

**ORDERS AND SIGNATURES OF HIGHER LEVEL
ON A COMMUTATIVE RING**

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ABSTRACT

One obtains orders of higher level in a commutative ring A by pulling back the higher level orders in the residue fields of its prime ideals. Since inclusion relationships can hold amongst the higher level orders in a field (unlike the level 1 situation), there may exist orders in the ring A which are not contained in a unique order maximal with respect to inclusion. However, if the specializations of an order P are defined to be those orders $Q \supseteq P$ such that $Q \setminus P \subseteq Q \cap -Q$, every higher level order in A is contained in a unique maximal specialization. The real spectrum of A relative to a higher level preorder T is defined to be the set $\text{Sper}_T A$ of all orders in A containing T . As with the ordinary real spectrum of Coste and Roy, $\text{Sper}_T A$ is given a compact topology in which the closed points are precisely the orders in A maximal with respect to specialization. For 2-primary level, we show that an abstract higher level version of the Hormander-Lojasiewicz Inequality holds and use it to characterize the basic sets in $\text{Sper}_T A$.

A higher level signature on a commutative ring A is a pull-back σ of a higher level signature on the residue field of some prime ideal \mathfrak{p} with $\sigma(\mathfrak{p}) = 0$. If T is a higher level preorder in A and $\sigma(T) = \{0, 1\}$ then σ is called a T -signature. Specializations of T -signatures are defined just as for orders and every T -signature is shown to have a unique maximal specialization. Each T -signature σ determines a unique order in A containing T which is maximal with respect to specialization iff σ is. Generalizing a result of M. Marshall, we show for a higher level preorder T in a commutative ring satisfying a certain simple axiom, the space X_T of all maximal T -signatures can be embedded in the character group of a suitable abelian group G_T of finite even exponent and under this embedding, the pair (X_T, G_T) is a space of signatures in the sense of Mulcahy and Marshall.

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LIST OF SYMBOLS

A^*	the group of units of the ring A
A^n	the set of elements of the form a^n , $a \in A$
$S^{-1}A$	the localization of A at the multiplicative set $S \subseteq A$
$A[1/a]$	the localization of A at the multiplicative set generated by the element $a \in A$
$A_{\mathfrak{p}}$	the localization of A at the multiplicative set $A \setminus \mathfrak{p}$, where $\mathfrak{p} \subseteq A$ is a prime ideal
$F(\mathfrak{p})$	the quotient field of A/\mathfrak{p} , where $\mathfrak{p} \subseteq A$ is a prime ideal
$\alpha_{\mathfrak{p}}$	the natural map $A \rightarrow A/\mathfrak{p} \subseteq F(\mathfrak{p})$, where $\mathfrak{p} \subseteq A$ is a prime ideal
A_v	the valuation ring associated with the valuation v of a field K
\mathfrak{m}_v	the unique maximal ideal of the valuation ring A_v
k_v	the residue field A_v/\mathfrak{m}_v of the valuation ring A_v
S_v	the push-down $(A_v \cap S + \mathfrak{m}_v)/\mathfrak{m}_v$ of a subset $S \subseteq K$ to k_v
\mathbb{N}	the natural numbers
\mathbb{Z}	the ring of integers
\mathbb{Q}	the field of rational numbers
\mathbb{R}	the reals
\mathbb{C}	the complex numbers

Chapter 1

n	the fixed exponent, 7
T^*	$T \cap A^*$, where $T \subseteq A$, 7
ΣA^n	the set of all finite sums Σx_i^n , $x_i \in A$, 8
M^e	the set of all x such that $(n!)^r x \in M$ for some integer $r \geq 0$, 8
$s(M)$	the level of the T -module M , 9
$S^{-n}M$	the extension of the T -module M to $S^{-1}A$, 10
$M[1/a^n]$	the extension of the T -module M to $A[1/a]$, 10
M/\mathfrak{a}	the extension of the T -module M to A/\mathfrak{a} , 10
$M(\mathfrak{p})$	the extension of the T -module M to $F(\mathfrak{p})$, 11
$\text{Hom}(A, \mathbb{R})$	the set of all ring homomorphisms of A to \mathbb{R} , 12
$X(M)$	the ring homomorphisms $\varphi : A \rightarrow \mathbb{R}$ such that $\varphi(M) \geq 0$, where $M \subseteq A$, 12
$\text{Arch}(M)$	the set of all elements $a \in A$ such that for all $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $m(1 + ka) \in M$, where $M \subseteq A$, 12
$T \sim A_v$	the preorder T is compatible with the valuation ring A_v , 16

T^v	the smallest preorder containing T fully compatible with v , 16
$T \wedge Q$	the wedge product of a preorder T with a proper preorder $Q \supseteq T_v$, 17
\mathbb{Q}^+	the positive rationals, 18
$A(M)$	the set of elements $x \in K$ such that $r \pm x \in M$ for some $r \in \mathbb{Q}^+$, where M is a T -module in a field K , 18
$I(M)$	the set of elements $x \in K$ such that $r \pm x \in M$ for all $r \in \mathbb{Q}^+$, where M is a T -module in a field K , 18
$P_{a-,s}$	the unique order in $\mathbb{R}(X)$ of level s compatible with $\mathbb{R}[X]_{(X-a)}$ containing $-(X-a)^s$, 29
$P_{a+,s}$	the unique order in $\mathbb{R}(X)$ of level s compatible with $\mathbb{R}[X]_{(X-a)}$ containing $(X-a)^s$, 29
$P_{\infty-,s}$	the unique order in $\mathbb{R}(X)$ of level s compatible with $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$ containing $-X^s$, 29
$P_{\infty+,s}$	the unique order in $\mathbb{R}(X)$ of level s compatible with $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$ containing X^s , 29
$T^{(m)}$	the set of all x such that $x^m \in T$, 31
$\text{supp } P$	the support of the order P , 34
$\text{Sper } A$	the set of all orders of exponent n in A , 34
$\text{Sper}_T A$	the set of all orders in $\text{Sper } A$ containing the preorder T , 34
$P_{(2)}$	the 2-primary part of an order P , 35
$P \prec Q$	the order Q specializes the order P , 35
$\text{Spermax } A$	the set of all maximal orders in the ring A , 39
$\text{Spermax}_T A$	$\text{Spermax } A \cap \text{Sper}_T A$, 39
P_a	the unique order in $\mathbb{R}[X]$ of level 1 with support $(X-a)$, 39

Chapter 2

$\text{Sig}_T A$	the set of all T -signatures of a ring A , 42
P_σ	the unique order associated with the signature σ , 42
$s(\sigma)$	the level of the signature σ , 43
$\sigma \prec \tau$	τ specializes the signature σ , 43
$\text{Sigmax}_T A$	the set of maximal T -signatures of the ring A , 44
$U(a_1, \dots, a_r; \alpha_0)$	the set of all T -signatures σ with $\sigma(a_i) = \alpha_0(a_i)$, for all i , 44
A_T	the set of all $a \in A$ such that $\sigma(a) \neq 0$ for all $\sigma \in \text{Sig}_T A$, 47
\tilde{T}	the intersection of all the orders lying over T , 47
G_T	the abelian group $\frac{A_T}{T \cap A_T}$, 47
$[a]$	the image of a in G_T , where $a \in A_T$, 47

Chapter 3

$C(X, \mathbb{C})$	the ring of all locally constant complex-valued functions on the topological space X , 52
X_T	$\text{Sigmax}_T A$, 53
\hat{a}	the map $X_T \rightarrow \mathbb{C}$ defined by $\sigma \mapsto \sigma(a)$ where $a \in A_T$, 53
$\hat{\varphi}$	the map $X_T \rightarrow \mathbb{C}$ defined by $\sigma \mapsto \sigma(a_1) + \cdots + \sigma(a_r)$ where $\varphi = \langle a_1, \dots, a_r \rangle$, 53
$W_T(A)$	the reduced Witt ring for the preorder T , 54
$\varphi \cong_T \psi$	the T -forms φ and ψ are T -isometric, 54
$D_T(\varphi)$	the set of elements represented by the T -form φ , 56
V_T	the valuations compatible with the preorder T , 57
$\chi(G)$	$\text{Hom}(G, \Omega)$ where G is an abelian group of exponent n , 57
$\bar{\varphi}_a$	the a -th residue class form of the T -form φ , 58
$X_T(a_1, \dots, a_r)$	the set of signatures $\sigma \in X_T$ such that $a_i \equiv a_j \pmod{A(P_\sigma)^* T^*}$ for all i, j , 60
$V_T(a_1, \dots, a_r)$	the set of valuations $v \in V_T$ such that $v(a_i) \not\equiv v(a_j) \pmod{v(T^*)}$ for some $i \neq j$, 60

Chapter 4

$\mathcal{W}_T(a)$	the set of orders $P \in \text{Sper}_T A$ with $a \in P$, 72
$\mathcal{W}_T^c(a)$	the set of orders $P \in \text{Sper}_T A$ with $a \notin P$, 72
$\mathcal{U}_T(a)$	the set orders $P \in \text{Sper}_T A$ with $a \in P \setminus \text{supp } P$, 72
\bar{S}	the closure of S in the Harrison topology, 72
$\mathcal{Z}_T(a)$	the set of orders $P \in \text{Sper}_T A$ with $a \in \text{supp } P$, 73
$\text{z-cl}(S)$	the Zariski-closure of the set S , 74
$\mathcal{U}_T(a; m)$	$\mathcal{U}_T(a^m) \cap \bigcap_{\substack{d m \\ d \neq m}} \mathcal{W}_T^c(a^d)$, 74
$\mathcal{W}_T(a; m)$	$\mathcal{U}_T(a; m) \cup \mathcal{Z}_T(a)$, 74
$T[a_1, \dots, a_r]$	the smallest preorder containing T and a_1, \dots, a_r , 76
$S(\mathfrak{p})$	$S \cap \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$, 77

INTRODUCTION

The concept of an ordered field originated with Hilbert's work on the foundations of geometry, around 1898, but it was Artin's solution to Hilbert's 17th Problem that led to the development of the theory of formally real fields beginning with the 1927 papers of Artin and Schreier. In these papers, it was shown that fields admitting orders were precisely those fields in which -1 is not a sum of squares, the formally real fields, and further, that an element of a field is a sum of squares if and only if it is positive at every order in that field. The important class of real closed fields, the formally real fields which have no formally real proper algebraic extension, were introduced and it was shown that every real closed field R admits the unique order $R^2 := \{x^2 \mid x \in R\}$ and every ordered field has a unique (up to isomorphism) real closed algebraic extension R where R^2 extends the given order. In 1931, van der Waerden acknowledged the importance of the Artin-Schreier theory of ordered fields by including it in his text "Moderne Algebra" and it has remained a part of standard algebra texts to this day.

The theory of ordered fields has played an important role in many areas of mathematics. Artin's proof of Hilbert's 17th Problem related for the first time the theory of ordered fields and real algebraic geometry. This relationship is clear in Lang's proof (1965) of the Hilbert problem using his Homomorphism Theorem which later leads to the Real Nullstellensatz of Dubois and Risler (1970). Tarski's discovery of his famous Tarski Principle in 1948 and Robinson's later proof of the model completeness of the elementary theory of real closed fields contributed significantly to the development of model theory. There is also the application of the Artin-Schreier theory to the algebraic theory of quadratic forms.

The relationship between orders and quadratic forms is defined by the notion of a signature: for any order P and any quadratic form φ over a formally real field F , one defines the signature $\text{sgn}_P(\varphi)$ just as Sylvester defined the signature of a quadratic form over \mathbb{R} . In 1966, Pfister showed that the signatures of the form φ

with respect to all the different orders in F completely determines the Witt class of φ up to torsion. Thus the study of the reduced Witt ring (the Witt ring modulo its torsion ideal) is intimately tied to the study of the space of orders of the underlying field. Specifically, we endow the set of orderings $\text{Sper } F$ with the smallest topology for which the non-zero field elements represent continuous functions into the discrete space $\{\pm 1\} \subseteq \mathbb{Z}$. The reduced Witt ring is simply the ring of continuous \mathbb{Z} -valued functions on $\text{Sper } F$ generated by the non-zero field elements.

The study of both the space of orders and the Witt ring of a field leads naturally to the consideration of the (Krull) valuations of that field. The importance of valuations to the study of orders seems to have been recognized from the start. Although the language of valuation theory was not yet available, the idea of pushing down an order P to the residue field of the valuation ring $A(P)$ already appears in the papers of Artin and Schreier [2]. In 1973, Prestel [45] introduced the concept of a semiorder in a field and showed that it, too, gives rise to a valuation ring. This relationship between valuations and semiorders is the key to the important Local-global Criterion for Isotropy [8, Theorem 3.3] due to Bröcker and to Prestel.

In [4], Becker extended the notions of orders and semiorders to higher level, replacing “sums of squares” with “sums of 2^m -th powers”, for arbitrary m , and later, in [5]–[7], with “sums of $2k$ -th powers” for arbitrary k . In a joint work with Rosenberg [12], this higher level theory of orders is shown to lead to a corresponding higher level reduced theory of forms. Signatures again establish the connection between the higher level orders and forms of higher degree but unlike the original level 1 situation, each order does not determine a unique signature.

Surprisingly, Becker was able to show that the same relationship between orders and valuations exists in the higher level setting. The results are presented here in sections 1.3–1.5. Even more surprising is the extension of the result of Prestel concerning semiorders and valuations (1.3.9). As observed by Becker in [7]: “there is no simple proof this time.” Essential to establishing this connection between the valuations of a field and the higher level orders and semiorders in that field is the Kadison-Dubois Representation Theorem [5], [21], [23] concerning archimedean

partial orders in commutative rings. In section 1.2, a simple, self-contained proof is given which is simply a translation of the one given by Becker and Schwartz in [13].

As Becker was developing his higher level theory, Knebusch [25], Kleinstein and Rosenberg [24] and others were extending the theory of quadratic forms to semi-local rings. Later it was shown to extend to rings with many units in [17] and [48]. In the series of papers [29]–[34], Marshall develops an axiomatic approach to the reduced theory of quadratic forms. Fields, semi-local rings and rings with many units all give rise to spaces of orderings in the terminology of [32].

Although signatures of a semi-local ring were already being considered as early as 1971 in [26], it was not until the joint paper of Coste and Roy [20] that the correct notion of a (non-higher level) order in an arbitrary commutative ring was formulated. Just as for fields in the non-higher level case, signatures and orders in a semi-local ring (or a ring with many units) are essentially the same thing. Thus the work of Coste and Roy did not produce anything new where the reduced theory of forms was concerned but the real spectrum introduced in [20] has had significant applications to real algebraic geometry. This is a result of the previously mentioned Tarski Principle: one may identify semi-algebraic subsets of a real algebraic variety with the constructible subsets of the real spectrum of the coordinate ring [15, Théorème 7.2.3].

For example, consider the following situation. Let F be a formally real field, P an order in F and R a real closed extension of (F, P) . Fix an algebraic set $V = \{x = (x_1, \dots, x_n) \in R^n \mid f_1(x) = \dots = f_k(x) = 0\}$, where f_1, \dots, f_k are polynomials with coefficients in F . A subset $S \subseteq V$ is semi-algebraic if it is defined by means of a finite number of polynomial inequalities. One is interested in determining bounds on the number of polynomial inequalities required. Using Tarski's Principle, this problem is equated to one of determining bounds on the number of inequalities required to define constructible subsets in the real spectrum of a commutative ring. The solution to this problem is Bröcker's theory of the complexity of constructible sets [16], [18], [28], [35], [47]. A more general problem is to determine bounds on

the number of inequalities required if we insist the polynomials take their coefficients from the base field F . In [39], bounds are determined for basic semi-algebraic sets defined over F by translating the problem to an equivalent one concerning the complexity of constructible sets in the real spectrum of a ring. Using the results of this thesis, a beginning has been made in the generalization of Bröcker's complexity theory to higher level. Whether this will have an application to real algebraic geometry is still an open question.

This thesis is concerned with developing a higher level theory for commutative rings. In chapter 1, Becker's higher level orders are extended to an arbitrary commutative ring. The results of the first section have already appeared in [38]. The main theorem of section 1.1 is the weak local-global principle (1.1.7) which is a generalization of a similar result in [19] both to higher level and to an arbitrary preorder.

Sections 1.2 through 1.5 are a survey of results contained in the papers of Becker [5]–[7], Becker–Harman–Rosenberg [10], Becker–Rosenberg [12] and Becker–Schwartz [13]. They are included not only to make this thesis as self-contained as possible but also to bring together, for the first time, all of these results in the same place.

Orders and maximal orders of higher level are defined for commutative rings in section 1.6. The notions of specialization and maximal orders for non-2-primary orders were not understood at the time the paper [38] was written. Applying (1.6.9) to a ring with many units, one can show that for any maximal order P and any preorder T in a ring with many units, $T^* \subseteq P^*$ iff $T \subseteq P$. (This was shown to hold for 2-primary preorders in [38] and is proved in a more general form in chapter 2.) This, together with the results of section 1.6, yields improved versions of results in [38] in several instances (see chapter 3.)

One of the consequences of Theorem 1.1.7 is the Positivstellensatz of section 1.7. A weaker version of (1.7.2) is due to Berr [14] who, in turn, was extending a result of Becker–Gondard in [9]. The full strength of the Positivstellensatz as it appears here is required in chapter 3.

The notion of a signature of higher level is defined for an arbitrary commutative ring in chapter 2. This generalizes the signatures of higher level already defined for fields in [10], [12] and for rings with many units in [38]. Just as for fields, the space of higher level signatures plays a prominent role in the reduced theory of higher degree forms (see chapter 3) and is distinct from the higher level real spectrum. Specializations and maximal signatures are defined in section 2.1 and in section 2.2, the set of signatures $\text{Sig } A$ of a ring A is given a suitable "Harrison" topology in which the maximal signatures are precisely the closed points of $\text{Sig } A$ thus arriving at one possible generalization of the real spectrum of Coste and Roy. In fact, $\text{Sig } A$ is essentially the real spectrum $R_n\text{-spec } A$ of S. Barton defined in [3].

Marshall's abstract theory of spaces of orderings has been successfully generalized to the higher level theory of spaces of signatures by Mulcahy and Marshall [37], [41]. (See also the joint work with Becker and Rosenberg [11] and the papers of Powers [43], [44].) In [36], Marshall gives a simple axiom for a level 1 preorder T in a commutative ring sufficient for T to give rise to a space of orderings. We show that a higher level version of this axiom is sufficient for higher level preorders to give rise to spaces of signatures. A necessary first step is to show the T -signatures of a ring A can be viewed as characters on an appropriate abelian group G_T whenever T satisfies Marshall's axiom. This is done in section 2.3. In chapter 3, a reduced theory of higher level forms is developed for preorders satisfying this axiom and we show for any such preorder T , (X_T, G_T) is a space of signatures, where X_T is the set of maximal T -signatures of A . Since this axiom holds for preorders in a ring with many units, we have new, somewhat simpler, proofs of the main results of [38].

In chapter 4, the task of generalizing Coste and Roy's real spectrum to higher level is completed by defining higher level analogues of the Tychonoff, Harrison and Zariski topologies. In section 4.1, it is shown that the higher level real spectrum with these topologies has the desired properties. In section 4.2, the constructible subsets of the real spectrum are considered. In the non-higher level real spectrum, a subset S is called constructible if it can be obtained from the Harrison sub-basic sets by means of a finite number of unions, intersections and complements. The

constructible subsets are then shown to be precisely the Tychonoff clopen (closed and open) subsets. We have precisely the same situation if we consider only orders of 2-power level. In order to achieve this characterization in the more general situation, the sub-base given in section 4.1 must be modified. In the last section, the characterizations of basic constructible sets given in [16], [35] are shown to extend to higher level in the 2-primary case. As in level 1, the proof requires an abstract version of the Hörmander-Lojasiewicz Inequality for semi-algebraic functions [15, Corollaire 2.6.7].

Chapter 1

Orders of Higher Level

In the sequence of papers [4]–[7], E. Becker has developed a theory of higher level orders for fields, generalizing the ordered fields of Artin and Schreier. In [38], a beginning was made in extending this theory to commutative rings. (See also [9] and [14]. Orders of higher level are also defined for commutative rings in [3] but the approach taken there corresponds more closely to the theory of higher level signatures developed in chapter 2 of this thesis.) Although the notions of preorders and orders of arbitrary level were defined for commutative rings in [38], the theory was really only developed for 2-primary level. In sections 1.6 and 1.7 below, this theory is extended to include the non-2-primary case.

The material of the first five sections of this chapter is drawn from [5]–[7],[12], [13] and [38]. Note that the terminology used here is not always consistent with the original papers.

1.1 Preorders and semiorders of higher level

Let A be a commutative ring and fix a positive even integer n called the *fixed exponent*. A *preorder* (of *exponent* n) in A is a subset $T \subseteq A$ satisfying

$$T + T \subseteq T, \quad T \cdot T \subseteq T, \quad A^n \subseteq T.$$

If, in addition, $-1 \notin T$, we say T is a *proper* preorder. We denote the group of units of the ring A by A^* and for any subset $T \subseteq A$, write T^* for $T \cap A^*$. If T is

a preorder then T^* is a subgroup of A^* and A^*/T^* is of exponent n . (An abelian group G is said to be of *exponent* k , where k is a positive integer, if $G^k = 0$. The smallest positive integer k for which $G^k = 0$ is called the *exact exponent* of G .)

Denote by ΣA^n the set of all finite sums Σx_i^n , $x_i \in A$. It is the unique smallest preorder in A of exponent n .

Let $T \subseteq A$ be a preorder. A T -module is a subset $M \subseteq A$ such that

$$M + M \subseteq M, \quad T \cdot M \subseteq M, \quad 1 \in M.$$

A T -module M is *proper* if $-1 \notin M$. A T -semiorder is a proper T -module S satisfying $S \cup -S = A$. Clearly T itself is a T -module and for any T -module M , $T \subseteq M$.

1.1.1 Proposition. *If $T - T = A$ then $M \cap -M$ is an ideal for any T -module M .*

Proof. Clearly, $M \cap -M$ is an additive group and $T(M \cap -M) \subseteq M \cap -M$. Since $T - T = A$, it follows that $A(M \cap -M) \subseteq M \cap -M$. \square

Using the identity ([22, Theorem 8.2.2])

$$(1.1.2) \quad n!x = \sum_{h=0}^{n-1} (-1)^{n-1-h} \binom{n-1}{h} [(x+h)^n - h^n],$$

it is clear every proper preorder T in a field K satisfies $T - T = K$. (Note that if $-1 \notin \Sigma K^n$ then the characteristic of K must be 0.) For the ring A , $n!$ need not be a unit so it may be necessary to slightly enlarge T in order to assume $T - T = A$.

For any T -module M , define

$$M^e := \{x \in A \mid (n!)^r x \in M \text{ for some } r \geq 0\}.$$

Then T^e is a preorder and M^e is a T^e -module which is proper iff M is proper. If $n! \in A^*$ then $M^e = M$. In particular, this holds for any proper T -module M in a field K .

1.1.3 Proposition. *For any preorder $T \subseteq A$, $T^e - T^e = A$.*

Proof. Let $x \in A$. By (1.1.2), $n!x = y - z$ where $y, z \in \Sigma A^n$ so $n!(x + z) = y - z + n!z = y + (n! - 1)z \in \Sigma A^n \subseteq T$ and therefore, $x = (x + z) - z \in T^e - T^e$. \square

1.1.4 Theorem. *If S is a maximal proper T -module then S is a T -semiorder and $S \cap -S$ is a prime ideal of A .*

Proof. Since S is maximal, $S = S^e$ so S is a T^e -module and therefore, $\mathfrak{p} := S \cap -S$ is an ideal of A .

Suppose $a \in A$ and $a \notin S \cup -S$. Then $-1 \in S + aT$ and $-1 \in S - aT$ so there exists $s_1, s_2 \in S, t_1, t_2 \in T$ such that

$$-1 = s_1 + at_1 \quad \text{and} \quad -1 = s_2 - at_2.$$

Then $-(t_1 + t_2) = t_1(s_2 - at_2) + t_2(s_1 + at_1) = t_1s_2 + t_2s_1 \in S$ so $-t_1 = -(t_1 + t_2) + t_2 \in S$ and therefore, $t_1 \in \mathfrak{p}$. But then $-1 = s_1 + at_1 \in S + \mathfrak{p} \subseteq S$, a contradiction. Therefore, S is a T -semiorder.

Suppose $a, b \in A, ab \in \mathfrak{p}$. Then $-a^n b^n = -(ab)^n \in \mathfrak{p} \subseteq S$. If $-a^n \notin S$ then $-1 \in S - a^n T$ so there exists $s \in S, t \in T$ such that $-1 = s - a^n t$ and therefore, $-b^n = b^n s - a^n b^n t \in S$. So at least one of a^n, b^n is in \mathfrak{p} , say $a^n \in \mathfrak{p}$. Pick m such that $n \leq 2^m$. Then $a^{2^m} \in \mathfrak{p}$ so it suffices to prove the following claim.

Claim. $a^2 \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$.

Suppose $a^2 \in \mathfrak{p}$ and $a \notin \mathfrak{p}$. Replacing a by $-a$ if necessary, we can assume $a \notin S$. By the maximality of S , $-1 \in S + aT$ so there exists $s \in S, t \in T$ with $-1 = s + at$, that is, $at = -(1 + s)$. Since $a^2 t^2 \in \mathfrak{p}$, $(1 + at)^n = 1 + nat + \cdots + (at)^n = 1 + nat + x$ for some $x \in \mathfrak{p}$. Then $1 - n(1 + s) = 1 + nat = (1 + at)^n - x \in S$ and therefore, $-1 = (1 - n(1 + s)) + ns + n - 2 \in S$, a contradiction. \square

Let M be a T -module. Since $\Sigma A^n \subseteq T$, M may also be viewed as a ΣA^n -module. As in [12], we define the *level* of M , denoted $s(M)$, to be the smallest positive integer ℓ such that M is a ΣA^{2^ℓ} -module. If $s(M)$ is a power of 2, we say M is *2-primary*.

1.1.5 Corollary. *If P is a maximal proper preorder of level 1 then*

- (i) $P \cup -P = A$,
- (ii) $P \cap -P$ is a prime ideal of A .

Proof. By (1.1.4), it suffices to show P is also maximal as a P -module. Suppose M is a proper P -module. For any $x \in M$, we have $P \subseteq P + xP \subseteq M$. Since $x^2 \in P$, $P + xP$ is a proper preorder so $P = P + xP$ and therefore, $x \in P$. Thus P is the only proper P -module and hence is certainly maximal. \square

Level 1 preorders $P \subseteq A$ satisfying (i) and (ii) of (1.1.5) are simply called *orders* in much of the literature. Here they will be referred to as *orders of level 1*. (Orders of higher level will be defined in sections 1.5 and 1.6.)

Let $\varphi : A \rightarrow B$ be a ring homomorphism. For any preorder $T \subseteq B$ and any T -module $M \subseteq B$, $\varphi^{-1}(T)$ is clearly a preorder in A and $\varphi^{-1}(M)$ is a $\varphi^{-1}(T)$ -module. Conversely, if T is a preorder in A and M is a T -module, we denote by $\Sigma B^n \varphi(M)$ the set of all finite sums $\Sigma y_i^n \varphi(x_i)$, $y_i \in B$, $x_i \in M$. Then $\Sigma B^n \varphi(T)$ is a preorder in B and $\Sigma B^n \varphi(M)$ is a $\Sigma B^n \varphi(T)$ -module. $\Sigma B^n \varphi(M)$ is called the *extension of M to B* and we say the ring homomorphism φ is *M -compatible* if $-1 \notin \Sigma B^n \varphi(M)$.

1.1.6 Remarks. Let $T \subseteq A$ be a preorder, M a T -module. If $S^{-1}A$ is the localization of A at some multiplicative set $S \subseteq A$ then the extension of M to $S^{-1}A$ is

$$S^{-n}M := \left\{ \frac{x}{s^n} \mid x \in M, s \in S \right\}.$$

If S is generated by an element $a \in A$ then $S^{-1}A$ is denoted by $A[1/a]$ and $S^{-n}M$ is denoted by $M[1/a^n]$.

For an ideal $\mathfrak{a} \subseteq A$, the extension of M to A/\mathfrak{a} is

$$M/\mathfrak{a} := \{x + \mathfrak{a} \mid x \in M\}.$$

If \mathfrak{p} is a prime ideal of A , the *residue field* of A at \mathfrak{p} (that is, the field of fractions of the domain A/\mathfrak{p}) is denoted by $F(\mathfrak{p})$ and $\alpha_{\mathfrak{p}} : A \rightarrow F(\mathfrak{p})$ denotes the natural map

$A \rightarrow A/\mathfrak{p} \subseteq F(\mathfrak{p})$. The extension of M to $F(\mathfrak{p})$ is denoted by $M(\mathfrak{p})$. We say \mathfrak{p} is *M-compatible* if $\alpha_{\mathfrak{p}}$ is *M-compatible*, that is, if $-1 \notin M(\mathfrak{p})$.

By Zorn's Lemma, every proper T -module M is contained in a maximal proper T -module S . By (1.1.4), $\mathfrak{p} := S \cap -S$ is a prime ideal. If $-1 \in M(\mathfrak{p})$ then there exists $a \in A \setminus \mathfrak{p}$ such that $-a^n \in M + \mathfrak{p} \subseteq S$. But then $a^n \in S \cap -S = \mathfrak{p}$, contradiction. Therefore, we have the following weak local-global principle.

1.1.7 Theorem ([38, Theorem 1.6]). *For any proper T -module M , there exists an M -compatible prime.*

We therefore have the following extension of [19, Theorem 1].

1.1.8 Corollary. *If U is a subset of A with $1 \in U$ then the following are equivalent:*

- (i) $-1 = \sum u_i a_i^n$, for some $u_i \in U$, $a_i \in A$.
- (ii) For each prime $\mathfrak{p} \subseteq A$, $-1 = \sum \alpha_{\mathfrak{p}}(u_i) x_i^n$, for some $u_i \in U$, $x_i \in F(\mathfrak{p})$.

Proof. (i) \Rightarrow (ii) is clear. For (ii) \Rightarrow (i), just apply (1.1.7) to the ΣA^n -module generated by U . \square

A field K is called *formally real* if $-1 \notin \Sigma K^2$.

1.1.9 Corollary. *For any commutative ring A , the following are equivalent:*

- (i) A admits a proper preorder of exponent n .
- (ii) A admits an order of level 1.
- (iii) $F(\mathfrak{p})$ is formally real for some prime $\mathfrak{p} \subseteq A$.

Proof. Taking $U = \{1\}$ in (1.1.8), we have $-1 \in \Sigma A^n$ iff $-1 \in \Sigma F(\mathfrak{p})^n$ for all primes $\mathfrak{p} \subseteq A$. Since n is an arbitrary positive even integer, this holds for $n = 2$ as well. By [22, Theorem 6.15] (or by (1.4.6) below), $-1 \notin \Sigma F(\mathfrak{p})^n$ iff $-1 \notin \Sigma F(\mathfrak{p})^2$. \square

1.2 Kadison-Dubois Representation Theorem

This section is essentially a translation of the paper [13] by E. Becker and N. Schwartz and provides a simple, self-contained proof of the Kadison-Dubois Representation Theorem (see [5], [21], [23].)

