NUMERICAL METHODS APPLIED TO POWER SYSTEM LOAD FLOW STUDIES

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ABSTRACT

Load flow analysis is an important aspect of power system studies in both planning and operating areas. This thesis studies several methods for load flow solution by developing corresponding digital computer programs and by applying them to different size systems. The conclusion was reached that the overrelaxed Newton-Raphson Method and the Optimum Accelerated Gauss-Seidel Method are the preferable methods for normal load flow studies. The speed of convergence and the optimum acceleration factor for different methods was studied using numerical analysis techniques. The relationship between the optimum acceleration factor and the maximum eigenvalue of the coefficient matrix for each different method was determined and applied to the iterative process.

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1. INTRODUCTION

1.1 General

The demand for electricity has rapidly increased due to industrial development and improvements in daily living. Power systems are becoming larger and more complex. In the planning of new networks and the extending of existing ones, numerous load-flow studies must be made. A load-flow study is the determination of the voltage, current, power and power factor or reactive power at various points in an electrical power system under existing or contemplated conditions of steady state operation. Satisfactory operation of the system depends upon knowing the effects of new loads, new generating stations, proposed transmission lines and interconnections with other power systems.

The bulk of all load flow studies are performed to determine the following:

(1) If the system is steady-state stable, i.e. the required amount of energy can be transported from the appropriate generating station to the load points.

(2) The system voltage levels are within predetermined limits and the real and reactive power flows in the system network are acceptable.

(3) Transmission load levels under normal and abnormal operating conditions.

(4) System restoration procedures under conditions of system disturbance.

At the present time, most of these studies are performed on a digital computer and numerous papers have been published describing various methods of computation.

The purpose of this thesis is to study these various methods from

the numerical analysis point of view and to determine the method which requires minimum computation time, minimum computer storage, and in general, minimum computation facilities for a normal load-flow study.

1.2 The problem

The load-flow solution can be treated as a combination of two problems: the solution of the network equations themselves and the satisfaction of nodal constraints at certain points in the system.

The current and voltage relationship in the network can be expressed in the general matrix form:

$$[\dot{Y}][\dot{V}] = [\dot{I}] \tag{1.1}$$

Or in the inverse form:

$$[\dot{V}] = [\dot{Y}]^{-1}[\dot{I}] = [\dot{Z}][\dot{I}] \tag{1.2}$$

where $[\dot{Y}]$ and $[\dot{Z}]$ are admittance and impedance matrices respectively.

The relationship between the voltages and complex power in the network is then:

$$[\dot{S}] = [\dot{V}][\dot{I}]^* \tag{1.3}$$

where $[\dot{I}]^*$ is the complex conjugate of matrix $[\dot{I}]$.

Many excellent papers have been written describing various methods for solving the above equations. Each of these methods possess advantages and disadvantages under certain conditions and particularly in situations where computing facilities are limited.

The following areas have been examined in detail in this thesis. (1) Digital computer programs for load flow studies have been developed using the following methods:

(a) Direct Method - Elimination Method

(b) Gauss-Seidel method .

(c) Newton-Raphson method

(d) Relaxation method

The digital computer flow charts for each of these programs are shown in the Appendix (7.3).

(2) The different methods have been compared for systems of different sizes and for systems containing special conditions.

(3) The following aspects are examined in detail for the iterative solutions using the Gauss-Seidel and the Newton-Raphson techniques.

(a) The derivation and determination of optimum acceleration factors.

(b) Speed of convergence comparisons for different solution tolerances and acceleration factors.

(c) Speed of convergence comparisons for solutions utilizing a variable acceleration factor.

1.3 Power system data

The data used to represent a practical power system were provided by the Saskatchewan Power Corporation. A single line diagram of this system and of a hypothetical simple system configuration also used are shown in the Appendix (7.1) and (7.2).

In general the necessary system data consists of the following:

(a) Transmission line and transformer impedances, line susceptances and transformer tap settings.

(b) Generation schedules.

(c) Regulated bus voltage schedules, i.e. generator bus and synchronous condenser busses.

(d) System loads.

(e) Static capacitor and shunt reactor admittance values.

1.4 Facilities for studies

The studies outlined in section 1.2 were performed using the IBM 7040 and 360 digital computers at the University of Saskatchewan. The digital computer programs used for the system load flow studies were developed by the author. In the following chapters, the details of the various studies and the results are presented.

2. METHODS FOR SOLVING LOAD-FLOW PROBLEMS

2.1 General

The importance of load flow studies in planning the future expansion of power systems and in determining the optimum operation of existing systems was noted in section 1.1. Load flow studies can be made using both digital computers and A.C. network analyzers.

(a) The A.C. Network Analyzer Method:

The power system to be studied is set up in miniature in direct analog fashion on the analyzer. Each generating station is represented by a generator unit and each transmission line by a circuit having the steady state electrical characteristics of the real line. Load units are used to represent the load points in the system. The currents, voltages and real and reactive power flows within the network can be readily measured on a single front panel instrument after the appropriate circuits have been selected and interconnected. Alterations to the network can easily be made and their effect examined.

(b) The Digital Computer Method:

Applying Kirchhoff's current law at the nodes of a power system network, the relationship between nodal current I_k and node-to-datum voltage V_k in a network of N nodes is given by the following linear equation. The formulation of this equation is shown in detail in page 14.

$$\dot{I}_k = \sum_{m=1}^{N} \dot{Y}_{km} \dot{V}_m \qquad (2.1)$$

where $\dot{Y}_{km}$ is an element of the admittance matrix.

$\dot{I}_k$, $\dot{Y}_{km}$ and $\dot{V}_m$ are complex quantities.

Complex power at node k is given by:

$$P_k + jQ_k = \dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^* \qquad (2.2)$$

where P_k and Q_k are the respective real and reactive powers entering node k.

In the digital computation method, it is necessary to solve a system of (N - 1) equations of the form shown in equation 2.2. In the normal load flow problem, there are three basic types of additional constraints which must be satisfied. At each node there are four variables: P_k, Q_k, V_k and δ_k. The three classes are therefore as follows:

(1) At the swing bus, V_k and δ_k are specified but not P_k and Q_k.

(2) At a generator or voltage controlled bus, V_k and P_k are specified but not Q_k and δ_k, where δ_k is the phase angle of $\dot{V}_k$ with respect to $\dot{V}_{swing}$.

(3) At a load bus, P_k and Q_k are specified, but not V_k and δ_k.

The solution proceeds by finding the complex magnitude of all the node voltages followed by the system currents and power flows.

Compared with the A.C. network analyzer method, the digital computer approach offers advantages of greater precision, speed and a reduction in the labour involved in preparing and executing a study. The bulk of the time required by a digital load flow study is in finding the solution to the sets of equations relating current and voltage. This thesis is restricted to digital computation methods and in particular to this latter problem.

Using the divisions commonly considered in linear algebra, the available techniques can be classed as being either a direct or an iterative method.

2.2 Direct method

The direct method for solving a system of linear equations is known

as the "Elimination Method". For any set of equations: $[Y][V] = [I]$. The elimination process consists of the following three steps:

(a) Triangularization

An appropriate multiple of the first equation is subtracted from each of the remaining equations yielding equations in which the coefficients of the first variables are zero. An appropriate multiple of the first new equation is then subtracted from each of the remaining equations yielding another set of equations in which the coefficients of the second variables are zero. The process is continued until only one new equation is produced. The resulting equations in matrix form display zero values in the lower left triangle of the coefficient matrix.

(b) Normalization

Each new equation is multiplied by the reciprocal of its first non-zero coefficient to reduce the first non-zero coefficient of each equation to unity.

(c) Back substitution

The solution of the linear equations are obtained by progressive substitution, commencing with the last equation which contains only a single diagonal non-zero coefficient. The equations are then solved successively for the remaining variables.

In the load flow solution for a power system, a set of linear equations can be formed giving the relationship between the small changes in the angle and in the magnitude of the voltage variable and in the small changes of the real and reactive power.

The equations can be formulated as follows:

$$\Delta P_k = \sum_{m=1}^{N} H_{km} \Delta \delta_m + \sum_{m=1}^{N} N_{km} \frac{\Delta V_m}{V_m}$$

$$\Delta Q_k = \sum_{m=1}^{N} J_{km} \Delta\delta_m + \sum_{m=1}^{N} L_{km} \frac{\Delta V_m}{V_m} \qquad (2.3)$$

The coefficients H_{km}, N_{km}, J_{km} and L_{km} can be evaluated by taking partial derivatives of the real and reactive power as follows:

Rewrite the equation (2.2) as

$$P_k + jQ_k = \dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^*$$

$$= V_k e^{j\delta_k} \sum_{m=1}^{N} Y_{km} e^{-j\theta_{km}} V_m e^{-j\delta_m} \qquad (2.4)$$

Take the partial derivative of equation (2.4) with respect to δ_m:

$$\frac{\partial P_k}{\partial \delta_m} + j\frac{\partial Q_k}{\partial \delta_m} = -j V_k e^{+j\delta_k} Y_{km} e^{-j\theta_{km}} V_m e^{-j\delta_m}$$

Expressed in rectangular form:

$$\frac{\partial P_k}{\partial \delta_m} + j \frac{\partial Q_k}{\partial \delta_m} = -j (e_k + jf_k)(a_m - jb_m) \qquad (2.5)$$

where $a_m + jb_m = (e_m + jf_m)(G_{km} + jB_{km})$

Equating real and imaginary parts of equation (2.5) gives:

$$H_{km} = \frac{\partial P_k}{\partial \delta_m} = a_m f_k - b_m e_k \qquad \text{for } k \neq m \qquad (2.6)$$

$$J_{km} = \frac{\partial Q_k}{\partial \delta_m} = -(a_m e_k + b_m f_k)$$

Similarly, take the partial derivatives of equation (2.4) with respect to V_m:

$$\frac{\partial P_k}{\partial V_m} + j\frac{\partial Q_k}{\partial V_m} = \frac{1}{V_m}(V_k e^{j\delta_k})(Y_{km}e^{-j\theta_{km}})(V_m e^{-j\delta_m})$$

Hence
$$N_{km} = \frac{\partial P_k}{\partial V_{m/V_m}} = a_m e_k + b_m f_k$$
$$L_{km} = \frac{\partial Q_k}{\partial V_{m/V_m}} = a_m f_k - b_m e_k \qquad \text{for } k \neq m \qquad (2.7)$$

when m is equal to k, the approach is as follows:

Take the partial derivative of equation (2.4) with respect to δ_k

$$\frac{\partial P_k}{\partial \delta_k} + j\frac{\partial Q_k}{\partial \delta_k} = jV_k e^{j\delta_k}\sum_{m=1}^{N} Y_{km}e^{-j\theta_{km}} V_m e^{-j\delta_m} - jV_k e^{j\delta_k} Y_{kk}e^{-j\theta_{kk}} V_k e^{-j\delta_k}$$

$$= j(P_k + jQ_k) - jV_k^2(G_{kk} - jB_{kk})$$

Hence
$$H_{kk} = \frac{\partial P_k}{\partial \delta_k} = -Q_k - B_{kk}V_k^2$$
$$J_{kk} = \frac{\partial Q_k}{\partial \delta_k} = P_k - G_{kk}V_k^2 \qquad (2.8)$$

Similarly

$$\frac{\partial P_k}{\partial V_k} + j\frac{\partial Q_k}{\partial V_k} = \frac{1}{V_k}(P_k + jQ_k) + V_k(G_{kk} - jB_{kk})$$

Therefore
$$N_{kk} = \frac{\partial P_k}{\partial V_{k/V_k}} = P_k + G_{kk}V_k^2$$
$$L_{kk} = \frac{\partial Q_k}{\partial V_{k/V_k}} = Q_k - B_{kk}V_k^2 \qquad (2.9)$$

In a brief general form

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} H & N \\ J & L \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \frac{\Delta V}{V} \end{bmatrix} \tag{2.10}$$

For a 5-bus network, the equations can be written in the following matrix form:

$$\left[\begin{array}{c} \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \\ \Delta P_5 \\ \hline \Delta Q_3 \\ \Delta Q_5 \end{array}\right] \quad \left[\begin{array}{cccc:cc} H_{22} & H_{23} & H_{24} & H_{25} & N_{23} & N_{25} \\ H_{32} & H_{33} & H_{34} & H_{35} & N_{33} & N_{35} \\ H_{42} & H_{43} & H_{44} & H_{45} & N_{43} & N_{45} \\ H_{52} & H_{53} & H_{54} & H_{55} & N_{53} & N_{55} \\ \hdashline J_{32} & J_{33} & J_{34} & J_{35} & L_{33} & L_{35} \\ J_{52} & J_{53} & J_{54} & J_{55} & L_{53} & L_{55} \end{array}\right] \quad \left[\begin{array}{c} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \\ \Delta\delta_5 \\ \hline \frac{\Delta V_3}{V_3} \\ \frac{\Delta V_5}{V_5} \end{array}\right] \tag{2.11}$$

It was assumed that the first node represents the swing bus for which both V_k and δ_k are given and therefore there will be no variation in their magnitudes. Nodes 2 and 4 represent generators for which V_k and P_k are given, and therefore only δ_k is variable. Nodes 3 and 5 represent loads where P_k and Q_k are specified and both V_k and δ_k are variables.

In the Elimination Method, the real and reactive power are computed for each node using estimated and known values of V and δ. At the same time, the coefficients H, L, J and N are computed and stored. The ΔP and ΔQ values are then found by comparing the P and Q values obtained using equation (2.2) with the initially specified values. The nodal voltage correction factors are then found from matrix equation (2.11) using the

elimination process. These correction factors, however, will not completely correct the voltage magnitudes and angles as they did not take into account the non-linearity of the original equation (2.2). The process is repeated until the desired accuracy is obtained. In the elimination method, the decrease in error at each step is independent of the system size. This is discussed in more detail in the next chapter.

The disadvantages of this method are:

(a) The storage requirement for the non-symmetric coefficient matrix increases rapidly as the size of the system increases.

(b) The coefficient matrix, which is a function of voltage, must be re-evaluated for each step.

(c) The number of multiplications and divisions required in the elimination process increases as the size of system increases. The necessary computation time therefore also increases.

(d) A round-off error is incorporated in this process which may be significant depending upon the available digital computer and the size of the system being studied.

Certain tehcniques are available to rearrange and/or reduce the storage requirement. These techniques generally increase the computation time. It is doubtful, however, if the reduced storage requirement is less than that required to solve the load flow problem using an iterative approach.

A modified elimination method[6] is discussed in section 2.4.

2.3 Iterative methods

The basic mathematical expression for the load flow problem was shown in equation (2.2) as:

$$P_k + jQ_k = \dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^*$$

These equations are non-linear. The solution can be reached only by an iterative procedure. This is carried out by assigning estimated values to the unknown nodal voltages (one per unit is often chosen). The voltage at the first node is calculated using the estimated values at the other nodes together with its own real and reactive power constraints. A corrected value is thus obtained for one node and this new value is then used to perform a similar calculation to obtain a corrected voltage at the next node. Each nodal voltage in turn around the system is corrected and the process is then repeated until corrections at each node are less than a specified tolerance value. The number of iterations required to reach an acceptable precision can be reduced considerably by introducing an acceleration factor to the voltage correction at each node. The speed of convergence of different methods and the acceleration factor value is discussed in detail in this thesis.

2.3.1 Gauss-Seidel Method

The Gauss-Seidel method which is also known as the method of successive displacement is a practical iterative scheme for solving a set of linear equations under certain circumstances. A typical set of linear equations are as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2' + \ldots\ldots\ldots + a_{nn}x_n &= b_n \end{aligned} \qquad (2.12)$$

The process of this method is as follows:

$x^{(0)}$ is arbitrary and $x^{(p+1)}$ is obtained from $x^{(p)}$ by finding the solution of the following triangular system:

$$
\begin{aligned}
a_{11}x_1^{(p+1)} + a_{12}x_2^{(p)} + \ldots\ldots + a_{1n}x_n^{(p)} &= b_1 \\
a_{21}x_1^{(p+1)} + a_{22}x_2^{(p+1)} + \ldots\ldots + a_{2n}x_n^{(p)} &= b_2 \qquad (2.13) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{n1}x_1^{(p+1)} + a_{n2}x_2^{(p+1)} + \ldots\ldots + a_{nn}x_n^{(p+1)} &= b_n
\end{aligned}
$$

where $x^{(p)}$ is the value of x of p^{th} iteration.

