SIR Infectious Disease Modelling with Vaccination

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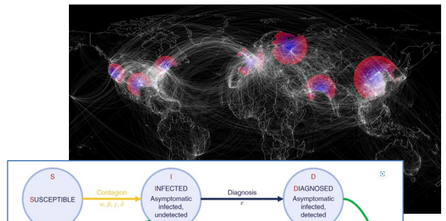
Susceptible-Infected-Recovered (SIR) Models: Applications

- Infectious diseases, in particular, pandemics, can have severe impact on human health and society
  - Acute respiratory syndrome (SARS)
  - Avian influenza (H5N1)
  - Swine flu (H1N1)
  - Coronavirus (COVID-19)
- Research on the prevention and control of infectious diseases is extremely important
- Mathematical models can reveal essential characteristics of infectious diseases



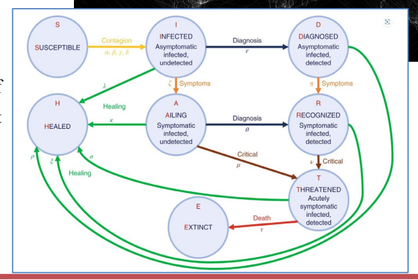


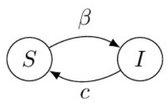
Figure on the right: An advanced SIR-type model diagram for COVID-19 (Ref. [2])

SI Model

- **Parameters:** $S(t)$ is the number of susceptible individuals at time t
  $I(t)$ is the number of infected individuals at time t
- **Assumption:** N is the fixed constant total population and we do not consider the birth rate and death rate of the population
  The basic model assumes no incubation period, no cure, and no immunity
- **Model equations:**

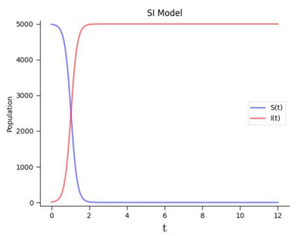
$$\frac{dS}{dt} = -\beta IS + cI$$

$$\frac{dI}{dt} = \beta IS - cI$$



- Independent variable time t is measured in months in months. β is the **infection rate**, c is the **recovery rate**
- Use $S = N - I$ to simplify and get a single differential equation on $I(t)$:

$$\frac{dI}{dt} = \beta I(N - I) - cI$$



- **Typical dynamics:** as the number of infected people increases, the number of susceptible people decreases until everyone gets infected. One can predict the climax of of infectious diseases.
- **Two equilibria:**
  - $S = N$, $I = 0$ (no disease at all): unstable
  - $S = c/\beta$, $I = N - c/\beta$ (endemic state): stable
- **An SI model computation example:** $N = 5000$ $I(0) = 10$
  $S(0) = N - I(0)$ $\beta = 0.2$
  after $t = 12$ months

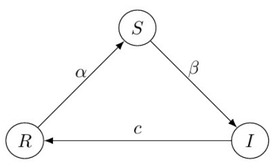
SIR Model

- **Compartments:** $S(t)$ is the number of susceptible individuals at time t
  $I(t)$ is the number of infected individuals at time t
  $R(t)$ is the number of recovered individuals at time t
- **Assumption:** N is the fixed constant total population and we do not consider the birth rate and death rate of the population
  There is no incubation period, no cure, and no immunity
  Recovered people can then become susceptible again and be infected
- Independent variable time t is measured in months. β is the **infection rate**, c is the **recovery rate** and α is the **infection descent rate**
- **Model equations:**

$$\frac{dS}{dt} = -\beta IS + \alpha R$$

$$\frac{dI}{dt} = \beta IS - cI$$

$$\frac{dR}{dt} = cI - \alpha R$$



- Use $R = N - I - S$ to simplify the equation and get:

