{\gamma\epsilon}(x') \left\{\psi_\alpha(x), \psi^\epsilon(x')\right\} \\ & - m\bar{\zeta}_{\dot\beta} (\bar\sigma_0)^{\dot\beta\gamma} \tilde{S}_{\gamma\delta}(x') \left\{\psi_\alpha(x), \chi^\delta(x')\right\} - 2i\bar{\zeta}^{\dot\beta} \tilde{R}_{\dot\beta\epsilon}(x') \left\{\psi_\alpha(x), \dot{\chi}^\epsilon(x')\right\} \\ & +2i\bar{\zeta}_{\dot\beta} (\bar\sigma_0)^{\dot\gamma\delta} \boldsymbol{\sigma}_\delta{}^{\dot\beta} \cdot \boldsymbol{\nabla}' \tilde{R}_{\dot\gamma\epsilon}(x') \left\{\psi_\alpha(x), \chi^\epsilon(x')\right\}\Big) \,.\end{aligned} \tag{5.20}$$

By comparison with equation (3.69) it can be seen that this time the only nonvanishing term is proportional to $\bar\zeta\tilde{R}$ while all other anticommutators have to vanish identically

$$\left\{\psi_\alpha(x), \dot{\psi}_\beta(x')\right\} = 0 \,, \tag{5.21}$$

$$\left\{\psi_\alpha(x), \psi_\beta(x')\right\} = 0 \,, \tag{5.22}$$

$$\left\{\psi_\alpha(x), \chi_\beta(x')\right\} = 0 \,. \tag{5.23}$$

The nonvanishing term satisfies

$$\bar\zeta^{\dot\beta}\tilde{R}_{\dot\beta\alpha} = 2i\bar\zeta^{\dot\beta} \int \mathrm{d}\mathbf{x}' \, \tilde{R}_{\dot\beta}{}^\gamma(x') \left\{\psi_\alpha(x), \dot\chi_\gamma(x')\right\} \,, \tag{5.24}$$

which implies an anticommutator of the form

$$\left\{\psi_\alpha(x), \dot\chi_\gamma(x')\right\} = a\epsilon_{\alpha\gamma}\delta(\mathbf{x} - \mathbf{x}') \,. \tag{5.25}$$

Inserting this solution into equation (5.24) yields

$$-\bar{\zeta}^{\dot{\beta}} R'_{\dot{\beta}\alpha} = -2i\bar{\zeta}^{\dot{\beta}} \int \mathrm{d}\mathbf{x}' \, \tilde{R}_{\dot{\beta}}{}^{\gamma}(x') a \delta(\mathbf{x} - \mathbf{x}') \epsilon_{\alpha\gamma}$$
$$= 2ia\bar{\zeta}^{\dot{\beta}} \tilde{R}_{\dot{\beta}\alpha}(x) \,. \tag{5.26}$$

This equation is satisfied for $a = i/2$ and the anticommutator is found to be

$$\left\{\psi_\alpha(x), \dot{\chi}_\gamma(x')\right\} = \frac{i}{2} \epsilon_{\alpha\gamma} \delta(\mathbf{x} - \mathbf{x}') \,. \tag{5.27}$$

5.1.3 Superfield Transformation of $\tilde{S}$

For the calculation of the variation of the bosonic second rank spinor field $\delta\tilde{S}$ it is no longer sufficient to adapt some intermediate result of the previous calculations for the fermionic component fields. This is due to the fact that the change from the variation of a fermionic to a bosonic field results in an exchange of all anticommutators with commutators

$$\begin{aligned}\delta\tilde{S}_{\beta\alpha}(x) = \int \mathrm{d}\mathbf{x}' \Big(& -m\zeta^\gamma (\sigma_0)_\gamma{}^{\dot{\epsilon}} \left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot{\epsilon}\delta}(x')\psi^\delta(x')\right] \\ & - 2i\zeta^\gamma (\sigma_0)^{\epsilon\dot{\delta}} \left[\tilde{S}_{\beta\alpha}(x), \tilde{S}_{\epsilon\kappa}(x')\bar{\partial\!\!\!/}_{\dot{\delta}\gamma}\psi^\kappa(x')\right] \\ & + m\bar{\zeta}_{\dot{\gamma}} (\bar{\sigma}_0)^{\dot{\gamma}\epsilon} \left[\tilde{S}_{\beta\alpha}(x), \tilde{S}_{\epsilon\delta}(x')\chi^\delta(x')\right] \\ & -2i\bar{\zeta}_{\dot{\gamma}} (\bar{\sigma}_0)^{\dot{\epsilon}\delta} \left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot{\epsilon}\kappa}(x')\partial\!\!\!/_\delta{}^{\dot{\gamma}}\chi^\kappa(x')\right]\Big) \,. \end{aligned} \tag{5.28}$$

The commutators involved in this expression each contain two bosonic and one fermionic component field and can be simplified using the commutator relation

$$[B_1, B_2F_2] = F_2 \, [B_1, B_2] \,. \tag{5.29}$$

The variation of the bosonic second rank spinor field $\tilde{S}$ then takes the form

$$\begin{aligned}\delta\tilde{S}_{\beta\alpha}(x) = \int \mathrm{d}\mathbf{x}' \Big(& -m\zeta^\gamma (\sigma_0)_\gamma{}^{\dot{\epsilon}} \psi^\delta(x') \left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot{\epsilon}\delta}(x')\right] \\ & - 2i\zeta^\gamma (\sigma_0)^{\epsilon\dot{\delta}} \, \bar{\partial\!\!\!/}_{\dot{\delta}\gamma}\psi^\kappa(x') \left[\tilde{S}_{\beta\alpha}(x), \tilde{S}_{\epsilon\kappa}(x')\right] \\ & + m\bar{\zeta}_{\dot{\gamma}} (\bar{\sigma}_0)^{\dot{\gamma}\epsilon} \chi^\delta(x') \left[\tilde{S}_{\beta\alpha}(x), \tilde{S}_{\epsilon\delta}(x')\right] \\ & -2i\bar{\zeta}_{\dot{\gamma}} (\bar{\sigma}_0)^{\dot{\epsilon}\delta} \, \partial\!\!\!/_\delta{}^{\dot{\gamma}}\chi^\kappa(x') \left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot{\epsilon}\kappa}(x')\right]\Big) \,. \end{aligned} \tag{5.30}$$

If this result is compared to the result for the superspace translation of $\tilde{S}$ in equation (3.72) it can be immediately read off that

$$\left[\tilde{S}_{\beta\alpha}(x), \tilde{S}_{\gamma\delta}\left(x'\right)\right] = 0\,. \tag{5.31}$$

The remaining two relations for the commutator between $\tilde{S}$ and $\tilde{R}$ should yield the same result. From the first relation

$$m\zeta_\beta\psi_\alpha = -m\zeta^\gamma \left(\sigma_0\right)_\gamma{}^{\dot\epsilon} \int \mathrm{d}\mathbf{x}'\psi^\delta\left(x'\right)\left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot\epsilon\delta}\left(x'\right)\right] \tag{5.32}$$

it is found that $\tilde{S}$ and $\tilde{R}$ satisfy a commutation relation of the form

$$\left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot\epsilon\delta}\left(x'\right)\right] = a\epsilon_{\alpha\delta}\left(\sigma^0\right)_{\beta\dot\epsilon}\delta\left(x - x'\right) \tag{5.33}$$

and thus

$$\begin{aligned} m\zeta_\beta\psi_\alpha(x) &= -m\zeta^\gamma\left(\sigma_0\right)_\gamma{}^{\dot\epsilon}\int \mathrm{d}\mathbf{x}'\psi^\delta\left(x'\right)a\epsilon_{\alpha\delta}\delta\left(x - x'\right)\left(\sigma^0\right)_{\beta\dot\epsilon} \\ &= -am\zeta_\beta\psi_\alpha(x)\,. \end{aligned} \tag{5.34}$$

This relation is satisfied if the prefactor is chosen to $a = -1$ and the commutator is given by

$$\left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot\epsilon\delta}\left(x'\right)\right] = -\epsilon_{\alpha\delta}\delta\left(x - x'\right)\left(\sigma^0\right)_{\beta\dot\epsilon}\,. \tag{5.35}$$

With this result for the commutator between the two second rank spinor fields it can be shown that the second relation is satisfied identically

$$\begin{aligned} 2i\bar\zeta^{\dot\gamma}\bar{\partial\!\!\!/}_{\dot\gamma\beta}\chi_\alpha(x) &= -2i\bar\zeta_{\dot\gamma}\left(\bar\sigma_0\right)^{\dot\epsilon\delta}\int \mathrm{d}\mathbf{x}'\bar{\partial\!\!\!/}'_\delta{}^{\dot\gamma}\chi^\kappa\left(x'\right)\left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot\epsilon\kappa}\left(x'\right)\right] \\ &= 2i\bar\zeta^{\dot\gamma}\bar{\partial\!\!\!/}_{\dot\gamma\beta}\chi_\alpha(x)\,. \end{aligned} \tag{5.36}$$

This indicates that a consistent second quantisation for the bosonic second rank component fields exists. However, it is still necessary to perform a similar derivation for $\delta\tilde{R}$ to arrive at a complete set of commutation relations for the second rank spinor fields.

5.1.4 Superfield Transformation of $\tilde{R}$

The last superfield transformation to be discussed is the transformation of the second rank spinor field $\tilde{R}$. Again it is possible to shorten the derivation and use an intermediate result of the derivation for $\tilde{S}$ from equation (5.30)

$$\begin{aligned}\delta \tilde{R}_{\dot{\beta}\alpha}(x) = \int \mathrm{d}\mathbf{x}' \Big(& -m\zeta^{\gamma} \left(\sigma_0\right)_{\gamma}{}^{\dot{\epsilon}} \psi^{\delta}\left(x'\right) \left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{R}_{\dot{\epsilon}\delta}\left(x'\right)\right] \\ & - 2i\zeta^{\gamma} \left(\sigma_0\right)^{\epsilon\dot{\delta}} \bar{\partial}'_{\dot{\delta}\gamma} \psi^{\kappa}\left(x'\right) \left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{S}_{\epsilon\kappa}\left(x'\right)\right] \\ & + m\bar{\zeta}_{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\gamma}\epsilon} \chi^{\delta}\left(x'\right) \left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{S}_{\epsilon\delta}\left(x'\right)\right] \\ & -2i\bar{\zeta}_{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\epsilon}\delta} \partial'_{\delta}{}^{\dot{\gamma}} \chi^{\kappa}\left(x'\right) \left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{R}_{\dot{\epsilon}\kappa}\left(x'\right)\right]\Big) \, . \end{aligned} \tag{5.37}$$

Together with the superfield translation from equation (3.71) this implies the commutation relation

$$\left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{R}_{\dot{\gamma}\delta}\left(x'\right)\right] = 0 \, . \tag{5.38}$$

Furthermore, the relation

$$m\bar{\zeta}_{\dot{\beta}}\chi_{\alpha}(x) = m\bar{\zeta}_{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\gamma}\epsilon} \int \mathrm{d}\mathbf{x}' \chi^{\delta}\left(x'\right) \left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{S}_{\epsilon\delta}\left(x'\right)\right] \tag{5.39}$$

results in a commutation relation of the form

$$\left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{S}_{\epsilon\delta}\left(x'\right)\right] = a\epsilon_{\alpha\delta}\delta\left(x-x'\right) \left(\bar{\sigma}^0\right)_{\dot{\beta}\epsilon} \, . \tag{5.40}$$

Therefore, it is found that

$$\begin{aligned} -m\bar{\zeta}_{\dot{\beta}}\chi_{\alpha}(x) &= m\bar{\zeta}_{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\gamma}\epsilon} \int \mathrm{d}\mathbf{x}' \chi^{\delta}\left(x'\right) a\epsilon_{\alpha\delta}\delta\left(x-x'\right) \left(\bar{\sigma}^0\right)_{\dot{\beta}\epsilon} \\ &= -am\bar{\zeta}_{\dot{\beta}}\chi_{\alpha}(x) \, , \end{aligned} \tag{5.41}$$

which determines the prefactor to $a = -1$. The commutator between $\tilde{R}$ and $\tilde{S}$ is then given by

$$\left[R_{\dot{\beta}\alpha}(x), S'_{\epsilon\delta}\left(x'\right)\right] = -\epsilon_{\alpha\delta} \left(\bar{\sigma}^0\right)_{\dot{\beta}\epsilon} \delta\left(x-x'\right) . \tag{5.42}$$

This solution should be equivalent to the commutation relation between $\tilde{S}$ and $\tilde{R}$ from equation (5.35). To show that they are identical it is necessary to commute $\tilde{R}$ and $\tilde{S}$ and rename the indices appropriately. It is found that they are indeed the same. As consistency check for equation (5.42) it is again possible to employ the second nonvanishing relation in equation (5.37)

$$\begin{aligned} -2i\zeta^{\gamma}\partial\!\!\!/_{\gamma\dot{\beta}}\psi_{\alpha}(x) &= -2i\zeta^{\gamma}\left(\sigma_0\right)^{\epsilon\dot{\delta}}\int \mathrm{d}\mathbf{x}'\bar{\partial\!\!\!/}'_{\dot{\delta}\gamma}\psi^{\kappa}(x')\left[\tilde{R}_{\dot{\beta}\alpha}(x),\tilde{S}_{\epsilon\kappa}(x')\right] \\ &= -2i\zeta^{\gamma}\partial\!\!\!/_{\gamma\dot{\beta}}\psi_{\alpha}(x)\,, \end{aligned} \tag{5.43}$$

which leads to a true statement. This confirms once again that the commutation and anticommutation relations derived in the previous section represent a consistent second quantisation in position space.

5.1.5 Variation of the Component Fields

Generally it is possible to repeat the calculations outlined in the previous sections for the Hermitian conjugate component fields. However, it is much easier to calculate the Hermitian conjugate of the previously derived commutation and anticommutation relations.

For the anticommutation relations between the spinor fields the Hermitian conjugation is straightforward and it is found that the two nonvanishing anticommutation relations between the barred component fields are

$$\left\{\bar{\chi}_{\dot{\alpha}}(x),\dot{\bar{\psi}}_{\dot{\gamma}}(x')\right\} = \frac{i}{2}\epsilon_{\dot{\alpha}\dot{\gamma}}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{5.44}$$

$$\left\{\bar{\psi}_{\dot{\alpha}}(x),\dot{\bar{\chi}}_{\dot{\gamma}}(x')\right\} = \frac{i}{2}\epsilon_{\dot{\alpha}\dot{\gamma}}\delta(\mathbf{x}-\mathbf{x}')\,. \tag{5.45}$$

The only difficulty that arises here is the fact that the second rank ϵ-tensor changes sign under Hermitian conjugation.

For the commutation relations of the bosonic second rank spinor fields the discussion is slightly more involved as the Hermitian conjugation inverts the ordering of the component fields which induces an additional sign flip for the commutators that didn't occur for the spinor fields that satisfied anticommutation relations. Therefore, the Hermitian conjugate of the right and left hand side of equation (5.35) will be discussed independently

$$\left[\tilde{S}_{\beta\alpha}(x), \tilde{R}_{\dot{\epsilon}\delta}(x')\right]^\dagger = -\left[\tilde{\bar{S}}_{\dot{\beta}\dot{\alpha}}(x), \tilde{\bar{R}}_{\epsilon\dot{\delta}}(x')\right] , \tag{5.46}$$

$$\left(-\epsilon_{\alpha\delta}\delta(\mathbf{x}-\mathbf{x}')\left(\sigma^0\right)_{\beta\dot{\epsilon}}\right)^\dagger = \epsilon_{\dot{\alpha}\dot{\delta}}\delta(\mathbf{x}-\mathbf{x}')\left(\bar{\sigma}^0\right)_{\dot{\beta}\epsilon} . \tag{5.47}$$

Under Hermitian conjugation the commutator between $\tilde{S}$ and $\tilde{R}$ changes sign as both bosonic component fields are replaced with their barred counterparts while their ordering is exchanged. For the Hermitian conjugate of the left hand side it must again be taken into account that the second rank ϵ-tensor changes sign under Hermitian conjugation. This implies a commutation relation for the barred component fields of

$$\left[\tilde{\bar{S}}_{\dot{\beta}\dot{\alpha}}(x), \tilde{\bar{R}}_{\epsilon\dot{\delta}}(x')\right] = -\epsilon_{\dot{\alpha}\dot{\delta}}\delta(\mathbf{x}-\mathbf{x}')\left(\bar{\sigma}^0\right)_{\dot{\beta}\epsilon} . \tag{5.48}$$

The calculation for the remaining nonvanishing commutation relation can then be performed in perfect analogy

$$\left[\tilde{R}_{\dot{\beta}\alpha}(x), \tilde{S}_{\epsilon\delta}(x')\right]^\dagger = -\left[\tilde{\bar{R}}_{\beta\dot{\alpha}}(x), \tilde{\bar{S}}_{\dot{\epsilon}\dot{\delta}}(x')\right] , \tag{5.49}$$

$$\left(-\epsilon_{\alpha\delta}\left(\bar{\sigma}^0\right)_{\dot{\beta}\epsilon}\delta(\mathbf{x}-\mathbf{x}')\right)^\dagger = \epsilon_{\dot{\alpha}\dot{\delta}}\left(\sigma^0\right)_{\beta\dot{\epsilon}}\delta(\mathbf{x}-\mathbf{x}') . \tag{5.50}$$

The same arguments apply as in the previous case and the second nonvanishing commutation relation is given by

$$\left[\tilde{\bar{R}}_{\beta\dot{\alpha}}(x), \tilde{\bar{S}}_{\dot{\epsilon}\dot{\delta}}(x')\right] = -\epsilon_{\dot{\alpha}\dot{\delta}}\left(\sigma^0\right)_{\beta\dot{\epsilon}}\delta(\mathbf{x}-\mathbf{x}') . \tag{5.51}$$

It has to be pointed out that this relation could also have been derived from the first commutation relation by commuting the second rank spinor fields and renaming the indices appropriately.

5.2 The Hamiltonian from the Supersymmetry Algebra

To derive an explicit equation for the Hamiltonian the supersymmetry generators in equation (5.1) have to be expressed in terms of the component fields. This can be achieved using the relations between the supersymmetry generators which are the conserved Noether charges of the system and the supercurrents which were defined in equations (5.3) and (5.4). Inserting the result for the supercurrent from equations (4.11) and (4.12) leads to the following expression of the supersymmetry generators in terms of the component fields

$$\begin{aligned} Q_\alpha = \int \mathrm{d}\mathbf{x} \Big(& -im\,(\sigma_0)_\alpha{}^{\dot\gamma} \tilde{R}_{\dot\gamma\beta}(x)\psi^\beta(x) + 2\,(\sigma_0)^{\gamma\dot\delta}\,\tilde{S}_{\gamma\beta}(x)\bar{\partial\!\!\!/}_{\dot\delta\alpha}\psi^\beta(x) \\ & -im\,(\sigma_0)_\alpha{}^{\dot\gamma}\tilde{\bar{S}}_{\dot\gamma\dot\beta}(x)\bar\chi^{\dot\beta}(x) + 2\,(\sigma_0)^{\gamma\dot\delta}\,\tilde{\bar{R}}_{\gamma\dot\beta}(x)\bar{\partial\!\!\!/}_{\dot\delta\alpha}\bar\chi^{\dot\beta}(x)\Big)\,, \end{aligned} \tag{5.52}$$

$$\begin{aligned} \bar{Q}_{\dot\alpha} = \int \mathrm{d}\mathbf{x} \Big(& im\,(\bar\sigma_0)_{\dot\alpha}{}^{\gamma}\tilde{S}_{\gamma\beta}(x)\chi^\beta(x) + 2\,(\bar\sigma_0)^{\dot\gamma\delta}\,\tilde{R}_{\dot\gamma\beta}(x)\partial\!\!\!/_{\delta\dot\alpha}\chi^\beta(x) \\ & +im\,(\bar\sigma_0)_{\dot\alpha}{}^{\gamma}\tilde{\bar{R}}_{\gamma\dot\beta}(x)\bar\psi^{\dot\beta}(x) + 2\,(\bar\sigma_0)^{\dot\gamma\delta}\,\tilde{\bar{S}}_{\dot\gamma\dot\beta}(x)\partial\!\!\!/_{\delta\dot\alpha}\bar\psi^{\dot\beta}(x)\Big)\,. \end{aligned} \tag{5.53}$$

To streamline the notation it proves useful to introduce the short notation

$$P\!\!\!/_{\alpha\dot\beta} = (\sigma^\mu)_{\alpha\dot\beta}\,P_\mu\,, \tag{5.54}$$

which is defined in analogy to the commonly used contraction of Dirac matrices with four derivatives. The momentum operator is then given by

$$\begin{aligned} 2P\!\!\!/_{\alpha\dot\beta} = \Big\{ & \int \mathrm{d}\mathbf{x} \Big(-im\,(\sigma_0)_\alpha{}^{\dot\gamma}\tilde{R}_{\dot\gamma\omega}(x)\psi^\omega(x) + 2\,(\sigma_0)^{\gamma\dot\delta}\,\tilde{S}_{\gamma\omega}(x)\bar{\partial\!\!\!/}_{\dot\delta\alpha}\psi^\omega(x) \\ & -im\,(\sigma_0)_\alpha{}^{\dot\gamma}\tilde{\bar{S}}_{\dot\gamma\dot\omega}(x)\bar\chi^{\dot\omega}(x) + 2\,(\sigma_0)^{\gamma\dot\delta}\,\tilde{\bar{R}}_{\gamma\dot\omega}(x)\bar{\partial\!\!\!/}'_{\dot\delta\alpha}\bar\chi^{\dot\omega}(x)\Big)\,, \\ & \int \mathrm{d}\mathbf{x}' \Big(im\,(\bar\sigma_0)_{\dot\beta}{}^{\kappa}\tilde{S}_{\kappa\epsilon}(x')\chi^\epsilon(x') + 2\,(\bar\sigma_0)^{\dot\kappa\tau}\,\tilde{R}_{\dot\kappa\epsilon}(x')\partial\!\!\!/'_{\tau\dot\beta}\chi^\epsilon(x') \\ & +im\,(\bar\sigma_0)_{\dot\beta}{}^{\kappa}\tilde{\bar{R}}_{\kappa\dot\epsilon}(x')\bar\psi^{\dot\epsilon}(x') + 2\,(\bar\sigma_0)^{\dot\kappa\tau}\,\tilde{\bar{S}}_{\dot\kappa\dot\epsilon}(x')\partial\!\!\!/_{\tau\dot\beta}\bar\psi^{\dot\epsilon}(x')\Big)\Big\}\,. \end{aligned} \tag{5.55}$$

The anticommutators containing two fermionic and two bosonic component fields can now be rewritten using the commutator relation

$$\{B_1F_1, B_2F_2\} = [B_1, B_2]\,F_1F_2 + B_2B_1\,\{F_1, F_2\}\,, \tag{5.56}$$

where it was assumed that the fermionic and bosonic fields commute. This assumption is justified by the previous derivation of the commutation and anticommutation relations of the component fields as well as the results of the superfield translations. The momentum operator can then be expressed as

$$
\begin{aligned}
2\not{P}_{\alpha\dot{\beta}} = \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}' \Big(& m^2 \left(\sigma_0\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_0\right)_{\dot{\beta}}{}^{\kappa} \psi^{\omega}(x)\chi^{\epsilon}(x') \left[\tilde{R}_{\dot{\gamma}\omega}(x), \tilde{S}_{\kappa\epsilon}(x')\right] \\
& - 2im \left(\sigma_0\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \tilde{R}_{\dot{\kappa}\epsilon}(x') \tilde{R}_{\dot{\gamma}\omega}(x) \left\{\psi^{\omega}(x), \not{\partial}'_{\tau\dot{\beta}}\chi^{\epsilon}(x')\right\} \\
& + 2im \left(\sigma_0\right)^{\gamma\dot{\delta}} \left(\bar{\sigma}_0\right)_{\dot{\beta}}{}^{\kappa} \tilde{S}_{\kappa\epsilon}(x') \tilde{S}_{\gamma\omega}(x) \left\{\bar{\not{\partial}}_{\dot{\delta}\alpha}\psi^{\omega}(x), \chi^{\epsilon}(x')\right\} \\
& + 4 \left(\sigma_0\right)^{\gamma\dot{\delta}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \bar{\not{\partial}}_{\dot{\delta}\alpha}\psi^{\omega}(x) \not{\partial}'_{\tau\dot{\beta}}\chi^{\epsilon}(x') \left[\tilde{S}_{\gamma\omega}(x), \tilde{R}_{\dot{\kappa}\epsilon}(x')\right] \\
& + 4 \left(\sigma_0\right)^{\gamma\dot{\delta}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \tilde{R}_{\dot{\kappa}\epsilon}(x') \tilde{S}_{\gamma\omega}(x) \left\{\bar{\not{\partial}}_{\dot{\delta}\alpha}\psi^{\omega}(x), \not{\partial}'_{\tau\dot{\beta}}\chi^{\epsilon}(x')\right\} \\
& + m^2 \left(\sigma_0\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_0\right)_{\dot{\beta}}{}^{\kappa} \bar{\chi}^{\dot{\omega}}(x)\bar{\psi}^{\dot{\epsilon}}(x') \left[\tilde{\bar{S}}_{\dot{\gamma}\dot{\omega}}(x), \tilde{\bar{R}}_{\kappa\dot{\epsilon}}(x')\right] \\
& - 2im \left(\sigma_0\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \tilde{\bar{S}}_{\dot{\kappa}\dot{\epsilon}}(x') \tilde{\bar{S}}_{\dot{\gamma}\dot{\omega}}(x) \left\{\bar{\chi}^{\dot{\omega}}(x), \not{\partial}'_{\tau\dot{\beta}}\bar{\psi}^{\dot{\epsilon}}(x')\right\} \\
& + 2im \left(\sigma_0\right)^{\gamma\dot{\delta}} \left(\bar{\sigma}_0\right)_{\dot{\beta}}{}^{\kappa} \tilde{\bar{R}}_{\kappa\dot{\epsilon}}(x') \tilde{\bar{R}}_{\gamma\dot{\omega}}(x) \left\{\bar{\not{\partial}}_{\dot{\delta}\alpha}\bar{\chi}^{\dot{\omega}}(x), \bar{\psi}^{\dot{\epsilon}}(x')\right\} \\
& + 4 \left(\sigma_0\right)^{\gamma\dot{\delta}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \bar{\not{\partial}}_{\dot{\delta}\alpha}\bar{\chi}^{\dot{\omega}}(x) \not{\partial}'_{\tau\dot{\beta}}\bar{\psi}^{\dot{\epsilon}}(x') \left[\tilde{\bar{R}}_{\gamma\dot{\omega}}(x), \tilde{\bar{S}}_{\dot{\kappa}\dot{\epsilon}}(x')\right] \\
& + 4 \left(\sigma_0\right)^{\gamma\dot{\delta}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \tilde{\bar{S}}_{\dot{\kappa}\dot{\epsilon}}(x') \tilde{\bar{R}}_{\gamma\dot{\omega}}(x) \left\{\bar{\not{\partial}}_{\dot{\delta}\alpha}\bar{\chi}^{\dot{\omega}}(x), \not{\partial}'_{\tau\dot{\beta}}\bar{\psi}^{\dot{\epsilon}}(x')\right\}\Big) \ ,
\end{aligned} \tag{5.57}
$$

where all vanishing contributions were omitted. This can be simplified even further by splitting the four derivative $\not{\partial}$ into its time and spatial components

$$
\not{\partial}_{\alpha\dot{\beta}} = \left(\sigma^{\mu}\right)_{\alpha\dot{\beta}} \partial_{\mu} = \left(\sigma^{0}\right)_{\alpha\dot{\beta}} \partial_0 + \boldsymbol{\sigma}_{\alpha\dot{\beta}} \cdot \boldsymbol{\nabla} \, . \tag{5.58}
$$

It is important to recall that there is a plus sign between the time and spatial components and not a minus sign. This is due to the fact that the derivative is a covariant three vector $\boldsymbol{\nabla} = (\partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ while all standard vectors, e. g., $\mathbf{p} = \left(p^1, p^2, p^3\right)$, are contravariant three vectors. Therefore, there is no sign change due to the raising of the index using the metric and the four derivative is given by $\partial_{\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_0, \boldsymbol{\nabla})$. However, the property $\Box = \partial^0\partial_0 - \boldsymbol{\nabla}\boldsymbol{\nabla}$ is preserved, as $\partial^{\mu} = \eta^{\mu\nu}\partial_{\nu} = \left(\partial^0, -\partial_1, -\partial_2, -\partial_3\right) = \left(\partial^0, -\boldsymbol{\nabla}\right)$.

After separation of the time and spatial derivatives as well as partial spatial integration the momentum operator is given by

$$
\begin{aligned}
2\not{P}_{\alpha\dot{\beta}} = \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}' \Big(& m^2 \left(\sigma_0\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_0\right)_{\dot{\beta}}{}^{\kappa} \psi^{\omega}(x)\chi^{\epsilon}(x') \left[\tilde{R}_{\dot{\gamma}\omega}(x), \tilde{S}_{\kappa\epsilon}(x')\right] \\
& - 2im \left(\sigma_0\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_0\right)^{\dot{\kappa}\tau} \left(\sigma^0\right)_{\tau\dot{\beta}} \tilde{R}_{\dot{\kappa}\epsilon}(x') \tilde{R}_{\dot{\gamma}\omega}(x) \left\{\psi^{\omega}(x), \dot{\chi}^{\epsilon}(x')\right\}
\end{aligned}
$$

$$+ 2im\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)_{\dot{\beta}}{}^{\kappa}\,\left(\bar{\sigma}^0\right)_{\dot{\delta}\alpha}\tilde{S}_{\kappa\epsilon}(x')\tilde{S}_{\gamma\omega}(x)\left\{\chi^{\epsilon}(x'),\dot{\psi}^{\omega}(x)\right\}$$

$$+ 4\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\bar{\partial\!\!\!/}_{\dot{\delta}\alpha}\psi^{\omega}(x)\partial\!\!\!/'_{\tau\dot{\beta}}\chi^{\epsilon}(x')\left[\tilde{S}_{\gamma\omega}(x),\tilde{R}_{\dot{\kappa}\epsilon}(x')\right]$$

$$- 4\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\left(\bar{\sigma}^0\right)_{\dot{\delta}\alpha}\boldsymbol{\sigma}_{\tau\dot{\beta}}\cdot\boldsymbol{\nabla}'\tilde{R}_{\dot{\kappa}\epsilon}(x')\tilde{S}_{\gamma\omega}(x)\left\{\chi^{\epsilon}(x'),\dot{\psi}^{\omega}(x)\right\}$$

$$- 4\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\left(\sigma^0\right)_{\tau\dot{\beta}}\tilde{R}_{\dot{\kappa}\epsilon}(x')\bar{\boldsymbol{\sigma}}_{\dot{\delta}\alpha}\cdot\boldsymbol{\nabla}\tilde{S}_{\gamma\omega}(x)\left\{\psi^{\omega}(x),\dot{\chi}^{\epsilon}(x')\right\}$$

$$+ m^2\,(\sigma_0)_{\alpha}{}^{\dot{\gamma}}\,(\bar{\sigma}_0)_{\dot{\beta}}{}^{\kappa}\bar{\chi}^{\dot{\omega}}(x)\bar{\psi}^{\dot{\epsilon}}(x')\left[\bar{\tilde{S}}_{\dot{\gamma}\dot{\omega}}(x),\bar{\tilde{R}}_{\kappa\dot{\epsilon}}(x')\right]$$

$$- 2im\,(\sigma_0)_{\alpha}{}^{\dot{\gamma}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\left(\sigma^0\right)_{\tau\dot{\beta}}\bar{\tilde{S}}_{\dot{\kappa}\dot{\epsilon}}(x')\bar{\tilde{S}}_{\dot{\gamma}\dot{\omega}}(x)\left\{\bar{\chi}^{\dot{\omega}}(x),\dot{\bar{\psi}}^{\dot{\epsilon}}(x')\right\}$$

$$+ 2im\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)_{\dot{\beta}}{}^{\kappa}\,\left(\bar{\sigma}^0\right)_{\dot{\delta}\alpha}\bar{\tilde{R}}_{\kappa\dot{\epsilon}}(x')\bar{\tilde{R}}_{\gamma\dot{\omega}}(x)\left\{\bar{\psi}^{\dot{\epsilon}}(x'),\dot{\bar{\chi}}^{\dot{\omega}}(x)\right\}$$

$$+ 4\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\bar{\partial\!\!\!/}_{\dot{\delta}\alpha}\bar{\chi}^{\dot{\omega}}(x)\partial\!\!\!/'_{\tau\dot{\beta}}\bar{\psi}^{\dot{\epsilon}}(x')\left[\bar{\tilde{R}}_{\gamma\dot{\omega}}(x),\bar{\tilde{S}}_{\dot{\kappa}\dot{\epsilon}}(x')\right]$$

$$- 4\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\left(\bar{\sigma}^0\right)_{\dot{\delta}\alpha}\boldsymbol{\sigma}_{\tau\dot{\beta}}\cdot\boldsymbol{\nabla}'\bar{\tilde{S}}_{\dot{\kappa}\dot{\epsilon}}(x')\bar{\tilde{R}}_{\gamma\dot{\omega}}(x)\left\{\bar{\psi}^{\dot{\epsilon}}(x'),\dot{\bar{\chi}}^{\dot{\omega}}(x)\right\}$$

$$-4\,(\sigma_0)^{\gamma\dot{\delta}}\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\left(\sigma^0\right)_{\tau\dot{\beta}}\bar{\tilde{S}}_{\dot{\kappa}\dot{\epsilon}}(x')\bar{\boldsymbol{\sigma}}_{\dot{\delta}\alpha}\cdot\boldsymbol{\nabla}\bar{\tilde{R}}_{\gamma\dot{\omega}}(x)\left\{\bar{\chi}^{\dot{\omega}}(x),\dot{\bar{\psi}}^{\dot{\epsilon}}(x')\right\}\Big)\,. \qquad (5.59)$$