The nodal equation used in this method for a power system network are derived as follows:

Consider a typical node p:

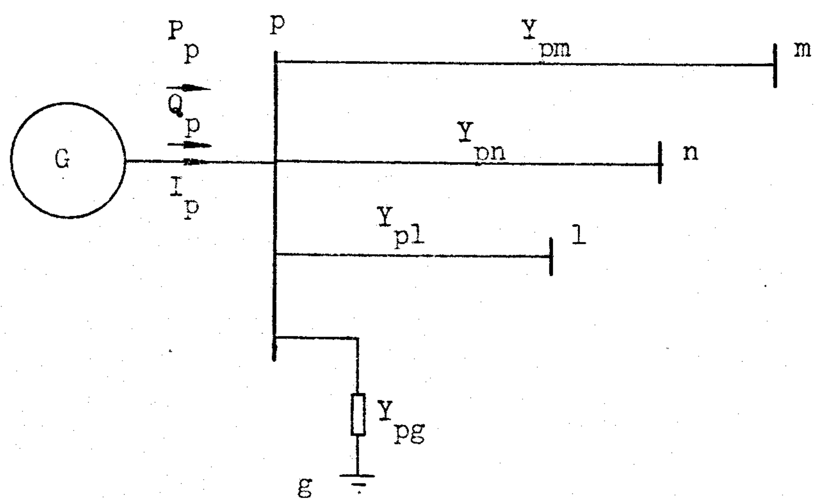

Figure 2.1 Interconnection of bus p with other busses in the system

The input current I_p, due to net power $P_p - jQ_p$ into the bus p can be written as:

$$\dot{I}_p = \dot{Y}_{pg}(\dot{V}_p - \dot{V}_g) + \dot{Y}_{pm}(\dot{V}_p - \dot{V}_m) + \dot{Y}_{pn}(\dot{V}_p - \dot{V}_n) + \dot{Y}_{pl}(\dot{V}_p - \dot{V}_l)$$

Also $P_p - jQ_p = \dot{V}_p^* \dot{I}_p$

Substitution yields:

$$\frac{P_p - jQ_p}{\dot{V}_p^*} = (\dot{Y}_{pg} + \dot{Y}_{pm} + \dot{Y}_{pn} + \dot{Y}_{pl})\dot{V}_p - \dot{Y}_{pm}\dot{V}_m - \dot{Y}_{pn}\dot{V}_n - \dot{Y}_{pl}\dot{V}_l$$

Solving for $\dot{V}_p$:

$$\dot{V}_p = \frac{\dfrac{P_p - jQ_p}{\dot{V}_p^*} + \dot{Y}_{pm}\dot{V}_m + \dot{Y}_{pn}\dot{V}_n + \dot{Y}_{pl}\dot{V}_l}{\dot{Y}_{pg} + \dot{Y}_{pm} + \dot{Y}_{pn} + \dot{Y}_{pl}}$$

In general,

$$\dot{V}_p = \frac{\dfrac{P_p - jQ_p}{\dot{V}_p^*} + \sum_{\substack{q=1 \\ q \neq p}}^{N} \dot{Y}_{pq}\dot{V}_q}{\sum_{q=1}^{N} \dot{Y}_{pq}} \qquad (2.14)$$

The Y_{pg} term in the denominator includes the admittances of line charging, static capacitors, constant admittance loads and any other fixed admittance to ground. Equation (2.14) is used to solve for $\dot{V}_p$ where P_p and Q_p are specified.

At a generator or voltage controlled bus, the voltage $\dot{V}_p$ and the

real power P_p are specified. Using a modified form of equation (2.14), the reactive power Q_p can be obtained as follows:

$$Q_p = - \mathcal{I}_m \left[\dot{V}_p \sum_{q=1}^{N} \dot{Y}_{pq} - \sum_{\substack{q=1 \\ q \neq p}}^{N} \dot{Y}_{pq} \dot{V}_q \right] \dot{V}_p^* \qquad (2.15)$$

where $\mathcal{I}_m [\ ]$ means the imaginary part of complex variable [ ].

The reactive power Q_p is evaluated using equation (2.15) for the best previous voltage values at the nodes. This value of Q_p is then substituted into equation (2.14) to find a new $\dot{V}_p$. At the swing bus node, the voltage magnitude and angle are known and therefore no equation is required. An acceleration factor ω can be introduced during the iterative process in the following manner:

$$\dot{V}^{(p+1)} = \omega \left[\dot{V}^{(p+1)}_{\text{calculated}} - \dot{V}^{(p)} \right] + \dot{V}^{(p)} \qquad (2.16)$$

The bus voltages are successively corrected until the predetermined tolerance limit is reached. A considerable number of studies have been done using this method and are discussed in detail in chapter 4. The results indicated that the Gauss-Seidel Method using an optimum acceleration factor has the following advantages:

(a) It requires the least computer storage of any technique and is therefore applicable to extremely large systems.

(b) The speed of solution is relatively fast. Using an optimum acceleration factor, the total computer time required by this method was less than that required by most of other methods studied.

The character of convergence and acceleration of this method is

analyzed in the next chapter.

The principal disadvantage of this method is that the solution may not converge under certain system conditions.

2.3.2 Newton-Raphson Method

This is an iterative process for solving non-linear equations utilizing their first order derivatives or partial-derivatives.

For a simple equation of the form:

$$f(x) = 0 \tag{2.17}$$

If $x^{(p)}$ is an approximation to the equation, with an error of $\Delta x^{(p)}$ at stage p of the iteration process.

$$f(x^{(p)} + \Delta x^{(p)}) = 0 \tag{2.18}$$

In the Newton-Raphson Method, a set of linear equations can be obtained by differentiating the system of non-linear equations describing the system.

$$f(x^{(p)}) + f'(x^{(p)})\ \Delta x^{(p)} = 0 \qquad x^{(p+1)} = x^{(p)} + \Delta x^{(p)} \tag{2.19}$$

The iterative process consists of repeating this calculation, obtaining the successive approximation $x^{(p+1)}$, $x^{(p+2)}$, ..., in turn until the final value is reached.

Equation (2.18) can also be expressed by Taylor's Theorem:

$$f(x^{(p)} + \Delta x^{(p)}) = f(x^{(p)}) + \Delta x^{(p)} f'(x^{(p)}) + \frac{\Delta x^{(p)^2}}{2} f''(x^{(p)}) + \ldots \tag{2.20}$$

Comparing equation (2.20) with equation (2.19), it can be seen that the error in finding the next approximation by using the Newton-Raphson method is at least a second order of $\Delta x^{(p)}$.

Applying this method to the load flow problem, equation(2.2) can be re-written as:

$$(P_k + jQ_k) = \dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^* \qquad (2.21)$$

At any node k where P_k and Q_k are specified, this equation can be written as follows, where $P_k + jQ_k = \dot{S}_k$

$$\Delta \dot{S}_k = \dot{S}_{k\ \text{specified}} - \dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^* \qquad (2.22)$$

$\Delta\dot{S}$ must eventually be zero for this equation to be satisfied.

At any step p in the iteration:

$$\Delta S_k^{(p)} = \Delta P_k^{(p)} + j\Delta Q_k^{(p)}$$

The voltage at node k is V_k and $V_k = e_k + jf_k$.

$$\Delta S_k^{(p)} = \left(\frac{\partial P_k}{\partial e_k} + j\frac{\partial Q_k}{\partial e_k}\right)^{(p)} \Delta e_k^{(p)} + \left(\frac{\partial P_k}{\partial f_k} + j\frac{\partial Q_k}{\partial f_k}\right)^{(p)} \Delta f_k^{(p)} \qquad (2.23)$$

Take the partial derivative of equation (2.21) with respect to $\Delta e_k^{(p)}$ and $\Delta f_k^{(p)}$, and let:

$\dot{V}_m$ = constant for $m \neq k$.

$$\frac{\partial}{\partial e_k}(P_k+jQ_k)^{(p)} = \frac{\partial}{\partial e_k}[\dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^*]^{(p)}$$

$$= \frac{\partial}{\partial e_k}[(e_k+jf_k)\sum_{m=1}^{N}(G_{km}-jB_{km})(e_m-jf_m)]^{(p)}$$

$$= [\sum_{m=1}^{N}(G_{km}e_m^{(p)}-B_{km}f_m^{(p)}) + e_k^{(p)}G_{kk} + f_k^{(p)}B_{kk}] +$$

$$j[\sum_{m=1}^{N}(-G_{km}f_m^{(p)}-B_{km}e_m^{(p)}) - e_k^{(p)}B_{kk} + f_k^{(p)}G_{kk}]$$

Equating real and imaginary parts of this equation gives

$$\frac{\partial P_k^{(p)}}{\partial e_k} = \sum_{m=1}^{N}(G_{km}e_m^{(p)} - B_{km}f_m^{(p)}) + e_k^{(p)}G_{kk} + f_k^{(p)}B_{kk}$$

$$\frac{\partial Q_k^{(p)}}{\partial e_k} = \sum_{m=1}^{N}(-G_{km}f_m^{(p)} - B_{km}e_m^{(p)}) - e_k^{(p)}B_{kk} + f_k^{(p)}G_{kk}$$

Similarly $$\frac{\partial}{\partial f_k}(P_k+jQ_k)^{(p)} = \frac{\partial}{\partial f_k}[\dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^*]^{(p)}$$

$$= [\sum_{m=1}^{N}(G_{km}f_m^{(p)} + B_{km}e_m^{(p)}) - e_k^{(p)}B_{kk} + f_k^{(p)}G_{kk}] +$$

$$j[\sum_{m=1}^{N}(G_{km}e_m^{(p)} - B_{km}f_m^{(p)}) - e_k^{(p)}G_{kk} - f_k^{(p)}B_{kk}]$$

Hence $$\frac{\partial P_k^{(p)}}{\partial f_k} = \sum_{m=1}^{N} (G_{km} f_m^{(p)} + B_{km} e_m^{(p)}) - e_k^{(p)} B_{kk} + f_k^{(p)} G_{kk}$$

$$\frac{\partial Q_k^{(p)}}{\partial f_k} = \sum_{m=1}^{N} (G_{km} e_m^{(p)} - B_{km} f_m^{(p)}) - e_k^{(p)} G_{kk} - f_k^{(p)} B_{kk}$$

Equation 2.23 can then be expressed in the following form:

$$\begin{bmatrix} \Delta P_k^{(p)} \\ \Delta Q_k^{(p)} \end{bmatrix} = \begin{bmatrix} \left\{\sum_{m=1}^{N} (G_{km} e_m^{(p)} - B_{km} f_m^{(p)}) + e_k^{(p)} G_{kk} + f_k^{(p)} B_{kk}\right\} \\ \left\{\sum_{m=1}^{N} (-G_{km} f_m^{(p)} - B_{km} e_m^{(p)}) - e_k^{(p)} B_{kk} + f_k^{(p)} G_{kk}\right\} \end{bmatrix}$$

$$\begin{bmatrix} \left\{\sum_{m=1}^{N} (G_{km} f_m^{(p)} + B_{km} e_m^{(p)}) - e_k^{(p)} B_{kk} + f_k^{(p)} G_{kk}\right\} \\ \left\{\sum_{m=1}^{N} (G_{km} e_m^{(p)} - B_{km} f_m^{(p)}) - e_k^{(p)} G_{kk} - f_k^{(p)} B_{kk}\right\} \end{bmatrix} \begin{bmatrix} \Delta e_k^{(p)} \\ \Delta f_k^{(p)} \end{bmatrix} \qquad (2.24)$$

For a voltage-regulated node k, where the voltage magnitude V_k is specified, the second row in equation (2.24) can be replaced at any point in the iteration by:

$$(\Delta V_k^2)^{(p)} = \left(\frac{\partial V_k^2}{\partial e_k}\right)^{(p)} \Delta e_k^{(p)} + \left(\frac{\partial V_k^2}{\partial f_k}\right)^{(p)} \Delta f_k^{(p)}$$

$$= 2e_k^{(p)} \Delta e_k^{(p)} + 2f_k^{(p)} \Delta f_k^{(p)}$$

The equation corresponding to equation (2.24) for a voltage-regualted node is as follows:

$$\begin{bmatrix} \Delta P_k^{(p)} \\ \Delta V_k^{2\,(p)} \end{bmatrix} = \begin{bmatrix} \left\{\sum_{m=1}^{N} (G_{km} e_m^{(p)} - B_{km} f_m^{(p)}) + e_k^{(p)} G_{kk} + f_k^{(p)} B_{kk}\right\} & \left\{\sum_{m=1}^{N} (G_{km} f_m^{(p)} + B_{km} e_m^{(p)}) - e_k^{(p)} B_{kk} + f_k^{(p)} G_{kk}\right\} \\ 2e_k^{(p)} & 2f_k^{(p)} \end{bmatrix} \begin{bmatrix} \Delta e_k^{(p)} \\ \Delta f_k^{(p)} \end{bmatrix} \tag{2.25}$$

The calculation can be carried out by an iterative process.

The convergence of this method can be improved by the use of appropriate acceleration factors. System studies indicate that when an optimum acceleration factor is used, the number of iterations required for this method is slightly less than that required for the Gauss-Seidel method. The difference is, however, not significant. The computer storage requirement is essentially the same as that required for the Gauss-Seidel method, as it is not necessary to store the coefficient matrix shown in equation (2.24).

2.3.3 Relaxation Method

The basic process used in the Relaxation method, to solve a set of simultaneous equations, can be described as follows, using the nodal notation:

$$[\dot{I}] = [\dot{Y}][\dot{V}]$$

The right side of this matrix equation will not equal the left side unless the correct answer sets are used. The assumption is made that the left side of the equation includes a factor $\dot{R}_k$ where $\dot{R}_k$ represents the error involved for a given set of values for $[\dot{V}]$. If $\dot{R}_k$ is the residual load current at node k, and $\dot{R}_k = 0$ for all values of k, then the set of voltages $[\dot{V}]$ must satisfy the original equations.

The relaxation process operates by dividing the negative of the largest residual $\dot{R}_k$ by the self-admittance at node k, $\dot{Y}_{kk}$, to obtain a new value of $\dot{V}_k$. This value is such that when substituted back into the equation reduces $\dot{R}_k$ to zero (at least momentarily). The same voltage correction is applied to the interconnected nodes and the corresponding current residuals are modified accordingly. The remaining nodal voltages are then corrected in turn and their current residuals modified. The next complete iteration commences with the selection of the largest residual and the process is repeated until all of the residuals become zero and a voltage set is produced which satisfies the original equations.

This method differs from the previous iterative approach in that the order in which the unknowns are calculated may be varied in each

cycle. Applying this method to the power system nodal equations, and using the initial estimated voltage $\dot{V}_k$ for node k, the current at node k (k is not a swing bus node) can be calculated from the following equation.

$$\dot{I}_k^* = \frac{\dot{S}_k}{\dot{V}_k} \tag{2.26}$$

The current at node k can also be calculated from the system admittance values and estimated voltages.

$$\dot{I}_k' = \sum_{m=1}^{N} \dot{Y}_{km} \dot{V}_m \tag{2.27}$$

The approximate values of real and reactive power at the swing bus are obtained by subtracting the total scheduled generation from the summation of the system loads and transmission losses.

This can be obtained from:

$$\dot{S}_1 = \sum_{m=1}^{N} \dot{V}_m \dot{I}_m^* - \sum_{m=2}^{N} \dot{S}_m \tag{2.28}$$

where bus 1 is the swing bus. $\dot{I}_m$ is the load or generator current at node m, and $\dot{S}_m$ is the specified load or generation at node m.

The swing bus voltage is specified and the current can be found as:

$$\dot{I}_1^* = \frac{\dot{S}_1}{\dot{V}_1} \tag{2.29}$$

With a set of estimated values of voltage $[\dot{V}_k]$, the residual currents can be found from:

$$\dot{R}_k = \dot{I}_k' - \dot{I}_k \tag{2.30}$$

The corresponding voltage residual is then:

$$\Delta \dot{V}_k = \frac{-\dot{R}_k}{\dot{Y}_{kk}} \tag{2.31}$$

This voltage correction affects the current at all of the nodes which are interconnected with node k, as seen from equation (2.27). These residual currents are then modified by:

$$\dot{\gamma}_{m(1)} = \dot{Y}_{km} \Delta \dot{V}_k \tag{2.32}$$

For the swing bus, the voltage residual is:

$$\Delta \dot{V}_1 = \frac{-\dot{R}_1}{\dot{Y}_{11}} \tag{2.33}$$

and must be corrected by the factor $\dot{K}$:

$$\dot{K} = \dot{V}_{1 \text{ reference}} - (\dot{V}_{1 \text{ specified}} + \Delta \dot{V}_1) = -\Delta \dot{V}_1 \tag{2.34}$$

The same voltage correction must be added to each nodal voltage. This will change each current residual by:

$$\dot{\gamma}_{k(2)} = \dot{K} \sum_{m=1}^{N} \dot{Y}_{km} \tag{2.35}$$

The total residual of any node k is then:

$$\dot{R}_k' = \dot{R}_k + \dot{\gamma}_{k(2)} + \sum \dot{\gamma}_{k(1)} \tag{2.36}$$

Each nodal voltage correction can be obtained by

$$\Delta \dot{V}_k' = - \frac{\dot{R}_k'}{\dot{Y}_{kk}}$$

and as each voltage is found, the residuals are modified accordingly. The voltage for the next iterative cycle is found from:

$$\dot{V}_k^{(p+1)} = \dot{V}_k^{(p)} + \Delta \dot{V}_k' + \dot{K} \qquad (2.37)$$

This method worked successfully on the small system, however, it did not give satisfactory results when applied to a large system such as that specified for the Saskatchewan Power Corporation. It is believed that the reason for the unsatisfactory performance of this approach in this case is due to the relatively large system susceptance associated with long high voltage transmission lines. During the iterative process, it appeared that the program continually operated on these busses which contained the largest charging MVA. After reducing these residuals to zero by changing of corresponding voltage values, the subsequent modificationsbrought the residuals up again.

This approach is not as effective as the Gauss-Seidel method or the Newton-Raphson method.

2.4 Review of other methods from available literature

Several other methods of analysis suitable for load flow studies have been discussed in the literature. This section briefly reviews the theoretical concepts used in some of these methods and generally compares their speed of convergence, time requirements, and storage requirement for

digital computer applications:

2.4.1 Jacobi iterative method

This method has the same basic formulation as the Gauss-Seidel method.

At any node K of a power system, equation (2.2) can be rewritten as follows:

$$P_k + jQ_k = \dot{V}_k \sum_{m=1}^{N} \dot{Y}_{km}^* \dot{V}_m^*$$

The iterative process then proceeds using:

$$\dot{V}_k^{(p+1)*} = \sum_{\substack{m=1 \\ m\neq k}}^{N} \frac{\dot{Y}_{km}^*}{\dot{Y}_{kk}^*} \dot{V}_m^{(p)*} + \frac{P_k + jQ_k}{\dot{V}_k^{(p)}} \cdot \frac{1}{\dot{Y}_{kk}^*} \qquad (2.38)$$

This derivation was shown in section 2.3.1.

The basic difference between the Jacobi method and the Gauss-Seidel method is that in the Jacobi method, no element of $\dot{V}^{(p+1)}$ is used in the iteration until every element of $\dot{V}^{(p)}$ has been calculated. $\dot{V}^{(p+1)}$ then replaced $\dot{V}^{(p)}$ for the next complete iteration cycle.

