Quantum Field Theoretic Approach to The Unruh Effect in Rindler-Anti de-Sitter and Rotating Rindler-Anti de-Sitter Spacetimes

A thesis submitted to the
College of Graduate and Postdoctoral Studies
in partial fulfillment of the requirements
for the degree of Master of Science
in the Department of Physics and Engineering Physics
University of Saskatchewan
Saskatoon

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Abstract

A peculiar prediction of quantum field theory in curved spacetime is that the existence of particles is observer dependent. This peculiarity is associated with the existence of a horizon, either the Rindler horizon due to the accelerating motion of an observer, which is the case of the Unruh effect, or the event horizon of a black hole, which is the case of the Hawking radiation. The Unruh effect predicts that an accelerating observer in the vacuum of flat spacetime will observe a thermal bath with temperature proportional to their acceleration. In this thesis, by means of the quantum field theoretic approach, we find the Unruh temperature in Rindler-AdS and rotating Rindler-AdS by calculating the expected number of particles in the AdS vacuum seen by Rindler observers. Our results for the Unruh temperatures confirm the temperatures obtained by dimensional analysis arguments in previous studies on the spacetime.

Acknowledgements

I would like to thank my supervisors, Dr. Masoud Ghezelbash and Dr. Kaori Tanaka, for their guidance, patience, and funding necessary for me to finish my thesis as well as for providing me the opportunity to pursue my graduate level study of physics. I would also like to thank the Department of Physics and Engineering Physics and the University of Saskatchewan for providing me with the funding and opportunity to learn through the Graduate Teaching Fellowship.

To my loving family.

Contents

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List of Abbreviations

QFT	Quantum Field Theory
CFT	Conformal Field Theory
dS	de-Sitter
AdS	Anti de-Sitter
PAdS	Poincaré Anti de-Sitter
RAdS	Rindler-Anti de-Sitter
RRAdS	Rotating Rindler-Anti de-Sitter
KG	Klein-Gordon

1 Introduction

General relativity and quantum field theory (QFT) are the two theoretical frameworks widely accepted. General relativity describes gravitation as spacetime curvature satisfying the Einstein field equations and free-falling matter within the spacetime follows its geodesics [1]. Three well known, arguably the simplest, solutions of the Einstein field equations are the Minkowski, de-Sitter (dS), and anti de-Sitter (AdS) spacetimes which respectively have cosmological constant of zero, positive, and negative. QFT describes matter and forces as quantized fields which could interact with each other, relativistic or otherwise. The Standard Model of particle physics models the quantum behaviour of electromagnetism, weak and strong nuclear interactions, as well as all known elementary particles with the framework of QFT. In studying the effects of gravity or spacetime curvature on quantum fields, the techniques of QFT can be performed in such curved spacetime [2] where gravity itself is not considered a quantum mechanical effect or a quantum field.

The Unruh effect, also known as the Fulling-Davies-Unruh effect, is a prediction of QFT that an accelerating observer in a vacuum will observe particles with temperature, called the Unruh temperature, proportional to their proper acceleration. This effect is often examined using the Rindler spacetime which describes the spacetime in the perspective of an observer with constant proper acceleration in Minkowski spacetime. In 1973, Fulling [3] stated that the notion of particles and vacuum state are dependent on the observers and illustrated this point using a scalar quantum field in Rindler spacetime. The temperature of the Minkowski vacuum state in Rindler spacetime was separately calculated by Davies [4] in 1975 and Unruh [5] in 1976. Unruh's publication was a comment on Hawking's radiation of black holes and, indeed, the Unruh effect is closely related to Hawking radiation. The physical meaning, the concept of temperature, applications as well as possible experiments, and a review in the context of the Unruh effect have been discussed in the past few decades [6, 7, 8, 9].

There has been a lot of research on the Unruh effect in the past few years. Arias et al. [10], in 2018, introduced a relativistic quantum analogue of the classical Otto engine which uses the quantum vacuum as a thermal bath through the Unruh effect, called the Unruh engine. The authors obtained the efficiency of such an engine and derived the conditions required to have a thermodynamic as well as a kinematic cycle which constrain the range of the engine's acceleration. In 2022, Mukherjee et al. [11] then expanded on the idea of the Unruh engine by adding a perfectly reflecting boundary and examined the effects that the reflecting boundary has on the engine's quantum thermodynamics. It was found that, interestingly, the addition of the reflecting boundary reduces the engine's work output but does not change its efficiency. Biermann et al. [12], in 2020, examined the Unruh temperatures for circular motion in $3+1$ and $2+1$ dimensional massless scalar field in comparison to the linear accelerated motion. In addition to that, the authors also translate their results to an analogue spacetime with non-relativistic field theory. This analogue Unruh effect provides the possibility to test the Unruh effect experimentally. In 2022, G.E. Volovik [13] examined the interconnection between the Schwinger effect where electron-positron pair is predicted to be created in the presence of an electric field, the particle creation in the Hawking radiation, and the particle creation in the Unruh effect. The authors showed that three of these processes combined obey the sum rules for the entropy and the inverse temperature. In examining the sum rules, it was found that some of the processes contribute negative entropy and temperature, which the authors concluded as a consequence of quantum entanglements between the created objects. In the early year of 2023, Chen [14] proposed a generalization of the Unruh effect in the vacuum to include arbitrary excited states. By using the generalization result, the author also proposed a potential resolution to the black hole information paradox. The potential resolution proposed is that information lost when a particle crosses a black hole's event horizon might not be lost after all and could be at least partially retrievable by measuring the deviation of the Hawking radiation from the black body radiation spectrum. In short, the Unruh effect holds a lot of potential for exciting research in uncovering the quantum nature of gravity, in experimental and theoretical physics alike.

The Unruh effect in dS and AdS spacetimes has also been studied in the past [2, 15]. The AdS spacetimes are particularly interesting because of the AdS/CFT correspondence proposed by Maldacena [16]. The AdS/CFT correspondence is a bulk/boundary correspondence that essentially states that the quantum gravity of string theory in the bulk of AdS spacetime can be fully described using non-gravitational conformal field theory (CFT) on the AdS spacetime boundary. Whether or not the assumptions used in the AdS/CFT correspondence apply to our Universe, the insights from the study of AdS/CFT have been used in studying gravity further, such as in Ref. [17], without depending on the assumptions of AdS/CFT. The AdS/CFT correspondence is one realization of the holographic principle which is the notion that quantum gravity theory is encoded in a non-gravitational theory one dimension lower. In the spirit of the holographic duality of the AdS/CFT, a new duality between extreme rotating black holes, where the magnitude of the black hole's angular momentum approaches its mass squared, and two-dimensional CFT was proposed in Ref. [18] known as the Kerr/CFT correspondence. The Kerr/CFT correspondence states that the theory of quantum gravity near the horizon of an extreme rotating (Kerr) black hole can be described by a particular non-gravitational CFT with one dimension lower. The authors of [18] demonstrated the usefulness of this black hole holography by investigating the quantum origin of the extreme Kerr black hole's macroscopic entropy. The Kerr/CFT correspondence, much like the AdS/CFT correspondence, holds potential insights about the quantum behaviour of gravity.

In 2011, the rotating Rindler-AdS spacetime was proposed by Parikh et al. [19]. The non-rotating Rindler-AdS spacetime, analogous to the Rindler spacetime, describes the spacetime from the perspective of an accelerating observer in AdS spacetime. The 'rotating' part of the rotating Rindler-AdS comes from a class of Lorentz transformations called the "rota-boosts," the generator of which is a linear combination of rotations and boosts that cannot be reduced to either pure rotations or pure boosts. The rotating Rindler-AdS is the isometry of these "rota-boost" transformations. The "rota-boost" transformations are parameterized by a rotation parameter β which is considered a constant and the value of $\beta = 0$ reduces the metric back to the non-rotating Rindler-AdS. In fact, the rotating Rindler-AdS and the non-rotating Rindler-AdS are related by a boost transformation. In the paper [19], the authors briefly stated that the Unruh temperature could be seen from its metric and written the temperature to be

$$T = \frac{1-\beta^2}{2\pi L} \tag{1.1}$$

where L is the AdS curvature scale and it is related to the cosmological constant Λ by $L^2 = -\frac{(n-1)(n-2)}{2\Lambda}$ for an n-dimensional spacetime. The Unruh temperature (1.1), in the case of $\beta = 0$, is the Unruh temperature in the non-rotating Rindler-AdS.

The goal of my thesis is to confirm the Unruh temperature in Rindler-AdS and rotating Rindler-AdS with the quantum field theoretic approach shown in references [4, 8, 9]. This thesis is the first step in the study of the Unruh effect in the context of black hole holography. Examining the singularity and the horizon of the rotating Rindler-AdS, one could also interpret the spacetime time as containing a black dot, i.e., a black hole with zero radius. With that interpretation, it might be possible to promote the rotating Rindler-AdS black dot to a black hole by surrounding the singularity with matter. This promotion then potentially allows one to examine the black hole's horizon in the context of black hole holography, in a similar way as the Kerr/CFT correspondence, and examine its thermodynamic properties. It would be interesting to also examine the Unruh temperature in the presence of such a black hole and how it relates to the CFT that the black hole is dual to. The natural first step then is to calculate the Unruh temperature for the rotating Rindler-AdS black dot case before promoting it to a black hole.

Chapter 2 contains reviews of existing studies of Anti de-Sitter spacetimes, quantum field theory of a scalar field in flat spacetime as well as in curved spacetime, the Bogoliubov transformation, and the Unruh effect in three-dimensional Rindler spacetime. Chapter 3 contains the derivations of the Unruh temperature in Rindler-AdS. In particular, section 3.1 reviews the massless Klein-Gordon field in Poincaré AdS which has been previously studied in references [25, 26], while sections 3.2 and 3.3 are my original contributions containing my calculations of the massless Klein-Gordon field and the Unruh temperature in Rindler-AdS, respectively. Chapter 4 contains my calculations for the massless KG field in rotating Rindler-AdS with normalization of its frequency modes and the Unruh temperature in rotating Rindler-AdS. Chapter 5 presents the summary of this thesis and related research possibilities. The appendix contains the units and conventions used in this thesis.

2 Theoretical Background

2.1 Anti de-Sitter Spacetime Review

The theory of General Relativity prescribes the relation between matter/energy and spacetime curvature as well as the behaviour of free-falling classical objects in spacetime. The latter is described by the geodesic equations while the latter is described by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.1}$$

where $g_{\mu\nu}$ is of course the metric, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci curvature scalar, Λ is the cosmological constant, κ is the Einstein's gravitational constant, and $T_{\mu\nu}$ is the energy-momentum tensor.

The Einstein field equations have three maximally symmetric vacuum solutions where $T_{\mu\nu} = 0$, namely the Minkowski spacetime with zero cosmological constant $\Lambda = 0$, de-Sitter (dS) spacetime with positive cosmological constant $\Lambda > 0$, and Anti de-Sitter (AdS) spacetime with negative cosmological constant $\Lambda < 0$. The AdS spacetime is of interest in the research of quantum gravity in terms of the AdS/CFT correspondence. The AdS/CFT correspondence is a bulk/boundary correspondence between the quantum gravity theory in the bulk AdS spacetime and conformal field theory, with no gravity, on the AdS boundary. The AdS/CFT correspondence is not in the scope of this thesis, however, it provides a motivation to study AdS spacetime further as there might be insights that can be obtained and used to develop the theory of quantum gravity in our Universe.

An n-dimensional AdS spacetime can be embedded in an $n+1$-dimensional Minkowski metric

$$\mathrm{d}s^2 = -\,\mathrm{d}{X_0}^2 + \sum_{i=1}^{n-1} \mathrm{d}{X_i}^2 - \mathrm{d}{X_n}^2\,, \tag{2.2}$$

which has two timelike coordinates X_0 and X_n. In this embedding, the AdS spacetime is the hyperboloid

$$-X_0^2+\sum_{i=1}^{n-1}X_i^2-X_n^2=-L^2,\quad L^2=-\frac{(n-1)(n-2)}{2\Lambda}. \tag{2.3}$$

For the case $n=3$, the 3-dimensional AdS is the hyperboloid

$$\begin{cases}-X_0^2+X_1^2+X_2^2-X_3^2=-L^2, & L^2=-\dfrac{1}{\Lambda},\\ \mathrm{d}s^2=-\,\mathrm{d}{X_0}^2+\mathrm{d}{X_1}^2+\mathrm{d}{X_2}^2-\mathrm{d}{X_3}^2\,. \end{cases} \tag{2.4}$$

There are different coordinates that can cover such a hyperboloid. Three coordinates of interest are the Poincaré AdS, Rindler-AdS, and rotating Rindler-AdS. Regardless of the choice of coordinates, the Ricci scalar R and the Kretchmann scalar K respectively are

$$R=-\frac{6}{L^2}=6\Lambda, \tag{2.5}$$

and

$$K=R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}=\frac{12}{L^4}=12\Lambda^2. \tag{2.6}$$

Both of these curvature scalars suggest that the AdS spacetime does not have any curvature singularity, i.e., points where the curvature scalars diverge to infinity.

2.1.1 Poincaré AdS

The Poincaré AdS coordinates are defined by [20]

$$\begin{cases} X_0=L\dfrac{t}{x},\\ X_1=L\dfrac{y}{x},\\ X_2=\dfrac{1}{2x}\left[L^2-\left(-t^2+x^2+y^2\right)\right],\\ X_3=\dfrac{1}{2x}\left[L^2+\left(-t^2+x^2+y^2\right)\right]. \end{cases} \tag{2.7}$$

The metric in this choice of coordinates is

$$\mathrm{d}s^2=\frac{L^2}{x^2}\left[-\,\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2\right],\quad -\infty<t<\infty,\ 0<x<\infty,\ -\infty<y<\infty, \tag{2.8}$$

which is *conformally flat*, meaning it is related to the flat Minkowski metric by a conformal transformation

$$g_{\mu\nu}=\Omega^2\eta_{\mu\nu}, \tag{2.9}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and Ω is a positive and smooth function of the coordinates, called the conformal factor. In this case, the conformal factor is $\Omega = L/x$. In the discussion of the Unruh effect in 3D Rindler-AdS and rotating Rindler-AdS, the Poincaré AdS will be analogous to the Minkowski metric in the 3D Rindler-Minkowski Unruh effect discussion in section 2.3. This conformal relation with the Minkowski metric motivates the choice of Poincaré AdS to fit that role.

In addition to $x \to \pm\infty$, the Poincaré AdS metric has a coordinate singularity $x = 0$. In studying the Poincare AdS, only the $x > 0$ patch is usually considered which is only half of AdS. The other half of AdS is covered by the $x < 0$ patch. The reason being the regions $x > 0$ and $x < 0$ are causally disconnected, i.e., using the geodesic equations of General Relativity, a test particle starting in one region will not cross $x = 0$ into the other region. This can be confirmed by solving the geodesic equations

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \tag{2.10}$$

where τ is the object's proper time and $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbols, the non-zero components of which are

$$\Gamma^t_{tx} = \Gamma^t_{xt} = \Gamma^x_{xx} = -\Gamma^x_{yy} = \Gamma^x_{tt} = \Gamma^y_{xy} = \Gamma^y_{yx} = -\frac{1}{x}. \tag{2.11}$$

The system of differential equations that arises from this is

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = \frac{2}{x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{2.12}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = \frac{1}{x}\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2\right], \tag{2.13}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{2}{x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau}, \tag{2.14}$$

the solution of which is

$$t(\tau) = C_1 - \frac{2C_2\cos(C_4\tau)}{\sin(C_4\tau) - \sin(C_4(\tau + 2C_5))}, \tag{2.15}$$

$$x(\tau) = \frac{\sqrt{2}C_3}{\sqrt{1 + \cos(2C_4(\tau + C_5))}}, \tag{2.16}$$

$$y(\tau) = C_6 \pm \frac{\sin(2C_4(\tau + C_5))\sqrt{2(C_2^2 - C_3^2)}}{\sqrt{3 + \cos(4C_4(\tau + C_5)) + 4\cos(2C_4(\tau + C_5))}}, \tag{2.17}$$

where $C_1,\ C_2,\ C_3,\ C_4,\ C_5$ and C_6 are arbitrary constants. Although complicated, it can be seen that whether $x > 0$ or $x < 0$ is solely dictated by the constant C_3 which depends on the starting condition and whether or not the particle in question is massive. Hence, $x(\tau)$ will not change its sign along the geodesics. With that said, for the Unruh effect discussion in this thesis, only the $x > 0$ patch will be considered as indicated in (2.8).

For completeness, the non-zero components of the Ricci tensor $R_{\mu\nu}$ of the Poincaré AdS spacetime are

$$R_{tt} = R_{xx} = R_{yy} = -\frac{2}{x^2}. \tag{2.18}$$

As stated above, the singularity at $x = 0$ is a coordinate singularity as the Ricci curvature scalar (2.5) and the Kretchmann scalar (2.6) are not singular at that point. Any curvature singularities, such as the singularity at the center of the Schwarzschild black hole, would translate into singularities in the Ricci scalar and the Kretchmann scalar.

2.1.2 Rindler-AdS

The Rindler-AdS metric can be obtained by the choice of coordinates

$$\begin{cases} X_0 = \xi \sinh\left(\frac{\tau}{L}\right), \\ X_1 = \xi \cosh\left(\frac{\tau}{L}\right), \\ X_2 = \sqrt{L^2+\xi^2}\sinh\left(\frac{\chi}{L}\right), \\ X_3 = \sqrt{L^2+\xi^2}\cosh\left(\frac{\chi}{L}\right), \end{cases} \tag{2.19}$$

yielding

$$\mathrm{d}s^2 = -\left(\frac{\xi}{L}\right)^2 \mathrm{d}\tau^2 + \frac{\mathrm{d}\xi^2}{1+\frac{\xi^2}{L^2}} + \left(1+\frac{\xi^2}{L^2}\right)\mathrm{d}\chi^2\,, \tag{2.20}$$

with $-\infty < (\tau, \chi) < \infty$ and $\xi > 0$. In the limit of vanishing cosmological constant $\Lambda \to 0$ or $\xi/L \ll 1$, the metric (2.20) reduces to flat Rindler metric

$$\mathrm{d}s^2 = -\left(\frac{\xi}{L}\right)^2 \mathrm{d}\tau^2 + \mathrm{d}\xi^2 + \mathrm{d}\chi^2\,. \tag{2.21}$$

For the discussion of the Unruh effect in Rindler-AdS later on, it is convenient to transform the coordinates with $\xi \to Le^{\xi/L}$, changing the Rindler-AdS metric to

$$\mathrm{d}s^2 = -e^{\frac{2\xi}{L}}\,\mathrm{d}\tau^2 + \frac{e^{\frac{2\xi}{L}}\,\mathrm{d}\xi^2}{1+e^{\frac{2\xi}{L}}} + \left(1+e^{\frac{2\xi}{L}}\right)\mathrm{d}\chi^2\,, \quad -\infty < (\tau, \xi, \chi) < \infty. \tag{2.22}$$

The coordinate transformation from the Poincaré AdS to Rindler-AdS is

$$\begin{cases} x = \dfrac{Le^{-\chi/L}}{\sqrt{1+e^{\frac{2\xi}{L}}}}, \\ y = \dfrac{Le^{-\chi/L}}{\sqrt{1+e^{\frac{2\xi}{L}}}}\, e^{\xi/L} \cosh\left(\dfrac{\tau}{L}\right) = x\, e^{\xi/L} \cosh\left(\dfrac{\tau}{L}\right), \\ t = \dfrac{Le^{-\chi/L}}{\sqrt{1+e^{\frac{2\xi}{L}}}}\, e^{\xi/L} \sinh\left(\dfrac{\tau}{L}\right) = x\, e^{\xi/L} \sinh\left(\dfrac{\tau}{L}\right). \end{cases} \tag{2.23}$$

It can be calculated from this relationship between Poincaré Ads and Rindler-AdS that an observer at constant ξ and χ is an accelerating observer in Poincaré AdS. To show that, let us examine an observer at $\xi = \chi = 0$ and calculate the 4-velocity u^μ of the observer as

$$u^\mu = [u^t, u^x, u^y] \equiv \left[\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}\right] = \frac{1}{\sqrt{2}}\left[\cosh\left(\frac{\tau}{L}\right), 0, \sinh\left(\frac{\tau}{L}\right)\right]. \tag{2.24}$$

Using the Poincaré AdS metric, the square-magnitude of the 4-velocity vector can be calculated as

$$u^\mu u_\mu = \frac{L^2}{x^2}\left[-\left(u^t\right)^2 + (u^y)^2 + (u^y)^2\right] = \frac{L^2}{L^2/2}\frac{1}{2}\left[-\cosh^2\left(\frac{\tau}{L}\right) + \sinh^2\left(\frac{\tau}{L}\right)\right] = -1, \tag{2.25}$$

which means the trajectory $\xi = \chi = 0$ in Poincaré AdS can be interpreted as a timelike worldline of an observer with τ being the observer's proper time. The proper acceleration of the observer can be found with the covariant derivative of the 4-velocity with respect to the proper time, i.e.,

$$a^\mu \equiv \frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta, \tag{2.26}$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols which have non-zero components of

$$\begin{cases} \Gamma^\tau_{\tau\xi} = \Gamma^\tau_{\xi\tau} = \dfrac{1}{L}, \\ \Gamma^\xi_{\tau\tau} = \dfrac{1+e^{\frac{2\xi}{L}}}{L}, \\ \Gamma^\xi_{\xi\xi} = \dfrac{1}{L\left(1+e^{\frac{2\xi}{L}}\right)}, \\ \Gamma^\xi_{\chi\chi} = -\dfrac{1+e^{\frac{2\xi}{L}}}{L}, \\ \Gamma^\chi_{\xi\chi} = \Gamma^\chi_{\chi\xi} = \dfrac{e^{\frac{2\xi}{L}}}{L\left(1+e^{\frac{2\xi}{L}}\right)}. \end{cases} \tag{2.27}$$

Carrying out the proper time derivative and the index summation with the Christoffel symbols (2.11) yields the proper acceleration

$$a^\mu = \left[a^t, a^x, a^y\right] = \frac{1}{L\sqrt{2}}\left[\sinh\left(\frac{\tau}{L}\right), -1, \cosh\left(\frac{\tau}{L}\right)\right], \tag{2.28}$$

which implies the square-magnitude of the proper acceleration is

$$a^2 = a^\mu a_\mu = \frac{L^2}{x^2}\left[-\left(a^t\right)^2 + (a^y)^2 + (a^y)^2\right] = \frac{L^2}{L^2/2}\frac{1}{2L^2}\left[-\sinh^2\left(\frac{\tau}{L}\right) + 1 + \cosh^2\left(\frac{\tau}{L}\right)\right] = \frac{2}{L^2}. \tag{2.29}$$

It is interesting to note that accelerating observers in AdS are also accelerating observers in the embedding Minkowski space [15].