Inserting the previously derived results for the commutation and anticommutation relations between the component fields in position space then yields

$$2P\!\!\!/_{\alpha\dot{\beta}} = \int \mathrm{d}\mathbf{x}\,\Big(-m^2\,(\sigma_0)_{\alpha\dot{\beta}}\,\psi_{\epsilon}(x)\chi^{\epsilon}(x) - m\,(\sigma_0)_{\alpha}{}^{\dot{\gamma}}\tilde{R}_{\dot{\beta}\epsilon}(x)\tilde{R}_{\dot{\gamma}}{}^{\epsilon}(x)$$

$$+ m\,(\bar{\sigma}_0)_{\dot{\beta}}{}^{\kappa}\tilde{S}_{\kappa}{}^{\omega}(x)\tilde{S}_{\alpha\omega}(x) - 4\,(\sigma_0)^{\gamma\dot{\delta}}\,\bar{\partial\!\!\!/}_{\dot{\delta}\alpha}\psi_{\epsilon}(x)\partial\!\!\!/_{\gamma\dot{\beta}}\chi^{\epsilon}(x)$$

$$+ 2i\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\boldsymbol{\sigma}_{\tau\dot{\beta}}\cdot\boldsymbol{\nabla}\tilde{R}_{\dot{\kappa}}{}^{\omega}(x)\tilde{S}_{\alpha\omega}(x) + 2i\,(\sigma_0)^{\gamma\dot{\delta}}\,\tilde{R}_{\dot{\beta}\epsilon}(x)\bar{\boldsymbol{\sigma}}_{\dot{\delta}\alpha}\cdot\boldsymbol{\nabla}\tilde{S}_{\gamma}{}^{\epsilon}(x)$$

$$+ m^2\,(\sigma_0)_{\alpha\dot{\beta}}\,\bar{\chi}_{\dot{\epsilon}}(x)\bar{\psi}^{\dot{\epsilon}}(x) + m\,(\sigma_0)_{\alpha}{}^{\dot{\gamma}}\bar{\tilde{S}}_{\dot{\beta}\dot{\epsilon}}(x)\bar{\tilde{S}}_{\dot{\gamma}}{}^{\dot{\epsilon}}(x)$$

$$- m\,(\bar{\sigma}_0)_{\dot{\beta}}{}^{\kappa}\bar{\tilde{R}}_{\kappa}{}^{\dot{\omega}}(x)\bar{\tilde{R}}_{\alpha\dot{\omega}}(x) + 4\,(\sigma_0)^{\gamma\dot{\delta}}\,\bar{\partial\!\!\!/}_{\dot{\delta}\alpha}\bar{\chi}_{\dot{\epsilon}}(x)\partial\!\!\!/_{\gamma\dot{\beta}}\bar{\psi}^{\dot{\epsilon}}(x)$$

$$-2i\,(\bar{\sigma}_0)^{\dot{\kappa}\tau}\,\boldsymbol{\sigma}_{\tau\dot{\beta}}\cdot\boldsymbol{\nabla}\bar{\tilde{S}}_{\dot{\kappa}}{}^{\dot{\omega}}(x)\bar{\tilde{R}}_{\alpha\dot{\omega}}(x) - 2i\,(\sigma_0)^{\gamma\dot{\delta}}\,\bar{\tilde{S}}_{\dot{\beta}\dot{\epsilon}}(x)\bar{\boldsymbol{\sigma}}_{\dot{\delta}\alpha}\cdot\boldsymbol{\nabla}\bar{\tilde{R}}_{\gamma}{}^{\dot{\epsilon}}(x)\Big)\,. \qquad (5.60)$$

To extract the Hamiltonian from the momentum operator it is necessary to contract it with the appropriate Pauli matrix

$$\mathcal{H} = \frac{1}{2}\,(\sigma_0)^{\alpha\dot{\beta}}\,P\!\!\!/_{\alpha\dot{\beta}}\,. \qquad (5.61)$$

The Hamiltonian is therefore given by

$$
\begin{aligned}
\mathcal{H} = \frac{1}{4}\int \mathrm{d}\mathbf{x}\, \Big(& 2m^2\psi(x)\chi(x) + m\tilde{R}_{\dot{\beta}\epsilon}(x)\tilde{R}^{\dot{\beta}\epsilon}(x) \\
& - m\tilde{S}^{\alpha\omega}(x)\tilde{S}_{\alpha\omega}(x) - 4\left(\sigma_0\bar{\sigma}^{\mu}\sigma_0\right)^{\gamma\dot{\beta}}\partial_\mu\psi_\epsilon(x)\partial\!\!\!/_{\gamma\dot{\beta}}\chi^\epsilon(x) \\
& + 2i\left(\bar{\sigma}_0\sigma^i\bar{\sigma}_0\right)^{\dot{\kappa}\alpha}\partial_i\tilde{R}_{\dot{\kappa}}{}^{\omega}(x)\tilde{S}_{\alpha\omega}(x) + 2i\left(\sigma_0\bar{\sigma}^i\sigma_0\right)^{\gamma\dot{\beta}}\tilde{R}_{\dot{\beta}\epsilon}(x)\partial_i\tilde{S}_{\gamma}{}^{\epsilon}(x) \\
& + 2m^2\bar{\chi}(x)\bar{\psi}(x) - m\tilde{\bar{S}}_{\dot{\beta}\dot{\epsilon}}(x)\tilde{\bar{S}}_{\dot{\gamma}}{}^{\dot{\beta}\dot{\epsilon}}(x) \\
& + m\tilde{\bar{R}}^{\alpha\dot{\omega}}(x)\tilde{\bar{R}}_{\alpha\dot{\omega}}(x) + 4\left(\sigma_0\bar{\sigma}^{\mu}\sigma_0\right)^{\gamma\dot{\beta}}\partial_\mu\bar{\chi}_{\dot{\epsilon}}(x)\partial\!\!\!/_{\gamma\dot{\beta}}\bar{\psi}^{\dot{\epsilon}}(x) \\
& -2i\left(\bar{\sigma}_0\sigma^i\bar{\sigma}_0\right)^{\dot{\kappa}\alpha}\partial_i\tilde{\bar{S}}_{\dot{\kappa}}{}^{\dot{\omega}}(x)\tilde{\bar{R}}_{\alpha\dot{\omega}}(x) - 2i\left(\sigma_0\bar{\sigma}^i\sigma_0\right)^{\gamma\dot{\beta}}\tilde{\bar{S}}_{\dot{\beta}\dot{\epsilon}}(x)\partial_i\tilde{\bar{R}}_{\gamma}{}^{\dot{\epsilon}}(x)\Big)\,.
\end{aligned} \tag{5.62}
$$

This expression for the Hamiltonian can be further simplified using relations (A.38) and (A.39) in the Appendix A.4 between σ-matrices for the special case where the first and last index are 0

$$\sigma^0\bar{\sigma}^\mu\sigma^0 + \sigma^0\bar{\sigma}^\mu\sigma^0 = 4\eta^{\mu 0}\sigma^0 - 2\sigma^\mu\,, \tag{5.63}$$

$$\bar{\sigma}^0\sigma^\mu\bar{\sigma}^0 + \bar{\sigma}^0\sigma^\mu\bar{\sigma}^0 = 4\eta^{\mu 0}\bar{\sigma}^0 - 2\bar{\sigma}^\mu\,. \tag{5.64}$$

This can also be written as

$$\sigma^0\bar{\sigma}^\mu\sigma^0 = 2\eta^{\mu 0}\sigma^0 - \sigma^\mu\,, \tag{5.65}$$

$$\bar{\sigma}^0\sigma^\mu\bar{\sigma}^0 = 2\eta^{\mu 0}\bar{\sigma}^0 - \bar{\sigma}^\mu\,. \tag{5.66}$$

The Hamiltonian then reduces to

$$
\begin{aligned}
\mathcal{H} = \int \mathrm{d}\mathbf{x}\, \Big(& 2\dot{\psi}(x)\dot{\chi}(x) + 2\boldsymbol{\nabla}\psi(x)\cdot\boldsymbol{\nabla}\chi(x) + \frac{m^2}{2}\psi(x)\chi(x) \\
& + 2\dot{\bar{\chi}}(x)\dot{\bar{\psi}}(x) + 2\boldsymbol{\nabla}\bar{\chi}(x)\cdot\boldsymbol{\nabla}\bar{\psi}(x) + \frac{m^2}{2}\bar{\chi}(x)\bar{\psi}(x) \\
& + \frac{m}{4}\mathrm{Tr}\Big(\tilde{R}^T(x)\tilde{R}(x)\Big) + \frac{m}{4}\mathrm{Tr}\Big(\tilde{S}^T(x)\tilde{S}(x)\Big) - i\mathrm{Tr}\Big(\tilde{R}^T(x)\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}\tilde{S}(x)\Big) \\
& + \frac{m}{4}\mathrm{Tr}\Big(\tilde{\bar{R}}^T(x)\tilde{\bar{R}}(x)\Big) + \frac{m}{4}\mathrm{Tr}\Big(\tilde{\bar{S}}^T(x)\tilde{\bar{S}}(x)\Big) - i\mathrm{Tr}\Big(\tilde{\bar{S}}^T(x)\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}\tilde{\bar{R}}(x)\Big)\Big)\,.
\end{aligned} \tag{5.67}
$$

It can be observed that the Hamiltonian in position space is perfectly symmetric between the component fields and their Hermitian conjugate counterparts. This is a very interesting feature as it implies that the sum of the corresponding unbarred and barred terms is real.

For the bosonic contributions to the Hamiltonian this property is sufficient to conclude that their contribution to the Hamiltonian is positive. However, for the fermionic terms this is no longer the case. The sum of unbarred spinors products and their barred counterparts are only restricted to be real but can be either positive or negative. This makes it impossible to draw a conclusion on the positive definiteness of the energy spectrum of the theory – even though it should be by construction. To ensure that this property is satisfied it is necessary to discuss quantisation and the resulting energy spectrum of the Hamiltonian in momentum space.

5.3 The Hamiltonian from Legendre Transformation

The derivation of the Hamiltonian using the supersymmetry algebra should by construction be positive definite and is founded in the fundamental properties of the algebra. However, it immediately raises the question whether this approach is equivalent to a construction of the Hamiltonian by Legendre transformation which doesn't require the Lagrangian to be supersymmetric.

The Hamiltonian from Legendre transformation is then defined as

$$\mathcal{H}_{c.q.} = \int \mathrm{d}^3\mathbf{x} \left(-\frac{\partial \mathcal{L}}{\partial \dot{\chi}^\tau} \dot{\chi}^\tau - \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\tau} \dot{\psi}^\tau + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{S}}^{\tau\omega}} \dot{\tilde{S}}^{\tau\omega} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{R}}^{\dot{\tau}\omega}} \dot{\tilde{R}}^{\dot{\tau}\omega} - \mathcal{L} \right) \tag{5.68}$$

It is convenient at this point to rewrite the unbarred part of the Lagrangian from equation (3.88) in the following way

$$\begin{aligned} \mathcal{L} = {} & 2\dot{\chi}^\alpha \dot{\psi}_\alpha - 2\boldsymbol{\nabla}\chi^\alpha \boldsymbol{\nabla}\psi_\alpha - 2\tilde{m}^2 \psi^\alpha \chi_\alpha + \frac{i}{2} \tilde{S}^\beta{}_\alpha \left(\sigma^0\right)_{\beta\dot{\gamma}} \dot{\tilde{R}}^{\dot{\gamma}\alpha} - \frac{i}{2} \tilde{S}^{\beta\alpha} \boldsymbol{\sigma}_{\beta\dot{\gamma}} \cdot \boldsymbol{\nabla} \tilde{R}^{\dot{\gamma}}{}_\alpha \\ & - \frac{i}{2} \tilde{R}_{\dot{\beta}\alpha} \left(\bar{\sigma}^0\right)^{\dot{\beta}}{}_\gamma \dot{\tilde{S}}^{\gamma\alpha} + \frac{i}{2} \tilde{R}_{\dot{\beta}\alpha} \bar{\boldsymbol{\sigma}}^{\dot{\beta}\gamma} \cdot \boldsymbol{\nabla} \tilde{S}_\gamma{}^\alpha + \frac{\tilde{m}}{2} \tilde{S}^{\beta\alpha} \tilde{S}_{\beta\alpha} - \frac{\tilde{m}}{2} \tilde{R}_{\dot{\beta}\alpha} \tilde{R}^{\dot{\beta}\alpha} \end{aligned} \tag{5.69}$$

Inserting the Lagrangian into the previous definition of the Hamiltonian from Legendre transformation then results in

$$
\begin{aligned}
\mathcal{H}_{c.q.} =& \int \mathrm{d}^3\mathbf{x} \Bigg(-\left(\frac{\partial}{\partial \dot{\chi}^{\tau}} 2\dot{\chi}^{\alpha}\dot{\psi}_{\alpha} \right) \dot{\chi}^{\tau} - \left(\frac{\partial}{\partial \dot{\psi}^{\tau}} 2\dot{\psi}^{\alpha}\dot{\chi}_{\alpha} \right) \dot{\psi}^{\tau} \\
&+ \left(\frac{\partial}{\partial \dot{\tilde{S}}^{\tau\omega}} \left(-\frac{i}{2} \tilde{R}_{\dot{\beta}\alpha} \left(\bar{\sigma}^0\right)^{\dot{\beta}}{}_{\gamma} \dot{\tilde{S}}^{\gamma\alpha} \right) \right) \dot{\tilde{S}}^{\tau\omega} - \left(\frac{\partial}{\partial \dot{\tilde{R}}^{\dot{\tau}\omega}} \left(-\frac{i}{2} \tilde{S}^{\beta}{}_{\alpha} \left(\sigma^0\right)_{\beta\dot{\gamma}} \dot{\tilde{R}}^{\dot{\gamma}\alpha} \right) \right) \dot{\tilde{R}}^{\dot{\tau}\omega} \\
&- \Bigg(2\dot{\chi}^{\alpha}\dot{\psi}_{\alpha} - 2\boldsymbol{\nabla}\chi^{\alpha}\boldsymbol{\nabla}\psi_{\alpha} - 2\tilde{m}^2\psi^{\alpha}\chi_{\alpha} + \frac{i}{2}\tilde{S}^{\beta}{}_{\alpha}\left(\sigma^0\right)_{\beta\dot{\gamma}} \dot{\tilde{R}}^{\dot{\gamma}\alpha} - \frac{i}{2}\tilde{S}^{\beta\alpha}\boldsymbol{\sigma}_{\beta\dot{\gamma}} \cdot \boldsymbol{\nabla}\tilde{R}^{\dot{\gamma}}{}_{\alpha} \\
&- \frac{i}{2}\tilde{R}_{\dot{\beta}\alpha}\left(\bar{\sigma}^0\right)^{\dot{\beta}}{}_{\gamma}\dot{\tilde{S}}^{\gamma\alpha} + \frac{i}{2}\tilde{R}_{\dot{\beta}\alpha}\bar{\boldsymbol{\sigma}}^{\dot{\beta}\gamma} \cdot \boldsymbol{\nabla}\tilde{S}_{\gamma}{}^{\alpha} + \frac{\tilde{m}}{2}\tilde{S}^{\beta\alpha}\tilde{S}_{\beta\alpha} - \frac{\tilde{m}}{2}\tilde{R}_{\dot{\beta}\alpha}\tilde{R}^{\dot{\beta}\alpha} \Bigg) \Bigg) \\
=& \int \mathrm{d}^3\mathbf{x} \Bigg(2\dot{\chi}\dot{\psi} + 2\boldsymbol{\nabla}\chi\boldsymbol{\nabla}\psi + 2\tilde{m}^2\psi\chi - \frac{i}{2}\mathrm{Tr}\Big(\tilde{S}^T\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\tilde{R}\Big) \\
&- \frac{i}{2}\mathrm{Tr}\Big(\tilde{R}^T\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}\tilde{S}\Big) + \frac{\tilde{m}}{2}\mathrm{Tr}\Big(\tilde{S}^T\tilde{S}\Big) + \frac{\tilde{m}}{2}\mathrm{Tr}\Big(\tilde{R}^T\tilde{R}\Big) \Bigg) \\
=& \mathcal{H} \qquad (5.70)
\end{aligned}
$$

It turns out that the Hamiltonian as derived from Legendre transformation is identical to the one derived using the supersymmetry algebra. This result is especially intriguing as it paves the way for a significantly simplified derivation of the Hamiltonian in position space involving fermionic fields with mass dimension one. It represents an extension of the commonly used formalism of Legendre transformations to component fields with non-standard mass dimensions.

Chapter 6

The Hamiltonian in Momentum Space

In Section 3.4 the on-shell Lagrangian was derived. A review of the final result in equation (3.88) reveals that the on-shell Lagrangian doesn't contain any cross terms connecting the component fields with their Hermitian conjugates. Furthermore, according to Sections 3.2.2 and 5.1 all commutators and anticommutators between barred and unbarred component fields vanish identically. Therfore, it is sufficient to restrict the following discussion to the unbarred terms of the on-shell Lagrangian

$$\begin{aligned}\mathcal{L} &= \partial_\mu \chi \partial^\mu \psi + \partial_\mu \psi \partial^\mu \chi - \frac{m^2}{4}\psi\chi - \frac{m^2}{4}\chi\psi \\ &\quad + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T \partial\!\!\!/ \tilde{R}\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T \bar{\partial\!\!\!/} \tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{S}^T \tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{R}^T \tilde{R}\right).\end{aligned} \tag{6.1}$$

To comply with the convention that the mass term is defined such that the component fields satisfy a Klein-Gordon operator of the form $\Box + m^2$ the mass needs to be rescaled appropriately. The new mass $\tilde{m}$ is introduced which is related to the old mass m by

$$\tilde{m} = \frac{m}{2}. \tag{6.2}$$

This results in the on-shell Lagrangian

$$\begin{aligned}\mathcal{L} &= \partial_\mu \chi \partial^\mu \psi + \partial_\mu \psi \partial^\mu \chi - \tilde{m}^2\psi\chi - \tilde{m}^2\chi\psi \\ &\quad + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T \partial\!\!\!/ \tilde{R}\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T \bar{\partial\!\!\!/} \tilde{S}\right) - \frac{\tilde{m}}{2}\mathrm{Tr}\left(\tilde{S}^T \tilde{S}\right) - \frac{\tilde{m}}{2}\mathrm{Tr}\left(\tilde{R}^T \tilde{R}\right).\end{aligned} \tag{6.3}$$

The remainder of this section discusses the derivation of the Hamiltonian in momentum space in detail. First, the on-shell equations of motions are derived in Section 6.1. It is followed by the calculation of the fermionic component fields in momentum space and their anticommutation relations in Section 6.2. Subsequently, in Section 6.3 similar calculations are repeated for the bosonic component fields. Afterwards, the Hamiltonian in momentum space is derived in Section 6.4. Finally, in Section 6.5 it is worked out which of the momentum space operators are creation operators and which ones are annihilation operators.

6.1 The On-shell Equations of Motion

The on-shell equations of motion are calculated in the usual way using the Euler-Lagrange equations. Starting from the Lagrangian in equation (6.3) It can then be shown that the fermionic component field ψ satisfies the Klein-Gordon equation

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \chi^\alpha \right)} \right) - \frac{\partial \mathcal{L}}{\partial \chi^\alpha} \\ &= \left(\Box + \tilde{m}^2 \right) \psi_\alpha \, . \end{aligned} \tag{6.4}$$

As the Lagrangian is symmetric under the exchange of χ and ψ it is obvious that χ satisfies a Klein-Gordon equation as well

$$0 = \left(\Box + \tilde{m}^2 \right) \chi_\alpha \, . \tag{6.5}$$

Similar calculations can now be repeated for the bosonic second rank spinor fields $\tilde{S}$ and $\tilde{R}$. The equation of motion for $\tilde{S}$ is found to be

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \tilde{S}^{\tau\omega} \right)} \right) - \frac{\partial \mathcal{L}}{\partial \tilde{S}^{\tau\omega}} \\ &= -i \partial\!\!\!/_{\tau}{}^{\dot{\beta}} \tilde{R}_{\dot{\beta}\omega} - \tilde{m} \tilde{S}_{\tau\omega} \, , \end{aligned} \tag{6.6}$$

while the equation of motion for $\tilde{R}$ is given by

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \tilde{R}^{\dot{\tau}\omega} \right)} \right) - \frac{\partial \mathcal{L}}{\partial \tilde{R}^{\dot{\tau}\omega}} \\ &= -i \bar{\partial\!\!\!/}_{\dot{\tau}}{}^{\beta} \tilde{S}_{\beta\omega} + \tilde{m} \tilde{R}_{\dot{\tau}\omega} \, . \end{aligned} \tag{6.7}$$

This means that both second rank spinor fields satisfy Dirac type equations of motion.

In standard field theory any field that satisfies the Dirac equation automatically obeys the Klein-Gordon equation as well. To complete the analogy – even though the role of fermionic and bosonic fields is exchanged – it remains to be shown that $\tilde{S}$ and $\tilde{R}$ satisfy the Klein-Gordon equation in addition to the Dirac equation. This can be done by inserting the two equations into each other

$$\begin{aligned} 0 &= \tilde{S}_{\beta\alpha} + \frac{i}{\tilde{m}} \partial\!\!\!/_{\beta}{}^{\dot{\gamma}} \tilde{R}_{\dot{\gamma}\alpha} \\ &= \tilde{S}_{\beta\alpha} + \frac{i}{\tilde{m}} \partial\!\!\!/_{\beta}{}^{\dot{\gamma}} \frac{i}{m} \bar{\partial\!\!\!/}_{\dot{\gamma}}{}^{\delta} \tilde{S}_{\delta\alpha} \\ &= \left(\Box + \tilde{m}^2\right) \tilde{S}_{\beta\alpha} \,, \end{aligned} \tag{6.8}$$

$$\begin{aligned} 0 &= \tilde{R}_{\dot{\tau}\omega} - \frac{i}{\tilde{m}} \bar{\partial\!\!\!/}_{\dot{\tau}}{}^{\beta} \tilde{S}_{\beta\omega} \\ &= \tilde{R}_{\dot{\tau}\omega} + \frac{i}{\tilde{m}} \bar{\partial\!\!\!/}_{\dot{\tau}}{}^{\beta} \frac{i}{m} \partial\!\!\!/_{\beta}{}^{\dot{\gamma}} \tilde{R}_{\dot{\gamma}\omega} \\ &= \left(\Box + \tilde{m}^2\right) \tilde{R}_{\dot{\tau}\omega} \,. \end{aligned} \tag{6.9}$$

These results confirm that both $\tilde{S}$ and $\tilde{R}$ not only satisfy the Dirac-equation but also the Klein-Gordon equation.

Usually fermionic fields satisfy the Dirac equation while bosonic fields satisfy the Klein-Gordon equation. However, the analysis in this section shows that in the present scenario the fermionic component fields satisfy the Klein Gordon equation while the bosonic component fields satisfy Dirac type equations. This means that the equations of motion for a theory based on the general multiplet with one free spinor index featuring fermionic fields with mass dimension one are of exactly opposite type to those in standard field theory.

6.2 Second Quantisation of the Fermionic Component Fields in Momentum Space

In equation (6.4) it was shown that ψ satisfies the Klein-Gordon equation in position space. Furthermore, the Fourier expansion of the component field in momentum space is given by

$$\psi_\alpha(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \psi_\alpha(\mathbf{p}, t) \,. \tag{6.10}$$

The Klein-Gordon equation of ψ in position space can then be expressed as

$$\begin{aligned}
0 &= \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \tilde{m}^2\right)\psi_\alpha(\mathbf{x},t) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \tilde{m}^2\right) e^{i\mathbf{p}\cdot\mathbf{x}}\psi_\alpha(\mathbf{p},t) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right)\psi_\alpha(\mathbf{p},t)\,,
\end{aligned} \tag{6.11}$$

where the energy $\omega_{\mathbf{p}}$ is given by

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + \tilde{m}^2}\,. \tag{6.12}$$

Therefore, ψ satisfies a second order differential equation in momentum space

$$0 = \left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right)\psi_\alpha(\mathbf{p},t)\,, \tag{6.13}$$

which has the general solution

$$\psi_\alpha(\mathbf{p},t) = u_\alpha(\mathbf{p})e^{i\omega_{\mathbf{p}}t} + v_\alpha(\mathbf{p})e^{-i\omega_{\mathbf{p}}t}\,. \tag{6.14}$$

This results in a momentum space expansion for the position space component field ψ of

$$\begin{aligned}
\psi_\alpha(\mathbf{x},t) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\left(u_\alpha(\mathbf{p})e^{i\omega_{\mathbf{p}}t} + v_\alpha(\mathbf{p})e^{-i\omega_{\mathbf{p}}t}\right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(u_\alpha(-\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})e^{-ip\cdot x}\right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(u^1_\alpha(\mathbf{p})e^{ip\cdot x} + v^1_\alpha(\mathbf{p})e^{-ip\cdot x}\right)\,.
\end{aligned} \tag{6.15}$$

To derive this expansion it was used that the integral over space doesn't change under the transformation $\mathbf{p} \to -\mathbf{p}$ and that the energy is solely dependent on the magnitude of the momentum but not its direction $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$. To adopt standard conventions for the momentum label of the component fields it is necessary to make the substitution $u_\alpha(-\mathbf{p}) = u^1_\alpha(\mathbf{p})$ and $v_\alpha(\mathbf{p}) = v^1_\alpha(\mathbf{p})$. The substitution for $v_\alpha(\mathbf{p})$ is not really required at this point and was only performed to denote all momentum space operators that correspond to ψ with superscript 1 while those corresponding to χ will be denoted with superscript 2. This distinction is important as all four momentum space operators are ad hoc independent.

Identical calculations can be performed for the second fermionic component field χ. As they are in perfect analogy to those for ψ it is sufficient to adapt the final result for ψ from equation (6.15) and modify it appropriately

$$\chi_\alpha(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(u^2_\alpha(\mathbf{p}) e^{ip\cdot x} + v^2_\alpha(\mathbf{p}) e^{-ip\cdot x} \right) . \tag{6.16}$$

6.2.1 The Fermionic Component Fields in Momentum Space

So far equations (6.15) and (6.16) relate the component fields in position space ψ and χ with the momentum space operators u^1, u^2, v^1, and v^2. These equations for the momentum space operators can be inverse Fourier transformed and the resulting equations are solvable for the momentum space operators. However, the two Fourier expansions in momentum space at hand contain four independent momentum space operators and thus cannot be solved uniquely. Therefore, two additional independent equations are needed. If it is recalled that the anticommutation relations in position space also contained the time derivatives of the fermionic component fields $\dot{\chi}$ and $\dot{\psi}$ it yields the sought after equations

$$\dot{\psi}_\alpha(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} i\omega_{\mathbf{p}} \left(u^1_\alpha(\mathbf{p}) e^{ip\cdot x} - v^1_\alpha(\mathbf{p}) e^{-ip\cdot x} \right) , \tag{6.17}$$

$$\dot{\chi}_\alpha(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} i\omega_{\mathbf{p}} \left(u^2_\alpha(\mathbf{p}) e^{ip\cdot x} - v^2_\alpha(\mathbf{p}) e^{-ip\cdot x} \right) . \tag{6.18}$$

Together with equations (6.15) and (6.16) they form a complete set of equations that has a unique solution. The inverse Fourier transformed of the four component field expansions in momentum space are then found to be

$$\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \psi_\alpha(\mathbf{x},t) = u^1_\alpha(-\mathbf{p}') e^{i\omega_{\mathbf{p}'}t} + v^1_\alpha(\mathbf{p}') e^{-i\omega_{\mathbf{p}'}t} , \tag{6.19}$$

$$\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \dot{\psi}_\alpha(\mathbf{x},t) = i\omega_{\mathbf{p}'} u^1_\alpha(-\mathbf{p}') e^{i\omega_{\mathbf{p}'}t} - i\omega_{\mathbf{p}'} v^1_\alpha(\mathbf{p}') e^{-i\omega_{\mathbf{p}'}t} , \tag{6.20}$$

$$\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \chi_\alpha(\mathbf{x},t) = u^2_\alpha(-\mathbf{p}') e^{i\omega_{\mathbf{p}'}t} + v^2_\alpha(\mathbf{p}') e^{-i\omega_{\mathbf{p}'}t} , \tag{6.21}$$

$$\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \dot{\chi}_\alpha(\mathbf{x},t) = i\omega_{\mathbf{p}'} u^2_\alpha(-\mathbf{p}') e^{i\omega_{\mathbf{p}'}t} - i\omega_{\mathbf{p}'} v^2_\alpha(\mathbf{p}') e^{-i\omega_{\mathbf{p}'}t} . \tag{6.22}$$

These equations are added and subtracted to solve for the momentum space operators

$$u^1_\alpha(\mathbf{p}) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\, e^{-ip\cdot x}\left(\psi_\alpha(\mathbf{x},t) - \frac{i}{\omega_\mathbf{p}}\dot{\psi}_\alpha(\mathbf{x},t)\right) , \tag{6.23}$$

$$v^1_\alpha(\mathbf{p}) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\, e^{ip\cdot x}\left(\psi_\alpha(\mathbf{x},t) + \frac{i}{\omega_\mathbf{p}}\dot{\psi}_\alpha(\mathbf{x},t)\right) , \tag{6.24}$$

$$u^2_\alpha(\mathbf{p}) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\, e^{-ip\cdot x}\left(\chi_\alpha(\mathbf{x},t) - \frac{i}{\omega_\mathbf{p}}\dot{\chi}_\alpha(\mathbf{x},t)\right) , \tag{6.25}$$

$$v^2_\alpha(\mathbf{p}) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\, e^{ip\cdot x}\left(\chi_\alpha(\mathbf{x},t) + \frac{i}{\omega_\mathbf{p}}\dot{\chi}_\alpha(\mathbf{x},t)\right) . \tag{6.26}$$

6.2.2 The Anticommutation Relations in Momentum Space

With the knowledge of the position space expansion of the momentum space operators the anticommutators between the component fields in momentum space can be evaluated. This is done using the position space expansion of the momentum space operators and evaluating the arising anticommutators between the position space component fields that are known from Section 5.1

$$\begin{aligned}\left\{u^1_\alpha(\mathbf{p}), u^1_\beta(\mathbf{p}')\right\} &= \frac{1}{4}\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{x}'\, e^{-i(p\cdot x + p'\cdot x')} \\ &\quad\times \left\{\psi_\alpha(\mathbf{x},t) - \frac{i}{\omega_\mathbf{p}}\dot{\psi}_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{x}',t) - \frac{i}{\omega_{\mathbf{p}'}}\dot{\psi}_\beta(\mathbf{x}',t)\right\} \\ &= \frac{1}{4}\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{x}'\, e^{-i(p\cdot x + p'\cdot x')}\left(\left\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{x}',t)\right\}\right. \\ &\quad - \frac{i}{\omega_{\mathbf{p}'}}\left\{\psi_\alpha(\mathbf{x},t), \dot{\psi}_\beta(\mathbf{x}',t)\right\} - \frac{i}{\omega_\mathbf{p}}\left\{\dot{\psi}_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{x}',t)\right\} \\ &\quad \left. - \frac{1}{\omega_\mathbf{p}\omega_{\mathbf{p}'}}\left\{\dot{\psi}_\alpha(\mathbf{x},t), \dot{\psi}_\beta(\mathbf{x}',t)\right\}\right) \\ &= 0\,,\end{aligned} \tag{6.27}$$

where it was used that $\{\psi, \psi\} = \left\{\psi, \dot{\psi}\right\} = 0$ implies

$$0 = \partial_0\left\{\psi, \dot{\psi}\right\} = \left\{\dot{\psi}, \dot{\psi}\right\} + \left\{\psi, \ddot{\psi}\right\} = \left\{\dot{\psi}, \dot{\psi}\right\} - \omega^2_\mathbf{p}\left\{\psi, \psi\right\} = \left\{\dot{\psi}, \dot{\psi}\right\} . \tag{6.28}$$

It has to be emphasised that the second last step is mathematically not exact, as the factor $\omega_\mathbf{p}$ is obviously dependent on the momentum and therefore cannot simply be pulled in front of the component field in momentum space which is an integral over the momentum. A mathematically exact treatment would involve writing down the Fourier expansion of both component fields and realizing that up to the prefactor of ω^2 the anticommutation relations are identical to those from the calculation of $\{\psi, \psi\}$. This implies that the anticommutator of $\left\{\psi, \ddot{\psi}\right\}$ must vanish identically as well.

The remaining anticommutation relations between the fermionic momentum space operators can be calculated in analogy to the first one

$$\left\{u^1_\alpha(\mathbf{p}), u^2_\beta(\mathbf{p}')\right\} = 0\,, \tag{6.29}$$

$$\left\{u^1_\alpha(\mathbf{p}), v^1_\beta(\mathbf{p}')\right\} = 0\,, \tag{6.30}$$

$$\left\{u^1_\alpha(\mathbf{p}), v^2_\beta(\mathbf{p}')\right\} = -\,(2\pi)^3\,\frac{1}{4\omega_{\mathbf{p}}}\epsilon_{\alpha\beta}\delta(\mathbf{p}-\mathbf{p}')\,, \tag{6.31}$$

$$\left\{u^2_\alpha(\mathbf{p}), u^2_\beta(\mathbf{p}')\right\} = 0\,, \tag{6.32}$$

$$\left\{u^2_\alpha(\mathbf{p}), v^1_\beta(\mathbf{p}')\right\} = -\,(2\pi)^3\,\frac{1}{4\omega_{\mathbf{p}}}\epsilon_{\alpha\beta}\delta(\mathbf{p}-\mathbf{p}')\,, \tag{6.33}$$

$$\left\{u^2_\alpha(\mathbf{p}), v^2_\beta(\mathbf{p}')\right\} = 0\,, \tag{6.34}$$

$$\left\{v^1_\alpha(\mathbf{p}), v^1_\beta(\mathbf{p}')\right\} = 0\,, \tag{6.35}$$

$$\left\{v^1_\alpha(\mathbf{p}), v^2_\beta(\mathbf{p}')\right\} = 0\,, \tag{6.36}$$

$$\left\{v^2_\alpha(\mathbf{p}), v^2_\beta(\mathbf{p}')\right\} = 0\,. \tag{6.37}$$

It is found that only two of the anticommutators $\left\{u^1, v^2\right\}$ and $\left\{u^2, v^1\right\}$ are nonzero while all other anticommutators vanish identically.

6.3 Second Quantisation of the Bosonic Component Fields in Momentum Space

From equation (6.8) it is known that $\tilde{S}$ satisfies the Klein-Gordon equation. Additionally, the second rank spinor field in position space can be expanded in terms of its Fourier modes in momentum space

$$\tilde{S}_{\beta\alpha}(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{S}_{\beta\alpha}(\mathbf{p}, t)\,. \tag{6.38}$$

Starting from the Klein-Gordon equation for the position space component field then yields

$$\begin{aligned}
0 &= \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \tilde{m}^2\right) \tilde{S}_{\beta\alpha}(\mathbf{x}, t) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \tilde{m}^2\right) e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{S}_{\beta\alpha}(\mathbf{p}, t) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right) \tilde{S}_{\beta\alpha}(\mathbf{p}, t)\,,
\end{aligned} \tag{6.39}$$

where the energy $\omega_{\mathbf{p}}$ is again given by equation (6.12). Therefore, $\tilde{S}$ satisfies a second order differential equation in momentum space as well

$$0 = \left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right) \tilde{S}_{\beta\alpha}(\mathbf{p}, t)\,, \tag{6.40}$$

which has the general solution

$$\tilde{S}_{\beta\alpha}(\mathbf{p}, t) = a^1_{\beta\alpha}(\mathbf{p}) e^{i\omega_{\mathbf{p}} t} + a^2_{\beta\alpha}(\mathbf{p}) e^{-i\omega_{\mathbf{p}} t}\,. \tag{6.41}$$

This results in a momentum space expansion for $\tilde{S}$ of

$$\begin{aligned}
\tilde{S}_{\beta\alpha}(\mathbf{x}, t) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \left(a^1_{\beta\alpha}(\mathbf{p}) e^{i\omega_{\mathbf{p}} t} + a^2_{\beta\alpha}(\mathbf{p}) e^{-i\omega_{\mathbf{p}} t}\right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(w^1_{\beta\alpha}(-\mathbf{p}) e^{i\omega_{\mathbf{p}} t - i\mathbf{p}\cdot\mathbf{x}} + w^2_{\beta\alpha}(\mathbf{p}) e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(w^1_{\beta\alpha}(\mathbf{p}) e^{ip\cdot x} + w^2_{\beta\alpha}(\mathbf{p}) e^{-ip\cdot x}\right)\,.
\end{aligned} \tag{6.42}$$

Here, the conventional notation was recovered by making the substitutions $a^1(-\mathbf{p}) = w^1(\mathbf{p})$ and $a^2(\mathbf{p}) = w^2(\mathbf{p})$ which is in perfect analogy to those for the fermionic momentum space operators that were discussed earlier. The same calculation can now be repeated for $\tilde{R}$ and it is found that

$$\tilde{R}_{\dot{\beta}\alpha}(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(y^1_{\dot{\beta}\alpha}(\mathbf{p}) e^{ip\cdot x} + y^2_{\dot{\beta}\alpha}(\mathbf{p}) e^{-ip\cdot x}\right)\,. \tag{6.43}$$

It was also shown in equations (6.6) and (6.7) that $\tilde{S}$ and $\tilde{R}$ satisfy Dirac type equations. These additional relations between the component fields can be thought of as boundary conditions or constraints, that restrict the general solutions of the Klein-Gordon equations.