It has been proved[1] that, theoretically, the asymptotic rate of convergence of the Gauss-Seidel method is twice as fast as that of the Jacobi method. This is shown in the Appendix (7-5) by using a numerical example.

2.4.2 Z - matrix iterative method

This method solves the load flow problem by inverting the network

nodal-admittance equation:

$$[\dot{V}][\dot{Y}] = [\dot{I}]$$

to obtain

$$[\dot{V}] = [\dot{Z}][\dot{I}]$$

The current values $[\dot{I}]$ are determined using known and estimated values of the nodal voltages and powers. A modified method was published[7] in 1963, in which the complete inverted admittance matrix was used. At each load bus, a fixed impedance to ground approximating the load on the basis of the estimated bus voltage was used. The actual specified load is obtained by a fringing current which is injected into each bus to correct for the difference between the actual and the estimated voltage.

At a generator bus, the injected current is calculated from:

$$\dot{I}_k^* = \frac{\dot{S}_k}{\dot{V}_k}$$

The basic iterative procedure is as follows: the load and generator injected currents are determined using known and estimated values of voltage and reactive power. Before the impedance matrix equation can be used to calculate better voltage values, it is necessary to determine the swing bus current using the following equation:

$$\dot{I}_1 = \frac{\dot{V}_1 - \sum_{n=2}^{N} \dot{Z}_{1n} \dot{I}_n}{\dot{Z}_{11}} \qquad (2.39)$$

The equation

$$\dot{V}_k = \sum_{m=1}^{N} \dot{Z}_{km} \dot{I}_m \qquad (2.40)$$

will now yield better voltage values.

As each voltage is corrected, its corresponding injecting current and the swing bus current can also be corrected. The voltage at a generator bus must be modified to the specified magnitudes before a correction is made to the generator current. The generator reactive power can be recalculated using:

$$Q_k = - \mathcal{I}_m.(\dot{V}_k \dot{I}_k^*)$$

where $\dot{V}_k$ is the corrected voltage value using $\dot{I}_k$.

This method has a rapid convergence characteristic which is obtained from inverting the admittance matrix. The time required, however, to invert the admittance matrix is a significant part of the total solution time and tends to eliminate any time advantage between this method and the optimum accelerated iterative method. This method also required additional storage to invert the matrix.

2.4.3 Modified elimination method

This method was published[6] in 1967. In this approach the coefficient matrix has the same form as shown for the Elimination method noted in section (2.2). The Elimination Method is a direct method and therefore has the advantage of a rapid convergence characteristic provided that the coefficient matrix has a unique

value and the process starts with a set of reasonable initial estimations. The principle disadvantage of the Elimination Method is that for large systems, the storage requirement increases as the order of N^2 and the total number of operations increases as the order of N^3 (N denotes the number of system nodes).

In order to avoid these difficulties, the authors of this method modified the elimination process by using an optimal ordered sequence to triangularize the coefficient matrix and thereby reduced the overall computing time. They also achieved a reduction in the storage requirement by employing a packed storage scheme for the modified matrix.

It is interesting to note the effect of those modifications upon the total time and storage requirements:

(1) Although the initial coefficient matrix may have a number of zeroes, each step of the elimination and triangularization process leads rapidly to a matrix with very few zero elements. Even using an optimal ordered sequence, the computing time required by this method will vary as N^k where the exponent k is greater than 1.

(2) The utilization of a packed storage scheme will save somewhat less than one half of the additional storage required by the coefficient matrix. Compared with the iterative method, however, this method still requires considerable extra storage. The storage problem can be overcome by the use of tapes or magnetic drums, but, the reading and writing times of these components on a large scale computer are considerably slower than internal calculating speeds.

Several other direct methods have been suggested in the published literature:

(a) Hybrid matrix technique with topological control[8]

(b) Diakoptics Techniques[9]

These and other techniques offer extremely interesting possibilities for solution of the load flow problem. It is virtually impossible to state that an iterative or a direct technique is the best approach under all conditions. Each method possesses different limitations when applied to a specific system and using a particular computer. A numerical analysis of the matrix characteristics and some of the limitations of direct and iterative methods are discussed in detail in the next chapter.

3. ACCELERATION AND CONVERGENCE

3.1 General

Basically the load flow problem involves the solution of a set of non-linear simultaneous equations. A set of linear variational equations can be found from the original equations and a solution obtained using one of the many different available methods. It is interesting to compare some of these techniques and to see if any one method is superior to the others. There are a number of criteria for evaluating the relative merits of the different methods. The main criteria, however, in the load flow problem is the total computer time required for an accurate solution. In most of the methods studied, this computer time requirement was largely dependent on the speed of convergence of the corresponding method. Using numerical analysis techniques, a number of studies on the convergence characteristics of different approaches have been carried out. The results indicate that the rate of convergence of each individual method is effected by the characteristic equation of the sets of linear equations for the specified load-flow problem and by the procedure of the method itself. The rate of convergence can be increased considerably by the use of acceleration factors. The particular optimum acceleration factor can be obtained from the roots of the characteristic equation or from certain approximations related to the elements of the corresponding linear equations. The optimum acceleration factor will vary with the size of the problem studied and with the different method used. It should be noted that each method has certain limitations which may affect the use of that

method for some special conditions.

3.2 Direct method

A Direct method is defined as a method which will give a solution of the problem using a finite number of elementary arithmetic operations.

For solving a system of linear equation

$$[\dot{A}][\dot{X}] = [\dot{K}] \tag{3.1}$$

[X] is obtained from:

$$[\dot{X}] = [\dot{A}]^{-1}[\dot{K}] \tag{3.2}$$

If the initial data defining the problem are given exactly and if the computations are carried out exactly, then the solution is also exact. The number of computational operations necessary to solve this system depend only on the type of computation scheme and on the order of the matrix defined by the given problem.

In a load flow problem, the above linear conditions do not exist. Approximately linear equations are obtained by modifying the original non-linear equations of the form shown in equation (2.2). Several iterations are therefore necessary to minimize the associated error.

In the direct method, the main problem is to invert matrix A of equation (3.1). As noted in section 2.2, the direct method usually takes less iterations than a purely iterative method. The total computation time, however, may not be less than that required by the

optimum iterative method due to the matrix inversion process. The storage requirement of the direct method is definitely more than that of the iterative method and will greatly increase as the size of the system increases.

In considering the convergence characteristic of the direct method:

(1) $[\dot{A}][\dot{X}] = [\dot{K}]$ has a unique solution if and only if $[\dot{A}]$ is a non-singular matrix[1] (i.e. if and only if $[\dot{A}]^{-1}$ exist).

$[\dot{A}]$ is non-singular if and only if $\det|\dot{A}| \neq 0$. ($\det|\dot{A}|$ is the determinent of $[\dot{A}]$).

In the direct approach, the initial elements of matrix $\dot{A}$ are only approximate values. During the process of arriving at the exact solution of the system, the determinent may turn out to be zero or nearly zero. The condition for convergence of this method is therefore defined that the inverse matrix of $[\dot{A}]$ should be stable. (i.e. for "small" changes in the initial matrix elements, the corresponding changes in the inverse matrix should also be "small".) For stability of an inverse matrix, the determinent of the matrix in every case should not be too small (or too large, which will make the determinent of the inverse matrix small). The "smallness" of the determinent may be produced by an ill-conditioned system or by unreasonable initial estimates if matrix $[\dot{A}]$ of equation (3.1) involves the initial estimated variables. It is difficult to determine the degree of "smallness" of the determinent which definitely rules out a stable inverse matrix. In order to characterize a matrix with regard to its condition, several papers have defined quantitative characteristics known as

condition numbers which are based on probability theory[3].

(2) Rounding-off error

The number of multiplications and divisions required by the direct method is a direct function of the system size. During the computation process, the results of such operations are rounded off to a fixed number of significant digits. The loss of subsequent figures can result in a significant reduction in solution accuracy. The condition number of a matrix is also important in round-off error studies.

3.3 Iterative method

The basic iterative process for solving linear equations can be described as follows:

Let $[\dot{A}] = [\dot{a}_{i,j}]$ be a n x n complex matrix. The system of equations is:

$$[\dot{A}][\dot{X}] = [\dot{K}]$$

where $[\dot{K}]$ is a given column vector. The solution vector $[\dot{X}]$ exists and is unique if and only if $[\dot{A}]$ is non-singular. The matrix $[\dot{A}]$ can be written as the sum of three sub-matrices.

$$[\dot{A}] = [\dot{D}] - [\dot{E}] - [\dot{F}] \qquad (3.3)$$

where $[\dot{D}]$ has the main-diagonal elements of $[\dot{A}]$, but is zero elsewhere.

$[\dot{E}]$ has the below-diagonal elements of $[\dot{A}]$, but is zero elsewhere.

$[\dot{F}]$ has the above-diagonal elements of $[\dot{A}]$, but is zero elsewhere.

The values in $[\dot{E}]$ and $[\dot{F}]$ are the negative of the respective elements in $[\dot{A}]$. Matrix $[\dot{A}]$ therefore has the following form:

$$[\dot{A}] = \begin{bmatrix} \dot{D} & -\dot{F} \\ -\dot{E} & \dot{D} \end{bmatrix} \tag{3.4}$$

Equation (3.1) can be written as:

$$[\dot{D}][\dot{X}] = [\dot{E} + \dot{F}][\dot{X}] + [\dot{K}] \tag{3.5}$$

The diagonal elements $\dot{a}_{i,i}$ of $[\dot{A}]$ are all non-zero elements. An iterative process can be obtained from equation (3.5) as follows:

$$\dot{a}_{i,i}\, \dot{x}_i^{(m+1)} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \dot{a}_{i,j}\, \dot{x}_j^{(m)} + \dot{K}_i \qquad 1 \leq i \leq n, \quad m \geq 0 \tag{3.6}$$

The $x_i^{(0)}$ values are initial component estimates of the unique solution of $[X]$ of equation (3.1). Rewriting equation (3.6)

$$\dot{x}_i^{(m+1)} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{\dot{a}_{i,i}}{\dot{a}_{i,i}} \right) \dot{x}_j^{(m)} + \frac{\dot{K}_i}{\dot{a}_{i,i}} \qquad 1 \leq i \leq n, \quad m \geq 0 \tag{3.7}$$

In matrix notation, equation (3.6) becomes:

$$[\dot{D}]\ \dot{x}^{(m+1)} = [\dot{E} + \dot{F}]\ \dot{x}^{(m)} + [\dot{K}] \qquad m \geq 0 \tag{3.8}$$

$[\dot{D}]$ is a non-singular matrix, the matrix analogue of equation (3.7) is:

$$\dot{x}^{(m+1)} = [\dot{D}]^{-1} [\dot{E} + \dot{F}] \dot{x}^{(m)} + [\dot{D}]^{-1}[\dot{K}] \qquad m \geq 0 \qquad (3.9)$$

This is the general expression of the Jacobi iterative method. The matrix $[\dot{B}]$ shown below is known as the Jacobi matrix:

$$[\dot{B}] = [\dot{D}]^{-1} [\dot{E}+\dot{F}] \qquad (3.10)$$

An examination of the Jacobi iterative method of equation (3.9) shows that in general, all the components of the vector $\dot{x}^{(m)}$ must be retained in order to compute the components of the vector $\dot{X}^{(m+1)}$. Intuitively it appears reasonable to use the latest estimates $\dot{X}_i^{(m+1)}$ of the $\dot{X}_i$ components in subsequent computations. This modification results in another iterative method:

$$\dot{a}_{i,i} \dot{x}_i^{(m+1)} = - \sum_{j=1}^{i-1} \dot{a}_{i,j} \dot{x}_j^{(m+1)} - \sum_{j=i+1}^{n} \dot{a}_{i,j} \dot{x}_j^{(m)} + \dot{K}_i$$

$$1 \leq i \leq n, \quad m \geq 0 \qquad (3.11)$$

This iterative method has the computational advantages that it does not require the simultaneous storage of the two approximations $\dot{x}_i^{(m+1)}$ and $\dot{x}_i^{(m)}$ in the course of computation as does the Jacobi iterative method. It is shown in the Appendix (7.5) that the rate of convergence of this method is theoretically twice as fast as that of the Jacobi method.

In matrix notation, equation (3.11) becomes:

$$[\dot{D} - \dot{E}]\ \dot{X}^{(m+1)} = [\dot{F}]\ \dot{X}^{(m)} + [\dot{K}] \qquad m \geq 0 \qquad (3.12)$$

$[\dot{D} - \dot{E}]$ is a non-singular lower triangular matrix and equation (3.12) can be rewritten as:

$$\dot{x}^{(m+1)} = [\dot{D} - \dot{E}]^{-1}\ [\dot{F}]\ \dot{X}^{(m)} + [\dot{D} - \dot{E}]^{-1}\ [\dot{K}] \quad m \geq 0 \qquad (3.13)$$

This is the general expression of the Gauss-Seidel method. The matrix $[\dot{C}]$ shown below is known as the "Gauss-Seidel" matrix.

$$[\dot{C}] = [\dot{D} - \dot{E}]^{-1}\ [\dot{F}] \qquad (3.14)$$

A third basic iterative approach is obtained by incorporating the concept of acceleration into the Gauss-Seidel method. The components of the auxiliary vector $\underline{\dot{x}}^{(m+1)}$ are defined as:

$$\dot{a}_{i,i}\ \underline{\dot{x}}_i^{(m+1)} = -\sum_{j=1}^{i-1} \dot{a}_{ij}\dot{x}_j^{(m+1)} - \sum_{j=i+1}^{n} \dot{a}_{ij}\dot{x}_j^{(m)} + \dot{k}_i$$

$$1 \leq i \leq n, \quad m \geq 0 \qquad (3.15)$$

These values are identical to those shown in equation (3.11). Prior to subsequent use in the next step of the process, these values are modified as shown in the following equation:

$$\dot{x}_i^{(m+1)} = \dot{x}_i^{(m)} + \omega\ (\ \underline{\dot{x}}_i^{(m+1)} - \dot{x}_i^{(m)}) = (1 - \omega)\ \dot{x}_i^{(m)} + \omega \underline{\dot{x}}_i^{(m+1)} \qquad (3.16)$$

The quantity ω is called the acceleration factor, or relaxation factor and can have values of $\omega>1$ (or $\omega<1$) corresponding to over (or under) acceleration or over(or under) relaxation.

Combining the two equations (3.15) and (3.16)

$$\dot{a}_{ii}\dot{x}_i^{(m+1)} = \dot{a}_{ii}\dot{x}_i^{(m)} + \omega[-\sum_{j=1}^{i-1}\dot{a}_{ij}\dot{x}_j^{(m+1)} - \sum_{j=i+1}^{n}\dot{a}_{ij}\dot{x}_j^{(m)} + \dot{k}_i - \dot{a}_{ii}\dot{x}_i^{(m)}]$$

In matrix notation:

$$[\dot{D} - \omega\dot{E}]\,\dot{x}^{(m+1)} = [(1-\omega)\dot{D} + \omega\dot{F}]\,\dot{x}^{(m)} + \omega[\dot{K}] \qquad m \geq 0 \tag{3.17}$$

$[\dot{D} - \omega\dot{E}]$ is non-singular for any choice of ω. Defining $[\dot{L}] = [\dot{D}]^{-1}[\dot{E}]$, $[\dot{U}] = [\dot{D}]^{-1}[\dot{F}]$, then equation (3.10) becomes:

$$[\dot{B}] = [\dot{L}] + [\dot{U}]$$

Equation (3.17) becomes:

$$\dot{x}^{(m+1)} = [\dot{I} - \omega\dot{L}]^{-1}[(1-\omega)\dot{I} + \omega\dot{U}]\,\dot{x}^{(m)} + \omega[\dot{I} - \omega\dot{L}]^{-1}[\dot{D}]^{-1}[\dot{K}] \tag{3.18}$$

$[\dot{I}]$ is the unity matrix. Equation (3.18) is the general expression for the accelerated Gauss-Seidel or successive relaxation iterative method. The $[\dot{H}\omega]$ matrix shown below is known as the accelerated Gauss-Seidel or successive relaxation matrix.

$$[\dot{H}\omega] = [\dot{I} - \omega\dot{L}]^{-1}\,[(1-\omega)\dot{I} + \omega\dot{U}] \tag{3.19}$$

when $\omega = 1$, this iterative approach is identical to the Gauss-Seidel method and like the Gauss-Seidel method, needs only one vector approximation in storage at any one time.

The Newton-Raphson method, as described in section 3.3.2, can be accelerated by the same process used for the successive relaxation method. The principal difference is that the original non-linear equations (2.2) linearized by the Newton-Raphson method exhibit a second-order convergence characteristic and those of the general iterative method exhibit a first-order convergence characteristic.