Let A be a commutative ring, $T \subseteq A$ a proper preorder. A proper T -module M is said to be *archimedean* if for all $a \in A$, there exists $k \in \mathbb{N}$ such that $k - a \in M$.

Fix an archimedean preorder $T \subseteq A$ and let M be a T -module. Set

$$X(M) := \{\varphi \in \text{Hom}(A, \mathbb{R}) \mid \varphi(M) \geq 0\}$$

and

$$\text{Arch}(M) := \{a \in A \mid \text{for all } k \in \mathbb{N}, m(1 + ka) \in M \text{ for some } m \in \mathbb{N}\}$$

where $\text{Hom}(A, \mathbb{R})$ is the set of all ring homomorphisms from A to \mathbb{R} .

1.2.1 Theorem. *For each T -semiorder $S \subseteq A$, there exists a unique ring homomorphism $\varphi : A \rightarrow \mathbb{R}$ such that $\varphi(S) \geq 0$. Moreover,*

- (i) $\ker \varphi = I(S) := \{a \in A \mid 1 \pm ka \in S \text{ for all } k \in \mathbb{N}\},$
- (ii) $\varphi^{-1}(\mathbb{R}^2) = S \cup I(S) = \{a \in A \mid 1 + ka \in S \text{ for all } k \in \mathbb{N}\}.$

Proof. (Uniqueness of φ) Let $a \in A$. Since T is archimedean, there exists $(r, s) \in \mathbb{Z} \times \mathbb{N}$ with $r - sa \in T \subseteq S$. For any $(r, s) \in \mathbb{Z} \times \mathbb{N}$ such that $r - sa \in S$, we have $r - s\varphi(a) \geq 0$. Thus,

$$\varphi(a) \leq \psi(a) := \inf \left\{ \frac{r}{s} \mid (r, s) \in \mathbb{Z} \times \mathbb{N} \text{ and } r - sa \in S \right\}.$$

Suppose $(u, v) \in \mathbb{Z} \times \mathbb{N}$ with $\frac{u}{v} < \psi(a)$. Then $u - va \notin S$ so $va - u \in S$ and therefore, $v\varphi(a) \geq u$, that is, $\varphi(a) \geq \frac{u}{v}$. Thus $\varphi = \psi$ which is uniquely determined by S .

(Existence of φ) For $a \in A$, define

$$\varphi(a) := \inf \left\{ \frac{r}{s} \mid (r, s) \in \mathbb{Z} \times \mathbb{N} \text{ and } r - sa \in S \right\}.$$

Pick $t \in \mathbb{N}$ such that $t + a \in T$. Suppose $(r, s) \in \mathbb{Z} \times \mathbb{N}$ and $r - sa \in S$. Then $st + sa \in T \subseteq S$ so $r + st \in S$. Since $-1 \notin S$, $r + st$ is a non-negative integer so $\frac{r}{s} \geq -t$. Thus, $\varphi(a) \geq -t$ and hence, is in \mathbb{R} . Note that if $a \in S$ then r is necessarily non-negative so $\varphi(a) \geq 0$.

Suppose $r - sa, u - v(-a) \in S, r, u \in \mathbb{Z}, s, v \in \mathbb{N}$. Then $rv + us \in S$ so $rv + us \geq 0$ and therefore, $\frac{u}{v} \geq -\frac{r}{s}$. Thus, $\varphi(-a) \geq -\varphi(a)$. Suppose $(u, v) \in \mathbb{Z} \times \mathbb{N}$ such that $\frac{u}{v} < \varphi(-a)$. Then $u + va = u - v(-a) \notin S$ so $-u - va \in S$ and therefore, $\varphi(a) \leq -\frac{u}{v}$. Thus $\varphi(-a) \leq -\varphi(a)$ and hence, $\varphi(-a) = -\varphi(a)$.

Suppose $r - sa, u - vb \in S, r, u \in \mathbb{Z}, s, v \in \mathbb{N}$. Then $(rv + us) - sv(a + b) \in S$ so

$$\varphi(a + b) \leq \frac{rv + us}{sv} = \frac{r}{s} + \frac{u}{v}$$

and therefore, $\varphi(a + b) \leq \varphi(a) + \varphi(b)$. Using $\varphi(-a) = -\varphi(a)$, it follows that $\varphi(a + b) = \varphi(a) + \varphi(b)$.

From the definition of $\varphi(1)$, we clearly have $\varphi(1) \leq 1$. If $r - s \cdot 1 \in S$ then $r - s$ must be a non-negative integer so $\frac{r}{s} \geq 1$. Thus $\varphi(1) = 1$.

In order to show $\varphi(ab) = \varphi(a)\varphi(b)$, it suffices to consider the case $b \in T$ (since $A = T - T$ and φ is additive.) Suppose $(r, s) \in \mathbb{Z} \times \mathbb{N}$, $r - sa \in S$. Then $rb - sab \in S$ so $r\varphi(b) - s\varphi(ab) \geq 0$ and therefore, $\varphi(ab) \leq \frac{r}{s}\varphi(b)$. Since $\varphi(b) \geq 0$, we have $\varphi(ab) \leq \varphi(a)\varphi(b)$. Similarly, for $-a$ we have $-\varphi(ab) \leq -\varphi(a)\varphi(b)$ so $\varphi(ab) = \varphi(a)\varphi(b)$. Thus, φ is a ring homomorphism with $\varphi(S) \geq 0$ as required.

(i) Note that if $a \notin S$ then $-\varphi(a) = \varphi(-a) \geq 0$ so $a \in S$ whenever $\varphi(a) > 0$. Suppose $\varphi(a) = 0$. Then for all $k \in \mathbb{N}$, $\varphi(1 \pm ka) = 1$ so $1 \pm ka \in S$. Conversely, suppose $1 \pm ka \in S$ for all $k \in \mathbb{N}$. Then $1 \geq k|\varphi(a)|$ for all k so $\varphi(a) = 0$.

(ii) If $\varphi(a) \geq 0$ then either $\varphi(a) = 0$ so $a \in I(S)$ or $\varphi(a) > 0$ so $a \in S$. Therefore, $\varphi^{-1}(\mathbb{R}^2) \subseteq S \cup I(S)$. Clearly, $S \cup I(S) \subseteq \{a \in A \mid 1 + ka \in S \text{ for all } k \in \mathbb{N}\}$. Now suppose $1 + ka \in S$ for all $k \in \mathbb{N}$. Then $1 + k\varphi(a) \geq 0$ for all k and therefore $\varphi(a) \geq 0$, which completes the proof. \square

1.2.2 Remark. Any archimedean level 1 order P in a field K can also be viewed as a P -semiorder. Applying (1.2.1) in this special case, we get a unique embedding

$\varphi : K \hookrightarrow \mathbb{R}$ such that $\varphi(P) \geq 0$. Thus we have the well-known result that every archimedean ordered field is order isomorphic to a subfield of \mathbb{R} .

1.2.3 Proposition. $\varphi \mapsto \varphi^{-1}(\mathbb{R}^2)$ gives a 1-1 correspondence between $X(M)$ and the set of maximal proper T -modules lying over M .

Proof. For each $\varphi \in X(M)$ the set $\varphi^{-1}(\mathbb{R}^2)$ is clearly a T -semiorder (in fact, a level 1 order) lying over M and by (1.2.1), φ is the unique ring homomorphism with $\varphi(\varphi^{-1}(\mathbb{R}^2)) \geq 0$. This shows $\varphi \mapsto \varphi^{-1}(\mathbb{R}^2)$ is 1-1 and $\varphi^{-1}(\mathbb{R}^2) \not\subseteq \psi^{-1}(\mathbb{R}^2)$ if $\varphi \neq \psi$. Let S be a maximal proper T -module lying over M . Since S is a T -semiorder, there exists a unique $\psi \in X(M)$ with $S \subseteq \psi^{-1}(\mathbb{R}^2)$. By the maximality of S , $S = \psi^{-1}(\mathbb{R}^2)$. \square

1.2.4 Proposition. $\text{Arch}(M) = \bigcap_{\varphi \in X(M)} \varphi^{-1}(\mathbb{R}^2)$.

Proof. Let S be a maximal proper T -module lying over M .

Claim 1. $S = \text{Arch}(S)$.

Clearly, $S \subseteq \text{Arch}(S)$. We show $\text{Arch}(S)$ is a proper T -module. Then $S = \text{Arch}(S)$ follows from the maximality of S .

Let $a, b \in \text{Arch}(S)$, $k \in \mathbb{N}$. Then there exists $l, m \in \mathbb{N}$ such that $l(1+2ka), m(1+2kb) \in S$ so $2lm(1+k(a+b)) \in S$ and therefore, $a+b \in \text{Arch}(S)$. Let $t \in T$. Pick $l, m \in \mathbb{N}$ such that $l-t \in T$ and $m(1+lka) \in S$. Then $lm(1+kta) = mt(1+lka) + m(l-t) \in S$ so $ta \in \text{Arch}(S)$. If $-1 \in \text{Arch}(S)$ then $m(1+2(-1)) = -m \in S$ for some $m \in \mathbb{N}$ and therefore, $-1 \in S$, a contradiction. Since 1 is clearly in $\text{Arch}(S)$, $\text{Arch}(S)$ is a proper T -module, which proves the claim.

Claim 2. If $a \notin \text{Arch}(M)$ there exists a maximal proper T -module $S \supseteq M$ with $-a \in S$.

If $-1 \in M - aT$ then there exists $t \in T$ such that $at - 1 \in M$. Pick $k \in \mathbb{N}$ such that $k - t \in T$. Consider the set

$$\Sigma := \left\{ \frac{r}{s} \mid r, s \in \mathbb{N} \text{ and } r + sa \in M \right\}.$$

Since T is archimedean, $\Sigma \neq \emptyset$. Suppose $\frac{r}{s} \in \Sigma$. Then $kr - s + ksa = (k - t)(r + sa) + s(ta - 1) + rt \in M$. If $\frac{r}{s} > \frac{1}{k}$ then $kr - s > 0$ so $\frac{kr-s}{ks} = \frac{r}{s} - \frac{1}{k} \in \Sigma$. Otherwise, $\frac{r}{s} \leq \frac{1}{k}$ so $s - kr + r > 0$ and hence, $r + ksa = kr - s + ksa + s - kr + r \in M$. Therefore, $\frac{r}{s} \frac{1}{k} \in \Sigma$. This shows $0 = \inf \Sigma$. But then for each $k \in \mathbb{N}$, there exists $\frac{r}{s} \in \Sigma$ with $\frac{r}{s} < \frac{1}{k}$ so $s(1 + ka) = k(r + sa) + s - rk \in M$. This shows $a \in \text{Arch}(M)$, a contradiction. Thus $-1 \notin M - aT$ so take S to be any maximal proper T -module lying over $M - aT$. This proves the claim.

By claim 1, $\text{Arch}(M) \subseteq \text{Arch}(S) = S$ for all maximal proper T -modules $S \supseteq M$. Conversely, suppose $a \in S$, for all maximal proper T -modules $S \supseteq M$. Let $k \in \mathbb{N}$. If $1 + (k + 1)a \notin \text{Arch}(M)$ then, by claim 2, there exists a maximal proper T -module $S \supseteq M$ with $-1 - (k + 1)a \in S$, a contradiction since $a \in S$. Thus, $1 + (k + 1)a \in \text{Arch}(M)$ so there exists $m \in \mathbb{N}$ such that $m(1 + k(1 + (k + 1)a)) = m(k + 1)(1 + ka) \in M$. Thus, $a \in \text{Arch}(M)$. The result now follows from (1.2.3). \square

For each $a \in A$, we denote the evaluation map $\varphi \mapsto \varphi(a)$ by \hat{a} . We give $X(M)$ the weakest topology such that the evaluation maps \hat{a} , $a \in A$, are continuous. Then $a \mapsto \hat{a}$ defines a ring homomorphism

$$\Phi_M : A \rightarrow C(X(M), \mathbb{R})$$

where $C(X(M), \mathbb{R})$ denotes the ring of all continuous \mathbb{R} -valued functions on $X(M)$.

1.2.5 Theorem (Kadison-Dubois Representation Theorem). *Suppose $T \subseteq A$ is an archimedean preorder. For any proper T -module M ,*

- (i) $X(M)$ is a non-empty compact Hausdorff space,
- (ii) $\text{Arch}(M) = \{a \in A \mid \hat{a}(X(M)) \geq 0\}$,
- (iii) $\ker \Phi_M = \text{Arch}(M) \cap -\text{Arch}(M)$,
- (iv) $\mathbb{Q} \cdot \Phi_M(A)$ is dense in $C(X(M), \mathbb{R})$.

Proof. $X(M)$ is non-empty by (1.2.3). For each $a \in A$, pick $k_a \in \mathbb{N}$ such that $k_a \pm a \in T$ and therefore, $\hat{a}(X(M)) \subseteq [-k_a, k_a]$. Thus, we have an embedding

$$X(M) \hookrightarrow \prod_{a \in A} [-k_a, k_a]$$

given by $\varphi \mapsto (\varphi(a))_{a \in A}$. This is a closed mapping so $X(M)$ is compact and Hausdorff. (ii) is just (1.2.4), (iii) follows from (ii) and (iv) follows from the Stone-Weierstrass theorem since $\Phi_M(A)$ clearly separates points of $X(M)$. \square

1.3 Compatible valuations

Let K be a field, v a Krull valuation of K (written additively.) We denote the valuation ring of v by A_v , the maximal ideal of A_v by \mathfrak{m}_v and the residue field A_v/\mathfrak{m}_v by k_v . For any $S \subseteq K$, the set $S_v := (A_v \cap S + \mathfrak{m}_v)/\mathfrak{m}_v$ in k_v is called the *push-down* of S (along v).

Let $T \subseteq K$ be a proper preorder. Clearly T_v is a preorder in the residue field k_v with $s(T_v) \leq s(T)$. Following [4] and [27], we say T is *compatible* with the valuation ring A_v (or with the valuation v), written $T \sim A_v$, if T_v is proper, that is, if $-1 \notin T_v$. T is said to be *fully compatible* with A_v (or with v) if $1 + \mathfrak{m}_v \subseteq T$. Clearly, if T is fully compatible with A_v then $T \sim A_v$. Denote by T^v the smallest preorder in K containing T which is fully compatible with v .

1.3.1 Proposition. *If $T \sim A_v$ then*

$$v(t_1 + \cdots + t_m) = \min \{v(t_i) \mid i = 1, \dots, m\}$$

for any $t_1, \dots, t_m \in T^$.*

Proof. Let $s = t_1 + \cdots + t_m$ and assume $v(t_1) \leq v(t_i)$ for all i . Then $s/t_1 \in 1 + (T \cap A_v)$. Since T is compatible with A_v , $s/t_1 \in A_v^*$ and therefore, $v(s) = v(t_1)$. \square

1.3.2 Proposition. *Suppose U is a subgroup of K^* with $1 + \mathfrak{m}_v \subseteq U$. Then $A_v^* \cap U$ is additively closed iff U is additively closed.*

Proof. Suppose $A_v^* \cap U$ is additively closed and $u \in U$. Clearly, $1 + u \in U$ if $u \in \mathfrak{m}_v$. If $1/u \in \mathfrak{m}_v$ then $1 + u = u(1 + \frac{1}{u}) \in U(1 + \mathfrak{m}_v) \subseteq U$. Otherwise, $u \in A_v^*$ so $1 + u \in A_v^* \cap U \subseteq U$. Thus $1 + U \subseteq U$ which, of course, implies U is additively closed. Conversely, suppose $U + U \subseteq U$ and $u \in A_v^* \cap U$. If $1 + u \notin A_v^*$ then

$1 + u \in \mathfrak{m}_v$ so $-u \in 1 + \mathfrak{m}_v \subseteq U$ and therefore, $0 = u + (-u) \in U$, a contradiction. Thus, $1 + u \in A_v^* \cap U$. \square

1.3.3 Corollary. *Suppose $Q \supseteq T_v$ is a proper preorder in k_v . Then the wedge product of T and Q is defined to be*

$$T \wedge Q := T \cdot \{a \in A_v^* \mid \bar{a} \in Q\}.$$

It is a proper preorder in K fully compatible with v which pushes down to Q .

Proof. Clearly $(T \wedge Q)^*$ is a subgroup of K^* containing $1 + \mathfrak{m}_v$ and $(T \wedge Q) \cap A_v^* = \{a \in A_v^* \mid \bar{a} \in Q\}$ is additively closed so, by (1.3.2), $T \wedge Q$ is additively closed and hence, a proper preorder fully compatible with v which pushes down to Q . \square

1.3.4 Remark. Suppose v is a valuation of K such that $-1 \notin \Sigma k_v^n$. For any $x \in (\Sigma K^n) \cap A_v^*$, write $x = x_1^n + \dots + x_k^n$ where $x_1, \dots, x_k \in K^*$ and $v(x_1) \leq v(x_i)$ for all i . If $x_1 \notin A_v$ then $x_1^{-n} \in \mathfrak{m}_v$ so $0 = 1 + \bar{a}_2^n + \dots + \bar{a}_k^n$ for some $a_2, \dots, a_k \in A_v$, a contradiction. Thus, $x_i \in A_v$ for all i and $\bar{x} = \bar{x}_1^n + \dots + \bar{x}_k^n \in \Sigma k_v^n$. It follows that $-1 \notin \Sigma K^n$ and the push-down of ΣK^n is Σk_v^n . If $Q \subseteq k_v$ is any proper preorder of exponent n then

$$T := \Sigma K^n \wedge Q$$

is a proper preorder in K which is fully compatible with v and $T_v = Q$.

1.3.5 Corollary. *If $T \sim A_v$ then $T^v = T \cdot (1 + \mathfrak{m}_v) = T \wedge T_v$ is a proper preorder and we have the exact sequence*

$$1 \rightarrow \frac{k_v^*}{T_v^*} \xrightarrow{i} \frac{K^*}{T^{v*}} \xrightarrow{v} \frac{v(K^*)}{v(T^*)} \rightarrow 0,$$

where $i(\bar{a}T_v^*) = aT^{v*}$ and $v(xT^{v*}) = v(x) + v(T^*)$.

Proof. $\{a \in A_v^* \mid \bar{a} \in T_v\} = (T \cap A_v^*) \cdot (1 + \mathfrak{m}_v)$ so $T \wedge T_v = T \cdot (1 + \mathfrak{m}_v) \subseteq T^v$. Since $T \wedge T_v$ is a proper preorder fully compatible with A_v , $T^v = T \wedge T_v$. It is now easily seen that the given sequence is exact. \square

For a proper T -module M , set

$$A(M) := \{x \in K \mid r \pm x \in M \text{ for some } r \in \mathbb{Q}^+\},$$

$$I(M) := \{x \in K \mid r \pm x \in M \text{ for all } r \in \mathbb{Q}^+\},$$

where \mathbb{Q}^+ denotes the positive rationals.

1.3.6 Theorem ([5, Theorem 3.7(i),(ii)]). *For any proper preorder $T \subseteq K$,*

- (i) $A(T)$ is a Prüfer domain with quotient field K ,
- (ii) $I(T)$ is a proper ideal of $A(T)$,
- (iii) $A(T)$ is generated as a ring by the elements $\frac{1}{1+t}$, $t \in T$.

Proof. Since

$$(1.3.7) \quad rs \pm xy = \frac{1}{2} [(r+x)(s \pm y) + (r-x)(s \mp y)],$$

it is clear $A(T)$ is a subring of K and $I(T)$ is a proper ideal. If $t \in T$, $1 \pm \frac{1}{1+t}$, $1 \pm \frac{t}{1+t} \in T$ so $\frac{1}{1+t}$, $\frac{t}{1+t} \in A(T)$. Now let $a \in A(T)$. There exists $k, m \in \mathbb{N}$ such that $k+a \in T^* \cap A(T)$ and $m - (k+a) \in T$. Then $\frac{m}{k+a} = 1+t$ for some $t \in T$ so $a = \frac{m}{1+t} - k$. This proves (iii). To prove (ii), we must show the localization $A(T)_{\mathfrak{p}}$ of $A(T)$ at any prime ideal \mathfrak{p} is a valuation ring.

Suppose $\mathfrak{p} \subseteq A(T)$ is prime. Let $t \in T^*$. If $\frac{1}{1+t} \notin \mathfrak{p}$ then $1+t \in A(T)_{\mathfrak{p}}$ and hence, $t \in A(T)_{\mathfrak{p}}$. Otherwise $\frac{t}{1+t} \notin \mathfrak{p}$ so $\frac{1}{t} = \frac{1}{1+t} \cdot \frac{1+t}{t} \in \mathfrak{p}A(T)_{\mathfrak{p}}$. In particular, for all $x \in K^*$,

$$(*) \quad x^n \notin A(T)_{\mathfrak{p}} \Rightarrow 1/x^n \in \mathfrak{p}A(T)_{\mathfrak{p}}.$$

Let B be the integral closure of $A(T)_{\mathfrak{p}}$ in K . It follows from (*) that B is a valuation ring. Let \mathfrak{m} be the unique maximal ideal of B . Then $\mathfrak{m} \cap A(T)_{\mathfrak{p}} = \mathfrak{p}A(T)_{\mathfrak{p}}$ so for any $x \in B$, $1/x^n \notin \mathfrak{p}A(T)_{\mathfrak{p}}$ and therefore, $x^n \in A(T)_{\mathfrak{p}}$. Thus, $\Sigma B^n \subseteq A(T)_{\mathfrak{p}}$. Since $\mathbb{Q} \subseteq B$, $B \subseteq A(T)_{\mathfrak{p}}$ by (1.1.2) and hence, $B = A(T)_{\mathfrak{p}}$. \square

1.3.8 Proposition. *For any valuation v of K , the following are equivalent:*

- (i) $T \sim A_v$.

(ii) T^v is a proper preorder.

(iii) $A(T) \subseteq A_v$.

Proof. (i) \Rightarrow (ii) by (1.3.5).

(ii) \Rightarrow (iii) For any $x \in \mathfrak{m}_v$, $r \pm x = r(1 \pm r^{-1}x) \in T^*(1 + \mathfrak{m}_v) = T^{v*}$ for all $r \in \mathbb{Q}^+$. Thus, $\mathfrak{m}_v \subseteq I(T^v)$. Suppose $a \in A(T)$. Pick $m \in \mathbb{N}$ such that $m - a^n \in T^*$. Then $\frac{1}{m} - \frac{1}{a^n} \notin T^v$ so $\frac{1}{a^n} \notin I(T^v)$ and therefore, $\frac{1}{a} \notin \mathfrak{m}_v$.

(iii) \Rightarrow (i) Let $t \in T \cap A_v$. Then $1 - \frac{1}{1+t} = \frac{t}{1+t} \in T$ so $\frac{1}{1+t} \in A(T) \subseteq A_v$ and therefore, $1+t \in A_v^*$. This shows $-1 \notin T_v$. \square

If $S \subseteq K$ is a T -semiorder and v is a valuation of K then the push-down S_v is clearly a T_v -module satisfying $k_v = S_v \cup -S_v$. Following [27] (rather than [7]), we say S is *compatible* with A_v and write $S \sim A_v$ if $-1 \notin S_v$. Since $T \subseteq S$, this implies $-1 \notin T_v$ so $S \sim A_v$ iff $T \sim A_v$ and S_v is a T_v -semiorder.

In [45], Prestel shows that for any ΣK^2 -semiorder S , the set $A(S)$ is a valuation ring compatible with S and, moreover, that the push-down of S to the residue field is an order of level 1. In [7], Becker shows this is also the case for any higher level semiorder. His proof is given below. One should note, however, that the level 1 case has a much easier proof. See, for example, [27, Theorems 15.5 and 15.6].

1.3.9 Theorem ([7, Theorem 1.2]). *If $S \subseteq K$ is a T -semiorder then*

(i) $A(S)$ is a valuation ring with maximal ideal $I(S)$,

(ii) $(1 + I(S))(A(S)^* \cap S) \subseteq S$,

(iii) *The push-down \overline{S} of S to the residue field k of $A(S)$ is a level 1 order in k .*

Proof. By (1.3.7), it is clear that $A(S)$ and $I(S)$ are both $A(T)$ -modules. Then $\mathfrak{p} := A(T) \cap I(S)$ is a proper ideal in $A(T)$. We show \mathfrak{p} is prime, $A(S) = A(T)_{\mathfrak{p}}$ and $I(S) = \mathfrak{p}A(T)_{\mathfrak{p}}$. Then, by (1.3.6), $A(S)$ is a valuation ring with maximal ideal $I(S)$.

Let $P := (A(T) \cap T) + \mathfrak{p}$. Suppose $-1 \in P$, say $-1 = t + x$, where $t \in A(T) \cap T$ and $x \in \mathfrak{p} \subseteq I(S)$. Since $-1 \notin I(S)$, $t \neq 0$. Then $-1 = t^{-1}(1 + x) \in TS \subseteq S$,

a contradiction. Therefore, P is a proper archimedean preorder in $A(T)$ so we can apply the Kadison-Dubois Representation Theorem (1.2.5) to P .

Suppose $a \in \text{Arch}(P) \cap -\text{Arch}(P) \subseteq A(T)$ and k is a positive integer. $\frac{1}{2k} + a \in P$ so there exists $t \in A(T) \cap T$, $x \in \mathfrak{p} \subseteq I(S)$ such that $\frac{1}{k} + a = t + \frac{1}{2k} + x \in T + S \subseteq S$. Similarly, $\frac{1}{k} - a \in S$ so $a \in A(T) \cap I(S) = \mathfrak{p}$. Since $\mathfrak{p} \subseteq P \subseteq \text{Arch}(P)$, we have $\mathfrak{p} = \text{Arch}(P) \cap -\text{Arch}(P)$. Suppose $a, b \in A(T) \setminus \mathfrak{p}$. By (1.2.5(iii)), $a^n, b^n \notin \mathfrak{p} = A(T) \cap I(S)$ so there exists $k \in \mathbb{N}$ such that $\frac{1}{k} - a^n \notin S$ and $\frac{1}{k} - b^n \notin S$. Then $a^n - \frac{1}{k}, b^n - \frac{1}{k} \in S$ and $a^n + \frac{1}{k}, b^n + \frac{1}{k} \in T$ so by (1.3.7), $a^n b^n - \frac{1}{k^2} \in S$ and therefore, $a^n b^n \notin I(S)$. It follows that $ab \notin \mathfrak{p}$. Thus, \mathfrak{p} is a prime ideal in $A(T)$.

Let $x \in A(T)_{\mathfrak{p}}$. Then there exists $a \in A(T)$, $s \in A(T) \setminus \mathfrak{p}$ such that $x = \frac{a}{s^n}$. Since $s^n \notin I(S)$, there exists $k \in \mathbb{N}$ such that $\frac{1}{k} - s^n \notin S$ and therefore, $k - \frac{1}{s^n} = \frac{k}{s^n}(s^n - \frac{1}{k}) \in S$. Let $b \in I(S)$. Since $I(S)$ is an $A(T)$ -module, $ab \in I(S)$ so for any $m > 0$, $\frac{1}{km} \pm ab \in S$ and therefore, $\frac{1}{m} \pm xb = \frac{1}{km}(k - \frac{1}{s^n}) + \frac{1}{s^n}(\frac{1}{km} \pm ab) \in S$ so $xb \in I(S)$. This shows $I(S)$ is an $A(T)_{\mathfrak{p}}$ -module. Similarly, one shows $A(S)$ is an $A(T)_{\mathfrak{p}}$ -module.

For any $a \in K$, $a^{-1}a = 1 \notin I(S)$ so, $a \in I(S) \Rightarrow a^{-1} \notin A(T)_{\mathfrak{p}}$. Since $A(T)_{\mathfrak{p}}$ is a valuation ring, this shows $I(S) \subseteq \mathfrak{p}A(T)_{\mathfrak{p}}$. Since $\mathfrak{p} \subseteq I(S)$ and $I(S)$ is a $A(T)_{\mathfrak{p}}$ -module, $\mathfrak{p}A(T)_{\mathfrak{p}} = I(S)$.

Suppose there exists $a \in A(S)$ with $a \notin A(T)_{\mathfrak{p}}$. Replacing a by $-a$ if necessary, we may assume $a \in S$. Consider the T -semiorder $S' := S \frac{1}{a}$. We can apply the above arguments to S' to get the prime ideal $\mathfrak{p}' := A(T) \cap I(S')$ and the valuation ring $A(T)_{\mathfrak{p}'}$ with maximal ideal $I(S') = \mathfrak{p}'A(T)_{\mathfrak{p}'}$.

Suppose $x \in I(S')$. For any $k > 0$, $ka \in A(S)$ so there exists $m \in \mathbb{N}$ such that $m \pm ka \in S$ and therefore, $\frac{1}{k} \pm ax = \frac{1}{km}(m - ka) + a(\frac{1}{m} \pm x) \in S + aS' \subseteq S$. This shows $aI(S') \subseteq I(S)$. Similarly, one shows $aA(S') \subseteq A(S)$.

Since $a \notin A(T)_{\mathfrak{p}}$, $\frac{1}{a} \in \mathfrak{p}A(T)_{\mathfrak{p}} = I(S)$. Thus, $I(S') \subseteq \frac{1}{a}I(S) \subseteq I(S)$ so $\mathfrak{p}' \subseteq \mathfrak{p}$. $A(S')$ is a $A(T)_{\mathfrak{p}'}$ -module so we have

$$A(T)_{\mathfrak{p}} \subseteq A(T)_{\mathfrak{p}'} \subseteq A(S') \subseteq \frac{1}{a}A(S) \subseteq A(S).$$

Since $\frac{1}{a^n} \in I(S)$, $r - \frac{1}{a^n} \in S$ for any $r \in \mathbb{Q}^+$ so $a^n \notin A(S)$. However, $a \in A(S)$

so there exists $m > 0$ such that $m - a \in S$ and therefore, $\frac{1}{a} - \frac{1}{m} \in S_a^1 = S'$. Then $\frac{1}{a} \notin I(S') = \mathfrak{p}'A(T)_{\mathfrak{p}'}$ so $a \in A(T)_{\mathfrak{p}'}$. But then $a^n \in A(T)_{\mathfrak{p}'} \subseteq A(S)$, a contradiction. Thus, $A(S) \subseteq A(T)_{\mathfrak{p}}$. Since $A(S)$ is a $A(T)_{\mathfrak{p}}$ -module, the reverse inclusion holds. Therefore, $A(S) = A(T)_{\mathfrak{p}}$ which proves (i).

Suppose $x \in I(S)$, $a \in A(S)^* \cap S$. Then $r - a \notin S$ for some $r \in \mathbb{Q}^+$ and $ax \in I(S)$ so $(1+x)a = a + ax = (a-r) + (r+ax) \in S + S \subseteq S$. This proves (ii).