Inserting the component field expansion in momentum space into the Dirac equation for $\tilde{S}$ yields the relation

$$
\begin{aligned}
0 &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(w^1_{\beta\alpha}(\mathbf{p}) e^{ip\cdot x} + w^2_{\beta\alpha}(\mathbf{p}) e^{-ip\cdot x} \right) \\
&\quad + \frac{i}{\tilde{m}} \partial\!\!\!/_{\beta}{}^{\dot{\gamma}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(y^1_{\dot{\gamma}\alpha}(\mathbf{p}) e^{ip\cdot x} + y^2_{\dot{\gamma}\alpha}(\mathbf{p}) e^{-ip\cdot x} \right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(\left(w^1_{\beta\alpha}(\mathbf{p}) - \frac{1}{\tilde{m}} p \cdot \sigma_{\beta}{}^{\dot{\gamma}} y^1_{\dot{\gamma}\alpha}(\mathbf{p}) \right) e^{ip\cdot x} \right. \\
&\quad \left. + \left(w^2_{\beta\alpha}(\mathbf{p}) + \frac{1}{\tilde{m}} p \cdot \sigma_{\beta}{}^{\dot{\gamma}} y^2_{\dot{\gamma}\alpha}(\mathbf{p}) \right) e^{-ip\cdot x} \right) .
\end{aligned}
\tag{6.44}
$$

This indicates that only two of the four bosonic second rank momentum space operators that appear in the momentum space expansion of the bosonic component fields are independent. A brief inspection of this relation immediately reveals that w^1 is proportional to y^1 while w^2 is proportional to y^2

$$
w^1_{\beta\alpha}(\mathbf{p}) = \frac{1}{\tilde{m}} p \cdot \sigma_{\beta}{}^{\dot{\gamma}} y^1_{\dot{\gamma}\alpha}(\mathbf{p}) \, , \tag{6.45}
$$

$$
w^2_{\beta\alpha}(\mathbf{p}) = -\frac{1}{\tilde{m}} p \cdot \sigma_{\beta}{}^{\dot{\gamma}} y^2_{\dot{\gamma}\alpha}(\mathbf{p}) \, . \tag{6.46}
$$

If the momentum space expansions of $\tilde{S}$ and $\tilde{R}$ are instead inserted into the Dirac equation for $\tilde{R}$ it leads to the relation

$$
\begin{aligned}
0 &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(y^1_{\dot{\tau}\omega}(\mathbf{p}) e^{ip\cdot x} + y^2_{\dot{\tau}\omega}(\mathbf{p}) e^{-ip\cdot x} \right) \\
&\quad - \frac{i}{\tilde{m}} \bar{\partial\!\!\!/}_{\dot{\tau}}{}^{\beta} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(w^1_{\beta\omega}(\mathbf{p}) e^{ip\cdot x} + w^2_{\beta\omega}(\mathbf{p}) e^{-ip\cdot x} \right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(\left(y^1_{\dot{\tau}\omega}(\mathbf{p}) + \frac{1}{\tilde{m}} p \cdot \bar{\sigma}_{\dot{\tau}}{}^{\beta} w^1_{\beta\omega}(\mathbf{p}) \right) e^{ip\cdot x} \right. \\
&\quad \left. + \left(y^2_{\dot{\tau}\omega}(\mathbf{p}) - \frac{1}{\tilde{m}} p \cdot \bar{\sigma}_{\dot{\tau}}{}^{\beta} w^2_{\beta\omega}(\mathbf{p}) \right) e^{-ip\cdot x} \right) .
\end{aligned}
\tag{6.47}
$$

Again, the solution of this equation can be read off immediately and yields expressions for y^1 in terms of w^1 and for y^2 in terms of w^2

$$
y^1_{\dot{\tau}\omega}(\mathbf{p}) = -\frac{1}{\tilde{m}} p \cdot \bar{\sigma}_{\dot{\tau}}{}^{\beta} w^1_{\beta\omega}(\mathbf{p}) \, , \tag{6.48}
$$

$$
y^2_{\dot{\tau}\omega}(\mathbf{p}) = \frac{1}{\tilde{m}} p \cdot \bar{\sigma}_{\dot{\tau}}{}^{\beta} w^2_{\beta\omega}(\mathbf{p}) \, . \tag{6.49}
$$

As there are no further guidelines on which of the component fields are more fundamental – with the w's having two undotted indices while the y's have one dotted and one undotted index – it is possible to choose one of the momentum space operators with index one and one with index 2 as independent one. The remaining momentum space operators can then be expressed in terms of the independent operators. However, it proves useful to make the specific choice with w^1 and w^2 as independent momentum space operators which streamlines the evaluation of the commutation relations significantly

$$\tilde{S}_{\beta\alpha}(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(w^1_{\beta\alpha}(\mathbf{p}) e^{ip\cdot x} + w^2_{\beta\alpha}(\mathbf{p}) e^{-ip\cdot x} \right) , \tag{6.50}$$

$$\tilde{R}_{\dot{\beta}\alpha}(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} \left(-w^1_{\gamma\alpha}(\mathbf{p}) e^{ip\cdot x} + w^2_{\gamma\alpha}(\mathbf{p}) e^{-ip\cdot x} \right) . \tag{6.51}$$

It can be seen that the resulting momentum space expansions look very similar besides an overall prefactor of $p\cdot\bar{\sigma}/\tilde{m}$ for $\tilde{R}$ as well as an additional sign flip of the first term.

6.3.1 The Bosonic Component Fields in Momentum Space

To solve for the component fields in momentum space it is again necessary to inverse Fourier transform the momentum space expansions of the component fields in position space

$$\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \tilde{S}_{\beta\alpha}(\mathbf{x},t) &= \int \frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}'\cdot\mathbf{x}} \left(w^1_{\beta\alpha}(\mathbf{p}) e^{ip\cdot x} + w^2_{\beta\alpha}(\mathbf{p}) e^{-ip\cdot x} \right) \\
&= w^1_{\beta\alpha}(-\mathbf{p}') e^{i\omega_{\mathbf{p}'}t} + w^2_{\beta\alpha}(\mathbf{p}') e^{-i\omega_{\mathbf{p}'}t} , \qquad (6.52) \\
\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \tilde{R}_{\dot{\beta}\alpha}(\mathbf{x},t) &= \int \frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} \left(-w^1_{\gamma\alpha}(\mathbf{p}) e^{ip\cdot x} + w^2_{\gamma\alpha}(\mathbf{p}) e^{-ip\cdot x} \right) \\
&= -\frac{1}{\tilde{m}} \left(\omega_{\mathbf{p}'} \left(\bar{\sigma}^0\right)_{\dot{\beta}}{}^{\gamma} + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot{\beta}}{}^{\gamma} \right) w^1_{\gamma\alpha}(-\mathbf{p}') e^{i\omega_{\mathbf{p}'}t} \\
&\quad + \frac{1}{\tilde{m}} p'\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} w^2_{\gamma\alpha}(\mathbf{p}') e^{-i\omega_{\mathbf{p}'}t} . \qquad (6.53)
\end{aligned}$$

As there are only two independent bosonic operators in position space, w^1 and w^2, the inverse Fourier transformed of $\tilde{R}$ and $\tilde{S}$ are sufficient to solve for the momentum space operators. A close look at these results reveals that w^2 can be eliminated by subtracting the inverse Fourier transformed of $\tilde{R}$ from the inverse Fourier transformed of $\tilde{S}$ times a prefactor $p' \cdot \bar{\sigma}/\tilde{m}$

$$\frac{2}{\tilde{m}}\omega_{\mathbf{p}'}\left(\bar{\sigma}^0\right)_{\dot{\beta}}{}^{\gamma} w^1_{\gamma\alpha}(-\mathbf{p}')e^{i\omega_{\mathbf{p}'}t} = \frac{1}{\tilde{m}}p' \cdot \bar{\sigma}_{\dot{\beta}}{}^{\gamma} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \tilde{S}_{\gamma\alpha}(\mathbf{x},t) - \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \tilde{R}_{\dot{\beta}\alpha}(\mathbf{x},t)\,. \tag{6.54}$$

If both sides are contracted with σ^0 with appropriate choice for the spinor index structure it is found that

$$-\frac{2}{\tilde{m}}\omega_{\mathbf{p}'}\epsilon_{\delta}{}^{\gamma} w^1_{\gamma\alpha}(-\mathbf{p}')e^{i\omega_{\mathbf{p}'}t} = \frac{1}{\tilde{m}}\left(\sigma^0\right)_{\delta}{}^{\dot{\beta}} p' \cdot \bar{\sigma}_{\dot{\beta}}{}^{\gamma} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} S_{\gamma\alpha}(\mathbf{x},t) - \left(\sigma^0\right)_{\delta}{}^{\dot{\beta}} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} R_{\dot{\beta}\alpha}(\mathbf{x},t)\,, \tag{6.55}$$

where it was used that

$$\left(\sigma^0\right)_{\delta}{}^{\dot{\beta}}\left(\bar{\sigma}^0\right)_{\dot{\beta}}{}^{\gamma} = -\left(\sigma^0\bar{\sigma}^0\right)_{\delta}{}^{\gamma} = -\epsilon_{\delta}{}^{\gamma}\,. \tag{6.56}$$

This equation can immediately be solved for w^1

$$w^1_{\delta\alpha}(-\mathbf{p}') = -\frac{\tilde{m}}{2\omega_{\mathbf{p}'}}\left(\sigma^0\right)_{\delta}{}^{\dot{\beta}} e^{-i\omega_{\mathbf{p}'}t} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \times \left(\frac{1}{\tilde{m}}p' \cdot \bar{\sigma}_{\dot{\beta}}{}^{\gamma} \tilde{S}_{\gamma\alpha}(\mathbf{x},t) - \tilde{R}_{\dot{\beta}\alpha}(\mathbf{x},t)\right)\,. \tag{6.57}$$

It can be seen that w^1 depends on $-\mathbf{p}'$ while it is conventional to write the operators such that they depend on the positive momentum. Replacing $-\mathbf{p}' \to \mathbf{p}'$ leads to a sign flip of all terms containing $\mathbf{p}'$ while the energy $\omega_{\mathbf{p}'}$ is preserved

$$w^1_{\delta\alpha}(\mathbf{p}') = -\frac{\tilde{m}}{2\omega_{\mathbf{p}'}}\left(\sigma^0\right)_{\delta}{}^{\dot{\beta}} \int \mathrm{d}^3\mathbf{x}\, e^{-ip'\cdot x} \times \left(\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}'}\left(\bar{\sigma}^0\right)_{\dot{\beta}}{}^{\gamma} + \mathbf{p}' \cdot \bar{\boldsymbol{\sigma}}_{\dot{\beta}}{}^{\gamma}\right) \tilde{S}_{\gamma\alpha}(\mathbf{x},t) - \tilde{R}_{\dot{\beta}\alpha}(\mathbf{x},t)\right)\,. \tag{6.58}$$

Alternatively it is possible to add equations (6.52) and (6.53) to eliminate w^1

$$\frac{2\omega_{\mathbf{p}'}}{\tilde{m}}\left(\bar{\sigma}^0\right)_{\dot{\beta}}{}^{\gamma} w^2_{\gamma\alpha}(\mathbf{p}')e^{-i\omega_{\mathbf{p}'}t} = \frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}'}\left(\bar{\sigma}^0\right)_{\dot{\beta}}{}^{\gamma} + \mathbf{p}' \cdot \bar{\boldsymbol{\sigma}}_{\dot{\beta}}{}^{\gamma}\right) \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \tilde{S}_{\gamma\alpha}(\mathbf{x},t) + \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot\mathbf{x}} \tilde{R}_{\dot{\beta}\alpha}(\mathbf{x},t)\,. \tag{6.59}$$

Again both sides need to be contracted with σ^0

$$\frac{2\omega_{\mathbf{p}'}}{\tilde{m}}\epsilon_\delta{}^\gamma w^2_{\gamma\alpha}(\mathbf{p}')e^{-i\omega_{\mathbf{p}'}t} = -\left(\sigma^0\right)_\delta{}^{\dot\beta}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}'\cdot x}$$
$$\times\left(\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}'}\left(\bar\sigma^0\right)_{\dot\beta}{}^\gamma + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot\beta}{}^\gamma\right)\tilde{S}_{\gamma\alpha}(\mathbf{x},t) + \tilde{R}_{\dot\beta\alpha}(\mathbf{x},t)\right). \quad (6.60)$$

Finally, this intermediate result can be solved for the remaining bosonic momentum space operator

$$w^2_{\delta\alpha}(\mathbf{p}') = -\frac{\tilde{m}}{2\omega_{\mathbf{p}'}}\left(\sigma^0\right)_\delta{}^{\dot\beta}\int \mathrm{d}^3\mathbf{x}\, e^{ip'\cdot x}$$
$$\times\left(\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}'}\left(\bar\sigma^0\right)_{\dot\beta}{}^\gamma + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot\beta}{}^\gamma\right)\tilde{S}_{\gamma\alpha}(\mathbf{x},t) + \tilde{R}_{\dot\beta\alpha}(\mathbf{x},t)\right). \quad (6.61)$$

6.3.2 The Commutation Relations in Momentum Space

The commutation relations in momentum space can now be determined by expressing the momentum space operators in terms of their position space expansions as derived in equations (6.58) and (6.61). It is then possible to use the commutation relations in position space to evaluate their momentum space counterparts.

$$\left[w^1_{\beta\alpha}(\mathbf{p}), w^1_{\gamma\delta}(\mathbf{p}')\right] = \frac{\tilde{m}^2}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}\left(\sigma^0\right)_\beta{}^{\dot\kappa}\left(\sigma^0\right)_\gamma{}^{\dot\tau}\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{x}'\, e^{-i(p\cdot x+p'\cdot x')}$$
$$\left[\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}}\left(\bar\sigma^0\right)_{\dot\kappa}{}^\epsilon + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}_{\dot\kappa}{}^\epsilon\right)S_{\epsilon\alpha}(\mathbf{x},t) - R_{\dot\kappa\alpha}(\mathbf{x},t),\right.$$
$$\left.\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}'}\left(\bar\sigma^0\right)_{\dot\tau}{}^\omega + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot\tau}{}^\omega\right)S_{\omega\delta}(\mathbf{x}',t) - R_{\dot\tau\delta}(\mathbf{x}',t)\right]. \quad (6.62)$$

The commutation relations for the bosonic second rank spinor fields in position space either vanish or are proportional to δ-functions in position space. Therefore, one of the spatial integrals in all remaining terms gets cancelled by a δ-function

$$\left[w^1_{\beta\alpha}(\mathbf{p}), w^1_{\gamma\delta}(\mathbf{p}')\right] = \frac{\tilde{m}^2}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}\left(\sigma^0\right)_\beta{}^{\dot\kappa}\left(\sigma^0\right)_\gamma{}^{\dot\tau}\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{x}'\, e^{-i(p\cdot x+p'\cdot x')}$$
$$\left(\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}}\left(\bar\sigma^0\right)_{\dot\kappa}{}^\epsilon + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}_{\dot\kappa}{}^\epsilon\right)\epsilon_{\alpha\delta}\left(\sigma^0\right)_{\epsilon\dot\tau}\delta(\mathbf{x}-\mathbf{x}')\right.$$
$$\left.+\frac{1}{\tilde{m}}\left(\omega_{\mathbf{p}'}\left(\bar\sigma^0\right)_{\dot\tau}{}^\omega + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot\tau}{}^\omega\right)\epsilon_{\alpha\delta}\left(\bar\sigma^0\right)_{\dot\kappa\omega}\delta(\mathbf{x}-\mathbf{x}')\right)$$

$$
\begin{aligned}
&= \frac{\tilde{m}^2}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}'}} \left(\sigma^0\right)_\beta{}^{\dot{\kappa}} \left(\sigma^0\right)_\gamma{}^{\dot{\tau}} \int \mathrm{d}^3\mathbf{x}\, e^{-i(p+p')\cdot x} \\
&\left(\frac{1}{\tilde{m}} \left(\omega_{\mathbf{p}} \left(\bar{\sigma}^0\right)_{\dot{\kappa}}{}^{\epsilon} + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}_{\dot{\kappa}}{}^{\epsilon}\right) \epsilon_{\alpha\delta} \left(\sigma^0\right)_{\epsilon\dot{\tau}} \right. \\
&\left. + \frac{1}{\tilde{m}} \left(\omega_{\mathbf{p}'} \left(\bar{\sigma}^0\right)_{\dot{\tau}}{}^{\omega} + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot{\tau}}{}^{\omega}\right) \epsilon_{\alpha\delta} \left(\bar{\sigma}^0\right)_{\dot{\kappa}\omega} \right) .
\end{aligned} \tag{6.63}
$$

The surviving spatial integral then yields a δ-function in momentum space

$$
\begin{aligned}
\left[w^1_{\beta\alpha}(\mathbf{p}), w^1_{\gamma\delta}(\mathbf{p}')\right] &= \frac{\tilde{m}^2}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}'}} \left(\sigma^0\right)_\beta{}^{\dot{\kappa}} \left(\sigma^0\right)_\gamma{}^{\dot{\tau}} (2\pi)^3 e^{-i\left(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}\right)t} \delta\left(\mathbf{p}+\mathbf{p}'\right) \\
&\left(\frac{1}{\tilde{m}} \left(\omega_{\mathbf{p}} \left(\bar{\sigma}^0\right)_{\dot{\kappa}}{}^{\epsilon} + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}_{\dot{\kappa}}{}^{\epsilon}\right) \epsilon_{\alpha\delta} \left(\sigma^0\right)_{\epsilon\dot{\tau}} \right. \\
&\left. + \frac{1}{\tilde{m}} \left(\omega_{\mathbf{p}'} \left(\bar{\sigma}^0\right)_{\dot{\tau}}{}^{\omega} + \mathbf{p}'\cdot\bar{\boldsymbol{\sigma}}_{\dot{\tau}}{}^{\omega}\right) \epsilon_{\alpha\delta} \left(\bar{\sigma}^0\right)_{\dot{\kappa}\omega} \right) .
\end{aligned} \tag{6.64}
$$

Using relations between σ-matrices and the fact that the momentum space δ-function relates the momenta $\mathbf{p} = -\mathbf{p}'$ it can be shown that the commutator vanishes identically

$$
\begin{aligned}
\left[w^1_{\beta\alpha}(\mathbf{p}), w^1_{\gamma\delta}(\mathbf{p}')\right] &= \frac{\tilde{m}}{4\omega^2_{\mathbf{p}}} (2\pi)^3 \epsilon_{\alpha\delta} e^{-2i\omega_{\mathbf{p}}t} \delta\left(\mathbf{p}+\mathbf{p}'\right) \left(p_i \left(\sigma^0\bar{\sigma}^i\right)_{\beta\gamma} - p_i \left(\sigma^0\bar{\sigma}^i\right)_{\gamma\beta}\right) \\
&= 0
\end{aligned} \tag{6.65}
$$

The remaining commutation relations are then found to be

$$
\left[w^1_{\beta\alpha}(\mathbf{p}), w^2_{\gamma\delta}(\mathbf{p}')\right] = \frac{\tilde{m}}{2\omega_{\mathbf{p}}} (2\pi)^3 \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} \delta\left(\mathbf{p}-\mathbf{p}'\right) , \tag{6.66}
$$

$$
\left[w^2_{\beta\alpha}(\mathbf{p}), w^2_{\gamma\delta}(\mathbf{p}')\right] = 0 . \tag{6.67}
$$

Therefore, only one of the commutators between the second rank momentum space operators yields a nonvanishing result.

6.4 The Hamiltonian in Momentum Space

The Hamiltonian in momentum space is derived from the Hamiltonian in position space in equation (5.67) by replacing all component fields in position space with their corresponding momentum space expansions. To simplify the discussion it is convenient to split the Hamiltonian into four parts which represent the contributions of the fermionic component $\mathcal{H}_F$ and its Hermitian conjugate $\mathcal{H}_{\bar{F}}$ as well as the contributions from the bosonic part $\mathcal{H}_B$

and its Hermitian conjugate $\mathcal{H}_{\bar{B}}$.

For the unbarred fermionic part the Fourier expansion of the component fields leads to

$$\begin{aligned}
\mathcal{H}_F =& \int \mathrm{d}\mathbf{x} \left(2\dot{\psi}(x)\dot{\chi}(x) + 2\nabla\psi(x)\cdot\nabla\chi(x) + 2\tilde{m}^2\psi(x)\chi(x)\right) \\
=& \int \frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{p}'}{(2\pi)^6} \left(2\partial_0 \left(u^{1\alpha}(\mathbf{p})e^{ip\cdot x} + v^{1\alpha}(\mathbf{p})e^{-ip\cdot x}\right)\partial^0 \left(u^2_\alpha(\mathbf{p}')e^{ip'\cdot x} + v^2_\alpha(\mathbf{p}')e^{-ip'\cdot x}\right)\right. \\
&+ 2\boldsymbol{\nabla}\left(u^{1\alpha}(\mathbf{p})e^{ip\cdot x} + v^{1\alpha}(\mathbf{p})e^{-ip\cdot x}\right)\boldsymbol{\nabla}\left(u^2_\alpha(\mathbf{p}')e^{ip'\cdot x} + v^2_\alpha(\mathbf{p}')e^{-ip'\cdot x}\right) \\
&\left.+2\tilde{m}^2\left(u^{1\alpha}(\mathbf{p})e^{ip\cdot x} + v^{1\alpha}(\mathbf{p})e^{-ip\cdot x}\right)\left(u^2_\alpha(\mathbf{p}')e^{ip'\cdot x} + v^2_\alpha(\mathbf{p}')e^{-ip'\cdot x}\right)\right) \\
=& \int \frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{p}'}{(2\pi)^6} \left(-\left(2\omega_\mathbf{p}\omega_{\mathbf{p}'} + 2\mathbf{p}\mathbf{p}'\right)\left(u^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{i(p+p')\cdot x} - u^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{i(p-p')\cdot x}\right.\right. \\
&\left.-v^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{-i(p-p')\cdot x} + v^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{-i(p+p')\cdot x}\right) \\
&+ 2\tilde{m}^2\left(u^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{i(p+p')\cdot x} + u^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{i(p-p')\cdot x}\right. \\
&\left.\left.+v^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{-i(p-p')\cdot x} + v^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{-i(p+p')\cdot x}\right)\right) \; . \qquad (6.68)
\end{aligned}$$

It can be seen that the spatial integral results in δ-functions in momentum space that can be used to eliminate one of the momentum space integrals

$$\begin{aligned}
\mathcal{H}_F =& \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} \left(-\left(2\omega_\mathbf{p}\omega_{\mathbf{p}'} + 2\mathbf{p}\mathbf{p}'\right)\left(u^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{i\left(\omega_\mathbf{p}+\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}+\mathbf{p}')\right.\right. \\
&- u^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{i\left(\omega_\mathbf{p}-\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}-\mathbf{p}') - v^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{-i\left(\omega_\mathbf{p}-\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}-\mathbf{p}') \\
&\left.+v^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{-i\left(\omega_\mathbf{p}+\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}+\mathbf{p}')\right) \\
&+ 2\tilde{m}^2\left(u^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{i\left(\omega_\mathbf{p}+\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}+\mathbf{p}') + u^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{i\left(\omega_\mathbf{p}-\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}-\mathbf{p}')\right. \\
&\left.\left.+v^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}')e^{-i\left(\omega_\mathbf{p}-\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}-\mathbf{p}') + v^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}')e^{-i\left(\omega_\mathbf{p}+\omega_{\mathbf{p}'}\right)t}\delta(\mathbf{p}+\mathbf{p}')\right)\right) \\
=& \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(-\left(2\omega^2_\mathbf{p} - 2\mathbf{p}^2 - 2\tilde{m}^2\right)u^{1\alpha}(\mathbf{p})u^2_\alpha(-\mathbf{p}')e^{2i\omega_\mathbf{p}t}\right. \\
&+ \left(2\omega^2_\mathbf{p} + 2\mathbf{p}^2 + 2\tilde{m}^2\right)u^{1\alpha}(\mathbf{p})v^2_\alpha(\mathbf{p}) + \left(2\omega^2_\mathbf{p} + 2\mathbf{p}^2 + 2\tilde{m}^2\right)v^{1\alpha}(\mathbf{p})u^2_\alpha(\mathbf{p}) \\
&\left.- \left(2\omega^2_\mathbf{p} - 2\mathbf{p}^2 - 2\tilde{m}^2\right)v^{1\alpha}(\mathbf{p})v^2_\alpha(-\mathbf{p})e^{-2i\omega_\mathbf{p}t}\right) \; . \qquad (6.69)
\end{aligned}$$

Using the equation for the relativistic energy it is straightforward to show that the time dependent terms cancel identically while the remaining terms are reduced to

$$\mathcal{H}_F = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} 4\omega^2_\mathbf{p}\left(u^1(\mathbf{p})v^2(\mathbf{p}) + v^1(\mathbf{p})u^2(\mathbf{p})\right) \; . \qquad (6.70)$$

The calculations for the barred fermionic part can be performed in perfect analogy and it is found that

$$\begin{aligned}\mathcal{H}_{\bar{F}} &= \int \mathrm{d}\mathbf{x}\left(2\dot{\bar{\chi}}(x)\dot{\bar{\psi}}(x) + 2\nabla\bar{\chi}(x)\cdot\nabla\bar{\psi}(x) + 2\tilde{m}^2\bar{\chi}(x)\bar{\psi}(x)\right)\\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}4\omega_{\mathbf{p}}^2\left(\bar{v}^2(\mathbf{p})\bar{u}^1(\mathbf{p}) + \bar{u}^2(\mathbf{p})\bar{v}^1(\mathbf{p})\right)\,. \end{aligned} \tag{6.71}$$

The calculation of the unbarred bosonic part of the Hamiltonian is more involved. Inserting the momentum space expansions into the position space Hamiltonian yields

$$\begin{aligned}\mathcal{H}_B &= \int \mathrm{d}^3\mathbf{x}\left(\frac{\tilde{m}}{2}\mathrm{Tr}\Big(\tilde{R}^T(x)\tilde{R}(x)\Big) + \frac{\tilde{m}}{2}\mathrm{Tr}\Big(\tilde{S}^T(x)\tilde{S}(x)\Big) - i\mathrm{Tr}\Big(\tilde{R}^T(x)\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}\tilde{S}(x)\Big)\right)\\ &= \int\frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{p}'}{(2\pi)^6}\left(\frac{1}{2\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma}p'\cdot\bar{\sigma}^{\dot{\beta}\delta}\left(w^1_{\gamma\alpha}(\mathbf{p})e^{ip\cdot x} - w^2_{\gamma\alpha}(\mathbf{p})e^{-ip\cdot x}\right)\right.\\ &\quad\left(w_\delta^{1\alpha}(\mathbf{p}')e^{ip'\cdot x} - w_\delta^{2\alpha}(\mathbf{p}')e^{-ip'\cdot x}\right)\\ &\quad -\frac{\tilde{m}}{2}\left(w^{1\beta\alpha}(\mathbf{p})e^{ip\cdot x} + w^{2\beta\alpha}(\mathbf{p})e^{-ip\cdot x}\right)\left(w^1_{\beta\alpha}(\mathbf{p}')e^{ip'\cdot x} + w^2_{\beta\alpha}(\mathbf{p}')e^{-ip'\cdot x}\right)\\ &\quad -\frac{i}{\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\left(w^1_{\delta\alpha}(\mathbf{p})e^{ip\cdot x} - w^2_{\delta\alpha}(\mathbf{p})e^{-ip\cdot x}\right)\\ &\quad\left.\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\cdot\boldsymbol{\nabla}\left(w^{1\gamma\alpha}(\mathbf{p}')e^{ip'\cdot x} + w^{2\gamma\alpha}(\mathbf{p}')e^{-ip'\cdot x}\right)\right)\,. \end{aligned} \tag{6.72}$$

Again the position space integral results in a δ-function in momentum space that eliminates one of the momentum space integrals

$$\begin{aligned}\mathcal{H}_B &= \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(\frac{1}{2\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma}\left(\omega_{\mathbf{p}}\left(\bar{\sigma}^0\right)^{\dot{\beta}\delta} + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}\delta}\right)w^1_{\gamma\alpha}(\mathbf{p})w_\delta^{1\alpha}(-\mathbf{p})e^{2i\omega_{\mathbf{p}}t}\right.\\ &\quad -\frac{1}{2\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma}p\cdot\bar{\sigma}^{\dot{\beta}\delta}w^1_{\gamma\alpha}(\mathbf{p})w_\delta^{2\alpha}(\mathbf{p}) - \frac{1}{2\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma}p\cdot\bar{\sigma}^{\dot{\beta}\delta}w^2_{\gamma\alpha}(\mathbf{p})w_\delta^{1\alpha}(\mathbf{p})\\ &\quad +\frac{1}{2\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma}\left(\omega_{\mathbf{p}}\left(\bar{\sigma}^0\right)^{\dot{\beta}\delta} + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}\delta}\right)w^2_{\gamma\alpha}(\mathbf{p})w_\delta^{2\alpha}(-\mathbf{p})e^{-2i\omega_{\mathbf{p}}t}\\ &\quad -\frac{\tilde{m}}{2}w^{1\beta\alpha}(\mathbf{p})w^1_{\beta\alpha}(-\mathbf{p})e^{2i\omega_{\mathbf{p}}t} - \frac{\tilde{m}}{2}w^{1\beta\alpha}(\mathbf{p})w^2_{\beta\alpha}(\mathbf{p}) - \frac{\tilde{m}}{2}w^{2\beta\alpha}(\mathbf{p})w^1_{\beta\alpha}(\mathbf{p})\\ &\quad -\frac{\tilde{m}}{2}w^{2\beta\alpha}(\mathbf{p})w^2_{\beta\alpha}(-\mathbf{p})e^{-2i\omega_{\mathbf{p}}t} + \frac{1}{\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\cdot\mathbf{p}w^1_{\delta\alpha}(\mathbf{p})w^{1\gamma\alpha}(-\mathbf{p})e^{2i\omega_{\mathbf{p}}t}\\ &\quad +\frac{1}{\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\cdot\mathbf{p}w^1_{\delta\alpha}(\mathbf{p})w^{2\gamma\alpha}(\mathbf{p}) + \frac{1}{\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\cdot\mathbf{p}w^2_{\delta\alpha}(\mathbf{p})w^{1\gamma\alpha}(\mathbf{p})\\ &\quad\left. +\frac{1}{\tilde{m}}p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\cdot\mathbf{p}w^2_{\delta\alpha}(\mathbf{p})w^{2\gamma\alpha}(-\mathbf{p})e^{-2i\omega_{\mathbf{p}}t}\right)\,. \end{aligned} \tag{6.73}$$

At this point it proves useful to restrict the discussion to a number of terms of interest which will be referred to as $\mathcal{I}$ which encompasses all terms proportional to $e^{2i\omega t}$. As the Hamiltonian should be time independent the terms proportional to $e^{2i\omega t}$ must vanish. Explicit calculations show that $\mathcal{I}$ satisfies

$$\begin{aligned}
\mathcal{I} &= \frac{1}{2\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma}\left(\omega_{\mathbf{p}}\left(\bar{\sigma}^{0}\right)^{\dot{\beta}\delta} + \mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}\delta}\right) w^{1}_{\gamma\alpha}(\mathbf{p}) w^{1\alpha}_{\delta}(-\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} \\
&\quad - \frac{\tilde{m}}{2} w^{1\beta\alpha}(\mathbf{p}) w^{1}_{\beta\alpha}(-\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} + \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\,\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma} w^{1}_{\delta\alpha}(\mathbf{p}) w^{1\gamma\alpha}(-\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} \\
&= \frac{1}{2\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} p\cdot\bar{\sigma}^{\dot{\beta}\delta} w^{1}_{\gamma\alpha}(\mathbf{p}) w^{1\alpha}_{\delta}(-\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} - \frac{\tilde{m}}{2} w^{1\beta\alpha}(\mathbf{p}) w^{1}_{\beta\alpha}(-\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} \\
&\quad \text{with } p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} p\cdot\bar{\sigma}^{\dot{\beta}\delta} = p^{2}\epsilon^{\gamma\delta} \\
&= \left(\frac{p^{2}}{2\tilde{m}} - \frac{\tilde{m}}{2}\right) w^{1}_{\gamma\alpha}(\mathbf{p}) w^{1\gamma\alpha}(-\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} \\
&= 0\,. \qquad (6.74)
\end{aligned}$$

which proves that all time dependent terms proportional to $e^{2i\omega t}$ indeed vanish. The calculation for the terms proportional to $e^{-2i\omega t}$ is nearly identical to this discussion with all w^1 replaced by w^2. Therefore, the terms proportional to $e^{-2i\omega t}$ vanish identically as well. The remaining Hamiltonian is then given by

$$\begin{aligned}
\mathcal{H}_B = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} &\left(-\frac{1}{2\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} p\cdot\bar{\sigma}^{\dot{\beta}\delta} w^{1}_{\gamma\alpha}(\mathbf{p}) w^{2\alpha}_{\delta}(\mathbf{p}) + \frac{1}{2\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} p\cdot\bar{\sigma}^{\dot{\beta}\delta}\epsilon^{\epsilon\alpha} w^{2}_{\gamma\alpha}(\mathbf{p}) w^{1}_{\delta\epsilon}(\mathbf{p})\right. \\
&- \frac{\tilde{m}}{2} w^{1\beta\alpha}(\mathbf{p}) w^{2}_{\beta\alpha}(\mathbf{p}) + \frac{\tilde{m}}{2}\epsilon^{\beta\epsilon}\epsilon^{\kappa\alpha} w^{2}_{\epsilon\kappa}(\mathbf{p}) w^{1}_{\beta\alpha}(\mathbf{p}) + \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\,\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma} w^{1}_{\delta\alpha}(\mathbf{p}) w^{2\gamma\alpha}(\mathbf{p}) \\
&\left.- \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\,\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\epsilon^{\gamma\epsilon}\epsilon^{\kappa\alpha} w^{2}_{\delta\alpha}(\mathbf{p}) w^{1}_{\epsilon\kappa}(\mathbf{p})\right). \qquad (6.75)
\end{aligned}$$