Two basic limitations of Newton-Raphson method are that:

1. Slow convergence if $\frac{d}{d\dot{v}} (p + jQ)$ at any node is small.

2. The selection of poor initial estimates may lead to divergence.

In each of the three iterative methods described, an associate error vector $\dot{\varepsilon}^{(m)}$ can be defined as:

$$\dot{\varepsilon}^{(m)} = \dot{x}^{(m)} - \dot{x}^{(\infty)} \qquad m \geq 0 \qquad (3.20)$$

where $\dot{x}^{(\infty)}$ is the unique vector solution of equation (3.1). For each of the iterative methods described by equations (3.10), (3.14) and (3.19), the error vector $\dot{\varepsilon}^{(m)}$ can be expressed as:

$$\dot{\varepsilon}^{(m)} = [\dot{M}]\, \dot{\varepsilon}^{(m-1)} \qquad m \geq 0 \qquad (3.21)$$

Hence

$$\dot{\varepsilon}^{(m)} = [\dot{M}]^m\, \dot{\varepsilon}^{(0)} \qquad m \geq 0 \qquad (3.22)$$

where $[\dot{M}]$ is the iterative matrix for the specified method. Equations (3.21) and (3.22) illustrate the linear behavior of the iterative process and the error vector $\dot{\varepsilon}^{(m)}$ in terms of the initial error. Using linear transformation theory[9], the condition for and speed of convergence for the iterative method can be obtained as follows:

For the matrix $[\dot{M}]$, there usually exists a set of eigenvalues, μ, and eigenvector, $\dot{Y}$, such that:

$$[\dot{M}][\dot{Y}] = \mu [\dot{Y}]$$

where $[\dot{Y}]$ is a column vector similar in form to the error vector $\dot{\varepsilon}$ of equation (3.22). μ is a set of real or complex roots of the characteristic equation of matrix $[\dot{M}]$ and can be obtained from:

$$\det \left| \dot{M} - \mu \dot{I} \right| = 0 \qquad (3.23)$$

where $[\dot{I}]$ is the unity matrix. Each eigenvalue has a corresponding eigenvector which can be used to form the solution of the matrix equation. The initial error vector $\dot{\varepsilon}^{(0)}$ of equation (3.22) is given by:

$$\dot{\varepsilon}^{(0)} = c_1\dot{Y}_1 + c_2\dot{Y}_2 + \cdots + c_n\dot{Y}_n = \sum_{i=1}^{N} c_i\dot{Y}_i$$

where the constants c_i are determined by the initial conditions. After m repeated multiplications by $[\dot{M}]$, equation (3.22) becomes:

$$\dot{\varepsilon}^{(m)} = [\dot{M}]^m \dot{\varepsilon}^{(0)} = \mu_1^m c_1\dot{Y}_1 + \mu_2^m c_2\dot{Y}_2 + \cdots + \mu_n^m c_n\dot{Y}_n$$

Assume $|\mu_1| > |\mu_2| \geq |\mu_3| \geq \ldots \ldots \geq |\mu_n|$. The resulting vector $\dot{\varepsilon}^{(m)}$ approaches a value corresponding to the largest absolute root μ_1.

i.e. $$\dot{\varepsilon}^{(m)} = c_1 \mu_1^m \dot{Y}_1 \qquad (3.24).$$

This equation illustrates the speed at which $\dot{\varepsilon}^{(m)}$ approaches zero. If all $|\mu_i| < 1$, then the iterative process will converge ($\dot{x}^{(m)} \rightarrow \dot{x}^{(\infty)}$). If any $|\mu_i| \geq 1$, the iterative process will diverge. If $|\mu_1| < 1$, each step reduces the length of $\dot{\varepsilon}^{(m)}$ to the fraction $|\mu_1|$ of itself. The smaller the value of μ_1, the faster the speed of convergence. In order for the process to converge, all of the eigenvalues must be less than 1. i.e. matrix $[\dot{A}]$ must be a diagonally dominant matrix. ($[\dot{A}]$ should be a non-singular matrix.) As the diagonal dominance increases, the largest eigenvalue decreases and the speed of convergence improves. In a practical problem, it is necessary to examine not only whether the system will converge but also the rate of convergence. The concept of vector and matrix norms are introduced for this purpose and their definitions are shown in the Appendix 7.4.

From equation (3.22), using matrix and vector norms, it can be shown that:

$$\| \dot{\varepsilon}^{(m)} \| \leq \| \dot{M}^m \| \, \| \dot{\varepsilon}^{(0)} \| \qquad m \geq 0 \qquad (3.25)$$

If $\dot{\varepsilon}^{(0)}$ is not the null vector, then $\| \dot{M}^m \|$ gives a upper-limit estimate for the ratio $\dfrac{\| \dot{\varepsilon}^{(m)} \|}{\| \dot{\varepsilon}^{(0)} \|}$ of the Euclidean Norms of $\| \dot{\varepsilon}^{(m)} \|$ and $\| \dot{\varepsilon}^{(0)} \|$ for m iterations. The initial vector

$\dot{\varepsilon}^{(0)}$ is unknown in practical problems and therefore $\| \dot{A}^m \|$ serves as a basis of comparison of different iterative methods.

The Average Rate of Convergence for m iterations of the matrix $[\dot{A}]$ is then defined as:

$$R(\dot{A}^m) = -\ln [\| \dot{A}^m \|^{1/m}] = - \frac{\ln \| \dot{A}^m \|}{m} \qquad (3.26)$$

The Asymptotic Rate of Convergence is:

$$\lim_{m \to \infty} R(\dot{A}^m) = R(\dot{A}^\infty) = - \ln \rho(\dot{A}) \qquad (3.27)$$

where $\rho(\dot{A})$ is the spectral radius of the matrix $\dot{A}$. (See Appendix 7.4.) This asymptotic rate of convergence is the simplest practical measure of speed of convergence of a matrix.

3.4 Determination of the optimum acceleration factor

For a non-singular diagonally dominant matrix $[\dot{A}]$ in the form of equation (3.1), the successive relaxation matrix (or accelerated Gauss-Seidel matrix) $[\dot{H}\omega]$ of equation (3.19) is convergent for $\omega = 1$ and by continuity of the matrix theorem, it is also convergent for some interval in ω containing unity.

The problem is to determine an acceleration factor (relaxation factor) ω_m which maximizes the average rate of convergence $R(\dot{H}\omega^m)$ where $R(\dot{H}\omega^m)$ is defined in equation (3.26). The solution of this problem is complicated by the fact that ω_m is a function of m.

It is simpler to investigate the optimum acceleration factor ω_b which maximizes the asymptotic rate of convergence $R(\dot{H}\omega^{\infty})$.

For a matrix $[\dot{A}]$ of equation (3.1). If $\omega \neq 0$ and if λ is a non-zero eigenvalue of $\dot{H}\omega$, and if μ satisfies:(*)

$$(\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda \tag{3.28}$$

then μ is an eigenvalue of matrix $[\dot{B}]$. If μ is a eigenvalue of matrix $[\dot{B}]$, and if λ satisfies the above equation, then λ is an eigenvalue of $[\dot{H}\omega]$. Where matrix $[\dot{B}]$ is defined in terms of $[\dot{A}]$ by equations (3.10). From equation (3.28)

$$\frac{\lambda + \omega - 1}{\omega} = \pm \lambda^{1/2} \mu \tag{3.29}$$

Defining the equation:

$$g_\omega(\lambda) = \frac{\lambda + \omega - 1}{\omega} \qquad \omega \neq 0 \tag{3.30}$$

$$m(\lambda) = \lambda^{1/2} \mu \qquad 0 \leq \mu \leq \rho(\dot{B}) < 1 \tag{3.31}$$

then $g_\omega(\lambda)$ is a straight line through the point (1,1) whose slope decreases monotonically with increasing ω. Equation (3.29) can be geometrically interpreted as the intersection of the curve $g_\omega(\lambda)$ and $\pm m(\lambda)$ as illustrated in Figure (3.1).

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* Varga, Richard S., "Matrix Iterative Analysis", p. 110.

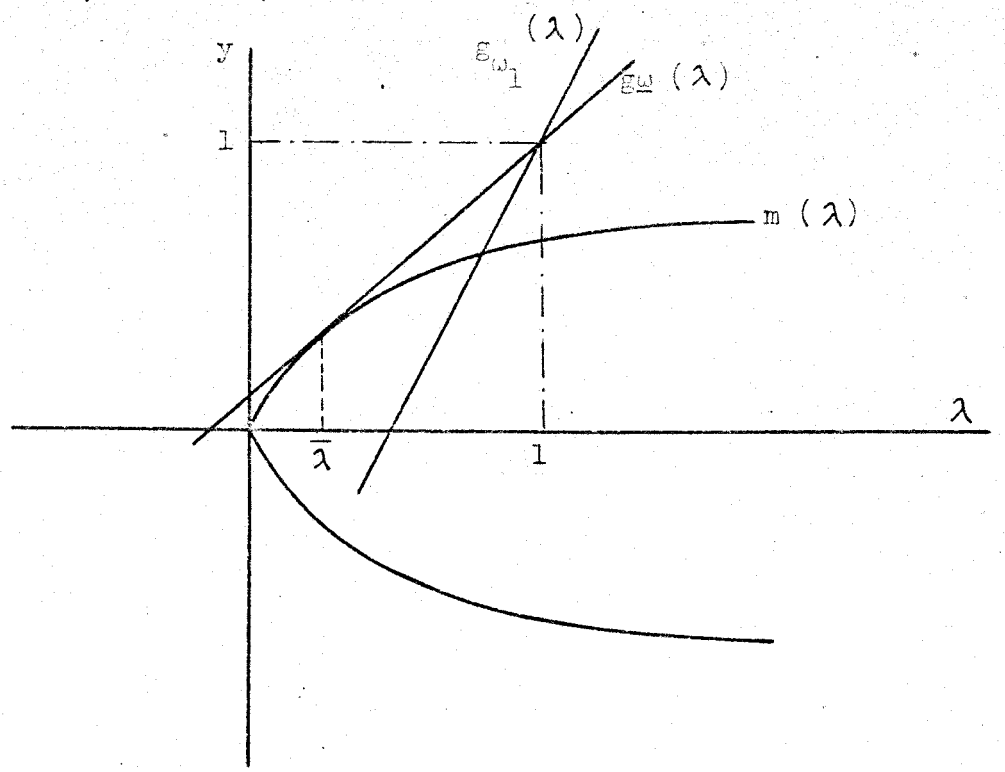

Figure 3.1 The relationship between acceleration factor and the eigenvalues of the iteration matrix

The largest λ value of the two points of intersection decreases with increasing ω until $g_\omega(\lambda)$ becomes tangent to $m(\lambda)$, this occurs when

$$\underline{\omega}_b = \frac{2}{1 + \sqrt{1 - \mu^2}} \qquad 0 < \omega < 2 \qquad (3.32)$$

Hence, the optimum acceleration factor:

$$\omega_b = \frac{2}{1 + \sqrt{1 - \mu_{max}^2}} \qquad 0 < \omega < 2 \qquad (3.33)$$

In the iterative solution of practical load flow problems, the eigenvalues of the Jacobi matrix or of the accelerated Gauss-Seidel

matrix are not available nor can they be exactly determined a priori. The optimum acceleration factor can only be estimated by trial and error techniques or from estimated maximum eigenvalues.

There are certain approximations concerning maximum eigenvalues of a matrix.

(1) The theorem of FROBENIUS[1] associates the maximum eigenvalue of a matrix with its maximum row or column sums: i.e.

$$|\mu_{imax}| \leq \max_i \sum_{j=1}^{N} |\dot{A}_{i,j}| \quad \text{or} \quad \leq \max_j \sum_{i=1}^{N} |\dot{A}_{i,j}| \tag{3.34}$$

where $\dot{A}_{i,j}$ is an element of the matrix.

(2) For an arbitrary n x n complex matrix $[\dot{A}]$[3]

$$\| \dot{A} \| \geq \rho(\dot{A})$$

Since $\rho(\dot{A}) = \max_{1 \leq i \leq n} |\mu_i|$

and $\| \dot{A} \| = \left(\sum_{i,j=1}^{N} |\dot{A}_{i,j}|^2 \right)^{1/2}$

$$\max_{1 \leq i \leq n} |\mu_i| \leq \left[\sum_{i,j=1}^{N} |\dot{A}_{i,j}|^2 \right]^{1/2} \tag{3.35}$$

The best estimate of the largest eigenvalue is the smallest value of

μ_i obtained from equation (3.34) and (3.35).

Experimental results indicate that overestimating μ_{max} by a small margin causes a smaller decrease in $R(H_\omega^\infty)$ than does underestimating μ_{max} by a comparable amount. This can be seen by plotting $\rho(H_\omega)$ as a function of ω for $0 \leq \omega \leq 2$ as shown in Figure (3.2)

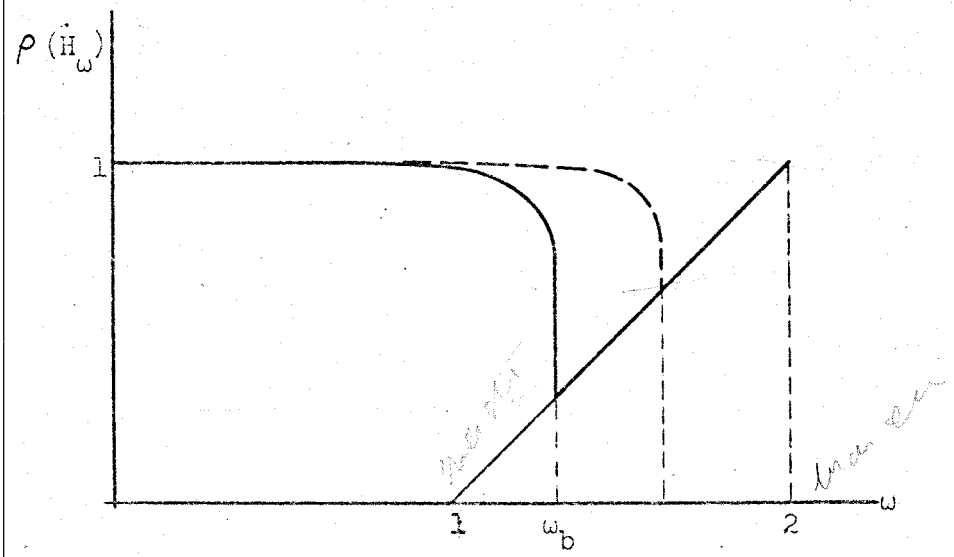

Figure 3.2 The relationship between acceleration factor and the maximum eigenvalues of the iteration matrix

It was noted earlier that for a convergent matrix equation, the maximum eigenvalue, μ_1, must be less than unity. Substituting this value into equation (3.33) shows that the maximum optimum acceleration factor must be less than 2. The optimum acceleration factor determined from studies of the Saskatchewan Power Corporation system was found to be 1.8 + j 1.8. These results are shown in the next chapter.

It should be realized that for any problem, the optimum acceleration factor may not be the same for different methods.

4. APPLICATIONS AND RESULTS

Four load flow digital computer programs have been developed. The flow chart for each program is shown in the Appendix (7.3). It was found that the accelerated Gauss-Seidel method and the over-relaxed Newton-Raphson method are preferable methods as compared to the Elimination method and the Relaxation method in terms of computer time, storage requirements and convergence characteristics.

The accelerated Gauss-Seidel method and the Newton-Raphson method were therefore applied to a model of the Saskatchewan Power Corporation system containing 46 busses, 54 lines, and also a typical small system with 5 busses and 6 lines. Various studies were performed and the results are presented in this chapter. These results indicate that the Newton-Raphson method is slightly better than the accelerated Gauss-Seidel method for the cases studied.

4.1 Convergence for different tolerance ranges

In load flow studies, the convergence is checked against a pre-determined bus voltage tolerance using both the accelerated Gauss-Seidel method and the Newton-Raphson method. The number of iterations required for convergence were found at selected tolerance values. These results are shown in Figure (4.1).

As obviously expected, the number of required iterations decrease as the tolerance is relaxed. For the specified S.P.C. system, the optimum acceleration factor for both methods was found to be $1.8 + j1.8$. The difference in the required number of iterations for the tolerance range considered is much less at the optimum acceleration factor than

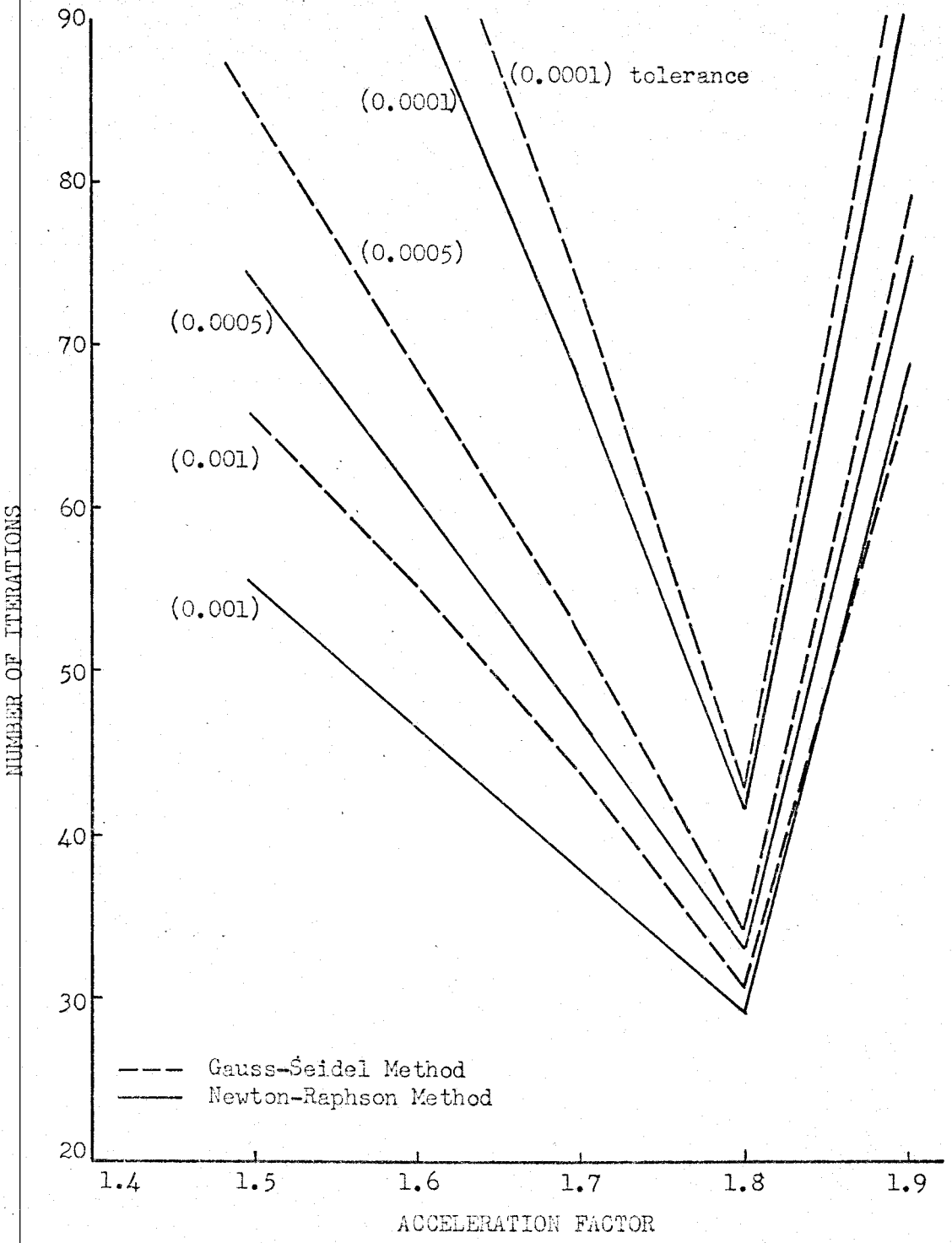

Figure 4.1 Convergence with different tolerance values for Saskatchewan Power Corporation System

that of other acceleration factors. If the optimum acceleration factor is known, then the gain in solution time by relaxing the tolerance may not be worthwhile compared to the increase in solution precision. The Newton-Raphson method shows a slightly better convergence characteristic than the Gauss-Seidel method when compared for equivalent acceleration factors. It is not valid to compare their speed of convergence at different acceleration factors.