Suppose $a, b \in A(S)^* \cap S$. Then $a, b \notin I(S)$ so there exists $r \in \mathbb{Q}^+$ such that $\frac{r}{2} - a, \frac{r}{2} - b \notin S$. Then $(a+b) - r = (a - \frac{r}{2}) + (b - \frac{r}{2}) \in S + S \subseteq S$ so $a+b \notin I(S)$. This shows $A(S)^* \cap S$ is additively closed and therefore, $-1 \notin \overline{S}$.

Suppose $a \in A(T) \setminus \mathfrak{p}$. By (1.2.5(ii)), $a^2 \subseteq \text{Arch}(P)$ so for any $k > 0$, there exists $t \in A(T) \cap T$, $x \in \mathfrak{p} \subseteq I(S)$ such that $\frac{1}{2k} + a^2 = t + x$ so $\frac{1}{k} + a^2 = \frac{1}{2k} + t + x = t + (\frac{1}{2k} + y) \in T + S \subseteq S$. If $a^2 \in -S$ then $\frac{1}{k} - a^2 \in S$ for all $k > 0$ and hence, $a^2 \in A(T) \cap I(S) = \mathfrak{p}$, a contradiction. Thus, $a^2 \in A(S)^* \cap S$. Since $A(S)^* \cap S$ is additively closed, $\frac{1}{k} + a^2 \in A(S)^* \cap S$. Therefore,

$$(\frac{1}{k} + a^2)(A(S)^* \cap S) \subseteq A(S)^* \cap S$$

for any $k > 0$.

Suppose $x \in A(S)^*$, $s, t \in A(S)^* \cap S$ such that $x^2s = -t$. Since $A(S) = A(T)_{\mathfrak{p}}$, there exists $a, b \in A(T) \setminus \mathfrak{p}$ such that $x = \frac{a}{b^n}$. Then $a^2s = -b^{2n}t$ and $b^{2n}t \notin I(S)$. Pick $k, m \in \mathbb{N}$ such that $k \pm s \in S$ and $\frac{1}{m} - b^{2n}t \notin S$. Then $b^{2n}t - \frac{s}{km} = (b^{2n}t - \frac{1}{m}) + \frac{1}{km}(k-s) \in S$ and $\frac{s}{km} - b^{2n}t = \frac{s}{km} + a^2s = (\frac{1}{km} + a^2)s \subseteq A(S)^* \cap S$, a contradiction. Therefore,

$$A(S)^{*2}(A(S)^* \cap S) \subseteq A(S)^* \cap S.$$

This shows \overline{S} is an archimedean Σk^2 -semiorder. It remains only to prove the following result due to Prestel [46, Theorem 1.20].

Claim. Any archimedean Σk^2 -semiorder is a level 1 order.

We need only show \overline{S} is closed under multiplication. Let $s, t \in \overline{S}^*$. Then $st = \frac{1}{4}[(t+s)^2 - (t-s)^2]$ so it suffices to show for any $a, b \in k$,

$$a, b - a \in \overline{S} \Rightarrow b^2 - a^2 \in \overline{S}.$$

If $a = 0$ or $a = b$ the result is trivial so assume $a, b - a \in \overline{S}^*$. Then

$$\begin{aligned}
 ab(b-a) &= a^2(b-a)^2 \frac{b}{a(b-a)} \\
 (*) \quad &= a^2(b-a)^2 \left[\frac{1}{a} + \frac{1}{b-a} \right] \\
 &= a^2(b-a)^2 \left[\frac{a}{a^2} + \frac{b-a}{(b-a)^2} \right]
 \end{aligned}$$

which is clearly in \overline{S} . Suppose $a \in \Sigma k^2$. Then, multiplying (*) by $\frac{1}{a}$, we have $b(b-a) \in \overline{S}$ and therefore, $b^2 - a^2 = b^2 - ab + ab - a^2 = b(b-a) + a(b-a) \in \overline{S} + (\Sigma k^2)\overline{S} \subseteq \overline{S}$. Similarly, if $b \in \Sigma k^2$, we get $b^2 - a^2 \in \overline{S}$. Thus, it suffices to show there exists $r \in \mathbb{Q}^+$ with $b-r, r-a \in \overline{S}$.

Since \overline{S} is archimedean we can pick $k > 0$ such that $k - \frac{1}{b-a} \in \overline{S}^*$. Since $\frac{1}{b-a} = \frac{b-a}{(b-a)^2} \in \overline{S}^*$, it follows from (*) that $\frac{k}{b-a}(k - \frac{1}{b-a}) \in \overline{S}$. Multiplying by $\frac{(b-a)^2}{k}$, we have $k(b-a) - 1 \in \overline{S}$. Pick $m \in \mathbb{Z}$ minimal with respect to $m - ka \in \overline{S}^*$. Then $kb - m = (k(b-a) - 1) + (ka - (m-1)) \in \overline{S}$ so $b - \frac{m}{k}, \frac{m}{k} - a \in \overline{S}$. This completes the proof. \square

1.3.10 Proposition. *For a valuation v of K and a T -semiorder $S \subseteq K$, the following are equivalent:*

- (i) $A(S) \subseteq A_v$.
- (ii) $S \sim A_v$.
- (iii) $(1 + \mathfrak{m}_v)(A_v^* \cap S) \subseteq S$.
- (iv) $1 + \mathfrak{m}_v \subseteq S$.

Proof. (i) \Rightarrow (ii) If $-1 \in S_v$ then there exists $s \in A_v^* \cap S$, $x \in \mathfrak{m}_v$ such that $-1 = s + x = (1 + s^{-1}x)s$. Since $\mathfrak{m}_v \subseteq I(S)$, $1 + s^{-1}x \in 1 + I(S) \subseteq A(S)^*$. Since $-1 \in A(S)^*$, we must have $s \in A(S)^*$. But then $-1 \in S$ by (1.3.9(ii)), a contradiction.

(ii) \Rightarrow (iii) Suppose there exists $s \in A_v^* \cap S$, $x \in \mathfrak{m}_v$ such that $(1+x)s \in -S$. Then $\bar{s} \in S_v \cap -S_v$ and $\bar{s} \neq 0$ so by (1.1.1), $S_v \cap -S_v = k_v$ and hence, $-1 \in S_v$, a contradiction.

(iii) \Rightarrow (iv) is clear.

(iv) \Rightarrow (i) For any $x \in \mathfrak{m}_v$, $\frac{1}{m} \pm x = \frac{1}{m}(1 \pm mx) \in T(1 + \mathfrak{m}_v) \subseteq S$ for all integers $m > 0$. Thus, $\mathfrak{m}_v \subseteq I(S)$ so $A(S) \subseteq A_v$. \square

1.3.11 Theorem ([12, Theorem 3.1]). *Suppose $S \subseteq K$ is a T -semiorder. If $a_1, \dots, a_r \in S^*$ such that $a_i \notin A(S)^*T^*$ for at least one i then there exists a valuation ring A compatible with S such that*

- (i) $(1 + \mathfrak{m})a_i \subseteq S$ for all i , where \mathfrak{m} is the maximal ideal of A ,
- (ii) $a_i \notin A^*T^*$ for some i .

Proof. Consider the family $\{A_\alpha\}$ of all valuation rings of K containing $A(S)$ with $a_i \notin A_\alpha^*T^*$ for some i . Since $A(S)$ is a valuation ring, this family is linearly ordered by inclusion so that $A := \cup A_\alpha$ is again a valuation ring of K containing $A(S)$ with $a_i \notin A^*T^*$ for at least one i . Let \mathfrak{m} denote the maximal ideal of A . We show $(1 + \mathfrak{m})a_i \in S$ for all i .

Suppose B is a valuation ring properly containing A . Then $A(S) \subseteq A \subseteq B$ and $a_i \in B^*T^* \cap S = (B^* \cap S)T^*$ for each i so, by (1.3.10), $(1 + \mathfrak{m}_B)a_i \subseteq (1 + \mathfrak{m}_B)(B^* \cap S)T^* \subseteq S$, where \mathfrak{m}_B is the maximal ideal of B .

Consider now the family $\{B_\beta\}$ of valuation rings of K which properly contain A and let $B = \cap B_\beta$. If $A = B$ we are done since then \mathfrak{m} is the union of the maximal ideals $\mathfrak{m}_\beta \subseteq B_\beta$. So assume $A \subsetneq B$. Then $B = B_\beta$ for some β so $a_i \in (B^* \cap S)T^*$ for all i . Let π denote the natural homomorphism of B onto its residue field k and let $\overline{S} := \pi(B \cap S)$. If $(1 + \pi(\mathfrak{m}))\overline{S} \subseteq \overline{S}$ then $(1 + \mathfrak{m})(B^* \cap S) \subseteq S$ so $(1 + \mathfrak{m})a_i \in S$ for each i . Therefore, it remains only to show $(1 + \pi(\mathfrak{m}))\overline{S} \subseteq \overline{S}$.

$\pi(A)$ is a valuation ring of k of (Krull) dimension 1 and $\pi(\mathfrak{m})$ is its maximal ideal. Since $1 + \mathfrak{m} \subseteq A^* \cap S \subseteq B^* \cap S$, $1 + \pi(\mathfrak{m}) \subseteq \overline{S}$. Let \hat{k} denote the completion of k with respect to $\pi(A)$ and \hat{S} denote the closure of \overline{S} in \hat{k} . Clearly, \hat{S} is a $\Sigma \hat{k}^n$ -module. Suppose $-1 \in \hat{S}$. Then there exists $s \in \overline{S}^*$ such that $|-1 - s| < 1$ so $-1 - s \in \pi(\mathfrak{m})$ and therefore, $-s \in 1 + \pi(\mathfrak{m}) \subseteq \overline{S}$, a contradiction. Thus, \hat{S} is a proper $\Sigma \hat{k}^n$ -module in \hat{k} (in fact, \hat{S} is a semiorder.) By (1.1.1), $\hat{S} \cap -\hat{S} = \{0\}$ so

$\hat{S} \cap k = \bar{S}$. For any $x \in \pi(\mathfrak{m})$, $|x| < 1$ so $(1+x)^{\frac{1}{n}} \in \hat{k}$. Then $(1+\pi(\mathfrak{m}))\hat{S} \subseteq \hat{k}^n \hat{S} \subseteq \hat{S}$ and therefore, $(1+\pi(\mathfrak{m}))\bar{S} \subseteq \hat{S} \cap k = \bar{S}$. \square

1.4 Complete preorders

If $G \neq 0$ is an abelian group of finite exponent, the *2-primary part* of G is defined to be the subgroup H_2 consisting of all elements whose order is a power of 2.

A proper preorder P in a field K is called *complete* if the 2-primary part of K^*/P^* is cyclic. For example, every order of level 1 is complete.

1.4.1 Proposition. *A proper preorder $P \subseteq K$ is complete iff for all $x \in K$,*

$$(*) \quad x^2 \in P \Rightarrow x \in P \cup -P.$$

Proof. If the 2-primary part of K^*/P^* is cyclic then K^*/P^* has a unique element of order 2 and hence, $(*)$ holds. Conversely, suppose $(*)$ holds. Pick $x \in K^*$ such that xP^* generates the 2-primary part of K^*/P^* . If $y \in K^*$ has order 2^{s+1} modulo P^* , $s \geq 0$, then $y^{2^s}P^*$ has order 2 so by $(*)$, $y^{2^s}P^* = -P^* = x^{2^r}P^*$, where 2^{r+1} is the order of xP^* , and hence, $(x^{2^{r-s}}y)^{2^s} \in P^*$. By induction on s , $x^{2^{r-s}}yP^* = x^kP^*$ and hence, $y = x^{k-2^{r-s}}P^*$. \square

1.4.2 Proposition. *Any maximal proper preorder is complete. Conversely, if P is a 2-primary complete preorder then P is maximal.*

Proof. Suppose there exists $x \in K$ such that $x^2 \in T$ and $x \notin T \cup -T$. Then $-1 \notin T + xT$ since otherwise, $-1 = s + xt$, where $s, t \in T$, $t \neq 0$, and hence, $-x = \frac{1}{t}(1+s) \in T$. Therefore, $T + xT$ is a proper preorder properly containing T so T is not maximal. Conversely, suppose $T \subseteq T'$ are proper 2-primary preorders and $x \in T' \setminus T$. Replacing x with a suitable power we can assume $x^2 \in T$, $x \notin T$. Since $T' \cap -T' = \{0\}$, $-x \notin T$ and hence, T is not complete. \square

1.4.3 Lemma. *If $T \subseteq K$ is a proper preorder which is not complete then*

$$T = \bigcap \{T + aT \mid a^2 \in T, a \notin T \cup -T\}.$$

Proof. Since T is not complete there exists $a \in K$ with $a^2 \in T$, $a \notin T \cup -T$.

Suppose $x \in T + aT$ for all $a^2 \in T$, $a \notin T \cup -T$. For any such a , there exists $s_1, s_2, t_1, t_2 \in T$ such that $x = s_1 + as_2$ and $x = t_1 - at_2$. Then $(s_1s_2 + t_1t_2)x^2 = s_1s_2(t_1^2 - 2at_1t_2 + a^2t_2^2) + t_1t_2(s_1^2 + 2as_1s_2 + a^2s_2^2) = s_1s_2t_1^2 + t_1t_2s_1^2 + a^2(s_1s_2t_2^2 + t_1t_2s_2^2) \in T$. Suppose $x \notin T$. Then none of the s_i, t_i are 0 so $s_1s_2 + t_1t_2 \neq 0$ and therefore, $x^2 \in T$. Since $a = \frac{t_1 - x}{t_2} \notin T$, $x \notin -T$. But then $x \in T - xT$ so $x = s - xt$ for some $s, t \in T$ and therefore, $x = \frac{s}{1+t} \in T$, a contradiction. \square

1.4.4 Theorem. *Every proper preorder in K is the intersection of the complete preorders lying over it.*

Proof. Suppose T is a proper preorder in K and $x \notin T$. Let $P \supseteq T$ be a proper preorder maximal with respect to $x \notin P$. Then $x \in P + aP$ for all $a \in K$ such that $a^2 \in P$, $a \notin P \cup -P$. By (1.4.3), P is complete since otherwise, $x \in \cap \{P + aP \mid a^2 \in P, a \notin P \cup -P\} = P$. \square

1.4.5 Theorem ([5, Theorem 3.4]). *If $P \subseteq K$ is a complete preorder then $A(P)$ is a valuation ring, $I(P)$ is the unique maximal ideal and the push-down of P to the residue field of $A(P)$ is an archimedean level 1 order.*

Proof. Since $P \cap A(P)$ is an archimedean preorder in $A(P)$, we can apply the Kadison-Dubois Representation Theorem (1.2.5). Set

$$X := X(P \cap A(P)) = \{\varphi \in \text{Hom}(A(P), \mathbb{R}) \mid \varphi(P \cap A(P)) \geq 0\}.$$

For $a \in A(P)$, \hat{a} denotes the evaluation map $X \rightarrow \mathbb{R}$ given by $\varphi \mapsto \varphi(a)$. Then $a \in I(P)$ iff $\hat{a} = 0$ on X and $\hat{a} \geq 0$ on X iff for all $k \in \mathbb{N}$, $1 + ka \in P$.

Suppose $a \in A(P)$, $a \notin I(P)$. Then $\varphi(a^2) > 0$ for some $\varphi \in X$, say $\varphi(a^2) > \frac{1}{k}$, $k \in \mathbb{N}$. Set $b = a^2 - \frac{1}{k} \in A(P)$. So $\varphi(b) > 0$. Suppose $s = 2^r m$, m odd, is the order of b in K^*/P^* . If $r \geq 1$ then $b^{2^{r-1}m} \in -P$ so $\varphi(b^{2^{r-1}m}) \leq 0$, a contradiction. Thus $b^m \in P$ so $\hat{b}^m \geq 0$ on X . Since m is odd, $\hat{b} \geq 0$ on X . Thus, $a^2 = \frac{1}{k} + b \in P$ and $a^2 - \frac{1}{2k} = \frac{1}{2k} + b \in P$ so $2k \pm a^{-2} \in P$ and hence, $a^{-2} \in A(P)$. It follows

that $a^{-1} \in A(P)$ so $a \in A(P)^*$. Therefore, $I(P)$ is the unique maximal ideal of the Prüfer ring $A(P)$ so $A(P) = A(P)_{I(P)}$ is a valuation ring. We have already seen $a \in A(P) \setminus I(P)$ implies $a^2 \in P$ so $A(P)^* \subseteq P \cup -P$ and hence, the push-down \bar{P} of P is an archimedean level 1 order. \square

In section 1.1, we used a result in [22] to show $-1 \notin \Sigma K^n$ iff K is formally real. We are now in a position to prove this directly.

1.4.6 Corollary. $-1 \notin \Sigma K^2$ iff $-1 \notin \Sigma K^n$.

Proof. If $-1 \notin \Sigma K^n$ then K has a complete preorder P of exponent n by (1.4.2). Let v be the valuation associated with $A(P)$. Since P_v is a level 1 order in k_v , $-1 \notin \Sigma k_v^2$. By (1.3.4), $-1 \notin \Sigma K^2$. The converse is clear. \square

1.4.7 Corollary. Let $P \subseteq K$ be a complete preorder. For any valuation v of K , $P \sim A_v$ iff P is fully compatible with A_v . If $P \sim A_v$ then the push-down P_v is complete.

Proof. By (1.3.8), if $P \sim A_v$ then $A(P) \subseteq A_v$ so $1 + \mathfrak{m}_v \subseteq 1 + I(P) \subseteq P$ and hence, $P = P_v$. By (1.3.5), k_v^*/P_v^* is embedded in K^*/P^* so k_v^*/P_v^* necessarily has a unique element of order 2 and hence, is complete. \square

1.4.8 Corollary. For any proper preorder $T \subseteq K$,

$$A(T) = \bigcap \{A(P) \mid P \supseteq T \text{ is a maximal proper preorder}\}.$$

Proof. Let \mathfrak{m} be a maximal ideal of $A(T)$. Since $A(T)$ is a Prüfer domain, $A(T)_{\mathfrak{m}}$ is a valuation ring. Let v denote the associated valuation. It follows from (1.3.8) that T^v is a proper preorder. Let P be a maximal proper preorder lying over T^v . Since $1 + \mathfrak{m}A(T)_{\mathfrak{m}} \subseteq T^v \subseteq P$, $P \sim A(T)_{\mathfrak{m}}$ and therefore, $A(P) \subseteq A(T)_{\mathfrak{m}}$. P is complete so $A(P)$ is a valuation ring and therefore, $\mathfrak{m} \subseteq \mathfrak{m}A(T)_{\mathfrak{m}} \subseteq I(P)$ so $\mathfrak{m} = I(P) \cap A(T)$ and hence, $A(P) = A(T)_{\mathfrak{m}}$. Since $A(T) = \bigcap A(T)_{\mathfrak{m}}$ where \mathfrak{m} runs through all maximal ideals of $A(T)$, this completes the proof. \square

Let $P \subseteq K$ be a complete preorder of level $s(P) = 2^s m$, $s \geq 0$, $m \geq 1$ odd, and let v be a valuation of K such that P_v is a level 1 order in k_v . (For example, take v to be the valuation associated with $A(P)$.) Let $G := v(K^*)$, $G_0 := v(P^*)$.

1.4.9 Lemma. *There exists a homomorphism $\mu : G/nG \rightarrow K^*/K^{*n}$ such that $\bar{v} \circ \mu = id$, where \bar{v} denotes the homomorphism $K^*/K^{*n} \rightarrow G/nG$ induced by v .*

Proof. Write $n = p_1^{r_1} \cdots p_k^{r_k}$ where p_1, \dots, p_k are distinct primes and $r_i > 0$ for all i . For any abelian group G , $G/nG \cong G/p_1^{r_1}G \times \cdots \times G/p_k^{r_k}G$ so it suffices to consider the case $k = 1$, that is, where $n = p^r$ for some prime p and $r > 0$.

Let $\{g_i + pG\}$ be a $\mathbb{Z}/p\mathbb{Z}$ -basis for G/pG . We show that G/p^rG is the direct sum of the cyclic subgroups generated by the $g_i + p^rG$. For then we can define μ by $g_i + p^rG \mapsto x_i K^{*p^r}$ where the $x_i \in K^*$ are chosen so that $v(x_i) = g_i$ for each i .

Clearly the $g_i + p^rG$ generate G/p^rG . Let $e_i \in \mathbb{Z}$, $h \in G$ such that $\sum e_i g_i = p^r h$ and suppose for some $0 \leq s < r$, $p^s | e_i$ for all i . Then $p^s(p^{r-s}h - \sum \frac{e_i}{p^s} g_i) = 0$. Since G is the value group of the valuation v , 0 is the only element of finite order in G . Thus, $\sum \frac{e_i}{p^s} g_i = p^{r-s}h \in pG$. Since the g_i are $\mathbb{Z}/p\mathbb{Z}$ -independent, $p | \frac{e_i}{p^s}$ so $p^{s+1} | e_i$ for all i . By induction, $p^r | e_i$ for all i and hence, $e_i g_i \equiv 0 \pmod{p^r G}$ for all i . \square

Choose a set of representatives $\mathfrak{A} \subseteq K^*$ for $\mu(G/nG)$, taking $1 \in \mathfrak{A}$. Note that for $a \in \mathfrak{A}$, $x \in K^*$, $\mu(v(x) + nG) = aK^{*n}$ iff $x = a\epsilon y^n$ for some $\epsilon \in A_v^*$, $y \in K^*$. For any $x \in P^*$, define

$$\chi_P : G_0 \rightarrow \{\pm 1\}$$

by $\chi_P(v(x)) = \bar{\epsilon} P_v^*$, where $\epsilon \in A_v^*$ such that $x = a\epsilon y^n$ for some $a \in \mathfrak{A}$, $y \in K^*$. Suppose $x_i = a_i \epsilon_i y_i^n \in P^*$, where $a_i \in \mathfrak{A}$, $\epsilon_i \in A_v^*$, $y_i \in K^*$, $i = 1, 2$. If $v(x_1) = v(x_2)$ then $a_1 K^{*n} = \mu(v(x_1) + nG) = \mu(v(x_2) + nG) = a_2 K^{*n}$ so $a_1 = a_2$ and $\epsilon_1/\epsilon_2 = x_1 y_2^n / x_2 y_1^n \in P^*$ so $\chi_P(v(x_1)) = \bar{\epsilon}_1 P_v^* = \bar{\epsilon}_2 P_v^* = \chi_P(v(x_2))$. There exists $b \in \mathfrak{A}$ such that $a_1 a_2 K^{*n} = \mu(v(x_1 x_2) + nG) = b K^{*n}$. Then $x_1 x_2 = b \epsilon_1 \epsilon_2 z^n$ for some $z \in K^*$ and hence, $\chi_P(v(x_1) + v(x_2)) = \bar{\epsilon}_1 \bar{\epsilon}_2 P_v^* = \chi_P(v(x_1)) \chi_P(v(x_2))$. For any $y \in K^*$, $\chi_P(v(y^n)) = 1$ since $1 \in \mathfrak{A}$. Therefore, χ_P is a character with $\chi_P(nG) = 1$.

P is completely determined by the group G_0 and the character χ_P since

$$P = \bigcup \{aM_a K^n \mid a \in \mathfrak{A} \cap v^{-1}(G_0)\}$$

where $M_a := \{\varepsilon \in A_v^* \mid a\varepsilon \in P\} = \{\varepsilon \in A_v^* \mid \chi_P(v(a)) = \bar{\varepsilon}P_v^*\}$.

By (1.3.5), the sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \frac{K^*}{P^*} \rightarrow \frac{G}{G_0} \rightarrow 0$$

is exact. Since the 2-primary part of K^*/P^* is cyclic of order 2^{s+1} , the 2-primary part of G/G_0 is cyclic of order 2^s . Suppose $s \geq 1$. Let $x \in K^*$ such that x has order 2^{s+1} modulo P^* . P is complete so $-x^{2^s} \in P^*$. Write $x^{2^{s-1}} = a\varepsilon y^n$ where $a \in \mathfrak{A}$, $\varepsilon \in A_v^*$, $y \in K^*$. Then $-x^{2^s} = -a^2\varepsilon^2 y^{2n} = -b\varepsilon^2 y^{2n} z^n$ for some $b \in \mathfrak{A}$, $z \in K^*$ so $\chi_P(2^s v(x)) = \chi_P(v(-x^{2^s})) = -\bar{\varepsilon}^2 P_v^* = -1$ and therefore, $\chi_P(G_0 \cap 2^s G) \neq 1$.

We have proved one half of the following.

1.4.10 Theorem ([6, Satz 2.4]). *Let v be a valuation of K , $G := v(K^*)$, $\bar{v} : K^*/K^{*n} \rightarrow G/nG$ the induced homomorphism. Let $\mu : G/nG \rightarrow K^*/K^{*n}$ be a homomorphism such that $\bar{v} \circ \mu = \text{id}$ and fix a set of representatives $\mathfrak{A} \subseteq K^*$ for $\mu(G/nG)$ with $1 \in \mathfrak{A}$.*

Suppose G_0 is a subgroup of G containing nG such that the 2-primary part of G/G_0 is cyclic of order 2^s , $s \geq 0$, and $\chi : G_0 \rightarrow \{\pm 1\}$ is a character such that $\chi(nG) = 1$ and $\chi(G_0 \cap 2^s G) \neq 1$ if $s \geq 1$.

Let \bar{P} be a level 1 order in k_v and set $M_a := \{\varepsilon \in A_v^ \mid \chi(v(a)) = \bar{\varepsilon}\bar{P}^*\}$ for each $a \in \mathfrak{A} \cap v^{-1}(G_0)$. Then*

$$P := \bigcup \{aM_a K^n \mid a \in \mathfrak{A} \cap v^{-1}(G_0)\}$$

is a complete preorder with $P_v = \bar{P}$ and $G_0 = v(P^)$ and every complete preorder in K pushing down to \bar{P} is obtained in this way.*

Proof. It is easy to see that $P \cdot P \subseteq P$ and $P \cap -P = \{0\}$. Since $1 \in \mathfrak{A}$, $K^n \subseteq P$ and $M_1 = \{\varepsilon \in A_v^* \mid \bar{\varepsilon} \in \bar{P}^*\} \subseteq P$. Then $\overline{A_v^* \cap P} = \bar{M}_1 = \bar{P}^*$ and $1 + \mathfrak{m}_v \subseteq P$. Suppose $x \in P^*$. If $x \in \mathfrak{m}_v$ then $1 + x \in 1 + \mathfrak{m}_v \subseteq P$. If $x \notin \mathfrak{m}_v$ then $1 + x =$

$x(1+x^{-1}) \in P^*(1+\mathfrak{m}_v) \subseteq P^*$. If $x \in A_v^*$ then $\bar{x} \in \bar{P}^*$ so $1+\bar{x} \in \bar{P}^*$ and therefore, $1+x \in M_1 \subseteq P$. Thus, $1+P \subseteq P$ and hence, $P+P \subseteq P$. So P is a proper preorder with $P_v = \bar{P}$.

Let $v(x) \in G_0$. Write $x = a\varepsilon y^n$, where $a \in \mathfrak{A}$, $\varepsilon \in A_v^*$ and $y \in K^*$. Then $v(a) \in G_0$ so for any $\eta \in M_a \subseteq A_v^*$, $a\eta y^n \in P^*$ and $v(x) = v(a\eta y^n)$. Thus, $G_0 \subseteq v(P^*)$. Clearly $v(P^*) \subseteq G_0$ so $v(P^*) = G_0$. Note that since $P_v = \bar{P}$ has level 1, $A_v^* \subseteq P^* \cup -P^*$ and therefore, $v(x) \in G_0$ iff $x \in P^* \cup -P^*$.

Suppose there exists $x \in K^*$ such that $x^2 \in P^*$ and $x \notin P^* \cup -P^*$. Then $v(x) + G_0$ has order 2 so the 2-primary part of G/G_0 is non-trivial. Let $u \in K^*$ such that $2^s v(u) \in G_0$ and $\chi(2^s v(u)) = -1$. Write $u^{2^{s-1}} = a\varepsilon y^n$ where $a \in \mathfrak{A}$, $\varepsilon \in A_v^*$, $y \in K^*$. Let $b \in \mathfrak{A}$, $z \in K^*$ such that $a^2 = bz^n$. Then $u^{2^s} = a^2 \varepsilon^2 y^{2n} = b \varepsilon^2 y^{2n} z^n$, $v(b) \in G_0$ and $\chi(v(b)) = \chi(2^s v(u)) = -1 = -\bar{\varepsilon}^2 \bar{P}^*$ so $-\varepsilon^2 \in M_b$ and $-u^{2^s} = -b \varepsilon^2 y^{2n} z^n \in P^*$. Thus, $u^{2^{s-1}} \notin P^* \cup -P^*$ so $2^{s-1} v(u) \notin G_0$ and therefore, $v(u) + G_0$ generates the 2-primary part of G/G_0 . Since $v(x) + G_0$ has order 2, $v(x) + G_0 = 2^{s-1} v(u) + G_0$ and hence, $v(xu^{2^{s-1}}) = v(x) + 2^{s-1} v(u) \equiv 0 \pmod{G_0}$. Therefore, $xu^{2^{s-1}} \in P^* \cup -P^*$ so $x^2 u^{2^s} \in P^*$. But $x^2 \in P^*$ so $u^{2^s} \in P^*$, a contradiction. Therefore, P is complete.

Since we have already seen that every complete preorder can be obtained in this way, this completes the proof. \square

1.5 Orders of higher level in fields

An *order* in a field K is a proper preorder $P \subseteq K$ such that K^*/P^* is cyclic. Clearly, orders are complete and any 2-primary complete preorder is an order.

The set of all orders of exponent n in K is called the (*higher level*) *real spectrum* of K and is denoted $\text{Sper } K$. If T is a preorder in K , $\text{Sper}_T K$ denotes the set of all orders in $\text{Sper } K$ containing T .

1.5.1 Example. In $\mathbb{R}(X)$, the only complete preorders are orders and $\mathbb{R}(X)$ has precisely the following orders:

- (i) For each even integer $s \geq 2$, $\mathbb{R}(X)$ has a unique order $P_{\infty-,s}$ of level s compatible with the valuation ring $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$ and for each $a \in \mathbb{R}$, a unique

order $P_{a-,s}$ of level s compatible with the valuation ring $\mathbb{R}[X]_{(X-a)}$.

- (ii) For each odd integer $m \geq 1$, there exists exactly two orders $P_{\infty-,m}$ and $P_{\infty+,m}$ of level m in $\mathbb{R}(X)$ compatible with the valuation ring $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$ and for each $a \in \mathbb{R}$, exactly two orders $P_{a-,m}$ and $P_{a+,m}$ of level m compatible with the valuation ring $\mathbb{R}[X]_{(X-a)}$.

Proof. Since $\mathbb{R}[X]_{(X-a)}$, $a \in \mathbb{R}$, and $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$ are the only valuation rings of $\mathbb{R}(X)$ with a formally real residue field, every order in $\mathbb{R}(X)$ must be compatible with one of these valuation rings and must push down to the order $\bar{P} := \mathbb{R}^2$ on the residue field. Let $s = 2^r m$, where $r \geq 0$ and $m \geq 1$ is odd, and set $n = 2s$. We use (1.4.10) to determine all the orders in $\mathbb{R}(X)$ of level s .