To further simplify this result the momentum space operators must be brought into the same order using the commutation relations between the momentum space operators

$$\begin{aligned}
\mathcal{H}_B = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} &\left(-\frac{1}{2\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} p\cdot\bar{\sigma}^{\dot{\beta}\delta} w^{1}_{\gamma\alpha}(\mathbf{p}) w^{2\alpha}_{\delta}(\mathbf{p})\right. \\
&+ \frac{1}{2\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\gamma} p\cdot\bar{\sigma}^{\dot{\beta}\delta}\epsilon^{\epsilon\alpha}\left(w^{1}_{\delta\epsilon}(\mathbf{p}) w^{2}_{\gamma\alpha}(\mathbf{p}) - (2\pi)^3\frac{\tilde{m}}{2\omega_{\mathbf{p}}}\epsilon_{\alpha\epsilon}\epsilon_{\gamma\delta}\delta(\mathbf{0})\right) \\
&- \frac{\tilde{m}}{2} w^{1\beta\alpha}(\mathbf{p}) w^{2}_{\beta\alpha}(\mathbf{p}) + \frac{\tilde{m}}{2}\epsilon^{\beta\epsilon}\epsilon^{\kappa\alpha}\left(w^{1}_{\beta\alpha}(\mathbf{p}) w^{2}_{\epsilon\kappa}(\mathbf{p}) - (2\pi)^3\frac{\tilde{m}}{2\omega_{\mathbf{p}}}\epsilon_{\kappa\alpha}\epsilon_{\epsilon\beta}\delta(\mathbf{0})\right) \\
&+ \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\,\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma} w^{1}_{\delta\alpha}(\mathbf{p}) w^{2\gamma\alpha}(\mathbf{p}) \\
&\left.- \frac{1}{\tilde{m}} p\cdot\bar{\sigma}_{\dot{\beta}}{}^{\delta}\,\mathbf{p}\cdot\bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma}\epsilon^{\gamma\epsilon}\epsilon^{\kappa\alpha}\left(w^{1}_{\epsilon\kappa}(\mathbf{p}) w^{2}_{\delta\alpha}(\mathbf{p}) - (2\pi)^3\frac{\tilde{m}}{2\omega_{\mathbf{p}}}\epsilon_{\alpha\kappa}\epsilon_{\delta\epsilon}\delta(\mathbf{0})\right)\right). \qquad (6.76)
\end{aligned}$$

After elimination of the zero point energy – all terms proportional to $\delta(\mathbf{0})$ – and using some relations between σ-matrices the Hamiltonian is found to be

$$\mathcal{H}_B = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \bigg(-\frac{1}{\tilde{m}} p^2 w^{1\beta\alpha}(\mathbf{p}) w^2_{\beta\alpha}(\mathbf{p}) - \tilde{m} w^{1\beta\alpha}(\mathbf{p}) w^2_{\beta\alpha}(\mathbf{p}) \\ + \frac{1}{\tilde{m}} \left(p \cdot \bar{\sigma}_{\dot{\beta}}{}^{\delta} \mathbf{p} \cdot \bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma} - p \cdot \bar{\sigma}_{\dot{\beta}\gamma} \mathbf{p} \cdot \bar{\boldsymbol{\sigma}}^{\dot{\beta}\delta} \right) w^1_{\delta\alpha}(\mathbf{p}) w^{2\gamma\alpha}(\mathbf{p}) \bigg) . \tag{6.77}$$

The third term can still be simplified using the relation

$$p \cdot \bar{\sigma}_{\dot{\beta}}{}^{\delta} \mathbf{p} \cdot \bar{\boldsymbol{\sigma}}^{\dot{\beta}}{}_{\gamma} - p \cdot \bar{\sigma}_{\dot{\beta}\gamma} \mathbf{p} \cdot \bar{\boldsymbol{\sigma}}^{\dot{\beta}\delta} = 2\mathbf{p}^2 \epsilon^{\delta}{}_{\gamma} . \tag{6.78}$$

This leads to a very compact result for the bosonic part of the Hamiltonian in momentum space

$$\begin{aligned} \mathcal{H}_B &= -\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{p^2 + \tilde{m}^2 + 2\mathbf{p}^2}{\tilde{m}} w^{1\beta\alpha}(\mathbf{p}) w^2_{\beta\alpha}(\mathbf{p}) \\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{2\omega_{\mathbf{p}}^2}{\tilde{m}} \mathrm{Tr}\big(w^{1T}(\mathbf{p}) w^2(\mathbf{p}) \big) . \end{aligned} \tag{6.79}$$

Again the calculations for the barred bosonic momentum space operators are perfectly aligned with those for the unbarred operators. It is then straightforward to show that the contribution of the barred momentum space operators to the momentum space Hamiltonian is given by

$$\begin{aligned} \mathcal{H}_B &= \int \mathrm{d}^3\mathbf{x} \left(\frac{m}{4} \mathrm{Tr}\Big(\tilde{\bar{R}}^T(x) \tilde{\bar{R}}(x) \Big) + \frac{m}{4} \mathrm{Tr}\Big(\tilde{\bar{S}}^T(x) \tilde{\bar{S}}(x) \Big) - i \mathrm{Tr}\Big(\tilde{\bar{S}}^T(x) \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} \tilde{\bar{R}}(x) \Big) \right) \\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{2\omega_{\mathbf{p}}^2}{\tilde{m}} \mathrm{Tr}\big(\bar{w}^{2T}(\mathbf{p}) \bar{w}^1(\mathbf{p}) \big) . \end{aligned} \tag{6.80}$$

The full Hamiltonian in momentum space is then found to be

$$\mathcal{H} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \bigg(4\omega_{\mathbf{p}}^2 \left(u^1(\mathbf{p}) v^2(\mathbf{p}) + v^1(\mathbf{p}) u^2(\mathbf{p}) + \bar{v}^2(\mathbf{p}) \bar{u}^1(\mathbf{p}) + \bar{u}^2(\mathbf{p}) \bar{v}^1(\mathbf{p}) \right) \\ + \frac{2\omega_{\mathbf{p}}^2}{\tilde{m}} \left(\mathrm{Tr}\big(w^{1T}(\mathbf{p}) w^2(\mathbf{p}) \big) + \mathrm{Tr}\big(\bar{w}^{2T}(\mathbf{p}) \bar{w}^1(\mathbf{p}) \big) \right) \bigg) . \tag{6.81}$$

This result is surprisingly compact containing only six terms of which four are fermionic and two are bosonic. Furthermore, it is perfectly symmetric between the component fields and their Hermitian conjugates. To derive the normal ordered Hamiltonian where all the annihilation operators stand on the left of all creation operators the properties of the operators still need to be determined. This will be discussed in the following section and will verify that the normal ordered Hamiltonian is given by

$$:\mathcal{H}: = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \Big(4\omega_{\mathbf{p}}^2 \left(v^2(\mathbf{p})u^1(\mathbf{p}) + v^1(\mathbf{p})u^2(\mathbf{p}) + \bar{u}^1(\mathbf{p})\bar{v}^2(\mathbf{p}) + \bar{u}^2(\mathbf{p})\bar{v}^1(\mathbf{p})\right) \\ + \frac{2\omega_{\mathbf{p}}^2}{\tilde{m}} \left(\mathrm{Tr}\big(w^2(\mathbf{p})w^{1T}(\mathbf{p})\big) + \mathrm{Tr}\big(\bar{w}^1(\mathbf{p})\bar{w}^{2T}(\mathbf{p})\big)\right)\Big) . \tag{6.82}$$

6.5 The Creation and Annihilation Operators

To be able to draw conclusions on the energy spectrum of the Hamiltonian in momentum space it has to be determined which of the momentum space operators are actually creation operators and which ones are annihilation operators.

6.5.1 The Fermionic Creation and Annihilation Operators

It is assumed that $|\Psi\rangle$ is an eigenstate of the Hamiltonian with eigenvalue E

$$\mathcal{H}\,|\Psi\rangle = E\,|\Psi\rangle . \tag{6.83}$$

Furthermore, it is known that the fermionic momentum space operators commute with their Hermitian conjugate counterparts and also with any bosonic operator in momentum space. This implies that the fermionic momentum space operator $u^1(\mathbf{p})$ commutes with all contributions to the Hamiltonian except the contribution from $H_{\mathcal{F}}$. Therefore, it proves to be convenient for the investigation of the properties of the fermionic momentum space operators to separate the full Hamiltonian into two parts

$$\mathcal{H} = \mathcal{H}_F + \mathcal{H}_{\perp F} . \tag{6.84}$$

Here $\mathcal{H}_{\perp F}$ denotes the part of the Hamiltonian that commutes with any unbarred fermionic operator in momentum space and is given by the sum $\mathcal{H}_{\bar{F}} + \mathcal{H}_B + \mathcal{H}_{\bar{B}}$. Based on these commutation properties it is sufficient to restrict the following discussion to calculating the effect of $\mathcal{H}_F$ acting on $u^1(\mathbf{p}') |\Psi\rangle$

$$\mathcal{H}_F u^1_\gamma(\mathbf{p}') |\Psi\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} 4\omega_{\mathbf{p}}^2 \left(u^1(\mathbf{p})v^2(\mathbf{p}) + v^1(\mathbf{p})u^2(\mathbf{p})\right) u^1_\gamma(\mathbf{p}') |\Psi\rangle \ , \tag{6.85}$$

which simplifies the notation significantly. Repeated application of the anticommutation relations for the fermionic momentum space operators to move $u^1(\mathbf{p}')$ all the way to the left leads to

$$\begin{aligned}\mathcal{H}_F u^1_\gamma(\mathbf{p}') |\Psi\rangle = u^1_\gamma(\mathbf{p}') \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} 4\omega_{\mathbf{p}}^2 \left(u^1(\mathbf{p})v^2(\mathbf{p}) + v^1(\mathbf{p})u^2(\mathbf{p})\right) |\Psi\rangle \\ + \int \mathrm{d}^3\mathbf{p}\,\omega_{\mathbf{p}} \epsilon_\gamma{}^\alpha \delta(\mathbf{p}-\mathbf{p}') u^1_\alpha(\mathbf{p}) |\Psi\rangle \ .\end{aligned} \tag{6.86}$$

The momentum space integral in the first term is exactly the unbarred fermionic part of the Hamiltonian while the second term contains a δ-function that cancels the momentum space integral. To be able to use the initial assumption that the full Hamiltonian $\mathcal{H}$ acting on ψ has eigenvalue E the perpendicular part which satisfies $\mathcal{H}_{\perp F} u_1(\mathbf{p}') = u_1(\mathbf{p}')\mathcal{H}_{\perp F}$ has to be added on both sides of the equation. The energy of the state $u^1 |\Psi\rangle$ is thus found to be

$$\begin{aligned}\mathcal{H} u^1_\gamma(\mathbf{p}') |\Psi\rangle &= u^1_\gamma(\mathbf{p}')\mathcal{H} |\Psi\rangle + \omega_{\mathbf{p}'} u^1_\gamma(\mathbf{p}') |\Psi\rangle \\ &= \left(E + \omega_{\mathbf{p}'}\right) u^1_\gamma(\mathbf{p}') |\Psi\rangle \ .\end{aligned} \tag{6.87}$$

The energy $\omega_{\mathbf{p}}$ is by definition positive and therefore the energy of $u^1 |\Psi\rangle$ is higher than for the state $|\Psi\rangle$. This clearly shows that u^1 is a creation operator.

Similar calculations can be repeated for the remaining fermionic momentum space operators

$$\mathcal{H}_F u^2_\gamma(\mathbf{p}') |\Psi\rangle = \left(E + \omega_{\mathbf{p}'}\right) u^2_\gamma(\mathbf{p}') |\Psi\rangle \ , \tag{6.88}$$

$$\mathcal{H}_F v^1_\gamma(\mathbf{p}') |\Psi\rangle = \left(E - \omega_{\mathbf{p}'}\right) v^1_\gamma(\mathbf{p}') |\Psi\rangle \ , \tag{6.89}$$

$$\mathcal{H}_F v^2_\gamma(\mathbf{p}') |\Psi\rangle = \left(E - \omega_{\mathbf{p}'}\right) v^2_\gamma(\mathbf{p}') |\Psi\rangle \ . \tag{6.90}$$

It is found that u^1 and u^2 are creation operators while v^1 and v^2 are annihilation operators.

In general it is possible to repeat the previous calculations for the barred fermionic momentum space operators. However, by definition they are the Hermitian conjugate of the already discussed unbarred momentum space operators. Furthermore, it is known that the Hermitian conjugate of a creation operator is an annihilation operator and vice versa. Therefore, it can immediately be concluded that $\bar{u}^1$ and $\bar{u}^2$ are annihilation operators while $\bar{v}^1$ and $\bar{v}^2$ are creation operators which satisfy

$$\mathcal{H}\bar{u}^1_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle = \left(E - \omega_{\mathbf{p}'}\right) \bar{u}^1_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle \ , \tag{6.91}$$

$$\mathcal{H}\bar{u}^2_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle = \left(E - \omega_{\mathbf{p}'}\right) \bar{u}^2_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle \ , \tag{6.92}$$

$$\mathcal{H}\bar{v}^1_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle = \left(E + \omega_{\mathbf{p}'}\right) \bar{v}^1_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle \ , \tag{6.93}$$

$$\mathcal{H}\bar{v}^2_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle = \left(E + \omega_{\mathbf{p}'}\right) \bar{v}^2_{\dot{\gamma}}(\mathbf{p}') |\Psi\rangle \ . \tag{6.94}$$

The complete set of momentum space operators thus contains four creation and four annihilation operators.

6.5.2 The Bosonic Creation and Annihilation Operators

The determination of the bosonic creation and annihilation operators in momentum space follows mostly the schematic for the fermionic operators outlined in the previous section. It is again assumed that $|\Psi\rangle$ is an eigenstate of the Hamiltonian with eigenvalue E. However, the separation of the Hamiltonian has to be redefined appropriately to represent a split into the bosonic part and the component perpendicular to it $\mathcal{H}_{\perp B}$

$$\mathcal{H} = \mathcal{H}_F + \mathcal{H}_{\perp F} \, . \tag{6.95}$$

As the same arguments for the commutation properties between the various fermionic and bosonic operators apply the perpendicular part $\mathcal{H}_{\perp B}$ is now given by the sum of $\mathcal{H}_F + \mathcal{H}_{\bar{F}} + \mathcal{H}_{\bar{B}}$ and it is sufficient to restrict all but the last step of the following discussion to the calculation of

$$\mathcal{H}_B w^1_{\gamma\delta}(\mathbf{p}') |\Psi\rangle = -\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{2\omega^2_{\mathbf{p}}}{\tilde{m}} w^{1\beta\alpha}(\mathbf{p}) w^2_{\beta\alpha}(\mathbf{p}) w^1_{\gamma\delta}(\mathbf{p}') |\Psi\rangle \ . \tag{6.96}$$

With the help of the commutation relations $w^1(\mathbf{p}')$ can be pulled all the way to the left

$$\mathcal{H}_B w^1_{\gamma\delta}(\mathbf{p}')\,|\Psi\rangle = -w^1_{\gamma\delta}(\mathbf{p}')\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{2\omega^2_{\mathbf{p}}}{\tilde{m}} w^{1\beta\alpha}(\mathbf{p})w^2_{\beta\alpha}(\mathbf{p})\,|\Psi\rangle$$
$$+\int \mathrm{d}^3\mathbf{p}\,\omega_{\mathbf{p}}\epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\delta(\mathbf{p}-\mathbf{p}')w^{1\beta\alpha}(\mathbf{p})\,|\Psi\rangle\ . \tag{6.97}$$

Here, the first integral corresponds the unbarred bosonic part of the Hamiltonian while the integral in the second term is cancelled by the momentum space δ-function that arises from the commutation of $w^1(\mathbf{p}')$ with the bosonic part of the Hamiltonian. Addition of the part perpendicular to $\mathcal{H}_B$ allows the use of the assumption on the eigenvalue of $\mathcal{H}$ and leads to

$$\begin{aligned}\mathcal{H} w^1_{\gamma\delta}(\mathbf{p}')\,|\Psi\rangle &= w^1_{\gamma\delta}(\mathbf{p}')\mathcal{H}\,|\Psi\rangle + \omega'_{\mathbf{p}} w^1_{\gamma\delta}(\mathbf{p}')\,|\Psi\rangle \\ &= \left(E+\omega'_{\mathbf{p}}\right) w^1_{\gamma\delta}(\mathbf{p}')\,|\Psi\rangle\ .\end{aligned} \tag{6.98}$$

Therefore, the state $w^1(\mathbf{p}')\,|\Psi\rangle$ has the eigenvalue $E+\omega'_{\mathbf{p}}$ which identifies it as creation operator.

Similarly it can be shown that $w^2(\mathbf{p}')\,|\Psi\rangle$ satisfies

$$\mathcal{H} w^2_{\gamma\delta}(\mathbf{p}')\,|\Psi\rangle = \left(E-\omega_{\mathbf{p}'}\right) w^2_{\gamma\delta}(\mathbf{p}')\,|\Psi\rangle\ . \tag{6.99}$$

It can be seen that the state $w^2(\mathbf{p}')\,|\Psi\rangle$ has the eigenvalue $E-\omega'_{\mathbf{p}}$ and thus is an annihilation operator.

The results for the barred bosonic operators in momentum space are found to be

$$\mathcal{H} \bar{w}^1_{\dot\gamma\dot\delta}(\mathbf{p}')\,|\Psi\rangle = \left(E-\omega'_{\mathbf{p}}\right) \bar{w}^1_{\dot\gamma\dot\delta}(\mathbf{p}')\,|\Psi\rangle\ , \tag{6.100}$$
$$\mathcal{H} \bar{w}^2_{\dot\gamma\dot\delta}(\mathbf{p}')\,|\Psi\rangle = \left(E+\omega_{\mathbf{p}'}\right) \bar{w}^2_{\dot\gamma\dot\delta}(\mathbf{p}')\,|\Psi\rangle\ . \tag{6.101}$$

This shows that $\bar{w}^1(\mathbf{p}')$ is an annihilation operator while $\bar{w}^2(\mathbf{p}')$ is a creation operator. As before this result for the barred bosonic momentum space operators is no surprise as they are by construction the Hermitian conjugate of their unbarred counterparts.

Chapter 7

Coupling to the O'Raifeartaigh Model

7.1 The O'Raifeartaigh Model

The modern formulation of supersymmetry was introduced by Wess and Zumino (1974b) and is based on a single chiral multiplet

$$\begin{aligned}\Phi &= \exp\left(-i\theta\partial\!\!\!/\bar{\theta}\right)\left(A + \theta^\alpha\phi_\alpha + \theta^2 F\right) \,, \\ &= A + \theta^\alpha\phi_\alpha + \theta^2 F - i\theta\partial\!\!\!/\bar{\theta}A + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}}{}^{\alpha}\phi_\alpha - \frac{1}{4}\theta^2\bar{\theta}^2\Box A \,. \end{aligned} \tag{7.1}$$

In the first line the chiral character of Φ as a product of an exponential function times a chiral term that is solely dependent on θ becomes clear while the second line represents the full superfield expansion of the chiral superfield. The Wess-Zumino Lagrangian which is the most general Lagrangian that can be constructed using a chiral superfield and covariant derivatives can be written in a very compact form involving the F-component of superfield products

$$L = \left(\frac{1}{2}\Phi \cdot T\Phi - \frac{m}{2}\Phi \cdot \Phi - \frac{g}{3}\Phi \cdot \Phi \cdot \Phi\right)_F + h.c.\,. \tag{7.2}$$

This first approach was then generalised by O'Raifeartaigh (1975). Instead of using three identical chiral superfields to construct the Lagrangian, O'Raifeartaigh utilised three distinguishable chiral multiplets Φ_a which will be denoted by italic subscripts. For the following discussion it proves useful to separate the Lagrangian into its super-kinetic part and the superpotential

$$L = \frac{1}{2}\left(\Phi_a \cdot T\Phi_a\right)_F - (V)_F + h.c.\,. \tag{7.3}$$

It has to be pointed out that the separation into super-kinetic term and superpotential is usually not identical to the separation into kinetic and potential terms, e. g., the super-kinetic term may still contain contributions to the potential and vice versa, which has to be addressed later on in the dicussion. The general superpotential in this equation is given by

$$V = \lambda_a \Phi_a + \frac{1}{2} m_{ab} \Phi_a \cdot \Phi_b + \frac{1}{3} g_{abc} \Phi_a \cdot \Phi_b \cdot \Phi_c \,, \tag{7.4}$$

where the structure constants λ_a, m_{ab}, and g_{abc} are real, totally symmetric, and the summation indices run over $a, b, c = 1 \ldots 3$. Comparison with equation (7.2) reveals that the superpotential of the O'Raifeartaigh model still contains a linear term in the superfield which is absent in the above Wess-Zumino Lagrangian. It can be eliminated from the Lagrangian by a redefinition of the superfields. However, it proves useful to retain the linear term as it conveniently splits off the part that is responsible for spontaneous supersymmetry breaking. This will be discussed in the next section.

The explicit calculation of the various superfield products that appear in the O'Raifeartaigh Lagrangian is straightforward using the definition of the kinetic multiplet and multiplication rules outlined in Appendix A.5. It is further simplified by the fact that only the F-component of the superfield products is needed to construct the Lagrangian. The super-kinetic term is found to be

$$\frac{1}{2} \left(\Phi_a \cdot \mathrm{T} \Phi_a \right)_F = -\frac{1}{2} A_a \Box A_a^\dagger + \frac{1}{2} F_a F_a^\dagger + \frac{i}{4} \phi_a \partial\!\!\!/ \bar{\phi}_a \,, \tag{7.5}$$

while the superpotential can be written as

$$\begin{aligned} (V)_F = {} & \lambda_a F_a + \frac{1}{2} m_{ab} \left(A_a F_b + F_a A_b - \frac{1}{2} \phi_a \phi_b \right) \\ & + \frac{1}{3} g_{abc} \left(A_a A_b F_c + A_a F_b A_c + F_a A_b A_c - \frac{3}{2} \phi_a \phi_b A_c \right) . \end{aligned} \tag{7.6}$$

Therefore, the Lagrangian for the O'Raifeartaigh model is given by

$$\begin{aligned} L = {} & -\frac{1}{2} A_a \Box A_a^\dagger + \frac{1}{2} F_a F_a^\dagger + \frac{i}{4} \phi_a \partial\!\!\!/ \bar{\phi}_a - \lambda_a F_a - \frac{1}{2} m_{ab} \left(A_a F_b + F_a A_b - \frac{1}{2} \phi_a \phi_b \right) \\ & - \frac{1}{3} g_{abc} \left(A_a A_b F_c + A_a F_b A_c + F_a A_b A_c - \frac{3}{2} A_a \phi_b \phi_c \right) + h.c. \,. \end{aligned} \tag{7.7}$$

As the upcoming derivation of the equations of motion for the component fields – especially for the fermionic component fields ϕ_a – requires knowledge of the Hermitian conjugate part, the full Lagrangian is found to be

$$
\begin{aligned}
L = &-\frac{1}{2}A_a\Box A_a^\dagger - \frac{1}{2}\Box A_a A_a^\dagger + F_a F_a^\dagger + \frac{i}{4}\phi_a \partial\!\!\!/ \bar{\phi}_a + \frac{i}{4}\bar{\phi}_a \bar{\partial\!\!\!/} \phi_a - \lambda_a\left(F_a + F_a^\dagger\right) \\
&- \frac{1}{2}m_{ab}\left(A_a F_b + F_a A_b + A_a^\dagger F_b^\dagger + F_a^\dagger A_b^\dagger - \frac{1}{2}\phi_a\phi_b - \frac{1}{2}\bar{\phi}_a\bar{\phi}_b\right) \\
&- \frac{1}{3}g_{abc}\Big(A_a A_b F_c + A_a F_b A_c + F_a A_b A_c + A_a^\dagger A_b^\dagger F_c^\dagger + A_a^\dagger F_b^\dagger A_c^\dagger + F_a^\dagger A_b^\dagger A_c^\dagger \\
&- \frac{3}{2}A_a\phi_b\phi_c - \frac{3}{2}A_a^\dagger\bar{\phi}_b\bar{\phi}_c\Big)\,.
\end{aligned}
\tag{7.8}
$$

This Lagrangian can be used to derive the equations of motion for the component fields of the chiral multiplet

$$F_d = \lambda_d + m_{da}A_a^\dagger + g_{dab}A_a^\dagger A_b^\dagger\,, \tag{7.9}$$

$$-i\partial\!\!\!/_{\dot{\alpha}}{}^{\beta}\phi_{d\beta} = m_{da}\bar{\phi}_{a\dot{\alpha}} + 2g_{dab}A_a^\dagger\bar{\phi}_{b\dot{\alpha}}\,, \tag{7.10}$$

$$-\Box A_d = m_{da}F_a^\dagger + g_{dab}\left(A_a^\dagger F_b^\dagger + F_a^\dagger A_b^\dagger - \frac{1}{2}\bar{\phi}_a\bar{\phi}_b\right)\,. \tag{7.11}$$

As the Lagrangian is by construction symmetric with respect to the component fields the equations of motion for the Hermitian conjugate component fields can be derived by Hermitian conjugation

$$F_d^\dagger = \lambda_d + m_{da}A_a + g_{dab}A_a A_b\,, \tag{7.12}$$

$$-i\bar{\partial\!\!\!/}_{\alpha\dot{\beta}}\phi_d^{\dot{\beta}} = m_{da}\phi_{a\alpha} + 2g_{dab}A_a\phi_{b\alpha}\,, \tag{7.13}$$

$$-\Box A_d^\dagger = m_{da}F_a + g_{dab}\left(A_a F_b + F_a A_b - \frac{1}{2}\phi_a\phi_b\right)\,. \tag{7.14}$$

The equations of motion (7.9) to (7.11) can also be rewritten using the multiplication rules from Appendix A.5

$$F_d = \lambda_d + m_{da}A_a^\dagger + g_{dab}A_{ab}^\dagger\,, \tag{7.15}$$

$$-i\partial\!\!\!/_{\dot{\alpha}}{}^{\beta}\phi_{d\beta} = m_{da}\bar{\phi}_{a\dot{\alpha}} + g_{dab}\bar{\phi}_{ab\dot{\alpha}}\,, \tag{7.16}$$

$$-\Box A_d = m_{da}F_a^\dagger + g_{dab}F_{ab}^\dagger\,, \tag{7.17}$$

which can be expressed in the even more compact form

$$\mathrm{T}\Phi_d = \lambda_d + m_{db}\Phi_b + g_{dbc}\Phi_b \cdot \Phi_c \,. \tag{7.18}$$

The equations of motion can then be used to express the superpotential in terms of the bosonic auxiliary fields F_a and the spinor fields ϕ_a. A closer look at the superpotential in equation (7.6) reveals that all the contributions to the superpotential are either linear in F_a or quadratic in ϕ_a. The same is valid for the Hermitian conjugate of the superpotential except that it is now dependent on the Hermitian conjugate superfields $F_a^\dagger$ and $\bar{\phi}_a$. Therefore, it is sufficient to restrict the discussion to the terms in equation (7.6) and it can be shown that the following scaling relation holds

$$[V(\phi)]_F = \left(F_d \frac{\partial}{\partial F_d} + \frac{1}{2}\phi_d^\alpha \frac{\partial}{\partial \phi_d^\alpha} \right) [V(\phi)]_F \,. \tag{7.19}$$

With the previous results for the equations of motion the derivatives with respect to F_a and ϕ_a can be written down immediately as the derivatives of the superpotential satisfy

$$\frac{\partial}{\partial F_d}[V(\phi)]_F = \lambda_d + m_{da}A_a + g_{dab}A_aA_b = F_d^\dagger \,, \tag{7.20}$$

$$\frac{\partial}{\partial \phi_d^\alpha}[V(\phi)]_F = m_{da}\phi_{a\alpha} + 2g_{dab}A_a\phi_{b\alpha} = -i\bar{\partial\!\!\!/}_{\alpha\dot{\beta}}\phi_d^{\dot{\beta}} \,. \tag{7.21}$$

Using these results the scaling relation simplifies to

$$[V(\phi)]_F = F_aF_a^\dagger - \frac{i}{2}\bar{\phi}_a\bar{\partial\!\!\!/}\phi_a \,. \tag{7.22}$$

As mentioned before, the identification of super-kinetic term and superpotential is not exactly the same as the separation into kinetic and potential term and the super-kinetic terms still contains a non-kinetic contribution while the superpotential contains a kinetic term. Identification of the potential terms is straightforward and the actual potential U is found to be

$$U = F_aF_a^\dagger \,. \tag{7.23}$$

This implies that the potential is always positive and vanishes only if $\langle F_a \rangle = 0$.

7.1.1 Spontaneous Supersymmetry Breaking

To determine whether supersymmetry is spontaneously broken within the framework of a given theory the fundamental condition for symmetry breaking has to be examined. If the F-term of a given theory has a vanishing expectation value

$$\langle F_a \rangle = 0 \tag{7.24}$$

for any choice of the structure constants, supersymmetry cannot be spontaneously broken. However, if there exist regions or points in parameter space for which F_a acquires a nonvanishing expectation value, supersymmetry is spontaneously broken for these parameters. In this case the superpotential becomes positive according to equation (7.23) and therefore, the ground state must acquire a nonzero energy as well. The supersymmetry algebra then implies that the ground state cannot be invariant under supersymmetry transformations, as $E = \langle 0| H |0\rangle > 0$ is not compatible with the condition for invariance under superspace translations $Q_\alpha |0\rangle = 0, \bar{Q}_{\dot{\alpha}} |0\rangle = 0$ which corresponds to an identically vanishing ground state energy.

A basic example for supersymmetry breaking was pointed out by O'Raifeartaigh (1975). Starting from the Lagrangian in equation (7.8) the structure constants were chosen such that

$$\lambda_3 = \Lambda\,, \qquad \text{all other}\,\Lambda_a = 0\,, \tag{7.25}$$

$$m_{12} = m_{21} = M\,, \qquad \text{all other}\,m_{ab} = 0\,, \tag{7.26}$$

$$g_{113} = g_{131} = g_{311} = g\,, \qquad \text{all other}\,g_{abc} = 0\,. \tag{7.27}$$

Following the original definition of the structure constant in equation (7.4), Λ, M, and g are real. Using this choice of constants, the equations of motions for the auxiliary fields F_a from equation (7.15) reduce to

$$F_1 = MA_2^\dagger + g\left(A_1^\dagger A_3^\dagger + A_3^\dagger A_1^\dagger\right)\,, \tag{7.28}$$

$$F_2 = MA_1^\dagger\,, \tag{7.29}$$

$$F_3 = \Lambda + gA_1^\dagger A_1^\dagger\,. \tag{7.30}$$

Due to the real, positive, and nonzero constant Λ that appears in the equation of motion for the auxiliary field F_3 it is impossible to set F_2 and F_3 to zero simultaneously. This means that at least one of them must have a nonvanishing expectation value. Conventionally it is assumed that $\langle F_3 \rangle \neq 0$.

To analyse the effect of this nonzero expectation value on the superpotential, the equations of motion for the auxiliary fields of the O'Raifeartaigh model have to be inserted into the superpotential that was derived in equation (7.23)

$$
\begin{aligned}
U = & \left(M A_2^\dagger + g \left(A_1^\dagger A_3^\dagger + A_3^\dagger A_1^\dagger \right) \right) \left(M A_2 + g \left(A_1 A_3 + A_3 A_1 \right) \right) + M^2 A_1^\dagger A_1 \\
& + \left(\Lambda + g A_1^\dagger A_1^\dagger \right) \left(\Lambda + g A_1 A_1 \right) .
\end{aligned} \tag{7.31}
$$

To minimise the potential and determine the parameter range for which the potential is always positive, the complex component fields A_a have to be replaced with their real components

$$
A_a = a_a + i b_a \, . \tag{7.32}
$$

Expressing all the complex component fields in the superpotential in terms of their two real contributions leads to

$$
\begin{aligned}
U = & \Lambda^2 + M^2 \left(a_2^2 + b_2^2 \right) + \left(M^2 + 2g\Lambda \right) a_1^2 + \left(M^2 - 2g\Lambda \right) b_1^2 + g^2 \left(a_1^2 + b_1^2 \right)^2 \\
& + 4Mg \left(a_1 a_2 + b_1 b_2 \right) a_3 + 4Mg \left(a_1 b_2 - a_2 b_1 \right) b_3 + 4g^2 \left(a_1^2 + b_1^2 \right) a_3^2 \\
& + 4g^2 \left(a_1^2 + b_1^2 \right) b_3^2 \, .
\end{aligned} \tag{7.33}
$$

As the structure parameters M and g are by definition real, the superpotential is always positive if the relations

$$
M^2 + 2g\Lambda \geq 0 \, , \tag{7.34}
$$

$$
M^2 - 2g\Lambda \geq 0 \, . \tag{7.35}
$$

are satisfied. These two relations for the structure constants can be combined into a single condition

$$M^2 \geq |2g\Lambda| \; . \tag{7.36}$$

The minimum value for the potential is then given by

$$U = \Lambda^2 \,, \tag{7.37}$$

and is achieved by choosing the expectation values for the fields to be

$$\langle a_1 \rangle = \langle a_2 \rangle = \langle b_1 \rangle = \langle b_2 \rangle = 0 \,. \tag{7.38}$$

It turns out that the expectation values for a_3 and b_3 are not constrained, since the superpotential is minimised without making any assumptions on these fields. Without loss of generality it can be assumed that only one of the two components of A_3 acquires a constant, real expectation value while the other one vanishes

$$\langle a_3 \rangle = \frac{\mu}{2g} \,, \tag{7.39}$$

$$\langle b_3 \rangle = 0 \,. \tag{7.40}$$

The new constant μ that sets the scale of the expectation value is real and may or may not be zero.

7.2 Dimensional Analysis

To achieve a coupling of the fermionic fields with mass dimension one to the O'Raifeartaigh model a dimensional analysis of all possible coupling terms is necessary to restrict the discussion to the most promising terms.