4.2 Convergence with variable acceleration factor

It was thought that a variable acceleration factor might give a better convergence characteristic than that obtained by using a fixed acceleration factor for the entire iterative process. A test program was prepared in which the acceleration factor was started at a large value, such as 2 + j2, and then was reduced as the solution progressed. The number of bus voltages within the predetermined tolerance range was checked at the end of each iteration. As soon as this number reached a certain value, such as 25% of the total, the acceleration factor automatically reduced a certain step, such as 0.2 + j0.2. The process was repeated to convergence. This method, however, did not give a very large improvement in the rate of convergence.

The reason for this was found by another test which compares the number of iterations required to bring an increasing number of bus voltages within the specified tolerance at different acceleration factors. Figure (4.2) shows that a relatively large number of iterations are required to bring the first few bus voltages within the tolerance

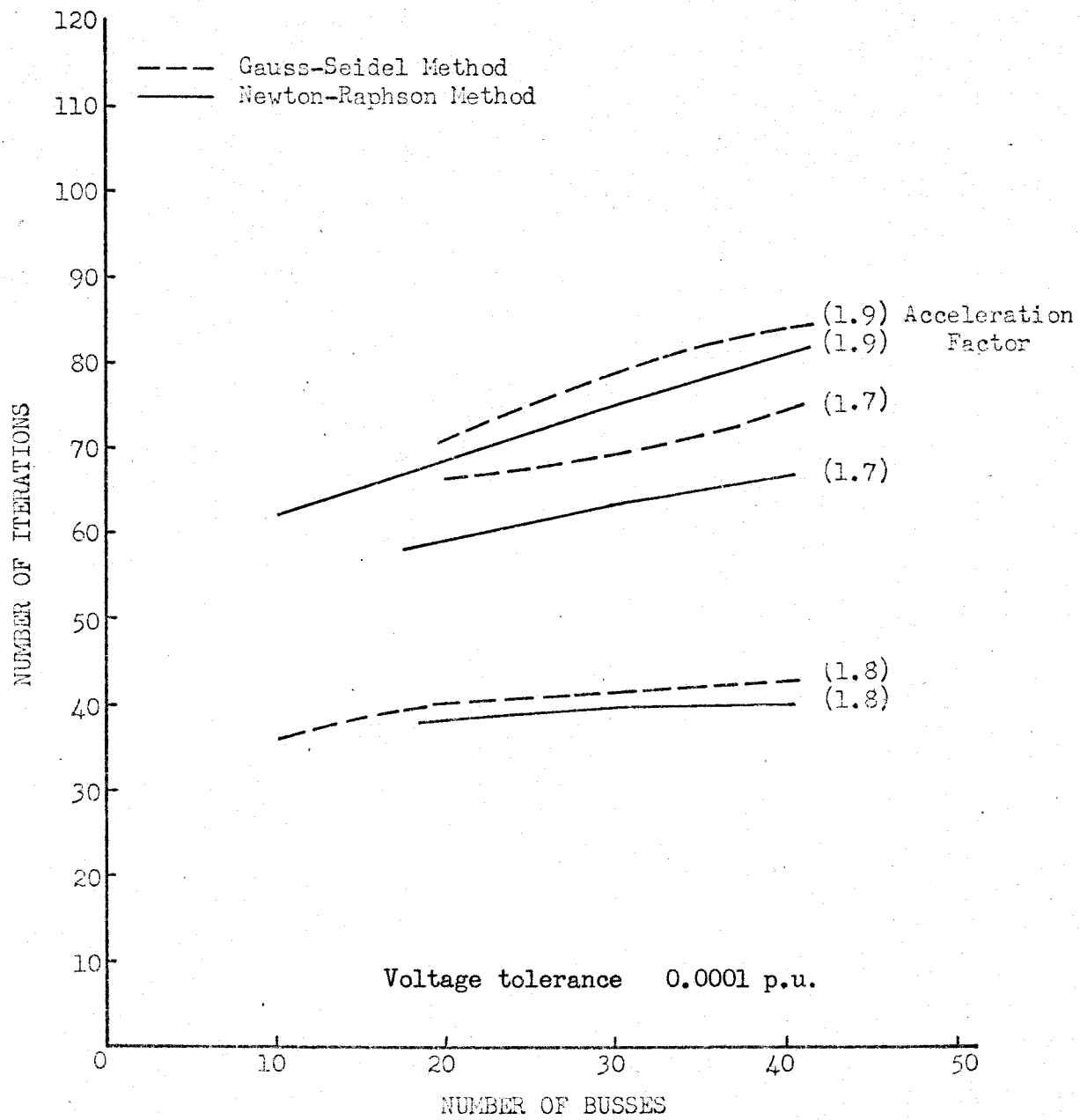

Figure 4.2 Number of iterations required for number of bus voltages converge within the tolerance

limit, however, only a few more iterations are needed for the remaining busses. This is especially true when the optimum acceleration factor is used. The results shown in Figure (4.2) agree with the theoretical analysis noted in Chapter 3. The convergence characteristic of an iterative process has a linear behaviour and is affected by the maximum eigenvalue of the coefficient matrix. The optimum acceleration factor will bring the bus voltages within the tolerance limits with the least number of iterations. The initial selection and continued application of the optimum acceleration factor gives a faster rate of convergence than any other fixed or variable selections. For the specified S.P.C. system, the optimum acceleration factor was found to be 1.8 + j1.8.

The real and imaginary components of the acceleration factor do not have to be equal in magnitude. The effect of varying these values is shown in Figures (4.3) and (4.4). It can clearly be seen in this case that a value of 1.8 + j1.8 gives the best rate of convergence.

4.3 Systems of different size

The accelerated Gauss-Seidel method and the overrelaxed Newton-Raphson method were applied to both the specified S.P.C. system and to a typical small system using different acceleration factors. The result showing the number of required iterations as a function of the acceleration factor are shown in Figures (4.5) and (4.6). The Newton-Raphson method shows a slightly better rate of convergence for both the large and the small system. These results also show that the optimum acceleration factor for a particular system may be different for each method. In the case of the S.P.C. system, the

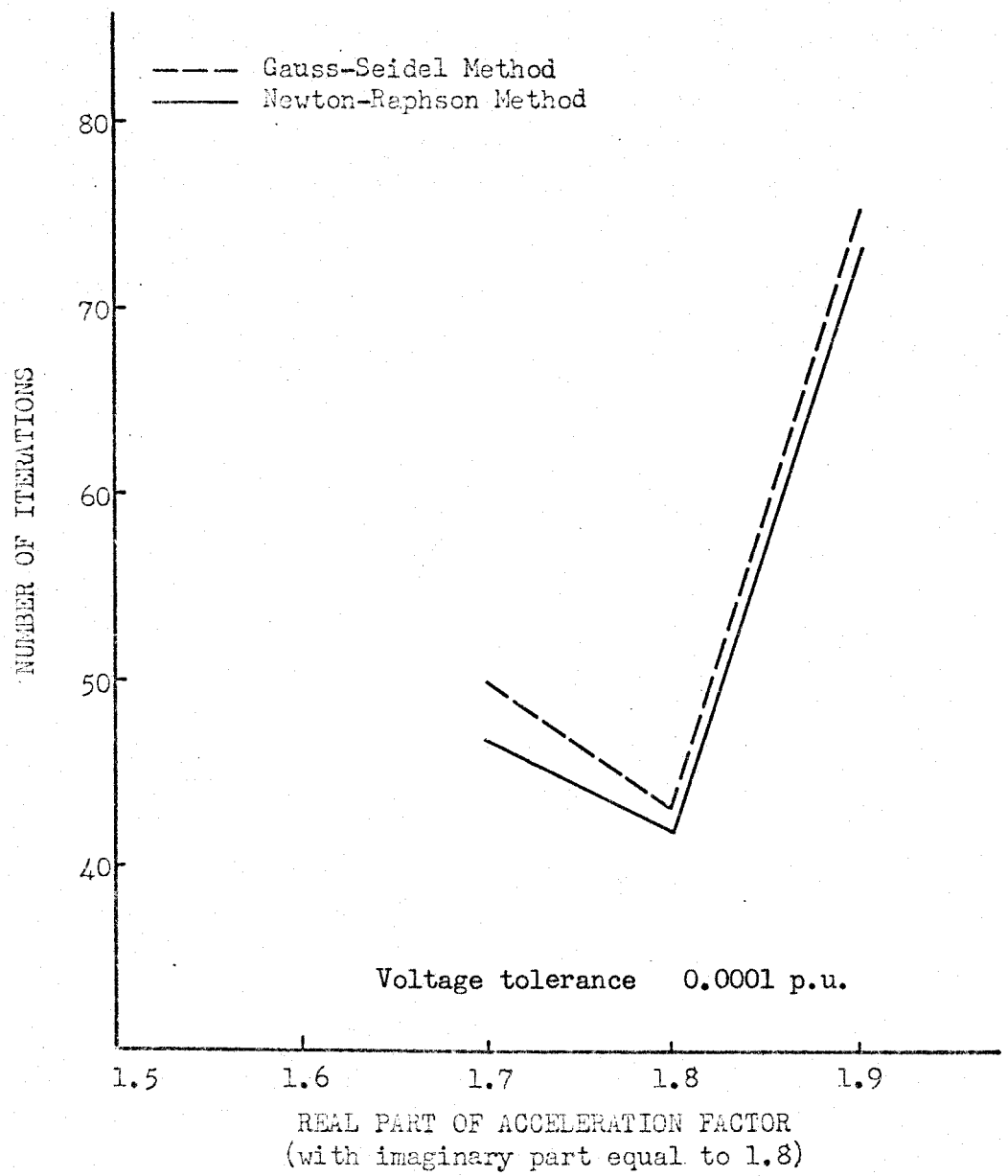

Figure 4.3 Convergence with acceleration factor having variable real component

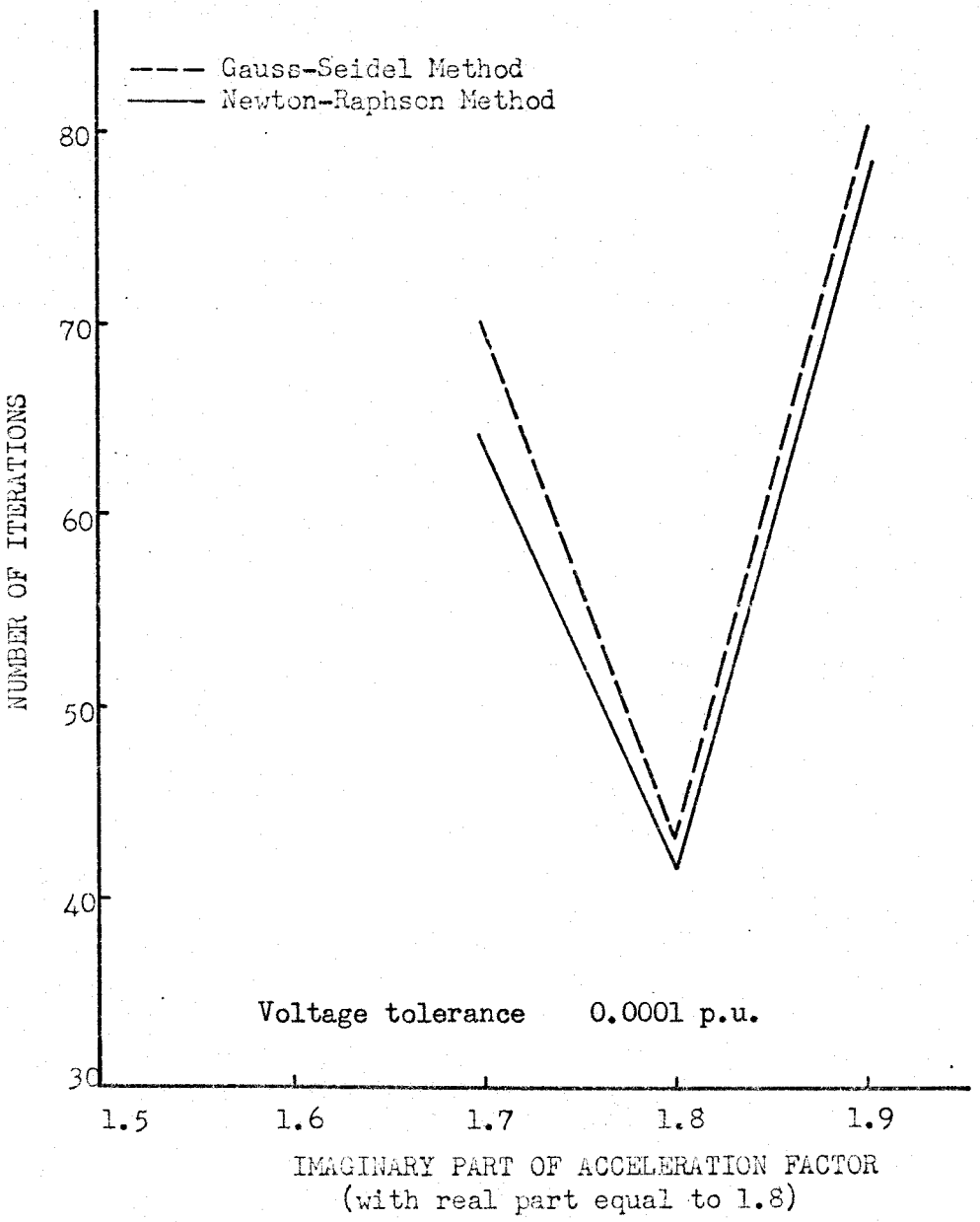

Figure 4.4 Convergence with acceleration factor having variable imaginary component

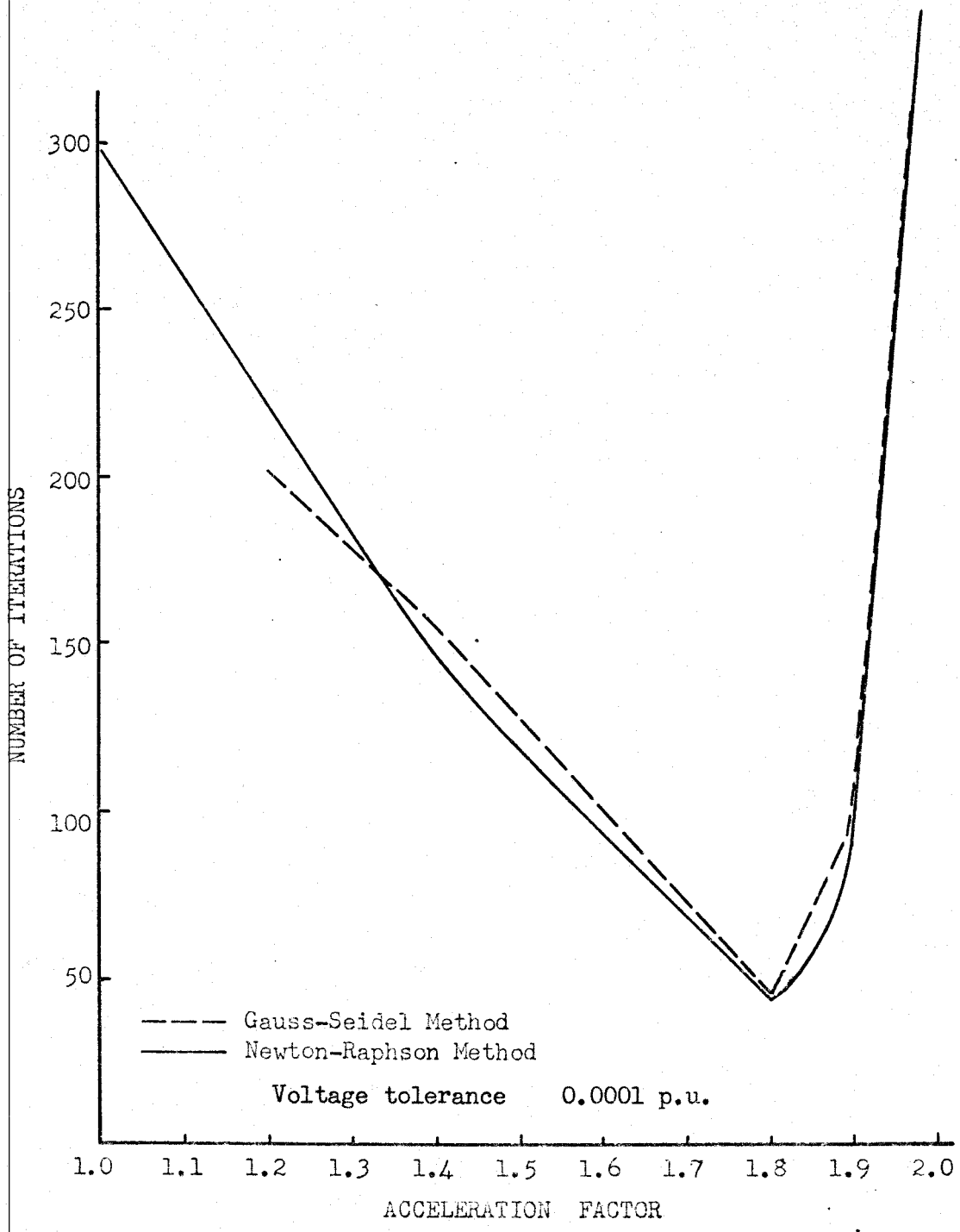

Figure 4.5 Convergence with difference acceleration factor for Saskatchewan Power Corporation System