Let $a \in \mathbb{R}$, $A_v := \mathbb{R}[X]_{(X-a)}$. A_v is the valuation ring of the valuation $v : \mathbb{R}(X)^* \rightarrow \mathbb{Z}$ defined by $v(f/g) = k$ iff $\frac{f}{g} = (X-a)^k \frac{f'}{g'}$ for some $f', g' \in \mathbb{R}[X]$ with $f'(a)g'(a) \neq 0$. Set

$$T_a := \left\{ \frac{f}{g} \mid f, g \in \mathbb{R}[X] \text{ and } f(a)g(a) > 0 \right\}$$

so $T_a = \{\varepsilon \in A_v^* \mid \bar{\varepsilon} \in \bar{P}^*\}$ and $A_v^* = T_a \cup -T_a$. Define $\mu : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{R}(X)^*/\mathbb{R}(X)^{*n}$ by $\mu(k + n\mathbb{Z}) = (X-a)^k + \mathbb{R}(X)^{*n}$ and take $\mathfrak{A} := \{1, X-a, \dots, (X-a)^{n-1}\}$.

To obtain a complete preorder of level s , the only possible choice for the subgroup G_0 is sG . Then $\mathfrak{A} \cap v^{-1}(G_0) = \{1, (X-a)^s\}$, G/G_0 is cyclic of order s and $G_0/nG \cong \mathbb{Z}/2\mathbb{Z}$. Let $\chi : G_0 \rightarrow \{\pm 1\}$ be the character for which $\chi(v((X-a)^s)) = \chi(2^r v((X-a)^m)) = -1$. Then $M_1 = T_a$ and $M_{(X-a)^s} = -T_a$ so

$$P_{a-,s} := (T_a \cup -(X-a)^s T_a) \mathbb{R}(X)^{2s}$$

is a complete preorder in $\mathbb{R}(X)$ of level s compatible with $\mathbb{R}[X]_{(X-a)}$. Since $G_0 = v(P_{a-,s}^*)$ and G/G_0 is cyclic, $P_{a-,s}$ is an order in $\mathbb{R}(X)$.

The only other character is the character χ' which is identically 1 on G_0 . If s is even, that is, $r > 0$, we cannot use χ' to define an order in $\mathbb{R}(X)$. However, if $r = 0$, the 2-primary part of G/G_0 is trivial so we can use χ' . In this case, $s = m$ is odd and $M_1 = M_{(X-a)^m} = T_a$ so

$$P_{a+,m} := (T_a \cup (X-a)^m T_a) \mathbb{R}(X)^{2m}$$

is a complete preorder in $\mathbb{R}(X)$ of level m compatible with $\mathbb{R}[X]_{(X-a)}$.

Now let $A_v := \mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$. A_v is valuation ring of the valuation $v : \mathbb{R}(X)^* \rightarrow \mathbb{Z}$ defined by $v(f/g) = \deg(g) - \deg(f)$ for any non-zero $f, g \in \mathbb{R}[X]$. Define

$$T_\infty := \left\{ \frac{f}{g} \mid f = \sum_{i=0}^d a_i X^i, g = \sum_{i=0}^d b_i X^i \text{ with } d \geq 0, a_d b_d > 0 \right\}$$

so $T_\infty = \{\varepsilon \in A_v^* \mid \bar{\varepsilon} \in \bar{P}^*\}$ and $A_v^* = T_\infty \cup -T_\infty$. Define $\mu : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{R}(X)^*/\mathbb{R}(X)^{*n}$ by $\mu(k + n\mathbb{Z}) = \frac{1}{X}k + \mathbb{R}(X)^{*n}$ and take $\mathfrak{A} := \{1, \frac{1}{X}, \dots, \frac{1}{X}^{n-1}\}$.

We again have $G = \mathbb{Z}$ and $G_0 = sG$ so $\mathfrak{A} \cap v^{-1}(G_0) = \{1, \frac{1}{X}s\}$, G/G_0 is cyclic of order s and $G_0/nG \cong \mathbb{Z}/2\mathbb{Z}$. For the character $\chi : G_0 \rightarrow \{\pm 1\}$ defined by $\chi(v(\frac{1}{X}s)) = \chi(2^r v(\frac{1}{X}m)) = -1$, $M_1 = T_\infty$ and $M_{\frac{1}{X}s} = -T_\infty$ so

$$P_{\infty-,s} := (T_\infty \cup -\frac{1}{X}s T_\infty) \mathbb{R}(X)^{2s}$$

is an order in $\mathbb{R}(X)$ of level s compatible with $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$.

If $r = 0$, the character which is identically 1 on G_0 also defines an order. In this case, $M_1 = M_{\frac{1}{X}m} = T_\infty$ so

$$P_{\infty+,m} := (T_\infty \cup \frac{1}{X}m T_\infty) \mathbb{R}(X)^{2m}$$

is an order in $\mathbb{R}(X)$ of level m compatible with $\mathbb{R}[\frac{1}{X}]_{(\frac{1}{X})}$. \square

For a preorder $T \subseteq K$ and an integer $m \geq 1$, we define

$$T^{(m)} = \{x \in K \mid x^m \in T\}.$$

Clearly for 2-primary preorders, $T^{(m)} = T$ for all odd integers m .

1.5.2 Theorem. *Suppose $T \subseteq K$ is a proper preorder and $m \geq 1$ is an odd integer.*

Then

- (i) $T^{(m)}$ is a proper preorder,
- (ii) $A(T) = A(T^{(m)})$,
- (iii) $T^{(m)}$ is complete iff T is complete,
- (iv) if $n = 2^s v$ where v is odd then $T^{(v)}$ is 2-primary.

Proof. (i) Clearly, $T \subseteq T^{(m)}$, $T^{(m)} \cdot T^{(m)} \subseteq T^{(m)}$ and since m is odd, $-1 \notin T^{(m)}$. We must show $T^{(m)*}$ is additively closed. By (1.4.4), we may assume T is complete.

Let $U = T^{(m)*}$ and consider the valuation ring $A(T)$. We have $1 + I(T) \subseteq T^* \subseteq U$ so (1.3.2) applies. Since $A(T)^* \cap T$ pushes down to a level 1 order on the residue field, $(A(T)^* \cap T) \cup -(A(T)^* \cap T) = A(T)^*$. It follows that $A(T)^* \cap U = A(T)^* \cap T$ (since $-1 \notin U$ and $T^* \subseteq U$.) Clearly $A(T)^* \cap T$ is additively closed so U is additively closed.

(ii) Clearly, $A(T) \subseteq A(T^{(m)})$ since $T \subseteq T^{(m)}$. If P is a maximal proper preorder containing T then $P = P^{(m)} \supseteq T^{(m)}$ so $A(P) \supseteq A(T^{(m)})$ and therefore, by (1.4.8), $A(T) \supseteq A(T^{(m)})$.

(iii) Every $x \in T^{(m)*}$ has odd order modulo T^* so the 2-primary parts of $K^*/T^{(m)*}$ and K^*/T^* are isomorphic.

(iv) For any $x \in K$, $(x^{2^s})^v = x^n \in T$ so $x^{2^s} \in T^{(v)}$. \square

1.5.3 Corollary. For orders P, Q in K , $P \subseteq Q$ iff $Q = P^{(m)}$ for some odd $m \geq 1$.

1.5.4 Corollary. For any proper preorder $T \subseteq K$, $\text{Sper}_T K \neq \emptyset$. In particular, T is contained in a 2-primary order.

Proof. By Zorn's lemma, T is contained in a maximal proper preorder P . By (1.4.2) and (1.5.2), P is complete and 2-primary and hence, an order. \square

1.5.5 Theorem ([6, Satz 2.17]). Every proper preorder in K is the intersection of the orders lying over it.

Proof. By (1.4.4), we need only consider complete preorders.

Let P be a complete preorder and let v be the valuation associated with $A(P)$. Set $G := v(K^*)$, $G_0 := v(P^*)$. Suppose $s(P) = 2^r m$, where $r \geq 0$, m odd. Then the 2-primary part of K^*/P^* is cyclic of order 2^{r+1} and G/G_0 decomposes as the direct sum

$$\frac{G}{G_0} = H_{2^r} \oplus H_m,$$

where H_{2^r} is cyclic of order 2^r and H_m is the subgroup consisting of all elements of odd order.

Let U be a subgroup of H_m such that H_m/U is cyclic and let $G'_0 \supseteq G_0$ be a subgroup of G such that $G'_0/G_0 = U$. Then $G/G'_0 = H_{2^r} \oplus H_m/U$ is cyclic. Define $\chi : G'_0 \rightarrow k^*/P_v^*$ by $\chi(v(x)) = \chi_P(v(x^m))$. Since P_v has level 1 and m is odd, $\chi(v(x)) = \chi_P(v(x))^m = \chi_P(v(x))$ for all $x \in P^*$. Therefore, $\chi(nG) = \chi_P(nG) = 1$ and $\chi(G'_0 \cap 2^r G) \supseteq \chi_P(G_0 \cap 2^r G)$. By (1.4.10), there exists a complete preorder Q in K with $Q_v = P_v$ and $v(Q^*) = G'_0$. Suppose $x = a\varepsilon y^n \in P^*$, where $a \in \mathfrak{A}$, $\varepsilon \in A_v^*$, $y \in K^*$. Then $v(a) \in G'_0$, $\chi(v(a)) = \chi(v(x)) = \chi_P(v(x)) = \bar{\varepsilon}P_v^*$ so $\varepsilon \in M_a$ and therefore, $x \in Q^*$. Thus, $P \subseteq Q$. Since the sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \frac{K^*}{Q^*} \rightarrow \frac{G}{G'_0} \rightarrow 0$$

is exact and Q is complete, the 2-primary part of K^*/Q^* is cyclic of order 2^{r+1} and its subgroup of elements of odd order is isomorphic to H_m/U which is also cyclic. Then K^*/Q^* is cyclic so Q is an order containing P such that $v(Q^*)/v(P^*) = U$.

Suppose the theorem is false. Then there exists $x \in K^*$ such that $x \in Q^*$ for all orders $Q \supseteq P$ and $x \notin P$. If $v(x) \in v(P^*)$ then $x \in P^*A_v^* \subseteq P^* \cup -P^*$ so $x^2 \in P$. But then $-x \in P^* \subseteq Q^*$ for any order $Q \supseteq P$, a contradiction. Thus, $v(x) \notin v(P^*)$. Therefore,

$$\bigcap \{U \mid U \text{ is a subgroup of } H_m \text{ with } H_m/U \text{ cyclic}\} \neq 0.$$

Suppose $H_m = \bigoplus_{i \in I} C_i$, where each C_i is a cyclic subgroup of H_m . If $U_i := \bigoplus_{j \neq i} C_j$ then $H_m/U_i \cong C_i$, for each i and therefore, $\bigcap_{i \in I} U_i = 0$, a contradiction. Since $mH_m = 0$, it suffices to prove the following.

Claim. Every abelian group of finite exponent is a direct sum of cyclic subgroups.

Suppose H is an abelian group and $mH = 0$. Write $m = p_1^{r_1} \dots p_k^{r_k}$, where p_1, \dots, p_k are distinct primes, $r_i > 0$. Then H is the direct sum of the subgroups $H_{p_i} := \{x \in H \mid p_i^{r_i} x = 0\}$ so we may as well assume $m = p^{r+1}$, for some prime p , $r \geq 0$. Let $P := \{x \in H \mid px = 0\}$. For each $i = 0, \dots, r$, $P \cap p^i H / P \cap p^{i+1} H$ is a $\mathbb{Z}/p\mathbb{Z}$ -vector space so we can find $x_{i,j} \in H$ such that $\{p^i x_{i,j}\}$ is a $\mathbb{Z}/p\mathbb{Z}$ -basis

of $P \cap p^i H$ modulo $P \cap p^{i+1} H$. Then H is the direct sum of the cyclic subgroups generated by the $x_{i,j}$. \square

1.6 Orders of higher level in commutative rings

As in [38], we define an *order* in a commutative ring A to be a proper preorder $P \subseteq A$ (of exponent n) such that there exists a prime $\mathfrak{p} \subseteq A$ and an order $\overline{P} \subseteq F(\mathfrak{p})$ with $P = \alpha_{\mathfrak{p}}^{-1}(\overline{P})$. We have $\mathfrak{p} := P \cap -P$, called the *support* of P and denoted by $\text{supp } P$, $\overline{P} := P(\mathfrak{p})$ and $s(P) = s(P(\mathfrak{p})) = \frac{1}{2}[F(\mathfrak{p})^* : P(\mathfrak{p})^*]$.

The set of all orders of exponent n in A is called the (*higher level*) *real spectrum* of A and is denoted $\text{Sper } A$. If T is a preorder in A , $\text{Sper}_T A$ denotes the set of all orders in $\text{Sper } A$ containing T .

1.6.1 Theorem. $\text{Sper}_T A \neq \emptyset$ for any proper preorder $T \subseteq A$. In particular, any maximal proper preorder is a 2-primary order.

Proof. Let T be a maximal proper preorder. By (1.1.7), there exists a prime $\mathfrak{p} \subseteq A$ such that $T(\mathfrak{p})$ is proper. By (1.5.4), there exists a 2-primary order $\overline{P} \subseteq F(\mathfrak{p})$ containing $T(\mathfrak{p})$. Since T is maximal, $T = \alpha_{\mathfrak{p}}^{-1}(\overline{P})$. \square

1.6.2 Remarks. Let $\varphi : A \rightarrow B$ be a ring homomorphism, $T \subseteq B$ a preorder. Then $\varphi^{-1}(T)$ is a preorder in A and $P \mapsto \varphi^{-1}(P)$ defines a map $\varphi^* : \text{Sper}_T B \rightarrow \text{Sper}_{\varphi^{-1}(T)} A$. For any preorder $T \subseteq A$, $\text{Sper}_{\Sigma B^n \varphi(T)} B = \varphi^{*-1}(\text{Sper}_T A)$.

If $B = S^{-1}A$ is a localization of A at some multiplicative set S then $\varphi^* : \text{Sper}_{S^{-1}T} S^{-1}A \rightarrow \text{Sper}_T A$ is 1-1 with image $\{P \in \text{Sper}_T A \mid S \cap \text{supp } P = \emptyset\}$. If $B = A/\mathfrak{a}$ where $\mathfrak{a} \subseteq A$ is an ideal, φ^* identifies $\text{Sper}_{\overline{T}} A/\mathfrak{a}$ with $\{P \in \text{Sper}_T A \mid \mathfrak{a} \subseteq \text{supp } P\}$.

For any prime ideal $\mathfrak{p} \subseteq A$, we identify $\text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$ with the set $\{P \in \text{Sper}_T A \mid \mathfrak{p} = \text{supp } P\}$. Then

$$\text{Sper}_T A = \bigcup_{\mathfrak{p}} \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$$

where \mathfrak{p} runs through all primes of A .

For any preorder T and for any odd integer $m \geq 1$, define

$$T^{(m)} := \{x \in A \mid x^m \in T\}.$$

1.6.3 Proposition. *Let P be an order in A and $m \geq 1$ is an odd integer.*

- (i) $P^{(m)}$ is an order in A with $\text{supp } P^{(m)} = \text{supp } P$ and $s(P^{(m)}) = \frac{s(P)}{(m, s(P))}$, where (a, b) denotes the greatest common divisor of a, b .
- (ii) If $Q \supseteq P$ is an order in A with $\text{supp } Q = \text{supp } P$ then $Q = P^{(m)}$, where $m = \frac{s(P)}{s(Q)}$.
- (iii) If $n = 2^s v$, v odd, then $P^{(v)}$ is a 2-primary order containing $P^{(m)}$ for all odd $m \geq 1$. If $Q \supseteq P$ is any 2-primary order then $Q \supseteq P^{(v)}$. We call $P^{(v)}$ the 2-primary part of P and denote it by $P_{(2)}$.

Proof. Let $\mathfrak{p} = \text{supp } P$.

- (i) By (1.5.2), $P(\mathfrak{p})^{(m)}$ is an order in $F(\mathfrak{p})$ so $P^{(m)} = \alpha_{\mathfrak{p}}^{-1}(P(\mathfrak{p})^{(m)})$ is an order in A with support \mathfrak{p} . Clearly, $2s(P^{(m)}) \mid \frac{2s(P)}{(m, s(P))}$ and $[P(\mathfrak{p})^{(m)*} : P(\mathfrak{p})^*] \mid (m, s(P))$ so $s(P) = s(P^{(m)})(m, s(P))$.
- (ii) Since $[Q(\mathfrak{p})^* : P(\mathfrak{p})^*] = m$, $Q(\mathfrak{p}) \subseteq P(\mathfrak{p})^{(m)}$ and from (i), we have $s(Q(\mathfrak{p})) = s(P(\mathfrak{p})^{(m)})$. Thus, $Q(\mathfrak{p}) = P(\mathfrak{p})^{(m)}$ and therefore, $Q = P^{(m)}$.
- (iii) Write $s(P) = 2^r u$ where $r < s$ and $u \mid v$. Then $P^{(v)}$ has level $\frac{s(P)}{(v, s(P))} = \frac{s(P)}{u} = 2^r$. If $Q \supseteq P$ is 2-primary then $(v, s(Q)) = 1$ so $Q = Q^{(v)} \supseteq P^{(v)}$. \square

Let P, Q be orders in A . We say Q specializes P and write $P \prec Q$ if $Q = P \cup \text{supp } Q$.

1.6.4 Theorem. *If $P \subseteq Q$ are 2-primary orders in A then $P \prec Q$.*

Proof. Suppose $a \in Q \setminus P$. Replacing a by a suitable power of a we may assume $a^2 \in P$. Let $\mathfrak{p} = \text{supp } P$. Then $\bar{a}^2 \in P(\mathfrak{p})$, $\bar{a} \notin P(\mathfrak{p})$. Since $P(\mathfrak{p})$ is complete, $-\bar{a} \in P(\mathfrak{p})$ so $-a \in P \subseteq Q$ and hence, $a \in \text{supp } Q$. \square

1.6.5 Theorem. *Let P be an order in A , $\mathfrak{p} = \text{supp } P$, \mathfrak{q} a P -compatible prime. Then there exists a place $\pi : F(\mathfrak{p}) \rightarrow k \cup \{\infty\}$ extending the natural map $A/\mathfrak{p} \rightarrow F(\mathfrak{q})$. The valuation ring $\pi^{-1}(k)$ is compatible with $P(\mathfrak{p})$ and if $P(\mathfrak{p})_v$ denotes the push-down of $P(\mathfrak{p})$ to k then*

$$(*) \quad Q := \alpha_{\mathfrak{q}}^{-1}(P(\mathfrak{p})_v \cap F(\mathfrak{q}))$$

is an order in A with support \mathfrak{q} which specializes P .

To prove (1.6.5), we require the following proposition.

1.6.6 Proposition. *Suppose K is a field, $P \subseteq K$ is an order and B is a subring of K . Define*

$$A(B, P) := \{x \in K \mid b \pm x \in P \text{ for some } b \in B \cap P\},$$

$$I(B, P) := \{x \in K \mid \frac{1}{1+b} \pm x \in P \text{ for all } b \in B \cap P\}.$$

Then

- (i) $A(B, P) = B \cdot A(P)$,
- (ii) $A(B, P)$ is a valuation ring compatible with P with $I(B, P)$ its maximal ideal,
- (iii) if B is a local ring whose maximal ideal \mathfrak{m} is P -convex (that is, $s, t \in B \cap P$ and $s + t \in \mathfrak{m} \Rightarrow s, t \in \mathfrak{m}$) then $B \cap I(B, P) = \mathfrak{m}$.

Proof. For convenience, set $\hat{B} = A(B, P)$, $\hat{\mathfrak{m}} = I(B, P)$. Using (1.3.7), it is easy to see that \hat{B} is a ring and since $A(P) = A(\mathbb{Z}, P) \subseteq \hat{B}$, \hat{B} is a valuation ring of K which is compatible with P by (1.3.8). If $b \in B$ then clearly $b^n \in \hat{B}$ and therefore, $b \in \hat{B}$. Thus, $B \cdot A(P) \subseteq \hat{B}$. Conversely, if $x \in \hat{B}$ then there exists $b \in B \cap P^*$ such that $b \pm x \in P$ and therefore, $x/b \in A(P)$ so $x \in B \cdot A(P)$. This proves (i).

Suppose $x \in \hat{\mathfrak{m}}$ and $y \in \hat{B}$. Let $c \in B \cap P$ such that $(1 + c) \pm y \in P$. For any $b \in B \cap P$, $b' := (1 + c)(1 + b) - 1 \in B \cap P$ so $\frac{1}{1+b'} \pm xy = \frac{1+c}{1+b'} \pm xy \in P$ by (1.3.7) and therefore, $xy \in \hat{\mathfrak{m}}$. Since $-1 \notin P$, $1 \notin \hat{\mathfrak{m}}$. Since $\hat{\mathfrak{m}}$ is clearly closed under addition, $\hat{\mathfrak{m}}$ is a proper ideal of \hat{B} . If $x \in K^*$ and $1/x \notin \hat{B}$ then, for any $b \in B \cap P$,

$\frac{1}{(1+b)x} \notin A(P)$ so $1 \pm (1+b)x \in P$. Since $1+b \in P^*$, $\frac{1}{1+b} \pm x \in P$ and therefore, $x \in \hat{m}$. Thus, \hat{m} is the maximal ideal of \hat{B} , which proves (ii).

Now suppose B is a local ring whose maximal ideal \mathfrak{m} is P -convex. Suppose $x \in \mathfrak{m}$, $x \neq 0$. Then, for any $b \in B \cap P$, $(bx^n - 1) + 1 = bx^n \in \mathfrak{m}$ and $1 \notin \mathfrak{m}$ so $bx^n - 1 \notin P$. Thus, $b - \frac{1}{x^n} \notin P$ for all $b \in B \cap P$ so $\frac{1}{x^n} \notin \hat{B}$. Therefore, $\mathfrak{m} \subseteq \hat{m} \cap B$. Since \mathfrak{m} is maximal, this proves (iii). \square

Proof of (1.6.5). Let B be the localization of A/\mathfrak{p} at the prime $\mathfrak{q}/\mathfrak{p}$ and let \mathfrak{m} be its maximal ideal. We show \mathfrak{m} is $P(\mathfrak{p})$ -convex. Since \mathfrak{q} is P -compatible, there exists an order $Q \supseteq P$ such that $\mathfrak{q} = \text{supp } Q$. Suppose $x, y \in B \cap P(\mathfrak{p})$, $x + y \in \mathfrak{m}$. Then there exists $s, s', t \in P$, $s \in \mathfrak{q}$, $t \notin \mathfrak{q}$ such that $x = \bar{s}'/\bar{t}$ and $x + y = \bar{s}/\bar{t}$. Since $y \in P(\mathfrak{p})$, $s - s' \in P \subseteq Q$. Since $s \in \mathfrak{q} = \text{supp } Q$, $\pm s' \in Q$ so $s' \in \mathfrak{q}$. Therefore, $x \in \mathfrak{m}$.

The existence of the place π follows now from (1.6.6). Define Q as in (*). Clearly Q is an order in A with support \mathfrak{q} . Since $Q = (\pi \circ \alpha_{\mathfrak{p}})^{-1}(P(\mathfrak{p})_v)$, it is also clear that $P \subseteq Q$. Suppose $a \in Q \setminus \text{supp } Q$. Then $\pi \circ \alpha_{\mathfrak{p}}(a) \in P(\mathfrak{p})_v^*$ so $\alpha_{\mathfrak{p}}(a) \in P(\mathfrak{p})$ and therefore, $a \in P$. \square

1.6.7 Corollary. *If $P \subseteq Q$ are orders in A then there exists an odd $m \geq 1$ such that $P^{(m)} \prec Q$.*

Proof. Since $\mathfrak{q} = \text{supp } Q$ is P -compatible, $P \cup \mathfrak{q}$ is an order in A . By (1.6.3(ii)), $Q = (P \cup \mathfrak{q})^{(m)} = P^{(m)} \cup \mathfrak{q}$, where $m = \frac{s(P \cup \mathfrak{q})}{s(Q)} = \frac{s(P(\mathfrak{q}))}{s(Q(\mathfrak{q}))}$. \square

1.6.8 Theorem. *The orders specializing a given order form a chain.*

Proof. Let Q, Q' be orders in A specializing P . Then $P_{(2)} \subseteq Q_{(2)}$, $P_{(2)} \subseteq Q'_{(2)}$. Suppose there exists $x \in Q_{(2)} \setminus Q'_{(2)}$, $y \in Q'_{(2)} \setminus Q_{(2)}$. Replacing x and y by suitable powers, we may assume $-x \in Q'_{(2)}$ and $-y \in Q_{(2)}$. Then $y - x \in Q'_{(2)}$. If $y - x \in P_{(2)}$ then $y = y - x + x \in Q_{(2)}$, a contradiction. By (1.6.4), $P_{(2)} \prec Q'_{(2)}$ so $y - x \in \text{supp } Q'_{(2)}$ and therefore, $x = y - (y - x) \in Q'_{(2)}$, another contradiction. Thus, we may assume $Q_{(2)} \subseteq Q'_{(2)}$. Then $\text{supp } Q \subseteq \text{supp } Q'$ so $Q \subseteq Q'$. \square

We say an order P in A is *maximal* if $P \prec Q$ implies $P = Q$. Of course, 2-primary orders are maximal iff they are maximal with respect to inclusion.

1.6.9 Theorem. *Suppose P is an order in A , $\mathfrak{p} = \text{supp } P$. Then the following are equivalent:*

- (i) $P_{(2)}$ is maximal.
- (ii) P is maximal.
- (iii) \mathfrak{p} is the only P -compatible prime.
- (iv) $F(\mathfrak{p}) = A/\mathfrak{p} \cdot A(P(\mathfrak{p}))$.

Proof. (i) \Rightarrow (ii) Suppose $P \prec Q$. Then $P_{(2)} \subseteq Q_{(2)}$. Since $P_{(2)}$ is maximal, $P_{(2)} = Q_{(2)}$ and therefore, $\text{supp } P = \text{supp } Q$ so $P = Q$.

(ii) \Rightarrow (i) Suppose $P_{(2)} \prec Q$. Then $\text{supp } Q$ is P -compatible. By (1.6.5), $P \cup \text{supp } Q$ is an order so, by the maximality of P , $\text{supp } Q \subseteq P \subseteq P_{(2)}$ and therefore, $P_{(2)} = Q$.

(ii) \Leftrightarrow (iii) follows from (1.6.5).

(iii) \Rightarrow (iv) Let $\pi : F(\mathfrak{p}) \rightarrow k \cup \{\infty\}$ be the place associated with the valuation ring $A/\mathfrak{p} \cdot A(P(\mathfrak{p}))$. By (1.3.8), π is compatible with $P(\mathfrak{p})$ so the kernel of $\pi \circ \alpha_{\mathfrak{p}}$ is a P -compatible prime. Since \mathfrak{p} is the only P -compatible prime, π is trivial.

(iv) \Rightarrow (iii) Suppose \mathfrak{q} is a P -compatible prime. By (1.6.5), there exists a place $\pi : F(\mathfrak{p}) \rightarrow k \cup \{\infty\}$ compatible with $P(\mathfrak{p})$ which extends $A/\mathfrak{p} \rightarrow F(\mathfrak{q})$. Then both A/\mathfrak{p} and $A(P(\mathfrak{p}))$ are contained in the valuation ring of π so π is trivial. It follows that $\mathfrak{q} = \mathfrak{p}$. \square

1.6.10 Theorem. *For each order $P \subseteq A$, there exists a unique maximal order specializing P .*

Proof. Let Q' be a maximal proper preorder containing $P_{(2)}$. By (1.6.1), Q' is a 2-primary order. Let $Q = P \cup \text{supp } Q'$. Then $Q_{(2)} = P_{(2)} \cup \text{supp } Q' = Q'$. By (1.6.9), Q is maximal. The uniqueness follows from (1.6.8). \square

Denote the set of all maximal orders in A by $\text{Spermax } A$ and set $\text{Spermax}_T A = \text{Sper}_T A \cap \text{Spermax } A$ for any preorder $T \subseteq A$. By (1.6.10), we have a canonical specialization map $\mu : \text{Sper } A \rightarrow \text{Spermax } A$. Note that for a field K , $\text{Sper } K = \text{Spermax } K$.

1.6.11 Example. $\mathbb{R}[X]$ has precisely the following orders of higher level:

(i) *The maximal orders*

$$P_a := \{f \in \mathbb{R}[X] \mid f(a) \geq 0\}$$

of level 1 with support $(X - a)$ for each $a \in \mathbb{R}$, the maximal orders

$$P_{\infty-,s} \cap \mathbb{R}[X]$$

of level s with support 0 for each integer $s \geq 1$ and the maximal orders

$$P_{\infty+,m} \cap \mathbb{R}[X]$$

of odd level m with support 0 for each odd integer $m \geq 1$.

(ii) *For each $a \in \mathbb{R}$, the following orders having support 0 and P_a as their unique maximal specialization. For each integer $s \geq 1$, the orders*

$$P_{a-,s} \cap \mathbb{R}[X]$$

of level s and for each odd integer $m \geq 1$, the orders

$$P_{a+,m} \cap \mathbb{R}[X]$$

of odd level m .

Proof. All the orders on $\mathbb{R}(X)$ intersect down to support 0 orders on $\mathbb{R}[X]$. The only other primes of $\mathbb{R}[X]$ compatible with $\Sigma \mathbb{R}[X]^n$ are the maximal ideals $(X - a)$ for each $a \in \mathbb{R}$. The residue field of $\mathbb{R}[X]$ at $(X - a)$ is \mathbb{R} which has only the level 1 order \mathbb{R}^2 and $\alpha_{\mathbb{P}}^{-1}(\mathbb{R}^2) = P_a$. These orders are clearly maximal since their supports are maximal ideals. Let $a \in \mathbb{R}$. For any $f \in \mathbb{R}[X]$, $f \notin P_a$ iff $f(a) < 0$ iff $-f \in T_a$ and $f \in P_a \setminus \text{supp } P_a$ iff $f(a) > 0$ iff $f \in T_a$. Therefore, P_a specializes the orders $P_{a\pm,s}$. If $b > a^n$ then $f(X) = X^n - b \notin P_a$. Since $X^n - b = X^n(1 - \frac{b}{X^n}) \in P_{\infty\pm,s}$, the orders $P_{\infty\pm,s} \cap \mathbb{R}[X]$ are maximal. \square

1.7 Null- and Positivstellensatz

1.7.1 Theorem (Nullstellensatz). *Let $T \subseteq A$ be a preorder, $a \in A$. Then $a \in \text{supp } P$ for all $P \in \text{Sper}_T A$ iff $-a^{nk} \in T$ for some integer $k \geq 0$.*

Proof. Consider the localization $A[1/a]$ and the preorder $T[1/a^n] \subseteq A[1/a]$ extending T . If $a \in \text{supp } P$ for all $P \in \text{Sper}_T A$ then $\text{Sper}_{T[1/a^n]} A[1/a] = \emptyset$. By (1.6.1), $T[1/a^n]$ is not proper so there exists $t \in T$, $k \geq 0$ such that $-a^{nk} = t$. The converse is clear. \square

In [14], Berr generalized the Positivstellensatz of Stengle to preorders of higher level. The following version is slightly stronger than the one appearing in [14].