From Section 3.2.1 it is known that the building blocks of the Wess-Zumino model and therefore also of the O'Raifeartaigh model have mass dimension

$$\dim V = 0 \,, \quad \dim \Phi = 1 \,, \quad \dim D_\alpha = \frac{1}{2} \,. \tag{7.41}$$

Furthermore, if χ is identified with a fermionic field with mass dimension one it can be

Contribution	Mass Dimension	Possible Contributions
$\Phi\Phi DV$	$\dim(\Phi\Phi DV) = 5/2$	mass dimension too large for D-component

Table 7.1: Contributions to the Lagrangian for a coupling between two chiral superfields of the O'Raifeartaigh model Φ and one general superfield with one free spinor index. In addition to the contributions built from products of unbarred superfields, the Hermitian conjugates are permitted as well.

shown that the corresponding superfields and covariant derivatives satisfy

$$\dim V_\alpha = 0\,, \quad \dim X_\alpha = 1\,, \quad \dim D_\alpha = \frac{1}{2}\,. \tag{7.42}$$

With these results for mass dimension of the building blocks of the Lagrangian all possible terms can be worked out. For convenience the following arguments are restricted to the unbarred superfields while obviously the Hermitian conjugates have to be considered for the Lagrangian as well.

All contributions to the Lagrangian which is a Lorentz scalar have to satisfy two basic requirements. First, they cannot contain any uncontracted spinor indices, and second, all normalisation constants must have positive mass dimension for the theory to be renormalizable. The maximally allowed mass dimension for contributions is 3 if the F-component can be utilised and 2 for contributions via the D-component.

The goal is to construct a coupling of the fermionic fields with mass dimension one to the O'Raifeartaigh model. Therefore, contributions that contain three superfields should be considered. This results in two possible scenarios – (1) two chiral superfields from the O'Raifeartaigh model and one general superfield from the discussion of fermionic fields with mass dimension one; (2) one superfield from the O'Raifeartaigh model and two superfields from the discussion of fermionic fields with mass dimension one.

In the first case the number of products which are presented in Table 7.1 is rather short. The two chiral superfields of the O'Raifeartaigh model each have mass dimension one, while the simplest contribution involving the general superfield V_α that has no uncontracted spinor indices is DV which has mass dimension $1/2$. As DV is not chiral, the F-component cannot be used and the accumulative mass dimension of $5/2$ is too large for a contribution via the D-component. Any other product that can be conceived using two Φ's and one V_α, as well as an appropriate number of covariant derivatives yields a mass dimension in excess of 3, which is required for the F-component. It is also worth mentioning that, at

Contribution	Mass Dimension	Possible Contributions
ΦVV	$\dim(\Phi VV) = 1$	$(m\Phi VV)_D$, $\left(m\bar{\Phi}VV\right)_D$
$\mathrm{T}\Phi VV$	$\dim(\mathrm{T}\Phi VV) = 2$	$(\mathrm{T}\Phi VV)_D$, $\left(\mathrm{T}\bar{\Phi}VV\right)_D$
ΦXV	$\dim(\Phi XV) = 2$	$(\Phi XV)_D$, $\left(\bar{\Phi}XV\right)_D$, $(\Phi YV)_D$, $\left(\bar{\Phi}YV\right)_D$
$\Phi DVDV$	$\dim(\Phi DVDV) = 2$	$(\Phi DVDV)_D$, $\left(\bar{\Phi}DVDV\right)_D$
ΦVX	$\dim(\Phi VX) = 2$	$(\Phi VX)_D$, $\left(\bar{\Phi}VX\right)_D$, $(\Phi VY)_D$, $\left(\bar{\Phi}VY\right)_D$
ΦXX	$\dim(\Phi XX) = 3$	$(\Phi XX)_F$, $\left(\bar{\Phi}YY\right)_F$

Table 7.2: Contributions to the Lagrangian for a coupling between one chiral superfields of the O'Raifeartaigh model Φ and two general superfield with one free spinor index. In addition to the contributions built from products of unbarred superfields, the Hermitian conjugates are permitted as well.

least in this scenario, it is impossible to construct a combination containing three fields and covariant derivatives that has no uncontracted indices as well as an integer valued mass dimension. Therefore, a structure constant would always have to be half-integer valued.

Dimensional analysis for the second case reveals that there are numerous possibilities for a coupling involving the D- and F-component. They can be categorised into three distinct groups based on the mass dimension of the superfield products without structure constants. It has to be emphasised that the contributions presented in Table 7.2 only discusses terms that can be constructed utilising the chiral superfield Φ, the kinetic superfield $\mathrm{T}\Phi$, and the general superfield V_α, as well as its covariant derivatives. Terms containing linear derivatives of Φ were ignored, because they are not chiral. This restriction was made to preserve the fact that the O'Raifeartaigh model is built solely using the chiral superfield Φ as well as the kinetic superfield $\mathrm{T}\Phi$.

The first group summarising all terms with mass dimension one contains only two terms as well as their Hermitian conjugates. The first term is the product of the chiral superfield Φ and two general superfields V_α, while the second one is the product of the anti-chiral superfield $\bar{\Phi}$ and two general superfields V_α. As V_α is neither chiral nor anti-chiral, only contributions via the D-component are possible.

The second group collects all terms with mass dimension two which means that it contains two covariant derivatives in addition to the field configuration of the first group. Within the outlined framework it is possible to construct 12 distinct terms which are all contributing via the D-component, as neither V_α nor DV are chiral or anti-chiral.

Of special interest is the third group summarising all terms with mass dimension 3. Due to its mass dimension it can only contain contributions via the F-component. Indeed, it is possible to construct two such contributions. They are very similar to the mass terms for the fermionic fields with mass dimension one $(mXX)_F$ and $(mYY)_F$ which potentially leads to a connection between the vacuum expectation value of the spontaneously broken superfield in the O'Raifeartaigh model and the mass of the fermionic fields with mass diemension one.

7.3 The Interaction Lagrangian

Even though there is a large number of potential contributions to the Lagrangian, the following discussion will be restricted to the most promising ones which are those contributing via the F-component. This reduces the number of terms from 32 to 4 if the Hermitian conjugates are considered as well. Of the 4 remaining terms two contain only chiral superfields, ΦXX and $\Phi\bar{Y}\bar{Y}$, while $\bar{\Phi}\bar{X}\bar{X}$ and $\bar{\Phi}YY$ are solely made up of anti-chiral superfields. At the present point the index that distinguishes the three distinct chiral superfields Φ_j of the O'Raifeartaigh model is treated generally. Later on in the discussion it will be restricted to describe coupling of the chiral superfields X_α and the anti-chiral superfields Y_α to a specific superfield of the O'Raifeartaigh model, e. g., Φ_3.

The product of the chiral superfield Φ_j from equation (7.1) with the two chiral superfields X_α as derived in equation (3.21) is given by

$$\begin{aligned}\Phi_j X^\alpha X_\alpha = &\left(A_j + \theta^\delta \phi_{j\delta} + \theta^2 F_j - i\theta\not{\partial}\bar{\theta} A_j + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\epsilon}}\bar{\not{\partial}}_{\dot{\epsilon}}{}^{\delta}\phi_{j\delta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box A_j\right)\\ &\left(\chi^\alpha + \theta^\beta \tilde{S}_\beta{}^\alpha + \theta^2\left(\tilde{\lambda}^\alpha + \frac{i}{2}\tilde{\omega}^\alpha\right) - i\theta\not{\partial}\bar{\theta}\chi^\alpha + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\tau}}\bar{\not{\partial}}_{\dot{\tau}}{}^{\beta}\tilde{S}_\beta{}^\alpha - \frac{1}{4}\theta^2\bar{\theta}^2\Box\chi^\alpha\right)\\ &\left(\chi_\alpha + \theta^\gamma \tilde{S}_{\gamma\alpha} + \theta^2\left(\tilde{\lambda}_\alpha + \frac{i}{2}\tilde{\omega}_\alpha\right) - i\theta\not{\partial}\bar{\theta}\chi_\alpha + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\kappa}}\bar{\not{\partial}}_{\dot{\kappa}}{}^{\gamma}\tilde{S}_{\gamma\alpha} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\chi_\alpha\right).\end{aligned} \tag{7.43}$$

On the first glance this product of three superfields looks rather difficult. However, if it is recalled that the term of interest is the F-component – or in other words all terms proportional to θ^2 – the calculation simplifies significantly. It is sufficient to restrict the discussion to the pure component fields as well as the terms linear and quadratic in θ. This results in 6 terms that are proportional to θ^2

$$(\Phi_j X^\alpha X_\alpha)_{\theta^2} = \theta^2 A_j \chi^\alpha \left(\tilde{\lambda}_\alpha + \frac{i}{2}\tilde{\omega}_\alpha\right) - \frac{1}{2}\epsilon^{\beta\gamma}\theta^2 A_j \tilde{S}_\beta{}^\alpha \tilde{S}_{\gamma\alpha} + \theta^2 A_j \left(\tilde{\lambda}^\alpha + \frac{i}{2}\tilde{\omega}^\alpha\right)\chi_\alpha$$
$$- \frac{1}{2}\epsilon^{\delta\gamma}\theta^2 \phi_{j\delta}\chi^\alpha \tilde{S}_{\gamma\alpha} + \frac{1}{2}\epsilon^{\delta\beta}\theta^2 \phi_{3\delta}\tilde{S}_\beta{}^\alpha \chi_\alpha + \theta^2 F_j \chi^\alpha \chi_\alpha \,. \tag{7.44}$$

After contraction of the spinor indices the F-component is found to be

$$(\Phi_j X^\alpha X_\alpha)_F = 2A_j \chi\tilde{\lambda} + iA_j \chi\tilde{\omega} - \frac{1}{2}A_j \mathrm{Tr}\left(\tilde{S}^T \tilde{S}\right) - \phi_j \tilde{S}\chi + F_j \chi\chi \,, \tag{7.45}$$

where it was assumed that the fermionic and bosonic component fields commute. Similar calculations can be repeated for the 3 remaining superfield products that contribute to the Lagrangian

$$\left(\bar{\Phi}_j \bar{X}_{\dot{\alpha}} \bar{X}^{\dot{\alpha}}\right)_F = 2A_j^\dagger \bar{\chi}\bar{\tilde{\lambda}} - iA_j^\dagger \bar{\chi}\bar{\tilde{\omega}} - \frac{1}{2}A_j^\dagger \mathrm{Tr}\left(\bar{\tilde{S}}^T \bar{\tilde{S}}\right) + \bar{\phi}_j \bar{\tilde{S}}\bar{\chi} + F_j^\dagger \bar{\chi}\bar{\chi} \,, \tag{7.46}$$

$$\left(\bar{\Phi}_j Y^\alpha Y_\alpha\right)_F = 2A_j^\dagger \psi\tilde{\lambda} - iA_j^\dagger \psi\tilde{\omega} - \frac{1}{2}A_j^\dagger \mathrm{Tr}\left(\tilde{R}^T \tilde{R}\right) - \bar{\phi}_j \tilde{R}\psi + F_j^\dagger \psi\psi \,, \tag{7.47}$$

$$\left(\Phi_j \bar{Y}_{\dot{\alpha}} \bar{Y}^{\dot{\alpha}}\right)_F = 2A_j \bar{\psi}\bar{\tilde{\lambda}} + iA_j \bar{\psi}\bar{\tilde{\omega}} - \frac{1}{2}A_j \mathrm{Tr}\left(\bar{\tilde{R}}^T \bar{\tilde{R}}\right) + \phi_j \bar{\tilde{R}}\bar{\psi} + F_j \bar{\psi}\bar{\psi} \,. \tag{7.48}$$

The Lagrangian describing the coupling of fermionic fields with mass dimension one to the O'Raifeartaigh model is then nothing but the sum of the O'Raifeartaigh Lagrangian from equation (7.8), the Lagrangian for fermionic fields with mass dimension one from equation (3.85), and the interaction terms in equations (7.45) to (7.48). In terms of superfield products the Lagrangian can be expressed in a very compact form

$$\mathcal{L} = \frac{1}{2}(\Phi_a \cdot \mathrm{T}\Phi_a)_F - \lambda_a (\Phi_a)_F - \frac{1}{2}M_{ab}(\Phi_a \cdot \Phi_b)_F - \frac{1}{3}g_{abc}(\Phi_a \cdot \Phi_b \cdot \Phi_c)_F + h.c.$$
$$+ (X^\alpha Y_\alpha)_D + (Y^\alpha X_\alpha)_D + \frac{m}{2}(X^\alpha X_\alpha)_F + \frac{m}{2}(Y^\alpha Y_\alpha)_F + h.c.$$
$$+ \xi(\Phi_j X^\alpha X_\alpha)_F + \xi\left(\bar{\Phi}_j \bar{X}_{\dot{\alpha}} \bar{X}^{\dot{\alpha}}\right)_F + \xi\left(\bar{\Phi}_j Y^\alpha Y_\alpha\right)_F + \xi\left(\Phi_j \bar{Y}_{\dot{\alpha}} \bar{Y}^{\dot{\alpha}}\right)_F \,, \tag{7.49}$$

where the strength of the interaction is encoded in the real coupling constant ξ. Alternatively, the Lagrangian can be written in terms of the superfield components

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}A_a\Box A_a^\dagger + \frac{1}{2}F_aF_a^\dagger + \frac{i}{4}\phi_a\bar{\partial\!\!\!/}\bar{\phi}_a - \lambda_aF_a - \frac{1}{2}M_{ab}\left(A_aF_b + F_aA_b - \frac{1}{2}\phi_a\phi_b\right) \\
&- \frac{1}{3}g_{abc}\left(A_aA_bF_c + A_aF_bA_c + F_aA_bA_c - \frac{3}{2}A_a\phi_b\phi_c\right) \\
&- \frac{1}{2}\Box A_aA_a^\dagger + \frac{1}{2}F_aF_a^\dagger + \frac{i}{4}\bar{\phi}_a\bar{\partial\!\!\!/}\phi_a - \lambda_aF_a^\dagger - \frac{1}{2}M_{ab}\left(A_a^\dagger F_b^\dagger + F_a^\dagger A_b^\dagger - \frac{1}{2}\bar{\phi}_a\bar{\phi}_b\right) \\
&- \frac{1}{3}g_{abc}\left(A_a^\dagger A_b^\dagger F_c^\dagger + A_a^\dagger F_b^\dagger A_c^\dagger + F_a^\dagger A_b^\dagger A_c^\dagger - \frac{3}{2}A_a^\dagger\bar{\phi}_b\bar{\phi}_c\right) \\
&+ 2\partial_\mu\chi\partial^\mu\psi + 2\tilde{\lambda}\tilde{\lambda} + \frac{1}{2}\tilde{\omega}\tilde{\omega} + m\chi\tilde{\lambda} + \frac{im}{2}\chi\tilde{\omega} + m\psi\tilde{\lambda} - \frac{im}{2}\psi\tilde{\omega} + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T\partial\!\!\!/\tilde{R}\right) \\
&+ \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T\bar{\partial\!\!\!/}\tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) - \frac{m}{4}\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) + 2\partial_\mu\bar{\chi}\partial^\mu\bar{\psi} + 2\bar{\tilde{\lambda}}\bar{\tilde{\lambda}} + \frac{1}{2}\bar{\tilde{\omega}}\bar{\tilde{\omega}} + m\bar{\chi}\bar{\tilde{\lambda}} \\
&- \frac{im}{2}\bar{\chi}\bar{\tilde{\omega}} + m\bar{\psi}\bar{\tilde{\lambda}} + \frac{im}{2}\bar{\psi}\bar{\tilde{\omega}} + \frac{i}{2}\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\partial\!\!\!/}\bar{\tilde{R}}\right) + \frac{i}{2}\mathrm{Tr}\left(\bar{\tilde{R}}^T\partial\!\!\!/\bar{\tilde{S}}\right) - \frac{m}{4}\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\tilde{S}}\right) \\
&- \frac{m}{4}\mathrm{Tr}\left(\bar{\tilde{R}}^T\bar{\tilde{R}}\right) + 2\xi A_j\chi\lambda + i\xi A_j\chi\tilde{\omega} - \frac{\xi}{2}A_j\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) - \xi\phi_j\tilde{S}\chi + \xi F_j\chi\chi + 2\xi A_j^\dagger\bar{\chi}\bar{\lambda} \\
&- i\xi A_j^\dagger\bar{\chi}\bar{\tilde{\omega}} - \frac{\xi}{2}A_j^\dagger\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\tilde{S}}\right) + \xi\bar{\phi}_j\bar{\tilde{S}}\bar{\chi} + \xi F_j^\dagger\bar{\chi}\bar{\chi} + 2\xi A_j^\dagger\psi\lambda - i\xi A_j^\dagger\psi\tilde{\omega} - \frac{\xi}{2}A_j^\dagger\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) \\
&- \xi\bar{\phi}_j\tilde{R}\psi + \xi F_j^\dagger\psi\psi + 2\xi A_j\bar{\psi}\bar{\lambda} + i\xi A_j\bar{\psi}\bar{\tilde{\omega}} - \frac{\xi}{2}A_j\mathrm{Tr}\left(\bar{\tilde{R}}^T\bar{\tilde{R}}\right) + \xi\phi_j\bar{\tilde{R}}\bar{\psi} + \xi F_j\bar{\psi}\bar{\psi}\,. \qquad (7.50)
\end{aligned}
$$

7.3.1 The Equations of Motion for the Auxiliary Fields

A brief look at the Lagrangian in equation (7.50) reveals that the fields F_a, $\tilde{\lambda}$, and $\tilde{\omega}$ are auxiliary fields and thus can be eliminated from the Lagrangian using their respective equations of motion. In addition to the equations of motion for the auxiliary fields the equations of motion for the spinor fields ϕ_j and the second rank spinor fields $\tilde{R}$ and $\tilde{S}$ were derived as well as they will prove useful for the derivation of the actual superpotential without kinetic terms. The equations of motion can then be summarised to

$$F_d = \lambda_d + M_{da}A_a^\dagger + g_{dab}A_a^\dagger A_b^\dagger - \xi\delta_{dj}\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\,, \qquad (7.51)$$

$$\tilde{\lambda}_\alpha = -\frac{m}{4}\left(\chi_\alpha + \psi_\alpha\right) - \xi\frac{1}{2}\left(A_j\chi_\alpha + A_j^\dagger\psi_\alpha\right)\,, \qquad (7.52)$$

$$\tilde{\omega}_\alpha = -\frac{im}{2}\left(\chi_\alpha - \psi_\alpha\right) - i\xi\left(A_j\chi_\alpha - A_j^\dagger\psi_\alpha\right)\,, \qquad (7.53)$$

$$\frac{i}{2}\bar{\partial\!\!\!/}_{\dot{\alpha}\beta}\phi_d^\beta = \frac{1}{2}M_{da}\bar{\phi}_{\dot{\alpha}a} + g_{dab}A_a^\dagger\bar{\phi}_{\dot{\alpha}b} + \xi\delta_{dj}\left(\bar{\tilde{S}}_{\dot{\alpha}\dot{\gamma}}\bar{\chi}^{\dot{\gamma}} + \tilde{R}_{\dot{\alpha}\gamma}\psi^\gamma\right)\,, \qquad (7.54)$$

$$\left(\frac{m}{2} + \xi A_j^\dagger\right)\tilde{R}_{\dot{\beta}\alpha} = -i\bar{\partial\!\!\!/}_{\dot{\beta}\gamma}\tilde{S}^\gamma{}_\alpha - \xi\bar{\phi}_{j\dot{\beta}}\psi_\alpha\,, \qquad (7.55)$$

$$\left(\frac{m}{2} + \xi A_j\right)\tilde{S}_{\beta\alpha} = i\partial\!\!\!/_{\beta\dot{\delta}}\tilde{R}^{\dot{\delta}}{}_\alpha - \xi\phi_{j\beta}\chi_\alpha\,. \qquad (7.56)$$

It is worth mentioning that it is not necessary to derive the equations of motion for the conjugate component fields as they are related to the unbarred fields by Hermitian conjugation

$$F_d^\dagger = \lambda_d + M_{da}A_a + g_{dab}A_aA_b - \xi\delta_{dj}\left(\chi\chi + \bar{\psi}\bar{\psi}\right) , \tag{7.57}$$

$$\tilde{\bar{\lambda}}_{\dot{\alpha}} = -\frac{m}{4}\left(\bar{\chi}_{\dot{\alpha}} + \bar{\psi}_{\dot{\alpha}}\right) - \frac{\xi}{2}\left(A_j^\dagger\bar{\chi}_{\dot{\alpha}} + A_j\bar{\psi}_{\dot{\alpha}}\right) , \tag{7.58}$$

$$\tilde{\bar{\omega}}_{\dot{\alpha}} = \frac{im}{2}\left(\bar{\chi}_{\dot{\alpha}} - \bar{\psi}_{\dot{\alpha}}\right) + i\xi\left(A_j^\dagger\bar{\chi}_{\dot{\alpha}} - A_j\bar{\psi}_{\dot{\alpha}}\right) , \tag{7.59}$$

$$-\frac{i}{2}\partial\!\!\!/_{\alpha\dot{\beta}}\bar{\phi}_d^{\dot{\beta}} = \frac{1}{2}M_{da}\phi_{\alpha a} + g_{dab}A_a\phi_{\alpha b} + \xi\delta_{dj}\left(\tilde{S}_{\alpha\gamma}\chi^\gamma + \tilde{\bar{R}}_{\alpha\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\right) , \tag{7.60}$$

$$\left(\frac{m}{2} + \xi A_j\right)\tilde{\bar{R}}_{\beta\dot{\alpha}} = i\partial\!\!\!/_{\beta\dot{\gamma}}\tilde{\bar{S}}^{\dot{\gamma}}{}_{\dot{\alpha}} + \xi\phi_{j\beta}\bar{\psi}_{\dot{\alpha}} , \tag{7.61}$$

$$\left(\frac{m}{2} + \xi A_j^\dagger\right)\tilde{\bar{S}}_{\dot{\beta}\dot{\alpha}} = -i\bar{\partial\!\!\!/}_{\dot{\beta}\delta}\tilde{\bar{R}}^\delta{}_{\dot{\alpha}} + \xi\bar{\phi}_{j\dot{\beta}}\bar{\chi}_{\dot{\alpha}} . \tag{7.62}$$

7.3.2 Spontaneous Symmetry Breaking

To determine whether or not supersymmetry is spontaneously broken the Lagrangian in equation (7.50) describing the coupling between the O'Raifeartaigh model and the fermionic sector has to be split up into the super kinetic term $\mathcal{L}_{\text{kin}}$ and the superpotential $\mathcal{L}_{\text{pot}}$. The super-kinetic term is given by

$$\begin{aligned}\mathcal{L}_{\text{kin}} = &-\frac{1}{2}A_a\Box A_a^\dagger - \frac{1}{2}\Box A_aA_a^\dagger + \frac{i}{4}\phi_a\partial\!\!\!/\bar{\phi}_a + \frac{i}{4}\bar{\phi}_a\bar{\partial\!\!\!/}\phi_a + 2\partial_\mu\chi\partial^\mu\psi + 2\partial_\mu\bar{\chi}\partial^\mu\bar{\psi}\\ &+ \frac{i}{2}\text{Tr}\left(\tilde{S}^T\partial\!\!\!/\tilde{R}\right) + \frac{i}{2}\text{Tr}\left(\tilde{R}^T\bar{\partial\!\!\!/}\tilde{S}\right) + \frac{i}{2}\text{Tr}\left(\tilde{\bar{S}}^T\bar{\partial\!\!\!/}\tilde{\bar{R}}\right) + \frac{i}{2}\text{Tr}\left(\tilde{\bar{R}}^T\partial\!\!\!/\tilde{\bar{S}}\right) .\end{aligned} \tag{7.63}$$

The remaining terms make up the superpotential

$$\begin{aligned}\mathcal{L}_{\text{pot}} = &-F_aF_a^\dagger + \lambda_aF_a + \lambda_aF_a^\dagger + \frac{1}{2}M_{ab}\left(A_aF_b + F_aA_b - \frac{1}{2}\phi_a\phi_b + A_a^\dagger F_b^\dagger + F_a^\dagger A_b^\dagger - \frac{1}{2}\bar{\phi}_a\bar{\phi}_b\right)\\ &+ \frac{1}{3}g_{abc}\left(A_aA_bF_c + A_aF_bA_c + F_aA_bA_c - \frac{3}{2}A_a\phi_b\phi_c + A_a^\dagger A_b^\dagger F_c^\dagger + A_a^\dagger F_b^\dagger A_c^\dagger\right.\\ &\left.+F_a^\dagger A_b^\dagger A_c^\dagger - \frac{3}{2}A_a^\dagger\bar{\phi}_b\bar{\phi}_c\right) - \left(2\tilde{\lambda} + m\chi + m\psi + 2\xi A_j\chi + 2\xi A_j^\dagger\psi\right)\tilde{\lambda}\\ &- \left(\frac{1}{2}\tilde{\omega} + \frac{im}{2}\chi - \frac{im}{2}\psi + i\xi A_j\chi - i\xi A_j^\dagger\psi\right)\tilde{\omega} + \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\text{Tr}\left(\tilde{S}^T\tilde{S}\right)\\ &+ \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\text{Tr}\left(\tilde{R}^T\tilde{R}\right) - \left(2\tilde{\bar{\lambda}} + m\bar{\chi} + m\bar{\psi} + 2\xi A_j^\dagger\bar{\chi} + 2\xi A_j\bar{\psi}\right)\tilde{\bar{\lambda}}\end{aligned}$$

$$- \left(\frac{1}{2}\tilde{\bar{\omega}} - \frac{im}{2}\bar{\chi} + \frac{im}{2}\bar{\psi} - i\xi A_j^\dagger \bar{\chi} + i\xi A_j \bar{\psi}\right)\tilde{\bar{\omega}} + \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\tilde{\bar{S}}^T \tilde{\bar{S}}\right)$$

$$+ \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\tilde{\bar{R}}^T \tilde{\bar{R}}\right) + \xi\phi_j \tilde{S}\chi - \xi F_j \chi\chi - \xi\bar{\phi}_j \tilde{\bar{S}}\bar{\chi} - \xi F_j^\dagger \bar{\chi}\bar{\chi}$$

$$+ \xi\bar{\phi}_j \tilde{R}\psi - \xi F_j^\dagger \psi\psi - \xi\phi_j \tilde{\bar{R}}\bar{\psi} - \xi F_j \bar{\psi}\bar{\psi}\,, \tag{7.64}$$

where the potential part of the Lagrangian is defined in analogy to the superpotential $(V)_F$ which corresponds to a sign convention of $\mathcal{L} = \mathcal{L}_{\text{kin}} - \mathcal{L}_{\text{pot}}$.

To determine whether supersymmetry is spontaneously broken the following discussion can be restricted to the superpotential. The superpotential has then to be expressed solely in terms of the auxiliary fields F_a, $\tilde{\lambda}$, and $\tilde{\omega}$. It turns out that the combination of component fields in the prefactors of $\tilde{\omega}$ and $\tilde{\lambda}$ correspond to the equations of motion of $\tilde{\omega}$ and $\tilde{\lambda}$ which simplifies the superpotential significantly

$$\begin{aligned}\mathcal{L}_{\text{pot}} = &-F_a F_a^\dagger + \lambda_a F_a + \lambda_a F_a^\dagger + M_{ab} A_a F_b - \frac{1}{4} M_{ab}\phi_a\phi_b + M_{ab} A_a^\dagger F_b^\dagger - \frac{1}{4} M_{ab}\bar{\phi}_a\bar{\phi}_b \\ &+ g_{abc} A_a A_b F_c - \frac{1}{2} g_{abc} A_a \phi_b \phi_c + g_{abc} A_a^\dagger A_b^\dagger F_c^\dagger - \frac{1}{2} g_{abc} A_a^\dagger \bar{\phi}_b \bar{\phi}_c + 2\tilde{\lambda}\tilde{\lambda} + \frac{1}{2}\tilde{\omega}\tilde{\omega} \\ &+ \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\tilde{S}^T \tilde{S}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\tilde{R}^T \tilde{R}\right) + 2\tilde{\bar{\lambda}}\tilde{\bar{\lambda}} + \frac{1}{2}\tilde{\bar{\omega}}\tilde{\bar{\omega}} \\ &+ \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\tilde{\bar{S}}^T \tilde{\bar{S}}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\tilde{\bar{R}}^T \tilde{\bar{R}}\right) + \xi\phi_j \tilde{S}\chi + \xi F_j \chi\chi \\ &- \xi\bar{\phi}_j \tilde{\bar{S}}\bar{\chi} - \xi F_j^\dagger \bar{\chi}\bar{\chi} + \xi\bar{\phi}_j \tilde{R}\psi - \xi F_j^\dagger \psi\psi - \xi\phi_j \tilde{\bar{R}}\bar{\psi} - \xi F_j \bar{\psi}\bar{\psi}\,.\end{aligned} \tag{7.65}$$

Collecting all terms proportional to F_a and $F_a^\dagger$ reveals that the combinations of component fields making up the prefactors correspond to $F_a^\dagger$ and F_a, respectively

$$\begin{aligned}\mathcal{L}_{\text{pot}} = &F_a F_a^\dagger - \frac{1}{4} M_{ab}\phi_a\phi_b - \frac{1}{4} M_{ab}\bar{\phi}_a\bar{\phi}_b - \frac{1}{2} g_{abc} A_a \phi_b \phi_c - \frac{1}{2} g_{abc} A_a^\dagger \bar{\phi}_b \bar{\phi}_c \\ &+ 2\tilde{\lambda}\tilde{\lambda} + \frac{1}{2}\tilde{\omega}\tilde{\omega} + \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\tilde{S}^T \tilde{S}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\tilde{R}^T \tilde{R}\right) \\ &+ 2\tilde{\bar{\lambda}}\tilde{\bar{\lambda}} + \frac{1}{2}\tilde{\bar{\omega}}\tilde{\bar{\omega}} + \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\tilde{\bar{S}}^T \tilde{\bar{S}}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\tilde{\bar{R}}^T \tilde{\bar{R}}\right) \\ &+ \xi\phi_j \tilde{S}\chi - \xi\bar{\phi}_j \tilde{\bar{S}}\bar{\chi} + \xi\bar{\phi}_j \tilde{R}\psi - \xi\phi_j \tilde{\bar{R}}\bar{\psi}\,.\end{aligned} \tag{7.66}$$

Up to now only the equations of motion for the auxiliary fields have been used to rewrite the superpotential in terms of the auxiliary fields. However, the intermediate result for the superpotential still depends on A_j, ϕ_j, $\tilde{S}$, and $\tilde{R}$. At this point the equations of motion for these fields have to be used to eliminate or rewrite the terms of interest. Using the equation of motion for ϕ_a and $\bar{\phi}_a$ the 8 terms containing ϕ_a and its Hermitian conjugate are reduced to two kinetic terms

$$\begin{aligned}\mathcal{L}_{\text{pot}} = {} & F_a F_a^\dagger + \frac{i}{2}\phi_a^\alpha \not{\partial}_{\alpha\dot{\beta}}\bar{\phi}_a^{\dot{\beta}} + \frac{i}{2}\bar{\phi}_a^{\dot{\alpha}}\bar{\not{\partial}}_{\dot{\alpha}\beta}\phi_a^\beta - \frac{\xi}{2}\delta_{ja}\phi_a^\alpha \tilde{S}_{\alpha\beta}\chi^\beta - \frac{\xi}{2}\delta_{ja}\phi_a^\alpha \bar{\tilde{R}}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} \\ & + \frac{\xi}{2}\delta_{ja}\bar{\phi}_a^{\dot{\alpha}}\bar{\tilde{S}}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}} + \frac{\xi}{2}\delta_{ja}\bar{\phi}_a^{\dot{\alpha}}\tilde{R}_{\dot{\alpha}\beta}\psi^\beta + 2\tilde{\lambda}\tilde{\lambda} + \frac{1}{2}\tilde{\omega}\tilde{\omega} \\ & + \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) + 2\bar{\tilde{\lambda}}\bar{\tilde{\lambda}} + \frac{1}{2}\bar{\tilde{\omega}}\bar{\tilde{\omega}} \\ & + \left(\frac{m}{4} + \frac{\xi}{2}A_j^\dagger\right)\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\tilde{S}}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_j\right)\mathrm{Tr}\left(\bar{\tilde{R}}^T\bar{\tilde{R}}\right). \end{aligned} \tag{7.67}$$

The same can be repeated using the equations of motion for the second rank spinor fields $\tilde{S}$ and $\tilde{R}$. The resulting superpotential has an especially simple form

$$\begin{aligned}\mathcal{L}_{\text{pot}} = {} & F_a F_a^\dagger + \frac{i}{2}\phi_a\not{\partial}\bar{\phi}_a + \frac{i}{2}\bar{\phi}_a\bar{\not{\partial}}\phi_a + 2\tilde{\lambda}\tilde{\lambda} + \frac{1}{2}\tilde{\omega}\tilde{\omega} + \frac{i}{2}\mathrm{Tr}\left(\tilde{R}^T\bar{\not{\partial}}\tilde{S}\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{S}^T\not{\partial}\tilde{R}\right) \\ & + 2\bar{\tilde{\lambda}}\bar{\tilde{\lambda}} + \frac{1}{2}\bar{\tilde{\omega}}\bar{\tilde{\omega}} + \frac{i}{2}\mathrm{Tr}\left(\bar{\tilde{R}}^T\not{\partial}\bar{\tilde{S}}\right) + \frac{i}{2}\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\not{\partial}}\bar{\tilde{R}}\right). \end{aligned} \tag{7.68}$$

It is obvious that it still contains numerous kinetic terms for the bosonic component fields $\tilde{R}$ and $\tilde{S}$ as well as the fermionic component fields ϕ_a. As the super-kinetic term in equation (7.63) is by construction free of contributions to the superpotential the actual superpotential U is found to be

$$U = F_a F_a^\dagger + 2\left(\tilde{\lambda}\tilde{\lambda} + \bar{\tilde{\lambda}}\bar{\tilde{\lambda}}\right) + \frac{1}{2}\left(\tilde{\omega}\tilde{\omega} + \bar{\tilde{\omega}}\bar{\tilde{\omega}}\right). \tag{7.69}$$

This superpotential is in perfect analogy to the superpotential for the O'Raifeartaigh model that was derived in equation (7.23). Besides the term induced by the bosonic auxiliary fields F_a, it includes two additional terms for the auxiliary fields $\tilde{\lambda}$ and $\tilde{\omega}$, which originate in the model for fermionic fields with mass dimension one that were used to extend the O'Raifeartaigh model. As the contributions from $\tilde{\lambda}$ and $\tilde{\omega}$ are given by a sum of spinor products, it is not immediately clear whether the superpotential is always positive. This is due to the fact that a sum of spinor products and their Hermitian conjugates is real; however, this is not sufficient to conclude that they are positive as well. Nevertheless, there are two arguments that should guarantee a positive superpotential. First, the construction of the fermionic sector using the supersymmetry algebra ensures a positive energy spectrum.

For the O'Raifeartaigh model this property is well established and the positivity of the energy spectrum for the fermionic fields with mass dimension one was shown in Chapter 6. Second, the coupling of two theories with positive energy spectrum should possess the same property. Therefore, if the expectation values for all auxiliary fields F_a, $\tilde{\lambda}$, and $\tilde{\omega}$ vanish, supersymmetry is preserved. Otherwise the superpotential acquires a finite positive minimum and supersymmetry is spontaneously broken.