The Rindler-AdS is only a portion of the Poincaré AdS, right wedge ($|t| < y$) region, due to the Rindler horizon at $t = y$ just as the flat Rindler is only a portion of Minkowski. The left wedge ($|t| < -y$) region is a copy of Rindler-AdS with the transformation

$$\begin{cases} x = \dfrac{Le^{-\bar{\chi}/L}}{\sqrt{1+e^{\frac{2\bar{\xi}}{L}}}}, \\ y = -\dfrac{Le^{-\bar{\chi}/L}}{\sqrt{1+e^{\frac{2\bar{\xi}}{L}}}}\, e^{\bar{\xi}/L}\cosh\left(\dfrac{\bar{\tau}}{L}\right) = -x\, e^{\bar{\xi}/L}\cosh\left(\dfrac{\bar{\tau}}{L}\right), \\ t = \dfrac{Le^{-\bar{\chi}/L}}{\sqrt{1+e^{\frac{2\bar{\xi}}{L}}}}\, e^{\bar{\xi}/L}\sinh\left(\dfrac{\bar{\tau}}{L}\right) = x\, e^{\bar{\xi}/L}\sinh\left(\dfrac{\bar{\tau}}{L}\right). \end{cases} \tag{2.30}$$

In flat Rindler spacetime, the Rindler horizon causes the Unruh effect to emerge. It has been established in [21] that the Unruh effect also emerges in Rindler-AdS. Fixing the spatial coordinates $\chi =$ const. and $\xi =$ const. describes hyperbolic trajectories in the $t - y$ plane, which can be seen in Figure 2.1, akin to the trajectories of accelerating Rindler observers in the Minkowski spacetime. These hyperbolic trajectories are therefore in agreement with the physical interpretation that an observer at constant ξ and χ is an accelerating observer in Poincaré AdS. The notable difference is that the x coordinate of each trajectory is determined by equation (2.23) rather than being arbitrary as in the Rindler-Minkowski case. For

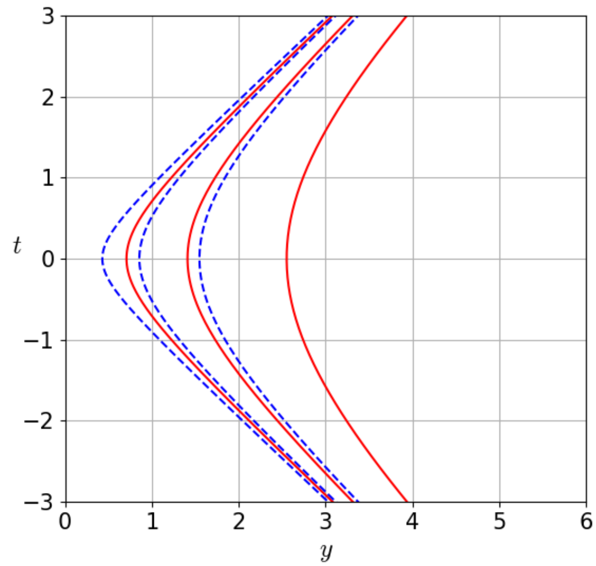


Figure 2.1: Hyperbolic trajectories in the $t-y$ plane Poincaré AdS for constant values of $\xi = (0, 1, 2)$ (left to right) at $\chi = 0$ (red line) and $\chi = 0.5$ (dashed blue line) with $L = 1$. Note that different values of ξ and χ have different constant x coordinates which are suppressed in this plot.

completeness, the non-zero components of the Ricci tensor $R_{\mu\nu}$ are

$$\begin{cases} R_{\tau\tau} = \dfrac{2e^{\frac{2\xi}{L}}}{L^2}, \\ R_{\xi\xi} = -\dfrac{2}{L^2}\dfrac{e^{\frac{2\xi}{L}}}{1+e^{\frac{2\xi}{L}}}, \\ R_{\chi\chi} = -\dfrac{2}{L^2}\left(1+e^{\frac{2\xi}{L}}\right). \end{cases} \tag{2.31}$$

2.1.3 Rotating Rindler-AdS Spacetime

The rotating Rindler-AdS metric can be obtained by the choice of coordinates

$$\begin{cases} X_0 = \xi \sinh\left(\dfrac{\tau - \beta\chi}{L}\right), \\ X_1 = \xi \cosh\left(\dfrac{\tau - \beta\chi}{L}\right), \\ X_2 = \sqrt{L^2+\xi^2}\sinh\left(\dfrac{\chi - \beta\tau}{L}\right), \\ X_3 = \sqrt{L^2+\xi^2}\cosh\left(\dfrac{\chi - \beta\tau}{L}\right), \end{cases} \tag{2.32}$$

yielding

$$\mathrm{d}s^2 = -\left[(\xi/L)^2\left(1-\beta^2\right)-\beta^2\right]\mathrm{d}\tau^2 - 2\beta\,\mathrm{d}\tau\,\mathrm{d}\chi + \frac{\mathrm{d}\xi^2}{1+(\xi/L)^2} + \left[1+(\xi/L)^2\left(1-\beta^2\right)\right]\mathrm{d}\chi^2 \tag{2.33}$$

where β is the rotation parameter with $-1 \leq \beta \leq 1$, $-\infty < (\tau, \chi) < \infty$, and $\xi > 0$. It can be seen that for $\beta = 0$, the metric reduces back to the non-rotating Rindler AdS metric (2.20). Also, the portion of the spacetime that rotating Rindler-AdS covers is identical to the non-rotating case and they can be related, from the non-rotating to rotating case, with the transformation

$$\tau \to \tau - \beta\chi, \qquad \chi \to \chi - \beta\tau. \tag{2.34}$$

This rotating Rindler-AdS spacetime was obtained in reference [19] by considering the possibilities of stationary vacuum states in AdS spacetime. It was found, in the aforementioned reference, the rotating Rindler-AdS can have a stationary vacuum state only if it is three-dimensional. It is also worth noting for $\beta \neq 0$, the metric becomes singular in the limit of $\Lambda \to 0$ or $\xi/L \ll 1$, which means there is no rotating Rindler metric in flat space. Just as for non-rotating Rindler-AdS, it is convenient to transform the coordinates further with $\xi \to Le^{\xi/L}$, changing the metric to

$$\mathrm{d}s^2 = -\left[e^{\frac{2\xi}{L}}\left(1-\beta^2\right)-\beta^2\right]\mathrm{d}\tau^2 - 2\beta\,\mathrm{d}\tau\,\mathrm{d}\chi + \frac{e^{\frac{2\xi}{L}}\,\mathrm{d}\xi^2}{1+e^{\frac{2\xi}{L}}} + \left[1+e^{\frac{2\xi}{L}}\left(1-\beta^2\right)\right]\mathrm{d}\chi^2\,, \tag{2.35}$$

with $-\infty < (\tau, \xi, \chi) < \infty$. The coordinate transformation from the Poincaré AdS to the rotating Rindler-AdS is

$$\begin{cases} x = \dfrac{L}{\sqrt{1+e^{\frac{2\xi}{L}}}}\exp\left(-\dfrac{\chi-\beta\tau}{L}\right), \\ y = x\,e^{\xi/L}\cosh\left(\dfrac{\tau-\beta\chi}{L}\right), \\ t = x\,e^{\xi/L}\sinh\left(\dfrac{\tau-\beta\chi}{L}\right). \end{cases} \tag{2.36}$$

An observer at $\xi = \chi = 0$ has a proper acceleration in Poincaré AdS

$$a^2 = a^\mu a_\mu = \frac{2}{L^2}\left(\frac{1-\beta^2}{1-2\beta^2}\right)^2, \tag{2.37}$$

which can be calculated in a similar manner as the proper acceleration in the non-rotating case. The non-zero components of the Christoffel symbols here are

$$\begin{cases} \Gamma^{\tau}_{\tau\xi} = \Gamma^{\tau}_{\xi\tau} = \dfrac{1 + e^{\frac{2\xi}{L}}(1-\beta^2)}{L\left(1 + e^{\frac{2\xi}{L}}\right)(1-\beta^2)}, \\ \Gamma^{\tau}_{\xi\chi} = \Gamma^{\tau}_{\chi\xi} = -\Gamma^{\chi}_{\tau\xi} = -\Gamma^{\chi}_{\xi\tau} = -\dfrac{\beta}{L\left(1 + e^{\frac{2\xi}{L}}\right)(1-\beta^2)}, \\ \Gamma^{\chi}_{\xi\chi} = \Gamma^{\chi}_{\chi\xi} = \dfrac{e^{\frac{2\xi}{L}}(1-\beta^2) - \beta^2}{L\left(1 + e^{\frac{2\xi}{L}}\right)(1-\beta^2)}, \\ \Gamma^{\xi}_{\tau\tau} = -\Gamma^{\xi}_{\chi\chi} = \dfrac{\left(1 + e^{\frac{2\xi}{L}}\right)(1-\beta^2)}{L}, \\ \Gamma^{\xi}_{\xi\xi} = \dfrac{1}{L\left(1 + e^{\frac{2\xi}{L}}\right)}, \end{cases} \tag{2.38}$$

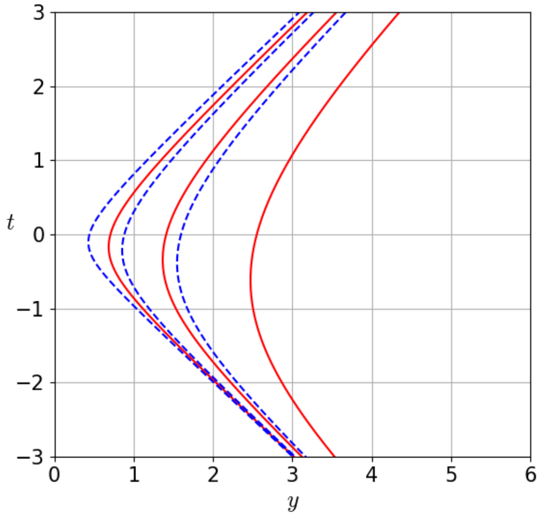
which reduces back to the Christoffel symbols of the non-rotating Rindler-AdS (2.27) if $\beta = 0$. Fixing the spatial coordinates $\chi = \chi_0 = \text{const.}$ and $\xi = \xi_0 = \text{const.}$ describes the trajectories, which can be seen in Figure 2.2, similar to the hyperbolic trajectories of accelerating Rindler observers in the Minkowski spacetime. It is worth noting that for the case of $\beta \neq 0$, the x coordinates of each trajectories are no longer constant as can be seen in Figure 2.2(b). For completeness, the non-zero components of the Ricci tensor $R_{\mu\nu}$ are

$$\begin{cases} R_{\tau\tau} = \dfrac{2}{L^2}\left[e^{\frac{2\xi}{L}}(1-\beta^2) - \beta^2\right], \\ R_{\tau\chi} = R_{\chi\tau} = \dfrac{2\beta}{L^2}, \\ R_{\xi\xi} = -\dfrac{2}{L^2}\dfrac{e^{\frac{2\xi}{L}}}{1 + e^{\frac{2\xi}{L}}}, \\ R_{\chi\chi} = -\dfrac{2}{L^2}\left[1 + e^{\frac{2\xi}{L}}(1-\beta^2)\right], \end{cases} \tag{2.39}$$

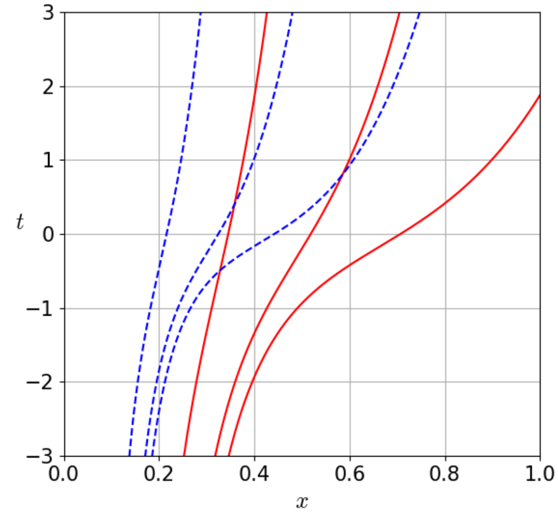
which reduces back to the Ricci tensor of the non-rotating Rindler-AdS (2.31) if $\beta = 0$.

2.2 Quantum Field Theory Review

Quantum field theory (QFT) is the framework of the Standard Model of particle physics that models all known elementary particles, electromagnetism, and nuclear interactions. In



(a) Plot of the $t - y$ coordinates



(b) Plot of the $t - x$ coordinates

Figure 2.2: Trajectories in Poincaré adS for constant values of $\xi = (0, 1, 2)$, left to right for (a) and right to left in (b), at $\chi = 0$ (red lines) and $\chi = 0.5$ (dashed blue lines) with $L = 1$ and $\beta = 0.25$. Note that the x coordinates of these rotating Rindler-AdS observers are no longer constants, which can be seen in (b).

quantum mechanics, quantization is performed by promoting classical observables such as the position and momentum of a particle into an operator. In QFT, the field is the classical observable and it is promoted to field operators in quantizing it. This section reviews QFT for a scalar field in flat spacetime [22] as well as in curved spacetime [2] to study the effects of spacetime curvature on quantum fields. The spacetime coordinates are denoted as $x = \{x^0, \mathbf{x}\}$ where x^0 is the time coordinate and $\mathbf{x}$ is the spatial coordinates.

2.2.1 Canonical Quantization and Scalar Field in Flat Spacetime

To illustrate the canonical quantization of fields, we begin with a classical real scalar field ϕ in Minkowski spacetime with metric $\eta_{\mu\nu} = \mathrm{Diag}(-1, 1, 1, 1)$. In the formalism of classical field theory, the fundamental quantity of the field ϕ is the action

$$S = \int \mathrm{d}^4x \, \mathscr{L}(\phi, \partial_\mu \phi), \tag{2.40}$$

where the integration d^4x is the integral over a spacetime volume and $\partial_\mu = \frac{\partial}{\partial x^\mu}$, with the Lagrangian

$$\mathscr{L}(\phi, \partial_\mu\phi) = -\frac{1}{2}\big[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\big]. \tag{2.41}$$

The Hamiltonian of the field is obtained from the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\eta_{\mu\nu}, \tag{2.42}$$

yielding

$$H = \int \mathrm{d}^3\mathbf{x}\, T_{00} = \int \mathrm{d}^3\mathbf{x}\, \frac{1}{2}\left[\dot{\phi}^2 + \sum_{i=1}^{3}(\partial_i\phi)^2 + m^2\phi^2\right]. \tag{2.43}$$

The principle of least action, with the restriction that the variation of the field $\delta\phi$ vanish at the boundary of the spacetime volume, yields the Euler-Lagrange equation for field theory

$$\begin{aligned} \delta S = \int \mathrm{d}^4x \left[\frac{\partial\mathscr{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\right] = 0, \\ \Longrightarrow \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathscr{L}}{\partial\phi} = 0. \end{aligned} \tag{2.44}$$

If the Lagrangian contains more than one field, there is one Euler-Lagrange equation for each field. The Euler-Lagrange (2.44) yields the Klein-Gordon (KG) field equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = 0, \tag{2.45}$$

and the solution can be expanded using the Fourier transform as

$$\phi(x) = \int \mathrm{d}^3\mathbf{k}\left[a_{\mathbf{k}}u_{\mathbf{k}} + a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}}\right], \tag{2.46}$$

where $u_{\mathbf{k}} = [2\omega_{\mathbf{k}}(2\pi)^3]^{-\frac{1}{2}}e^{-i\omega x^0 + i\mathbf{k}\cdot\mathbf{x}}$ is the positive-frequency modes[1] and $\omega_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$. The conjugate momentum of the field ϕ is defined as

$$\pi(x) \equiv \frac{\partial\mathscr{L}}{\partial\dot{\phi}(x)} = \dot{\phi}(x), \tag{2.47}$$

where $\dot{\phi} = \partial_0\phi$ is the partial derivative of the field with respect to the time coordinate. In the canonical quantization procedure, we interpret the field $\phi(x)$ and its conjugate momentum

[1] The factors preceding the exponent term is chosen for normalization and for later convenience in the commutation relations of the Fourier coefficients $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$.

$\pi(x)$ as operators obeying the equal-time commutation relations

$$\begin{aligned} [\phi(t,\mathbf{x}),\pi(t,\mathbf{y})] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \\ [\phi(t,\mathbf{x}),\phi(t,\mathbf{y})] &= [\pi(t,\mathbf{x}),\pi(t,\mathbf{y})] = 0. \end{aligned} \tag{2.48}$$

Therefore, the Fourier coefficients $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$ also need to be interpreted as operators with commutation relations

$$\begin{aligned} \left[a_\mathbf{k}, a_{\mathbf{k}'}^\dagger\right] &= \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \\ [a_\mathbf{k}, a_{\mathbf{k}'}] &= \left[a_\mathbf{k}^\dagger, a_{\mathbf{k}'}^\dagger\right] = 0. \end{aligned} \tag{2.49}$$

The Hamiltonian then, after normal ordering to remove problematic divergence, can be written in terms of these Fourier coefficients as [22, 2]

$$H = \int \mathrm{d}^3\mathbf{k}\; a_\mathbf{k}^\dagger a_\mathbf{k}\, \omega_\mathbf{k}. \tag{2.50}$$

Analogous to the quantum mechanics of the simple harmonic oscillator, the operators $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$ are respectively interpreted as the annihilation and creation operators. The no-particle or vacuum eigenstate $|0\rangle$ is defined such that

$$a_\mathbf{k}\,|0\rangle = 0 \quad \text{for all } \mathbf{k}, \tag{2.51}$$

and all the other orthonormal eigenstates of the Hamiltonian can be built by acting on $|0\rangle$ with the creation operator repeatedly

$$\left|n_{1\mathbf{k}_1}, n_{2\mathbf{k}_2}, \cdots, n_{j\mathbf{k}_j}\right\rangle = \frac{1}{\sqrt{n_1!n_2!\cdots n_j!}}(a_{\mathbf{k}_1}^\dagger)^{n_1}(a_{\mathbf{k}_2}^\dagger)^{n_2}\cdots(a_{\mathbf{k}_j}^\dagger)^{n_j}\,|0\rangle\,. \tag{2.52}$$

2.2.2 Scalar Klein-Gordon Field in Curved Spacetime

For a scalar Klein-Gordon (KG) field ϕ in spacetime with metric $g_{\mu\nu}$ and line-element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{2.53}$$

the Lagrangian is

$$\mathscr{L} = -\frac{1}{2}\sqrt{|g|}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right], \tag{2.54}$$

where g is the determinant of the metric and m is the mass of the field quanta. The Euler-Lagrange equation (2.44) for ϕ yields the KG equation

$$\partial_\mu\Big(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\Big) + \sqrt{|g|}m^2\phi = \sqrt{|g|}\left(\nabla_\mu\nabla^\mu\phi + m^2\phi\right) = 0. \tag{2.55}$$

If the metric is independent of the time coordinate x^0, the solution of the KG equation (2.55) can be expanded in terms of the frequency modes as

$$\phi(x) = \int \mathrm{d}\omega \left[a_\omega f_\omega(x) + a_\omega^\dagger f_\omega^*(x)\right], \tag{2.56}$$

where $f_\omega(x) \propto e^{-i\omega x^0}$. The expansion (2.56) can be seen as the linear combination of a set of basis $f_\omega(x)$, exactly the same as the notion of a set of basis in linear algebra. Furthermore, the inner product of $f_\omega(x)$ or any given solutions to the KG equation can be defined by utilizing the global phase symmetry of the KG equation.