1.7.2 Theorem (Positivstellensatz). *Suppose $T \subseteq A$ is a preorder, $a \in A$.*

- (i) *$a \in P \setminus \text{supp } P$ for all $P \in \text{Sper}_T A$ iff $a(1+s) = 1+t$ for some $s, t \in T$.*
- (ii) *$a \in P$ for all $P \in \text{Sper}_T A$ iff $(a^{nk} + s)a = a^{nk} + t$ for some $s, t \in T$, $k \geq 0$.*

Proof. (i) Suppose $a \in P \setminus \text{supp } P$ for all $P \in \text{Sper}_T A$. Consider the ΣA^n -module $M := T - a\Sigma A^n$. Suppose $-1 \notin M$. By (1.1.7), there exists an M -compatible prime $\mathfrak{p} \subseteq A$. Since $T \subseteq M$, \mathfrak{p} is also T -compatible so $\text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p}) \neq \emptyset$. Then $\bar{a} \in Q$ for all $Q \in \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$ so, by (1.5.5), $\bar{a} \in T(\mathfrak{p}) \subseteq M(\mathfrak{p})$. But $-a \in M$ so $\bar{a} \in M(\mathfrak{p}) \cap -M(\mathfrak{p})$ and hence, $a \in \mathfrak{p}$, a contradiction. Therefore, $-1 \in M$ so there exists $s' \in \Sigma A^n$ such that $as' \in 1 + T$. Clearly $s' \notin \text{supp } P$ for all $P \in \text{Sper}_T A$ so the preorder $T - s'T$ must contain -1 . Then $-(1 - s') \in T - s'T$ so there exists $s, t \in T$ such that $-(1 - s') = s - s't$ and therefore, $a(1 + s) = as'(1 + t) \in 1 + T$.

(ii) If $a \in P$ for all $P \in \text{Sper}_T A$ then $a \in Q \setminus \text{supp } Q$ for all $Q \in \text{Sper}_{T[1/a^n]} A[1/a]$. By (i), there exists $s', t' \in T[1/a^n]$ such that $a(1+s') = 1+t'$. Clearing denominators, we get $s, t \in T$, $k \geq 0$, such that $a(a^{nk} + s) = a^{nk} + t$. \square

Chapter 2

Signatures of Higher Level

Given an order P of level 1 in a field K , one defines a character $\text{sgn}_P : K^* \rightarrow \{\pm 1\}$ by $\text{sgn}_P(x) = 1$ iff $x \in P^*$. sgn_P is called a signature of K . Generalizing this to higher level, a signature was defined in [10] to be a character $\sigma : K^* \rightarrow \Omega$ such that $\ker \sigma \cup \{0\}$ is an order of exponent n , where Ω denotes the group of n -th roots of unity. If P is any order of exponent n in K then K^*/P^* is cyclic of order dividing n so there exists signatures of K with kernel P^* .

In [38], signatures were defined for a certain class of commutative rings called rings with many units. A polynomial $f \in A[X_1, \dots, X_r]$ is said to have *unit values* if there exists $x_1, \dots, x_r \in A$ such that $f(x_1, \dots, x_r) \in A^*$. f is said to have *local unit values* if for every maximal ideal $\mathfrak{m} \subseteq A$, there exists $x_1, \dots, x_r \in A$ such that $f(x_1, \dots, x_r) \notin \mathfrak{m}$. The ring A is called a *ring with many units* if every polynomial with local unit values also has unit values. Examples include semi-local rings and von Neumann regular rings, see [40], [48]. If A is a ring with many units, a signature of A is a character $\sigma : A^* \rightarrow \Omega$ such that $\ker \sigma = P^*$ for some order $P \in \text{Sper } A$. The problem with this definition for arbitrary commutative rings is that the unit group may be much too small. Consider, for example, the ring $\mathbb{R}[X]$. The unit group of $\mathbb{R}[X]$ is \mathbb{R}^* and \mathbb{R} has exactly one order $P := \mathbb{R}^2$ so $Q^* = P^*$ for every order $Q \in \text{Sper } \mathbb{R}[X]$. We need a signature to be defined on the whole ring, not just for units.

Returning to our ring with many units, let σ be a signature of A . Pick $P \in$

$\text{Sper } A$ such that $\ker \sigma = P^*$ and let $\mathfrak{p} := \text{supp } P$. Then the signature σ lifts to a character $\bar{\sigma} : F(\mathfrak{p})^* \rightarrow \Omega$ with $\ker \bar{\sigma} = P(\mathfrak{p})^*$ so the signatures of A are precisely those characters $\sigma : A^* \rightarrow \Omega$ for which there exists a prime $\mathfrak{p} \subseteq A$ and a signature $\bar{\sigma}$ of the residue field $F(\mathfrak{p})$ such that σ is the composite map $\bar{\sigma} \circ \alpha_{\mathfrak{p}} : A^* \rightarrow \Omega$. We extend this definition to an arbitrary commutative ring (cf. the higher level orders defined in [3].)

2.1 Higher level signatures of a commutative ring

Let A be a commutative ring and $T \subseteq A$ a proper preorder. Denote by Ω the group of n -th roots of unity in \mathbb{C} and set $\Omega_0 := \Omega \cup \{0\}$. A T -signature of A is a map $\sigma : A \rightarrow \Omega_0$ such that $\sigma(T) = \{0, 1\}$, $\mathfrak{p} := \sigma^{-1}(0)$ is a T -compatible prime ideal and there exists a character $\bar{\sigma} : F(\mathfrak{p})^* \rightarrow \Omega$ with $\ker \bar{\sigma}$ additively closed and $\sigma = \bar{\sigma} \circ \alpha_{\mathfrak{p}}$ on $A \setminus \mathfrak{p}$. Note that if we extend $\bar{\sigma}$ to $F(\mathfrak{p})$ by $\bar{\sigma}(0) = 0$ then $\bar{\sigma}$ is a $T(\mathfrak{p})$ -signature on $F(\mathfrak{p})$. We denote by $\text{Sig}_T A$ the set of all T -signatures of A .

For any T -compatible prime \mathfrak{p} , we have the injection $\alpha_{\mathfrak{p}}^* : \text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p}) \rightarrow \text{Sig}_T A$ given by $\bar{\sigma} \mapsto \bar{\sigma} \circ \alpha_{\mathfrak{p}}$ and

$$(2.1.1) \quad \text{Sig}_T A = \bigcup_{\mathfrak{p}} \alpha_{\mathfrak{p}}^*(\text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p}))$$

where \mathfrak{p} runs through all T -compatible primes.

2.1.2 Theorem. *For every $\sigma \in \text{Sig}_T A$, there exists a unique order $P_{\sigma} \in \text{Sper}_T A$ with*

$$P_{\sigma} = \sigma^{-1}(\{0, 1\}).$$

The map $\text{Sig}_T A \rightarrow \text{Sper}_T A$ defined by $\sigma \mapsto P_{\sigma}$ is surjective. In particular, $\text{Sig}_T A \neq \emptyset$ for any proper preorder T .

Proof. Let $\sigma \in \text{Sig}_T A$, say $\sigma = \bar{\sigma} \circ \alpha_{\mathfrak{p}}$ where \mathfrak{p} is a T -compatible prime and $\bar{\sigma} \in \text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p})$. Since the kernel of the character $\bar{\sigma}$ is additively closed, $P_{\bar{\sigma}} := \bar{\sigma}^{-1}(1) \cup \{0\}$ is an order in $F(\mathfrak{p})$ with $T(\mathfrak{p}) \subseteq P_{\bar{\sigma}}$. Then $P_{\sigma} := \sigma^{-1}(\{0, 1\}) = \alpha_{\mathfrak{p}}^{-1}(P_{\bar{\sigma}})$ and hence, is an order in A containing T .

Conversely, for any order P in A with support \mathfrak{p} , $F(\mathfrak{p})^*/P(\mathfrak{p})^*$ is cyclic so there exists a character on $F(\mathfrak{p})^*$ with kernel $P(\mathfrak{p})^*$ and therefore, a signature $\sigma \in \text{Sig}_T A$ with $P_\sigma = P$. \square

2.1.3 Theorem. Suppose $\sigma \in \text{Sig}_T A$ and $m \geq 1$ is an odd integer. Then $\sigma^m \in \text{Sig}_T A$ and $P_{\sigma^m} = P_\sigma^{(m)}$.

Proof. Suppose $\bar{\sigma} \in \text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p})$ such that $\sigma = \bar{\sigma} \circ \alpha_{\mathfrak{p}}$. Clearly, $P_{\bar{\sigma}}^{(m)*} = \ker \bar{\sigma}^m$ so $\bar{\sigma}^m$ is a $T(\mathfrak{p})$ -signature and therefore, $\sigma^m = \bar{\sigma}^m \circ \alpha_{\mathfrak{p}}$ is a T -signature with $P_{\sigma^m} = P_\sigma^{(m)}$. \square

We define the *level* $s(\sigma)$ of a signature σ to be the level of the order P_σ .

2.1.4 Proposition. For $\sigma, \tau \in \text{Sig}_T A$, the following are equivalent:

- (i) $P_\sigma = P_\tau$,
- (ii) $\tau = \sigma^m$ for some odd integer m relatively prime to $s(\sigma)$.

Proof. Suppose $\bar{\sigma} \in \text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p})$, $\bar{\tau} \in \text{Sig}_{T(\mathfrak{q})} F(\mathfrak{q})$ such that $\sigma = \bar{\sigma} \circ \alpha_{\mathfrak{p}}$, $\tau = \bar{\tau} \circ \alpha_{\mathfrak{q}}$.

(i) \Rightarrow (ii) We have $\mathfrak{p} = \mathfrak{q}$ and $P_{\bar{\sigma}} = P_{\bar{\tau}}$ so $\ker \bar{\sigma} = \ker \bar{\tau}$. Therefore, there exists an odd integer m relatively prime to $s(\bar{\sigma}) = s(\sigma)$ such that $\bar{\tau} = \bar{\sigma}^m$ and therefore, $\tau = \sigma^m$.

(ii) \Rightarrow (i) $P_\tau = P_{\sigma^m} = P_\sigma^{(m)} = P_\sigma$ since m is relatively prime to $s(\sigma) = s(P_\sigma)$. \square

Let $\sigma, \tau \in \text{Sig}_T A$. We say τ *specializes* σ and write $\sigma \prec \tau$ if for all $a \in A$, $\tau(a) \neq 0 \Rightarrow \tau(a) = \sigma(a)$. If $\sigma \prec \tau$ then clearly $P_\sigma \prec P_\tau$.

2.1.5 Proposition. Suppose $\sigma \in \text{Sig}_T A$ and \mathfrak{q} is a P_σ -compatible prime. Define $\tau : A \rightarrow \Omega_0$ by $\tau(a) = \sigma(a)$ if $a \in A \setminus \mathfrak{q}$, $\tau(a) = 0$ otherwise. Then $\tau \in \text{Sig}_T A$ and $\sigma \prec \tau$.

Proof. Set $Q = P_\sigma \cup \mathfrak{q}$. Suppose $a, a' \in A \setminus \mathfrak{q}$ and $a - a' \in \mathfrak{q}$. Then $\bar{a}' \bar{a}^{n-1} = \bar{a}^n \in Q(\mathfrak{q})$ so $a' a^{n-1} \in Q \setminus \mathfrak{q} \subseteq P_\sigma$ and therefore, $\sigma(a) = \sigma(a')$. Define $\bar{\tau} : F(\mathfrak{q})^* \rightarrow \Omega$ by $\bar{\tau}(\frac{a}{b}) = \frac{\sigma(a)}{\sigma(b)}$. Clearly, $\bar{\tau} \in \text{Hom}(F(\mathfrak{q})^*, \Omega)$ and $\ker \bar{\tau} = Q(\mathfrak{q})^*$ so $\bar{\tau}$ is a $T(\mathfrak{q})$ -signature and $\tau = \bar{\tau} \circ \alpha_{\mathfrak{q}} \in \text{Sig}_T A$. \square

2.1.6 Theorem. *The signatures specializing a given signature form a chain.*

Proof. Suppose $\sigma \prec \tau_1$ and $\sigma \prec \tau_2$. Then $P_\sigma \prec P_{\tau_1}$ and $P_\sigma \prec P_{\tau_2}$. By (1.6.8), we may assume $P_{\tau_1} \prec P_{\tau_2}$ so $\tau_1^{-1}(0) \subseteq \tau_2^{-1}(0)$. If $\tau_2(a) \neq 0$ then $\tau_1(a) \neq 0$ and $\tau_2(a) = \sigma(a) = \tau_1(a)$. \square

A signature $\sigma \in \text{Sig}_T A$ is said to be *maximal* if $\sigma \prec \tau \Rightarrow \sigma = \tau$, for all $\tau \in \text{Sig}_T A$. Define

$$\text{Sigmax}_T A := \{\sigma \in \text{Sig}_T A \mid \sigma \text{ is maximal}\}.$$

2.1.7 Theorem. *$\sigma \in \text{Sigmax}_T A$ iff P_σ is maximal. If σ is maximal then so is σ^m for any odd integer $m > 0$.*

Proof. Let $\sigma \in \text{Sigmax}_T A$ and suppose $P_\sigma \prec Q$. Define τ as in (2.1.5), where $\mathfrak{q} = \text{supp } Q$. Then $\sigma \prec \tau$ so $\sigma = \tau$ and therefore, $\text{supp } P_\sigma = \text{supp } Q$ so $P_\sigma = Q$. Conversely, suppose P_σ is maximal and $\sigma \prec \tau$. We have $P_\sigma \prec P_\tau$ so $P_\sigma = P_\tau$ and therefore, $\sigma^{-1}(0) = \tau^{-1}(0)$. Since $\sigma \prec \tau$, for all $a \in A$ such that $\tau(a) \neq 0$, $\tau(a) = \sigma(a)$ and thus, $\sigma = \tau$.

By (1.6.9), P_σ is maximal iff $P_{\sigma(2)}$ is maximal and $P_{\sigma(2)} = P_\sigma^{(m)}(2) = P_{\sigma^m(2)}$ for any odd integer $m > 0$. \square

2.1.8 Theorem. *For all $\sigma \in \text{Sig}_T A$, there exists a unique maximal signature τ such that $\sigma \prec \tau$. Hence, we have a well-defined specialization map $\mu : \text{Sig}_T A \rightarrow \text{Sigmax}_T A$.*

Proof. Let Q be the unique maximal order specializing P_σ and let $\mathfrak{q} = \text{supp } Q$. Define τ as in (2.1.5). Then $\sigma \prec \tau$ and $P_\tau = P_\sigma \cup \mathfrak{q} = Q$. By (2.1.7), τ is maximal. The uniqueness follows from (2.1.6). \square

2.2 Topologies on $\text{Sig}_T A$

Give Ω_o^A the product topology (where Ω_o has the discrete topology.) The sets

$$U(a_1, \dots, a_r; \alpha_0) := \{\sigma \in \text{Sig}_T A \mid \sigma(a_i) = \alpha_0(a_i) \text{ for } i = 1, \dots, r\},$$

where $a_1, \dots, a_r \in A$, $\alpha_0 \in \Omega_0^A$, form a basis for the subspace topology on $\text{Sig}_T A$.

2.2.1 Theorem. $\text{Sig}_T A$ is closed in Ω_0^A .

Proof. Suppose $\tau \in \Omega_0^A$ is in the closure of $\text{Sig}_T A$. For any $a, b \in A$, $t \in T$, there exists $\sigma \in U(a, b, ab, a + b, t, -1; \tau) \cap \text{Sig}_T A$. Clearly, $\tau(ab) = \sigma(ab) = \sigma(a)\sigma(b) = \tau(a)\tau(b)$ and if $\tau(b) = 0$ then $\sigma(b) = 0$ and $\tau(a + b) = \sigma(a + b) = \sigma(a) = \tau(a)$. It follows that $\mathfrak{p} := \tau^{-1}(0)$ is a prime ideal, $T \subseteq \tau^{-1}(\{0, 1\})$ and $\bar{\tau} : F(\mathfrak{p})^* \rightarrow \Omega$ defined by $\bar{\tau}(\frac{a}{b}) = \frac{\tau(a)}{\tau(b)}$ is a well-defined character with $\bar{\tau}(-1) = -1$ and $\tau = \bar{\tau} \circ \alpha_{\mathfrak{p}}$. Since $T(\mathfrak{p})^* \subseteq \bar{\tau}^{-1}(1)$, $-1 \notin T(\mathfrak{p})^*$ so \mathfrak{p} is T -compatible. It remains only to show that $\ker \bar{\tau}$ is additively closed.

Let $a, b, x, y \in A \setminus \mathfrak{p}$ such that $\bar{\tau}(\frac{a}{b}) = \bar{\tau}(\frac{x}{y}) = 1$. Then $\tau(a) = \tau(b) \neq 0$ and $\tau(x) = \tau(y) \neq 0$. Pick $\sigma = \bar{\sigma} \circ \alpha_{\mathfrak{p}} \in U(a, b, x, y, ay + bx, by; \tau) \cap \text{Sig}_T A$. Clearly, $\bar{\sigma}(\frac{a}{b}) = 1$, $\bar{\sigma}(\frac{x}{y}) = 1$ so $\bar{\sigma}(\frac{a}{b} + \frac{x}{y}) = 1$. Then $\tau(ay + bx) = \sigma(ay + bx) = \sigma(by) = \tau(by)$ and therefore, $\bar{\tau}(\frac{a}{b} + \frac{x}{y}) = 1$. \square

We define the *Harrison topology* on $\text{Sig}_T A$ by taking as a sub-base the sets $U(a; \sigma)$, where $a \in A$ and $\sigma \in \text{Sig}_T A$ such that $\sigma(a) \neq 0$. This is coarser than the product topology on $\text{Sig}_T A$ so $\text{Sig}_T A$ is compact in the Harrison topology. (Of course, if K is a field then the Harrison topology coincides with the product topology.) Unless otherwise stated, the topology on $\text{Sig}_T A$ will be assumed to be the Harrison topology. For a subset $U \subseteq \text{Sig}_T A$, we denote the closure of U (in the Harrison topology) by \bar{U} .

2.2.2 Theorem.

- (i) For $\sigma, \tau \in \text{Sig}_T A$, $\tau \in \overline{\{\sigma\}}$ iff $\sigma \prec \tau$.
- (ii) The maximal signatures are precisely the closed points of $\text{Sig}_T A$.

Proof. (i) $\tau \in \overline{\{\sigma\}}$ iff $\sigma \in U(a; \tau)$ for all $a \in A$ such that $\tau(a) \neq 0$ iff for all $a \in A$, $\tau(a) \neq 0$ implies $\sigma(a) = \tau(a)$. (ii) follows from (i). \square

2.2.3 Proposition. For $\sigma, \tau \in \text{Sig}_T A$, the following are equivalent:

(i) $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$.

(ii) There exists disjoint open sets U, V in $\text{Sig}_T A$ such that $\sigma \in U, \tau \in V$.

Proof. (i) \Rightarrow (ii) Since $\sigma \not\prec \tau$, there exists $a \in A$ such that $\tau(a) \neq 0$ and $\tau(a) \neq \sigma(a)$. If $\sigma(a) \neq 0$ then take $U = U(a; \sigma)$ and $V = U(a; \tau)$. So assume $\sigma(b) = 0$ whenever $\tau(b) \neq 0$ and $\tau(b) \neq \sigma(b)$. Since $\tau \not\prec \sigma$, there exists $b \in A$ such that $\sigma(b) \neq 0$ and $\sigma(b) \neq \tau(b)$ and hence, $\tau(b) = 0$. Then $\sigma(a^n - b^n) = \sigma(-b^n) = -1$ and $\tau(a^n - b^n) = \tau(a^n) = 1$ so take $U = U(a^n - b^n; \sigma)$ and $V = U(a^n - b^n; \tau)$. (ii) \Rightarrow (i) follows from (2.2.2). \square

2.2.4 Corollary. If C, D are disjoint closed sets in $\text{Sig}_T A$ then there exists disjoint opens sets U, V in $\text{Sig}_T A$ such that $C \subseteq U$ and $D \subseteq V$.

2.2.5 Theorem. $\text{Sigmax}_T A$ is compact and Hausdorff. The specialization map μ is a closed mapping.

Proof. Suppose $\sigma_0 \in \text{Sig}_T A, \tau_0 = \mu(\sigma_0)$. Let U_0 be an open neighborhood of τ_0 and set $C = \text{Sig}_T A \setminus U_0$. Since τ_0 is maximal, $\{\tau_0\}$ is closed so by (2.2.4) we can find disjoint open sets U, V in $\text{Sig}_T A$ such that $\tau_0 \in U$ and $C \subseteq V$. By (2.2.2), $\sigma_0 \in U$ and $\mu(U) \subseteq U_0$. Therefore, μ is continuous. $\text{Sigmax}_T A$ is Hausdorff by (2.2.3). It follows that $\text{Sigmax}_T A$ is compact and μ is a closed mapping. \square

2.3 The group G_T

Let T be a proper preorder in A and consider the restriction map from $\text{Sig}_T A$ to $\text{Hom}(A^*, \Omega)$. If $\sigma, \tau \in \text{Sig}_T A$ and $\sigma \prec \tau$ then σ and τ agree on A^* . Thus, the restriction map factors through $\text{Sigmax}_T A$. If A is a ring with many units (in particular, if A is a field), the image of this map is the set of signatures defined in [10], [12] and [38]. We show for rings with many units, this restriction map is also 1-1 on $\text{Sigmax}_T A$ and hence, the signatures defined in [38] are precisely the maximal signatures defined here. More generally, we show for a certain class of preorders (including the preorders in rings with many units) $\text{Sigmax}_T A$ can be

embedded into a character group $\text{Hom}(G_T, \Omega)$, where G_T is an abelian group of exponent n depending on A and T . Inspired by [36], we define G_T as follows.

Let A be a commutative ring and T a proper preorder of A . Set

$$A_T := \{a \in A \mid \sigma(a) \neq 0 \text{ for all } \sigma \in \text{Sig}_T A\}$$

and

$$\tilde{T} := \{a \in A \mid a \in P_\sigma \text{ for all } \sigma \in \text{Sig}_T A\}.$$

\tilde{T} is a proper preorder in A containing T . (Note that since $T^e \subseteq \tilde{T}$, $\tilde{T} - \tilde{T} = A$ by (1.1.3).) A_T is a multiplicative semigroup and $1 + T \subseteq T \cap A_T \subseteq \tilde{T} \cap A_T$ are subsemigroups of A_T . Suppose $a \in \tilde{T} \cap A_T$. Then $a \in P \setminus \text{supp } P$ for all $P \in \text{Sper}_T A$ so, by the Positivstellensatz, there exists $s, t \in T$ such that $a(1 + s) = 1 + t$. It follows that

$$G_T := \frac{A_T}{1 + T} = \frac{A_T}{T \cap A_T} = \frac{A_T}{\tilde{T} \cap A_T}.$$

Since $a^n \in T \cap A_T$ for all $a \in A_T$, G_T is a group of exponent n . If $a \in A_T$, we let $[a]$ denote the class of a in G_T .

For any $\sigma \in \text{Sig}_T A$, $\sigma(\tilde{T} \cap A_T) = 1$ so we get a natural map

$$\text{Sig}_T A \rightarrow \text{Hom}(G_T, \Omega)$$

which factors through $\text{Sigmax}_T A$.

2.3.1 Remarks.

- (i) We can always replace A by its localization at the multiplicative set $1 + T$. This leaves $\text{Sig}_T A$ unchanged but A_T gets identified with A^* . If $1 + T \subseteq A^*$ then $\tilde{T}^* = T^*$ by the Positivstellensatz so $G_T = A^*/T^*$ and $A = T^* - T^*$.
- (ii) For a level 1 preorder T , $\text{Spermax}_T A$ is identified with $\text{Sigmax}_T A$ via $P \mapsto \text{sgn}_P$. In [36], it is shown the restriction map $\text{Sigmax}_T A \rightarrow \text{Hom}(G_T, \{\pm 1\})$ is an embedding if the natural maps $G_T \rightarrow G_{T'}$ are surjective for every proper preorder $T' \supseteq T$.

2.3.2 Proposition. *If A is a ring with many units then for any $a \in A$, there exists $b \in A^*$ such that for all $\sigma \in \text{Sig}_T A$,*

$$a^n - 1 \in P_\sigma \Rightarrow \sigma(a) = \sigma(b).$$

Proof. Let $a \in A$ and consider the polynomial

$$(*) \quad f(X, Y) = (a - 1)X^n + aY^n.$$

For any maximal ideal $\mathfrak{m} \subseteq A$, there exists $x, y \in A$ such that $f(x, y) \notin \mathfrak{m}$. (In fact, we can always take $x, y \in \{0, 1\}$.) Since A has many units, there exists $x, y \in A$ such that $b := f(x, y) \in A^*$.

Let $\sigma \in \text{Sig}_T A$ and suppose $a^n - 1 \in P_\sigma$. If $a \in P_\sigma$ then factoring $a^n - 1$, we see $a - 1 \in P_\sigma$ and therefore, $a, b \in P_\sigma^*$ so $\sigma(a) = 1 = \sigma(b)$. If $-a \in P_\sigma$ then $-b = -a(x^n + y^n) + x^n \in P_\sigma^*$ so $\sigma(a) = -1 = \sigma(b)$. Assume $a^2 \notin P_\sigma$. Let $\mathfrak{p} = \text{supp } P$. The pushdown of $P_\sigma(\mathfrak{p})$ is a level 1 order so $a \notin A(P_\sigma(\mathfrak{p}))^*$. Since $a^n \pm 1 \in P_\sigma$, $1 \pm \frac{1}{a^n} \in P_\sigma(\mathfrak{p})$ and therefore, $\frac{1}{a} \in I(P_\sigma(\mathfrak{p}))$. Then $1 - \frac{1}{a} \in P_\sigma(\mathfrak{p})$ so $\frac{b}{a} = \bar{y}^n + \bar{x}^n(1 - \frac{1}{a}) \in P_\sigma(\mathfrak{p})$ and hence, $\sigma(a) = \sigma(b)$ in this case as well. \square

2.3.3 Corollary. *If A is a ring with many units then $G_T \cong A^*/\tilde{T}^*$. If $T - T = A$ then $T^* = \tilde{T}^*$ so $G_T \cong A^*/T^*$.*

Proof. Let $a \in A_T$. By the Positivstellensatz, there exists $s \in T$ such that $a^n(1+s) \in 1+T$. Replacing a with $a(1+s)$ we may assume $a^n \in 1+T$. By (2.3.2), there exists a unit b such that $\sigma(a) = \sigma(b)$ for all $\sigma \in \text{Sig}_T A$ and therefore, $ab^{-1} \in \tilde{T} \cap A_T$. Thus, the natural injection $A^*/\tilde{T}^* \hookrightarrow G_T$ is surjective.

Suppose $T - T = A$ and $a \in \tilde{T}^*$. Let $t_1, t_2 \in T$ such that $-a = t_1 - t_2$. By the Positivstellensatz, there exists $s_1 \in 1+T$, $s_2 \in T$ such that $as_1 = 1 + s_2$. Consider the polynomial

$$f(X) := s_1 X^n + s_1 t_1 + 1.$$

Let $\mathfrak{m} \subseteq A$ be a maximal ideal. If $s_1 t_1 + 1 \notin \mathfrak{m}$ then $f(0) \notin \mathfrak{m}$; otherwise, $s_1 \notin \mathfrak{m}$ so $f(1) \notin \mathfrak{m}$. Since A has many units, there exists $x \in A$ such that $u := s_1 x^n + s_1 t_1 + 1 \in T^*$. Since $au = t_2 + x^n + s_2(t_1 + x^n) \in T^*$, $a \in T^*$. \square

Proposition 2.3.2 suggests the following generalization of [36, Theorem 2.1].

2.3.4 Theorem. *For any proper preorder $T \subseteq A$, the following are equivalent:*

- (i) *The natural map $G_T \rightarrow G_{T'}$ is surjective for each proper preorder $T' \subseteq A$ containing T .*
- (ii) *For all $a \in A$, there exists $b \in A_T$ such that for each (maximal) signature $\sigma \in \text{Sig}_T A$, $a^n - 1 \in P_\sigma \Rightarrow \sigma(a) = \sigma(b)$.*

Proof. (i) \Rightarrow (ii) Let T' be the smallest preorder of A containing T and $a^n - 1$. (ii) is vacuous if T' is not proper so assume T' is a proper preorder. By (i), there exists $b \in A_T$ such that $[a] = [b]$ in $G_{T'}$. If $a^n - 1 \in P_\sigma$ then $\sigma \in \text{Sig}_{T'} A$ and therefore, $\sigma(a) = \sigma(b)$.

(ii) \Rightarrow (i) Let $a \in A_{T'}$. By the Positivstellensatz, there exists $s, t \in T'$ such that $a^n(1+s) = 1+t$. Replacing a by $a(1+s)$ we may assume $a^n - 1 \in T'$. By (ii), there exists $b \in A_T$ such that for all $\sigma \in \text{Sig}_{T'} A$, $\sigma(a) = \sigma(b)$. Then $ab^{n-1} \in \tilde{T}' \cap A_{T'}$ so $[a] = [b]$ in $G_{T'}$. \square

2.3.5 Examples.

- (1) By (2.3.2), any proper preorder in a ring with many units satisfies (2.3.4(ii)). Of course, if $A = K$ is a field then for any proper preorder T , $K_T = K^*$ and $G_T = K^*/T^*$ so (2.3.4(i)) is obvious.
- (2) If T is a proper preorder such that only finitely many primes occur as the support of a maximal order then T satisfies the equivalent conditions of (2.3.4). This follows from (1) by semi-localizing A at this finite set of supports.
- (3) If we replace the hypothesis that A has many units in Proposition 2.3.2 with Bröcker's U1 axiom of [17],

$$(a, b) = A \quad \Rightarrow \quad (aT + bT) \cap A^* \neq \emptyset$$

then the same proof shows $G_T \cong A^*/\tilde{T}^*$ and T satisfies (2.3.4(ii)). Note, however, that we really only require the polynomial f in (*) to have values

in A_T so the full-strength of Bröcker's U1 is not needed. Any preorder T in a commutative ring satisfying

$$(aT + (a - 1)T) \cap A_T \neq \emptyset$$

for every $a \in A$ also satisfies (2.3.4(ii)). It is not clear under what conditions the converse is true.