7.3.3 The On-shell Lagrangian

The calculation of the on-shell Lagrangian is very similar to the discussion in the previous section. However, this time the equations of motion for the auxiliary fields are used to eliminate the auxiliary fields from the superpotential. This can be done starting from the original equation for the superpotential in equation (7.64). The calculation can be simplified significantly if the intermediate result from equation (7.66) is used as a starting point. It is derived from equation (7.64) using solely the equations of motion for the auxiliary fields and the contribution of the auxiliary fields is restricted to only three terms. Subsequent results for the discussion of supersymmetry breaking cannot be employed as the use of the equations of motion for ϕ_a, $\tilde{R}$, and $\tilde{S}$ eliminates important contributions to the on-shell Lagrangian which are irrelevant for the previous discussion of supersymmetry breaking. Inserting the equations of motion for the auxiliary fields into equation (7.66) immediately leads to

$$\begin{aligned}
\mathcal{L}_{\text{pot}} = & -\lambda_a\lambda_a - \lambda_a M_{ad} A_d - \lambda_a g_{ade} A_d A_e + \xi\lambda_a\delta_{a3}\left(\chi\chi + \bar{\psi}\bar{\psi}\right) - \lambda_a M_{ab} A_b^\dagger \\
& - M_{ab} M_{ad} A_b^\dagger A_d - M_{ab} g_{ade} A_b^\dagger A_d A_e + \xi M_{ab} A_b^\dagger \delta_{aj}\left(\chi\chi + \bar{\psi}\bar{\psi}\right) - \lambda_a g_{abc} A_b^\dagger A_c^\dagger \\
& - M_{ad} g_{abc} A_b^\dagger A_c^\dagger A_d - g_{abc} g_{ade} A_b^\dagger A_c^\dagger A_d A_e + \xi g_{abc} A_b^\dagger A_c^\dagger \delta_{aj}\left(\chi\chi + \bar{\psi}\bar{\psi}\right) \\
& + \xi\lambda_a\delta_{aj}\left(\bar{\chi}\bar{\chi} + \psi\psi\right) + \xi M_{ad}\delta_{aj}\left(\bar{\chi}\bar{\chi} + \psi\psi\right) A_d + \xi g_{ade}\delta_{aj}\left(\bar{\chi}\bar{\chi} + \psi\psi\right) A_d A_e \\
& - \xi^2\delta_{aj}\delta_{aj}\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\left(\chi\chi + \bar{\psi}\bar{\psi}\right) + \frac{1}{4} M_{ab}\phi_a\phi_b + \frac{1}{4} M_{ab}\bar{\phi}_a\bar{\phi}_b + \frac{1}{2} g_{abc} A_a \phi_b \phi_c \\
& + \frac{1}{2} g_{abc} A_a^\dagger \bar{\phi}_b \bar{\phi}_c - 2\left(\frac{m}{2} + \xi A_j\right)\left(\frac{m}{2} + \xi A_j^\dagger\right)\chi\psi - \left(\frac{m}{4} + \frac{\xi}{2} A_j\right)\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) \\
& - \left(\frac{m}{4} + \frac{\xi}{2} A_j^\dagger\right)\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) - 2\left(\frac{m}{2} + \xi A_j\right)\left(\frac{m}{2} + \xi A_j^\dagger\right)\bar{\chi}\bar{\psi}
\end{aligned}$$

$$
- \left(\frac{m}{4} + \frac{\xi}{2} A_j^\dagger\right) \mathrm{Tr}\left(\tilde{\bar{S}}^T \tilde{\bar{S}}\right) - \left(\frac{m}{4} + \frac{\xi}{2} A_j\right) \mathrm{Tr}\left(\tilde{\bar{R}}^T \tilde{\bar{R}}\right) - \xi \phi_j \tilde{S} \chi + \xi \bar{\phi}_j \tilde{\bar{S}} \bar{\chi}
$$

$$
- \xi \bar{\phi}_j \tilde{R} \psi + \xi \phi_j \tilde{\bar{R}} \bar{\psi} \,. \tag{7.70}
$$

Therefore the on-shell Lagrangian is given by

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2} A_a \Box A_a^\dagger - \frac{1}{2} \Box A_a A_a^\dagger + \frac{i}{4} \phi_a \partial\!\!\!/ \bar{\phi} + \frac{i}{4} \bar{\phi}_a \bar{\partial\!\!\!/} \phi_a + \partial_\mu \chi \partial^\mu \psi + \partial_\mu \psi \partial^\mu \chi + \partial_\mu \bar{\chi} \partial^\mu \bar{\psi} \\
& + \partial_\mu \bar{\psi} \partial^\mu \bar{\chi} + \frac{i}{2} \mathrm{Tr}\left(\tilde{S}^T \partial\!\!\!/ \tilde{R}\right) + \frac{i}{2} \mathrm{Tr}\left(\tilde{R}^T \bar{\partial\!\!\!/} \tilde{S}\right) + \frac{i}{2} \mathrm{Tr}\left(\tilde{\bar{S}}^T \bar{\partial\!\!\!/} \tilde{\bar{R}}\right) + \frac{i}{2} \mathrm{Tr}\left(\tilde{\bar{R}}^T \partial\!\!\!/ \tilde{\bar{S}}\right) \\
& - \lambda_a \lambda_a - \lambda_a M_{ad} A_d - \lambda_a g_{ade} A_d A_e + \xi \lambda_a \delta_{aj} \left(\chi\chi + \bar{\psi}\bar{\psi}\right) - \lambda_a M_{ab} A_b^\dagger - M_{ab} M_{ad} A_b^\dagger A_d \\
& - M_{ab} g_{ade} A_b^\dagger A_d A_e + \xi M_{ab} A_b^\dagger \delta_{aj} \left(\chi\chi + \bar{\psi}\bar{\psi}\right) - \lambda_a g_{abc} A_b^\dagger A_c^\dagger - M_{ad} g_{abc} A_b^\dagger A_c^\dagger A_d \\
& - g_{abc} g_{ade} A_b^\dagger A_c^\dagger A_d A_e + \xi g_{abc} A_b^\dagger A_c^\dagger \delta_{aj} \left(\chi\chi + \bar{\psi}\bar{\psi}\right) + \xi \lambda_a \delta_{aj} \left(\bar{\chi}\bar{\chi} + \psi\psi\right) \\
& + \xi M_{ad} \delta_{aj} \left(\bar{\chi}\bar{\chi} + \psi\psi\right) A_d + \xi g_{ade} \delta_{aj} \left(\bar{\chi}\bar{\chi} + \psi\psi\right) A_d A_e \\
& - \xi^2 \delta_{aj} \delta_{aj} \left(\bar{\chi}\bar{\chi} + \psi\psi\right) \left(\chi\chi + \bar{\psi}\bar{\psi}\right) + \frac{1}{4} M_{ab} \phi_a \phi_b + \frac{1}{4} M_{ab} \bar{\phi}_a \bar{\phi}_b + \frac{1}{2} g_{abc} A_a \phi_b \phi_c \\
& + \frac{1}{2} g_{abc} A_a^\dagger \bar{\phi}_b \bar{\phi}_c - 2 \left(\frac{m}{2} + \xi A_j\right) \left(\frac{m}{2} + \xi A_j^\dagger\right) \chi\psi - \left(\frac{m}{4} + \frac{\xi}{2} A_j\right) \mathrm{Tr}\left(\tilde{S}^T \tilde{S}\right) \\
& - \left(\frac{m}{4} + \frac{\xi}{2} A_j^\dagger\right) \mathrm{Tr}\left(\tilde{R}^T \tilde{R}\right) - 2 \left(\frac{m}{2} + \xi A_j\right) \left(\frac{m}{2} + \xi A_j^\dagger\right) \bar{\chi}\bar{\psi} - \left(\frac{m}{4} + \frac{\xi}{2} A_j^\dagger\right) \mathrm{Tr}\left(\tilde{\bar{S}}^T \tilde{\bar{S}}\right) \\
& - \left(\frac{m}{4} + \frac{\xi}{2} A_j\right) \mathrm{Tr}\left(\tilde{\bar{R}}^T \tilde{\bar{R}}\right) - \xi \phi_j \tilde{S} \chi + \xi \bar{\phi}_j \tilde{\bar{S}} \bar{\chi} - \xi \bar{\phi}_j \tilde{R} \psi + \xi \phi_j \tilde{\bar{R}} \bar{\psi} \,.
\end{aligned} \tag{7.71}
$$

This is the most general Lagrangian describing the coupling of the O'Raifeartaigh model to a fermionic sector as discussed in Chapters 3 to 6. Up to now no assumptions besides the usual symmetry properties were made regarding the structure constants of the O'Raifeartaigh model. Furthermore, the coupling of the fermionic sector to the O'Raifeartaigh is not restricted to a specific superfield.

7.4 Coupling to the Field with Nonzero Expectation Value

The O'Raifeartaigh model contains three distinguishable chiral superfields of which only one obtains a nonvanishing expectation value. Therefore, there are two possibilities to couple the fermionic sector to the O'Raifeartaigh model. It can either be coupled to the superfield with nonvanishing expectation value – for the specific choice of structure constants outlined in equations (7.25) to (7.27) this corresponds to a coupling to Φ_3 – or to one of the superfields with vanishing expectation value. Without loss of generality it is sufficient to discuss one of the possible scenarios as the results for the other case can be obtained in perfect analogy. It can be shown that differences between the scenarios are restricted to the matrix components of the mass matrices while the fundamental properties, e. g., spontaneous supersymmetry breaking, are preserved.

For convenience it was chosen to discuss the coupling of the fermionic fields with mass dimension one to the chiral superfield of the O'Raifeartaigh model with nonvanishing expectation value. The structure constants remain those introduced in equations (7.25) to (7.27) for the discussion of the O'Raifeartaigh model. For this specific choice the on-shell Lagrangian from equation (7.71) is given by

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}A_1\Box A_1^\dagger - \frac{1}{2}A_2\Box A_2^\dagger - \frac{1}{2}A_3\Box A_3^\dagger - \frac{1}{2}\Box A_1 A_1^\dagger - \frac{1}{2}\Box A_2 A_2^\dagger - \frac{1}{2}\Box A_3 A_3^\dagger + \frac{i}{4}\phi_1\partial\!\!\!/\bar{\phi}_1 \\
&+ \frac{i}{4}\phi_2\partial\!\!\!/\bar{\phi}_2 + \frac{i}{4}\phi_3\partial\!\!\!/\bar{\phi}_3 + \frac{i}{4}\bar{\phi}_1\bar{\partial\!\!\!/}\phi_1 + \frac{i}{4}\bar{\phi}_2\bar{\partial\!\!\!/}\phi_2 + \frac{i}{4}\bar{\phi}_3\bar{\partial\!\!\!/}\phi_3 + 2\partial_\mu\chi\partial^\mu\psi + 2\partial_\mu\bar{\chi}\partial^\mu\bar{\psi} \\
&+ \frac{i}{2}\mathrm{Tr}\Big(\tilde{S}^T\partial\!\!\!/\tilde{R}\Big) + \frac{i}{2}\mathrm{Tr}\Big(\tilde{R}^T\bar{\partial\!\!\!/}\tilde{S}\Big) + \frac{i}{2}\mathrm{Tr}\Big(\bar{\tilde{S}}^T\bar{\partial\!\!\!/}\bar{\tilde{R}}\Big) + \frac{i}{2}\mathrm{Tr}\Big(\bar{\tilde{R}}^T\partial\!\!\!/\bar{\tilde{S}}\Big) - \Lambda^2 \\
&- \Lambda g A_1 A_1 + \xi\Lambda\left(\chi\chi + \bar{\psi}\bar{\psi}\right) - M^2 A_2^\dagger A_2 - M^2 A_1^\dagger A_1 - 2MgA_2^\dagger A_1 A_3 - \Lambda g A_1^\dagger A_1^\dagger \\
&- MgA_2^\dagger A_1^\dagger A_3 - MgA_2^\dagger A_3^\dagger A_1 - 4g^2 A_1^\dagger A_3^\dagger A_1 A_3 - g^2 A_1^\dagger A_1^\dagger A_1 A_1 + \xi g A_1^\dagger A_1^\dagger\left(\chi\chi + \bar{\psi}\bar{\psi}\right) \\
&+ \xi\Lambda\left(\bar{\chi}\bar{\chi} + \psi\psi\right) + \xi g\left(\bar{\chi}\bar{\chi} + \psi\psi\right)A_1 A_1 - \xi^2\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\left(\chi\chi + \bar{\psi}\bar{\psi}\right) \\
&+ \frac{1}{2}M\phi_1\phi_2 + \frac{1}{2}M\bar{\phi}_1\bar{\phi}_2 + gA_1\phi_1\phi_3 + \frac{1}{2}gA_3\phi_1\phi_1 + gA_1^\dagger\bar{\phi}_1\bar{\phi}_3 + \frac{1}{2}gA_3^\dagger\bar{\phi}_1\bar{\phi}_1 \\
&- 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\chi\psi - \left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\mathrm{Tr}\Big(\tilde{S}^T\tilde{S}\Big) - \left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\mathrm{Tr}\Big(\tilde{R}^T\tilde{R}\Big) \\
&- 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\bar{\chi}\bar{\psi} - \left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\mathrm{Tr}\Big(\bar{\tilde{S}}^T\bar{\tilde{S}}\Big) - \left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\mathrm{Tr}\Big(\bar{\tilde{R}}^T\bar{\tilde{R}}\Big) \\
&- \xi\phi_3\tilde{S}\chi + \xi\bar{\phi}_3\bar{\tilde{S}}\bar{\chi} - \xi\bar{\phi}_3\tilde{R}\psi + \xi\phi_3\bar{\tilde{R}}\bar{\psi}\,.
\end{aligned} \tag{7.72}
$$

Based on this Lagrangian the equations of motion for the auxiliary fields can be derived in the usual way using the Euler-Lagrange equations. Alternatively, the general equations of motion from equations (7.51) to (7.53) can be adapted according to the specific choice of structure constants and coupling to Φ_3

$$F_1 = MA_2^\dagger + g\left(A_1^\dagger A_3^\dagger + A_3^\dagger A_1^\dagger\right)\,, \tag{7.73}$$

$$F_2 = MA_1^\dagger\,, \tag{7.74}$$

$$F_3 = \Lambda + gA_1^\dagger A_1^\dagger - \xi\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\,, \tag{7.75}$$

$$\tilde{\lambda}_\alpha = -\frac{\xi}{2}\left(\frac{m}{2\xi} + A_3\right)\chi_\alpha - \frac{\xi}{2}\left(\frac{m}{2\xi} + A_3^\dagger\right)\psi_\alpha\,, \tag{7.76}$$

$$\tilde{\omega}_\alpha = -i\xi\left(\frac{m}{2\xi} + A_3\right)\chi_\alpha + i\xi\left(\frac{m}{2\xi}\psi_\alpha + A_3^\dagger\right)\psi_\alpha\,. \tag{7.77}$$

7.4.1 Review of the Case for Vanishing Interaction

Before the coupling of the fermionic fields with mass dimension one to the O'Raifeartaigh model is discussed in detail it is important to verify whether the previous results for the Lagrangian and the equations of motion are reasonable. This can be achieved by discussing the special case for a vanishing coupling constant. In this limit the equations of motion for the auxiliary fields of the O'Raifeartaigh model and the model for fermionic fields with mass dimension one should decouple and reduce to the originally derived equations of motions for the individual models. For $\xi \to 0$ the equations of motion reduce to

$$F_1 = MA_2^\dagger + g\left(A_1^\dagger A_3^\dagger + A_3^\dagger A_1^\dagger\right)\,, \tag{7.78}$$

$$F_2 = MA_1^\dagger\,, \tag{7.79}$$

$$F_3 = \Lambda + gA_1^\dagger A_1^\dagger\,, \tag{7.80}$$

$$\tilde{\lambda}_\alpha = -\frac{m}{4}\left(\chi_\alpha + \psi_\alpha\right)\,, \tag{7.81}$$

$$\tilde{\omega}_\alpha = -\frac{im}{2}\left(\chi_\alpha - \psi_\alpha\right)\,. \tag{7.82}$$

It is clear that in this limit the equations of motion for F_a decouple from those for $\tilde{\lambda}$ and $\tilde{\omega}$. Furthermore, it can be seen that the equations of motion for F_a reproduce the equations of motion of the O'Raifeartaigh model from equations (7.28) to (7.30) while the equations of motion for $\tilde{\lambda}$ and $\tilde{\omega}$ are exactly those derived in equations (3.86) and (3.87) for the model describing fermionic fields with mass dimension one.

As the equations of motion are exactly those of the two individual models the discussion of the expectation values is straightforward. For the bosonic component fields A_a it is found that

$$\langle A_1 \rangle = 0\,, \tag{7.83}$$

$$\langle A_2 \rangle = 0\,, \tag{7.84}$$

$$\langle A_3 \rangle = c_A\,, \tag{7.85}$$

where c_A is a real constant. Therefore, the auxiliary field F_3 acquires a nonvanishing expectation value which implies that supersymmetry is again spontaneously broken. Furthermore, the only solution for the fermionic auxiliary fields $\tilde{\lambda}$ and $\tilde{\omega}$ that does not spontaneously break supersymmetry is the trivial solution

$$\langle \chi_\alpha \rangle = 0\,, \tag{7.86}$$

$$\langle \psi_\alpha \rangle = 0\,. \tag{7.87}$$

This means that in the limit of vanishing coupling any spontaneous supersymmetry breaking originates in the O'Raifeartaigh model while the fermionic sector preserves supersymmetry.

7.4.2 Expectation Values for Nonzero Interaction

To calculate the expectation values for the component fields all auxiliary fields have to be eliminated from the superpotential. As before it proves to be easier to start from the intermediate result in equation (7.66) where the auxiliary fields appear in only three terms. It is important to note that even though the equations following equation (7.66) have an even simpler structure they cannot be used to derive the superpotential without auxiliary fields as the use of the equations of motion for ϕ, $\tilde{R}$ and $\tilde{S}$ eliminate terms that are important for the further discussion. For the specific choice of structure constants in equations (7.25) to (7.27) the superpotential is given by

$$\begin{aligned} U = {}& F_1 F_1^\dagger + F_2 F_2^\dagger + F_3 F_3^\dagger - \frac{1}{2} M \phi_1 \phi_2 - \frac{1}{2} M \bar{\phi}_1 \bar{\phi}_2 - g A_1 \phi_1 \phi_3 - \frac{1}{2} g A_3 \phi_1 \phi_1 - g A_1^\dagger \bar{\phi}_1 \bar{\phi}_3 \\ & - \frac{1}{2} g A_3^\dagger \bar{\phi}_1 \bar{\phi}_1 + 2 \tilde{\lambda} \tilde{\lambda} + \frac{1}{2} \tilde{\omega} \tilde{\omega} + \left(\frac{m}{4} + \frac{\xi}{2} A_3 \right) \mathrm{Tr}\Big(\tilde{S}^T \tilde{S} \Big) + \left(\frac{m}{4} + \frac{\xi}{2} A_3^\dagger \right) \mathrm{Tr}\Big(\tilde{R}^T \tilde{R} \Big) \\ & + 2 \bar{\tilde{\lambda}} \bar{\tilde{\lambda}} + \frac{1}{2} \bar{\tilde{\omega}} \bar{\tilde{\omega}} + \left(\frac{m}{4} + \frac{\xi}{2} A_3^\dagger \right) \mathrm{Tr}\Big(\bar{\tilde{S}}^T \bar{\tilde{S}} \Big) + \left(\frac{m}{4} + \frac{\xi}{2} A_3 \right) \mathrm{Tr}\Big(\bar{\tilde{R}}^T \bar{\tilde{R}} \Big) + \xi \phi_3 \tilde{S} \chi \\ & - \xi \bar{\phi}_3 \bar{\tilde{S}} \bar{\chi} + \xi \bar{\phi}_3 \tilde{R} \psi - \xi \phi_3 \bar{\tilde{R}} \bar{\psi}\,. \end{aligned} \tag{7.88}$$

Inserting the equations of motion for the auxiliary fields F_a, $\tilde{\lambda}$, and $\tilde{\omega}$, as well as their Hermitian conjugates leads to the on-shell superpotential

$$
\begin{aligned}
U &= \left(MA_2^\dagger + 2gA_1^\dagger A_3^\dagger\right)\left(MA_2 + 2gA_1A_3\right) + MA_1^\dagger MA_1 \\
&\quad + \left(\Lambda + gA_1^\dagger A_1^\dagger - \xi\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\right)\left(\Lambda + gA_1A_1 - \xi\left(\chi\chi + \bar{\psi}\bar{\psi}\right)\right) \\
&\quad - \frac{1}{2}M\phi_1\phi_2 - \frac{1}{2}M\bar{\phi}_1\bar{\phi}_2 - gA_1\phi_1\phi_3 - \frac{1}{2}gA_3\phi_1\phi_1 - gA_1^\dagger\bar{\phi}_1\bar{\phi}_3 - \frac{1}{2}gA_3^\dagger\bar{\phi}_1\bar{\phi}_1 \\
&\quad + 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\chi\psi + \left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) \\
&\quad + 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\bar{\chi}\bar{\psi} + \left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\tilde{S}}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\mathrm{Tr}\left(\bar{\tilde{R}}^T\bar{\tilde{R}}\right) \\
&\quad + \xi\phi_3\tilde{S}\chi - \xi\bar{\phi}_3\bar{\tilde{S}}\bar{\chi} + \xi\bar{\phi}_3\tilde{R}\psi - \xi\phi_3\bar{\tilde{R}}\bar{\psi} \\
&= M^2A_2A_2^\dagger + 2MgA_1A_2^\dagger A_3 + 2MgA_1^\dagger A_2A_3^\dagger + 4g^2A_1A_1^\dagger A_3A_3^\dagger + M^2A_1A_1^\dagger + \Lambda^2 \\
&\quad + \Lambda gA_1A_1 - \xi\Lambda\left(\chi\chi + \bar{\psi}\bar{\psi}\right) + \Lambda gA_1^\dagger A_1^\dagger + g^2A_1A_1A_1^\dagger A_1^\dagger - \xi gA_1^\dagger A_1^\dagger\left(\chi\chi + \bar{\psi}\bar{\psi}\right) \\
&\quad - \xi\Lambda\left(\bar{\chi}\bar{\chi} + \psi\psi\right) - \xi gA_1A_1\left(\bar{\chi}\bar{\chi} + \psi\psi\right) + \xi^2\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\left(\chi\chi + \bar{\psi}\bar{\psi}\right) - \frac{1}{2}M\phi_1\phi_2 \\
&\quad - \frac{1}{2}M\bar{\phi}_1\bar{\phi}_2 - gA_1\phi_1\phi_3 - \frac{1}{2}gA_3\phi_1\phi_1 - gA_1^\dagger\bar{\phi}_1\bar{\phi}_3 - \frac{1}{2}gA_3^\dagger\bar{\phi}_1\bar{\phi}_1 \\
&\quad + 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\chi\psi + \left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\mathrm{Tr}\left(\tilde{S}^T\tilde{S}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\mathrm{Tr}\left(\tilde{R}^T\tilde{R}\right) \\
&\quad + 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\bar{\chi}\bar{\psi} + \left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\mathrm{Tr}\left(\bar{\tilde{S}}^T\bar{\tilde{S}}\right) + \left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\mathrm{Tr}\left(\bar{\tilde{R}}^T\bar{\tilde{R}}\right) \\
&\quad + \xi\phi_3\tilde{S}\chi - \xi\bar{\phi}_3\bar{\tilde{S}}\bar{\chi} + \xi\bar{\phi}_3\tilde{R}\psi - \xi\phi_3\bar{\tilde{R}}\bar{\psi}\,. \qquad (7.89)
\end{aligned}
$$

The most general way to derive the expectation values for the component fields is to minimise the superpotential. In the present scenario this corresponds to solving a system of equations with 10 complex or 20 real variables

$$
\begin{aligned}
0 &= 2MgA_2^\dagger A_3 + 4g^2A_1^\dagger A_3A_3^\dagger + M^2A_1^\dagger + 2\Lambda gA_1 + 2g^2A_1A_1^\dagger A_1^\dagger \\
&\quad - 2\xi gA_1\left(\bar{\chi}\bar{\chi} + \psi\psi\right) - g\phi_1\phi_3\,, \qquad (7.90) \\
0 &= 2MgA_2A_3^\dagger + 4g^2A_1A_3A_3^\dagger + M^2A_1 + 2\Lambda gA_1^\dagger + 2g^2A_1A_1A_1^\dagger \\
&\quad - 2\xi gA_1^\dagger\left(\chi\chi + \bar{\psi}\bar{\psi}\right) - g\bar{\phi}_1\bar{\phi}_3\,, \qquad (7.91) \\
0 &= M^2A_2^\dagger + 2MgA_1^\dagger A_3^\dagger\,, \qquad (7.92) \\
0 &= M^2A_2 + 2MgA_1A_3\,, \qquad (7.93)
\end{aligned}
$$

$$0 = 2MgA_1A_2^\dagger + 4g^2A_1A_1^\dagger A_3^\dagger - \frac{1}{2}g\phi_1\phi_1 + 2\xi\left(\frac{m}{2} + \xi A_3^\dagger\right)\chi\psi + \frac{\xi}{2}\mathrm{Tr}\Big(\tilde{S}^T\tilde{S}\Big)$$
$$+ 2\xi\left(\frac{m}{2} + \xi A_3^\dagger\right)\bar{\chi}\bar{\psi} + \frac{\xi}{2}\mathrm{Tr}\Big(\bar{\tilde{R}}^T\bar{\tilde{R}}\Big)\,, \tag{7.94}$$
$$0 = 2MgA_1^\dagger A_2 + 4g^2A_1A_1^\dagger A_3 - \frac{1}{2}g\bar{\phi}_1\bar{\phi}_1 + 2\xi\left(\frac{m}{2} + \xi A_3\right)\chi\psi + \frac{\xi}{2}\mathrm{Tr}\Big(\tilde{R}^T\tilde{R}\Big)$$
$$+ 2\xi\left(\frac{m}{2} + \xi A_3\right)\bar{\chi}\bar{\psi} + \frac{\xi}{2}\mathrm{Tr}\Big(\bar{\tilde{S}}^T\bar{\tilde{S}}\Big)\,, \tag{7.95}$$
$$0 = -\frac{1}{2}M\phi_{2\alpha} - gA_1\phi_{3\alpha} - gA_3\phi_{1\alpha}\,, \tag{7.96}$$
$$0 = \frac{1}{2}M\bar{\phi}_{2\dot{\alpha}} + gA_1^\dagger\bar{\phi}_{3\dot{\alpha}} + gA_3^\dagger\bar{\phi}_{1\dot{\alpha}}\,, \tag{7.97}$$
$$0 = -\frac{1}{2}M\phi_{1\alpha}\,, \tag{7.98}$$
$$0 = \frac{1}{2}M\bar{\phi}_{1\dot{\alpha}}\,, \tag{7.99}$$
$$0 = -gA_1\phi_{1\alpha} - \xi\tilde{S}_{\alpha\gamma}\chi^\gamma - \xi\bar{\tilde{R}}_{\alpha\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\,, \tag{7.100}$$
$$0 = gA_1^\dagger\bar{\phi}_{1\dot{\alpha}} + \xi\bar{\tilde{S}}_{\dot{\alpha}\dot{\gamma}}\bar{\chi}^{\dot{\gamma}} + \xi\tilde{R}_{\dot{\alpha}\gamma}\psi^\gamma\,, \tag{7.101}$$
$$0 = -2\xi\Lambda\chi_\alpha - 2\xi gA_1^\dagger A_1^\dagger\chi_\alpha + 2\xi^2\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\chi_\alpha$$
$$+ 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\psi_\alpha + \xi\phi_3^\beta\tilde{S}_{\beta\alpha}\,, \tag{7.102}$$
$$0 = 2\xi\Lambda\bar{\chi}_{\dot{\alpha}} + 2\xi gA_1A_1\bar{\chi}_{\dot{\alpha}} - 2\xi^2\bar{\chi}_{\dot{\alpha}}\left(\chi\chi + \bar{\psi}\bar{\psi}\right)$$
$$- 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\bar{\psi}_{\dot{\alpha}} - \xi\bar{\phi}_3^{\dot{\beta}}\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}}\,, \tag{7.103}$$
$$0 = -2\xi\Lambda\psi_\alpha - 2\xi gA_1A_1\psi_\alpha + 2\xi^2\psi_\alpha\left(\chi\chi + \bar{\psi}\bar{\psi}\right)$$
$$+ 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\chi_\alpha - \xi\bar{\phi}_3^{\dot{\beta}}\tilde{R}_{\dot{\beta}\alpha}\,, \tag{7.104}$$
$$0 = 2\xi\Lambda\bar{\psi}_{\dot{\alpha}} + 2\xi gA_1^\dagger A_1^\dagger\bar{\psi}_{\dot{\alpha}} - 2\xi^2\left(\bar{\chi}\bar{\chi} + \psi\psi\right)\bar{\psi}_{\dot{\alpha}}$$
$$- 2\left(\frac{m}{2} + \xi A_3\right)\left(\frac{m}{2} + \xi A_3^\dagger\right)\bar{\chi}_{\dot{\alpha}} + \xi\phi_3^\beta\bar{\tilde{R}}_{\beta\dot{\alpha}}\,, \tag{7.105}$$
$$0 = 2\left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\tilde{S}_{\beta\alpha} + \xi\phi_{3\beta}\chi_\alpha\,, \tag{7.106}$$
$$0 = 2\left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\bar{\tilde{S}}_{\dot{\beta}\dot{\alpha}} - \xi\bar{\phi}_{3\dot{\beta}}\bar{\chi}_{\dot{\alpha}}\,, \tag{7.107}$$
$$0 = 2\left(\frac{m}{4} + \frac{\xi}{2}A_3^\dagger\right)\tilde{R}_{\dot{\beta}\alpha} + \xi\bar{\phi}_{3\dot{\beta}}\psi_\alpha\,, \tag{7.108}$$
$$0 = 2\left(\frac{m}{4} + \frac{\xi}{2}A_3\right)\bar{\tilde{R}}_{\beta\dot{\alpha}} - \xi\phi_{3\beta}\bar{\psi}_{\dot{\alpha}}\,. \tag{7.109}$$

Alternatively, it is possible to use the equations of motion to derive the expectation values. Even though the equations of motion lead to the same expectation values they don't result in a complete set of solutions. Specifically, they don't constrain the component fields

ϕ, $\tilde{R}$, and $\tilde{S}$. Therefore, a combination of the two approaches proves to be very powerful as the equations of motion immediately lead to expectation values for A_a, χ, and ψ. The remaining expectation values for ϕ_a, $\tilde{R}$, and $\tilde{S}$ can then be derived from the reduced set of equations that is implied by equations (7.90) to (7.109) and the expectation values for A_a, χ, and ψ.

A close look at the equations of motion for $\tilde{\lambda}$ and $\tilde{\omega}$ reveals that there are two possible solutions that lead to vanishing expectation values. First, there is the trivial solution with $\langle\chi_\alpha\rangle = \langle\psi_\alpha\rangle = 0$ which results in the same equations of motion for F_a as the O'Raifeartaigh model and thus leads to a nonvanishing expectation value for either F_2 or F_3. Therefore supersymmetry is spontaneously broken. It can be shown that this scenario also has the same minimum of the superpotential as the O'Raifeartaigh model. This minimum is a local minimum as the superpotential acquires the finite positive value Λ^2. Second, if the expectation value of A_3 is chosen such that $\langle A_3\rangle = -\frac{m}{2\xi}$ the expectation values for the auxiliary fields $\tilde{\lambda}$ and $\tilde{\omega}$ vanish identically without making any assumptions on the component fields ψ and χ. Therefore, one or both of them can acquire a nonvanishing expectation value such that the expectation value for F_3 vanishes and supersymmetry is preserved. In this case the superpotential vanishes identically and thus represents a global minimum. It will be shown later on that restoring supersymmetry using nonvanishing expectation values of spinor products comes at the cost of breaking Lorentz invariance.

As the first case doesn't yield any new results the following discussion will be resticted to the second scenario. Starting from the equations of motion it can be shown that the component fields A_a have the expectation values

$$\left\langle A_1^\dagger\right\rangle = \langle A_1\rangle = 0\,, \tag{7.110}$$

$$\left\langle A_2^\dagger\right\rangle = \langle A_2\rangle = 0\,, \tag{7.111}$$

$$\left\langle A_3^\dagger\right\rangle = \langle A_3\rangle = -\frac{m}{2\xi}\,. \tag{7.112}$$

Furthermore, the equations of motion imply a relation for the spinor fields χ and ψ

$$\langle\bar{\chi}\bar{\chi} + \psi\psi\rangle = \left\langle\chi\chi + \bar{\psi}\bar{\psi}\right\rangle = \frac{\Lambda}{\xi}\,. \tag{7.113}$$

These results can then be used to simplify the system of equations (7.90) to (7.109). The reduced set of equations implies that the expectation values for the spinor fields ϕ_a vanish identically

$$\langle\phi_1\rangle = \langle\bar{\phi}_1\rangle = 0\,, \tag{7.114}$$

$$\langle\phi_2\rangle = \langle\bar{\phi}_2\rangle = 0\,, \tag{7.115}$$

$$\langle\phi_3\rangle = \langle\bar{\phi}_3\rangle = 0\,. \tag{7.116}$$

This doesn't come as a surprise as the pure O'Raifeartaigh model doesn't produce nonvanishing expectation values for the ϕ_a either. The remaining relations are then summarised to

$$0 = \mathrm{Tr}\left(\tilde{S}^T\tilde{S} + \bar{\tilde{R}}^T\bar{\tilde{R}}\right)\,, \tag{7.117}$$

$$0 = \mathrm{Tr}\left(\tilde{R}^T\tilde{R} + \bar{\tilde{S}}^T\bar{\tilde{S}}\right)\,, \tag{7.118}$$

$$0 = \tilde{S}_{\alpha\beta}\chi^\beta + \bar{\tilde{R}}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\,, \tag{7.119}$$

$$0 = \bar{\tilde{S}}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}} + \tilde{R}_{\dot{\alpha}\beta}\psi^\beta\,. \tag{7.120}$$

Therefore, the second rank spinor fields $\tilde{R}$ and $\tilde{S}$ are not restricted and may acquire a nonzero expectation value

$$\left\langle\tilde{R}_{\alpha\dot{\beta}}\right\rangle = c_{R\alpha\dot{\beta}}\,, \tag{7.121}$$

$$\left\langle\tilde{S}_{\alpha\beta}\right\rangle = c_{S\alpha\beta}\,, \tag{7.122}$$

$$\tag{7.123}$$

where $c_{R\alpha\dot{\beta}}$ and $c_{S\alpha\beta}$ are constant second rank spinors.