The Lagrangian (2.54), hence also the action, is invariant under a global phase transformation $\phi \to e^{i\alpha}\phi$. Therefore, given A and B are complex solutions to the KG equation and through Noether's theorem, one can find the conserved current

$$J^\mu_{(A,B)} = -i[A^*(\nabla^\mu B) - (\nabla^\mu A^*)B] \tag{2.57}$$

associated with this symmetry and these complex solutions A and B, which satisfies

$$\nabla_\mu J^\mu_{(A,B)} = 0. \tag{2.58}$$

Through Stokes' theorem, one can define the KG inner product

$$\begin{aligned} \langle A, B\rangle &\equiv \int \mathrm{d}\mathbf{x}\, \sqrt{|g|} J^0_{(A,B)} \\ &= -i\int \mathrm{d}\mathbf{x}\, \sqrt{|g|} g^{0\mu}[A^*(\nabla_\mu B) - (\nabla_\mu A^*)B], \end{aligned} \tag{2.59}$$

which is a conserved quantity, i.e., independent of the time coordinate x^0. The complex conjugate of the KG inner product satisfies the property

$$\langle A, B\rangle^* = -\langle A^*, B^*\rangle = \langle B, A\rangle. \tag{2.60}$$

The KG inner product enables us to define the notion of orthonormality for the field modes. For example, the frequency modes $u_\mathbf{k}$ and $u_\mathbf{k}^*$ in Minkowski spacetime are orthonormal in this KG inner product, i.e.,

$$\left.\begin{aligned} \langle u_\mathbf{k}, u_{\mathbf{k}'}\rangle &= \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ \langle u_\mathbf{k}^*, u_{\mathbf{k}'}^*\rangle &= -\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ \langle u_\mathbf{k}, u_{\mathbf{k}'}^*\rangle &= \langle u_\mathbf{k}^*, u_{\mathbf{k}'}\rangle = 0, \end{aligned}\right\} \tag{2.61}$$

where δ is the Dirac delta function. The idea is that these orthonormal modes can be seen as a complete orthonormal basis for the field $\phi(x)$.

Suppose there are two observers in the same underlying spacetime and field ϕ. Each of the observers has different coordinate systems and hence expands the field ϕ in terms of their separate set of orthonormal frequency modes, say u_i and $\bar{u}_j$. Both u_i and $\bar{u}_j$ are complete sets of orthonormal basis, so the relations between the modes u_i and $\bar{u}_j$ can be seen as a change of basis transformation. This change of basis is known as the Bogoliubov transformation.

2.2.3 Bogoliubov Transformation

Suppose, in the same underlying spacetime, there are two observers, say observer $\mathcal{O}$ and $\bar{\mathcal{O}}$. Each of these two observers has their own preferred spacetime coordinates, say x and $\bar{x}$, and there exist coordinate transformations between the two coordinates. The line-element can be written in terms of the x coordinates and the $\bar{x}$ coordinates as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \bar{g}_{\mu\nu}\,\mathrm{d}\bar{x}^\mu\,\mathrm{d}\bar{x}^\nu\,. \tag{2.62}$$

The KG equation (2.55) can be then solved for each set of coordinates. Observer $\mathcal{O}$ and $\bar{\mathcal{O}}$ expands the field ϕ in terms of their separate frequency modes

$$\phi = \int \mathrm{d}\omega\,\left[a_\omega f_\omega(x) + a_\omega^\dagger f_\omega^*(x)\right] = \int \mathrm{d}\bar{\omega}\,\left[b_{\bar{\omega}} g_{\bar{\omega}}(\bar{x}) + b_{\bar{\omega}}^\dagger g_{\bar{\omega}}^*(\bar{x})\right], \tag{2.63}$$

where $f_\omega(x) \propto e^{-i\omega t}$ and $g_{\bar{\omega}}(\bar{x}) \propto e^{-i\bar{\omega}\bar{t}}$ are respectively the positive-frequency modes for $\mathcal{O}$ and $\bar{\mathcal{O}}$, while a_ω and $b_{\bar{\omega}}$ are respectively the annihilation operator for $\mathcal{O}$ and $\bar{\mathcal{O}}$. Since both f_ω and $g_{\bar{\omega}}$ are complete sets of orthonormal basis of the field operator ϕ, i.e.,

$$\begin{aligned}
\langle f_\omega, f_{\omega'}\rangle &= \delta(\omega - \omega'),\\
\langle f_\omega^*, f_{\omega'}^*\rangle &= -\delta(\omega - \omega'),\\
\langle f_\omega, f_{\omega'}^*\rangle &= 0,
\end{aligned} \tag{2.64}$$

and

$$\begin{aligned}
\langle g_{\bar{\omega}}, g_{\bar{\omega}'}\rangle &= \delta(\bar{\omega} - \bar{\omega}'),\\
\langle g_{\bar{\omega}}^*, g_{\bar{\omega}'}^*\rangle &= -\delta(\bar{\omega} - \bar{\omega}'),\\
\langle g_{\bar{\omega}}, g_{\bar{\omega}'}^*\rangle &= 0,
\end{aligned} \tag{2.65}$$

the frequency modes $g_{\bar{\omega}}$ can be expanded in terms of f_ω as

$$g_{\bar{\omega}}(\bar{x}) = \int \mathrm{d}\omega \, [\alpha_{\omega\bar{\omega}} \, f_\omega(x) + \beta_{\omega\bar{\omega}} \, f^*_\omega(x)], \tag{2.66}$$

where $\alpha_{\omega\bar{\omega}}$ and $\beta_{\omega\bar{\omega}}$ are called Bogoliubov coefficients and can be evaluated using the KG inner product as

$$\begin{aligned} \langle f_\omega, g_{\bar{\omega}} \rangle &= \alpha_{\omega\bar{\omega}}, \\ \langle f^*_\omega, g_{\bar{\omega}} \rangle &= -\beta_{\omega\bar{\omega}}. \end{aligned} \tag{2.67}$$

Conversely, since

$$\begin{aligned} \langle g_{\bar{\omega}}, f_\omega \rangle &= \langle f_\omega, g_{\bar{\omega}} \rangle^* = \alpha^*_{\omega\bar{\omega}}, \\ \langle g^*_{\bar{\omega}}, f_\omega \rangle &= - \langle f^*_\omega, g_{\bar{\omega}} \rangle = \beta_{\omega\bar{\omega}}, \end{aligned} \tag{2.68}$$

it can be deduced that f_ω can be expanded in terms of $g_{\bar{\omega}}$ as

$$f_\omega(x) = \int \mathrm{d}\bar{\omega} \, [\alpha^*_{\omega\bar{\omega}} \, g_{\bar{\omega}}(\bar{x}) - \beta_{\omega\bar{\omega}} \, g^*_{\bar{\omega}}(\bar{x})]. \tag{2.69}$$

From the equality in (2.63) and making use of the Bogoliubov transformations (2.66), (2.69), and the orthonormality of the frequency modes, one can obtain the relations between the annihilation operators a_ω and $b_{\bar{\omega}}$ as

$$a_\omega = \int \mathrm{d}\bar{\omega} \left[\alpha_{\omega\bar{\omega}} b_{\bar{\omega}} + \beta^*_{\omega\bar{\omega}} b^\dagger_{\bar{\omega}} \right], \tag{2.70}$$

and

$$b_{\bar{\omega}} = \int \mathrm{d}\omega \, [\alpha^*_{\omega\bar{\omega}} a_\omega - \beta^*_{\omega\bar{\omega}} a^\dagger_\omega]. \tag{2.71}$$

These relations allow us to examine the vacuum state of the f_ω modes, $|0\rangle$, in the $g_{\bar{\omega}}$ mode. From (2.71), it can be immediately seen that the vacuum state of f_ω modes, unless in a special case where $\beta_{\omega\bar{\omega}} = 0$, will not be annihilated by $b_{\bar{\omega}}$:

$$b_{\bar{\omega}} \, |0\rangle = \int \mathrm{d}\omega \, \beta_{\omega\bar{\omega}} a^\dagger_\omega \, |0\rangle \, . \tag{2.72}$$

Therefore, the expectation value of the operator $b^\dagger_{\bar{\omega}} b_{\bar{\omega}}$ for the number of $g_{\bar{\omega}}$-mode particles in the state $|0\rangle$ is

$$\langle 0 | b^\dagger_{\bar{\omega}} b_{\bar{\omega}} | 0 \rangle = \int \mathrm{d}\omega \, |\beta_{\omega\bar{\omega}}|^2, \tag{2.73}$$

which in some cases will be proportional to the Bose-Einstein distribution with a certain temperature for a bosonic field[2]. The Bogoliubov coefficients themselves have the following properties

$$\begin{aligned} \int \mathrm{d}\omega \, [\alpha_{\omega\bar{\omega}}\alpha^*_{\omega\bar{\omega}'} - \beta_{\omega\bar{\omega}}\beta^*_{\omega\bar{\omega}'}] &= \delta(\bar{\omega} - \bar{\omega}'), \\ \int \mathrm{d}\omega \, [\alpha_{\omega\bar{\omega}}\beta_{\omega\bar{\omega}'} - \beta_{\omega\bar{\omega}'}\alpha_{\omega\bar{\omega}'}] &= 0. \end{aligned} \tag{2.74}$$

2.3 The Unruh Effect in 3-Dimensional Rindler-Minkowski Spacetime

In the following discussion, the field examined is the massless KG field. A massless KG field is chosen for its simplicity but still able to demonstrate the field theory approach of the Unruh effect. The same approach for a 2-dimensional massless Dirac field, along with other discussions, was demonstrated in [23]. The dimensionality of 3 (1 time + 2 space) is chosen in anticipation of the discussion about the Unruh effect in rotating Rindler-AdS which is 3-dimensional. The discussion of the Unruh effect in this section is adapted from reference [9] which reviews the Unruh effect in 2-dimensional spacetime and reference [8], in one of the sections, discusses the Unruh effect in 4-dimensional spacetime.

Suppose there is an inertial observer in Minkowski spacetime with coordinates (t, x, y),

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \,, \tag{2.75}$$

and there is also an observer, called the Rindler observer, with constant proper acceleration a in the x-direction that sees the spacetime as the Rindler metric using Rindler coordinates (τ, ξ, y). The relationship between the Minkowski coordinates with the Rindler coordinates is given as

$$\begin{aligned} t &= \frac{e^{a\xi}}{a}\sinh(a\tau), \\ x &= \frac{e^{a\xi}}{a}\cosh(a\tau), \end{aligned} \tag{2.76}$$

[2] For a fermionic field, this expectation value would be proportional to the Fermi-Dirac distribution. However, fermionic fields are outside of the scope of this thesis.

whereby the Rindler metric can be found by applying these coordinates transformation to the Minkowski metric, yielding

$$\mathrm{d}s^2 = e^{2a\xi}\left[-\,\mathrm{d}\tau^2 + \mathrm{d}\xi^2\right] + \mathrm{d}y^2\,. \tag{2.77}$$

This transformation (2.76) only applies for the region $|t| < x$ called the right Rindler wedge and there is a Rindler coordinate singularity at $t = x = 0$ or $\xi \to -\infty$. A different transformation which yields the same Rindler metric for the region $|t| < -x$, called the left Rindler wedge, is given as

$$\begin{aligned} t &= \frac{e^{a\bar{\xi}}}{a}\sinh(a\bar{\tau}),\\ x &= -\frac{e^{a\bar{\xi}}}{a}\cosh(a\bar{\tau}). \end{aligned} \tag{2.78}$$

The relevance of this left Rindler wedge is that the field quantization in both right and left Rindler wedges combined is equivalent to field quantization on the whole Minkowski space. Meaning, the frequency modes of the fields on both wedges are complete on Minkowski space although, separately, neither is. From the Rindler observer's point of view, however, there is a Rindler horizon at $t = x$ which cuts off the frequency modes on the left Rindler wedge. This Rindler horizon is the cause of the Unruh effect. [2, 8]

The KG equation (2.55) in the Minkowski coordinates can be written as

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi(t,x,y) = 0, \tag{2.79}$$

the solution of which can be expanded as

$$\phi(t,x,y) = \int \mathrm{d}k_x \int \mathrm{d}k_y \left[a_{k_x,k_y} f_{k_x,k_y} + a^\dagger_{k_x,k_y} f^*_{k_x,k_y}\right], \tag{2.80}$$

where a_{k_x,k_y} ($a^\dagger_{k_x,k_y}$) is the annihilation (creation) operator and f_{k_x,k_y} are the positive-frequency modes given by

$$f_{k_x,k_y}(t,x,y) = \frac{1}{2\pi\sqrt{2k_0}}\, e^{-ik_0 t + ik_x x + ik_y y}, \tag{2.81}$$

with $k_0 \equiv \sqrt{k_x^2 + k_y^2}$. The quantization of the field ϕ and its conjugate momentum can be done by imposing, on the annihilation operator a_{k_x,k_y} and creation operator $a^\dagger_{k_x,k_y}$, the commutation relations

$$\begin{aligned} \left[a_{k_x,k_y}, a^\dagger_{k'_x,k'_y}\right] &= \delta(k_x - k'_x)\delta(k_y - k'_y),\\ \left[a_{k_x,k_y}, a_{k'_x,k'_y}\right] &= \left[a^\dagger_{k_x,k_y}, a^\dagger_{k'_x,k'_y}\right] = 0. \end{aligned} \tag{2.82}$$

The factor next to the exponential factor is chosen such that the positive-frequency modes are normalized with respect to the KG inner product, i.e.,

$$\langle f_{k_x,k_y}, f_{k'_x,k'_y}\rangle = i\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\left[f^*_{k_x,k_y}\left(\partial_t f_{k'_x,k'_y}\right) - \left(\partial_t f^*_{k_x,k_y}\right)f_{k'_x,k'_y}\right] = \delta(k_x - k'_x)\delta(k_y - k'_y). \tag{2.83}$$

Meanwhile, the KG equation (2.55) in the right Rindler wedge is

$$\left[-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\xi^2} + e^{2a\xi}\frac{\partial^2}{\partial y^2}\right]\Phi^R(\tau,\xi,y) = 0 \tag{2.84}$$

the solution of which can be expanded as

$$\Phi^R(\tau,\xi,y) = \int_0^{\infty}\mathrm{d}\omega\int\mathrm{d}k_y\left[b^R_{\omega,k_y}g^R_{\omega,k_y} + b^{R\dagger}_{\omega,k_y}g^{R*}_{\omega,k_y}\right], \tag{2.85}$$

where b_{ω,k_y} ($b^{\dagger}_{\omega,k_y}$) is the annihilation (creation) operator with the commutation relations

$$\begin{aligned}\left[b^R_{\omega,k_y}, b^{R\dagger}_{\omega',k'_y}\right] &= \delta(\omega-\omega')\delta(k_y - k'_y),\\ \left[b^R_{\omega,k_y}, b^R_{\omega',k'_y}\right] = \left[b^{R\dagger}_{\omega,k_y}, b^{R\dagger}_{\omega',k'_y}\right] &= 0,\end{aligned} \tag{2.86}$$

the positive-frequency modes are

$$g^R_{\omega,k_y}(\tau,\xi,y) = \frac{1}{\sqrt{4\pi\omega}}\,e^{-i\omega\tau + ik_y y}G^R_{\omega,k_y}(\xi) \tag{2.87}$$

in the right Rindler wedge and $g^R_{\omega,k_y} \equiv 0$ in the left Rindler wedge[3]. The function $G^R_{\omega,k_y}(\xi)$ can be calculated by substituting the positive-frequency modes (2.89) into the KG equation (2.84), yielding

$$\begin{aligned}\left[-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\xi^2} + e^{2a\xi}\frac{\partial^2}{\partial y^2}\right]\left(\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\tau+ik_y y}G^R_{\omega,k_y}\right) &= 0,\\ \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\tau+ik_y y}\left[\omega^2 + \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - e^{2a\xi}k_y^2\right]G^R_{\omega,k_y} &= 0.\end{aligned}$$

Therefore, the function $G^R_{\omega,k_y}(\xi)$ satisfies the differential equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + e^{2a\xi}k_y^2\right]G^R_{\omega,k_y}(\xi) = \omega^2 G^R_{\omega k_y}(\xi), \tag{2.88}$$

[3]This definition in the left Rindler wedge is made so that the KG inner product can also be evaluated in Minkowski coordinates.

and can be written as

$$G^R_{\omega,k_y}(\xi) = \sqrt{\frac{2\omega\sinh\left(\frac{\pi\omega}{a}\right)}{\pi^2 a}}K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right), \tag{2.89}$$

where $K_\nu(x)$ is the modified Bessel function.

The positive-frequency modes g^R_{ω,k_y} are normalized under the KG inner product. To show this normalization, let us begin by substituting in the positive-frequency modes (2.87) into (2.59) with the Rindler metric (2.77), that is

$$\begin{aligned}
\left\langle g^R_{\omega,k_y}, g^R_{\omega',k'_y}\right\rangle &= i\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{-\infty}^{\infty}\mathrm{d}y\left[g^{R*}_{\omega,k_y}\left(\partial_\tau g^R_{\omega',k'_y}\right) - \left(\partial_\tau g^{R*}_{\omega,k_y}\right)g^R_{\omega',k'_y}\right] \\
&= \int_{-\infty}^{\infty}\mathrm{d}\xi\int_{-\infty}^{\infty}\mathrm{d}y\,\frac{1}{4\pi}\frac{\omega+\omega'}{\sqrt{\omega\omega'}}e^{i(\omega-\omega')\tau}e^{-i(k_y-k'_y)y}G^{R*}_{\omega,k_y}G^R_{\omega',k'_y} \\
\left\langle g^R_{\omega,k_y}, g^R_{\omega',k'_y}\right\rangle &= \frac{\omega+\omega'}{2\sqrt{\omega\omega'}}e^{i(\omega-\omega')\tau}\delta(k_y-k'_y)\int_{-\infty}^{\infty}\mathrm{d}\xi\,G^{R*}_{\omega,k_y}G^R_{\omega',k_y}.
\end{aligned} \tag{2.90}$$

To calculate this integral, let us start by defining

$$S_A(\omega,\omega') = \int_{-A}^{\infty}\mathrm{d}\xi\,G^{R*}_{\omega,k_y}(\xi)\,G^R_{\omega',k_y}(\xi), \tag{2.91}$$

so the integral is precisely S_A in the limit $A\to\infty$. Multiplying S_A by $(\omega^2-\omega'^2)$ and using the differential equation (2.88) yield

$$\begin{aligned}
(\omega^2-\omega'^2)S_A(\omega,\omega') &= \int_{-A}^{\infty}\mathrm{d}\xi\left[\left(\omega^2 G^{R*}_{\omega,k_y}\right)G^R_{\omega',k_y} - G^{R*}_{\omega,k_y}\left(\omega'^2 G^R_{\omega',k_y}\right)\right] \\
&= \int_{-A}^{\infty}\mathrm{d}\xi\left\{\left[-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}G^{R*}_{\omega,k_y} + e^{2a\xi}G^{R*}_{\omega,k_y}\right]G^R_{\omega',k_y} - G^{R*}_{\omega,k_y}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}G^R_{\omega',k_y} + e^{2a\xi}G^R_{\omega',k_y}\right]\right\} \\
&= \int_{-A}^{\infty}\mathrm{d}\xi\left[-\left(\frac{\mathrm{d}^2 G^{R*}_{\omega,k_y}}{\mathrm{d}\xi^2}\right)G^R_{\omega',k_y} + G^{R*}_{\omega,k_y}\left(\frac{\mathrm{d}^2 G^R_{\omega',k_y}}{\mathrm{d}\xi^2}\right)\right] \\
&= \int_{-A}^{\infty}\mathrm{d}\xi\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left[-\frac{\mathrm{d}G^{R*}_{\omega,k_y}}{\mathrm{d}\xi}G^R_{\omega',k_y} + G^{R*}_{\omega,k_y}\frac{\mathrm{d}}{\mathrm{d}\xi}G^R_{\omega',k_y}\right] \\
(\omega^2-\omega'^2)S_A(\omega,\omega') &= \left[-\frac{\mathrm{d}G^{R*}_{\omega,k_y}}{\mathrm{d}\xi}G^R_{\omega',k_y} + G^{R*}_{\omega,k_y}\frac{\mathrm{d}}{\mathrm{d}\xi}G^R_{\omega',k_y}\right]_{-A}^{\infty} \\
\implies \lim_{A\to\infty}S_A(\omega,\omega') &= \frac{1}{\omega^2-\omega'^2}\lim_{A\to\infty}\left[-\frac{\mathrm{d}G^{R*}_{\omega,k_y}}{\mathrm{d}\xi}G^R_{\omega',k_y} + G^{R*}_{\omega,k_y}\frac{\mathrm{d}}{\mathrm{d}\xi}G^R_{\omega',k_y}\right]_{-A}^{\infty}.
\end{aligned} \tag{2.92}$$

To calculate these limits, let us focus on the expression of G^R_{ω,k_y} (2.89). In the limit $\xi\to\infty$, the modified Bessel function $K_\nu(x)\to 0$. This leaves the limit $\xi = -A\to-\infty$ to be examined. The argument of the modified Bessel function $K_\nu(x)$ approaches zero in this limit, therefore the small argument approximation of the modified Bessel function is needed. The modified Bessel function $K_\nu(x)$ is defined by

$$K_\nu(x) \equiv -\frac{\pi}{2}\frac{i^{-\nu}J_\nu(ix) - i^\nu J_{-\nu}(ix)}{\sin(\nu\pi)}, \tag{2.93}$$

where $J_\nu(x)$ is the Bessel function, which in series representation can be expressed as

$$J_\nu(x) = \frac{x^\nu}{2^\nu}\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{2^{2k}k!\Gamma(\nu+k+1)}, \quad |\arg(x)| < \pi. \tag{2.94}$$