2.3.6 Proposition. *If \mathfrak{p} is the only T -compatible prime occurring as the support of a maximal order containing T then the natural map $G_T \rightarrow F(\mathfrak{p})^*/T(\mathfrak{p})^*$ is an isomorphism, $\text{Sigmax}_T A = \alpha_{\mathfrak{p}}^*(\text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p}))$ and T satisfies the equivalent conditions of Theorem 2.3.4.*

Proof. $A_T = A \setminus \mathfrak{p}$ so the natural map $G_T \rightarrow F(\mathfrak{p})^*/T(\mathfrak{p})^*$ is surjective. Suppose $a \in A_T$ and $\bar{a} \in T(\mathfrak{p})^*$. Let $Q \in \text{Sper}_T A$ and let P be the unique maximal order specializing Q . Then $\bar{a} \in T(\mathfrak{p})^* \subseteq P(\mathfrak{p})^*$ so $a \in P \setminus \mathfrak{p} \subseteq Q$. Thus, $a \in \tilde{T} \cap A_T$ and therefore, $G_T \cong F(\mathfrak{p})^*/T(\mathfrak{p})^*$.

T satisfies (2.3.4(ii)) by (2.3.5(2)) but it is obvious in this case: if $a \in A$ and $a^n - 1 \in P_{\sigma}$ for some $\sigma \in \text{Sig}_T A$ then necessarily $a \in A \setminus \mathfrak{p} = A_T$. $\alpha_{\mathfrak{p}}^*(\text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p})) = \text{Sigmax}_T A$ follows from (2.1.1). \square

2.3.7 Theorem. *Let $T \subseteq A$ be a proper preorder satisfying the equivalent conditions of Theorem 2.3.4.*

(i) *For any preorder $T' \supseteq T$ and any $P \in \text{Spermax}_T A$,*

$$\tilde{T}' \cap A_T \subseteq P \Rightarrow \tilde{T}' \subseteq P.$$

(ii) *The natural map $\text{Sigmax}_T A \rightarrow \text{Hom}(G_T, \Omega)$ is an embedding.*

Proof. (i) Suppose $a \in T' \setminus P_{(2)}$. Replacing a by a suitable power we may assume $-a \in P_{(2)}$. By (1.6.4) and (1.6.9), $P_{(2)}$ is maximal with respect to inclusion so, by the Positivstellensatz, there exists $s \in P_{(2)}$ such that $-a(1+s)^n \in 1 + P_{(2)}$. Set $u := 2a(1+s)^n + 1$. Then $u \in 1 + T'$, $-u \in 1 + P_{(2)}$, $u^n - 1 \in T' \cap P_{(2)}$. By (2.3.4),

there exists $b \in A_T$ such that $-b \in P_{(2)}$ and $\sigma(b) = \sigma(u) = 1$ for all $\sigma \in \text{Sig}_{T'} A$ so $b \in \tilde{T}' \cap A_T \subseteq P \subseteq P_{(2)}$, a contradiction. Therefore, $T' \subseteq P_{(2)}$.

Suppose $a \in T' \setminus P$. Let $\mathfrak{p} := \text{supp } P$. By (1.6.9), there exists $b \in A \setminus \mathfrak{p}$, $x \in A(P(\mathfrak{p}))$ such that $1/\bar{a} = x\bar{b}$. Since $T' \subseteq P_{(2)}$, $-a \notin P$ so $\bar{a}^2 \notin P(\mathfrak{p})$ and therefore, $\bar{a} \notin A(P(\mathfrak{p}))^*$. If $\bar{a} \in I(P(\mathfrak{p}))$ then $\bar{b} \notin A(P(\mathfrak{p}))$ so $\bar{a}x\bar{b}^n = \bar{b}^{n-1} \notin A(P(\mathfrak{p}))$ and hence, $\bar{a}\bar{b}^n \notin A(P(\mathfrak{p}))$. Replacing a by ab^n if necessary, we may assume $\bar{a} \notin A(P(\mathfrak{p}))$. Then $1 \pm \frac{1}{1+\bar{a}}, 1 \pm \frac{1}{(1+\bar{a})^n} \in P(\mathfrak{p})$ so $1+a \notin P$ and $(1+a)^n - 1 \in P$. Since T' also contains $(1+a)^n - 1$, there exists $b \in A_T$ such that $b \notin P$ and $\sigma(b) = \sigma(1+a) = 1$ for all $\sigma \in \text{Sig}_{T'} A$, that is, $b \in \tilde{T}' \cap A_T \subseteq P$, a contradiction.

(ii) Let $\sigma, \tau \in \text{Sigmax}_T A$ such that $\sigma|_{A_T} = \tau|_{A_T}$. Then $P_\sigma \cap A_T = P_\tau \cap A_T$ so $P := P_\sigma = P_\tau$ by (i). Let $\mathfrak{p} := \text{supp } P$. By (2.3.6), $G_P \cong F(\mathfrak{p})^*/P(\mathfrak{p})^*$ so by (2.3.4), we can pick $a \in A_T$ such that $\bar{a}P(\mathfrak{p})^*$ generates $F(\mathfrak{p})^*/P(\mathfrak{p})^*$. Since $a \in A_T$, $\sigma(a) = \tau(a)$ and hence, $\sigma = \tau$. \square

Just as for fields, we call T *complete* if the 2-primary part of G_T is cyclic.

2.3.8 Corollary. *Suppose T is a complete preorder satisfying the equivalent conditions of Theorem 2.3.4. Then T is contained in a unique maximal 2-primary order P , $G_T \cong F(\mathfrak{p})^*/T(\mathfrak{p})^*$ where $\mathfrak{p} = \text{supp } P$, $\text{Sigmax}_T A = \alpha_{\mathfrak{p}}^*(\text{Sig}_{T(\mathfrak{p})} F(\mathfrak{p}))$ and $T(\mathfrak{p})$ is a complete preorder in $F(\mathfrak{p})$.*

Proof. Assume the 2-primary part of G_T is cyclic. Let P be a maximal 2-primary order containing T . If $a \in A_T$ such that $[a]$ has order 2^r in G_T then $[a]^{2^{r-1}} = [-1]$ in G_T so $-a^{2^{r-1}} \in \tilde{T} \subseteq P$ and therefore, $[a]$ has order 2^r in G_P as well. Since G_P is cyclic of 2-power order, the kernel $A_T \cap P$ of the natural map $G_T \rightarrow G_P$ is the subgroup H of all elements of G_T of odd order. If P' is another maximal 2-primary order containing T then $A_T \cap P = H = A_T \cap P'$ so, by (2.3.7(i)), $P = P'$. Thus, P is the only maximal 2-primary order containing T . For any maximal order $Q \supseteq T$, we have $Q_{(2)} = P$ so $\text{supp } Q = \text{supp } Q_{(2)} = \text{supp } P$. The result now follows from (2.3.6). \square

Chapter 3

A Reduced Theory of Higher Level Forms

There is a natural map of the Witt ring $W(K)$ of a field K to the ring $C(X, \mathbb{C})$ of all locally constant functions $f : X \rightarrow \mathbb{C}$ where $X = \text{Sper}_{\Sigma K^2} K$ (the set of all level 1 orders in K .) If $a \in K^*$, the 1-dimensional form determined by a is mapped to the function $\hat{a} : X \rightarrow \{\pm 1\}$ defined by $\hat{a}(P) = \text{sgn}_P(a)$. Pfister's famous Local-global Principle states that the kernel of this map is the torsion ideal of $W(K)$ so the reduced Witt ring can be identified with the subring of $C(X, \mathbb{C})$ generated by the \hat{a} , $a \in K^*$.

In [12], Becker and Rosenberg used the higher level signatures of [10] to develop an analogous theory of higher level reduced forms on a field. For each $a \in K^*$, we have the map $\hat{a} : \text{Sig } K \rightarrow \mathbb{C}$ given by $\sigma \mapsto \sigma(a)$ and the reduced Witt ring of higher level is defined to be the subring of $C(\text{Sig } K, \mathbb{C})$ generated by the \hat{a} , $a \in K^*$. Replacing $\text{Sig } K$ with $\text{Sig}_T K$, one defines the reduced Witt ring $W_T(K)$ for an arbitrary preorder $T \subseteq K$.

Reduced Witt rings for level 1 preorders in a semi-local ring or a ring with many units were defined in [17], [25], [48] and for higher level preorders in a ring with many units in [38].

Marshall's theory of spaces of orderings provides an axiomatic approach to the (level 1) reduced theory of quadratic forms. In [37] and [41], spaces of orderings are generalized to the higher level *spaces of signatures* with higher level preorders in fields and rings with many units providing examples. Recently Marshall has

shown $(\text{Spermax}_T A, G_T)$ is a space of orderings whenever T is a level 1 preorder in a commutative ring A satisfying condition (i) of Theorem 2.3.4. In this chapter, we show this is also the case for higher level preorders. Specifically, if $T \subseteq A$ is a preorder of higher level satisfying the equivalent conditions of Theorem 2.3.4 then $(\text{Sigmax}_T A, G_T)$ is a space of signatures in the sense of [37]. Since preorders in a ring with many units always satisfy these conditions, the results of [38] are obtained as a special case.

The results of the first section of this chapter hold for any preorder in a commutative ring. The proofs are modeled on those given in [12] and [38]. The field-theoretic results of section 3.2 have been drawn from [10] and [12] and are included in order to make this thesis self-contained.

3.1 Reduced forms and reduced Witt rings of higher level

Let A be a commutative ring and T a proper preorder of A . A T -form of dimension r over A is an r -tuple $\varphi = \langle a_1, \dots, a_r \rangle$, where $a_1, \dots, a_r \in A_T$. The *sum* and *product* of the T -forms $\varphi = \langle a_1, \dots, a_r \rangle$ and $\psi = \langle b_1, \dots, b_s \rangle$ are given by

$$\varphi \oplus \psi := \langle a_1, \dots, a_r, b_1, \dots, b_s \rangle$$

and

$$\varphi \otimes \psi := \langle a_1 b_1, \dots, a_1 b_s, \dots, a_r b_1, \dots, a_r b_s \rangle.$$

The T -form $\varphi \oplus \dots \oplus \varphi$ (k times) is denoted $k \times \varphi$ and for $c \in A_T$, the T -form $\langle c \rangle \otimes \varphi$ is denoted $c\varphi$.

Set $X_T = \text{Sigmax}_T A$. Let $C(X_T, \mathbb{C})$ denote the ring of locally constant functions $f : X_T \rightarrow \mathbb{C}$ (that is, the ring of continuous functions where \mathbb{C} is given the discrete topology.) For $a \in A_T$, define $\hat{a} : X_T \rightarrow \mathbb{C}$ by $\hat{a}(\sigma) = \sigma(a)$. For each $\sigma \in X_T$, $\hat{a}^{-1}(\hat{a}(\sigma)) = U(a; \sigma) \cap \text{Sigmax}_T A$ is a basic open set in X_T so $\hat{a} \in C(X_T, \mathbb{C})$. For a T -form $\varphi = \langle a_1, \dots, a_r \rangle$, define $\hat{\varphi} := \sum \hat{a}_i$. (If φ is the empty form $\langle \rangle$ then $\hat{\varphi} := 0$.) For any T -forms φ, ψ ,

$$\widehat{\varphi \oplus \psi} = \hat{\varphi} + \hat{\psi} \quad \text{and} \quad \widehat{\varphi \psi} = \hat{\varphi} \hat{\psi}$$

so the set

$$W_T(A) := \{\hat{\varphi} \mid \varphi \text{ is a } T\text{-form}\}$$

is a subring of $C(X_T, \mathbb{C})$ which we call the *reduced Witt ring of higher level for T*.

3.1.1 Remark. Let $\mathfrak{p} \subseteq A$ be a T -compatible prime. For a T -form $\varphi = \langle a_1, \dots, a_r \rangle$, we define $\alpha_{\mathfrak{p}}(\varphi)$ to be the $T(\mathfrak{p})$ -form $\langle \alpha_{\mathfrak{p}}(a_1), \dots, \alpha_{\mathfrak{p}}(a_r) \rangle$. Clearly, for any T -forms φ, ψ , $\hat{\varphi} = \hat{\psi}$ in $W_T(A)$ iff $\widehat{\alpha_{\mathfrak{p}}(\varphi)} = \widehat{\alpha_{\mathfrak{p}}(\psi)}$ in $W_{T(\mathfrak{p})}(F(\mathfrak{p}))$ for every T -compatible prime \mathfrak{p} in A . Thus, we have a natural injective ring homomorphism

$$W_T(A) \hookrightarrow \prod_{\mathfrak{p}} W_{T(\mathfrak{p})}(F(\mathfrak{p}))$$

where \mathfrak{p} runs through all T -compatible primes.

If φ, ψ are T -forms with $\hat{\varphi} = \hat{\psi}$ and $\dim \varphi = \dim \psi$, we say φ and ψ are *T -isometric* and write $\varphi \cong_T \psi$.

3.1.2 Proposition. If φ, ρ are T -forms with $\hat{\varphi} = \hat{\rho}$ and $\dim \varphi \geq \dim \rho$ then $\dim \varphi \equiv \dim \rho \pmod{2}$ and $\varphi \cong_T \rho \oplus m \times \langle 1, -1 \rangle$, for some $m \geq 0$.

Proof. Pick $\sigma \in X_T$. There exists an odd integer $v > 0$ such that $P_{\sigma}(2) = P_{\sigma}^{(v)} = P_{\sigma^v}$ so $\sigma^v \in X_T$ has 2-primary level, say $s(\sigma^v) = 2^t$. Let ω be a primitive 2^{t+1} -th root of unity. Then there exists $m_k \in \mathbb{Z}$, $m_k \geq 0$ such that

$$(\hat{\varphi} - \hat{\rho})(\sigma^v) = \sum_{k=0}^{2^{t+1}-1} m_k \omega^k \quad \text{and} \quad \dim(\varphi - \rho) = \sum_{k=0}^{2^{t+1}-1} m_k.$$

Since $\omega^{2^t} = -1$,

$$\begin{aligned} (\hat{\varphi} - \hat{\rho})(\sigma^v) &= \sum_{k=0}^{2^t-1} m_k \omega^k + \sum_{k=0}^{2^t-1} m_{2^t+k} \omega^{2^t+k} \\ &= \sum_{k=0}^{2^t-1} (m_k - m_{2^t+k}) \omega^k. \end{aligned}$$

The minimal polynomial of ω over \mathbb{Q} is $X^{2^t} + 1$ so $1, \omega, \dots, \omega^{2^t-1}$ are independent over \mathbb{Q} . Therefore, if $\hat{\varphi} = \hat{\rho}$, we must have $m_k = m_{2^t+k}$ for all $k = 0, \dots, 2^t - 1$ and hence,

$$\dim(\varphi - \rho) = \sum_{k=0}^{2^t-1} 2m_k \equiv 0 \pmod{2}.$$

□

3.1.3 Lemma. *Let K be a field.*

(i) *For any $x \in K^*$, $x \neq -1$, and for any $l, m \in \mathbb{N}$,*

$$\frac{x^{2m} + 1}{x^{2l} + 1} \cdot \frac{x^{2l-1} + 1}{x^{2m-1} + 1} \in \Sigma K^n.$$

(ii) *For any $a, b \in K^*$, $a + b \neq 0$, and for any $l, m \in \mathbb{N}$,*

$$\sigma \left(\frac{a^{2l}b + ab^{2l}}{a^{2l} + b^{2l}} \right) = \sigma \left(\frac{a^{2m}b + ab^{2m}}{a^{2m} + b^{2m}} \right).$$

Proof. (ii) follows from (i) using $x = a^{-1}b$. To prove (i), let

$$u := \frac{x^{2m} + 1}{x^{2l} + 1} \cdot \frac{x^{2l-1} + 1}{x^{2m-1} + 1}.$$

By (1.5.5), it suffices to show that $u \in P$ for all $P \in \text{Sper } K$. If $x \in I(P)$ then $1 + x^i \in 1 + I(P) \subseteq P^*$ for all $i > 0$ so $u \in P^*$. Since

$$\frac{x^{2m} + 1}{x^{2l} + 1} \cdot \frac{x^{2l-1} + 1}{x^{2m-1} + 1} = \frac{(x^{-1})^{2m} + 1}{(x^{-1})^{2l} + 1} \cdot \frac{(x^{-1})^{2l-1} + 1}{(x^{-1})^{2m-1} + 1},$$

$x^{-1} \in I(P)$ also implies $u \in P^*$. Assume $x \in A(P)^*$. Clearly $u \in P^*$ if $x \in P^*$ so assume $x \notin P$. Then $x \in -P^*$ and

$$u = \frac{x^{2m} + 1}{x^{2l} + 1} \cdot \frac{(-x)^{2l-1} - 1}{(-x)^{2m-1} - 1} \in P.$$

□

3.1.4 Proposition.

(i) $\langle a \rangle \cong_T \langle ta \rangle$ for all $a \in A_T$, $t \in T \cap A_T$.

(ii) $\langle a, b \rangle \cong_T \langle a + b, a^n b + ab^n \rangle$ for all $a, b \in A_T$ with $a + b \in A_T$.

Proof. (i) is clear. To prove (ii), we may assume $A = K$ is a field by (3.1.1). Scaling by a^{-1} , we may assume $a = 1$ and $b \neq -1$. Let $\sigma \in X_T$. It suffices to show

$$(*) \quad 1 + \sigma(b) = \sigma(1 + b) + \sigma(b)\sigma(1 + b^{n-1}).$$

If $b \in I(P_\sigma)$ then $\sigma(1 + b) = \sigma(1 + b^{n-1}) = 1$ so $(*)$ is clear. If $b^{-1} \in I(P_\sigma)$ then $\sigma(1 + b) = \sigma(b)$ and $\sigma(1 + b^{n-1}) = \sigma(b^{n-1})$ so again $(*)$ is clear. Finally, suppose $b \in A(P_\sigma)^*$. Then $b^2 \in P_\sigma$ so $\sigma(b) = \pm 1$ and $\sigma(1 + b^2) = 1$. By (3.1.3), $\sigma(b + b^2) = \sigma(b + b^n)$ so $\sigma(1 + b) = \sigma(1 + b^{n-1})$ and therefore, $(*)$ holds in this case as well. □

For $\varphi = \langle a_1, \dots, a_r \rangle$, the set of elements of A represented by φ is $D_T(\varphi) := a_1T + \dots + a_rT$.

3.1.5 Theorem. *Let $b \in A_T$. The following are equivalent:*

- (i) *There exists $t \in T$ such that $b(1+t) \in D_T\langle a_1, \dots, a_r \rangle$.*
- (ii) *$\alpha_{\mathfrak{p}}(b) \in D_{T(\mathfrak{p})}\langle \alpha_{\mathfrak{p}}(a_1), \dots, \alpha_{\mathfrak{p}}(a_r) \rangle$ for all T -compatible primes $\mathfrak{p} \subseteq A$.*

Proof. Suppose (ii) holds. Then, for every T -compatible prime \mathfrak{p} ,

$$\alpha_{\mathfrak{p}}(\pm b^n) \in T(\mathfrak{p}) + \alpha_{\mathfrak{p}}(-a_1 b^{n-1})T(\mathfrak{p}) + \dots + \alpha_{\mathfrak{p}}(-a_r b^{n-1})T(\mathfrak{p})$$

so, by (1.1.7), $-1 \in T + (-a_1 b^{n-1})T + \dots + (-a_r b^{n-1})T$ and therefore, $b \in -bT + a_1T + \dots + a_rT$. (i) \Rightarrow (ii) is clear. \square

A T -form $\varphi = \langle a_1, \dots, a_r \rangle$ is said to be T -isotropic if there exists $t_1, \dots, t_r \in T$ such that $a_1 t_1 + \dots + a_r t_r = 0$ and at least one t_i is in A_T . φ is called T -anisotropic if it is not T -isotropic.

3.1.6 Theorem. *For a T -form φ , the following are equivalent:*

- (i) *φ is T -isotropic.*
- (ii) *$\alpha_{\mathfrak{p}}(\varphi)$ is $T(\mathfrak{p})$ -isotropic for all T -compatible primes $\mathfrak{p} \subseteq A$.*
- (iii) *For all $a \in A_T$, there exists $t \in T$ such that $a(1+t) \in D_T(\varphi)$.*
- (iv) *There exists $a \in A_T$ such that $a, -a \in D_T(\varphi)$.*

Proof. Let $\varphi = \langle a_1, \dots, a_r \rangle$ where $a_1, \dots, a_r \in A_T$. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii) Let $a \in A_T$, $\mathfrak{p} \subseteq A$ a T -compatible prime. By (3.1.5), it suffices to show that $\alpha_{\mathfrak{p}}(a) \in D_{T(\mathfrak{p})}(\alpha_{\mathfrak{p}}(\varphi))$. Since $\alpha_{\mathfrak{p}}(\varphi)$ is $T(\mathfrak{p})$ -isotropic, there exists $t_1, \dots, t_r \in T$ such that $\alpha_{\mathfrak{p}}(a_1 t_1) + \dots + \alpha_{\mathfrak{p}}(a_r t_r) = 0$ and $t_i \notin \mathfrak{p}$ for some i . It follows that $-\alpha_{\mathfrak{p}}(a_i) \in D_{T(\mathfrak{p})}(\alpha_{\mathfrak{p}}(\varphi))$. By (1.1.3), $\alpha_{\mathfrak{p}}(a_i)^{-1} \alpha_{\mathfrak{p}}(a) \in T(\mathfrak{p}) - T(\mathfrak{p})$ so $\alpha_{\mathfrak{p}}(a) \in \alpha_{\mathfrak{p}}(a_i)T(\mathfrak{p}) - \alpha_{\mathfrak{p}}(a_i)T(\mathfrak{p}) \subseteq D_{T(\mathfrak{p})}(\alpha_{\mathfrak{p}}(\varphi))$.

(iii) \Rightarrow (iv) is clear.

(iv) \Rightarrow (i) Let $a \in A_T$ such that $a, -a \in D_T(\varphi)$. Since $\tilde{T} - \tilde{T} = A$, there exists $s, t \in 1 + \tilde{T}$ such that $a^{n-1}a_1 = s - t$. By the Positivstellensatz, we can assume $s, t \in 1 + T$ by scaling a by an element of $1 + T$ if necessary. Then $a(1 + s) = a_1a^n + a(1 + t)$. Let $s_1, \dots, s_r \in T$ such that $a = \sum a_i s_i$. Then $a(1 + s) = a_1(a^n + s_1(1 + t)) + a_2s_2(1 + t) + \dots + a_rs_r(1 + t)$ and $a^n + s_1(1 + t) \in A_T$ so we may assume $s_1 \in A_T$. Let $t_1, \dots, t_r \in T$ such that $-a = \sum a_i t_i$. Then $0 = \sum a_i(s_i + t_i)$ and clearly, $s_1 + t_1 \in A_T$. \square

3.2 T-forms and compatible valuations

Let K be a field, $T \subseteq K$ a proper preorder. Denote by V_T the set of all valuations of K compatible with T .

For an abelian group G of exponent n , $\chi(G)$ denotes the dual group $\text{Hom}(G, \Omega)$. If H is a subgroup of G , we identify $\chi(G/H)$ with the subgroup of $\chi(G)$ consisting of all characters χ with $\chi(H) = 1$.

Since the group of all complex roots of unity is divisible, the functor $G \mapsto \chi(G)$ is exact in the category of abelian groups of exponent n . Therefore, for any $v \in V_T$, (1.3.5) induces the exact sequence

$$(3.2.1) \quad 1 \rightarrow \chi\left(\frac{v(K^*)}{v(T^*)}\right) \xrightarrow{v^*} \chi\left(\frac{K^*}{T^{v^*}}\right) \xrightarrow{i^*} \chi\left(\frac{k_v^*}{T_v^*}\right) \rightarrow 1$$

where $v^*(\gamma) := \gamma \circ v$ and $i^*(\chi)(\bar{a}) := \chi(a)$ for $a \in A_v^*$.

We say a character $\chi \in \chi(K^*/T^*)$ is *compatible* with a valuation v , written $\chi \sim A_v$, if $1 + \mathfrak{m}_v \subseteq \ker \chi$, that is, if χ is in the subgroup $\chi(K^*/T^{v^*})$ of $\chi(K^*/T^*)$. The character $i^*(\chi)$, where $\chi \in \chi(K^*/T^{v^*})$, is called the *push down* of χ (along v). Conversely, suppose $\chi \in \chi(K^*/T^*)$ and $\xi \in \chi(k_v^*/T_v^*)$ with $\chi(a) = \xi(\bar{a})$, for all $a \in A^*$. Then $1 + \mathfrak{m}_v \subseteq \ker \chi$ so χ is compatible with v and $i^*(\chi) = \xi$. In this case, we call χ a *pull-back* of ξ (along v).

3.2.2 Theorem. For all $\xi \in \chi(k_v^*/T_v^*)$,

- (i) ξ has a pull-back along v ,

- (ii) if χ is a fixed pull-back of ξ , all other pull-backs are given by $\chi \cdot (\gamma \circ v)$ with γ running through $\chi(v(K^*)/v(T^*))$.

Proof. This follows from the exactness of (3.2.1). \square

3.2.3 Theorem. A character $\chi \in \chi(K^*/T^{v*})$ induces a T -signature on K iff its push-down induces a T_v -signature on k .

Proof. Let $\bar{\chi} = i^*(\chi)$. $\bar{\chi}$ induces a T_v -signature iff $\ker \bar{\chi}$ is additively closed iff $A^* \cap \ker \chi = \{a \in A^* \mid \chi(a) = \bar{\chi}(\bar{a}) = 1\}$ is additively closed. By (1.3.2), this is the case iff $\ker \chi$ is additively closed. \square

3.2.4 Corollary. There is a (non-canonical) bijection

$$\text{Sig}_{T^v} K \longrightarrow \text{Sig}_{T_v} k \times \chi \left(\frac{v(K^*)}{v(T^*)} \right).$$

Let φ be a T -form, $v \in V_T$. For any $a \in K^*$, we define the a -th residue class T_v -form $\bar{\varphi}_a$ as follows. Let a_1, \dots, a_k be the entries in φ with $v(a_i) \equiv v(a) \pmod{v(T^*)}$ and set

$$\varphi_a = \langle a^{-1}a_1, \dots, a^{-1}a_k \rangle.$$

If no such entries exist then φ_a is the empty form $\langle \rangle$. For $i = 1, \dots, k$, pick $u_i \in A_v^*$ such that $a^{-1}a_i \equiv u_i \pmod{T^*}$ and set

$$\bar{\varphi}_a = \langle \bar{u}_1, \dots, \bar{u}_k \rangle.$$

Note that the T_v -isometry class of $\bar{\varphi}_a$ does not depend on the choice of u_i .

3.2.5 Theorem. Let $v \in V_T$. If φ, ρ are T -forms with $\hat{\varphi} = \hat{\rho}$ on X_{T^v} then $\hat{\bar{\varphi}}_a = \hat{\bar{\rho}}_a$ on X_{T_v} for all $a \in K^*$.

Proof. Since $(-\overline{\varphi \oplus \rho})_a = -\bar{\varphi}_a \oplus \bar{\rho}_a$, it suffices to show that $\hat{\varphi} = 0$ implies $\hat{\bar{\varphi}}_a = 0$ for all $a \in K^*$. Let $c_1, \dots, c_s \in K^*$ such that $\bar{\varphi}_{c_1}, \dots, \bar{\varphi}_{c_s}$ are the distinct non-empty residue class T_v -forms of φ . Then $\hat{\varphi} = \sum \hat{c}_i \hat{\varphi}_{c_i}$. Let $\bar{\sigma} \in X_{T_v}$ and let $\sigma \in X_{T^v}$ be

a pull-back of $\bar{\sigma}$. For any $\gamma \in \chi(v(K^*)/v(T^*))$, $\sigma \cdot (\gamma \circ v)$ is a T^v -signature with push-down $\bar{\sigma}$ and

$$\begin{aligned} 0 &= \hat{\varphi}(\sigma \cdot (\gamma \circ v)) \\ &= \left(\sum \hat{c}_i \hat{\varphi}_{c_i} \right) (\sigma \cdot (\gamma \circ v)) \\ &= \sum \sigma(c_i) \gamma(g_i) \hat{\varphi}_{c_i}(\bar{\sigma}) \\ &= \left(\sum \sigma(c_i) \hat{\varphi}_{c_i}(\bar{\sigma}) \hat{g}_i \right) (\gamma) \end{aligned}$$

where $g_i := v(c_i) + v(T^*)$ and \hat{g}_i denotes the image of g_i in the double dual $\chi(\chi(\frac{v(K^*)}{v(T^*)}))$. Since distinct characters are linearly independent over \mathbb{C} , $\hat{\varphi}_{c_i}(\bar{\sigma}) = 0$ for all i . \square

3.2.6 Theorem ([12, Theorem 2.11]). *Let $v \in V_T$. A T -form φ is T^v -isotropic iff at least one residue class form $\bar{\varphi}_a$ is T_v -isotropic.*

Proof. Suppose $\varphi = \langle a_1, \dots, a_r \rangle$ and there exists $t_1, \dots, t_r \in T^v$, not all zero, such that $\sum t_i a_i = 0$. Assume $t_{i_1} \neq 0$ and $v(t_{i_1} a_{i_1}) = \min\{v(t_j a_j)\}$. We show $\bar{\varphi}_{a_{i_1}}$ is T_v -isotropic. $\varphi_{a_{i_1}} = \langle 1, a_{i_1}^{-1} a_{i_2}, \dots, a_{i_1}^{-1} a_{i_p} \rangle$, where a_{i_2}, \dots, a_{i_p} are the entries of φ with $v(a_{i_k}) \equiv v(a_{i_1}) \pmod{v(T^*)}$. If $v(a_j) \not\equiv v(a_{i_1}) \pmod{v(T^*)}$ then $t_j a_j / t_{i_1} a_{i_1} \in \mathfrak{m}_v$ so

$$\sum_{k=1}^p \frac{t_{i_k} a_{i_k}}{t_{i_1} a_{i_1}} \in \mathfrak{m}_v.$$

Let $u_1, \dots, u_p \in A_v^*$ such that $a_{i_1}^{-1} a_{i_k} \equiv u_k \pmod{T^*}$. Then there exists $s_1, \dots, s_p \in A_v \cap T^v$ such that $s_1 \in A_v^*$ and

$$\sum_{k=1}^p s_k u_k \in \mathfrak{m}_v.$$

Since $\bar{\varphi}_{a_{i_1}} = \langle \bar{u}_1, \dots, \bar{u}_p \rangle$, this shows $\bar{\varphi}_{a_{i_1}}$ is T_v -isotropic.

Conversely, assume $\bar{\varphi}_a$ is T_v -isotropic. $\varphi_a = \langle a^{-1} a_{i_1}, \dots, a^{-1} a_{i_p} \rangle$, where $v(a_{i_k}) \equiv v(a) \pmod{v(T^*)}$. Let $u_k \in A_v^*$, $t_k \in T^*$ such that $a^{-1} a_{i_k} = t_k^{-1} u_k$ and $\bar{\varphi}_a = \langle \bar{u}_1, \dots, \bar{u}_p \rangle$. Pick $s_1, \dots, s_p \in A_v \cap T$, not all in \mathfrak{m}_v , such that

$$x := \sum_{k=1}^p s_k u_k = \sum_{k=1}^p \frac{s_k t_k a_{i_k}}{a} \in \mathfrak{m}_v.$$

If $s_j \in A_v^*$ then

$$0 = s_j t_j a_{i_j} - ax + \sum_{k \neq j} s_k t_k a_{i_k}.$$

$s_j t_j a_{i_j} - ax = s_j t_j (1 - s_j^{-1} u_j^{-1} x) a_{i_j} \in T^*(1 + \mathfrak{m}_v) a_{i_j} = T^{v*} a_{i_j}$ so $\langle a_{i_1}, \dots, a_{i_p} \rangle$, and hence, φ , is T^v -isotropic. \square

A T -form $\varphi = \langle a_1, \dots, a_r \rangle$ is said to be σ -definite for $\sigma \in X_T$ if $\sigma(a_i) = \sigma(a_j)$ for all $1 \leq i \leq j \leq r$.