7.4.3 The Mass Terms

The mass matrix is defined as the quadratic terms of the Lagrangian in the component fields after expanding them around their expectation values. In the previous section it was shown that A_3, χ, ψ, $\tilde{S}$, and $\tilde{R}$ can acquire nonzero expectation values. Therefore, each of these fields can be separated into its expectation value and an excitation from the expectation value. To distinguish the excitations from the component fields the bosonic fields are denoted by the corresponding small case italic letters, e. g., $A_1 \rightarrow a_1$, while the notation for fermionic excitations is extended by a hat, e. g., $\chi \to \hat{\chi}$. The expectation values, as long as they are not replaced by otherwise specified constants, are denoted by a subscript 0. For consistency of notation the notation for the remaining component fields with vanishing expectation values was adapted accordingly. The transition is straightforward as for vanishing expectation values the excitations from the expectation values are identical to the component fields. This expansion of the component fields around their expectation values leads to the superpotential

$$\begin{aligned}
U = {} & M^2 a_2 a_2^\dagger + 2Mga_1a_2^\dagger\left(-\frac{m}{2\xi} + a_3\right) + 2Mga_1^\dagger a_2\left(-\frac{m}{2\xi} + a_3^\dagger\right) \\
& + 4g^2 a_1 a_1^\dagger \left(-\frac{m}{2\xi} + a_3\right)\left(-\frac{m}{2\xi} + a_3^\dagger\right) + M^2 a_1 a_1^\dagger + \Lambda^2 + \Lambda g a_1 a_1 + \xi\Lambda\left(\chi\chi + \bar{\psi}\bar{\psi}\right) \\
& + \Lambda g a_1^\dagger a_1^\dagger + g^2 a_1 a_1 a_1^\dagger a_1^\dagger - \xi g a_1^\dagger a_1^\dagger\left(\left(\chi_0 + \hat{\chi}\right)\left(\chi_0 + \hat{\chi}\right) + \left(\bar{\psi}_0 + \hat{\bar{\psi}}\right)\left(\bar{\psi}_0 + \hat{\bar{\psi}}\right)\right) \\
& - \xi\Lambda\left(\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right)\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right) + \left(\psi_0 + \hat{\psi}\right)\left(\psi_0 + \hat{\psi}\right)\right) \\
& - \xi g a_1 a_1\left(\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right)\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right) + \left(\psi_0 + \hat{\psi}\right)\left(\psi_0 + \hat{\psi}\right)\right) \\
& + \xi^2\left(\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right)\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right) + \left(\psi_0 + \hat{\psi}\right)\left(\psi_0 + \hat{\psi}\right)\right) \\
& \left(\left(\chi_0 + \hat{\chi}\right)\left(\chi_0 + \hat{\chi}\right) + \left(\bar{\psi}_0 + \hat{\bar{\psi}}\right)\left(\bar{\psi}_0 + \hat{\bar{\psi}}\right)\right) - \frac{1}{2}M\hat{\phi}_1\hat{\phi}_2 - \frac{1}{2}M\hat{\bar{\phi}}_1\hat{\bar{\phi}}_2 - ga_1\hat{\phi}_1\hat{\phi}_3 \\
& - \frac{1}{2}g\left(-\frac{m}{2\xi} + a_3\right)\hat{\phi}_1\hat{\phi}_1 - ga_1^\dagger\hat{\bar{\phi}}_1\hat{\bar{\phi}}_3 - \frac{1}{2}g\left(-\frac{m}{2\xi} + a_3^\dagger\right)\hat{\bar{\phi}}_1\hat{\bar{\phi}}_1 \\
& + 2\xi^2 a_3 a_3^\dagger\left(\chi_0 + \hat{\chi}\right)\left(\psi_0 + \hat{\psi}\right) + \frac{\xi}{2}a_3\mathrm{Tr}\Big(\left(\tilde{S}_0^T + \hat{\tilde{S}}^T\right)\left(\tilde{S}_0 + \hat{\tilde{S}}\right)\Big) \\
& + \frac{\xi}{2}a_3^\dagger\mathrm{Tr}\Big(\left(\tilde{R}_0^T + \hat{\tilde{R}}^T\right)\left(\tilde{R}_0 + \hat{\tilde{R}}\right)\Big) + 2\xi^2 a_3 a_3^\dagger\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right)\left(\bar{\psi}_0 + \hat{\bar{\psi}}\right) \\
& + \frac{\xi}{2}a_3^\dagger\mathrm{Tr}\Big(\left(\bar{\tilde{S}}_0^T + \hat{\bar{\tilde{S}}}^T\right)\left(\bar{\tilde{S}}_0 + \hat{\bar{\tilde{S}}}\right)\Big) + \frac{\xi}{2}a_3\mathrm{Tr}\Big(\left(\bar{\tilde{R}}_0^T + \hat{\bar{\tilde{R}}}^T\right)\left(\bar{\tilde{R}}_0 + \hat{\bar{\tilde{R}}}\right)\Big) \\
& + \xi\hat{\phi}_3\left(\tilde{S}_0 + \hat{\tilde{S}}\right)\left(\chi_0 + \hat{\chi}\right) - \xi\hat{\bar{\phi}}_3\left(\bar{\tilde{S}}_0 + \hat{\bar{\tilde{S}}}\right)\left(\bar{\chi}_0 + \hat{\bar{\chi}}\right) \\
& + \xi\hat{\bar{\phi}}_3\left(\tilde{R}_0 + \hat{\tilde{R}}\right)\left(\psi_0 + \hat{\psi}\right) - \xi\hat{\phi}_3\left(\bar{\tilde{R}} + \hat{\bar{\tilde{R}}}\right)\left(\bar{\psi}_0 + \hat{\bar{\psi}}\right) \; . \qquad (7.124)
\end{aligned}$$

For the mass terms only the terms to second order in the component fields are of interest while all terms to different order can be ignored. Therefore, the relevant terms are given by

$$\begin{aligned}
U_{\mathcal{O}^2} = {} & M^2 a_2 a_2^\dagger - \frac{mMg}{\xi}a_1 a_2^\dagger - \frac{mMg}{\xi}a_1^\dagger a_2 + \frac{m^2 g^2}{\xi^2}a_1 a_1^\dagger + M^2 a_1 a_1^\dagger + \Lambda g a_1 a_1 \\
& - \xi\Lambda\left(\hat{\chi}\hat{\chi} + \hat{\bar{\psi}}\hat{\bar{\psi}}\right) + \Lambda g a_1^\dagger a_1^\dagger - \xi g a_1^\dagger a_1^\dagger\left(\chi_0\chi_0 + \bar{\psi}_0\bar{\psi}_0\right) - \xi\Lambda\left(\hat{\bar{\chi}}\hat{\bar{\chi}} + \hat{\psi}\hat{\psi}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \xi g a_1 a_1 \left(\bar{\chi}_0\bar{\chi}_0 + \psi_0\psi_0\right) + \xi^2 \bar{\chi}_0\bar{\chi}_0\hat{\chi}\hat{\chi} + \xi^2 \bar{\chi}_0\bar{\chi}_0\hat{\bar{\psi}}\hat{\bar{\psi}} + \xi^2 \psi_0\psi_0\hat{\chi}\hat{\chi} + \xi^2 \psi_0\psi_0\hat{\bar{\psi}}\hat{\bar{\psi}} \\
&+ 4\xi^2 \bar{\chi}_0\hat{\bar{\chi}}\chi_0\hat{\chi} + 4\xi^2 \bar{\chi}_0\hat{\bar{\chi}}\bar{\psi}_0\hat{\bar{\psi}} + 4\xi^2 \psi_0\hat{\psi}\chi_0\hat{\chi} + 4\xi^2 \psi_0\hat{\psi}\bar{\psi}_0\hat{\bar{\psi}} + \xi^2 \hat{\bar{\chi}}\hat{\bar{\chi}}\chi_0\chi_0 + \xi^2 \hat{\bar{\chi}}\hat{\bar{\chi}}\bar{\psi}_0\bar{\psi}_0 \\
&+ \xi^2 \hat{\psi}\hat{\psi}\chi_0\chi_0 + \xi^2 \hat{\psi}\hat{\psi}\bar{\psi}_0\bar{\psi}_0 - \frac{1}{2} M \hat{\phi}_1\hat{\phi}_2 - \frac{1}{2} M \hat{\bar{\phi}}_1\hat{\bar{\phi}}_2 + \frac{mg}{4\xi}\hat{\phi}_1\hat{\phi}_1 + \frac{mg}{4\xi}\hat{\bar{\phi}}_1\hat{\bar{\phi}}_1 \\
&+ 2\xi^2 a_3 a_3^\dagger \chi_0\psi_0 + \frac{\xi}{2} a_3 \mathrm{Tr}\Big(\tilde{S}_0^T\hat{\tilde{S}} + \hat{\tilde{S}}^T\tilde{S}_0\Big) + \frac{\xi}{2} a_3^\dagger \mathrm{Tr}\Big(\tilde{R}_0^T\hat{\tilde{R}} + \hat{\tilde{R}}^T\tilde{R}_0\Big) + 2\xi^2 a_3 a_3^\dagger \bar{\chi}_0\bar{\psi}_0 \\
&+ \frac{\xi}{2} a_3^\dagger \mathrm{Tr}\left(\bar{\tilde{S}}_0^T\hat{\bar{\tilde{S}}} + \hat{\bar{\tilde{S}}}^T\bar{\tilde{S}}_0\right) + \frac{\xi}{2} a_3 \mathrm{Tr}\left(\bar{\tilde{R}}_0^T\hat{\bar{\tilde{R}}} + \hat{\bar{\tilde{R}}}^T\bar{\tilde{R}}_0\right) + \xi\hat{\phi}_3\tilde{S}_0\hat{\chi} + \xi\hat{\phi}_3\hat{\tilde{S}}\chi_0 - \xi\hat{\bar{\phi}}_3\bar{\tilde{S}}_0\hat{\bar{\chi}} \\
&- \xi\hat{\bar{\phi}}_3\hat{\bar{\tilde{S}}}\bar{\chi}_0 + \xi\hat{\bar{\phi}}_3\tilde{R}_0\hat{\psi} + \xi\hat{\bar{\phi}}_3\hat{\tilde{R}}\psi_0 - \xi\hat{\phi}_3\bar{\tilde{R}}_0\hat{\bar{\psi}} - \xi\hat{\phi}_3\hat{\bar{\tilde{R}}}\bar{\psi}_0 \,.
\end{aligned}
\tag{7.125}
$$

Using the equation of motion for F_3 from equation (7.75) which gives a relation for the sum of spinor products simplifies the second order terms of the superpotential significantly

$$
\begin{aligned}
U_{\mathcal{O}^2} = {}& M^2 a_2 a_2^\dagger - \frac{mMg}{\xi} a_1 a_2^\dagger - \frac{mMg}{\xi} a_1^\dagger a_2 + \frac{m^2 g^2}{\xi^2} a_1 a_1^\dagger + M^2 a_1 a_1^\dagger + 4\xi^2 \bar{\chi}_0\hat{\bar{\chi}}\chi_0\hat{\chi} \\
&+ 4\xi^2 \bar{\chi}_0\hat{\bar{\chi}}\bar{\psi}_0\hat{\bar{\psi}} + 4\xi^2 \psi_0\hat{\psi}\chi_0\hat{\chi} + 4\xi^2 \psi_0\hat{\psi}\bar{\psi}_0\hat{\bar{\psi}} - \frac{1}{2} M \hat{\phi}_1\hat{\phi}_2 - \frac{1}{2} M \hat{\bar{\phi}}_1\hat{\bar{\phi}}_2 + \frac{mg}{4\xi}\hat{\phi}_1\hat{\phi}_1 \\
&+ \frac{mg}{4\xi}\hat{\bar{\phi}}_1\hat{\bar{\phi}}_1 + 2\xi^2 a_3 a_3^\dagger \chi_0\psi_0 + \frac{\xi}{2} a_3 \mathrm{Tr}\Big(\tilde{S}_0^T\hat{\tilde{S}} + \hat{\tilde{S}}^T\tilde{S}_0\Big) + \frac{\xi}{2} a_3^\dagger \mathrm{Tr}\Big(\tilde{R}_0^T\hat{\tilde{R}} + \hat{\tilde{R}}^T\tilde{R}_0\Big) \\
&+ 2\xi^2 a_3 a_3^\dagger \bar{\chi}_0\bar{\psi}_0 + \frac{\xi}{2} a_3^\dagger \mathrm{Tr}\left(\bar{\tilde{S}}_0^T\hat{\bar{\tilde{S}}} + \hat{\bar{\tilde{S}}}^T\bar{\tilde{S}}_0\right) + \frac{\xi}{2} a_3 \mathrm{Tr}\left(\bar{\tilde{R}}_0^T\hat{\bar{\tilde{R}}} + \hat{\bar{\tilde{R}}}^T\bar{\tilde{R}}_0\right) + \xi\hat{\phi}_3\tilde{S}_0\hat{\chi} \\
&+ \xi\hat{\phi}_3\hat{\tilde{S}}\chi_0 - \xi\hat{\bar{\phi}}_3\bar{\tilde{S}}_0\hat{\bar{\chi}} - \xi\hat{\bar{\phi}}_3\hat{\bar{\tilde{S}}}\bar{\chi}_0 + \xi\hat{\bar{\phi}}_3\tilde{R}_0\hat{\psi} + \xi\hat{\bar{\phi}}_3\hat{\tilde{R}}\psi_0 - \xi\hat{\phi}_3\bar{\tilde{R}}_0\hat{\bar{\psi}} - \xi\hat{\phi}_3\hat{\bar{\tilde{R}}}\bar{\psi}_0 \,.
\end{aligned}
\tag{7.126}
$$

A thorough investigation reveals an unpleasant multiplet structure. It turns out that there are three multiplets. Two of them are doublets, a_1 and a_2 as well as $\hat{\phi}_1$ and $\hat{\phi}_2$, containing fields with the same properties. A different picture arises for the third multiplet which turns out to be a sextuplet containing a_3, $\hat{\chi}$, $\hat{\psi}$, $\hat{\phi}_3$, $\tilde{S}$ and $\tilde{R}$. This multiplet suffers of its large size as well as the mix of component fields it contains – scalar fields, spinor fields and second rank spinor fields – which makes it nearly impossible to reconcile.

A solution to the structure and size problem of this multiplet presents itself if it is recalled that the second rank spinor fields $\tilde{R}$ and $\tilde{S}$ were restricted by a relation involving the trace that implied a constant expectation which may or may not be zero. If it is assumed that the expectation values of both second rank spinor fields vanish identically the multiplet structure problem is resolved. Using this assumption reduces the superpotential in second order of the component fields to

$$
\begin{aligned}
U_{\mathcal{O}^2} = {} & M^2 a_2 a_2^\dagger - \frac{mMg}{\xi} a_1 a_2^\dagger - \frac{mMg}{\xi} a_1^\dagger a_2 + \frac{m^2 g^2}{\xi^2} a_1 a_1^\dagger + M^2 a_1 a_1^\dagger + 4\xi^2 \bar{\chi}_0 \hat{\bar{\chi}} \chi_0 \hat{\chi} \\
& + 4\xi^2 \bar{\chi}_0 \hat{\bar{\chi}} \bar{\psi}_0 \hat{\bar{\psi}} + 4\xi^2 \psi_0 \hat{\psi} \chi_0 \hat{\chi} + 4\xi^2 \psi_0 \hat{\psi} \bar{\psi}_0 \hat{\bar{\psi}} - \frac{1}{2} M \hat{\phi}_1 \hat{\phi}_2 - \frac{1}{2} M \hat{\bar{\phi}}_1 \hat{\bar{\phi}}_2 + \frac{mg}{4\xi} \hat{\phi}_1 \hat{\phi}_1 \\
& + \frac{mg}{4\xi} \hat{\bar{\phi}}_1 \hat{\bar{\phi}}_1 + 2\xi^2 a_3 a_3^\dagger \chi_0 \psi_0 + 2\xi^2 a_3 a_3^\dagger \bar{\chi}_0 \bar{\psi}_0 + \xi \hat{\phi}_3 \hat{\tilde{S}} \chi_0 - \xi \hat{\bar{\phi}}_3 \hat{\bar{\tilde{S}}} \bar{\chi}_0 + \xi \hat{\bar{\phi}}_3 \hat{\tilde{R}} \psi_0 \\
& - \xi \hat{\phi}_3 \hat{\bar{\tilde{R}}} \bar{\psi}_0 \,.
\end{aligned}
\tag{7.127}
$$

These terms result in a simpler multiplet structure which groups the component fields into four multiplets – two doublets and two triplets. Three of them – the doublets containing $\hat{\phi}_1$ and $\hat{\phi}_2$, $\hat{\chi}$ and $\hat{\psi}$, as well as the triplet containing a_1, a_2, and a_3 – are easily explained and are solely made up of either scalar or spinor fields. The remaining triplet is slightly more involved as it groups a spinor field together with two second rank spinor fields. However, a brief investigation of the terms of interest

$$
U_{\phi_3,\tilde{S},\tilde{R}} = \xi \hat{\phi}_3 \hat{\tilde{S}} \chi_0 - \xi \hat{\bar{\phi}}_3 \hat{\bar{\tilde{S}}} \bar{\chi}_0 + \xi \hat{\bar{\phi}}_3 \hat{\tilde{R}} \psi_0 - \xi \hat{\phi}_3 \hat{\bar{\tilde{R}}} \bar{\psi}_0 \tag{7.128}
$$

reveals that it can easily be resolved by the introduction of the fields

$$
\hat{\zeta}_{1\alpha} = -\hat{\tilde{S}}_{\alpha\beta} \chi_0^\beta \,, \tag{7.129}
$$

$$
\hat{\zeta}_{2\alpha} = \hat{\bar{\tilde{R}}}_{\alpha\dot{\beta}} \bar{\psi}_0^{\dot{\beta}} \,. \tag{7.130}
$$

The terms involving ϕ_3, $\tilde{R}$ and $\tilde{S}$ can then be expressed as

$$
U_{\phi_3,\tilde{S},\tilde{R}} = \xi \hat{\phi}_3 \hat{\zeta}_{1\alpha} + \xi \hat{\bar{\phi}}_3 \hat{\bar{\zeta}}_{1\alpha} - \xi \hat{\bar{\phi}}_3 \hat{\bar{\zeta}}_{2\alpha} - \xi \hat{\phi}_3 \hat{\zeta}_{2\alpha} \tag{7.131}
$$

and the superpotential to second order in the component fields is given by

$$
\begin{aligned}
U_{\mathcal{O}^2} = {} & M^2 a_2 a_2^\dagger - \frac{mMg}{\xi} a_1 a_2^\dagger - \frac{mMg}{\xi} a_1^\dagger a_2 + \frac{m^2 g^2}{\xi^2} a_1 a_1^\dagger + M^2 a_1 a_1^\dagger + 4\xi^2 \bar{\chi}_0 \hat{\bar{\chi}} \chi_0 \hat{\chi} \\
& + 4\xi^2 \bar{\chi}_0 \hat{\bar{\chi}} \bar{\psi}_0 \hat{\bar{\psi}} + 4\xi^2 \psi_0 \hat{\psi} \chi_0 \hat{\chi} + 4\xi^2 \psi_0 \hat{\psi} \bar{\psi}_0 \hat{\bar{\psi}} - \frac{1}{2} M \hat{\phi}_1 \hat{\phi}_2 - \frac{1}{2} M \hat{\bar{\phi}}_1 \hat{\bar{\phi}}_2 + \frac{mg}{4\xi} \hat{\phi}_1 \hat{\phi}_1 \\
& + \frac{mg}{4\xi} \hat{\bar{\phi}}_1 \hat{\bar{\phi}}_1 + 2\xi^2 a_3 a_3^\dagger \chi_0 \psi_0 + 2\xi^2 a_3 a_3^\dagger \bar{\chi}_0 \bar{\psi}_0 + \xi \hat{\phi}_3 \hat{\zeta}_{1\alpha} + \xi \hat{\bar{\phi}}_3 \hat{\bar{\zeta}}_{1\alpha} - \xi \hat{\bar{\phi}}_3 \hat{\bar{\zeta}}_{2\alpha} \\
& - \xi \hat{\phi}_3 \hat{\zeta}_{2\alpha} \,.
\end{aligned}
\tag{7.132}
$$

At the same time the previously mixed multiplet containing a spinor and two second rank spinor fields is transformed into a multiplet that contains the three spinor fields $\hat{\phi}_3$, $\hat{\zeta}_1$, and $\hat{\zeta}_2$. It has to be emphasised that this redefinition of fields is only successful after setting the expectation values for the second rank spinor fields to zero. Without this assumption on the expectation values the redefinition of fields fails to produce a meaningful multiplet structure.

To determine the mass matrices it is still necessary to separate the complex scalar fields into their two real components

$$a = \hat{a} + i\hat{b}\,. \tag{7.133}$$

The superpotential to second order in the component fields is then found to be

$$\begin{aligned}
U_{\mathcal{O}^2} = {} & M^2\left(\hat{a}_2 + i\hat{b}_2\right)\left(\hat{a}_2 - i\hat{b}_2\right) - \frac{mMg}{\xi}\left(\hat{a}_1 + i\hat{b}_1\right)\left(\hat{a}_2 - i\hat{b}_2\right) \\
& - \frac{mMg}{\xi}\left(\hat{a}_1 - i\hat{b}_1\right)\left(\hat{a}_2 + i\hat{b}_2\right) + \frac{m^2g^2}{\xi^2}\left(\hat{a}_1 + i\hat{b}_1\right)\left(\hat{a}_1 - i\hat{b}_1\right) \\
& + M^2\left(\hat{a}_1 + i\hat{b}_1\right)\left(\hat{a}_1 - i\hat{b}_1\right) + 4\xi^2\bar{\chi}_0\hat{\bar{\chi}}\chi_0\hat{\chi} + 4\xi^2\bar{\chi}_0\hat{\bar{\chi}}\bar{\psi}_0\hat{\bar{\psi}} + 4\xi^2\psi_0\hat{\psi}\chi_0\hat{\chi} \\
& + 4\xi^2\psi_0\hat{\psi}\bar{\psi}_0\hat{\bar{\psi}} - \frac{1}{2}M\hat{\phi}_1\hat{\phi}_2 - \frac{1}{2}M\hat{\bar{\phi}}_1\hat{\bar{\phi}}_2 + \frac{mg}{4\xi}\hat{\phi}_1\hat{\phi}_1 + \frac{mg}{4\xi}\hat{\bar{\phi}}_1\hat{\bar{\phi}}_1 \\
& + 2\xi^2\left(\hat{a}_3 + i\hat{b}_3\right)\left(\hat{a}_3 - i\hat{b}_3\right)\chi_0\psi_0 + 2\xi^2\left(\hat{a}_3 + i\hat{b}_3\right)\left(\hat{a}_3 - i\hat{b}_3\right)\bar{\chi}_0\bar{\psi}_0 + \xi\hat{\phi}_3\hat{\zeta}_{1\alpha} \\
& + \xi\hat{\bar{\phi}}_3\hat{\bar{\zeta}}_{1\alpha} - \xi\hat{\bar{\phi}}_3\hat{\bar{\zeta}}_{2\alpha} - \xi\hat{\phi}_3\hat{\zeta}_{2\alpha} \\
= {} & \left(M^2 + \frac{m^2g^2}{\xi^2}\right)\hat{a}_1\hat{a}_1 + M^2\hat{a}_2\hat{a}_2 + 2\xi^2\hat{a}_3\hat{a}_3\left(\chi_0\psi_0 + \bar{\chi}_0\bar{\psi}_0\right) - \frac{2mMg}{\xi}\hat{a}_1\hat{a}_2 \\
& + \left(M^2 + \frac{m^2g^2}{\xi^2}\right)\hat{b}_1\hat{b}_1 + M^2\hat{b}_2\hat{b}_2 + 2\xi^2\hat{b}_3\hat{b}_3\left(\chi_0\psi_0 + \bar{\chi}_0\bar{\psi}_0\right) - \frac{2mMg}{\xi}\hat{b}_1\hat{b}_2 \\
& + 4\xi^2\bar{\chi}_0\hat{\bar{\chi}}\chi_0\hat{\chi} + 4\xi^2\bar{\chi}_0\hat{\bar{\chi}}\bar{\psi}_0\hat{\bar{\psi}} + 4\xi^2\psi_0\hat{\psi}\chi_0\hat{\chi} + 4\xi^2\psi_0\hat{\psi}\bar{\psi}_0\hat{\bar{\psi}} - \frac{1}{2}M\hat{\phi}_1\hat{\phi}_2 \\
& - \frac{1}{2}M\hat{\bar{\phi}}_1\hat{\bar{\phi}}_2 + \frac{mg}{4\xi}\hat{\phi}_1\hat{\phi}_1 + \frac{mg}{4\xi}\hat{\bar{\phi}}_1\hat{\bar{\phi}}_1 + \xi\hat{\phi}_3\hat{\zeta}_{1\alpha} + \xi\hat{\bar{\phi}}_3\hat{\bar{\zeta}}_{1\alpha} - \xi\hat{\bar{\phi}}_3\hat{\bar{\zeta}}_{2\alpha} - \xi\hat{\phi}_3\hat{\zeta}_{2\alpha}\,.
\end{aligned} \tag{7.134}$$

All multiplets but the one containing $\hat{\psi}$ and $\hat{\chi}$ are in a simple form. The remaining multiplet, however, does not have a straightforward solution, as it involves the products of four spinor fields including terms of the form $\chi_0\chi_0\chi\chi$ as well as cross terms $\chi_0\chi\chi_0\chi$. Due to the additional spinor indices that are contracted it is not immediately clear how to rewrite them such that a proper mass term can be formulated.

Fortunately, it is possible to simplify the superpotential even further. During the derivation of the expectation values it was found that at least one of the spinor fields χ and ψ acquires a nonvanishing expectation value and they furthermore satisfy the following relation involving the sum of spinor products $\Lambda/\xi = \chi\chi + \bar{\psi}\bar{\psi}$. Without loss of generality it is possible to choose the two spinors such that χ acquires a finite expectation value while the expectation value for ψ vanishes. Therefore, all but one term involving the product of four spinor fields vanish identically. Furthermore, the spinor field $\hat{\zeta}_2$ which is by definition proportional to the constant spinor field ψ_0 vanishes identically and the superpotential is simplified to

$$\begin{aligned} U_{\mathcal{O}^2} = &\left(M^2 + \frac{m^2 g^2}{\xi^2}\right)\hat{a}_1\hat{a}_1 + M^2\hat{a}_2\hat{a}_2 - \frac{2mMg}{\xi}\hat{a}_1\hat{a}_2 + \left(M^2 + \frac{m^2 g^2}{\xi^2}\right)\hat{b}_1\hat{b}_1 \\ &+ M^2\hat{b}_2\hat{b}_2 - \frac{2mMg}{\xi}\hat{b}_1\hat{b}_2 + 4\xi^2\bar{\chi}_0\hat{\bar{\chi}}\chi_0\hat{\chi} - \frac{1}{2}M\hat{\phi}_1\hat{\phi}_2 - \frac{1}{2}M\hat{\bar{\phi}}_1\hat{\bar{\phi}}_2 \\ &+ \frac{mg}{4\xi}\hat{\phi}_1\hat{\phi}_1 + \frac{mg}{4\xi}\hat{\bar{\phi}}_1\hat{\bar{\phi}}_1 + \xi\hat{\phi}_3\hat{\zeta}_{1\alpha} + \xi\hat{\bar{\phi}}_3\hat{\bar{\zeta}}_{1\alpha}\,. \end{aligned} \tag{7.135}$$

This final form of the superpotential to second order in the component fields can then be utilised to extract the mass matrices. For the remainder of this section the various contributions to $U_{\mathcal{O}^2}$ will be discussed independently.

The bosonic component fields $\hat{a}_a$ and $\hat{b}_a$ can be grouped into two triplets of the form $a = (\hat{a}_1, \hat{a}_2, \hat{a}_3)$ and $b = \left(\hat{b}_1, \hat{b}_2, \hat{b}_3\right)$. The bosonic terms in equation (7.135) then correspond to the mass matrices

$$M_a = \begin{pmatrix} M^2 + \frac{m^2 g^2}{\xi^2} & -\frac{mMg}{\xi} & 0 \\ -\frac{mMg}{\xi} & M^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.136}$$

$$M_b = \begin{pmatrix} M^2 + \frac{m^2 g^2}{\xi^2} & -\frac{mMg}{\xi} & 0 \\ -\frac{mMg}{\xi} & M^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.137}$$

They are mostly in agreement with the results for the O'Raifeartaigh model which can be found in the reference literature, e. g., Sohnius (1985). The sole and important difference to these results is that the corrections to the mass in the bosonic mass matrices are no longer proportional to the scale parameter μ that sets the scale of the spontaneous supersymmetry breaking expectation value in the O'Raifeartaigh model. Instead the corrections in the mass term are proportional to the coupling strength ξ between the O'Raifeartaigh model and the fermionic sector, as well as the mass scale m of the fermionic sector. This behaviour was expected from dimensional analysis as the coupling terms via the F-component suggested a connection between the expectation value of the spontaneously broken superfield, the coupling strength, and the mass scale of the fermionic sector.

The derivation of the fermionic mass matrices is slightly more involved than for the bosonic mass matrices. To formulate the mass matrix the two-spinors and their Hermitian conjugates need to be grouped into four-spinors of the form

$$\Phi'_1 = \begin{pmatrix} \phi_{1\alpha} \\ \bar{\phi}_1^{\dot{\alpha}} \end{pmatrix} . \tag{7.138}$$

This leads to a mass matrix for the fermionic doublet $\Phi' = (\Phi'_1, \Phi'_2)$ of

$$M_{\Phi'} = \begin{pmatrix} \frac{mg}{4\xi} & -\frac{M}{4} \\ -\frac{M}{4} & 0 \end{pmatrix} . \tag{7.139}$$

This deviates from the results for the O'Raifeartaigh model that contains a spinor triplet $\Phi'' = (\Phi''_1, \Phi''_2, \Phi''_3)$. The spinor field Φ'_3 that is missing form the previously outlined doublet is massless in the O'Raifeartaigh model and forms a spinor triplet together with the two spinor fields $\hat{\zeta}_1$ and $\hat{\zeta}_2$ of the fermionic extension.

The remaining spinor triplet ζ contains the three spinor fields $\hat{\zeta}_1$, $\hat{\zeta}_2$, and $\hat{\phi}_3$. It can be shown that the superpotential from equation (7.135) leads to a matrix

$$C_\zeta = \begin{pmatrix} 0 & \frac{\xi}{2} & 0 \\ \frac{\xi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{7.140}$$

This matrix has only off diagonal entries and thus cannot represent a mass matrix. A closer look at the mass dimensions of the spinor fields that are contained in this multiplet reveals that two of them, ζ_1 and ζ_2 have mass dimension $5/2$ while the remaining spinor field ϕ_3 has mass dimension $3/2$. This means that the multiplet containing components with different mass dimensions does not represent a mass matrix but rather a coupling matrix between the component fields of the O'Raifeartaigh model and the fermionic sector. To make this distinction between mass and coupling matrices even more apparent the matrix was denoted with a capital C.

This nearly concludes the discussion of the terms in equation (7.135) and leaves only one last term, the one containing the product of four spinor fields, to explain

$$U_{\chi} = 4\xi^2 \bar{\chi}_0 \hat{\bar{\chi}} \chi_0 \hat{\chi} \,. \tag{7.141}$$

This term, even though it is to second order in the component fields, is not a mass term. Instead, it is responsible for the breaking of Lorentz invariance as the contraction over the dotted and undotted spinor indices results in a term that is proportional to a vector-field while the spinor indices are absorbed into a σ-matrix. This results in a preferred direction which can conveniently be chosen in the time direction or in other words proportional to σ^0.

Chapter 8

Conclusions

The primary objective of this thesis was to construct a supersymmetric model for fermionic fields with mass dimension one and to discuss its coupling to the O'Raifeartaigh model.

To achieve this goal it has been investigated whether it is possible to obtain a model based on a general scalar superfield commonly used in supersymmetric models. Subsequently, it has been shown that such a model cannot be formulated due to problems constructing a Lagrangian containing kinetic terms for the fermionic fields with mass dimension one. This eliminated all but the trivial solution which corresponds to a constant non-dynamical background spinor field and is not appealing considering the scope of this thesis. In addition, no consistent second quantisation of the component fields can be constructed.

This motivated the formulation of a model for fermionic fields with mass dimension one based on a general superfield with one free spinor index. Up to now few explicit calculations for the general superfield with one spinor index exists in the literature, hence, necessitating the derivation of a model from the ground up. This included the calculation of all chiral and anti-chiral superfields up to the third order in covariant derivatives. To the second oder in covariant derivatives there are one chiral and one anti-chiral spinor field, while to the third order there are one chiral and one anti-chiral second rank spinor field. Interestingly, the chiral second rank spinor field admits a special case if the two spinor indices are summed over, while the anti-chiral second-rank spinor field cannot contain a special case due to the mixed index structure.

Dimensional analysis revealed that there is a large number of possible contributions to the mass and kintic terms. Therefore, it proved useful to restrict the discussion to terms built from chiral and anti-chiral superfields. The resulting Lagrangian contains three spinor fields, two second-rank spinor fields and one spinor-vector field that is equivalent to a third-rank spinor field. It has turned out that one of the spinor fields as well as the spinor-vector field are auxiliary fields and thus their equations of motion were used to eliminate them from the Lagrangian. The resulting on-shell Lagrangian depends solely on two spinor fields and two second-rank spinor fields which corresponds to 8 fermionic and 8 bosonic degrees of freedom.

As it was not a priori clear that the Hamiltonian can be derived from the Lagrangian by Legendre transformation, a conservative approach based on the supersymmetry algebra was utilised. The algebra provides an anticommutation relation among the supersymmetry generators which is proportional to the momentum operator that contains the Hamiltonian as 0-th component. This is then related to the Lagrangian via the position space representation of the generators that are proportional to the spacetime integral of the supercurrent which can be derived from the Lagrangian. This process ensures a Hamiltonian that is consistent with the initial Lagrangian as well as the supersymmetry algebra.

There are two possible ways to derive the supercurrent from the Lagrangian. One of them relies solely on the Lagrangian without Hermitian conjugate, where both the supercurrent and its Hermitian conjugate have to be calculated. Alternatively, the full Lagrangian can be used which requires only the derivation of the supercurrent. The supercurrent was then used to express the supersymmetry generators in position space. Inserting the generators into the supersymmetry algebra resulted in a momentum operator that is proportional to two spacetime integrals containing the commutation and anti-commutation relations among the component fields. To proceed further, these commutators had to be derived in agreement with the variation of the general superfield with one free spinor index. In other words, a consistent second quantisation of the component fields had to be derived. It has been shown that unlike for the discussion of the general scalar superfield, it is possible to construct a consistent set of commutation and anticommutation relations. These commutation and anticommutation relations are utilised to find an expression for the Hamiltonian in position space. By construction this Hamiltonian is positive and finite, although, this property is not immediately obvious. This is caused by the sum of spinor products which can be shown to be real, but in principle could be negative.

To verify that the Hamiltonian indeed only admits positive energies, it had to be Fourier transformed into momentum space. This has been achieved by expanding the position space component fields in terms of momentum space operators. Afterwards, the commutation and anticommutation relations of the momentum space operators were derived in agreement with the position space relations. It has then been analysed which of the operators are creation and which ones are annihilation operators. With this knowledge it is straightforward to write down the normal ordered Hamiltonian in momentum space.