For small $|x|$, $J_\nu(x)$ can then be approximated as

$$J_\nu(x) \approx \frac{(x/2)^\nu}{\Gamma(1+\nu)} \tag{2.95}$$

since the terms with x^{2k} for $k > 0$ approaches zero and can be ignored in the limit. With this approximation then, the small argument limit of $K_\nu(x)$ is

$$\begin{aligned} K_\nu(x) &\approx -\frac{\pi}{2\sin(\nu\pi)}\left[i^{-\nu}\frac{(ix/2)^\nu}{\Gamma(1+\nu)} - i^\nu\frac{(ix/2)^{-\nu}}{\Gamma(1-\nu)}\right] \\ &\approx -\frac{\pi}{2\sin(\nu\pi)}\left[\frac{(x/2)^\nu}{\Gamma(1+\nu)} - \frac{(x/2)^{-\nu}}{\Gamma(1-\nu)}\right] \end{aligned} \tag{2.96}$$

$$\begin{aligned} \Longrightarrow \quad K_{i\omega/a}(x) &\approx -\frac{\pi}{2\sin\left(\frac{i\omega\pi}{a}\right)}\left[\frac{(x/2)^{i\omega/a}}{\Gamma(1+\frac{i\omega}{a})} - \frac{(x/2)^{-i\omega/a}}{\Gamma(1-\frac{i\omega}{a})}\right] \\ &\approx \frac{i\pi}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\frac{(x/2)^{i\omega/a}}{\Gamma(1+\frac{i\omega}{a})} - \frac{(x/2)^{-i\omega/a}}{\Gamma(1-\frac{i\omega}{a})}\right]. \end{aligned} \tag{2.97}$$

To simplify this small argument approximation further, let us rewrite the gamma functions as

$$\begin{aligned} \Gamma\left(1+\frac{i\omega}{a}\right) &= \left|\Gamma\left(1+\frac{i\omega}{a}\right)\right| e^{i\alpha}, \\ \Gamma\left(1-\frac{i\omega}{a}\right) &= \left|\Gamma\left(1-\frac{i\omega}{a}\right)\right| e^{-i\alpha}, \end{aligned} \tag{2.98}$$

where $\alpha = \arg\left[\Gamma\left(1+\frac{i\omega}{a}\right)\right]$. By noting that

$$\left|\Gamma\left(1+\frac{i\omega}{a}\right)\right|^2 = \left|\Gamma\left(1-\frac{i\omega}{a}\right)\right|^2 = \frac{\pi\omega}{a\sinh\left(\frac{\pi\omega}{a}\right)}, \tag{2.99}$$

the gamma functions can be written as

$$\begin{aligned} \Gamma\left(1+\frac{i\omega}{a}\right) &= \sqrt{\frac{\pi\omega}{a\sinh\left(\frac{\pi\omega}{a}\right)}}\; e^{i\alpha}, \\ \Gamma\left(1-\frac{i\omega}{a}\right) &= \sqrt{\frac{\pi\omega}{a\sinh\left(\frac{\pi\omega}{a}\right)}}\; e^{-i\alpha}. \end{aligned} \tag{2.100}$$

Therefore, the modified Bessel function $K_{i\omega/a}(x)$ can be written as

$$\begin{aligned} K_{i\omega/a}(x) &\approx \frac{i\pi}{2\sinh\left(\frac{\pi\omega}{a}\right)}\sqrt{\frac{a\sinh\left(\frac{\pi\omega}{a}\right)}{\pi\omega}}\left[e^{-i\alpha}(x/2)^{i\omega/a} - e^{i\alpha}(x/2)^{-i\omega/a}\right] \\ &\approx \frac{i}{2}\sqrt{\frac{\pi a}{\omega\sinh\left(\frac{\pi\omega}{a}\right)}}\left[e^{-i\alpha}(x/2)^{i\omega/a} - e^{i\alpha}(x/2)^{-i\omega/a}\right]. \end{aligned} \tag{2.101}$$

By substituting $x = k_y e^{a\xi}/a$, G^R_{ω,k_y} in the $\xi = -A \to -\infty$ limit is

$$\begin{aligned} \lim_{A\to\infty} G^R_{\omega,k_y}\bigg|_{\xi=-A} &= \sqrt{\frac{2\omega\sinh\left(\frac{\pi\omega}{a}\right)}{\pi^2 a}}\frac{i}{2}\sqrt{\frac{\pi a}{\omega\sinh\left(\frac{\pi\omega}{a}\right)}}\lim_{A\to\infty}\left[e^{-i\alpha}\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}e^{i\omega\xi} - e^{i\alpha}\left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}e^{-i\omega\xi}\right]_{\xi=-A} \\ &= \frac{i}{\sqrt{2\pi}}\lim_{A\to\infty}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}e^{-i\omega A}e^{-i\alpha} - \left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}e^{i\alpha}e^{i\omega A}\right]. \end{aligned} \tag{2.102}$$

For the limit $\xi = -A \to -\infty$ of the derivative of G^R_{ω,k_y}, let us first consider the derivative of the modified Bessel function. The recursion formula relevant for us is [24]

$$\frac{\mathrm{d}K_\nu(z)}{\mathrm{d}z} = \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z). \tag{2.103}$$

Using $z = k_y e^{a\xi}/a$, $\nu = i\omega/a$, and the chain-rule yields

$$\begin{aligned} \left(\frac{e^{-a\xi}}{k_y}\right)\frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right] &= \frac{i\omega e^{-a\xi}}{k_y}K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right) - K_{1+i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right) \\ \frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right] &= i\omega K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right) - k_y e^{a\xi}K_{1+i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right). \end{aligned} \tag{2.104}$$

Utilizing (2.97) for the first term and (2.96) for the second term yields the small argument approximation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right] \approx{}& (i\omega)\frac{i\pi}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma\left(1+\frac{i\omega}{a}\right)}-\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma\left(1-\frac{i\omega}{a}\right)}\right] \\
&- k_y e^{a\xi}\left[\frac{(-\pi)}{2\sin\left(\left(\frac{i\omega}{a}+1\right)\pi\right)}\right] \\
&\times\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}+1}\frac{\left(e^{a\xi}\right)^{\frac{i\omega}{a}+1}}{\Gamma\left(2+\frac{i\omega}{a}\right)}-\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}-1}\frac{\left(e^{a\xi}\right)^{-\frac{i\omega}{a}-1}}{\Gamma\left(-\frac{i\omega}{a}\right)}\right] \\
\approx{}& -\frac{\pi\omega}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma\left(1+\frac{i\omega}{a}\right)}-\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma\left(1-\frac{i\omega}{a}\right)}\right] \\
&+\frac{i\pi a}{\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}+2}\frac{e^{i\omega\xi}e^{2a\xi}}{\Gamma\left(2+\frac{i\omega}{a}\right)}-\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\left(-\frac{a}{i\omega}\right)\Gamma\left(1-\frac{i\omega}{a}\right)}\right] \\
\approx{}& -\frac{\pi\omega}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma\left(1+\frac{i\omega}{a}\right)}-\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma\left(1-\frac{i\omega}{a}\right)}\right] \\
&-\frac{\pi\omega}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[-\frac{2a}{\omega}\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}+2}\frac{e^{i\omega\xi}e^{2a\xi}}{\Gamma\left(2+\frac{i\omega}{a}\right)}+2\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma\left(1-\frac{i\omega}{a}\right)}\right] \\
\approx{}& -\frac{\pi\omega}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma\left(1+\frac{i\omega}{a}\right)}+\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma\left(1-\frac{i\omega}{a}\right)}\right. \\
&\qquad\left.-\frac{2a}{\omega}\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}+2}\frac{e^{i\omega\xi}e^{2a\xi}}{\Gamma\left(2+\frac{i\omega}{a}\right)}\right] \\
\frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right] \approx{}& -\frac{\pi\omega}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma\left(1+\frac{i\omega}{a}\right)}+\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma\left(1-\frac{i\omega}{a}\right)}\right].
\end{aligned}
\tag{2.105}
$$

Substituting in the expressions (2.100) for the gamma functions then yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right]_{\xi=-A} &\approx -\sqrt{\frac{a\pi\omega}{4\sinh\left(\frac{\pi\omega}{a}\right)}}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}e^{i\omega\xi}e^{-i\alpha}+\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}e^{-i\omega\xi}e^{i\alpha}\right]_{\xi=-A} \\
\lim_{A\to\infty}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right]_{\xi=-A} &= -\sqrt{\frac{a\pi\omega}{4\sinh\left(\frac{\pi\omega}{a}\right)}}\lim_{A\to\infty}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}e^{-i\omega A}e^{-i\alpha}+\left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}}e^{i\omega A}e^{i\alpha}\right],
\end{aligned}
\tag{2.106}
$$

and the derivative of G^R_{ω,k_y} in the limit $\xi = -A \to -\infty$ is then

$$\lim_{A\to\infty} \frac{\mathrm{d}G^R_{\omega,k_y}}{\mathrm{d}\xi}\bigg|_{\xi=-A} = \sqrt{\frac{2\omega\sinh\left(\frac{\pi\omega}{a}\right)}{\pi^2 a}} \lim_{A\to\infty} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[K_{i\omega/a}\left(\frac{k_y e^{a\xi}}{a}\right)\right]_{\xi=-A}$$
$$= -\frac{\omega}{\sqrt{2\pi}} \lim_{A\to\infty}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}} e^{-i\omega A}e^{-i\alpha} + \left(\frac{k_y}{2a}\right)^{-\frac{i\omega}{a}} e^{i\omega A}e^{i\alpha}\right]. \tag{2.107}$$

Substituting in the limits of G^R_{ω,k_y} and its derivative back to S_A (2.92) then yields

$$\lim_{A\to\infty} S_A(\omega,\omega') = \frac{1}{\omega^2-\omega'^2}\lim_{A\to\infty}\left[-\frac{\mathrm{d}G^{R*}_{\omega,k_y}}{\mathrm{d}\xi}G^R_{\omega',k_y} + G^{R*}_{\omega,k_y}\frac{\mathrm{d}}{\mathrm{d}\xi}G^R_{\omega',k_y}\right]^{\infty}_{\xi=-A}$$
$$= \frac{1}{\omega^2-\omega'^2}\lim_{A\to\infty}\left[\frac{\mathrm{d}G^{R*}_{\omega,k_y}}{\mathrm{d}\xi}G^R_{\omega',k_y} - G^{R*}_{\omega,k_y}\frac{\mathrm{d}}{\mathrm{d}\xi}G^R_{\omega',k_y}\right]_{\xi=-A}$$
$$= -\frac{i}{2\pi(\omega^2-\omega'^2)}\lim_{A\to\infty}\left\{\omega\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}} e^{-i(\omega A+\alpha)} + \left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}} e^{i(\omega A+\alpha)}\right]\right.$$
$$\left.\times\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega'}{a}} e^{-i(\omega' A+\alpha')} - \left(\frac{k_y}{2a}\right)^{\frac{-i\omega'}{a}} e^{i(\omega' A+\alpha')}\right] + (\omega\leftrightarrow\omega')\right\}, \tag{2.108}$$

where $\alpha' = \arg\left[\Gamma(1+i\omega'/a)\right]$ and $(\omega\leftrightarrow\omega')$ is the same term with ω and ω' interchanged. Let us then express

$$\left(\frac{k_y}{2a}\right)^{\pm\frac{i\omega}{a}} = e^{\pm\frac{i\omega}{a}\ln\left(\frac{k_y}{2a}\right)}, \tag{2.109}$$

and define

$$\gamma(\omega) \equiv \alpha - \frac{i\omega}{a}\ln\left(\frac{k_y}{2a}\right) \tag{2.110}$$

to simplify the expression of S_A to

$$\begin{aligned}
\lim_{A\to\infty} S_A(\omega,\omega') &= -\frac{i}{2\pi(\omega^2-\omega'^2)}\lim_{A\to\infty}\Bigg\{\omega\Big[e^{-i(\omega A+\gamma(\omega))}+e^{i(\omega A+\gamma(\omega))}\Big] \\
&\qquad\qquad \times\Big[e^{-i(\omega' A+\gamma(\omega'))}-e^{i(\omega' A+\gamma(\omega'))}\Big]+(\omega\leftrightarrow\omega')\Bigg\} \\
&= -\frac{i}{2\pi(\omega^2-\omega'^2)}\lim_{A\to\infty}\Bigg\{\omega\Big[\Big(e^{-i[A(\omega+\omega')+\gamma(\omega)+\gamma(\omega')]}-\text{ c.c. }\Big) \\
&\qquad\qquad +\Big(e^{i[A(\omega-\omega')+\gamma(\omega)-\gamma(\omega')]}-\text{ c.c. }\Big)\Big]+(\omega\leftrightarrow\omega')\Bigg\} \\
&= -\frac{i}{2\pi(\omega^2-\omega'^2)}\lim_{A\to\infty}\Bigg\{\omega\Big[-2i\sin\big(A(\omega+\omega')+\gamma(\omega)+\gamma(\omega')\big) \\
&\quad +2i\sin\big(A(\omega-\omega')+\gamma(\omega)-\gamma(\omega')\big)\Big] \\
&\qquad\qquad +\omega'\Big[-2i\sin\big(A(\omega+\omega')+\gamma(\omega)+\gamma(\omega')\big) \\
&\qquad\qquad\qquad -2i\sin\big(A(\omega-\omega')+\gamma(\omega)-\gamma(\omega')\big)\Big]\Bigg\} \\
&= \frac{1}{\pi(\omega^2-\omega'^2)}\lim_{A\to\infty}\Bigg\{(\omega-\omega')\sin\big(A(\omega-\omega')+\gamma(\omega)-\gamma(\omega')\big) \\
&\qquad\qquad +(\omega+\omega')\sin\big(A(\omega+\omega')+\gamma(\omega)+\gamma(\omega')\big)\Bigg\} \\
\lim_{A\to\infty} S_A(\omega,\omega') &= \lim_{A\to\infty}\Bigg\{\frac{\sin\big(A(\omega-\omega')+\gamma(\omega)-\gamma(\omega')\big)}{\pi(\omega-\omega')}-\frac{\sin\big(A(\omega+\omega')+\gamma(\omega)+\gamma(\omega')\big)}{\pi(\omega+\omega')}\Bigg\}.
\end{aligned} \tag{2.111}$$

Applying the limit $A \to \infty$, the γ terms in the sine functions can be ignored. Additionally, the limit representation of the Dirac delta function

$$\lim_{A\to\infty}\frac{\sin(Ax)}{\pi x}=\delta(x) \tag{2.112}$$

can be used, yielding

$$\lim_{A\to\infty} S_A(\omega,\omega')=\delta(\omega-\omega')-\delta(\omega+\omega')=\delta(\omega-\omega'). \tag{2.113}$$

The $\delta(\omega+\omega')$ vanishes due to the restriction $\omega > 0$. Therefore, going all the way back to

(2.90), the inner product of two positive-frequency modes is

$$\left\langle g^R_{\omega,k_y}, g^R_{\omega',k'_y}\right\rangle = \frac{\omega+\omega'}{2\sqrt{\omega\omega'}} e^{i(\omega-\omega')\tau}\delta(k_y-k'_y)\lim_{A\to\infty} S_A(\omega,\omega')$$
$$= \frac{\omega+\omega'}{2\sqrt{\omega\omega'}} e^{i(\omega-\omega')\tau}\delta(k_y-k'_y)\delta(\omega-\omega')$$
$$\left\langle g^R_{\omega,k_y}, g^R_{\omega',k'_y}\right\rangle = \delta(k_y-k'_y)\delta(\omega-\omega'), \tag{2.114}$$

where the property of the Dirac delta

$$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0) \tag{2.115}$$

has been used. This shows that the positive-frequency modes g^R_{ω,k_y} are normalized under the KG inner product.

The KG equation in the left Rindler wedge has the same solution as the right Rindler wedge solution with $\tau \to \bar{\tau}$ and $\xi \to \bar{\xi}$, i.e.,

$$\Phi^L(\bar{\tau},\bar{\xi},y) = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}k_y \left[b^L_{\omega,k_y} g^L_{\omega,k_y} + b^{L\dagger}_{\omega,k_y} g^{L*}_{\omega,k_y}\right], \tag{2.116}$$

with positive-frequency modes

$$g^L_{\omega,k_y}(\bar{\tau},\bar{\xi},y) = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega\bar{\tau}+ik_y y} G^L_{\omega,k_y}(\bar{\xi}), \tag{2.117}$$

$$G^L_{\omega,k_y}(\bar{\xi}) = \sqrt{\frac{2\omega\sinh\left(\frac{\pi\omega}{a}\right)}{\pi^2 a}} K_{i\omega/a}\left(\frac{k_y e^{a\bar{\xi}}}{a}\right), \tag{2.118}$$

in the left Rindler wedge and $g^L_{\omega,k_y} \equiv 0$ in the right Rindler wedge. Just as the case in the right wedge, the positive-frequency modes g^L_{ω,k_y} are also normalized under the KG inner product as

$$\left\langle g^L_{\omega,k_y}, g^L_{\omega',k'_y}\right\rangle = i\int_{-\infty}^{\infty}\mathrm{d}\bar{\xi}\int_{-\infty}^{\infty}\mathrm{d}\bar{\chi}\left[g^{L*}_{\omega,k_y}\left(\partial_{\bar{\tau}} g^L_{\omega',k'_y}\right) - \left(\partial_{\bar{\tau}} g^{L*}_{\omega,k_y}\right) g^L_{\omega',k'_y}\right] = \delta(\omega-\omega')\delta(k_y-k'_y). \tag{2.119}$$

The KG field ϕ can then be expanded in the left and right Rindler wedges as

$$\phi = \Phi^R + \Phi^L = \int_0^\infty \mathrm{d}\omega\int \mathrm{d}k_y\left[b^R_{\omega,k_y} g^R_{\omega,k_y} + b^{R\dagger}_{\omega,k_y} g^{R*}_{\omega,k_y} + b^L_{\omega,k_y} g^L_{\omega,k_y} + b^{L\dagger}_{\omega,k_y} g^{L*}_{\omega,k_y}\right]. \tag{2.120}$$

The modes in right and left Rindler wedges respectively can be written in terms of the modes in Minkowski as

$$\begin{aligned} g^R_{\omega,k_y} &= \int \mathrm{d}k_x \int \mathrm{d}k'_y \left[\alpha^R_{\omega k_x k_y}\delta(k'_y-k_y) f_{k_x,k'_y} + \beta^R_{\omega k_x k_y}\delta(k'_y+k_y) f^*_{k_x,k'_y}\right] \\ &= \int \mathrm{d}k_x\, \frac{e^{ik_y y}}{2\pi\sqrt{2k_0}}\left[\alpha^R_{\omega k_x k_y} e^{-ik_0 t+ik_x x} + \beta^R_{\omega k_x k_y} e^{ik_0 t - ik_x x}\right] \end{aligned} \tag{2.121}$$

and

$$
\begin{aligned}
g^L_{\omega,k_y} &= \int \mathrm{d}k_x \int \mathrm{d}k'_y \left[\alpha^L_{\omega k_x k_y}\delta(k'_y - k_y) f_{k_x,k'_y} + \beta^L_{\omega k_x k_y}\delta(k'_y + k_y) f^*_{k_x,k'_y}\right] \\
&= \int \mathrm{d}k_x \, \frac{e^{ik_y y}}{2\pi\sqrt{2k_0}} \left[\alpha^L_{\omega k_x k_y} e^{-ik_0 t + ik_x x} + \beta^L_{\omega k_x k_y} e^{ik_0 t - ik_x x}\right]
\end{aligned}
\tag{2.122}
$$

through the Bogoliubov transformation, which can be seen as a change of basis transformation. The coefficients $\left\{\alpha^R_{\omega k_x k_y}, \beta^R_{\omega k_x k_y}, \alpha^L_{\omega k_x k_y}, \beta^L_{\omega k_x k_y}\right\}$ are called the Bogoliubov coefficients and, using (2.83), it can be seen that

$$\left\langle f_{k'_x,k'_y}, g^R_{\omega,k_y}\right\rangle = \alpha^R_{\omega k'_x k_y}\delta(k_y - k'_y), \tag{2.123}$$

$$\left\langle f^*_{k'_x,k'_y}, g^R_{\omega,k_y}\right\rangle = -\beta^R_{\omega k'_x k_y}\delta(k_y + k'_y), \tag{2.124}$$

$$\left\langle f_{k'_x,k'_y}, g^L_{\omega,k_y}\right\rangle = \alpha^L_{\omega k'_x k_y}\delta(k_y - k'_y), \tag{2.125}$$

$$\left\langle f^*_{k'_x,k'_y}, g^L_{\omega,k_y}\right\rangle = -\beta^L_{\omega k'_x k_y}\delta(k_y + k'_y). \tag{2.126}$$