3.2.7 Lemma. *A T -form φ is σ -definite iff $\hat{\varphi}(\sigma) = \dim \varphi$.*

Proof. Suppose $\varphi = \langle a_1, \dots, a_r \rangle$. Set $\omega_i = \sigma(a_i) \in \Omega$ for $i = 1, \dots, r$ and assume $|\Sigma \omega_i| = r$. If $r = 2$ then $(\omega_1 + \omega_2)(\bar{\omega}_1 + \bar{\omega}_2) = 4$ so $2 = \omega_1 \bar{\omega}_2 + \bar{\omega}_1 \omega_2 = 2 \operatorname{Re}(\omega_1 \bar{\omega}_2) = 2 \operatorname{Re}(\omega_1 \omega_2^{-1})$ and therefore, $\omega_1 \omega_2^{-1} = 1$. For $r > 2$, we have for any $i \neq j$

$$r = \left| \sum_i \omega_i \right| \leq |\omega_i + \omega_j| + \left| \sum_{k \neq i, j} \omega_k \right| \leq |\omega_i + \omega_j| + r - 2.$$

Thus, $2 \leq |\omega_i + \omega_j| \leq 2$ so $\omega_i = \omega_j$ by the case $r = 2$. The converse is clear. \square

For $a_1, \dots, a_r \in K^*$, define

$$X_T(a_1, \dots, a_r) := \{ \sigma \in X_T \mid a_i \equiv a_j \pmod{A(P_\sigma)^* T^*} \text{ for all } i, j \},$$

$$V_T(a_1, \dots, a_r) := \{ v \in V_T \mid v(a_i) \not\equiv v(a_j) \pmod{v(T^*)} \text{ for some } i \neq j \}.$$

3.2.8 Theorem ([12, Theorem 3.3]). *A T -form $\varphi = \langle a_1, \dots, a_r \rangle$ is T -isotropic iff*

- (i) φ is σ -indefinite for all $\sigma \in X_T(a_1, \dots, a_r)$ and
- (ii) φ is T^v -isotropic for all $v \in V_T(a_1, \dots, a_r)$.

Proof. Suppose $0 = a_1 t_1 + \dots + a_r t_r$, where $t_1, \dots, t_r \in T$ and $t_1 \neq 0$. Clearly φ is T^v -isotropic for all $v \in V_T$. If φ is σ -definite for some $\sigma \in X_T$ then $\sigma(a_1^{-1} t_1^{-1} a_i t_i) \in \{0, 1\}$ for each i so $0 = \Sigma a_1^{-1} t_1^{-1} a_i t_i \in 1 + P_\sigma \subseteq P_\sigma^*$, a contradiction.

Conversely, suppose φ is T -anisotropic. Scaling by a_1^{-1} we may assume $a_1 = 1$. By (1.1.4), there exists a T -semiorder with $a_1, \dots, a_r \in S$. Let v be the valuation

associated with the valuation ring $A(S)$. Clearly $v \in V_T$. Since S_v is a level 1 order on the residue field k_v of $A(S)$, there is a unique signature $\bar{\sigma}$ on k_v with $P_{\bar{\sigma}} = S_v$. Pull it back along v to get a T -signature $\sigma \in \text{Sig}_{T^v} K$. Since σ is compatible with $A(S)$, $A(P_{\sigma}) \subseteq A(S)$ and $A(P_{\bar{\sigma}}) = A(P_{\sigma})/I(S)$. But $P_{\bar{\sigma}} = S_v$ is archimedean so $k_v = A(P_{\bar{\sigma}})$ and therefore, $A(P_{\sigma}) = A(S)$.

Suppose $\sigma \in X_T(1, a_2, \dots, a_r)$. Then, for all $i = 1, \dots, r$, $a_i = u_i t_i$ for some $t_i \in T^*$, $u_i \in A(S)^* \cap S$ and therefore, $\sigma(a_i) = \sigma(u_i) = \bar{\sigma}(\bar{u}_i) = 1$. Thus, φ is σ -definite.

Assume now that $\sigma \notin X_T(1, a_2, \dots, a_r)$. Then there exists $i \neq j$ such that $v(a_i) \not\equiv v(a_j) \pmod{v(T^*)}$ so $v \in V_T(1, a_2, \dots, a_r)$. In particular, $v(a_i) \notin v(T^*)$ for some i . By (1.3.11), there exists a valuation $v' \in V_T(1, a_2, \dots, a_r)$ with $(1 + \mathfrak{m}')a_i \subseteq S$ for all i , where \mathfrak{m}' is the maximal ideal of the valuation ring associated with v' . Then $a_i T^{v'} \subseteq S$ for all i so $D_{T^{v'}}(\varphi) \subseteq S$ and therefore, $-1 \notin D_{T^{v'}}(\varphi)$. It follows from (3.1.6) that φ is $T^{v'}$ -anisotropic. \square

A preorder $T' \subseteq K$ is said to be of *finite index* if $[K^* : T^*] < \infty$. If T' is of finite index and $v \in V_{T'}$ then the exact sequence (1.3.5) shows the push-down T'_v is also of finite index and $[v(K^*) : v(T'^*)] < \infty$.

3.2.9 Proposition. *Let $v \in V_T$ and $a_1, \dots, a_s \in K^*$ such that $v(a_i) \not\equiv v(a_j) \pmod{v(T^*)}$ for all $i \neq j$. Then there exists a preorder $T' \supseteq T^v$ of finite index with $v(a_i) \not\equiv v(a_j) \pmod{v(T'^*)}$ for all $i \neq j$.*

Proof. For each $i \neq j$, there exists characters $\gamma_{i,j} \in \chi(v(K^*)/v(T^*))$ such that $\gamma_{i,j}(v(a_i)) \neq \gamma_{i,j}(v(a_j))$. Fix $\sigma \in \text{Sig}_{T^v} K$ and set $\sigma_{i,j} = \sigma \cdot (\gamma_{i,j} \circ v) \in \text{Sig}_{T^v} K$ by (3.2.2) and (3.2.3). Let $T' = \cap \{P_{\sigma_{i,j}} \mid i < j\} \cap P_{\sigma}$. Clearly, $T^v \subseteq T'$ and since K^*/T'^* embeds in $\prod_{i < j} K^*/P_{\sigma_{i,j}}^* \times K^*/P_{\sigma}^*$ which is finite, T' is of finite index. Suppose $v(a_i) \equiv v(a_j) \pmod{v(T'^*)}$ for some $i \neq j$. Let $t \in T'$, $u \in A_v^*$ such that $a_i = a_j t u$. Then $\sigma(a_i) \gamma_{i,j}(v(a_i)) = \sigma_{i,j}(a_i) = \sigma_{i,j}(a_j t u) = \sigma_{i,j}(a_j) \sigma_{i,j}(u) = \sigma(a_j) \gamma_{i,j}(v(a_j)) \sigma(u) = \sigma(a_i) \gamma_{i,j}(v(a_j))$ so $\gamma_{i,j}(v(a_i)) = \gamma_{i,j}(v(a_j))$, a contradiction. \square

3.2.10 Theorem ([12, Theorem 3.6]). A T -form $\varphi = \langle a_1, \dots, a_r \rangle$ is T -isotropic iff φ is T' -isotropic for all preorders $T' \supseteq T$ of finite index.

Proof. Clearly if φ is T -isotropic then φ is T' -isotropic for every preorder $T' \supseteq T$. Suppose φ is T -anisotropic. If φ is σ -definite for some $\sigma \in X_T(a_1, \dots, a_r)$ then, for all $t_1, \dots, t_r \in T$ with $t_1 \neq 0$, $\Sigma a_1^{-1} t_1^{-1} a_i t_i \in 1 + P_\sigma \subseteq P_\sigma^*$ so φ is P_σ -anisotropic and we are done. So assume φ is σ -indefinite for all $\sigma \in X_T(a_1, \dots, a_r)$. By (3.2.8), φ is T^v -anisotropic for some $v \in V_T(a_1, \dots, a_r)$. Let $\bar{\varphi}_1, \dots, \bar{\varphi}_s$ be the distinct non-trivial residue class T_v -forms of φ . We have $s \geq 2$ so $\dim \bar{\varphi}_i < r$ for each i . By (3.2.6), each $\bar{\varphi}_i$ is T_v -anisotropic. By induction on the dimension of φ , there exists preorders $\bar{T}_i \supseteq T_v$ of finite index in the residue field k of v with $\bar{\varphi}_i$ \bar{T}_i -anisotropic. Set $\bar{T} = \cap \bar{T}_i$. Then \bar{T} is a preorder of finite index in k containing T_v and each $\bar{\varphi}_i$ is \bar{T} -anisotropic. For each $\bar{\sigma}_j \in \text{Sig}_{\bar{T}} k \subseteq \text{Sig}_{T_v} k$, choose a pull-back $\sigma_j \in \text{Sig}_{T^v} K$ and set $T'' = \cap P_{\sigma_j}$. Then T'' is a preorder of finite index in K containing T^v and $T''_v = \bar{T}$. Let $i_1, \dots, i_s \in \{1, \dots, r\}$ be such that $v(a_{i_1}) + v(T^*), \dots, v(a_{i_s}) + v(T^*)$ are all distinct. By (3.2.9), there exists a preorder $T''' \supseteq T^v$ of finite index with $v(a_{i_1}) + v(T'''), \dots, v(a_{i_s}) + v(T''')$ still all distinct. Let $T' = T'' \cap T'''$ which is clearly a preorder of finite index containing T . The residue class forms of the T' -form φ are still $\bar{\varphi}_1, \dots, \bar{\varphi}_s$ which are T'_v -anisotropic since $T'_v \subseteq \bar{T}$. By (3.2.6), φ is T' -anisotropic. This completes the proof. \square

3.2.11 Theorem. Let φ, ρ be T -forms. If $\hat{\varphi} = \hat{\rho}$ and $\dim \varphi > \dim \rho$ then φ is T -isotropic.

Proof. We proceed by induction on $\dim \varphi$. By (3.1.2), the first case is where $\rho = \langle \rangle$ and $\dim \varphi = 2$, say $\varphi = \langle a, b \rangle$. Then $\hat{\varphi} = 0$ so $\sigma(-a^{-1}b) = 1$ for all $\sigma \in X_T$ and therefore, $-a^{-1}b \in T$. This shows φ is T -isotropic. Suppose the result holds for T -forms of dimension less than r and $\varphi = \langle a_1, \dots, a_r \rangle$. We use (3.2.8) to show φ is T -isotropic.

For any $\sigma \in X_T(a_1, \dots, a_r)$, $|\hat{\varphi}(\sigma)| = |\hat{\rho}(\sigma)| \leq \dim \rho < \dim \varphi$ and therefore, φ is σ -indefinite. Suppose $v \in V_T(a_1, \dots, a_r)$. Let $c_1, \dots, c_s \in K^*$ such that $\bar{\varphi}_{c_1}, \dots, \bar{\varphi}_{c_s}$

are the distinct non-empty residue class T_v -forms of φ . We have $s \geq 2$ so $\dim \bar{\varphi}_{c_i} < \dim \varphi$ for all i . By (3.2.5), $\hat{\bar{\varphi}}_{c_i} = \hat{\bar{\rho}}_{c_i}$ for all i . Since

$$\sum_i \dim \bar{\varphi}_{c_i} = \dim \varphi > \dim \rho \geq \sum_i \dim \bar{\rho}_{c_i},$$

$\dim \bar{\varphi}_{c_i} > \dim \bar{\rho}_{c_i}$ for some i . By induction, $\bar{\varphi}_{c_i}$ is T_v -isotropic and therefore, by (3.2.6), φ is T^v -isotropic. Thus, φ satisfies both (i) and (ii) of (3.2.8) so φ is T -isotropic. \square

3.2.12 Corollary. *If φ and ρ are T -isometric forms then $D_T(\varphi) = D_T(\rho)$.*

Proof. Let $\varphi = \langle a_1, \dots, a_r \rangle$, $\rho = \langle b_1, \dots, b_r \rangle$. Then

$$\rho \oplus \langle -a_r \rangle = \langle a_1, \dots, a_{r-1} \rangle$$

so by (3.2.11), $\rho \oplus \langle -a_r \rangle$ is T -isotropic and therefore, there exists $s_1, \dots, s_r, t \in T$, not all zero, such that $ta_r = \sum s_i b_i \in D_T(\rho)$. If $t = 0$ then ρ is T -isotropic so $D_T(\rho) = K$ by (3.1.6). In any case, $a_r \in D_T(\rho)$. Similarly, $a_i \in D_T(\rho)$ for all i so $D_T(\varphi) \subseteq D_T(\rho)$ and reversing the roles of φ and ρ , we get $D_T(\rho) \subseteq D_T(\varphi)$. \square

3.3 The space of signatures of certain preorders

Throughout this section, we assume that T satisfies the equivalent conditions of Theorem 2.3.4.

3.3.1 Lemma. *Suppose K is a field, $x \in K$. For any order $P \in \text{Sper } K$, either $x \in -P$ or*

$$(1+x)^n \in \left(1 - \frac{1}{m}\right) + P,$$

for any integer $m > 0$.

Proof. Let $P \in \text{Sper } K$ and consider the valuation ring $A(P)$. If $x \in I(P)$ then for any integer $k > 0$, $1+x = \left(1 - \frac{1}{k}\right) + \frac{1}{k}(1+kx) \in \left(1 - \frac{1}{k}\right) + P$ so $(1+x)^n \in \left(1 - \frac{1}{k}\right)^n + P$. Given $m > 0$, choose k such that $1 - \left(1 - \frac{1}{k}\right)^n < \frac{1}{m}$. If $x \notin A(P)$ then $1+x \notin A(P)$ so $\frac{1}{(1+x)^n} \in I(P)$ and therefore, $(1+x)^n - 1 = (1+x)^n \left(1 - \frac{1}{(1+x)^n}\right) \in P$. Finally, if $x \in A(P)^*$ and $x \notin -P$ then $x \in P$ and therefore, $(1+x)^n \in (1+P)^n \subseteq 1+P$. \square

3.3.2 Theorem. If $a_1, \dots, a_r \in A_T$, $r \geq 2$, and $b \in D_T\langle a_1, \dots, a_r \rangle \cap A_T$ then there exists $t \in T$, $s \in T \cap A_T$ and $x \in D_T\langle a_2, \dots, a_r \rangle \cap A_T$ such that $b(1+t) = sa_1 + x$.

Proof. By (2.3.1(i)), we can pass to the localization $(1+T)^{-1}A$ and therefore, assume $A_T = A^*$ and $A = T^* - T^*$. Scaling by b^{-1} , we may assume $b = 1$. Scaling each a_i by $1 + (\frac{2}{a_i})^n$, we may assume $a_i^n \in 2^n + T$ for all i .

Write $1 = s'a_1 + y$, where $s' \in T$ and $y \in D_T\langle a_2, \dots, a_r \rangle$. By (2.3.1(i)), there exists $s, t \in T^*$ such that $a_1 + a_2 = s - t$. Then $s = a_1 + t + a_2 = a_1 + t(s'a_1 + y) + a_2 = (1 + ts')a_1 + a_2 + ty$. Replacing a_1 by $(1 + ts')a_1$ and y by ty , we may assume

$$s = a_1 + a_2 + y$$

for some $s \in T^*$, $y \in D_T\langle a_2, \dots, a_r \rangle$. (Note we still have $a_i^n \in 2^n + T$.) Let $z := a_2 + y$. Using (2.3.4(ii)), pick $a \in A^*$ such that for any $P \in \text{Sper}_T A$, $z^n - 1 \in P$ implies $z/a \in P$.

Claim. For each T -compatible prime $\mathfrak{p} \subseteq A$,

$$(*) \quad \bar{a} \in D_{T(\mathfrak{p})}\langle 1, -\bar{a}_1 \rangle \cap D_{T(\mathfrak{p})}\langle \bar{a}_2, \dots, \bar{a}_r \rangle.$$

If $z \in \mathfrak{p}$ then both forms are $T(\mathfrak{p})$ -isotropic so $(*)$ follows from (3.1.6). If $y \in \mathfrak{p}$ then, for all orders $P \in \text{Sper}_T A$ with $\mathfrak{p} = \text{supp } P$, $z^n \in a_2^n + \mathfrak{p} \subseteq 2^n + T + \mathfrak{p} \subseteq 1 + P$ so $z/a \in P$ and therefore, $\bar{z}/\bar{a} \in T(\mathfrak{p})$. Thus $(*)$ holds in this case as well.

Assume $\bar{z} \neq 0$ and $\bar{y} \neq 0$. By (3.1.4(ii)),

$$\langle \bar{a}_2, \bar{y} \rangle \cong_{T(\mathfrak{p})} \langle \bar{z}, \alpha \bar{z} \rangle \quad \text{and} \quad \langle 1, -\bar{a}_1 \rangle \cong_{T(\mathfrak{p})} \langle \bar{s}, -\bar{a}_1 \rangle \cong_{T(\mathfrak{p})} \langle \bar{z}, \beta \bar{z} \rangle$$

for some $\alpha, \beta \in F(\mathfrak{p})^*$. Let $\sigma \in X_{T(\mathfrak{p})}$ and suppose $\sigma(\bar{a}_2) + \sigma(\bar{y}) \neq 0$. By (3.3.1), $(1 + \frac{\bar{y}}{\bar{a}_2})^n \in \frac{1}{2^n} + P_\sigma$. Then $\bar{z}^n = \bar{a}_2^n (1 + \frac{\bar{y}}{\bar{a}_2})^n \in 1 + P_\sigma$ so $\bar{z}/\bar{a} \in P_\sigma$ and therefore, $\sigma(\bar{z}) = \sigma(\bar{a})$. If $\sigma(\bar{a}_2) + \sigma(\bar{y}) = 0$ then $\sigma(\alpha) = -1$ so $\sigma(\bar{z}) + \sigma(\alpha)\sigma(\bar{z}) = \sigma(\bar{a}) + \sigma(\alpha)\sigma(\bar{a})$. Thus,

$$\langle \bar{a}_2, \bar{y} \rangle \cong_{T(\mathfrak{p})} \langle \bar{z}, \alpha \bar{z} \rangle \cong_{T(\mathfrak{p})} \langle \bar{a}, \alpha \bar{a} \rangle.$$

Similarly one shows

$$\langle 1, -\bar{a}_1 \rangle \cong_{T(\mathfrak{p})} \langle \bar{z}, \beta \bar{z} \rangle \cong_{T(\mathfrak{p})} \langle \bar{a}, \beta \bar{a} \rangle$$

which proves the claim.

Since $1 + T \subseteq A^*$, it follows from (3.1.5) that $a \in D_T\langle 1, -a_1 \rangle \cap D_T\langle a_2, \dots, a_r \rangle$. Let $s', t' \in T$ such that $a = s' - t'a_1$. Write $\frac{1-a_1}{a} = u - v$ where $u, v \in T^*$. Then $ua = 1 - a_1 + va = 1 - a_1 + v(s' - t'a_1) = (1 + vs') - (1 + vt')a_1$ so $1 = sa_1 + x$ where $s = \frac{1+vt'}{1+vs'} \in T^*$ and $x = \frac{u}{1+vs'}a \in D_T\langle a_2, \dots, a_r \rangle \cap A^*$. \square

3.3.3 Theorem. *For any $b \in D_T(\varphi \oplus \psi) \cap A_T$, there exists $t \in T$, $x \in D_T(\varphi) \cap A_T$ and $y \in D_T(\psi) \cap A_T$ such that $b(1+t) = x + y$.*

Proof. Let $\varphi = \langle a_1, \dots, a_r \rangle$. By (3.3.2), we can write $b(1+t') = s'a_1 + c$ where $t' \in T$, $s' \in T \cap A_T$ and $c \in D_T(\langle a_2, \dots, a_r \rangle \oplus \psi) \cap A_T$. If $r = 1$ we are done so assume $r \geq 2$. By induction, there exists $t'' \in T$, $u \in D_T\langle a_2, \dots, a_r \rangle$, $y' \in D_T(\psi) \cap A_T$ such that $c(1+t'') = u + y'$. Then $b(1+t')(1+t'') = s'(1+t'')a_1 + u + y' \in D_T(\varphi \oplus \langle y' \rangle) \cap A_T$. Applying (3.3.2) again, we get $t \in T$, $s \in T \cap A_T$ and $x \in D_T(\varphi) \cap A_T$ such that $b(1+t) = x + sy'$ so take $y = ty' \in D_T(\psi) \cap A_T$. \square

3.3.4 Theorem. *Let $a_1, \dots, a_r, b_1 \in A_T$. The following are equivalent:*

- (i) $b_1(1+t) \in D_T\langle a_1, \dots, a_r \rangle$ for some $t \in T$.
- (ii) *There exists $b_2, \dots, b_r \in A_T$ such that $\langle a_1, \dots, a_r \rangle \cong_T \langle b_1, \dots, b_r \rangle$.*

Proof. Suppose (i) holds. If $r = 1$ this is clear so assume $r \geq 2$. By (3.3.2), there exists $s \in T \cap A_T$, $t' \in T$, $x \in D_T\langle a_2, \dots, a_r \rangle \cap A_T$ such that $b_1(1+t') = sa_1 + x$. By induction, there exists $b_3, \dots, b_r \in A_T$ such that $\langle a_2, \dots, a_r \rangle \cong_T \langle x, b_3, \dots, b_r \rangle$. By (3.1.4), $\langle a_1, x \rangle \cong_T \langle b_1, b_2 \rangle$ where $b_2 := (sa_1)^n x + (sa_1)x^n$ and therefore,

$$\langle a_1, \dots, a_r \rangle \cong_T \langle a_1, x, b_3, \dots, b_r \rangle \cong_T \langle b_1, b_2, \dots, b_r \rangle.$$

The converse follows from (3.2.12) and (3.1.5). \square

3.3.5 Corollary. *A T -form $\varphi = \langle a_1, \dots, a_r \rangle$ is T -isotropic iff $\hat{\varphi} = \hat{\rho}$ for some T -form ρ with $\dim \rho < \dim \varphi$.*

Proof. Suppose φ is T -isotropic. By (3.1.6), $-a_1(1+t') \in D_T(\varphi)$ for some $t' \in T$. Let $s_i \in T$ be such that $-a_1(1+t') = \sum s_i a_i$ so $-a_1(1+t) \in D_T\langle a_2, \dots, a_r \rangle$ for some $t \in T$. By (3.3.4), there exists a form ρ of dimension $r-2$ such that $\varphi \cong_T \langle a_1, -a_1 \rangle \oplus \rho \cong_T \langle 1, -1 \rangle \oplus \rho$ and hence, $\hat{\varphi} = \hat{\rho}$.

Conversely, if $\hat{\varphi} = \hat{\rho}$ and $\dim \rho < \dim \varphi$ then, by (3.1.2), $\varphi \cong_T \rho \oplus m \times \langle 1, -1 \rangle$ for some integer $m > 0$. Then, by (3.2.12), $\pm 1 \in D_{T(\mathfrak{p})}(\alpha_{\mathfrak{p}}(\varphi))$ for all T -compatible primes $\mathfrak{p} \subseteq A$ and therefore, φ is T -isotropic by (3.1.6). \square

Putting it all together, we have proved the following generalization of [36, Corollary 2.3] (terminology as in [37].)

3.3.6 Theorem. *For every preorder $T \subseteq A$ satisfying the equivalent conditions of Theorem 2.3.4, the pair (X_T, G_T) is a space of signatures.*

Proof. We need only check S_0 – S_4 as given in [37]. S_0 : If $\sigma \in X_T$ then $\sigma^m \in X_T$ for all odd integers m . This is (2.1.7). S_1 : X_T is closed in $\text{Hom}(G_T, \Omega)$. This is (2.3.7(ii)) together with (2.2.5). S_2 : $\sigma(-1) = -1$ for all $\sigma \in X_T$. S_3 : If $\sigma(a) = 1$ for all $\sigma \in X_T$ then $[a] = 1$ in G_T . These are clear. S_4 : If $a \in D_T(\varphi \oplus \psi) \cap A_T$ then there exists $b \in D_T(\varphi) \cap A_T$, $c \in D_T(\psi) \cap A_T$ such that $a \in D_T\langle b, c \rangle$. This is (3.3.3) together with (3.3.4). \square

As a consequence of (3.3.3), we have the following description of the image of X_T in the character group $\text{Hom}(G_T, \Omega)$.

3.3.7 Theorem. *For $\sigma \in \text{Hom}(G_T, \Omega)$, the following are equivalent:*

- (i) $\sigma \in X_T$.
- (ii) $\sigma([-1]) = -1$ and for every $a, b, c \in A_T$ with $a = b + c$, $\sigma([b]) = 1 = \sigma([c])$ implies $\sigma([a]) = 1$.

Proof. Suppose $\sigma \in \text{Hom}(G_T, \Omega)$ satisfies (ii). Let T_σ be the smallest preorder in A containing T and the set $\{a \in A_T \mid \sigma([a]) = 1\}$. Suppose $a \in T_\sigma \cap A_T$. We show $\sigma([a]) = 1$. Write $a = t_1 a_1 + \dots + t_r a_r$ where $r \geq 1$, $t_i \in T$ and $a_i \in A_T$

such that $\sigma([a_i]) = 1$. Then $a \in D_T\langle a_1, \dots, a_r \rangle \cap A_T$. If $r = 1$ then $t_1 \in T \cap A_T$ so $\sigma([a]) = \sigma([a_1]) = 1$. Suppose $r > 1$. By (3.3.3) and induction on r , $a = b + c$ where $b, c \in A_T$ and $\sigma([b]) = \sigma([c]) = 1$ and hence, $\sigma([a]) = 1$. Thus, $T_\sigma \cap A_T = \{a \in A_T \mid \sigma([a]) = 1\}$ so $G_{T_\sigma} \cong G_T / \ker \sigma$ which is cyclic. (Note T_σ is proper since $\sigma(-1) = -1$.) By (2.3.8), there exists a T -compatible prime \mathfrak{p} such that $A_{T_\sigma} = A \setminus \mathfrak{p}$ and $G_{T_\sigma} \cong F(\mathfrak{p})^* / T_\sigma(\mathfrak{p})^*$. Since G_{T_σ} is cyclic, $T_\sigma(\mathfrak{p})$ is an order in $F(\mathfrak{p})$. Thus, the character $\bar{\sigma}$ on $F(\mathfrak{p})^*$ induced by σ is a $T(\mathfrak{p})$ -signature and $\sigma = \bar{\sigma} \circ \alpha_{\mathfrak{p}}$ in $\text{Hom}(G_T, \Omega)$. The converse is clear. \square

We also get the usual inductive description of isometry and therefore, the usual description of the Witt ring $W_T(A)$ as a quotient of the integral group ring $\mathbb{Z}[G_T]$.

3.3.8 Proposition.

- (i) $\langle a \rangle \cong_T \langle b \rangle$ iff $[a] = [b]$ in G_T .
- (ii) $\langle a, b \rangle \cong_T \langle c, d \rangle$ iff there exists $s, t \in T \cap A_T$ such that $[c] = [sa + tb]$ and $[d] = [sa(tb)^n + (sa)^n tb]$ in G_T .
- (iii) $\langle a_1, \dots, a_r \rangle \cong_T \langle b_1, \dots, b_r \rangle$ where $r \geq 3$ iff there exists $a, b, c_3, \dots, c_r \in A_T$ such that $\langle a_1, a \rangle \cong_T \langle b_1, b \rangle$, $\langle a_2, \dots, a_r \rangle \cong_T \langle a, c_3, \dots, c_r \rangle$ and $\langle b_2, \dots, b_r \rangle \cong_T \langle b, c_3, \dots, c_r \rangle$.

3.3.9 Corollary. *The kernel of the natural ring epimorphism $\mathbb{Z}[G_T] \rightarrow W_T(A)$ is additively generated by $[1] + [-1]$ and all the elements of the form*

$$[a] + [b] - [a + b] - [ab^n + a^n b]$$

where $a, b, a + b \in A_T$.

Let Y be a non-empty subset of X_T . Set

$$\Delta := \{[a] \in G_T \mid \sigma(a) = 1 \text{ for all } \sigma \in Y\}$$

and

$$\Delta^\perp := \{\chi \in \text{Hom}(G_T, \Omega) \mid \chi(\Delta) = 1\}.$$

Clearly, $Y \subseteq X_T \cap \Delta^\perp$. If $Y = X_T \cap \Delta^\perp$, we say Y is a *subspace* of X_T . We show the subspaces of X_T are precisely the subsets $X_{T'}$ where T' is a proper preorder containing T . Note this implicitly requires the natural homomorphism $G_T \rightarrow G_{T'}$ to be surjective so our hypothesis that T satisfies (2.3.4(i)) is not particularly restrictive.

3.3.10 Proposition. *Every subspace $Y \subseteq X_T$ is of the form $X_{T'}$ for some proper preorder $T' \supseteq T$. Conversely, for any proper preorder $T' \supseteq T$, $X_{T'}$ is a subspace of X_T .*

Proof. Suppose T' is a proper preorder containing T . By (2.3.7(i)), $\sigma(T' \cap A_T) = 1$ iff $T' \subseteq P_\sigma$ for any $\sigma \in X_T$. Thus,

$$\Delta = \frac{\tilde{T}' \cap A_T}{T \cap A_T}$$

for $Y = X_{T'}$ and therefore, $X_T \cap \Delta^\perp = \{\sigma \in X_T \mid \tilde{T}' \subseteq P_\sigma\} = X_{T'}$.

Conversely, for a subspace $Y \subseteq X_T$, define $T' := \cap\{P_\sigma \mid \sigma \in Y\}$. Then $\Delta = T' \cap A_T / T \cap A_T$ so $Y = X_T \cap \Delta^\perp = \{\sigma \in X_T \mid T' \subseteq P_\sigma\} = X_{T'}$. \square

We say a locally constant function $f : X_T \rightarrow \mathbb{C}$ is *represented by a form over T* if there exists a T -form φ such that $f = \hat{\varphi}$ on X_T . The following theorem characterizes $W_T(A)$ as a subring of $C(X_T, \mathbb{C})$. It is a straightforward generalization of [38, Theorem 4.4].