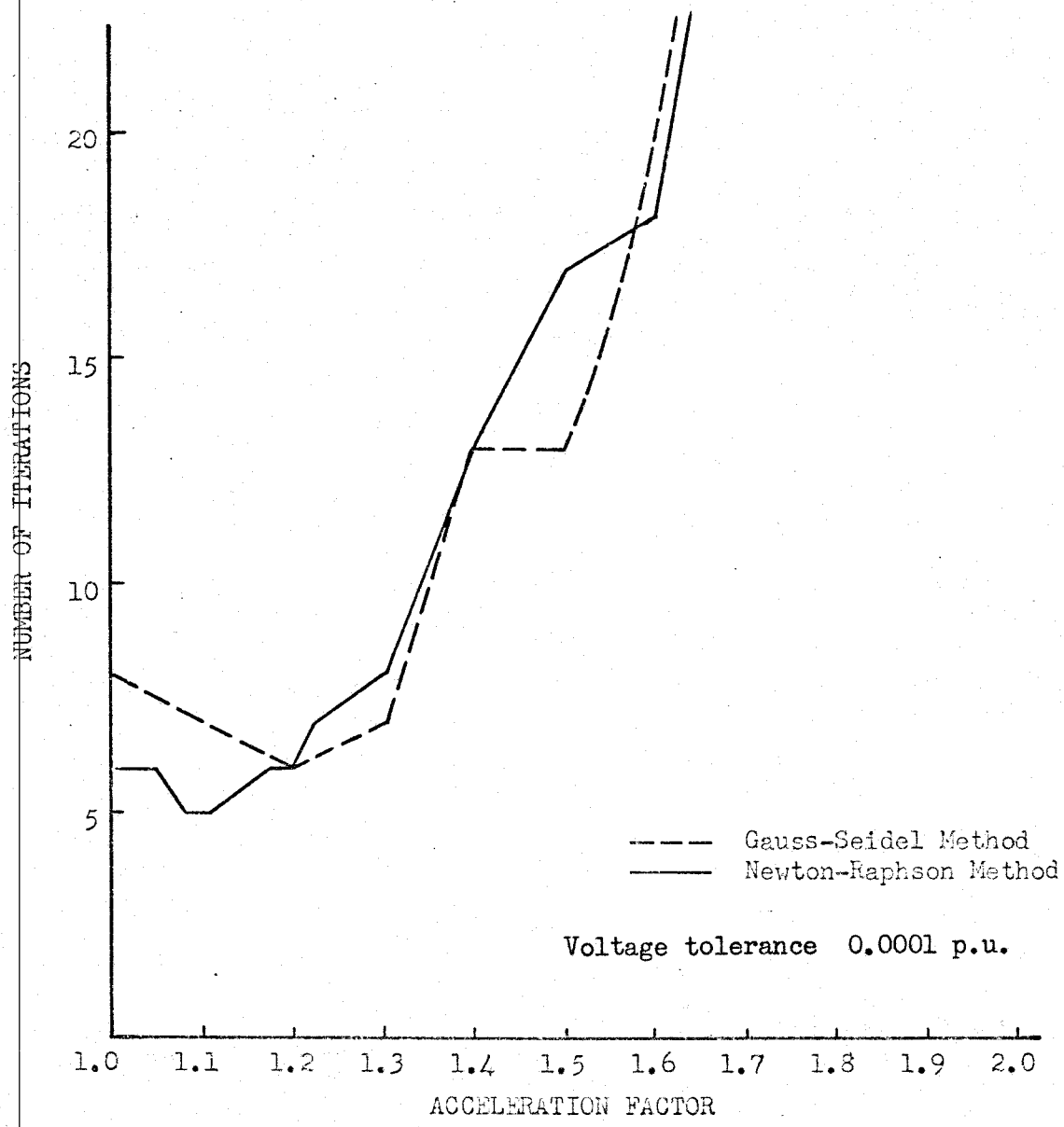

Figure 4.6 Convergence with different acceleration factor for hypothetical system

optimum acceleration factor is 1.8 + j1.8 in each case. In the small system, the optimum values are 1.1 + j1.1 and 1.2 + j1.2 for the Newton-Raphson and Gauss-Seidel methods respectively. No variation in real and imaginary components were considered for these cases.

4.4 System with series compensation

As noted in Chapter 3, one of the conditions for convergence of an iterative approach is that all of the diagonal terms of the coefficient matrix must be dominant. This condition is violated in the case of series capacitor compensation and the iterative process does not converge. This problem does not arise when a direct method of solution such as the elimination approach is used. This difficulty can be overcome by combining the series capacitor with the associated transmission line using network manipulation. The negative term then disappears from the resultant coefficient matrix as the series compensation is, in a practical case, always less than 100% of the total line inductance involved. After the transmission line terminal voltages are found, the bus voltage and current at the compensated section can then be obtained[21].

5. CONCLUSION

Several methods for solving power system load flow problems have been studied using numerical analysis techniques and by the application of these methods to an actual system model and to a hypothetical model. These methods can be classified into two types, direct methods and iterative methods. The load flow problem can be represented by a set of non-linear equations and therefore solution can be reached only by an iterative procedure. The so-called direct methods also require certain iterative loops to achieve a final solution.

Any comparison of methods can be based only on the accuracy of solution in relation to the total computer time. Methods cannot be compared on the basis of the time taken to achieve a certain voltage tolerance between successive iterations. Direct methods usually require less iterative loops than pure iterative methods. The total computer time required by the direct approach may not be less than that required for the optimum accelerated iterative method due to the necessary matrix reduction. As the system size increases, the time required for the necessary matrix manipulations will greatly increase. Certain mathematical techniques could be used for optimizing and then reducing this computation time. However, the degree of improvement is limited by the system conditions, and varies with individual situations. When computer facilities and programming efficiency are taken into account, the direct method definitely requires more computer storage than an iterative method. This requirement increases with system size. It is not possible to compare programming techniques, however, it is obvious that the direct method is more

complicated than the iterative method.

It is not possible to generally select any method as being entirely preferable to another. Due to the different computational procedures used, each method possesses definite advantages and disadvantages under certain system conditions. It is also not possible to select a method purely on the basis of network size without considering the computer facility, the programming technique and the condition of the problem. In general, however, iterative methods improve in comparison with direct methods as the problem size increases. It was found that the best iterative methods are the overrelaxed Newton-Raphson method and the optimum accelerated Gauss-Seidel method. Newton's method is slightly better than the Gauss-Seidel method for normal load-flow problems.

From the studies of convergence, it was found that a linear acceleration scheme is the most suitable acceleration technique for speeding up the rate of convergence of the iterative process. The relationship between the optimum acceleration factor and the maximum eigenvalue of the system characteristic equation has been illustrated. The analysis of the rate of convergence in terms of the eigenvalues and norms of the iteration matrix could lead to further methods for solving power system problems. There is need for more work in this area.

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7. APPENDICES

7.1 Input data and single line diagram of S.P.C. system

The entire digital computer program was written in parts using subroutines and controlled by the main program. The reading of input data and the calculating of and printing of the load flow results was accomplished using three separate subroutines. These subroutines remain unchanged for all of the four methods studied. The subroutines of different methods were programmed to find only the voltage magnitude and angle at every bus using the input data information.

The input data of S.P.C. system and the hypothetical system and the load flow study results of the hypothetical system are presented on the following pages with the respective single line diagrams.

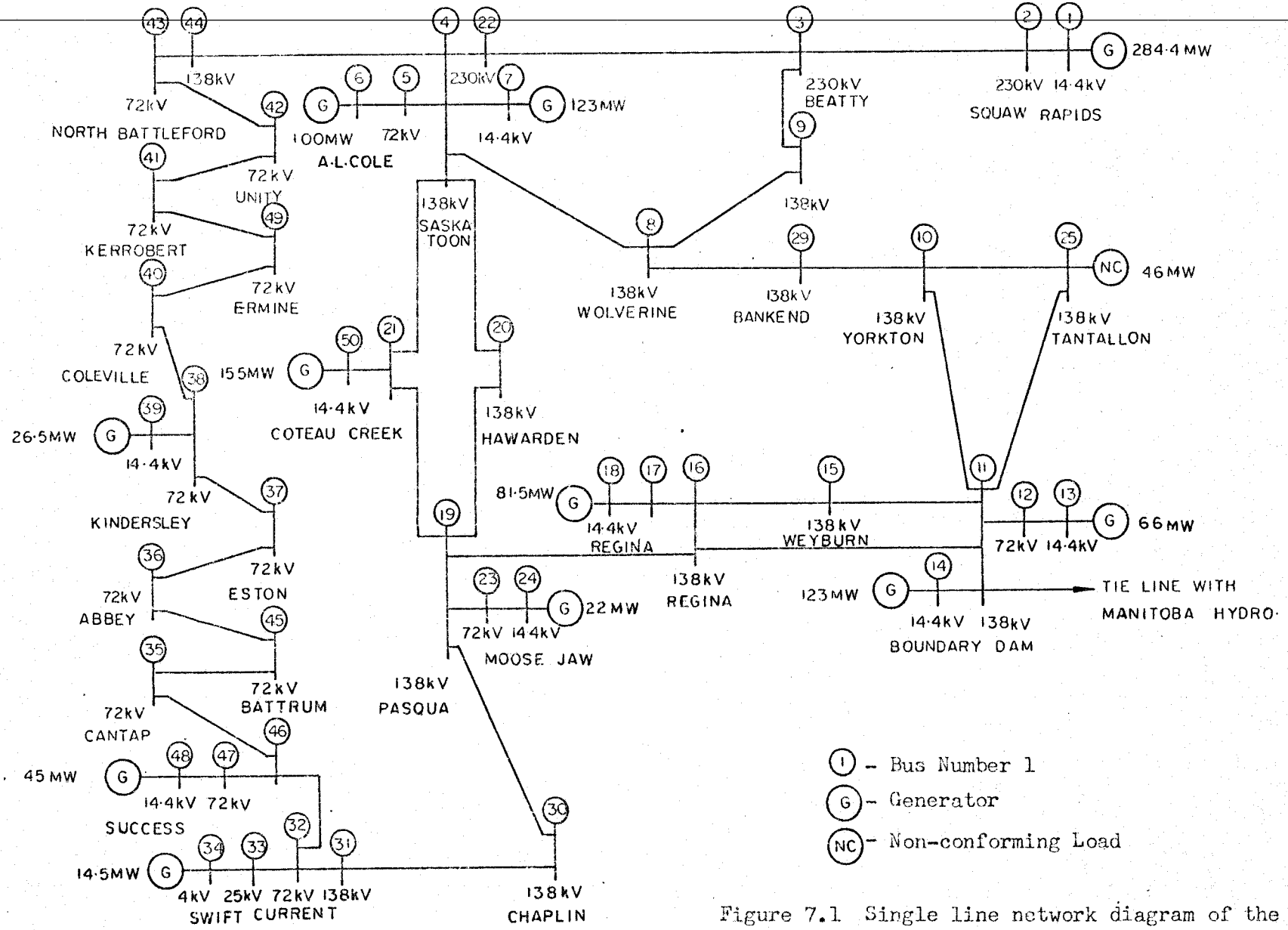

Figure 7.1 Single line network diagram of the Saskatchewan Power Corporation System

BUS DATA INFORMATION OF THE SASKATCHEWAN POWER CORPORATION SYSTEM

NBUS	BUSNAME	T	VOLT	ANG	PGEN	QGEN	PLOAD	QLOAD	QMIN	QMAX	REACT
1	SR144	1	0.970	0.0	200.00	0.0	0.0	0.0	-100.00	100.00	0.0
2	SR230	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	BEA230	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	QE138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-30000.0
5	ALC72	0	1.000	0.0	0.0	0.0	79.69	10.78	0.0	0.0	0.0
6	ALC144	1	1.000	0.0	20.00	0.0	36.11	11.14	-100.00	100.00	0.0
7	QE144	2	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	WOL138	0	1.000	0.0	0.0	0.0	35.27	0.0	0.0	0.0	0.0
9	BEA138	0	1.000	0.0	0.0	0.0	38.52	0.0	0.0	0.0	-45000.0
10	YOP138	0	1.000	0.0	0.0	0.0	24.50	4.27	0.0	0.0	0.0
11	BD138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	EST72	0	1.000	0.0	0.0	0.0	34.01	10.53	0.0	0.0	0.0
13	EST144	1	1.000	0.0	65.00	0.0	2.71	0.90	-100.00	100.00	0.0
14	BD144	1	1.000	0.0	120.00	0.0	0.0	0.0	-100.00	100.00	0.0
15	WEY138	0	1.000	0.0	0.0	0.0	27.27	8.43	0.0	0.0	0.0
16	REG138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
17	REG72	0	1.000	0.0	0.0	0.0	44.48	10.77	0.0	0.0	0.0
18	REG144	1	1.000	0.0	65.00	0.0	64.16	29.49	-100.00	100.00	0.0
19	PAS138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	HAW138	0	1.000	0.0	0.0	0.0	26.66	0.0	0.0	0.0	0.0
21	CC138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	QE230	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	PAS72	0	1.000	0.0	0.0	0.0	25.64	3.07	0.0	0.0	0.0
24	PAS144	0	1.000	0.0	0.0	0.0	5.84	2.83	0.0	0.0	0.0
25	TAN138	0	1.000	0.0	0.0	0.0	37.98	6.56	0.0	0.0	0.0
29	BAM138	0	1.040	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
30	CHP138	0	1.000	0.0	0.0	0.0	8.13	0.0	0.0	0.0	0.0
31	SC138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
32	SC72	0	1.000	0.0	0.0	0.0	18.36	2.23	0.0	0.0	0.0
33	SC25	0	1.000	0.0	0.0	0.0	5.84	2.71	0.0	0.0	0.0
34	SC416	1	1.000	0.0	12.00	0.0	0.0	0.0	-100.00	100.00	0.0
35	CAN72	0	1.000	0.0	0.0	0.0	6.56	3.79	0.0	0.0	0.0
36	AB872	0	1.000	0.0	0.0	0.0	2.23	0.36	0.0	0.0	0.0
37	EST72	0	1.000	0.0	0.0	0.0	2.83	0.48	0.0	0.0	0.0
38	KIND72	0	1.000	0.0	0.0	0.0	3.37	0.84	0.0	0.0	0.0
39	KIND14	1	1.000	0.0	20.00	0.0	0.0	0.0	-100.00	100.00	0.0
40	COL72	0	1.000	0.0	0.0	0.0	1.20	0.06	0.0	0.0	0.0
41	KER72	0	1.000	0.0	0.0	0.0	4.48	0.0	0.0	0.0	0.0
42	UN72	0	1.000	0.0	0.0	0.0	4.27	0.0	0.0	0.0	0.0
43	NB72	0	1.000	0.0	0.0	0.0	20.50	0.0	0.0	0.0	0.0
44	NB138	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	BAT72	0	1.000	0.0	0.0	0.0	1.20	0.66	0.0	0.0	0.0
46	SUCT72	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	SUC72	0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
48	SUC144	1	1.050	0.0	0.0	0.0	0.0	0.0	-45.00	45.00	0.0
49	ERM72	0	1.000	0.0	0.0	0.0	8.43	2.59	0.0	0.0	0.0