Using (2.60), then it can also be seen that

$$\left\langle g^R_{\omega,k_y}, f_{k'_x,k'_y}\right\rangle = \left\langle f_{k'_x,k'_y}, g^R_{\omega,k_y}\right\rangle^* = \alpha^{R*}_{\omega k'_x k_y}\delta(k_y - k'_y), \tag{2.127}$$

$$\left\langle g^{R*}_{\omega,k_y}, f_{k'_x,k'_y}\right\rangle = -\left\langle f^*_{k'_x,k'_y}, g^R_{\omega,k_y}\right\rangle = \beta^R_{\omega k'_x k_y}\delta(k_y + k'_y), \tag{2.128}$$

$$\left\langle g^L_{\omega,k_y}, f_{k'_x,k'_y}\right\rangle = \left\langle f_{k'_x,k'_y}, g^L_{\omega,k_y}\right\rangle^* = \alpha^{L*}_{\omega k'_x k_y}\delta(k_y - k'_y), \tag{2.129}$$

$$\left\langle g^{L*}_{\omega,k_y}, f_{k'_x,k'_y}\right\rangle = -\left\langle f^*_{k'_x,k'_y}, g^L_{\omega,k_y}\right\rangle = \beta^L_{\omega k'_x k_y}\delta(k_y + k'_y), \tag{2.130}$$

which implies that the Minkowski frequency modes can be expanded as

$$
\begin{aligned}
f_{k'_x,k'_y} = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}k_y \Big[&\alpha^{R*}_{\omega k'_x k_y}\delta(k_y - k'_y) g^R_{\omega,k_y} - \beta^R_{\omega k'_x k_y}\delta(k_y + k'_y) g^{R*}_{\omega,k_y} \\
&+ \alpha^{L*}_{\omega k'_x k_y}\delta(k_y - k'_y) g^L_{\omega,k_y} - \beta^L_{\omega k'_x k_y}\delta(k_y + k'_y) g^{L*}_{\omega,k_y}\Big].
\end{aligned}
\tag{2.131}
$$

Substituting this expansion (2.131) into the Minkowski field expansion (2.80) and comparing terms with (2.120), it can be concluded that the annihilation operators in right and left Rindler wedges respectively can be expressed in terms of the Minkowski operators as

$$b^R_{\omega,k_y} = \int \mathrm{d}k'_x \int \mathrm{d}k'_y \left[\alpha^{R*}_{\omega k'_x k_y}\delta(k_y - k'_y) a_{k'_x,k'_y} - \beta^{R*}_{\omega k'_x k_y}\delta(k_y + k'_y) a^\dagger_{k'_x,k'_y}\right], \tag{2.132}$$

$$b^L_{\omega,k_y} = \int \mathrm{d}k'_x \int \mathrm{d}k'_y \left[\alpha^{L*}_{\omega k'_x k_y}\delta(k_y - k'_y) a_{k'_x,k'_y} - \beta^{L*}_{\omega k'_x k_y}\delta(k_y + k'_y) a^\dagger_{k'_x,k'_y}\right]. \tag{2.133}$$

Additionally, substituting the same expansion (2.131) into the right Rindler wedge modes (2.121) yields the properties of the Bogoliubov coefficients, one of which that is relevant for the Unruh effect here is

$$\int \mathrm{d}k_x' \left[\alpha^R_{\omega k_x' k_y}\alpha^{R*}_{\omega' k_x' k_y} - \beta^R_{\omega k_x' k_y}\beta^{R*}_{\omega' k_x' k_y}\right] = \delta(\omega' - \omega). \tag{2.134}$$

With the Bogoliubov transformations of the frequency-modes (2.121) and operators (2.132) as well as the property (2.134) in hand, the discussion can continue without referring to the left Rindler wedge at all.

Let us now consider the normalized Minkowski vacuum state $|0_M\rangle$ which is defined by the annihilation operator in Minkowski a_{k_x,k_y} as

$$a_{k_x,k_y}|0_M\rangle = 0, \quad \langle 0_M|0_M\rangle = 1. \tag{2.135}$$

The number of particles with momentum k_x and k_y observed by an inertial observer in the Minkowski vacuum is of course

$$\langle 0_M|a^\dagger_{k_x,k_y}a_{k_x,k_y}|0_M\rangle = 0. \tag{2.136}$$

However, for a Rindler (accelerated) observer in the right wedge, the number of particles at energy ω and momentum k_y in the Minkowski vacuum can be calculated with

$$\begin{aligned}\langle 0_M|b^{R\dagger}_{\omega,k_y}b^R_{\omega,k_y}|0_M\rangle = \langle 0_M|\int \mathrm{d}k_x'' \int \mathrm{d}k_y'' \left[\alpha^R_{\omega k_x'' k_y}\delta(k_y - k_y'')a^\dagger_{k_x'',k_y''} - \beta^R_{\omega k_x'' k_y}\delta(k_y + k_y'')a_{k_x'',k_y''}\right] \\ \int \mathrm{d}k_x' \int \mathrm{d}k_y' \left[\alpha^{R*}_{\omega k_x' k_y}\delta(k_y - k_y')a_{k_x',k_y'} - \beta^{R*}_{\omega k_x' k_y}\delta(k_y + k_y')a^\dagger_{k_x',k_y'}\right]|0_M\rangle.\end{aligned} \tag{2.137}$$

Using the definition of the Minkowski vacuum (2.135) and the commutation relations (2.82) then yields

$$\langle 0_M|b^{R\dagger}_{\omega,k_y}b^R_{\omega,k_y}|0_M\rangle = \int \mathrm{d}k_x' \left|\beta^R_{\omega k_x' k_y}\right|^2 \delta(k_y - k_y) = \int \mathrm{d}k_x' \left|\beta^R_{\omega k_x' k_y}\right|^2 \delta(0). \tag{2.138}$$

The diverging factor $\delta(k_y - k_y) = \delta(0)$ in this expectation value can be interpreted as proportional to the spatial volume in the y-direction as

$$\delta(k_y - k_y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi} e^{i(k_y - k_y)y} = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi}. \tag{2.139}$$

Therefore, the expectation value (2.138) is expected to contain the particle number density. The amount of particles seen by the Rindler observer in the Minkowski vacuum ultimately depends on the Bogoliubov coefficient $\beta^R_{\omega k'_x k_y}$ which mixes the Minkowski positive-frequency modes $f_{k'_x,k'_y}$ with negative-frequency modes $f^*_{k'_x,k'_y}$ under the Bogoliubov transformation. The goal is to show that the particle number density contained in (2.138) is in the form of Bose-Einstein distribution with Unruh temperature.

To find the expressions for the Bogoliubov coefficients, let us change the Minkowski coordinates to

$$\begin{aligned} u \equiv t - x &= -\frac{e^{-a(\tau-\xi)}}{a} = -\frac{e^{-a\bar{u}}}{a}, \\ v \equiv t + x &= \frac{e^{a(\tau+\xi)}}{a} = \frac{e^{a\bar{v}}}{a}, \end{aligned} \tag{2.140}$$

where $\bar{u} \equiv \tau - \xi$ and $\bar{v} \equiv \tau + \xi$. Since the Bogoliubov coefficients do not depend on the coordinates, it is convenient to examine the frequency modes near the Rindler horizon $u \to 0^-$ or $\bar{u} \to \infty$. To calculate g^R_{ω,k_y} (2.87) in the limit $\bar{u} \to \infty$, we first need the limit $\xi \to -\infty$ of G^R_{ω,k_y} (2.89). To do that, we need the small argument approximation of the modified Bessel function $K_{i\omega/a}(k_y e^{a\xi}/a)$ and its derivative which have been discussed in the previous section and the results are in equations (2.97) and (2.105). Using those results, G^R_{ω,k_y} (2.89) can be approximated as

$$\begin{aligned} G^R_{\omega,k_y}(\xi)\Big|_{\xi\to-\infty} &\to \sqrt{\frac{2\omega\sinh\left(\frac{\pi\omega}{a}\right)}{\pi^2 a}}\frac{i\pi}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma(1+\frac{i\omega}{a})} - \left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma(1-\frac{i\omega}{a})}\right] \\ &\to \frac{i\sqrt{\omega}}{\sqrt{2a\sinh\left(\frac{\pi\omega}{a}\right)}}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{i\omega\xi}}{\Gamma(1+\frac{i\omega}{a})} - \left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}\frac{e^{-i\omega\xi}}{\Gamma(1-\frac{i\omega}{a})}\right], \end{aligned} \tag{2.141}$$

and g^R_{ω,k_y} (2.87) becomes

$$\begin{aligned} g^R_{\omega,k_y}\Big|_{\bar{u}\to\infty} &\to \frac{ie^{ik_y y}}{\sqrt{8\pi a\sinh\left(\frac{\pi\omega}{a}\right)}}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{-i\omega(\tau-\xi)}}{\Gamma(1+\frac{i\omega}{a})} - \left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}\frac{e^{-i\omega(\tau+\xi)}}{\Gamma(1-\frac{i\omega}{a})}\right] \\ &\to \frac{ie^{ik_y y}}{\sqrt{8\pi a\sinh\left(\frac{\pi\omega}{a}\right)}}\left[\left(\frac{k_y}{2a}\right)^{\frac{i\omega}{a}}\frac{e^{-i\omega\bar{u}}}{\Gamma(1+\frac{i\omega}{a})} - \left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}\frac{e^{-i\omega\bar{v}}}{\Gamma(1-\frac{i\omega}{a})}\right] \\ &\to -\frac{ie^{ik_y y}}{\sqrt{8\pi a\sinh\left(\frac{\pi\omega}{a}\right)}}\left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}}\frac{e^{-i\omega\bar{v}}}{\Gamma(1-\frac{i\omega}{a})}. \end{aligned} \tag{2.142}$$

The term with the factor $e^{-i\omega\bar{u}}$ as $\bar{u} \to \infty$ is regarded as zero since it oscillates infinitely many times around zero and is bounded in the limit. Meanwhile, for the limit $u \to 0^-$, the positive-frequency modes in the right wedge (2.121) is

$$g^R_{\omega,k_y}\Big|_{u\to 0^-} = \int \mathrm{d}k_x \,\frac{e^{ik_y y}}{2\pi\sqrt{2k_0}}\left[\alpha^R_{\omega k_x k_y} e^{-i\frac{(k_0-k_x)}{2}v} + \beta^R_{\omega k_x k_y} e^{i\frac{(k_0-k_x)}{2}v}\right] = g^R_{\omega,k_y}\Big|_{\bar{u}\to\infty}. \tag{2.143}$$

With this, the Bogoliubov coefficient $\alpha^R_{\omega k_x k_y}$ can be obtained in a similar manner as obtaining Fourier coefficients, i.e., by multiplying $g^R_{\omega,k_y}\big|_{\bar{u}\to\infty}$ with e^{iKv} where $K = (k_0 - k_x)/2 \geq 0$ and then integrating over v as

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}v\, e^{iKv} g^R_{\omega,k_y}\Big|_{\bar{u}\to\infty} &= \int_{-\infty}^{\infty} \mathrm{d}v \int \mathrm{d}k'_x \,\frac{e^{ik_y y}}{2\pi\sqrt{2k_0}}\left[\alpha^R_{\omega k'_x k_y} e^{i(K-K')v} + \beta^R_{\omega k'_x k_y} e^{i(K+K')v}\right] \\
&= \int \mathrm{d}k'_x \,\frac{e^{ik_y y}}{2\pi\sqrt{2k_0}}\left[\alpha^R_{\omega k'_x k_y}(2\pi)\delta(K-K') + \beta^R_{\omega k'_x k_y}(2\pi)\delta(K+K')\right] \\
&= \int \mathrm{d}k'_x \left(\frac{e^{ik_y y}}{\sqrt{2k_0}}\right)\alpha^R_{\omega k'_x k_y}\left(\frac{2k_0}{k_0-k_x}\right)\delta(k'_x - k_x) \\
\int_{-\infty}^{\infty} \mathrm{d}v\, e^{iKv} g^R_{\omega,k_y}\Big|_{\bar{u}\to\infty} &= e^{ik_y y}\frac{\sqrt{2k_0}}{k_0-k_x}\alpha^R_{\omega k_x k_y} \\
\implies \alpha^R_{\omega k_x k_y} &= \frac{k_0-k_x}{\sqrt{2k_0}} e^{-ik_y y}\int_{-\infty}^{\infty} \mathrm{d}v\, e^{iKv} g^R_{\omega,k_y}\Big|_{\bar{u}\to\infty},
\end{aligned} \tag{2.144}$$

where $K' = (k'_0 - k'_x)/2 \geq 0$, $k'^2_0 = k'^2_x + k^2_y$, and the advanced scaling property of the Dirac delta

$$\delta(g(x)) = |g'(x_0)|^{-1}\delta(x - x_0), \quad \text{for } g(x_0) = 0 \tag{2.145}$$

were used. The Bogoliubov coefficient $\beta^R_{\omega k_x k_y}$ can also be obtained in the same way with

$$\beta^R_{\omega k_x k_y} = \frac{k_0-k_x}{\sqrt{2k_0}} e^{-ik_y y}\int_{-\infty}^{\infty} \mathrm{d}v\, e^{-iKv} g^R_{\omega,k_y}\Big|_{\bar{u}\to\infty}. \tag{2.146}$$

To explicitly calculate the integral (2.144), recall that $g^R_{\omega,k_y} \equiv 0$ in the left Rindler wedge

$v < 0$ and $v = e^{a\bar{v}}/a$. Therefore, substituting in (2.142) into the integral yields

$$\begin{aligned}
\alpha^R_{\omega k_x k_y} &= -\frac{i(k_0 - k_x)}{4\sqrt{\pi a k_0 \sinh\left(\frac{\pi\omega}{a}\right)}}\left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}} \int_0^\infty \mathrm{d}v \, \frac{e^{-i\omega\bar{v}} e^{iKv}}{\Gamma\left(1 - \frac{i\omega}{a}\right)} \\
&= -\frac{i(k_0 - k_x)}{4\sqrt{\pi a k_0 \sinh\left(\frac{\pi\omega}{a}\right)}}\left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}} \int_0^\infty \mathrm{d}v \, \frac{(av)^{\frac{-i\omega}{a}} e^{iKv}}{\Gamma\left(1 - \frac{i\omega}{a}\right)} \\
&= -\frac{i(k_0 - k_x)}{4\sqrt{\pi a k_0 \sinh\left(\frac{\pi\omega}{a}\right)}}\left(\frac{k_y}{2a}\right)^{\frac{-i\omega}{a}} \frac{i}{K}\left(\frac{a}{K}\right)^{\frac{-i\omega}{a}} e^{\frac{\pi\omega}{2a}} \\
\alpha^R_{\omega k_x k_y} &= \frac{e^{\frac{\pi\omega}{2a}}}{2\sqrt{\pi a k_0 \sinh\left(\frac{\pi\omega}{a}\right)}}\left(\frac{k_0 + k_x}{k_0 - k_x}\right)^{\frac{-i\omega}{2a}}.
\end{aligned}$$

A similar calculation for the Bogoliubov coefficient $\beta^R_{\omega k_x k_y}$ yields

$$\beta^R_{\omega k_x k_y} = -\frac{e^{-\frac{\pi\omega}{2a}}}{2\sqrt{\pi a k_0 \sinh\left(\frac{\pi\omega}{a}\right)}}\left(\frac{k_0 + k_x}{k_0 - k_x}\right)^{\frac{-i\omega}{2a}}. \tag{2.147}$$

These expressions for the Bogoliubov coefficients are exactly the same as in the 4-dimensional Minkowski-Rindler spacetime case which was reviewed in [8]. Having these expressions, it can be seen that the coefficient $\alpha^R_{\omega k_x k_y}$ is related to the coefficient $\beta^R_{\omega k_x k_y}$ by

$$\alpha^R_{\omega k_x k_y} = -e^{\frac{\pi\omega}{a}} \beta^R_{\omega k_x k_y}. \tag{2.148}$$

With this relation, the property of the Bogoliubov coefficients (2.134) is then

$$\int \mathrm{d}k'_x \left(e^{\frac{\pi(\omega+\omega')}{a}} - 1\right)\beta^R_{\omega k'_x k_y}\beta^{R*}_{\omega' k'_x k_y} = \delta(\omega' - \omega), \tag{2.149}$$

which implies that the expectation value (2.138) is

$$\langle 0_M | b^{R\dagger}_{\omega,k_y} b^R_{\omega,k_y} | 0_M \rangle = \int \mathrm{d}k'_x \left|\beta^R_{\omega k'_x k_y}\right|^2 \delta(0) = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}\delta(0)^2. \tag{2.150}$$

The factor $\delta(0)^2$ again can be interpreted as proportional to the total spatial volume. Therefore, the quantity

$$n(\omega) = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \tag{2.151}$$

is the particle number density observed by the Rindler observer which is in the form of Bose-Einstein distribution with energy $E = \omega$ and temperature proportional to the observer's acceleration $T = \frac{a}{2\pi}$. This temperature is exactly the Unruh temperature [5].

3 The Unruh Effect in Rindler-AdS Spacetime

The discussion of the Unruh effect in Rindler-AdS in this chapter is sectioned into the discussions of the massless KG field in Poincaré AdS and Rindler-AdS, the Bogoliubov transformation, and the Unruh effect. Just as in the Unruh effect in Rindler-Minkowski, the Poincaré AdS and Rindler-AdS considered are 3-dimensional in anticipation of the 3-dimensional rotating Rindler-AdS.

It has been shown, alongside other discussions and with a different approach, in references [19, 21] that an RAdS observer sees Unruh temperature of $T = 1/(2\pi L)$. The goal of this chapter is to obtain the Unruh temperature with the approach outlined in section 2.3 in preparation to perform the same calculations in 3-dimensional rotating Rindler-AdS.

3.1 Massless KG field in Poincaré AdS (PAdS)

The quantization of a massive scalar field in PAdS spacetime has been previously studied in references [25, 26]. As such, this section is a review on the special case of massless scalar field in PAdS spacetime. The discussions that follow after this section, i.e., sections 3.2 and 3.3, as well as chapter 4, are my original contributions.

As discussed in subsection 2.1.1, the PAdS metric is given by

$$\mathrm{d}s^2 = \frac{L^2}{x^2}\left[-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right], \quad -\infty < (t, y) < \infty,\ 0 < x < \infty,\ L > 0. \tag{3.1}$$

With this metric, the massless KG equation (2.55) becomes

$$-\frac{\partial^2\phi}{\partial t^2} + \left[\frac{\partial^2\phi}{\partial x^2} - \frac{1}{x}\frac{\partial\phi}{\partial x}\right] + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{3.2}$$

The solution of this KG equation can be expanded as [25, 26]

$$\phi(t, x, y) = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[a_{k_0,k_y} f_{k_0,k_y}(x) + a^\dagger_{k_0,k_y} f^*_{k_0,k_y}(x)\right], \tag{3.3}$$

where a_{k_0,k_y} ($a^\dagger_{k_0,k_y}$) is the annihilation (creation) operator and f_{k_0,k_y} are the positive-frequency modes given by

$$f_{k_0,k_y}(t,x,y) = \frac{1}{\sqrt{4\pi L}} e^{-ik_0 t + ik_y y}\ x\ J_1(Kx), \tag{3.4}$$

where $K \equiv \sqrt{k_0^2 - k_y^2} \geq 0$ and $J_\nu(x)$ is the Bessel function. The quantization of the field ϕ and its conjugate momentum can be done by imposing, on the operators a_{k_0,k_y} and $a^\dagger_{k_0,k_y}$, the commutation relations

$$\begin{aligned} \left[a_{k_0,k_y}, a^\dagger_{k_0',k_y'}\right] &= \delta(k_0 - k_0')\delta(k_y - k_y'), \\ \left[a_{k_0,k_y}, a_{k_0',k_y'}\right] &= \left[a^\dagger_{k_0,k_y}, a^\dagger_{k_0',k_y'}\right] = 0. \end{aligned} \tag{3.5}$$

The factor $(4\pi L)^{1/2}$ in (3.4) is chosen so that the positive-frequency modes are normalized under the KG inner product. To show that is the case, let us calculate the inner product between two positive-frequency modes with (2.59), yielding

$$\begin{aligned} \left\langle f_{k_0,k_y}, f_{k_0',k_y'}\right\rangle &= i\int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}y\, \frac{L}{x}\left[f^*_{k_0,k_y}\left(\partial_t f_{k_0',k_y'}\right) - \left(\partial_t f^*_{k_0,k_y}\right) f_{k_0',k_y'}\right] \\ &= \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \frac{\mathrm{d}y}{4\pi}\left[k_0' e^{i(k_0-k_0')t} e^{-i(k_y-k_y')y} + k_0 e^{-i(k_0-k_0')t} e^{i(k_y-k_y')y}\right] x J_1(Kx) J_1(K'x) \\ &= \frac{1}{2}\int_0^\infty \mathrm{d}x\, \delta(k_y - k_y')\left[k_0' e^{i(k_0-k_0')t} + k_0 e^{-i(k_0-k_0')t}\right] x J_1(Kx) J_1(K'x) \\ \left\langle f_{k_0,k_y}, f_{k_0',k_y'}\right\rangle &= \frac{1}{2}\delta(k_y - k_y')\left[k_0' e^{i(k_0-k_0')t} + k_0 e^{-i(k_0-k_0')t}\right] \int_0^\infty \mathrm{d}x\, x J_1(Kx) J_1(K'x), \end{aligned} \tag{3.6}$$

with $K' \equiv \sqrt{k_0'^2 - k_y'^2} \geq 0$. This integral can be evaluated using [24]

$$\int_0^\infty \mathrm{d}x\, x J_n(ax) J_n(bx) = \frac{\delta(a-b)}{a}, \quad [n = 0, 1, ...]. \tag{3.7}$$

Therefore, the inner product simplifies to

$$\begin{aligned} \left\langle f_{k_0,k_y}, f_{k_0',k_y'}\right\rangle &= \frac{1}{2}\delta(k_y - k_y')\left[k_0' e^{i(k_0-k_0')t} + k_0 e^{-i(k_0-k_0')t}\right]\frac{1}{K}\delta(K - K') \\ &= \frac{1}{2}\delta(k_y - k_y')\left[k_0' e^{i(k_0-k_0')t} + k_0 e^{-i(k_0-k_0')t}\right]\frac{\delta(k_0 - k_0')}{k_0} \\ \left\langle f_{k_0,k_y}, f_{k_0',k_y'}\right\rangle &= \delta(k_0 - k_0')\delta(k_y - k_y), \end{aligned} \tag{3.8}$$

showing the positive-frequency modes are normalized. The properties of the Dirac delta (2.145) and

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0), \tag{3.9}$$

as well as the constraint $k_0 \geq 0$ have been used in obtaining (3.8).