3.3.11 Theorem. *Suppose $f : X_T \rightarrow \mathbb{C}$ is locally constant. Then f is represented by a form over T iff*

(*) *for each T -compatible prime $\mathfrak{p} \subseteq A$, $f \circ \alpha_{\mathfrak{p}}^* : X_T(\mathfrak{p}) \rightarrow \mathbb{C}$ is represented by a form $\varphi_{\mathfrak{p}}$ over $T(\mathfrak{p})$ and $\dim \varphi_{\mathfrak{p}} \equiv \dim \varphi_{\mathfrak{q}} \pmod{2}$ for all T -compatible primes $\mathfrak{p}, \mathfrak{q}$ in A .*

Proof. Suppose $f : X_T \rightarrow \mathbb{C}$ is a locally constant function satisfying (*) and f is not represented by a form over T . Consider the family of subspaces $Y \subseteq X_T$ such that

f is not represented by a form on Y . Let $\{Y_i\}$ be a family of such subspaces linearly ordered by inclusion. Then $Y = \bigcap_i Y_i$ is a subspace of X_T . Suppose $f = \hat{\varphi}$ on Y for some T -form φ . By the continuity of $f - \hat{\varphi}$, there exists an open set $U \supseteq Y$ such that $f - \hat{\varphi} = 0$ on U . Since the Y_i are closed in X_T and X_T is compact, $U \supseteq Y_j$ for some j . But this means $f = \hat{\varphi}$ on Y_j , a contradiction. Thus, f is not represented by a form on Y either. By Zorn's Lemma, there exists some subspace $Y \subseteq X_T$ minimal with respect to the property that f is not represented on Y . By (3.3.10), $Y = X_{T'}$ for some proper preorder $T' \supsetneq T$. Replacing T by T' , we may assume f is not represented by a form over T but f is represented by a form on every proper subspace of X_T . (Since $X_T = X_{\tilde{T}}$, we may assume $T = \tilde{T}$.)

It follows from (2.3.8) that T is not complete, that is, the 2-primary part of G_T is not cyclic. Therefore, G_T has at least two elements of order 2 (see the claim in the proof of (1.5.5).) Let $a \in A_T$ such that $[a]^2 = [1]$ but $[a] \neq [\pm 1]$. Then $a^2 \in T$ but $as \notin T \cup -T$ for all $s \in T \cap A_T$. It follows that $T + aT$, $T - aT$ are both proper preorders properly containing T so f is represented by a form ψ_1 on $X_{T+aT} = \{\sigma \in X_T \mid \sigma(a) = 1\}$ and by a form ψ_2 on $X_{T-aT} = \{\sigma \in X_T \mid \sigma(a) = -1\}$. On X_T , we have $f \cdot (1 + \hat{a}) = \hat{\psi}_1 \cdot (1 + \hat{a})$ and $f \cdot (1 - \hat{a}) = \hat{\psi}_2 \cdot (1 - \hat{a})$ so $2f$ is represented by the form $\rho := \psi_1 \otimes \langle 1, a \rangle \oplus \psi_2 \otimes \langle 1, -a \rangle$ over T .

Let $\mathfrak{p} \subseteq A$ be a T -compatible prime. Since $f \circ \alpha_{\mathfrak{p}}^* = \hat{\varphi}_{\mathfrak{p}}$ on $X_{T(\mathfrak{p})}$, $\widehat{\alpha_{\mathfrak{p}}(\rho)} = \widehat{2 \times \varphi_{\mathfrak{p}}}$ on $X_{T(\mathfrak{p})}$. By (3.3.5), we may assume $\varphi_{\mathfrak{p}}$ is $T(\mathfrak{p})$ -anisotropic by replacing it, if necessary, by a lower dimensional form. Thus, there exists an integer $m_{\mathfrak{p}} \geq 0$ such that

$$\alpha_{\mathfrak{p}}(\rho) \cong_{T(\mathfrak{p})} 2 \times \varphi_{\mathfrak{p}} \oplus m_{\mathfrak{p}} \times \langle 1, -1 \rangle.$$

By (*), $m_{\mathfrak{p}} \equiv m_{\mathfrak{q}} \pmod{2}$ for all T -compatible primes $\mathfrak{p}, \mathfrak{q}$. If $m_{\mathfrak{p}}$ is odd then $\alpha_{\mathfrak{p}}(\rho)$ is $T(\mathfrak{p})$ -isotropic for all T -compatible primes \mathfrak{p} so by (3.1.6), ρ is T -isotropic. By (3.3.5), we may replace ρ by a lower dimensional T -anisotropic form and hence, assume $m_{\mathfrak{p}}$ is even for all T -compatible primes \mathfrak{p} . Therefore, there exists $s_{\mathfrak{p}} \geq 0$ such that

$$\alpha_{\mathfrak{p}}(\rho) \cong_{T(\mathfrak{p})} 2 \times (\varphi_{\mathfrak{p}} \oplus s_{\mathfrak{p}} \times \langle 1, -1 \rangle).$$

Suppose $b_1 \in D_T(\rho) \cap A_T$. By (3.3.4),

$$\rho \cong_T \langle b_1 \rangle \oplus \rho'$$

for some T -form ρ' . Let \mathfrak{p} be a T -compatible prime. Then

$$\alpha_{\mathfrak{p}}(\rho) \cong_{T(\mathfrak{p})} \langle \alpha_{\mathfrak{p}}(b_1) \rangle \oplus \alpha_{\mathfrak{p}}(\rho').$$

Since $\alpha_{\mathfrak{p}}(b_1) \in D_{T(\mathfrak{p})}(\alpha_{\mathfrak{p}}(\rho)) = D_{T(\mathfrak{p})}(\varphi_{\mathfrak{p}} \oplus s_{\mathfrak{p}} \times \langle 1, -1 \rangle)$,

$$\alpha_{\mathfrak{p}}(\rho) \cong_{T(\mathfrak{p})} 2 \times (\varphi_{\mathfrak{p}} \oplus s_{\mathfrak{p}} \times \langle 1, -1 \rangle) \cong_{T(\mathfrak{p})} \langle \alpha_{\mathfrak{p}}(b_1), \dots \rangle \oplus \langle \alpha_{\mathfrak{p}}(b_1), \dots \rangle.$$

Therefore, $\alpha_{\mathfrak{p}}(b_1) \in D_{T(\mathfrak{p})}(\alpha_{\mathfrak{p}}(\rho'))$ for every T -compatible prime \mathfrak{p} . By (3.1.5), $b_1(1+t) \in D_T(\rho')$ for some $t \in T$ and hence,

$$\rho \cong_T \langle b_1 \rangle \oplus \rho' \cong_T \langle b_1, b_1 \rangle \oplus \rho''$$

for some T -form ρ'' . Thus, $2(f - \hat{b}_1)$ is represented by ρ'' on X_T . Repeating this argument, we eventually obtain $b_1, \dots, b_r \in A_T$ such that $2(f - (\hat{b}_1 + \dots + \hat{b}_r)) = 0$ on X_T and therefore, f is represented by the form $\langle b_1, \dots, b_r \rangle$ over T , a contradiction. \square

3.3.12 Remark. To determine whether the function $f \circ \alpha_{\mathfrak{p}}^* : X_{T(\mathfrak{p})} \rightarrow \mathbb{C}$ is represented over $T(\mathfrak{p})$, one can use the valuation-theoretic criteria in [11] or [12].

Chapter 4

The Higher Level Real Spectrum

As already seen in chapter 1, the theory of higher level orders on a commutative ring A is remarkably similar to Coste and Roy's theory of the real spectrum in [20]. In this chapter, we continue the process of generalizing this theory to higher level by appropriately defining the Tychonoff, Harrison and Zariski topologies for $\text{Sper}_T A$. We then consider the constructible (Tychonoff clopen) sets in $\text{Sper}_T A$. Justifying the term "constructible", we show that there is indeed a sub-base for the Harrison topology for which the constructible sets are precisely the sets obtained from this sub-base using a finite number of Boolean operations. In the last section, we generalize the characterizations of basic sets given in [16] and [35] for level 1 to arbitrary 2-primary level.

The question of whether Bröcker's theory of the complexity of constructible sets in the level 1 real spectrum can be successfully generalized to higher level remains open. Some preliminary results have been obtained jointly with Marshall that suggest this may be possible. Whether this will have any application to real algebraic geometry remains to be seen.

4.1 Topologies on $\text{Sper}_T A$

Let A be a commutative ring, $T \subseteq A$ a proper preorder. $\text{Sper}_T A$ is naturally identified with a subset of the product space $\{0,1\}^A$ by the map that sends an order P to its characteristic function. The induced subspace topology is called the

Tychonoff topology on $\text{Sper}_T A$. The sets of the form

$$\mathcal{W}_T(a) := \{P \in \text{Sper}_T A \mid a \in P\}, \quad a \in A$$

together with their complements $\mathcal{W}_T^c(a)$ form a sub-base for this topology.

4.1.1 Theorem. *The map $\text{Sig}_T A \rightarrow \text{Sper}_T A$ given by $\sigma \mapsto P_\sigma$ is closed. $\text{Sper}_T A$ with the Tychonoff topology is compact, Hausdorff and totally disconnected.*

Proof. Immediate from (2.2.1). \square

We consider a second topology on $\text{Sper}_T A$: the *Harrison topology*. It is defined by taking as a sub-base the sets $\mathcal{W}_T^c(a)$ together with the sets

$$\mathcal{U}_T(a) := \{P \in \text{Sper}_T A \mid a \in P \setminus \text{supp } P\}, \quad a \in A.$$

Since $\mathcal{U}_T(a) = \mathcal{W}_T(a) \cap \mathcal{W}_T^c(-a)$, the Harrison topology is coarser than the Tychonoff topology so $\text{Sper}_T A$ is also compact in the Harrison topology. Of course, if A is a field then the Harrison and the Tychonoff topologies coincide. Unless otherwise stated, the topology on $\text{Sper}_T A$ will be assumed to be the Harrison topology.

For any subset $S \subseteq \text{Sper}_T A$, \bar{S} denotes the closure of S in the Harrison topology.

4.1.2 Theorem. *Let $P, Q \in \text{Sper}_T A$. Then $Q \in \overline{\{P\}}$ iff $P \prec Q$. The closed points in $\text{Sper}_T A$ are precisely the maximal orders.*

Proof. Suppose $Q \in \overline{\{P\}}$. If $a \in P$ then $P \notin \mathcal{W}_T^c(a)$ so $Q \notin \mathcal{W}_T^c(a)$. Thus, $P \subseteq Q$. If $a \in Q \setminus \text{supp } Q$ then $Q \in \mathcal{U}_T(a)$ so $P \in \mathcal{U}_T(a)$ and hence, $a \in P$. Therefore, $P \prec Q$. The converse is clear. \square

4.1.3 Theorem. *If $S \subseteq \text{Sper}_T A$ is Tychonoff closed then*

$$\bar{S} = \{Q \in \text{Sper}_T A \mid P \prec Q \text{ for some } P \in S\}.$$

Proof. Suppose $Q \in \bar{S}$. Since S is compact,

$$\bigcap_{a \in Q \setminus \text{supp } Q} \mathcal{U}_T(a) \cap \bigcap_{b \notin Q} \mathcal{W}_T^c(b) \cap S \neq \emptyset.$$

Thus, there exists $P \in S$ such that $P \subseteq Q$ and $Q \setminus \text{supp } Q \subseteq P$, that is, $P \prec Q$. The reverse inclusion is clear. \square

4.1.4 Proposition. *For $P, Q \in \text{Sper}_T A$, the following are equivalent:*

- (i) $P \not\prec Q$ and $Q \not\prec P$.
- (ii) *There exists disjoint (sub-basic) open sets U, V in $\text{Sper}_T A$ such that $P \in U$ and $Q \in V$.*

Proof. (i) \Rightarrow (ii) Without loss of generality, assume $P \not\subseteq Q$. Fix $a \in P \setminus Q$. If $a \notin \text{supp } P$ then $P \in \mathcal{U}_T(a)$ and $Q \in \mathcal{W}_T^c(a)$ so assume $P \setminus Q \subseteq \text{supp } P$. Since $Q \not\prec P$, $Q \not\subseteq P$. Pick $b \in Q \setminus P$. If $b \notin \text{supp } Q$ then $Q \in \mathcal{U}_T(b)$ and $P \in \mathcal{W}_T^c(b)$. Otherwise, $Q \in \mathcal{U}_T(a^n - b^n)$ and $P \in \mathcal{W}_T^c(a^n - b^n)$.

(ii) \Rightarrow (i) follows from (4.1.2). \square

4.1.5 Corollary. *If $C, D \subseteq \text{Sper}_T A$ are disjoint closed sets then there exists disjoint open sets U, V in $\text{Sper}_T A$ such that $C \subseteq U$ and $D \subseteq V$.*

4.1.6 Theorem. *$\text{Spermax}_T A$ is compact and Hausdorff. The specialization map $\mu : \text{Sper}_T A \rightarrow \text{Spermax}_T A$ is a closed mapping.*

Proof. Suppose $P_0 \in \text{Sper}_T A$, $Q_0 = \mu(P_0)$. Let U_0 be an open neighborhood of Q_0 and set $C = \text{Sper}_T A \setminus U_0$. Since Q_0 is maximal, $\{Q_0\}$ is closed. Therefore, by (4.1.5), there exists disjoint open sets U, V such that $Q_0 \in U$ and $C \subseteq V$. By (4.1.2), we must have $P_0 \in U$ and $\mu(U) \subseteq U_0$ so μ is continuous. $\text{Spermax}_T A$ is Hausdorff by (4.1.4). It follows that $\text{Spermax}_T A$ is compact and μ is a closed mapping. \square

We also have the *Zariski topology* on $\text{Sper}_T A$. For $a \in A$, define

$$\mathcal{Z}_T(a) := \{P \in \text{Sper}_T A \mid a \in \text{supp } P\}.$$

The sets $\text{Sper}_T A \setminus \mathcal{Z}_T(a) = \mathcal{U}_T(a^n)$, $a \in A$, form a basis for this topology and closed sets have the form $\mathcal{Z}_T(\mathfrak{a}) := \{P \in \text{Sper}_T A \mid \mathfrak{a} \subseteq \text{supp } P\}$, where \mathfrak{a} is an ideal of A .

For $S \subseteq \text{Sper}_T A$, the Zariski-closure of S is denoted by $\text{z-cl}(S)$. Clearly,

$$\text{z-cl}(S) = \mathcal{Z}_T \left(\bigcap_{P \in S} \text{supp } P \right).$$

4.1.7 Theorem. *If $S \subseteq \text{Sper}_T A$ is Tychonoff closed then any minimal prime \mathfrak{p} lying over the ideal $\bigcap \{\text{supp } P \mid P \in S\}$ is of the form $\mathfrak{p} = \text{supp } P$ for some $P \in S$. Thus,*

$$\text{z-cl}(S) = \{Q \in \text{Sper}_T A \mid \text{supp } Q \supseteq \text{supp } P \text{ for some } P \in S\}.$$

Proof. Let \mathfrak{p} be a minimal prime lying over $\mathfrak{a} := \bigcap_{P \in S} \text{supp } P$. If $a \in A \setminus \mathfrak{p}$ then $a \notin \mathfrak{a}$ so $S \cap \mathcal{U}_T(a^n) \neq \emptyset$. By compactness,

$$\bigcap_{a \in A \setminus \mathfrak{p}} \mathcal{U}_T(a^n) \cap S \neq \emptyset.$$

Any P in this intersection satisfies $\mathfrak{a} \subseteq \text{supp } P \subseteq \mathfrak{p}$ so $\mathfrak{p} = \text{supp } P$ by the minimality of \mathfrak{p} . \square

4.2 Constructible sets

Let A be a commutative ring, $T \subseteq A$ a proper preorder. A subset $S \subseteq \text{Sper}_T A$ is called *constructible* if it is clopen (closed and open) in the Tychonoff topology. For all $a, a_1, \dots, a_r \in A$ and all positive integers m, m_1, \dots, m_r , define

$$\mathcal{U}_T(a; m) := \mathcal{U}_T(a^m) \cap \bigcap_{\substack{d \mid m \\ d \neq m}} \mathcal{W}_T^c(a^d)$$

and

$$\mathcal{U}_T(a_1, \dots, a_r; m_1, \dots, m_r) := \bigcap_i \mathcal{U}_T(a_i; m_i).$$

These sets form a basis for the Harrison topology on $\text{Sper}_T A$ and are called *basic open sets*. Similarly, we define

$$\mathcal{W}_T(a; m) := \mathcal{W}_T(a^m) \cap \bigcap_{\substack{d \mid m \\ d \neq m}} \text{Sper}_T A \setminus \mathcal{U}_T(a^d) = \mathcal{U}_T(a; m) \cup \mathcal{Z}_T(a)$$

and

$$\mathcal{W}_T(a_1, \dots, a_r; m_1, \dots, m_r) := \bigcap_i \mathcal{W}_T(a_i; m_i)$$

for $a, a_1, \dots, a_r \in A$, m, m_1, \dots, m_r positive integers. These sets are called *basic closed sets*. A set $S \subseteq \text{Sper}_T A$ is said to be *basic* if

$$S = \mathcal{W}_T(a_1, \dots, a_r; m_1, \dots, m_r) \cap \mathcal{U}_T(b_1, \dots, b_s; n_1, \dots, n_s)$$

where $a_1, \dots, a_r, b_1, \dots, b_s \in A$ and $m_1, \dots, m_r, n_1, \dots, n_s$ are positive integers. The basic sets form a basis for the Tychonoff topology on $\text{Sper}_T A$.

4.2.1 Remarks.

- (i) Note that for any $a \in A$, $m > 0$, $\mathcal{U}_T(a; m)$ is the set of orders $P \in \text{Sper}_T A$ such that $a \notin \mathfrak{p}$ and $\bar{a}P(\mathfrak{p})^*$ has order m in $F(\mathfrak{p})^*/P(\mathfrak{p})^*$ where $\mathfrak{p} = \text{supp } P$. For even m

$$\mathcal{U}_T(a; m) = \mathcal{U}_T(-a^{\frac{m}{2}}; 1)$$

so we may always assume m is an odd integer.

- (ii) If T is a 2-primary preorder then $\mathcal{U}_T(a; m) = \emptyset$ for all odd $m > 1$. Thus, in the 2-primary case we may always take $m = 1$.

4.2.2 Theorem.

- (i) Any open constructible set is a finite union of basic open sets.
(ii) Any closed constructible set is a finite union of basic closed sets.
(iii) Any constructible set is a finite union of sets of the form

$$\mathcal{U}_T(a_1, \dots, a_r; m_1, \dots, m_r) \cap \mathcal{Z}_T(b)$$

where $a_1, \dots, a_r, b \in A$ and m_1, \dots, m_r are positive integers.

Proof. (i) follows from the compactness of constructible sets. For (ii), note that

$$\text{Sper}_T A \setminus \mathcal{U}_T(a; m) = \bigcup_{\substack{d|n \\ d \neq m}} \mathcal{W}_T(a; d)$$

for any $a \in A$ and integer $m > 0$. Since

$$\mathcal{W}_T(a_1, \dots, a_r; m_1, \dots, m_r) = \mathcal{U}_T(a_1, \dots, a_r; m_1, \dots, m_r) \cup \mathcal{Z}_T(a_1 \dots a_r),$$

(iii) is clear from the compactness of constructible sets. \square

4.3 Hormander-Łojasiewicz Inequality and characterizing basic sets

Let A be a commutative ring, $T \subseteq A$ a proper 2-primary preorder. By (4.2.1(ii)), the sets of the form

$$\mathcal{U}_T(a_1, \dots, a_r) := \mathcal{U}_T(a_1) \cap \dots \cap \mathcal{U}_T(a_r)$$

are the basic open sets in $\text{Sper}_T A$ and the sets of the form

$$\mathcal{W}_T(a_1, \dots, a_r) := \mathcal{W}_T(a_1) \cap \dots \cap \mathcal{W}_T(a_r)$$

are the basic closed sets in $\text{Sper}_T A$. Note that any basic set $\mathcal{U}_T(a_1, \dots, a_r) \cap \mathcal{W}_T(b_1, \dots, b_s)$ can be expressed as

$$\mathcal{U}_T(a_1^n \dots a_r^n) \cap \mathcal{W}_T(a_1, \dots, a_r, b_1, \dots, b_s).$$

If $a_1, \dots, a_r \in A$ denote by $T[a_1, \dots, a_r]$ the smallest preorder in A containing T and a_1, \dots, a_r . Then $\mathcal{W}_T(a_1, \dots, a_r) = \text{Sper}_{T[a_1, \dots, a_r]} A$ so any closed constructible set in $\text{Sper}_T A$ is a finite union of sets of the form $\text{Sper}_{T'} A$ where T' is a preorder lying over T .

We extend the characterizations of basic sets given in [16] and [35] to 2-primary preorders. Just as for level 1, we use an abstract version of the Hormander-Łojasiewicz Inequality for semi-algebraic functions [15, Corollaire 2.6.7]. (See also [1], [16], [35].)

4.3.1 Theorem. *Suppose $S \subseteq \text{Sper}_T A$ is a closed constructible set, $f, g \in A$ such that $S \cap \mathcal{Z}_T(g) \subseteq \mathcal{Z}_T(f)$. Then there exists $a \in T$, $m \geq 0$ such that for all $P \in S$,*

$$ag^n + f^{nm+1} \in P.$$

Proof. Write $S = \text{Sper}_{T_1} A \cup \dots \cup \text{Sper}_{T_r} A$ where T_1, \dots, T_r are preorders lying over T . Applying the Nullstellensatz to the preorder $\overline{T}_i \subseteq A/(g^n)$ induced by T_i , we have $-\overline{f}^{nm_i} \in \overline{T}_i$ for some $m_i \geq 0$. Multiplying by suitable powers of \overline{f}^n , we may assume $m := m_1 = \dots = m_r$. Then for each i , there exists $s_i \in T_i$, $a_i \in A$ such that

$$-f^{nm} = s_i - a_i g^n.$$

By (1.1.3), there exists $s, t, p_i, q_i \in T^e$ such that $f = s - t$ and $a_i = p_i - q_i$ for each i . Let $a' = t \sum p_j \in T^e$. Then

$$\begin{aligned} a'g^n + f^{nm+1} &= t \left(\sum p_j g^n - f^{nm} \right) + f^{nm}(t + f) \\ &= t \left[\left(\sum p_j - a_i \right) g^n + a_i g^n - f^{nm} \right] + s f^{nm} \\ &= t \left[\left(\sum_{j \neq i} p_j + q_i \right) g^n + s_i \right] + s f^{nm} \end{aligned}$$

which is an element of T_i^e for each i . Let $c > 0$ be an integer such that $a := ca' \in T$. Then $ag^n + f^{nm+1} = (c-1)a'g^n + a'g^n + f^{nm+1} \in T_i^e$ for each i . For any $P \in S$, $T_i^e \subseteq P^e = P$ for some i and therefore, $ag^n + f^{nm+1} \in P$. \square

4.3.2 Proposition. *Suppose $S \subseteq \text{Sper}_T A$ is a closed constructible set, $f, g \in A$ such that $S \cap \mathcal{Z}_T(g) \subseteq \mathcal{W}_T(f)$. Then there exists $f_1 \in A$ such that $S \subseteq \mathcal{W}_T(f_1)$ and $\mathcal{W}_T(f) \cap \mathcal{Z}_T(g) = \mathcal{W}_T(f_1) \cap \mathcal{Z}_T(g)$.*

Proof. Let $S' = S \setminus \mathcal{U}_T(f)$. Then S' is a closed constructible set such that $S' \cap \mathcal{Z}_T(g) \subseteq \mathcal{Z}_T(f)$. By (4.3.1), there exists $a \in T$, $m \geq 0$ such that $f_1 := ag^n + f^{nm+1} \in P$ for all $P \in S'$. Clearly if $P \in \mathcal{U}_T(f)$ then $f_1 \in P$ so $S \subseteq \mathcal{W}_T(f_1)$. The remaining statement is clear. \square

If \mathfrak{p} is a prime ideal of A we identify $\text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$ with the set of orders in $\text{Sper}_T A$ with support \mathfrak{p} . For any set $S \subseteq \text{Sper}_T A$ we define

$$S(\mathfrak{p}) := S \cap \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p}).$$

4.3.3 Proposition. *Suppose $S \subseteq \text{Sper}_T A$ is a closed constructible set, $\mathfrak{p} \subseteq A$ is a prime ideal and $f \in A$ such that $S(\mathfrak{p}) \subseteq \mathcal{W}_T(f)$. Then there exists $f_1 \in A$ such that $S \subseteq \mathcal{W}_T(f_1)$ and $\mathcal{W}_T(f) \cap \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p}) = \mathcal{W}_T(f_1) \cap \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$.*

Proof. Note that

$$S(\mathfrak{p}) = S \cap \text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p}) = S \cap \bigcap_{g \in \mathfrak{p}} \mathcal{Z}_T(g) \cap \bigcap_{h \notin \mathfrak{p}} \mathcal{U}_T(h^n).$$

Since $S(\mathfrak{p}) \subseteq \mathcal{W}_T(f)$, by compactness, there exists $g \in \mathfrak{p}$, $h \notin \mathfrak{p}$ such that $S \cap \mathcal{Z}_T(g) \cap \mathcal{U}_T(h^n) \subseteq \mathcal{W}_T(f)$. Replacing f by fh^n we may assume $h = 1$. Now apply (4.3.2). \square

4.3.4 Theorem. *For any constructible set $S \subseteq \text{Sper}_T A$, the following are equivalent:*

- (i) S is basic closed in $\text{Sper}_T A$.
- (ii) S is basic and closed in $\text{Sper}_T A$.
- (iii) S is closed in $\text{Sper}_T A$ and $S(\mathfrak{p})$ is basic in $\text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$ for each prime $\mathfrak{p} \subseteq A$.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) Suppose $P \in \text{Sper}_T A$, $P \notin S$. Let $\mathfrak{p} = \text{supp } P$. By (iii), there exists $f \in A$ such that $S(\mathfrak{p}) \subseteq \mathcal{W}_T(f)$ and $f \notin P$. By (4.3.3), there exists $f_1 \in A$ with $S \subseteq \mathcal{W}_T(f_1)$ and $f_1 \notin P$. Thus $S = \bigcap \mathcal{W}_T(f)$ where f runs through all elements $f \in A$ such that $S \subseteq \mathcal{W}_T(f)$. By compactness, there exists $f_1, \dots, f_r \in A$ such that $S = \mathcal{W}_T(f_1, \dots, f_r)$. \square

4.3.5 Theorem. *For any constructible set $S \subseteq \text{Sper}_T A$, the following are equivalent:*

- (i) S is basic in $\text{Sper}_T A$.
- (ii) $S \cap \text{z-cl}(\overline{S} \setminus S) = \emptyset$ and $S(\mathfrak{p})$ is basic in $\text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$ for each prime $\mathfrak{p} \subseteq A$.

Proof. (i) \Rightarrow (ii) Write $S = \mathcal{U}_T(a^n) \cap \mathcal{W}_T(c_1, \dots, c_k)$. If $P \in \overline{S} \setminus S$ then $P \notin \mathcal{U}_T(a^n)$. Thus, $\overline{S} \setminus S \subseteq \mathcal{Z}_T(a)$ so $S \cap \text{z-cl}(\overline{S} \setminus S) \subseteq S \cap \mathcal{Z}_T(a)$ and $S \cap \mathcal{Z}_T(a) = \emptyset$. The remaining assertion is clear.

(ii) \Rightarrow (i) Set $\mathfrak{a} = \bigcap_{P \in \overline{S} \setminus S} \text{supp } P$. Then $\text{z-cl}(\overline{S} \setminus S) = \mathcal{Z}_T(\mathfrak{a}) = \bigcap_{a \in \mathfrak{a}} \mathcal{Z}_T(a)$. By compactness, there exists $a_1, \dots, a_n \in \mathfrak{a}$ such that $\bigcap_i \mathcal{Z}_T(a_i) \cap S = S \cap \text{z-cl}(\overline{S} \setminus S) = \emptyset$. Set $a = \sum a_i^n$. Then $\mathcal{Z}_T(a) \cap S = \emptyset$.

Consider the preorder $T[1/a^n] \subseteq A[1/a]$ induced by T . We have $S \subseteq \mathcal{U}_T(a^n)$ so we can identify S with a closed set in $\text{Sper}_{T[1/a^n]} A[1/a]$. By (4.3.4), there exists

$c'_1, \dots, c'_k \in A[1/a]$ such that S is identified with $\mathcal{W}_{T[1/a^n]}(c'_1, \dots, c'_k)$. After clearing denominators, we get $S = \mathcal{U}_T(a^n) \cap \mathcal{W}_T(c_1, \dots, c_k)$ for some $c_1, \dots, c_k \in A$. \square

4.3.6 Theorem. *For any constructible set $S \subseteq \text{Sper}_T A$, the following are equivalent:*

- (i) S is basic open in $\text{Sper}_T A$.
- (ii) S is basic and open in $\text{Sper}_T A$.
- (iii) S is open in $\text{Sper}_T A$, $S \cap \text{z-cl}(\overline{S} \setminus S) = \emptyset$ and $S(\mathfrak{p})$ is basic in $\text{Sper}_{T(\mathfrak{p})} F(\mathfrak{p})$ for each prime $\mathfrak{p} \subseteq A$.

Proof. (i) \Rightarrow (ii) is clear and (ii) \Leftrightarrow (iii) follows from (4.3.5).

(ii) \Rightarrow (i) By compactness, it suffices to show for each $P \in \text{Sper}_T A \setminus S$, there exists $a \in A$ such that $S \subseteq \mathcal{U}_T(a)$ and $P \notin \mathcal{U}_T(a)$. Since S is basic, we can write $S = \mathcal{U}_T(b^n) \cap \mathcal{W}_T(c_1, \dots, c_k)$ for some $b, c_1, \dots, c_k \in A$. If $P \notin \mathcal{U}_T(b^n)$, take $a = b^n$ so assume $P \in \mathcal{U}_T(b^n)$.

Consider the localization $A[1/b]$ and the preorder $T[1/b^n] \subseteq A[1/b]$. $S = \mathcal{W}_{T[1/b^n]}(c_1, \dots, c_k)$ is a clopen set in $\text{Sper}_{T[1/b^n]} A[1/b]$ and $P \in \text{Sper}_{T[1/b^n]} A[1/b] \setminus S$ since $\text{Sper}_{T[1/b^n]} A[1/b] = \mathcal{U}_T(b^n)$. Let $Q \in \text{Sper}_{T[1/b^n]} A[1/b]$ be the unique maximal order specializing P . Since $\text{Sper}_{T[1/b^n]} A[1/b] \setminus S$ is closed, $Q \notin S$. Then $c_i \notin Q$ for some i . Since Q is 2-primary, $-c_i^m \in Q \setminus \text{supp } Q$ for some $m \geq 1$. By the Positivstellensatz, there exists $s, t \in Q$ such that

$$-c_i^m(1+s) = 1+t.$$

Let $a' := 1 + 2c_i^m(1+s)^n = -[2(1+s)^{n-1}(1+t) - 1]$. Clearly $S \subseteq \mathcal{U}_{T[1/b^n]}(a')$ and $Q \in \mathcal{U}_{T[1/b^n]}(-a')$. Since $P \prec Q$, $P \in \mathcal{U}_{T[1/b^n]}(-a')$. Let $a \in A$, $k \geq 0$ such that $a' = a/b^{nk}$. Then $S \subseteq \mathcal{U}_T(a)$ and $P \in \mathcal{U}_T(-a)$. \square

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