TRANSMISSION LINE AND TRANSFORMER DATA INFORMATION OF THE SASKATCHEWAN POWER CORPORATION SYSTEM

FROM BUS	TO BUS	R PCT	X PCT	CHG KVA	R PU	X PU	BC/2	G PU	B PU	TAP	MAX	MIN	TYPE
2	3	0.725	5.960	50840.00	0.007	0.059	0.254	-2.011	16.534	0.0	0.0	0.0	0
3	22	0.860	7.070	60340.00	0.009	0.071	0.302	-1.595	13.938	0.0	0.0	0.0	0
4	20	5.120	19.860	5290.00	0.051	0.199	0.026	-1.217	4.721	0.0	0.0	0.0	0
4	21	6.110	23.790	6250.00	0.061	0.238	0.031	-1.013	3.943	0.0	0.0	0.0	0
4	44	8.610	34.740	8530.00	0.086	0.347	0.043	-0.672	2.712	0.0	0.0	0.0	0
4	8	11.400	25.900	5860.00	0.114	0.259	0.029	-1.424	3.234	0.0	0.0	0.0	0
8	9	14.590	33.130	8040.00	0.146	0.331	0.040	-1.113	2.528	0.0	0.0	0.0	0
8	29	7.010	27.160	7230.00	0.070	0.272	0.036	-0.891	3.452	0.0	0.0	0.0	0
10	11	16.060	62.210	16560.00	0.161	0.622	0.083	-0.389	1.507	0.0	0.0	0.0	0
10	25	5.130	19.970	5240.00	0.051	0.200	0.026	-1.207	4.698	0.0	0.0	0.0	0
10	29	6.680	25.880	6890.00	0.067	0.259	0.034	-0.935	3.623	0.0	0.0	0.0	0
11	15	6.080	23.570	6280.00	0.061	0.236	0.031	-1.026	3.978	0.0	0.0	0.0	0
11	16	11.690	48.200	11400.00	0.117	0.482	0.057	-0.475	1.959	0.0	0.0	0.0	0
15	16	6.190	24.010	6390.00	0.062	0.240	0.032	-1.007	3.905	0.0	0.0	0.0	0
16	19	3.900	15.050	4020.00	0.039	0.150	0.020	-1.613	6.226	0.0	0.0	0.0	0
19	20	9.210	35.690	9500.00	0.092	0.357	0.047	-0.678	2.627	0.0	0.0	0.0	0
19	21	9.060	35.300	9280.00	0.091	0.353	0.046	-0.682	2.658	0.0	0.0	0.0	0
19	30	9.410	23.610	5940.00	0.094	0.236	0.030	-1.339	3.359	0.0	0.0	0.0	0
30	31	8.740	24.530	6190.00	0.087	0.245	0.031	-1.289	3.617	0.0	0.0	0.0	0
32	46	10.320	19.800	360.00	0.103	0.198	0.002	-2.070	3.972	0.0	0.0	0.0	0
35	45	7.820	17.590	345.00	0.078	0.176	0.002	-2.110	4.747	0.0	0.0	0.0	0
35	46	4.620	8.910	162.00	0.046	0.089	0.001	-4.596	8.845	0.0	0.0	0.0	0
36	37	19.920	44.830	878.00	0.199	0.448	0.004	-0.828	1.863	0.0	0.0	0.0	0
36	45	15.710	35.340	693.00	0.157	0.353	0.003	-1.050	2.363	0.0	0.0	0.0	0
37	38	18.160	40.850	802.00	0.182	0.408	0.004	-0.909	2.044	0.0	0.0	0.0	0
38	40	15.230	20.160	424.00	0.152	0.202	0.002	-2.396	3.158	0.0	0.0	0.0	0
40	49	11.980	18.480	330.00	0.119	0.185	0.002	-2.461	3.829	0.0	0.0	0.0	0
41	49	8.530	13.270	237.00	0.085	0.133	0.001	-3.428	5.332	0.0	0.0	0.0	0
41	42	38.500	65.100	944.00	0.385	0.651	0.005	-0.673	1.138	0.0	0.0	0.0	0
42	43	46.870	72.910	1302.00	0.469	0.729	0.007	-0.624	0.970	0.0	0.0	0.0	0
47	46	4.690	10.560	207.00	0.047	0.106	0.001	-3.513	7.910	0.0	0.0	0.0	0
2	1	0.0	3.320	0.0	0.0	0.033	0.0	-0.0	30.120	1.050	0.0	0.0	2
3	9	0.0	1.510	0.0	0.0	0.015	0.0	-0.0	66.225	1.050	0.0	0.0	2
4	6	0.0	31.400	0.0	0.0	0.314	0.0	-0.0	3.185	1.000	0.0	0.0	2
4	7	0.0	7.850	0.0	0.0	0.078	0.0	-0.0	12.739	1.012	1.050	0.850	1
5	4	0.0	4.590	0.0	0.0	0.046	0.0	-0.0	21.786	1.010	1.175	0.950	1
5	6	0.0	43.100	0.0	0.0	0.431	0.0	-0.0	2.320	1.000	0.0	0.0	2
11	14	0.0	8.250	0.0	0.0	0.082	0.0	-0.0	12.121	1.010	1.050	0.950	1
12	11	0.0	18.300	0.0	0.0	0.183	0.0	-0.0	5.464	1.022	1.175	0.950	1
12	13	0.0	14.950	0.0	0.0	0.148	0.0	-0.0	6.734	1.000	0.0	0.0	2
17	16	0.0	10.000	0.0	0.0	0.100	0.0	-0.0	10.000	1.041	1.175	0.950	1
18	17	0.0	12.200	0.0	0.0	0.122	0.0	-0.0	8.197	1.000	0.0	0.0	2
22	4	0.0	1.310	0.0	0.0	0.013	0.0	-0.0	76.336	1.050	0.0	0.0	2
23	19	0.0	6.250	0.0	0.0	0.062	0.0	-0.0	16.000	1.024	1.175	0.950	1
23	24	0.0	31.600	0.0	0.0	0.316	0.0	-0.0	3.165	1.000	0.0	0.0	2
32	31	0.0	11.850	0.0	0.0	0.118	0.0	-0.0	8.439	1.000	1.175	0.950	1
32	33	0.0	64.700	0.0	0.0	0.647	0.0	-0.0	1.546	1.000	0.0	0.0	2
33	34	0.0	41.600	0.0	0.0	0.416	0.0	-0.0	2.404	1.000	0.0	0.0	2
38	39	0.0	29.200	0.0	0.0	0.292	0.0	-0.0	3.425	1.000	0.0	0.0	2
43	44	0.0	11.500	0.0	0.0	0.115	0.0	-0.0	8.696	1.025	1.175	0.950	1
47	48	0.0	15.000	0.0	0.0	0.150	0.0	-0.0	6.667	1.000	0.0	0.0	2

7.2 INPUT DATA OF THE HYPOTHETICAL SYSTEM

Transmission Lines And Transformers Data List:

Number of the from bus	Number of the to bus	Resistance per cent	Reactance per cent	Charging KVA	Tap Setting	Min. Tap Setting	Max. Tap Setting	Type of Tap
1	2	2.0	6.0	2000.0				
1	3	4.0	11.0	2400.0				
2	3	3.0	6.0	1600.0				
3	4	0.0	4.0	0.0	1.03			2
5	3	0.0	6.0	0.0	0.97			2
4	5	3.0	8.0	0.0				

BUS DATA INFORMATION

Number of bus	Name of bus	Bus Type	Voltage	Angle	P Generation	Q Generation	P Load	Q Load	Min. VAR	Max. VAR	Shunt Reactor
1	Bus 001	0	1.000	0.0	0.0	0.0	117.20	97.20	0.0	0.0	50000.0
2	Bus 002	1	1.030	0.0	237.90	0.0	100.00	50.00	-300.00	300.00	0.0
3	Bus 003	2	1.000	0.0	0.0	0.0	100.00	50.00	0.0	0.0	0.0
4	Bus 004	0	0.950	0.0	0.0	0.0	92.10	26.70	0.0	0.0	0.0
5	Bus 005	0	0.950	0.0	0.0	0.0	32.50	5.90	0.0	0.0	0.0

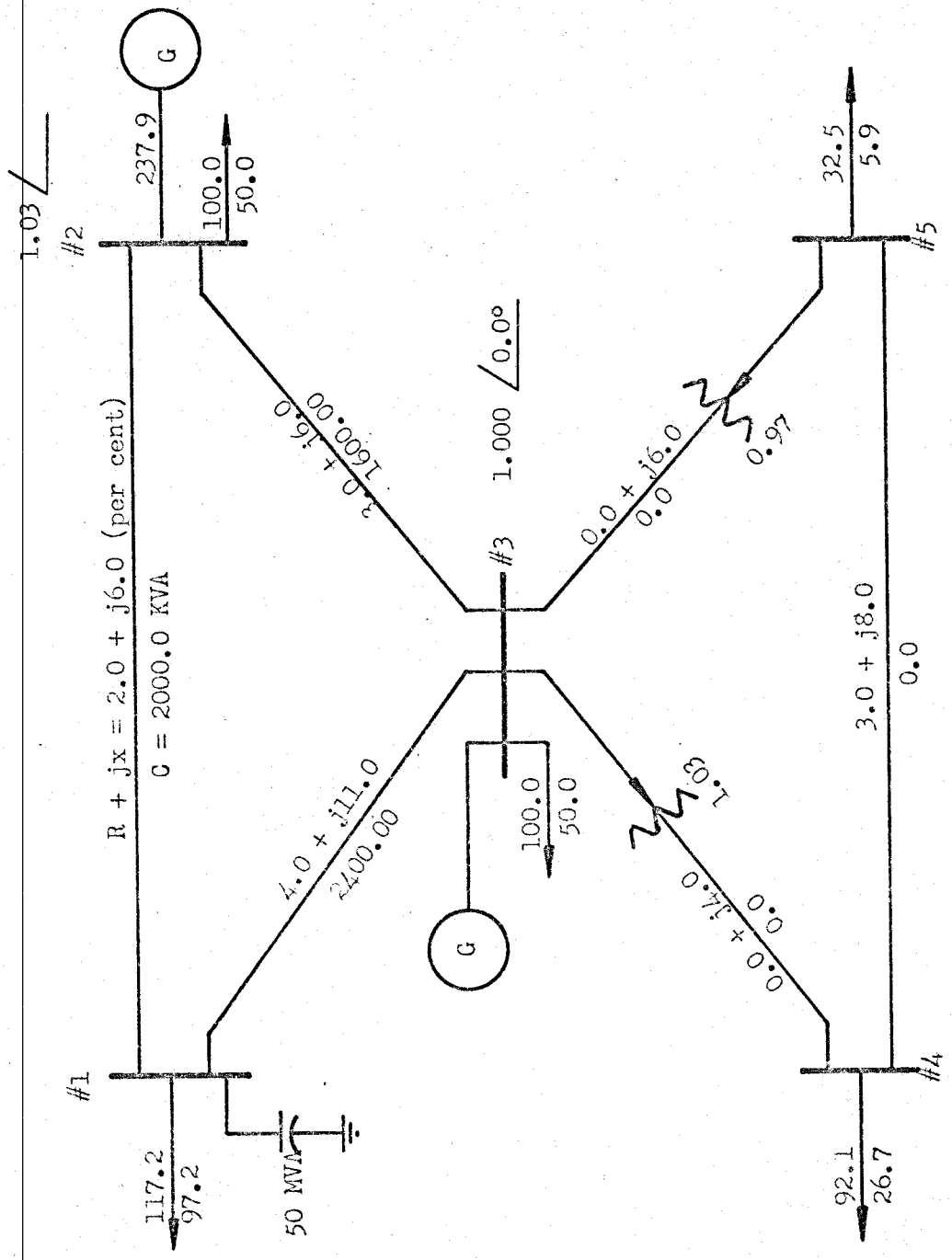

Figure 7.2 The hypothetical system configuration

LOAD FLOW STUDY DONE BY DAVID CHAN

REPORT ON LOAD-FLOW CALCULATION

NUMBER OF ITERATIONS IS 5 THE SWING BUS IS BUS NUMBER 3

********** B U S D A T A **********

BUS NO	BUS NAME	VOLTS %PU	ANGLE %DEG	GENERATION MW	GENERATION MVAR	LOAD MW	LOAD MVAR	CAP/REACT MVAR
1	BUS001	0.984	-1.6	0.0	0.0	117.2	97.2	48.5
2	BUS002	1.030	1.0	237.9	126.3	100.0	50.0	
3	BUS003	1.000	0.0	207.1	62.1	100.0	50.0	
4	BUS004	0.960	-2.0	0.0	0.0	92.1	26.7	
5	BUS005	0.965	-1.5	0.0	0.0	32.5	5.9	
				******	*******	*******	******	
	TOTALS			445.03	188.38	441.80	229.80	

**********&&&&&& L I N E F L O W

BUS NO	BUS NAME	MW	MVAR	TAP SETTING
2	BUS0 02	-90.9	-43.6	
3	BUS0 03	-26.3	-5.1	
1	BUS0 01	93.0	47.8	
3	BUS0 03	44.9	28.4	
1	BUS0 01	26.6	3.6	
2	BUS0 02	-44.1	-29.5	
4	BUS0 04	81.3	27.9	1.03
5	BUS0 05	43.4	9.1	
3	BUS0 03	-81.3	-24.8	
5	BUS0 05	-10.8	-1.9	
3	BUS0 03	-43.4	-8.0	0.97
4	BUS0 04	10.9	2.1	