3.2 Massless KG field in Rindler-AdS (RAdS)

As discussed in subsection 2.1.2, the RAdS metric can be obtained using the transformation (2.23) on the PAdS as

$$\mathrm{d}s^2 = -e^{\frac{2\xi}{L}}\,\mathrm{d}\tau^2 + \frac{e^{\frac{2\xi}{L}}\,\mathrm{d}\xi^2}{1+e^{\frac{2\xi}{L}}} + \left(1+e^{\frac{2\xi}{L}}\right)\mathrm{d}\chi^2\,, \quad -\infty < (\tau,\xi,\chi) < \infty. \tag{3.10}$$

With this metric, the massless KG equation (2.55) is

$$-\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial}{\partial\xi}\left[\left(1+e^{\frac{2\xi}{L}}\right)\frac{\partial\phi}{\partial\xi}\right] + \left(\frac{e^{\frac{2\xi}{L}}}{1+e^{\frac{2\xi}{L}}}\right)\frac{\partial^2\phi}{\partial\chi^2} = 0. \tag{3.11}$$

The solution of this KG equation can then be expanded as

$$\phi(\tau,\xi,\chi) = \int_0^\infty \mathrm{d}\omega \int \mathrm{d}k\,\left[b_{\omega,k}g_{\omega,k}(\tau,\xi,\chi) + b^\dagger_{\omega,k}g^*_{\omega,k}(\tau,\xi,\chi)\right], \tag{3.12}$$

where $b_{\omega,k}$ ($b^\dagger_{\omega,k}$) is the annihilation (creation) operator and $g_{\omega,k}$ are the positive-frequency modes given by

$$g_{\omega,k}(\tau,\xi,\chi) = \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\tau+ik\chi}G_{\omega,k}(\xi), \quad \omega^2-k^2>0,\ \omega>0. \tag{3.13}$$

The function $G_{\omega,k}(\xi)$, from the KG equation (3.11), satisfies the differential equation

$$\omega^2 G_{\omega,k}(\xi) = -\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(1+e^{\frac{2\xi}{L}}\right)\frac{\mathrm{d}G_{\omega,k}(\xi)}{\mathrm{d}\xi}\right] + \frac{e^{\frac{2\xi}{L}}k^2}{1+e^{\frac{2\xi}{L}}}G_{\omega,k}(\xi) \tag{3.14}$$

and can be written as

$$\begin{aligned} G_{\omega,k}(\xi) = \ & \frac{1}{\sqrt{2\pi}}\left(1+e^{\frac{2\xi}{L}}\right)^{\frac{iLk}{2}}\left[e^{-i\omega\xi}F\left(-\frac{iL}{2}(\omega-k), 1-\frac{iL}{2}(\omega-k); 1-iL\omega; -e^{\frac{2\xi}{L}}\right)\right. \\ & \left. + e^{i\omega\xi}F\left(\frac{iL}{2}(\omega+k), 1+\frac{iL}{2}(\omega+k); 1+iL\omega; -e^{\frac{2\xi}{L}}\right)\right] \end{aligned} \tag{3.15}$$

where $F(a,b;c;z)$ is the hypergeometric function [24]. The field quantization imposes, on the operators $b_{\omega,k}$ and $b^\dagger_{\omega,k}$, the commutations relations

$$\begin{aligned} \left[b_{\omega,k}, b^\dagger_{\omega',k'}\right] &= \delta(\omega-\omega')\delta(k-k'), \\ \left[b_{\omega,k}, b_{\omega',k'}\right] &= \left[b^\dagger_{\omega,k}, b^\dagger_{\omega',k'}\right] = 0. \end{aligned} \tag{3.16}$$

The positive-frequency modes $g_{\omega,k}$ are normalized under the KG inner product (2.59). To show that, let us calculate the inner product between two positive-frequency modes $g_{\omega,k}$ and $g_{\omega',k'}$, yielding

$$\begin{aligned}
\langle g_{\omega,k}, g_{\omega',k'}\rangle &= i\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\chi \left[g^*_{\omega,k}(\partial_\tau g_{\omega',k'}) - \left(\partial_\tau g^*_{\omega,k}\right) g_{\omega',k'}\right] \\
&= i\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\chi \, \frac{-i(\omega+\omega')}{4\pi\sqrt{\omega\omega'}} e^{i(\omega-\omega')\tau} e^{-i(k-k')\chi} G^*_{\omega,k}(\xi) G_{\omega',k'}(\xi) \\
\langle g_{\omega,k}, g_{\omega',k'}\rangle &= \frac{\omega+\omega'}{2\sqrt{\omega\omega'}} e^{-(\omega-\omega')\tau} \delta(k-k') \int_{-\infty}^{\infty} \mathrm{d}\xi \, G^*_{\omega,k}(\xi) G_{\omega',k}(\xi).
\end{aligned} \tag{3.17}$$

To calculate this integral, let us start by defining

$$S_A(\omega,\omega') = \int_{-A}^{\infty} \mathrm{d}\xi \, G^*_{\omega,k}(\xi) G_{\omega',k}(\xi) \tag{3.18}$$

so the integral is precisely S_A in the limit $A \to \infty$. Multiplying S_A by $(\omega^2 - \omega'^2)$ and using the differential equation (3.14) yield

$$\begin{aligned}
\lim_{A\to\infty} (\omega^2 - \omega'^2) S_A(\omega,\omega') &= \lim_{A\to\infty} \int_{-A}^{\infty} \left[\left(\omega^2 G^*_{\omega,k}\right) G_{\omega',k} - G^*_{\omega,k}\left(\omega'^2 G_{\omega',k}\right)\right] \\
&= \lim_{A\to\infty} \left[\left(1 + e^{\frac{2\xi}{L}}\right)\left(G^*_{\omega,k} \frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi} - \frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi} G_{\omega',k}\right)\right]_{\xi=-A}^{\infty}.
\end{aligned} \tag{3.19}$$

To evaluate these limits, let us focus on the lower limit $\xi \to -\infty$ of $G_{\omega,k}$ (3.15) first. In that case, the hypergeometric function $F \to 1$ because the argument approaches zero as $\xi \to -\infty$ and $F(a,b;c;0) = 1$. Additionally, the factor $(1 + \exp(2\xi/L))^{iLk/2} \to 1$ as $\xi \to -\infty$. Therefore, we have

$$G_{\omega,k} \overset{\xi\to-\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}}\left(e^{i\omega\xi} + e^{-i\omega\xi}\right). \tag{3.20}$$

Similarly, we also have

$$\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi} \overset{\xi\to-\infty}{\longrightarrow} \frac{i\omega}{\sqrt{2\pi}}\left(e^{i\omega\xi} - e^{-i\omega\xi}\right). \tag{3.21}$$

For the upper limit $\xi \to \infty$, using the identities of the hypergeometric function

$$F(a,b;c;z) = (1-z)^{-a} F[a, c-b; c; \frac{z}{z-1}], \tag{3.22}$$

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, \tag{3.23}$$

simplifies $G_{\omega,k}$ to

$$\lim_{\xi\to\infty} G_{\omega,k} = \frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma(1+iL\omega)}{\Gamma\left(1+\frac{iL}{2}(\omega+k)\right)\Gamma\left(1+\frac{iL}{2}(\omega-k)\right)} + \frac{\Gamma(1-iL\omega)}{\Gamma\left(1-\frac{iL}{2}(\omega+k)\right)\Gamma\left(1-\frac{iL}{2}(\omega-k)\right)}\right]. \tag{3.24}$$

And similarly, the derivative $\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi}$ simplifies to

$$\begin{aligned}\lim_{\xi\to\infty}\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi} =& -\frac{2}{L}\frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma(1+iL\omega)}{\Gamma\left(1+\frac{iL}{2}(\omega+k)\right)\Gamma\left(1+\frac{iL}{2}(\omega-k)\right)}\right.\\ &\left.+\frac{\Gamma(1-iL\omega)}{\Gamma\left(1-\frac{iL}{2}(\omega+k)\right)\Gamma\left(1-\frac{iL}{2}(\omega-k)\right)}\right]\\ =& -\frac{2}{L}\lim_{\xi\to\infty}G_{\omega,k}. \end{aligned}\tag{3.25}$$

Therefore, the limit (3.19) becomes

$$\begin{aligned}\lim_{A\to\infty}\left(\omega^2-\omega'^2\right)S_A(\omega,\omega') &= \lim_{A\to\infty}-\left[\left(1+e^{\frac{2\xi}{L}}\right)\left(G^*_{\omega,k}\frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi}-\frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi}G_{\omega',k}\right)\right]_{\xi=-A}\\ &= \lim_{A\to\infty}\frac{1}{\pi}[(\omega+\omega')\sin(A(\omega-\omega'))+(\omega-\omega')\sin(A(\omega+\omega'))]\\ \lim_{A\to\infty}S_A(\omega,\omega') &= \lim_{A\to\infty}\left[\frac{\sin(A(\omega-\omega'))}{\pi(\omega-\omega')}+\frac{\sin(A(\omega+\omega'))}{\pi(\omega+\omega')}\right]\\ &= \delta(\omega-\omega')+\delta(\omega+\omega')\\ \lim_{A\to\infty}S_A(\omega,\omega') &= \delta(\omega-\omega'), \end{aligned}\tag{3.26}$$

where the constraint $\omega > 0$ which implies $\delta(\omega+\omega') = 0$ and the limit representation of the Dirac delta

$$\lim_{A\to\infty}\frac{\sin(Ax)}{\pi x} = \delta(x) \tag{3.27}$$

have been used. With this, then, the inner product (3.17) is

$$\langle g_{\omega,k}, g_{\omega',k'}\rangle = \frac{\omega+\omega'}{2\sqrt{\omega\omega'}}e^{-(\omega-\omega')\tau}\delta(k-k')\delta(\omega-\omega') = \delta(\omega-\omega')\delta(k-k'), \tag{3.28}$$

showing that the positive-frequency modes $g_{\omega,k}$ is normalized under the KG inner product.

3.3 Bogoliubov Transformation and the Unruh effect in RAdS

Using the Bogoliubov transformation, the RAdS frequency modes $g_{\omega,k}$ can be written in terms of the PAdS frequency modes f_{k_0,k_y} as

$$g_{\omega,k} = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha_{\omega k_0,kk_y}f_{k_0,k_y}+\beta_{\omega k_0,kk_y}f^*_{k_0,k_y}\right]. \tag{3.29}$$

This RAdS positive-frequency modes $g_{\omega,k}$ only exist in the right wedge region with $|t| < y$, i.e., $g_{\omega,k} \equiv 0$ in the left wedge region $|t| < -y$. The left wedge region can be obtained with the transformation (2.30) which yields the same metric, field, and positive-frequency modes. This situation is similar to that in Rindler-Minkowski in which, recall, the field quantizations in both right and left Rindler wedges combined are equivalent to the field quantization in Minkowski. This then allowed us to obtain the Bogoliubov transformation of the Rindler operator (2.132) and the property of the Bogoliubov coefficients (2.134). With the assumption that field quantizations in both right and left Rindler-AdS wedges combined are equivalent to the field quantization in PAdS, we can obtain the Bogoliubov transformation of the operator $b_{\omega,k}$ as

$$b_{\omega,k} = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha^*_{\omega k_0,kk_y} a_{k_0,k_y} - \beta^*_{\omega k_0,kk_y} a^\dagger_{k_0,k_y}\right], \tag{3.30}$$

and the property of the Bogoliubov coefficients as

$$\int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha_{\omega k_0,kk_y} \alpha^*_{\omega' k_0,k' k_y} - \beta_{\omega k_0,kk_y} \beta^*_{\omega' k_0,k' k_y}\right] = \delta(\omega - \omega')\delta(k - k'). \tag{3.31}$$

Let us now consider the vacuum state in PAdS $|0_{\mathrm{PAdS}}\rangle$ defined by

$$a_{k_0,k_y}\,|0_{\mathrm{PAdS}}\rangle = 0, \quad \langle 0_{\mathrm{PAdS}}|0_{\mathrm{PAdS}}\rangle = 1. \tag{3.32}$$

The number of particles with energy k_0 and momentum k_y observed by an observer in PAdS is of course

$$\langle 0_{\mathrm{PAdS}}|a^\dagger_{k_0,k_y} a_{k_0,k_y}|0_{\mathrm{PAdS}}\rangle = 0. \tag{3.33}$$

However, for an observer in RAdS, the number of particles with energy ω and momentum k observed in the PAdS vacuum can be calculated with the expected value

$$\begin{aligned}\langle 0_{\mathrm{PAdS}}|b^\dagger_{\omega,k} b_{\omega,k}|0_{\mathrm{PAdS}}\rangle = \langle 0_{\mathrm{PAdS}}| &\int_0^\infty \mathrm{d}k'_0 \int \mathrm{d}k'_y \left[\alpha_{\omega k'_0,kk'_y} a^\dagger_{k'_0,k'_y} - \beta_{\omega k'_0,kk'_y} a_{k'_0,k'_y}\right] \\ &\int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha^*_{\omega k_0,kk_y} a_{k_0,k_y} - \beta^*_{\omega k_0,kk_y} a^\dagger_{k_0,k_y}\right] |0_{\mathrm{PAdS}}\rangle\,. \end{aligned} \tag{3.34}$$

Using the definition of the PAdS vacuum (3.32) and the commutation relations (3.5) then simplifies the expectation value to

$$\langle 0_{\mathrm{PAdS}}|b^\dagger_{\omega,k} b_{\omega,k}|0_{\mathrm{PAdS}}\rangle = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left|\beta_{\omega k_0,kk_y}\right|^2. \tag{3.35}$$

Drawing the analogue with the Rindler-Minkowski Unruh effect, (3.35) is expected to contain the particle number density. The goal now is to show that this particle number density is in the form of Bose-Einstein distribution with Unruh temperature.

Just as in the Rindler-Minkowski Unruh effect calculation, to find the expressions for the Bogoliubov coefficients, let us change the coordinates to

$$\begin{aligned} x &= \frac{L}{\sqrt{1+e^{2\xi/L}}} e^{-\chi/L} = \frac{L}{\sqrt{1+e^{(\bar{v}-\bar{u})/L}}} e^{-\chi/L}, \\ u \equiv t - y &= -x e^{-(\tau-\xi)/L} = -x e^{-\bar{u}/L}, \\ v \equiv t + y &= x e^{(\tau+\xi)/L} = x e^{\bar{v}/L}, \end{aligned} \tag{3.36}$$

where $\bar{u} \equiv \tau - \xi$ and $\bar{v} \equiv \tau + \xi$. Since the Bogoliubov coefficients do not depend on the coordinates, it is convenient to examine the frequency modes $g_{\omega,k}$ (3.13) near the horizon $u \to 0^-$ or $\bar{u} \to \infty$ with $x > 0$. For the limit $\bar{u} \to \infty$, using (3.20), the frequency modes become

$$\begin{aligned} g_{\omega,k}\Big|_{\bar{u}\to\infty} &\to \frac{1}{2\pi\sqrt{2\omega}} e^{-i\omega\tau + ik\chi}\left(e^{i\omega\xi} + e^{-i\omega\xi}\right) \\ &\to \frac{1}{2\pi\sqrt{2\omega}} e^{ik\chi}\left(e^{-i\omega\bar{u}} + e^{-i\omega\bar{v}}\right) \\ g_{\omega,k}\Big|_{\bar{u}\to\infty} &\to \frac{1}{2\pi\sqrt{2\omega}} e^{ik\chi} e^{-i\omega\bar{v}}. \end{aligned} \tag{3.37}$$

The term with the factor $e^{-i\omega\bar{u}}$ as $\bar{u} \to \infty$ is regarded as zero since it oscillates infinitely many times around zero and is bounded. Meanwhile, for the limit $u \to 0^-$, the positive-frequency modes (3.29) become

$$g_{\omega,k}\Big|_{u\to 0^-} = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \, \frac{x J_1(Kx)}{\sqrt{4\pi L}} \left[\alpha_{\omega k_0, k k_y} e^{-i\frac{(k_0-k_y)}{2}v} + \beta_{\omega k_0, k k_y} e^{i\frac{(k_0-k_y)}{2}v}\right] = g_{\omega,k}\Big|_{\bar{u}\to\infty}. \tag{3.38}$$

The Bogoliubov coefficient $\alpha_{\omega k_0, k k_y}$ can be obtained by evaluating the integral

$$\alpha_{\omega k_0, k k_y} = (k_0 - k_y)\sqrt{\frac{L}{4\pi}} \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}v \, e^{i\frac{(k_0-k_y)}{2}v} J_1(Kx) g_{\omega,k}\Big|_{\bar{u}\to\infty}. \tag{3.39}$$

To evaluate this integral, let us first express the frequency modes $g_{\omega,k}\big|_{\bar{u}\to\infty}$ (3.37) in (u, v, x) coordinates using (3.36), i.e.,

$$g_{\omega,k}\Big|_{\bar{u}\to\infty} \to \frac{1}{2\pi\sqrt{2\omega}} \left(\frac{x}{L}\right)^{-ikL} \left(\frac{v}{x}\right)^{-i\omega L} = \frac{1}{2\pi\sqrt{2\omega}} L^{ikL} x^{i(\omega-k)L} v^{-i\omega L}. \tag{3.40}$$

By noting that $v<0$ is in the left wedge region where the frequency modes $g_{\omega,k}\equiv 0$ and by substituting in (3.40), the integral (3.39) yields

$$
\begin{aligned}
\alpha_{\omega k_0,kk_y} &= (k_0-k_y)\sqrt{\frac{L}{4\pi}}\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}v\, e^{i\frac{(k_0-k_y)}{2}v}J_1(Kx)\left[\frac{1}{2\pi\sqrt{2\omega}}L^{ikL}x^{i(\omega-k)L}v^{-i\omega L}\right]\\
&= \sqrt{\frac{L}{2\pi\omega}}\frac{k_0-k_y}{8\pi}L^{ikL}\left[\int_0^\infty \mathrm{d}x\, x^{i(\omega-k)L}J_1(Kx)\right]\left[\int_0^\infty \mathrm{d}v\, v^{-i\omega L}e^{i\frac{(k_0-k_y)}{2}v}\right]\\
&= \sqrt{\frac{L}{2\pi\omega}}\frac{k_0-k_y}{8\pi}L^{ikL}\left[-\frac{(\omega-k)L}{2K\pi}\left(\frac{K}{2}\right)^{-iL(\omega-k)}\Gamma\left(\frac{iL}{2}(\omega-k)\right)^2\sinh\left(\frac{\pi L}{2}(\omega-k)\right)\right]\\
&\qquad\times\left[i\left(\frac{k_0-k_y}{2}\right)^{i\omega L-1}e^{\frac{\pi\omega L}{2}}\Gamma(1-i\omega L)\right]
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{\omega k_0,kk_y} = &-i\sqrt{\frac{L}{2\pi\omega}}\left(\frac{k_0-k_y}{2}\right)^{i\omega L}\frac{L^{1+ikL}}{8K\pi^2}\left(\frac{K}{2}\right)^{-iL(\omega-k)}(\omega-k)\sinh\left(\frac{\pi L}{2}(\omega-k)\right)\\
&\times\Gamma\left(\frac{iL}{2}(\omega-k)\right)^2\Gamma(1-i\omega L)e^{\frac{\pi\omega L}{2}}.
\end{aligned}
\tag{3.41}
$$

Similarly, the Bogoliubov coefficient $\beta_{\omega k_0,kk_y}$ can be obtained by evaluating the integral

$$
\beta_{\omega k_0,kk_y} = (k_0-k_y)\sqrt{\frac{L}{4\pi}}\int_0^\infty \mathrm{d}x\int_{-\infty}^\infty \mathrm{d}v\, e^{-i\frac{(k_0-k_y)}{2}v}J_1(Kx)g_{\omega,k}\Big|_{\bar{u}\to\infty}
\tag{3.42}
$$

$$
\begin{aligned}
&= (k_0-k_y)\sqrt{\frac{L}{4\pi}}\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}v\, e^{-i\frac{(k_0-k_y)}{2}v}J_1(Kx)\left[\frac{1}{2\pi\sqrt{2\omega}}L^{ikL}x^{i(\omega-k)L}v^{-i\omega L}\right]\\
&= \sqrt{\frac{L}{2\pi\omega}}\frac{k_0-k_y}{8\pi}L^{ikL}\left[\int_0^\infty \mathrm{d}x\, x^{i(\omega-k)L}J_1(Kx)\right]\left[\int_0^\infty \mathrm{d}v\, v^{-i\omega L}e^{-i\frac{(k_0-k_y)}{2}v}\right]\\
&= \sqrt{\frac{L}{2\pi\omega}}\frac{k_0-k_y}{8\pi}L^{ikL}\left[-\frac{(\omega-k)L}{2K\pi}\left(\frac{K}{2}\right)^{-iL(\omega-k)}\Gamma\left(\frac{iL}{2}(\omega-k)\right)^2\sinh\left(\frac{\pi L}{2}(\omega-k)\right)\right]\\
&\qquad\times\left[-i\left(\frac{k_0-k_y}{2}\right)^{i\omega L-1}e^{-\frac{\pi\omega L}{2}}\Gamma(1-i\omega L)\right]
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\omega k_0,kk_y} = &\, i\sqrt{\frac{L}{2\pi\omega}}\left(\frac{k_0-k_y}{2}\right)^{i\omega L}\frac{L^{1+ikL}}{8K\pi^2}\left(\frac{K}{2}\right)^{-iL(\omega-k)}(\omega-k)\sinh\left(\frac{\pi L}{2}(\omega-k)\right)\\
&\times\Gamma\left(\frac{iL}{2}(\omega-k)\right)^2\Gamma(1-i\omega L)e^{-\frac{\pi\omega L}{2}}.
\end{aligned}
\tag{3.43}
$$

Having these expressions for the Bogoliubov coefficients, it can be seen that they are related by

$$
\alpha_{\omega k_0,kk_y} = -e^{\pi\omega L}\beta_{\omega k_0,kk_y},
\tag{3.44}
$$

and the property of the Bogoliubov coefficients (3.31) is then

$$
\int_0^\infty \mathrm{d}k_0\int \mathrm{d}k_y\left(e^{\pi(\omega+\omega')L}-1\right)\beta_{\omega k_0,kk_y}\beta^*_{\omega' k_0,k'k_y} = \delta(\omega-\omega')\delta(k-k').
\tag{3.45}
$$

This then implies that the expected number of particles (3.35) is

$$\langle 0_{\text{PAdS}}|b^{\dagger}_{\omega,k}b_{\omega,k}|0_{\text{PAdS}}\rangle = \int_0^{\infty} \mathrm{d}k_0 \int \mathrm{d}k_y \left|\beta_{\omega k_0,kk_y}\right|^2 = \frac{1}{e^{2\pi\omega L}-1}\delta(0)^2 \tag{3.46}$$

where the factor $\delta(0)^2$ can be interpreted as proportional to the total spatial volume and the quantity

$$n(\omega) = \frac{1}{e^{2\pi\omega L}-1} \tag{3.47}$$

is the particle number density observed by the RAdS observer. This particle number density is in the form of Bose-Einstein distribution with energy $E = \omega$ and Unruh temperature $T = 1/(2\pi L)$, in agreement with [19, 21].