Up to now the discussion was concentrated on the formulation of a supersymmetric model for fermionic fields with mass dimension one. As it was intended not to find an alternate formulation of the Standard Model in terms of spinor fields but rather to extend it, the objective for the remaining discussion was to couple the model describing fermionic fields with mass dimension one to one of the well-known models. Due to the fact that up to now no supersymmetric partner of the Standard Model particles has been found it can be assumed that any realistic theory capable of describing physics below the TeV scale must break supersymmetry either spontaneously or explicitly. Therefore, the O'Raifeartaigh model presented itself as a perfect candidate to formulate a minimal model. It is the simplest model that spontaneously breaks supersymmetry and avoids the complexity of the Standard Model that arises from the inclusion of the gauge groups. Subsequently, it has been shown that the only possible couplings of the fermionic sector to the O'Raifeartaigh model involving three fields contain one chiral superfield from the O'Raifeartaigh model as well as two superfields from the fermionic sector. Interestingly, contributions via the F-component are possible, while various other terms have been neglected for the discussion. Moreover, these terms are similar to the mass terms that were used to construct the Lagrangian for the fermionic sector, thus hinting at a possible connection between the vacuum expectation value of the spontaneously broken superfield and the mass scale of the fermionic sector. Detailed calculations have revealed that the potential of the coupled model in terms of the auxiliary fields contains the terms expected from the O'Raifeartaigh model as well as two additional terms from the auxiliary fields of the fermionic sector.

As a simple and especially interesting example, the coupling of the fermionic sector to the field with nonvanishing expectation value has been discussed in detail. It has been shown that the equations of motion for the auxiliary fields are very similar to those derived for the individual models. However, the F-term that corresponds to the field with nonvanishing expectation value acquires an additional contribution that is proportional to the coupling strength, while the equations of motion for the fermionic sector acquire additional terms as well. A simple consistency check was performed by assuming a vanishing coupling strength. In this scenario it has been shown that the equations of motion of the coupled model reduce exactly to those of the individual models.

To find the expectation values for the component fields of the coupled model, the superpotential had to be minimised. This results in a system of 20 equations. Alternatively, the equations of motion for the auxiliary fields can be employed. They generally do not yield a complete set of solutions but are easier to solve than the full set of equations. Therefore, a combined approach was used to derive a complete set of expectation values. A brief look at the equations of motion for $\tilde{\lambda}$ and $\tilde{\omega}$ revealed that two distinct solutions exist. The first solution is the trivial solution $\langle\chi\rangle = \langle\psi\rangle = 0$ that leads to exactly the same equations of motion for F_i as the O'Raifeartaigh model and therefore, supersymmetry is spontaneously broken. The second solution where $\langle A_3\rangle = -\frac{m}{2\xi}$ is more intriguing. For this choice of expectation value the equations of motion for $\tilde{\lambda}$ and $\tilde{\omega}$ are satisfied without making any assumptions on χ and ψ. If it is assumed that the two spinor fields have the finite expectation value such that $\langle\bar{\chi}\bar{\chi} + \psi\psi\rangle = \frac{\Lambda}{\xi}$, supersymmetry is restored. At the same time the introduction of a nonvanishing expectation value for at least one of the spinor fields results in the introduction of a preferred direction and therefore breaks Lorentz invariance. Overall, it has been found that the scalar field A_3, the spinor fields χ and ψ, as well as the second-rank spinor fields $\tilde{R}$ and $\tilde{S}$ could acquire nonvanishing expectation values.

To determine the mass matrices the component fields in the superpotential were expanded around their expectation values. It was found that a reasonable multiplet structure exists if and only if the expectation values for the second-rank spinor fields vanish identically. In this case the multiplet structure reduces to two fermionic doublets, of which one is massless, and two bosonic triplets. Furthermore, a fermionic triplet arises that combines spinor fields with different mass dimensions. This identifies it as an interaction multiplet.

If the results for the coupled model are compared to those of the O'Raifeartaigh model a number of interesting changes have to be pointed out. The coupling to the fermionic sector restores supersymmetry at the cost of breaking Lorentz invariance. It is also found that the bosonic mass terms are now proportional to the coupling strength as well as the mass scale of the fermionic sector and no longer dependent on the arbitrary scale parameter of the O'Raifeartaigh model. The fermionic triplet of the O'Raifeartaigh model is replaced by a fermionic doublet that also depends on the coupling strength and the mass scale of the fermionic sector. Furthermore, there is an additional massless fermionic doublet. Finally, a fermionic triplet containing spinors with different mass dimensions exist, resulting in a coupling matrix that doesn't have an equivalent in the O'Raifeartaigh model.

At this point the motivation behind the specific choice of coupling between the fermionic sector and the O'Raifeartaigh model becomes clear. Up to now no superpartners have been detected experimentally; therfore, supersymmetry must be broken at energies currently accessible by experiments. Furthermore, the coupling to the fermionic sector mimics a coupling to the Higgs field of the Standard Model. An extension of the presented formalism to an extension of the Minimal Supersymmetric Standard Model should in general be possible in a perfect analogy and potentially result in similar effects. Due to mass dimensional arguments, the coupling of the fermionic sector to the Higgs field would dominate, while all other couplings are suppressed. Therefore, the presented model for fermionic fields with mass dimension one provides a good candidate for supersymmetric dark matter. Provided that the Higgs particle is detected at the LHC, potential deviations from the expected branching ratios of the Higgs particle can then, at least in principle, be used to predict the mass scale and coupling strength of the fermionic sector. This will provide experimental constraints on the amount of supersymmetric dark matter.

These results show that the presented model for fermionic fields with mass dimension one is a viable candidate for supersymmetric dark matter that can be accessed by experiments in the near future.

Appendix A

Mathematical Background

In the following sections the mathematical conventions and concepts used in this thesis are briefly summarised. They coincide with those employed by Peskin and Schroeder (1995) in their introduction to field theory, as well as by Sohnius (1985) in his report on supersymmetry.

A.1 General Conventions

The flat-space metric was chosen to be

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)\,. \tag{A.1}$$

The γ-matrices in the Weyl representation are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,, \tag{A.2}$$

where σ^μ are the Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{A.3}$$

and $\bar{\sigma}^\mu = \left(\mathbb{I}, -\boldsymbol{\sigma}^i\right)$ is defined as usual. Furthermore γ_5 is defined as

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\,. \tag{A.4}$$

This definition of γ^i coincides with the notation used by Peskin and Schroeder (1995), however, it differs by a minus sign from the notation used by Landau and Lifshitz (1982), while the notation employed by Weinberg (1995) deviates by factors of $-i$ and i for γ^0 and γ^i, respectively. Therefore, care has to be taken when equations and results from the reference literature are used or compared.

The Dirac spinors that correspond to the γ-matrices outlined in equation (A.2) can then be expressed as

$$u^s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \quad , \quad v^s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}\,, \tag{A.5}$$

where ξ^s and η^s are numerical two-spinors with spin index s. Additionally, the convention $\bar{\sigma}^\mu = (\mathbb{I}, -\boldsymbol{\sigma})$ was introduced to shorten the notation. This notation for the Dirac spinors was chosen because it simplifies the calculation of the transformation behaviour under discrete symmetry transformations. It is similar to the one used in Peskin and Schroeder (1995), where the difference in sign for the γ^i results in an exchange of σ and $\bar{\sigma}$.

Wigner's spin-1/2 time reversal operator is given by

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.6}$$

and the space inversion operator is defined as

$$\mathcal{R} \equiv (|\mathbf{p}| \to |\mathbf{p}|\,, \theta \to \pi - \theta, \varphi \to \varphi + \pi)\;. \tag{A.7}$$

Furthermore, the boost matrices in the $(1/2, 0)$ and $(0, 1/2)$ representations are given by

$$\kappa^{(1/2,0)} = \exp\left(\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\phi}\right) = \sqrt{\frac{E+m}{2m}} \left(\mathbb{I} + \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\right)\,, \tag{A.8}$$

$$\kappa^{(0,1/2)} = \exp\left(-\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\phi}\right) = \sqrt{\frac{E+m}{2m}} \left(\mathbb{I} - \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\right)\,, \tag{A.9}$$

where $\boldsymbol{\phi}$ is the boost parameter. These two boost matrices make up the diagonal, and are the only nonzero components of the boost matrix in the $(1/2, 0) \oplus (0, 1/2)$ representation.

Finally, the sign of the 4-dimensional ϵ-tensor is chosen such that

$$\epsilon_{0123} = 1\,. \tag{A.10}$$

A.2 ELKO Background

The following ELKO specific equations and notation correspond mostly to those used in the fundamental papers on ELKO spinors by Ahluwalia-Khalilova and Grumiller (2005a,b). However, if necessary, it was adapted to correspond with the general conventions for the metric and Dirac-matrices as outlined in Appendix A.1.

The rest frame ELKO spinor are defined as

$$\lambda^S_{\{-,+\}}(\mathbf{0}) = \begin{pmatrix} +i\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^* \\ \phi^+_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}\,, \tag{A.11}$$

$$\lambda^S_{\{+,-\}}(\mathbf{0}) = \begin{pmatrix} +i\Theta\left[\phi^-_{\mathrm{L}}(\mathbf{0})\right]^* \\ \phi^-_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}\,, \tag{A.12}$$

$$\lambda^A_{\{-,+\}}(\mathbf{0}) = \begin{pmatrix} -i\Theta\left[\phi^+_{\mathrm{L}}(\mathbf{0})\right]^* \\ \phi^+_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}\,, \tag{A.13}$$

$$\lambda^A_{\{+,-\}}(\mathbf{0}) = \begin{pmatrix} -i\Theta\left[\phi^-_{\mathrm{L}}(\mathbf{0})\right]^* \\ \phi^-_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}\,, \tag{A.14}$$

where S denotes self-conjugate and A anti-self-conjugate ELKO spinors and the helicity eigenstates are denoted with $\phi^\pm_L(\mathbf{0})$ which should not be confused with the previously introduced boost parameter $\boldsymbol{\phi}$. To explicitly write down the helicity eigenstates it is necessary to fix the up to now arbitrary direction of the momentum. For convenience the unit vector is chosen along $\mathbf{p}$ in spherical coordinates

$$\hat{\mathbf{p}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\;. \tag{A.15}$$

This choice of $\hat{\mathbf{p}}$ results in helicity eigenstates

$$\phi_{\mathrm{L}}^{+}(\mathbf{0}) = \sqrt{m}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\varphi/2}\\ \sin\frac{\theta}{2}e^{i\varphi/2}\end{pmatrix}, \tag{A.16}$$

$$\phi_{\mathrm{L}}^{-}(\mathbf{0}) = \sqrt{m}\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\varphi/2}\\ -\cos\frac{\theta}{2}e^{i\varphi/2}\end{pmatrix}. \tag{A.17}$$

All boosted spinors are then derived using

$$\lambda_{\{\pm,\mp\}}^{S/A}(\mathbf{p}) = \begin{pmatrix}\kappa^{(1/2,0)} & 0\\ 0 & \kappa^{(0,1/2)}\end{pmatrix}\lambda_{\{\pm,\mp\}}^{S/A}(\mathbf{0}). \tag{A.18}$$

It can be shown that this general equation for the boosted ELKO spinors leads to

$$\lambda_{\{-,+\}}^{S/A}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\left(1-\frac{|\mathbf{p}|}{E+m}\right)\lambda_{\{-,+\}}^{S}(\mathbf{0}), \tag{A.19}$$

$$\lambda_{\{+,-\}}^{S/A}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\left(1+\frac{|\mathbf{p}|}{E+m}\right)\lambda_{\{+,-\}}^{S}(\mathbf{0}). \tag{A.20}$$

A.3 Two Spinor Notation

For the supersymmetric and superfield formalism the notation by Sohnius (1985) was adopted. The two dimensional ϵ-tensors are chosen to be

$$\epsilon_{12} = \epsilon^{12} = -\epsilon_{\dot{1}\dot{2}} = -\epsilon^{\dot{1}\dot{2}} = 1. \tag{A.21}$$

Furthermore the raising and lowering of the spinor indices is defined such that

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta} \quad , \quad \psi_{\alpha} = \psi^{\beta}\epsilon_{\beta\alpha}, \tag{A.22}$$

$$\bar{\psi}^{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}} \quad , \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}. \tag{A.23}$$

Therefore, the two dimensional ϵ-tensors are related to the Kronecker-δ in the following way

$$\epsilon_{\alpha}{}^{\beta} = -\epsilon^{\beta}{}_{\alpha} = \delta_{\alpha}^{\beta}, \tag{A.24}$$

$$\epsilon^{\dot{\alpha}}{}_{\dot{\beta}} = -\epsilon_{\dot{\beta}}{}^{\dot{\alpha}} = \delta_{\dot{\beta}}^{\dot{\alpha}}. \tag{A.25}$$

In addition, the one and two dimensional sigma matrices are related by

$$\sigma^{\mu}\bar{\sigma}^{\nu} = \eta^{\mu\nu} - i\sigma^{\mu\nu}, \tag{A.26}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu} = \eta^{\mu\nu} - i\bar{\sigma}^{\mu\nu}. \tag{A.27}$$

It can then be shown that the matrices $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are symmetric under the exchange of spinor indices

$$(\sigma^{\mu\nu})_{\alpha\beta} = (\sigma^{\mu\nu})_{\beta\alpha}, \tag{A.28}$$

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}}, \tag{A.29}$$

and antisymmetric under exchange of Lorentz indices

$$\sigma^{\mu\nu} = -\sigma^{\nu\mu}\,, \tag{A.30}$$
$$\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu}\,. \tag{A.31}$$

A.4 Relations Between σ-matrices

A very good source for relations between σ-matrices can be found in the appendix of Wess and Bagger (1982). However, it is necessary to determine the appropriate phase factors as their choice of conventions for the metric and Dirac matrices differs from the one used in this thesis.

In the following section numerous relations involving two, three, or four σ-matrices as well as relations involving $\sigma^{\mu\nu}$ are summarised.

$$(\bar{\sigma}^{\mu})^{\dot{\gamma}\alpha}\,(\sigma^{\nu})_{\alpha\dot{\gamma}} = (\bar{\sigma}^{\mu}\sigma^{\nu})^{\dot{\gamma}}{}_{\dot{\gamma}} = \mathrm{Tr}(\bar{\sigma}^{\mu}\sigma^{\nu}) = 2\eta^{\mu\nu} \tag{A.32}$$
$$(\sigma^{\mu})_{\alpha\dot{\gamma}}\,(\bar{\sigma}^{\nu})^{\dot{\gamma}\alpha} = (\sigma^{\mu}\bar{\sigma}^{\nu})_{\alpha}{}^{\alpha} = \mathrm{Tr}(\sigma^{\mu}\bar{\sigma}^{\nu}) = 2\eta^{\mu\nu} \tag{A.33}$$

$$(\sigma^{\mu})_{\alpha\dot{\beta}}\,(\bar{\sigma}_{\mu})^{\dot{\gamma}\delta} = -2\epsilon_{\alpha}{}^{\delta}\epsilon_{\dot{\beta}}{}^{\dot{\gamma}} = 2\delta^{\delta}_{\alpha}\delta^{\dot{\gamma}}_{\dot{\beta}} \tag{A.34}$$
$$(\sigma^{\mu})_{\alpha\dot{\beta}}\,(\bar{\sigma}_{\mu})^{\dot{\beta}\alpha} = 2\delta^{\alpha}_{\alpha}\delta^{\dot{\beta}}_{\dot{\beta}} = 8 \tag{A.35}$$

$$(\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}} = 2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{A.36}$$
$$(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}{}^{\beta} = 2\eta^{\mu\nu}\delta^{\beta}_{\alpha} \tag{A.37}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} + \sigma^{\rho}\bar{\sigma}^{\nu}\sigma^{\mu} = 2\eta^{\nu\rho}\sigma^{\mu} - 2\eta^{\mu\rho}\sigma^{\nu} + 2\eta^{\mu\nu}\sigma^{\rho} \tag{A.38}$$
$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} + \bar{\sigma}^{\rho}\sigma^{\nu}\bar{\sigma}^{\mu} = 2\eta^{\nu\rho}\bar{\sigma}^{\mu} - 2\eta^{\mu\rho}\bar{\sigma}^{\nu} + 2\eta^{\mu\nu}\bar{\sigma}^{\rho} \tag{A.39}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} - \sigma^{\rho}\bar{\sigma}^{\nu}\sigma^{\mu} = -2i\epsilon^{\mu\nu\rho\tau}\sigma_{\tau} \tag{A.40}$$
$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} - \bar{\sigma}^{\rho}\sigma^{\nu}\bar{\sigma}^{\mu} = 2i\epsilon^{\mu\nu\rho\tau}\bar{\sigma}_{\tau} \tag{A.41}$$

$$\mathrm{Tr}(\sigma^{\mu\nu}) = 0 \tag{A.42}$$
$$\mathrm{Tr}(\bar{\sigma}^{\mu\nu}) = 0 \tag{A.43}$$

$$\mathrm{Tr}(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\sigma}) = 2\eta^{\rho\sigma}\eta^{\mu\nu} - 2\eta^{\nu\sigma}\eta^{\mu\rho} + 2\eta^{\nu\rho}\eta^{\mu\sigma} - 2i\epsilon^{\mu\nu\rho\sigma} \tag{A.44}$$
$$\mathrm{Tr}(\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\sigma}) = 2\eta^{\rho\sigma}\eta^{\mu\nu} - 2\eta^{\nu\sigma}\eta^{\mu\rho} + 2\eta^{\nu\rho}\eta^{\mu\sigma} + 2i\epsilon^{\mu\nu\rho\sigma} \tag{A.45}$$

$$\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = 2\eta^{\nu\sigma}\eta^{\mu\rho} - 2\eta^{\nu\rho}\eta^{\mu\sigma} + 2i\epsilon^{\mu\nu\rho\sigma} \tag{A.46}$$
$$\mathrm{Tr}(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}) = 2\eta^{\nu\sigma}\eta^{\mu\rho} - 2\eta^{\nu\rho}\eta^{\mu\sigma} - 2i\epsilon^{\mu\nu\rho\sigma} \tag{A.47}$$

$$\left(\sigma^{\mu}\bar{\sigma}^{\nu\rho}\right)_{\alpha\dot{\beta}} = -i\eta^{\mu\rho}\left(\sigma^{\nu}\right)_{\alpha\dot{\beta}} + i\eta^{\mu\nu}\left(\sigma^{\rho}\right)_{\alpha\dot{\beta}} + \epsilon^{\mu\nu\rho\sigma}\left(\sigma_{\sigma}\right)_{\alpha\dot{\beta}} \tag{A.48}$$

$$\left(\sigma^{\mu\nu}\sigma^{\rho}\right)_{\alpha\dot{\beta}} = i\eta^{\nu\rho}\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}} - i\eta^{\mu\rho}\left(\sigma^{\nu}\right)_{\alpha\dot{\beta}} + \epsilon^{\mu\nu\rho\sigma}\left(\sigma_{\sigma}\right)_{\alpha\dot{\beta}} \tag{A.49}$$

$$\left(\bar{\sigma}^{\mu}\sigma^{\nu\rho}\right)_{\dot{\alpha}\beta} = -i\eta^{\mu\rho}\left(\bar{\sigma}^{\nu}\right)_{\dot{\alpha}\beta} + i\eta^{\mu\nu}\left(\bar{\sigma}^{\rho}\right)_{\dot{\alpha}\beta} - \epsilon^{\mu\nu\rho\sigma}\left(\bar{\sigma}_{\sigma}\right)_{\dot{\alpha}\beta} \tag{A.50}$$

$$\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho}\right)_{\dot{\alpha}\beta} = i\eta^{\nu\rho}\left(\bar{\sigma}^{\mu}\right)_{\dot{\alpha}\beta} - i\eta^{\mu\rho}\left(\bar{\sigma}^{\nu}\right)_{\dot{\alpha}\beta} - \epsilon^{\mu\nu\rho\sigma}\left(\bar{\sigma}_{\sigma}\right)_{\dot{\alpha}\beta} \tag{A.51}$$

$$\begin{aligned}\left(\sigma^{\mu\nu}\sigma^{\rho\sigma}\right)_{\alpha}{}^{\beta} = &-\eta^{\nu\rho}\eta^{\mu\sigma}\epsilon_{\alpha}{}^{\beta} + i\eta^{\nu\rho}\left(\sigma^{\mu\sigma}\right)_{\alpha}{}^{\beta} + \eta^{\mu\rho}\eta^{\nu\sigma}\epsilon_{\alpha}{}^{\beta} - i\eta^{\mu\rho}\left(\sigma^{\nu\sigma}\right)_{\alpha}{}^{\beta} + i\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha}{}^{\beta}\\ &+ \epsilon^{\mu\nu\rho}{}_{\tau}\left(\sigma^{\tau\sigma}\right)_{\alpha}{}^{\beta} - i\eta^{\rho\sigma}\left(\sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\end{aligned} \tag{A.52}$$

$$\begin{aligned}\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}} = &-\eta^{\nu\rho}\eta^{\mu\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} + i\eta^{\nu\rho}\left(\bar{\sigma}^{\mu\sigma}\right)_{\dot{\alpha}\dot{\beta}} + \eta^{\mu\rho}\eta^{\nu\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} - i\eta^{\mu\rho}\left(\bar{\sigma}^{\nu\sigma}\right)_{\dot{\alpha}\dot{\beta}} - i\epsilon^{\mu\nu\rho\sigma}\epsilon_{\dot{\alpha}\dot{\beta}}\\ &- \epsilon^{\mu\nu\rho}{}_{\tau}\left(\bar{\sigma}^{\tau\sigma}\right)_{\dot{\alpha}\dot{\beta}} - i\eta^{\rho\sigma}\left(\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}}\end{aligned} \tag{A.53}$$

$$\begin{aligned}\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}} + \left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\beta}\dot{\alpha}} = &\, 2i\eta^{\nu\rho}\left(\bar{\sigma}^{\mu\sigma}\right)_{\dot{\alpha}\dot{\beta}} - 2i\eta^{\mu\rho}\left(\bar{\sigma}^{\nu\sigma}\right)_{\dot{\alpha}\dot{\beta}} - 2\epsilon^{\mu\nu\rho}{}_{\tau}\left(\bar{\sigma}^{\tau\sigma}\right)_{\dot{\alpha}\dot{\beta}}\\ &- 2i\eta^{\rho\sigma}\left(\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}}\end{aligned} \tag{A.54}$$

$$\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}} - \left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\beta}\dot{\alpha}} = -2\eta^{\nu\rho}\eta^{\mu\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} + 2\eta^{\mu\rho}\eta^{\nu\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} - 2i\epsilon^{\mu\nu\rho\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{A.55}$$

$$\begin{aligned}\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}} + \left(\bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}} = &-2\eta^{\nu\rho}\eta^{\mu\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} + i\eta^{\nu\rho}\left(\bar{\sigma}^{\mu\sigma}\right)_{\dot{\alpha}\dot{\beta}} + 2\eta^{\mu\rho}\eta^{\nu\sigma}\epsilon_{\dot{\alpha}\dot{\beta}} - 2i\epsilon^{\mu\nu\rho\sigma}\epsilon_{\dot{\alpha}\dot{\beta}}\\ &- \epsilon^{\mu\nu\rho}{}_{\tau}\left(\bar{\sigma}^{\tau\sigma}\right)_{\dot{\alpha}\dot{\beta}} - i\eta^{\rho\sigma}\left(\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}} + i\eta^{\sigma\mu}\left(\bar{\sigma}^{\rho\nu}\right)_{\dot{\alpha}\dot{\beta}}\\ &- \epsilon^{\rho\sigma\mu}{}_{\tau}\left(\bar{\sigma}^{\tau\nu}\right)_{\dot{\alpha}\dot{\beta}} - i\eta^{\mu\nu}\left(\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}}\end{aligned} \tag{A.56}$$

$$\begin{aligned}\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}} - \left(\bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}} = &\, i\eta^{\nu\rho}\left(\bar{\sigma}^{\mu\sigma}\right)_{\dot{\alpha}\dot{\beta}} - 2i\eta^{\mu\rho}\left(\bar{\sigma}^{\nu\sigma}\right)_{\dot{\alpha}\dot{\beta}} - \epsilon^{\mu\nu\rho}{}_{\tau}\left(\bar{\sigma}^{\tau\sigma}\right)_{\dot{\alpha}\dot{\beta}} - i\eta^{\rho\sigma}\left(\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}}\\ &- i\eta^{\sigma\mu}\left(\bar{\sigma}^{\rho\nu}\right)_{\dot{\alpha}\dot{\beta}} + \epsilon^{\rho\sigma\mu}{}_{\tau}\left(\bar{\sigma}^{\tau\nu}\right)_{\dot{\alpha}\dot{\beta}} + i\eta^{\mu\nu}\left(\bar{\sigma}^{\rho\sigma}\right)_{\dot{\alpha}\dot{\beta}}\end{aligned} \tag{A.57}$$

A.5 Superfield Calculus

Superfields can be separated into two groups – general superfields V as well as chiral Φ and anti-chiral $\bar{\Phi}$ superfields. The general superfields are as their name indicates the most general superfield that can be written down while chiral and anti chiral superfields satisfy additional symmetry conditions. It is usually possible to derive the chiral and anti chiral superfields by repeated operation of covariant derivatives form the corresponding general superfield.

These superfields can then be multiplied using one of the four products defined in the following equations

$$\Phi_{12} = \Phi_1 \cdot \Phi_2 \quad \Leftrightarrow \quad \Phi_{12} = \Phi_1 \Phi_2 \,, \tag{A.58}$$
$$V_{12} = V_1 \cdot V_2 \quad \Leftrightarrow \quad V_{12} = V_1 V_2 \,, \tag{A.59}$$
$$V = \Phi_1 \times \Phi_2 \quad \Leftrightarrow \quad V = \frac{1}{2}\left(\Phi_1 \bar{\Phi}_2 + \bar{\Phi}_1 \Phi_2\right) \,, \tag{A.60}$$
$$\Phi_{12} = \Phi_1 \wedge \Phi_2 \quad \Leftrightarrow \quad \Phi_{12} = \frac{i}{2}\left(\Phi_1 \bar{\Phi}_2 - \bar{\Phi}_1 \Phi_2\right) \,. \tag{A.61}$$

Here, equations (A.58) and (A.59) represent dot or scalar products. The scalar product between two superfields of the same type always results in a superfield with the same properties, e. g. the product of two chiral superfields is chiral and the product of two general superfields is another general superfield. However, the product of a chiral superfield with a general superfield is usually a general superfield and thus obviously cannot preserve the type of the initial superfield.

The vector product between two chiral superfields as defined in equation (A.60) no longer preserves the symmetry properties of the original superfields and the resulting product multiplet is in general not a chiral multiplet but rather a general multiplet instead. A brief look at the definition reveals that the vector product is symmetric under the exchange of the two chiral superfields.

The last product that can be formulated is the exterior product and is given in equation (A.61). Similar to the vector product the exterior product of two chiral superfields is a general superfield, however, due to the relative sign flip between the first and second term in the definition the exterior product is antisymmetric under the exchange of the two chiral superfields.

In the following subsections the fields and products that are relevant for this thesis will be discussed in detail. A discussion of the remaining superfields can be found in the book by Wess and Bagger (1982) and the report by Sohnius (1985).

A.5.1 The Chiral Multiplet

A chiral multiplet

$$\Phi_j = (A_j; \phi_j; F_j) \tag{A.62}$$

contains the two complex scalar fields A_j and F_j as well as one spinor field ϕ_j.

The Taylor expansion of a chiral superfield is especially simple as it can be separated into a part that solely depends on θ and an exponential function containing both θ and $\bar{\theta}$. The calculation is furthermore simplified, as the Taylor expansion of the exponential terminates rapidly and only the first two series terms – besides the constant term – can actually contribute while all higher order terms in the Grassmann variables vanish by definition

$$\begin{aligned}\Phi &= \exp\left(-i\theta\partial\!\!\!/\bar{\theta}\right)\left(A + \theta^\alpha \phi_\alpha + \theta^2 F\right) \\ &= A + \theta^\alpha \phi_\alpha + \theta^2 F - i\theta\partial\!\!\!/\bar{\theta} A + \frac{i}{2}\theta^2 \bar{\theta}^{\dot{\gamma}} \bar{\partial\!\!\!/}_{\dot{\gamma}}{}^{\alpha} \phi_\alpha - \frac{1}{4}\theta^2 \bar{\theta}^2 \Box A \,.\end{aligned} \tag{A.63}$$

The Hermitian conjugate of the chiral superfield which is then an anti-chiral superfield is found to be

$$\bar{\Phi} = \Phi^\dagger = A^\dagger + \bar{\theta}\bar{\phi} + \bar{\theta}^2 F^\dagger + i\theta\partial\!\!\!/\bar{\theta}A^\dagger + \frac{i}{2}\bar{\theta}^2\theta^\gamma \partial\!\!\!/_\gamma{}^{\dot{\alpha}}\bar{\phi}_{\dot{\alpha}} - \frac{1}{4}\theta^2\bar{\theta}^2\Box A^\dagger\,. \tag{A.64}$$

A.5.2 The Scalar Product

Using the explicit form for the chiral superfield from equation (A.63) it is possible to show that the dot-product of two chiral multiplets is again a chiral multiplet

$$\begin{aligned}\Phi_{12} &= \Phi_1 \cdot \Phi_2 \\ &= A_1A_2 + \theta^\alpha\left(A_1\phi_{2\alpha} + \phi_{1\alpha}A_2\right) + \theta^2\left(A_1F_2 + F_1A_2 - \frac{1}{2}\phi_1\phi_2\right) - i\theta\partial\!\!\!/\bar{\theta}\left(A_1A_2\right) \\ &\quad + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\beta}}\bar{\partial\!\!\!/}_{\dot{\beta}}{}^\alpha\left(A_1\phi_{2\alpha} + \phi_{1\alpha}A_2\right) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\left(A_1A_2\right)\,,\end{aligned} \tag{A.65}$$

where the superfield components of the product superfield Φ_{12} are given by the following combinations of the superfields components of Φ_1 and Φ_2

$$A_{12} = A_1A_2\,, \tag{A.66}$$

$$\phi_{12\alpha} = A_1\phi_{2\alpha} + \phi_{1\alpha}A_2\,, \tag{A.67}$$

$$F_{12} = A_1F_2 + F_1A_2 - \frac{1}{2}\phi_1\phi_2\,. \tag{A.68}$$

Higher order superfield products can then be calculated by repeated application of the product rule for two chiral superfields. For the product of three chiral superfields for example the component fields are given by

$$\begin{aligned}A_{123} &= A_{12}A_3 \\ &= A_1A_2A_3\,,\end{aligned} \tag{A.69}$$

$$\begin{aligned}\phi_{123} &= A_{12}\phi_{3\alpha} + \phi_{12\alpha}A_3 \\ &= A_1A_2\phi_{3\alpha} + A_1\phi_{2\alpha}A_3 + \phi_{1\alpha}A_2A_3\,,\end{aligned} \tag{A.70}$$

$$\begin{aligned}F_{123} &= A_{12}F_3 + F_{12}A_3 - \frac{1}{2}\phi_{12}\phi_3 \\ &= A_1A_2F_3 + A_1F_2A_3 + F_1A_2A_3 - \frac{1}{2}\phi_1\phi_2A_3 - \frac{1}{2}A_1\phi_2\phi_3 - \frac{1}{2}\phi_1A_2\phi_3\,.\end{aligned} \tag{A.71}$$

A.5.3 The Kinetic Multiplet

The kinetic multiplet belongs to the previously discussed group of chiral multiplets. However, it satisfies an additional condition that relates the kinetic multiplet $\mathrm{T}\Phi$ to another chiral multiplet Φ by

$$\mathrm{T}\Phi = -\frac{1}{4}\bar{D}^2\bar{\Phi}. \tag{A.72}$$

For the specific choice for Φ from equation (A.63) it can be shown that the kinetic multiplet is found to be

$$\mathrm{T}\Phi = F^\dagger - i\theta\partial\!\!\!/\bar{\phi} - \theta^2\Box A^\dagger - i\theta\partial\!\!\!/\bar{\theta}F^\dagger - \frac{1}{2}\theta^2\bar{\theta}\Box\bar{\phi} - \frac{1}{4}\theta^2\bar{\theta}^2\Box F^\dagger\,. \tag{A.73}$$

To bring this intermediate result into a form that represents a chiral superfield the term $-\frac{1}{2}\theta^2\bar{\theta}\Box\bar{\phi}$ needs to be rewritten using the identity

$$\bar{\theta}\Box\bar{\phi} = \bar{\theta}^{\dot{\alpha}}\epsilon_{\dot{\alpha}}{}^{\dot{\beta}}\Box\bar{\phi}_{\dot{\beta}} = \bar{\theta}^{\dot{\alpha}}\left(\bar{\partial\!\!\!/}\partial\!\!\!/\right)_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\phi}_{\dot{\beta}} = \bar{\theta}^{\dot{\alpha}}\bar{\partial\!\!\!/}_{\dot{\alpha}}{}^{\gamma}\partial\!\!\!/_{\gamma}{}^{\dot{\beta}}\bar{\phi}_{\dot{\beta}} = \bar{\theta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}}{}^{\alpha}\partial\!\!\!/_{\alpha}{}^{\dot{\beta}}\bar{\phi}_{\dot{\beta}}\,. \tag{A.74}$$

The kinetic multiplet can then be expressed as

$$\mathrm{T}\Phi = F^{\dagger} + \theta^{\alpha}\left(-i\partial\!\!\!/_{\alpha\dot{\beta}}\bar{\phi}^{\dot{\beta}}\right) - \theta^2\Box A^{\dagger} - i\theta\partial\!\!\!/\bar{\theta}F^{\dagger} + \frac{i}{2}\theta^2\bar{\theta}^{\dot{\gamma}}\bar{\partial\!\!\!/}_{\dot{\gamma}}{}^{\alpha}\left(-i\partial\!\!\!/_{\alpha\dot{\beta}}\bar{\phi}^{\dot{\beta}}\right) - \frac{1}{4}\theta^2\bar{\theta}^2\Box F^{\dagger}\,. \tag{A.75}$$

It is straightforward to identify the component fields of the kinetic multiplet with those of a chiral multiplet. It results in a relation between the component fields of the kinetic multiplet and the chiral multiplet of

$$A_{T\Phi} = F^{\dagger}\,, \tag{A.76}$$

$$\phi_{\alpha T\Phi} = -i\partial\!\!\!/_{\alpha\dot{\beta}}\bar{\phi}^{\dot{\beta}}\,, \tag{A.77}$$

$$F_{T\Phi} = -\Box A^{\dagger}\,. \tag{A.78}$$

Using this correspondence the kinetic multiplet can be abbreviated as

$$\mathrm{T}\Phi = \left(F^{\dagger}; -i\partial\!\!\!/\bar{\phi}; -\Box A^{\dagger}\right)\,. \tag{A.79}$$

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