TOTAL LOSSES 3.2247 MW 13.0860 MVAR

TOTAL MISMATCH 0.0024 MW -0.0008 MVAR

THE END OF REPORT FOR THIS CASE

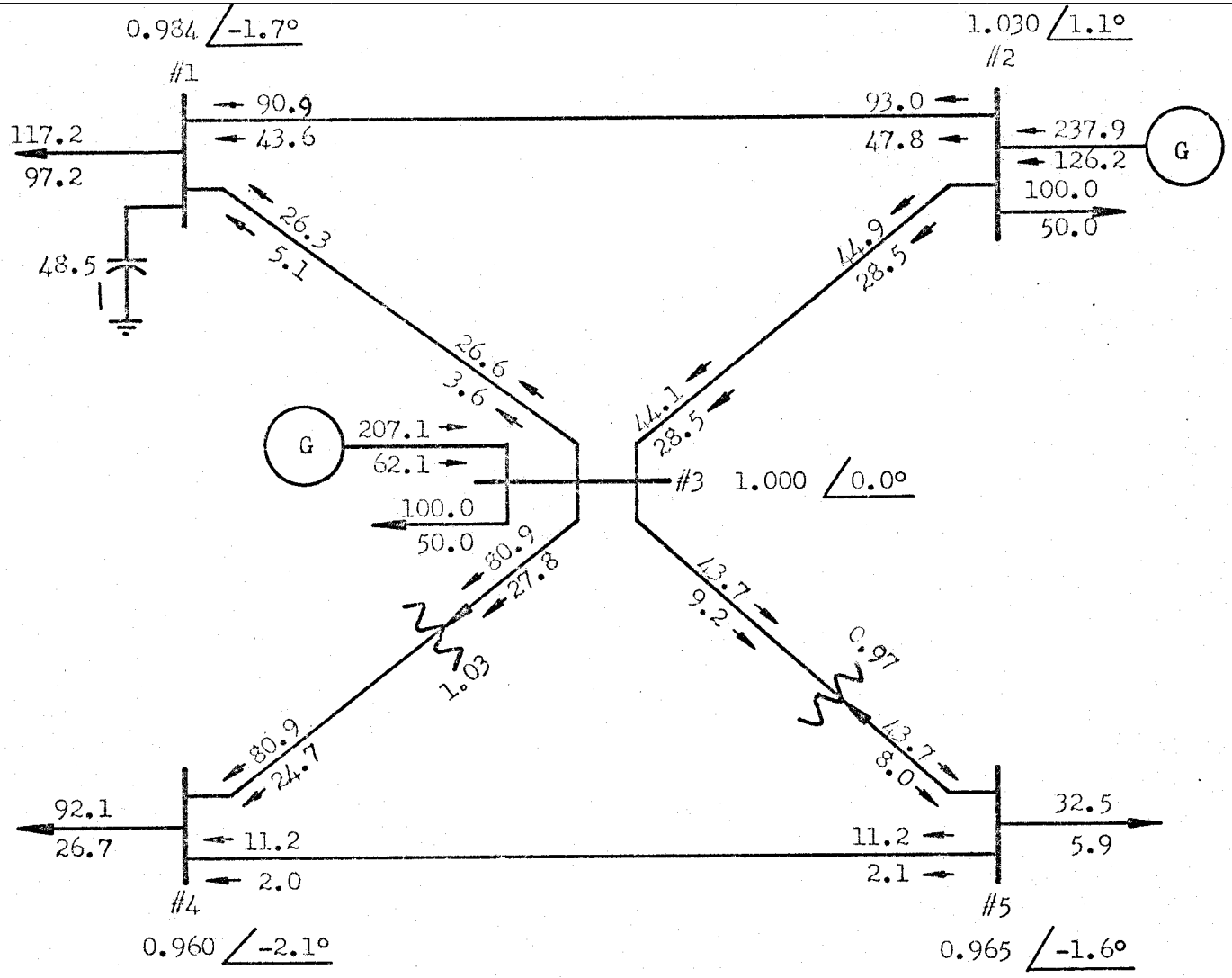

Figure 7.3 Bus voltage and power flow of the hypothetical system

7.3 Computer flow-chart for methods studied

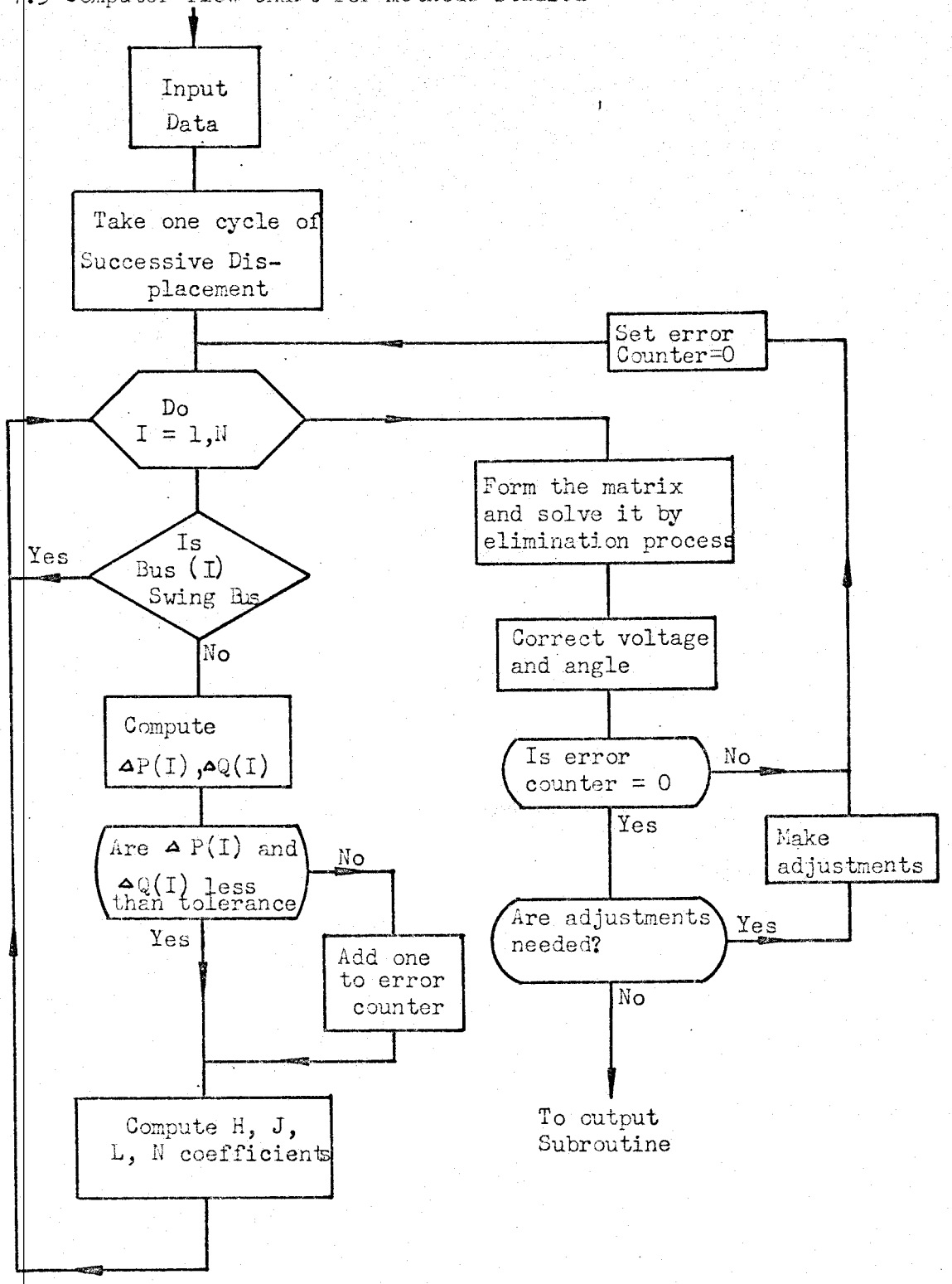

Figure 7.4 Digital computer flow chart of the Elimination Method

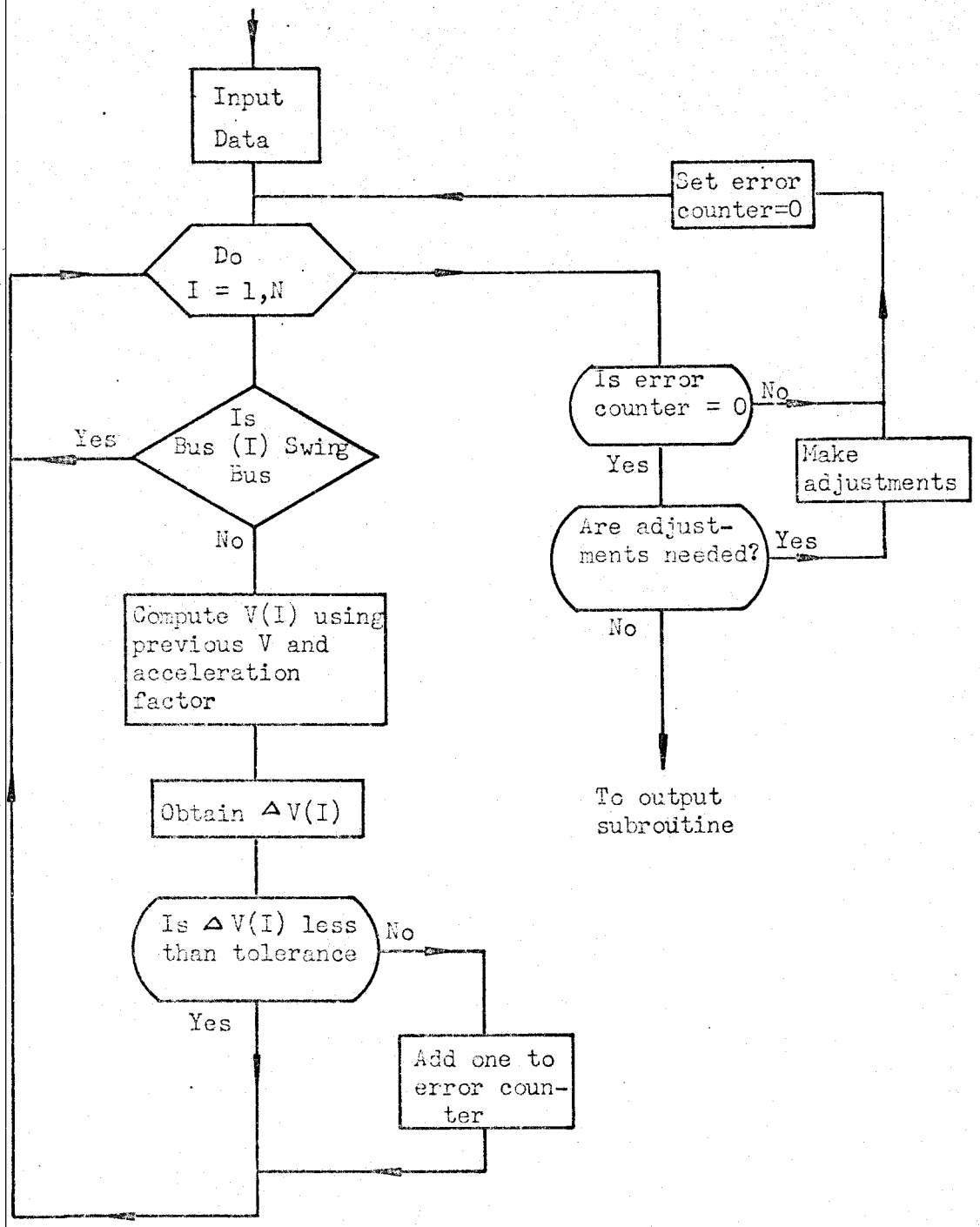

Figure 7.5 Digital computer flow chart of the accelerated Gauss-Seidel Method

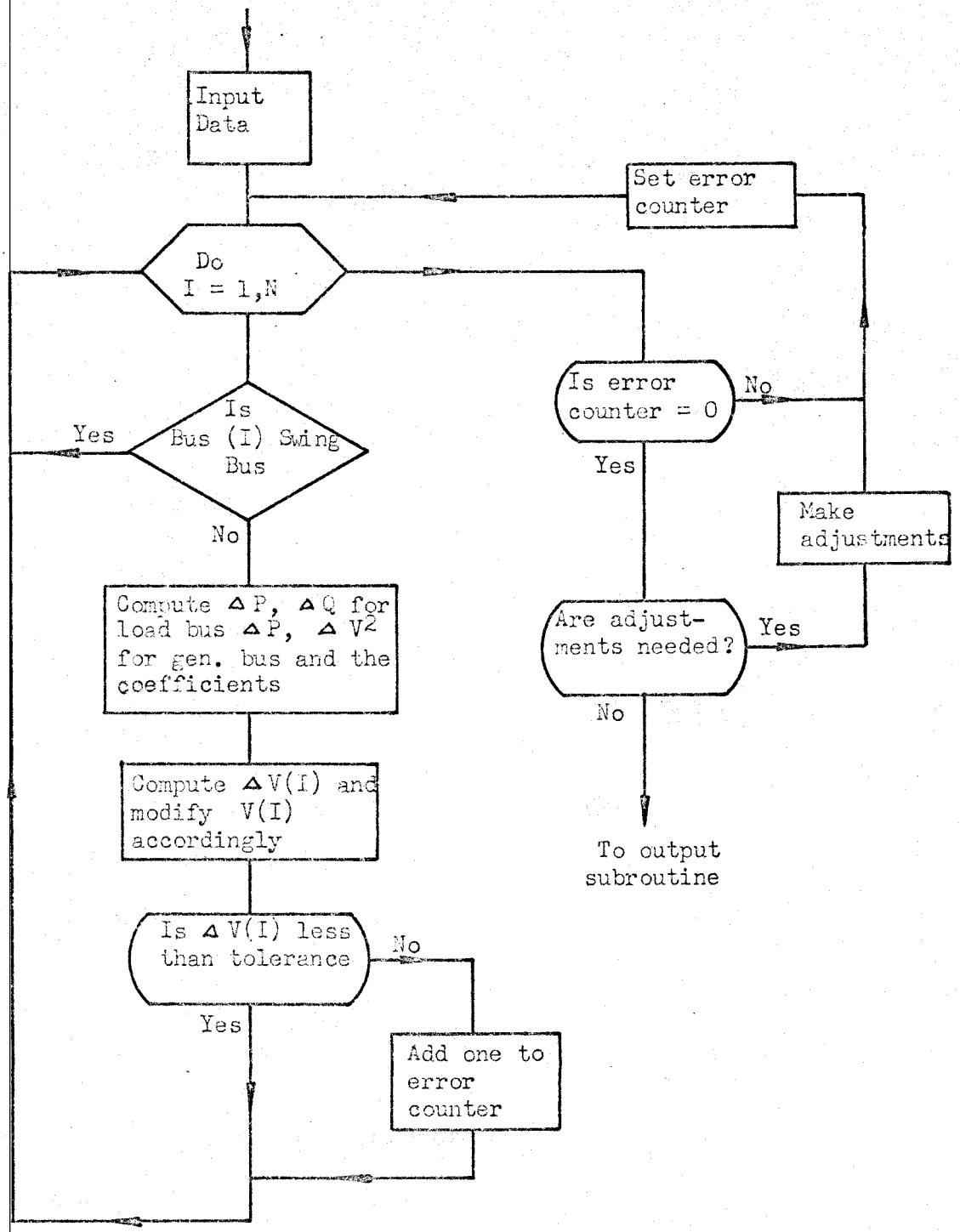

Figure 7.6 Digital computer flow chart of the Newton-Raphson Method

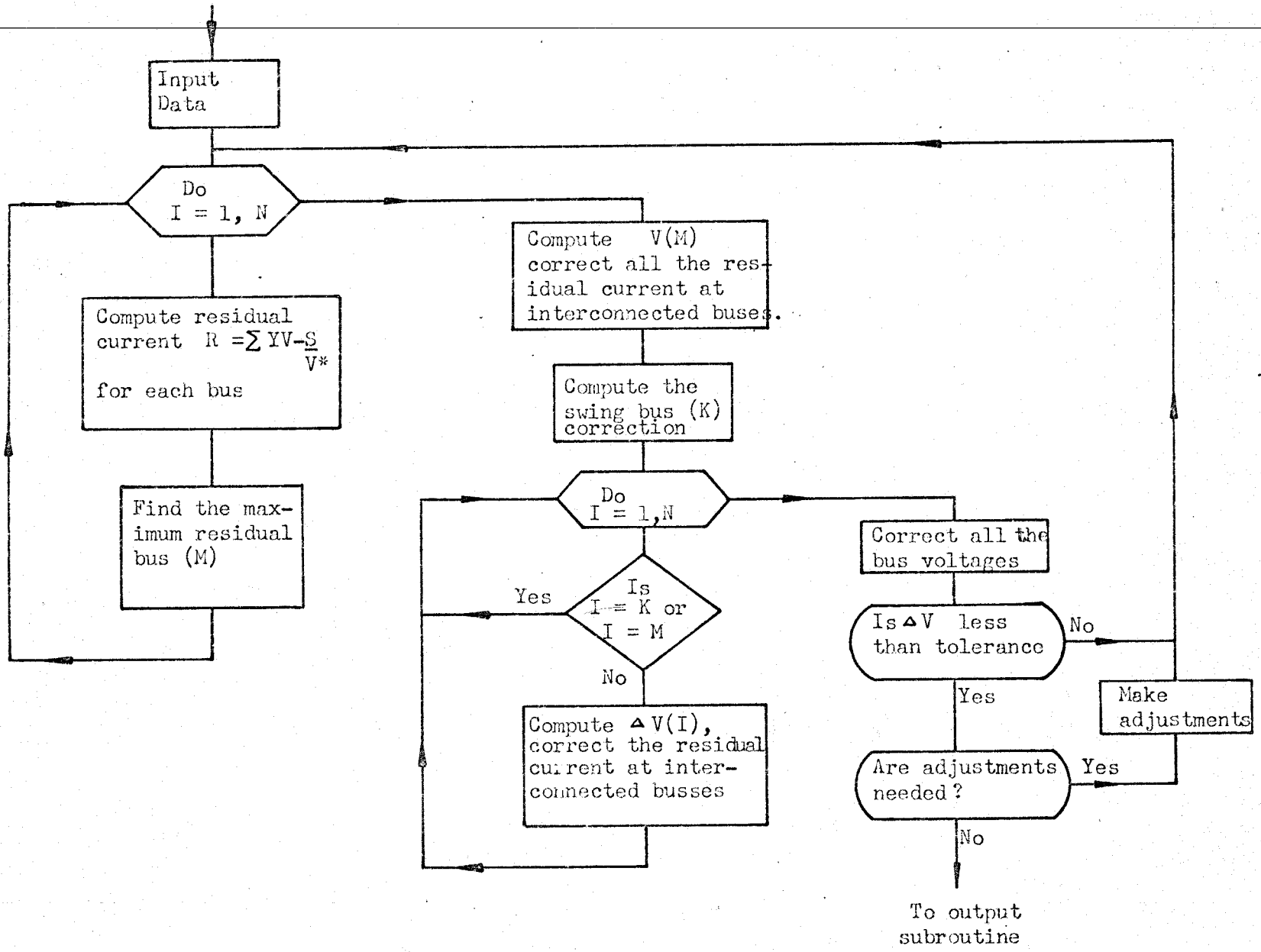

Figure 7.7 Digital computer flow chart of the relaxation method

7.4 Norms of vectors and matrices

The concepts of vector norms, matrix norms, and the spectral radius of matrices play an important role in iterative numerical analysis. As it is convenient to compare two vectors in terms of their lengths, two matrices can be compared by their norms. This comparison will be the basis for deciding which of the two iterative methods is more rapidly convergent. Norms of vectors and matrices are introduced by different definitions as shown below, and in individual cases one or another norm may turn out to be more convenient.

Norms of a vector

The norms of a vector $[\dot{X}]$ are non-negative numbers $\| \dot{X} \|$ and defined as follows:

(a) First norm

$$\| \dot{X} \|_I = \max_i | \dot{x}_i |$$

(b) Second norm $$\| \dot{X} \|_{II} = |\dot{x}_1| + |\dot{x}_2| + \ldots\ldots + |\dot{x}_n| = \sum_{i=1}^{N} |\dot{x}_i|$$

(c) Third norm $$\| \dot{X} \|_{III} = |\dot{x}| = \sqrt{|\dot{x}_1|^2 + |\dot{x}_2|^2 + \ldots + |\dot{x}_n|^2} = \sqrt{\sum_{i=1}^{N} |\dot{x}_i|^2}$$

This norm is also called the length of the matrix $[\dot{X}]$.

Norms of a matrix:

The norms of a matrix $\dot{A}$ are non-negative numbers $\| \dot{A} \|$ and defined as follows according to their corresponding vector norms.

(a) For $\| \dot{X} \|_I = \max_i |\dot{x}_i|$ The norm of matrix corresponding to this vector norm is defined as

$$\| \dot{A} \|_I = \max_i \sum_{k=1}^{n} |\dot{a}_{ik}|$$

(b) For $\| \dot{X} \|_{II} = \sum_{i=1}^{N} |\dot{x}_i|$ The corresponding matrix norm is defined as

$$\| \dot{A} \|_{II} = \max_K \sum_{i=1}^{n} |\dot{a}_{ik}|$$

(c) For $\| \dot{X} \|_{III} = \sqrt{\sum_{i=1}^{n} |\dot{x}_i|^2}$ The corresponding matrix norm is defined as

$$\| \dot{A} \|_{III} = \sqrt{\sum_{i,k=1}^{n} |\dot{a}_{i,k}|^2}$$

It was established[2] that all eigenvalues are less in modulus than any norm of the matrix.

Spectral radius of a matrix

Let $\dot{A} = [\dot{a}_{i,j}]$ be a [n x n] complex matrix with eigenvalue λ_i, $1 \leq i \leq n$, then $\rho(\dot{A}) = \max_{1 \leq i \leq n} |\lambda_i|$ is defined as the spectral radius of the matrix $\dot{A}$.

Geometrically, if all the eigenvalues λ_i of $\dot{A}$ are plotted in the complex $\dot{Z}$-plane, then $\rho(\dot{A})$ is the radius of the smallest circle $|\dot{Z}| \leq \rho(\dot{A})$, with center at the origin, which includes all the eigenvalues of the matrix $\dot{A}$.

For an arbitrary n x n complex matrix $\dot{A}$, it is found that

$$\| \dot{A} \| \geq \rho(\dot{A})$$

7.5 Comparison of the asymptotic rate of convergence of the Gauss-Seidel method and the Jacobi method

Example $AX = K$

$$\begin{pmatrix} 1 & 0 & -1/4 & -1/4 \\ 0 & 1 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1 & 0 \\ -1/4 & -1/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The unique vector solution of this matrix equation is obviously $x_1 = x_2 = x_3 = x_4 = 1$. For this case, the Jacobi matrix B and the Gauss-Seidel matrix L_ω are shown as follows:

$$B = 1/4 \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$L_\omega = \begin{pmatrix} \sigma & 0 & \tau & \tau \\ 0 & \sigma & \tau & \tau \\ \sigma\tau & \sigma\tau & 2\tau^2+\sigma & 2\tau^2 \\ \sigma\tau & \sigma\tau & 2\tau^2 & 2\tau^2+\sigma \end{pmatrix} = 1/8 \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

where $\tau = \omega/4$, $\sigma = 1 - \omega$ in this case $\omega = 1$.

If $^{(m)}$ denotes the errors for the Jacobi iterative method, and $^{(m)}$ denotes the errors for the Gauss-Seidel method. If the initial error vector is

$$\varepsilon^{(0)} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{then}$$

$$\alpha^{(m)} = B^m \varepsilon^{(0)} = -\frac{1}{2^m} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\beta^{(m)} = L_1^m \varepsilon^{(0)} = -\frac{1}{4^m} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

Thus, as

$$\| \alpha^{(m)} \| = \frac{1}{2^{m-1}} > \| \beta^{(m)} \| = \frac{\sqrt{10}}{4^m} \quad \text{for all } m \geq 1$$

it indicates that the asymptotic rate of convergence of the Gauss-Seidel method is faster than that of the Jacobi iterative method.