4 The Unruh Effect in Rotating Rindler-AdS Spacetime

The Unruh effect in rotating Rindler-AdS discussion in this chapter relies on section 3.1 which discusses the massless KG field in PAdS. This chapter starts with the discussion of the massless KG field in rotating Rindler-AdS and then moves on to the Bogoliubov transformation and the Unruh effect. The Unruh temperature in rotating Rindler-AdS, stated in [19], is $T = (1-\beta^2)/(2\pi L)$. The goal of this chapter is to obtain this Unruh temperature with the approach used in section 2.3 and chapter 3.

4.1 Massless KG Field in Rotating Rindler-AdS (RRAdS)

As discussed in subsection 2.1.3, the RRAdS metric can be obtained using the transformation (2.36) on the PAdS metric as

$$\mathrm{d}s^2 = -\left[e^{\frac{2\xi}{L}}\left(1-\beta^2\right)-\beta^2\right]\mathrm{d}\tau^2 - 2\beta\,\mathrm{d}\tau\,\mathrm{d}\chi + \frac{e^{\frac{2\xi}{L}}\,\mathrm{d}\xi^2}{1+e^{\frac{2\xi}{L}}} + \left[1+e^{\frac{2\xi}{L}}\left(1-\beta^2\right)\right]\mathrm{d}\chi^2\,, \tag{4.1}$$

with $-\infty < (\tau, \xi, \chi) < \infty$. The expressions of the determinant of the metric g and the inverse metric $g^{\mu\nu}$ are needed to express the massless KG equation (2.55) in this spacetime. To calculate the metric determinant it is convenient to write the metric $g_{\mu\nu}$ in matrix form, i.e.,

$$[g_{\mu\nu}] = \begin{bmatrix} -\left[e^{\frac{2\xi}{L}}(1-\beta^2)-\beta^2\right] & 0 & -\beta \\ 0 & \frac{e^{\frac{2\xi}{L}}}{1+e^{\frac{2\xi}{L}}} & 0 \\ -\beta & 0 & \left[1+e^{\frac{2\xi}{L}}(1-\beta^2)\right] \end{bmatrix}. \tag{4.2}$$

The metric determinant is then can be calculated using the technique of calculating the determinant of a matrix, which yields

$$
\begin{aligned}
g &= \frac{e^{\frac{2\xi}{L}}}{1+e^{\frac{2\xi}{L}}}\left\{-\left[e^{\frac{2\xi}{L}}\left(1-\beta^2\right)-\beta^2\right]\left[1+e^{\frac{2\xi}{L}}\left(1-\beta^2\right)\right]-\beta^2\right\} \\
&= -\frac{e^{\frac{2\xi}{L}}}{1+e^{\frac{2\xi}{L}}}\left[e^{\frac{2\xi}{L}}\left(1-\beta^2\right)+e^{\frac{4\xi}{L}}\left(1-\beta^2\right)^2-\beta^2-\beta^2 e^{\frac{2\xi}{L}}\left(1-\beta^2\right)+\beta^2\right] \\
&= -\frac{e^{\frac{2\xi}{L}}}{1+e^{\frac{2\xi}{L}}}\left[e^{\frac{2\xi}{L}}\left(1-\beta^2\right)^2+e^{\frac{4\xi}{L}}\left(1-\beta^2\right)^2\right] \\
&= -\frac{e^{\frac{4\xi}{L}}\left(1-\beta^2\right)^2}{1+e^{\frac{2\xi}{L}}}\left[1+e^{\frac{2\xi}{L}}\right] \\
g &= -e^{\frac{4\xi}{L}}\left(1-\beta^2\right)^2 \\
\implies \sqrt{|g|} &= e^{\frac{2\xi}{L}}\left(1-\beta^2\right).
\end{aligned} \tag{4.3}
$$

Meanwhile, the inverse metric in matrix form can be calculated as the inverse matrix of the metric, yielding

$$
\left[g^{\mu\nu}\right] = \begin{bmatrix}
-\frac{1+e^{-\frac{2\xi}{L}}-\beta^2}{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2} & 0 & -\frac{\beta e^{-\frac{2\xi}{L}}}{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2} \\
0 & 1+e^{-\frac{2\xi}{L}} & 0 \\
-\frac{\beta e^{-\frac{2\xi}{L}}}{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2} & 0 & \frac{1-\beta^2\left(1+e^{-\frac{2\xi}{L}}\right)}{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2}
\end{bmatrix}. \tag{4.4}
$$

Performing the index summation in (2.55) and noting that the inverse metric components $g^{\tau\xi} = g^{\xi\tau} = g^{\xi\chi} = g^{\chi\xi} = 0$, the massless KG equation is then

$$
\begin{aligned}
0 &= \partial_\tau\left[\sqrt{|g|}g^{\tau\tau}\partial_\tau\phi + \sqrt{|g|}g^{\tau\chi}\partial_\chi\phi\right]\partial_\chi\left[\sqrt{|g|}g^{\chi\chi}\partial_\chi\phi + \sqrt{|g|}g^{\chi\tau}\partial_\tau\phi\right]\partial_\xi\left[\sqrt{|g|}g^{\xi\xi}\partial_\xi\phi\right] \\
&= \sqrt{|g|}g^{\tau\tau}\partial_\tau^2\phi + 2\sqrt{|g|}g^{\tau\chi}\partial_\tau\partial_\chi\phi + \sqrt{|g|}g^{\chi\chi}\partial_\chi^2\phi + \partial_\xi\left[\sqrt{|g|}g^{\xi\xi}\partial_\xi\phi\right] \\
\to 0 &= -\frac{\partial^2\phi}{\partial\tau^2} - 2\frac{g^{\tau\chi}}{g^{\tau\tau}}\frac{\partial^2\phi}{\partial\tau\partial\chi} - \frac{g^{\chi\chi}}{g^{\tau\tau}}\frac{\partial^2\phi}{\partial\chi^2} - \frac{1}{\sqrt{|g|}g^{\tau\tau}}\frac{\partial}{\partial\xi}\left[\sqrt{|g|}g^{\xi\xi}\frac{\partial\phi}{\partial\xi}\right].
\end{aligned} \tag{4.5}
$$

Substituting in the metric determinant and components of the inverse metric yields

$$
\begin{aligned}
0 = &-\frac{\partial^2\phi}{\partial\tau^2} - 2\frac{\beta e^{-\frac{2\xi}{L}}}{1+e^{-\frac{2\xi}{L}}-\beta^2}\frac{\partial^2\phi}{\partial\tau\partial\chi} + \frac{1-\beta^2\left(1+e^{-\frac{2\xi}{L}}\right)}{1+e^{-\frac{2\xi}{L}}-\beta^2}\frac{\partial^2\phi}{\partial\chi^2} \\
&+ \frac{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2}{e^{\frac{2\xi}{L}}(1-\beta^2)\left(1+e^{-\frac{2\xi}{L}}-\beta^2\right)}\frac{\partial}{\partial\xi}\left[e^{\frac{2\xi}{L}}\left(1-\beta^2\right)\left(1+e^{-\frac{2\xi}{L}}\right)\frac{\partial\phi}{\partial\xi}\right]
\end{aligned} \tag{4.6}
$$

which then simplifies to

$$-\frac{\partial^2\phi}{\partial\tau^2}-\left[\frac{2\beta}{1+e^{\frac{2\xi}{L}}(1-\beta^2)}\right]\frac{\partial^2\phi}{\partial\chi\partial\tau}+\left[\frac{e^{\frac{2\xi}{L}}(1-\beta^2)-\beta^2}{1+e^{\frac{2\xi}{L}}(1-\beta^2)}\right]\frac{\partial^2\phi}{\partial\chi^2}$$
$$+\left[\frac{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2}{1+e^{\frac{2\xi}{L}}(1-\beta^2)}\right]\frac{\partial}{\partial\xi}\left[\left(1+e^{\frac{2\xi}{L}}\right)\frac{\partial\phi}{\partial\xi}\right]=0. \quad (4.7)$$

The solution of this KG equation can then be expanded as

$$\phi(\tau,\xi,\chi)=\int_0^\infty \mathrm{d}\omega\int \mathrm{d}k\,\left[b_{\omega k}g_{\omega k}(\tau,\xi,\chi)+b^\dagger_{\omega k}g^*_{\omega k}(\tau,\xi,\chi)\right], \quad (4.8)$$

where $b_{\omega,k}(b^\dagger_{\omega,k})$ is the annihilation (creation) operator and $g_{\omega,k}$ are the positive-frequency modes given by

$$g_{\omega,k}(\tau,\xi,\chi)=\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega\tau+ik\chi}G_{\omega,k}(\xi),\quad \omega^2-k^2>0,\ \omega>0. \quad (4.9)$$

From the KG equation (4.7), the function $G_{\omega,k}$ satisfies the differential equation

$$\omega^2G_{\omega,k}=-\left[\frac{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)^2}{1+e^{\frac{2\xi}{L}}(1-\beta^2)}\right]\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(1+e^{\frac{2\xi}{L}}\right)\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi}\right]$$
$$+\frac{2\beta\omega k+k^2\left[e^{\frac{2\xi}{L}}(1-\beta^2)-\beta^2\right]}{1+e^{\frac{2\xi}{L}}(1-\beta^2)}G_{\omega,k} \quad (4.10)$$

and can be written as

$$G_{\omega,k}(\xi)=\sqrt{\frac{\omega}{2\pi(\omega-\beta k)}}\left(1+e^{\frac{2\xi}{L}}\right)^{\frac{iL}{2}\frac{k-\beta\omega}{1-\beta^2}}$$
$$\times\left[e^{-\frac{i\xi(\omega-\beta k)}{1-\beta^2}}F\left(-\frac{iL}{2}\frac{\omega-k}{1-\beta},1-\frac{iL}{2}\frac{\omega-k}{1-\beta};1-\frac{iL(\omega-\beta k)}{1-\beta^2};-e^{\frac{2\xi}{L}}\right)\right.$$
$$\left.+e^{\frac{i\xi(\omega-\beta k)}{1-\beta^2}}F\left(\frac{iL}{2}\frac{\omega+k}{1+\beta},1+\frac{iL}{2}\frac{\omega+k}{1+\beta};1+\frac{iL(\omega-\beta k)}{1-\beta^2};-e^{\frac{2\xi}{L}}\right)\right] \quad (4.11)$$

where $F(a,b;c;z)$ is the hypergeometric function [24]. The field quantization imposes the commutation relations

$$\begin{aligned}\left[b_{\omega,k},b^\dagger_{\omega',k'}\right]&=\delta(\omega-\omega')\delta(k-k'),\\ \left[b_{\omega,k},b_{\omega',k'}\right]&=\left[b^\dagger_{\omega,k},b^\dagger_{\omega',k'}\right]=0.\end{aligned} \quad (4.12)$$

The positive-frequency modes $g_{\omega,k}$ are normalized under the KG inner product (2.59). Let us calculate the inner product between two positive frequency modes $g_{\omega,k}$ and $g_{\omega',k'}$ to show this normalization, yielding

$$\begin{aligned}
\langle g_{\omega,k}, g_{\omega',k'}\rangle &= \frac{i}{1-\beta^2}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{-\infty}^{\infty}\mathrm{d}\chi\left(\frac{1}{1+e^{\frac{2\xi}{L}}}\right)\Big\{\Big[1+e^{\frac{2\xi}{L}}\big(1-\beta^2\big)\Big] \\
&\quad\times\left[g^*_{\omega,k}(\partial_\tau g_{\omega',k'})-\big(\partial_\tau g^*_{\omega,k}\big)g_{\omega',k'}\right]+\beta\big[g^*_{\omega,k}(\partial_\chi g_{\omega',k'})-\big(\partial_\chi g^*_{\omega,k}\big)g_{\omega',k'}\big]\Big\} \\
&= \frac{e^{i(\omega-\omega')\tau}}{2(1-\beta^2)\sqrt{\omega\omega'}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\chi}{2\pi}e^{-i(k-k')\chi}\int_{-\infty}^{\infty}\mathrm{d}\xi\left[\frac{1}{1+e^{\frac{2\xi}{L}}}\right]\Big\{\Big[1+e^{\frac{2\xi}{L}}\big(1-\beta^2\big)\Big] \\
&\qquad\times(\omega+\omega')-\beta(k+k')\Big\}G^*_{\omega,k}G_{\omega',k'} \\
\langle g_{\omega,k}, g_{\omega',k'}\rangle &= \frac{e^{i(\omega-\omega')\tau}\delta(k-k')}{2(1-\beta^2)\sqrt{\omega\omega'}}\int_{-\infty}^{\infty}\mathrm{d}\xi\left[\frac{1}{1+e^{\frac{2\xi}{L}}}\right]\Big\{\Big[1+e^{\frac{2\xi}{L}}\big(1-\beta^2\big)\Big](\omega+\omega')-2\beta k\Big\}G^*_{\omega,k}G_{\omega',k}.
\end{aligned} \tag{4.13}$$

To evaluate this inner product, let us focus on the first term of the integral and define

$$S_A(\omega,\omega')\equiv\int_{-A}^{\infty}\mathrm{d}\xi\,(\omega+\omega')\left[\frac{1+e^{\frac{2\xi}{L}}(1-\beta^2)}{\Big(1+e^{\frac{2\xi}{L}}\Big)(1-\beta^2)}\right]G^*_{\omega,k}G_{\omega',k}, \tag{4.14}$$

so that the first term of the integral is S_A in the limit $A\to\infty$. Multiplying S_A by $(\omega^2-\omega'^2)$ and using the differential equation (4.10) yield

$$\begin{aligned}
(\omega^2-\omega'^2)S_A(\omega,\omega') &= \int_{-A}^{\infty}\mathrm{d}\xi\,(\omega+\omega')\left[\frac{1+e^{\frac{2\xi}{L}}(1-\beta^2)}{\Big(1+e^{\frac{2\xi}{L}}\Big)(1-\beta^2)}\right]\Big[\big(\omega^2G^*_{\omega,k}\big)G_{\omega',k}-G^*_{\omega,k}\big(\omega'^2G_{\omega',k}\big)\Big] \\
&= -\int_{-A}^{\infty}\mathrm{d}\xi\,(\omega+\omega')\big(1-\beta^2\big)\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\Big(1+e^{\frac{2\xi}{L}}\Big)\left[\frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi}G_{\omega',k}-G^*_{\omega,k}\frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi}\right]\right\} \\
&\qquad+\int_{-A}^{\infty}\frac{2\beta k(\omega^2-\omega'^2)}{\Big(1+e^{\frac{2\xi}{L}}\Big)(1-\beta^2)}G^*_{\omega,k}G_{\omega',k} \\
(\omega^2-\omega'^2)S_A(\omega,\omega') &= -(\omega+\omega')\big(1-\beta^2\big)\left[\Big(1+e^{\frac{2\xi}{L}}\Big)\left(\frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi}G_{\omega',k}-G^*_{\omega,k}\frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi}\right)\right]_{\xi=-A}^{\infty} \\
&\qquad+\int_{-A}^{\infty}\frac{2\beta k(\omega^2-\omega'^2)}{\Big(1+e^{\frac{2\xi}{L}}\Big)(1-\beta^2)}G^*_{\omega,k}G_{\omega',k}
\end{aligned} \tag{4.15}$$

$$\implies S_A(\omega,\omega') = -\frac{1-\beta^2}{\omega-\omega'}\left[\left(1+e^{\frac{2\xi}{L}}\right)\left(\frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi}G_{\omega',k} - G^*_{\omega,k}\frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi}\right)\right]_{\xi=-A}^{\infty} + \int_{-A}^{\infty}\mathrm{d}\xi\,\frac{2\beta k\,G^*_{\omega,k}G_{\omega',k}}{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)}. \quad (4.16)$$

Applying the limit $A\to\infty$ to S_A and substituting it into the inner product (4.13) yield

$$\begin{aligned}\langle g_{\omega,k}, g_{\omega',k'}\rangle &= \frac{e^{i(\omega-\omega')\tau}\delta(k-k')}{2\sqrt{\omega\omega'}}\left[\lim_{A\to\infty} S_A(\omega,\omega') - \int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{2\beta k\,G^*_{\omega,k}G_{\omega',k}}{\left(1+e^{\frac{2\xi}{L}}\right)(1-\beta^2)}\right]\\ &= -\frac{e^{i(\omega-\omega')\tau}\delta(k-k')}{2\sqrt{\omega\omega'}}\left[\frac{1-\beta^2}{\omega-\omega'}\right]\lim_{A\to\infty}\left[\left(1+e^{\frac{2\xi}{L}}\right)\left(\frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi}G_{\omega',k} - G^*_{\omega,k}\frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi}\right)\right]_{\xi=-A}^{\infty}.\end{aligned} \quad (4.17)$$

To evaluate these limits, let us focus on the lower limit $\xi\to-\infty$ of $G_{\omega,k}$ (4.11) first. In that case, similar to the non-rotating RAdS case, the hypergeometric function $F\to 1$ because its argument approaches zero and $F(a,b;c;0)=1$. Additionally, the factor $(1+\exp(2\xi/L))^{iL(k-\beta\omega)/[2(1-\beta^2)]}\to 1$ as $\xi\to-\infty$. Therefore, we have

$$G_{\omega,k} \overset{\xi\to-\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\omega}{\omega-\beta k}}\left(e^{\frac{i\xi(\omega-\beta k)}{1-\beta^2}} + e^{-\frac{i\xi(\omega-\beta k)}{1-\beta^2}}\right) \quad (4.18)$$

and similarly,

$$\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi} \overset{\xi\to-\infty}{\longrightarrow} \frac{i(\omega-\beta k)}{1-\beta^2}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\omega}{\omega-\beta k}}\left(e^{\frac{i\xi(\omega-\beta k)}{1-\beta^2}} - e^{-\frac{i\xi(\omega-\beta k)}{1-\beta^2}}\right). \quad (4.19)$$

For the upper limit $\xi\to\infty$, using the identities of the hypergeometric function (3.22) and (3.23) simplifies $G_{\omega,k}$ to

$$\lim_{\xi\to\infty} G_{\omega,k} = \sqrt{\frac{\omega}{2\pi(\omega-\beta k)}}\left[\frac{\Gamma\left(1+iL\frac{\omega-\beta k}{1-\beta^2}\right)}{\Gamma\left(1+\frac{iL}{2}\frac{\omega-k}{1-\beta}\right)\Gamma\left(1+\frac{iL}{2}\frac{\omega+k}{1+\beta}\right)} + \frac{\Gamma\left(1-iL\frac{\omega-\beta k}{1-\beta^2}\right)}{\Gamma\left(1-\frac{iL}{2}\frac{\omega-k}{1-\beta}\right)\Gamma\left(1-\frac{iL}{2}\frac{\omega+k}{1+\beta}\right)}\right]. \quad (4.20)$$