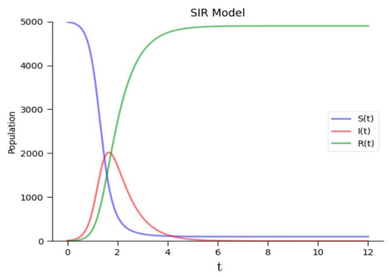
$$\frac{dS}{dt} = -\beta IS + \alpha(N - S - I)$$

$$\frac{dI}{dt} = \beta IS - cI$$

SIR Model: Equilibrium & Stability Analysis

- **SIR typical dynamics:**
  $N = 5000$, $R(0) = 0$, $I(0) = 10$
  $S(0) = N - I(0)$, $\beta = 0.2$
  $c = 0.05$ after $t = 12$ months
- **Two equilibria:**
  - $S = N$, $I = 0$ (disease free equilibrium)
  - $S = c/\beta$, $I = \alpha(\beta N\text{-}c)/\beta(c+a)$ (endemic state)
- To analyze the system, Jacobian matrix and eigenvalues are calculated

$$J(N,0) = \begin{bmatrix} -\alpha & -\beta N - \alpha \\ 0 & \beta N - c \end{bmatrix}$$



- $\lambda_1 = -\alpha$
- $\lambda_2 = N\beta - c$
- $(N, 0)$ is **stable** if eigenvalues all have negative real part, so this point is stable if

$$\beta N - c < 0$$

$$\beta N / c < 1$$

- For **endemic state**: $\begin{bmatrix} -\beta I - \alpha & -\beta S - \alpha \\ \beta I & \beta S - c \end{bmatrix}$
  - characteristic polynomial: $\lambda^2 + (-N\beta + \alpha + c)\lambda - (N\beta - c)\alpha = 0$

Think of this as: $\lambda^2 + C_1\lambda - C_0\alpha = 0$

$C_1 > 0$, so the sign of C_0 is not fixed. Now consider $\lambda = 1/2(-C_1 \pm \sqrt{C_1^2 - 4C_0})$

- If $C_0 > 0$, the real parts of both eigenvalues are negative
- If $C_0 < 0$, then the quantity under the square root is greater than C_1^2 , eigenvalues is positive
- The equilibrium $(c/\beta, \alpha(\beta N\text{-}c)/\beta(c+a))$ is stable if $\beta N\text{-}c > 0$

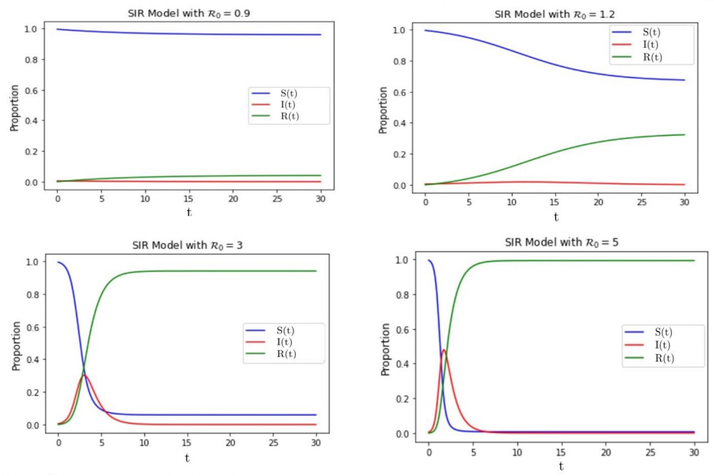
Summary:

	(N,0)	$\left(\frac{c}{\beta}, \frac{\alpha(\beta N - c)}{\beta(c+a)}\right)$
Case 1: $\beta N/c > 1$	stable	unstable
Case 2: $\beta N/c < 1$	unstable	stable

- Disease becomes endemic if morality is relatively smaller than transmission.
- Two equilibrium points exchange stability when they pass through each other, this is a **transcritical bifurcation**.