And similarly, the derivative $\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi}$ simplifies to

$$\begin{aligned}\lim_{\xi\to\infty}\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi} &= -\frac{2}{L}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\omega}{\omega-\beta k}}\left[\frac{\Gamma\left(1+iL\frac{\omega-\beta k}{1-\beta^2}\right)}{\Gamma\left(1+\frac{iL}{2}\frac{\omega-k}{1-\beta}\right)\Gamma\left(1+\frac{iL}{2}\frac{\omega+k}{1+\beta}\right)}\right.\\ &\qquad\left.+\frac{\Gamma\left(1-iL\frac{\omega-\beta k}{1-\beta^2}\right)}{\Gamma\left(1-\frac{iL}{2}\frac{\omega-k}{1-\beta}\right)\Gamma\left(1-\frac{iL}{2}\frac{\omega+k}{1+\beta}\right)}\right]\\ \lim_{\xi\to\infty}\frac{\mathrm{d}G_{\omega,k}}{\mathrm{d}\xi} &= -\frac{2}{L}\lim_{\xi\to\infty} G_{\omega,k}.\end{aligned} \quad (4.21)$$

Substituting in these limits (4.18), (4.19), (4.20), and (4.21) into the inner product (4.17) yields

$$\begin{aligned}\langle g_{\omega,k}, g_{\omega',k'}\rangle =& -\frac{e^{i(\omega-\omega')\tau}\delta(k-k')}{2\sqrt{\omega\omega'}}\left(\frac{1-\beta^2}{\omega-\omega'}\right)\lim_{A\to\infty}\left[\left(1+e^{\frac{2\xi}{L}}\right)\left(\frac{\mathrm{d}G^*_{\omega,k}}{\mathrm{d}\xi}G_{\omega',k}-G^*_{\omega,k}\frac{\mathrm{d}G_{\omega',k}}{\mathrm{d}\xi}\right)\right]_{\xi=-A}^{\infty}\\
=& \frac{e^{i(\omega-\omega')\tau}\delta(k-k')}{2(1-\beta^2)}\frac{\omega+\omega'-2\beta k}{\sqrt{(\omega-\beta k)(\omega'-\beta k)}}\lim_{A\to\infty}\left[\frac{1}{\pi}\frac{1-\beta^2}{\omega-\omega'}\sin\left(A\frac{\omega-\omega'}{1-\beta^2}\right)\right.\\
&\left.+\frac{1}{\pi}\frac{1-\beta^2}{\omega+\omega'+2\beta k}\sin\left(A\frac{\omega+\omega'-2\beta k}{1-\beta^2}\right)\right]\\
=& \frac{e^{i(\omega-\omega')\tau}\delta(k-k')}{2(1-\beta^2)}\frac{\omega+\omega'-2\beta k}{\sqrt{(\omega-\beta k)(\omega'-\beta k)}}\left[\delta\left(\frac{\omega-\omega'}{1-\beta^2}\right)+\delta\left(\frac{\omega+\omega'-2\beta k}{1-\beta^2}\right)\right]\\
\langle g_{\omega,k}, g_{\omega',k'}\rangle =& \,\delta(\omega-\omega')\delta(k-k'), \end{aligned}\tag{4.22}$$

where the limit representation of the Dirac delta (3.27) and the properties

$$\delta(kx) = \frac{\delta(x)}{|k|}, \tag{4.23}$$

$$x\delta(x) = 0, \tag{4.24}$$

have been used. This shows that the positive-frequency modes $g_{\omega,k}$ are normalized under the KG inner product.

4.2 Bogoliubov Transformation and the Unruh effect in RRAdS

Through the Bogoliubov transformation, the RRAdS positive-frequency modes $g_{\omega,k}$ can be expressed in terms of the PAdS positive-frequency modes f_{k_0,k_y} (3.4) as

$$g_{\omega,k} = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha_{\omega k_0,kk_y} f_{k_0,k_y} + \beta_{\omega k_0,kk_y} f^*_{k_0,k_y}\right], \tag{4.25}$$

exactly the same form as (3.29). Classically, in terms of General Relativity, the spacetime covered by the RRAdS is the same as the RAdS, namely the right wedge region $|t| < y$. Just as in the non-rotating RAdS, the positive-frequency modes in the left wedge region $|t| < -y$ is defined to be $g_{\omega,k} \equiv 0$ and positive-frequency modes in the left wedge region is a copy of that in the right wedge. With the assumption that field quantization in both right and left

RRAdS wedges combined are equivalent to the field quantization in PAdS, the Bogoliubov transformation of the operator $b_{\omega,k}$ can be obtained as

$$b_{\omega,k} = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha^*_{\omega k_0,kk_y} a_{k_0,k_y} - \beta^*_{\omega k_0,kk_y} a^\dagger_{k_0,k_y}\right], \tag{4.26}$$

and the property of the Bogoliubov coefficients can be also obtained as

$$\int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left[\alpha_{\omega k_0,kk_y} \alpha^*_{\omega' k_0,k'k_y} - \beta_{\omega k_0,kk_y} \beta^*_{\omega' k_0,k'k_y}\right] = \delta(\omega-\omega')\delta(k-k'). \tag{4.27}$$

Considering the vacuum state in PAdS $|0_{\mathrm{PAdS}}\rangle$ defined by (3.32), the number of particles with energy k_0 and momentum k_y observed by an observer in PAdS is zero (3.33). The number of particles with energy ω and momentum k observed in the PAdS vacuum as observed in RRAdS can be calculated with the expectation value

$$\langle 0_{\mathrm{PAdS}}| b^\dagger_{\omega,k} b_{\omega,k} |0_{\mathrm{PAdS}}\rangle = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left|\beta_{\omega k_0,kk_y}\right|^2 \tag{4.28}$$

which expected to contain the particle number density in the form of Bose-Einstein distribution with Unruh temperature.

To evaluate the expressions for the Bogoliubov coefficients, let us perform the same calculations and first change the coordinates to

$$\begin{aligned} x &= \frac{L}{\sqrt{1+e^{\frac{2\xi}{L}}}} \exp\left(-\frac{\chi-\beta\tau}{L}\right) = \frac{L}{\sqrt{1+e^{(\bar{v}-\bar{u})/L}}}\, e^{-\bar{\chi}/L}, \\ u &\equiv t-y = -xe^{-(\tau-\beta\chi-\xi)/L} = -xe^{-\bar{u}/L}, \\ v &\equiv t+y = xe^{(\tau-\beta\chi+\xi)/L} = xe^{\bar{v}/L}, \end{aligned} \tag{4.29}$$

where $\bar{u} \equiv \tau - \beta\chi - \xi$, $\bar{v} \equiv \tau - \beta\chi + \xi$, and $\bar{\chi} \equiv \chi - \beta\tau$. Since the Bogoliubov coefficients do not depend on the coordinates, it is again convenient to examine the positive-frequency modes $g_{\omega,k}$ (4.9) near the horizon $u \to 0^-$ or $\bar{u} \to \infty$ with $x > 0$. For the limit $\bar{u} \to \infty$, with (4.18), the frequency modes becomes

$$\begin{aligned} g_{\omega,k}\Big|_{\bar{u}\to\infty} &\to \frac{e^{-i\omega\tau+ik\chi}}{2\pi\sqrt{2(\omega-\beta k)}}\left(e^{i\xi\frac{\omega-\beta k}{1-\beta^2}} + e^{-\xi\frac{\omega-\beta k}{1-\beta^2}}\right) \\ &\to \frac{e^{i\frac{k-\beta\omega}{1-\beta^2}\bar{\chi}}}{2\pi\sqrt{2(\omega-\beta k)}}\left(e^{-i\frac{\omega-\beta k}{1-\beta^2}\bar{u}} + e^{-i\frac{\omega-\beta k}{1-\beta^2}\bar{v}}\right) \\ &\to \frac{e^{i\bar{k}\bar{\chi}}}{2\pi\sqrt{2(\omega-\beta k)}}\left(e^{-i\bar{\omega}\bar{u}} + e^{-i\bar{\omega}\bar{v}}\right) \\ g_{\omega,k}\Big|_{\bar{u}\to\infty} &\to \frac{e^{i\bar{k}\bar{\chi}} e^{-i\bar{\omega}\bar{v}}}{2\pi\sqrt{2(\omega-\beta k)}}, \end{aligned} \tag{4.30}$$

where $\bar{\omega} = (\omega - \beta k)/(1-\beta^2) > 0$ and $\bar{k} = (k - \beta\omega)/(1-\beta^2)$ are defined simply to shorten the expression. The term with the factor $e^{-i\bar{\omega}\bar{u}}$ as $\bar{u} \to \infty$ is regarded as zero as it oscillates infinitely many times around zero and is bounded. Meanwhile, for the limit $u \to 0^-$, the positive frequency modes (4.25) become

$$g_{\omega,k}\Big|_{u\to 0^-} = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \, \frac{1}{\sqrt{4\pi L}} \Big[\alpha_{\omega k_0,kk_y} e^{-i\frac{(k_0-k_y)}{2}v} + \beta_{\omega k_0,kk_y} e^{i\frac{(k_0-k_y)}{2}v}\Big] x J_1(Kx) = g_{\omega,k}\Big|_{\bar{u}\to\infty}. \tag{4.31}$$

The expression (4.31) is in the same form as the non-rotating RAdS case (3.38). Therefore, the Bogoliubov coefficients $\alpha_{\omega k_0,kk_y}$ and $\beta_{\omega k_0,kk_y}$ can be obtained with the same way with

$$\alpha_{\omega k_0,kk_y} = (k_0 - k_y)\sqrt{\frac{L}{4\pi}} \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}v \, e^{i\frac{(k_0-k_y)}{2}v} J_1(Kx) g_{\omega,k}\Big|_{\bar{u}\to\infty} \tag{4.32}$$

and

$$\beta_{\omega k_0,kk_y} = (k_0 - k_y)\sqrt{\frac{L}{4\pi}} \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}v \, e^{-i\frac{(k_0-k_y)}{2}v} J_1(Kx) g_{\omega,k}\Big|_{\bar{u}\to\infty}. \tag{4.33}$$

Let us first express $g_{\omega,k}\big|_{\bar{u}\to\infty}$ (4.30) in (u, v, x) coordinates using (4.29), i.e.,

$$g_{\omega,k}\Big|_{\bar{u}\to\infty} \to \frac{1}{2\pi\sqrt{2(\omega - \beta k)}} \left(\frac{x}{L}\right)^{-i\bar{k}L} \left(\frac{v}{x}\right)^{-i\bar{\omega}L} = \frac{L^{i\bar{k}L}}{2\pi\sqrt{2(\omega-\beta k)}} x^{i(\bar{\omega}-\bar{k})L} v^{-i\bar{\omega}L}. \tag{4.34}$$

Substituting in this expression into (4.32) and noting that $g_{\omega,k} \equiv 0$ in the left wedge $v < 0$ region yield a very similar integral to that in the non-rotating RAdS case (3.41). Therefore, the Bogoliubov coefficient $\alpha_{\omega k_0,kk_y}$ can be quickly found as

$$\begin{aligned}\alpha_{\omega k_0,kk_y} = -i\sqrt{\frac{L}{2\pi(\omega-\beta k)}} \left(\frac{k_0-k_y}{2}\right)^{i\bar{\omega}L} \frac{L^{1+i\bar{k}L}}{8K\pi^2} \left(\frac{K}{2}\right)^{-iL(\bar{\omega}-\bar{k})} (\bar{\omega} - \bar{k}) \\ \times \sinh\left(\frac{\pi L}{2}(\bar{\omega}-\bar{k})\right) \Gamma\left(\frac{iL}{2}(\bar{\omega}-\bar{k})\right)^2 \Gamma(1 - i\bar{\omega}L) e^{\pi\bar{\omega}L/2}.\end{aligned} \tag{4.35}$$

Similarly, the Bogoliubov coefficient $\beta_{\omega k_0,kk_y}$ can be found as

$$\begin{aligned}\beta_{\omega k_0,kk_y} = i\sqrt{\frac{L}{2\pi(\omega-\beta k)}} \left(\frac{k_0-k_y}{2}\right)^{i\bar{\omega}L} \frac{L^{1+i\bar{k}L}}{8K\pi^2} \left(\frac{K}{2}\right)^{-iL(\bar{\omega}-\bar{k})} (\bar{\omega} - \bar{k}) \\ \times \sinh\left(\frac{\pi L}{2}(\bar{\omega}-\bar{k})\right) \Gamma\left(\frac{iL}{2}(\bar{\omega}-\bar{k})\right)^2 \Gamma(1 - i\bar{\omega}L) e^{-\pi\bar{\omega}L/2}.\end{aligned} \tag{4.36}$$

Having these expressions for the Bogoliubov coefficients, it can be seen that they are related by

$$\alpha_{\omega k_0,kk_y} = -e^{\pi\bar{\omega}L} \beta_{\omega k_0,kk_y} = -e^{\frac{\pi L(\omega-\beta k)}{1-\beta^2}} \beta_{\omega k_0,kk_y}, \tag{4.37}$$

and the property of the Bogoliubov coefficients (4.27) is

$$\int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left(e^{\pi(\bar{\omega}+\bar{\omega}')L} - 1\right)\beta_{\omega k_0,kk_y}\beta^*_{\omega' k_0,k'k_y} = \delta(\omega-\omega')\delta(k-k'), \tag{4.38}$$

where $\bar{\omega}' = (\omega' - \beta k')/(1-\beta^2)$. This then implies that the expected number of particles (4.28) is

$$\langle 0_{\text{PAdS}}|b^\dagger_{\omega,k}b_{\omega,k}|0_{\text{PAdS}}\rangle = \int_0^\infty \mathrm{d}k_0 \int \mathrm{d}k_y \left|\beta_{\omega k_0,kk_y}\right|^2 = \frac{1}{e^{2\pi\bar{\omega}L}-1}\delta(0)^2 = \frac{1}{e^{\frac{2\pi(\omega-\beta k)L}{1-\beta^2}}-1}\delta(0)^2 \tag{4.39}$$

where the factor $\delta(0)^2$ can be interpreted as proportional to the total spatial volume and the quantity

$$n(\omega,k) = \frac{1}{e^{\frac{2\pi(\omega-\beta k)L}{1-\beta^2}}-1} \tag{4.40}$$

is the particle number density observed by an observer in RRAdS. The particle density of the Bose-Einstein distribution is

$$n = \frac{1}{e^{(E-\mu)/T}-1}, \tag{4.41}$$

where E is the energy, μ is the chemical potential, and T is the temperature. The resulting particle number density (4.40) is in the same form as (4.41) with Unruh temperature

$$T = \frac{1-\beta^2}{2\pi L}, \tag{4.42}$$

which is in agreement with references [19, 21].

5 Summary

In the process of obtaining the Unruh temperature, we have established several techniques because they are required or, at least, convenient for the calculations. Namely, the Bogoliubov transformations of the frequency modes and field operators, the equivalence of field quantizations in the two spacetimes, the notion of inner product and orthonormality between fields or frequency modes, and the coordinate transformations between the two spacetimes involved. The Bogoliubov transformation allows us to express the quantum field in one spacetime in terms of the field in the other spacetime, which is used to illustrate the Unruh effect. The equivalence of field quantizations in the two spacetimes involved is crucial in obtaining the Bogoliubov transformations of the annihilation and creation operators, which is needed to calculate the Unruh temperature with this approach and it was assumed to be true in this thesis. The inner product used in this thesis is the Klein-Gordon inner product which, combined with the orthonormality of the frequency modes, allows us to express the inverse Bogoliubov transformation in a straightforward way. Lastly, the coordinate transformation between the two spacetimes involved allows us to calculate the explicit expressions of the Bogoliubov coefficients, which is instrumental in calculating the Unruh temperature.

The Unruh temperature in the rotating Rindler-AdS spacetime is obtained with the field theoretic method by employing the techniques described above. The resulting Unruh temperature is consistent with the dimensional analysis made in a previous study on the spacetime [21].

Further work will be needed to prove that the field quantization in the right and left Rindler wedge of rotating and non-rotating Rindler-AdS is equivalent to that in Poincaré AdS by calculating the two-point function of the field $\langle 0_{\mathrm{PAdS}}|\phi(x)\phi(x')|0_{\mathrm{PAdS}}\rangle$. The field theoretic method, which was outlined in this thesis, can also be extended for higher spin fields as well as excited states for future research. Additionally, a different approach to examining the

Unruh effect in rotating Rindler-AdS such as the path integral formulation or the response of an Unruh detector is also possible for future research in this subject. As discussed in chapter 1, the rotating Rindler-AdS spacetime can potentially be promoted to a black hole. If that is the case, it is possible to examine the black hole in the context of black hole holography and find the associated holographic CFT quantity to the Unruh temperature. Lastly, it should not be overlooked that it is interesting to examine the rotating Rindler-AdS in the context of AdS/CFT correspondence. Both the black hole holography and the AdS/CFT correspondence hold the potential to further our knowledge of the quantum nature of gravity.

References

[1] A. Einstein, "The formal foundation of the general theory of relativity," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1914**, 1030 (1914).

[2] N. D. Birrell and P. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge university press, Cambridge (GB), 1989).

[3] S. A. Fulling, "Nonuniqueness of canonical field quantization in Riemannian space-time," Physical Review D **7**, 2850 (1973).

[4] P. C. Davies, "Scalar production in Schwarzschild and Rindler metrics," Journal of Physics A: Mathematical and General **8**, 609 (1975).

[5] W. G. Unruh, "Notes on black-hole evaporation," Phys. Rev. D **14**, 870 (1976).

[6] E. T. Akhmedov and D. Singleton, "On the physical meaning of the Unruh effect," JETP Letters **86**, 615 (2008).

[7] D. Buchholz and C. Solveen, "Unruh effect and the concept of temperature," Class. Quantum Grav. **30**, 085011 (2013).

[8] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, "The Unruh effect and its applications," Rev. Mod. Phys. **80**, 787 (2008), arXiv:0710.5373 [gr-qc].

[9] G. C. V. Mera, "A review on the Unruh effect and the thermofield-double state," (2020), arXiv:2001.09869v1 [hep-th].

[10] E. Arias, T. R. de Oliveira, and M. S. Sarandy, "The unruh quantum otto engine," J. High Energ. Phys. **2018** (2).

[11] A. Mukherjee, S. Gangopadhyay, and A. S. Majumdar, "Unruh quantum Otto engine in the presence of a reflecting boundary," J. High Energ. Phys. **2022** (9).

[12] S. Biermann, S. Erne, C. Gooding, J. Louko, J. Schmiedmayer, W. G. Unruh, and S. Weinfurtner, "Unruh and analogue Unruh temperatures for circular motion in $3+1$ and $2+1$ dimensions," Phys. Rev. D **102**, 10.1103/physrevd.102.085006 (2020).

[13] G. E. Volovik, "Particle Creation: Schwinger + Unruh + Hawking," JETP Letters **116**, 595 (2022).

[14] A. Chen, "Generalized Unruh effect: A potential resolution to the black hole information paradox," Phys. Rev. D **107**, 10.1103/physrevd.107.056014 (2023).

[15] S. Deser and O. Levin, "Accelerated detectors and temperature in (anti-) de Sitter spaces," Class. Quant. Grav. **14**, L163 (1997), arXiv:gr-qc/9706018.

[16] J. Maldacena, "The large-N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998), arXiv:hep-th/9711200.

[17] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, "Replica wormholes and the black hole interior," J. High Energy Phys. **2022** (3), 1.

[18] M. Guica, T. Hartman, W. Song, and A. Strominger, "The Kerr/CFT correspondence," Phys. Rev. D **80**, 10.1103/physrevd.80.124008 (2009).

[19] M. Parikh, P. Samantray, and E. Verlinde, "Rotating Rindler-AdS Space," Phys. Rev. D **86**, 024005 (2012), arXiv:1112.3433 [hep-th].

[20] J. B. Griffiths and J. Podolsky, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, 2009).

[21] M. Parikh and P. Samantray, "Rindler-AdS/CFT," J. High Energ. Phys. **2018** (10).

[22] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Pub. Co, Reading, Mass, 1995).

[23] J.-W. van Holten, "Conformal Symmetry and Supersymmetry in Rindler Space," Universe **6**, 144 (2020).

[24] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Elsevier Academic Press, 2007).

[25] I. Fujisawa and R. Nakayama, "Quantization of a scalar field in two Poincaré patches of anti-de Sitter space and AdS/CFT," Nucl. Phys. B **886**, 135 (2014).

[26] C. Dappiaggi and H. R. C. Ferreira, "On the algebraic quantization of a massive scalar field in anti-de Sitter spacetime," Rev. Math. Phys. **30**, 1850004 (2018).

Appendix A

Units and Convention

The units used throughout this thesis is the natural units such that

$$G = c = \hbar = k_B = 1, \tag{A.1}$$

where G is the Newton's gravitational constant, c is the speed of light, $\hbar$ is the reduced Planck constant, and k_B is the Boltzmann constant. The metric signature used throughout is the mostly-positive signature.