SIR Model: Basic Reproductive Ratio

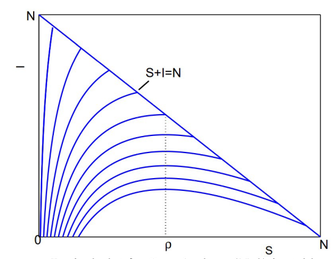
- The behavior of the SIR system depends the **reproductive ratio** $R_0 = \beta N/c$
- R_0 is an estimate of the number of individuals that are affected by a primary infective individual. If $R_0 > 1$, the infectious disease will develop into an epidemic
- **SIR simulations with different R_0 values:** All other parameters that are fixed except R_0



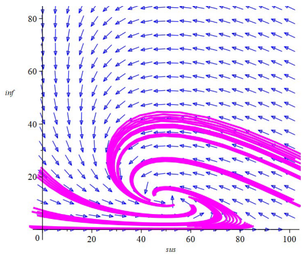
- When $R_0 < 1$, the speed and scale of spread of the disease are limited
- When $R_0 > 1$, the speed and scale of spread of the disease will rise quick
- When $R_0 = 3$, more than 90% of the population will get infected (and later recovered)
- When $R_0 = 5$, almost everyone will get infected (and subsequently recovered)

SIR Model: Phase Trajectories and Phase Portrait

- Phase portrait is a useful tool in studying dynamical systems



Typical plot for Case 1 where $(N, 0)$ is stable

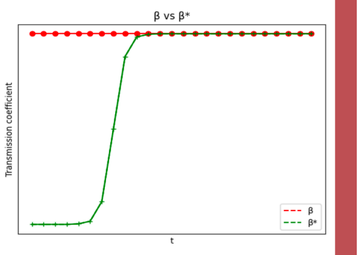


Typical plot for Case 2 (endemic state)

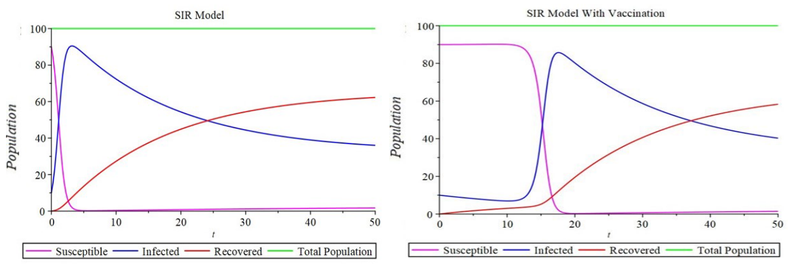
SIR Model with Vaccination

- **Variable infection rate** that accounts for vaccination is added to the model
- **Assumption:**
  - The total population $N = 100$
  - $R(0) = 0$ $I(0) = 10$
  - $S(0) = N - I(0)$ $\beta = 0.2$
  - $c = 0.05$
  - Instead of β = constant for a regular SIR, choose

$$\beta^* = \left(\beta_0 - \frac{a_0}{2}\right)\left(1 + \tanh\left(t - \frac{\omega}{T}\right)\right)$$



- Independent variable time t is measured in months. β is the **infection rate**, c is the **recovery rate**, α is the **infection descent rate**, ω is **vaccine effective time** and T is **typical transition time**. (Set ω = 14 months, $T = 2$ months)
- The graph β vs. β^* and the comparison below show the effects of immunization on comparing it to beta
- Comparison of dynamics for **SIR Model** vs. **SIR Model with vaccination:**



Conclusions

- The extended SIR model demonstrates how the vaccine can effectively postpone the disease outbreak.
- Preventive vaccination gives time to look for treatments for infectious diseases while also temporarily slowing the spread.
- For different diseases, different models can be used according to characteristics of diseases.
- Infectious disease modeling can point out epidemic time and outbreak scales.
- The simple SIR model and its natural extensions can capture the essential dynamics of the infectious disease spread and the extrapolate to asymptotic final states.
- MATLAB and MAPLE packages are useful aids in modeling; they help in symbolic and analytical computations and have substantial graphic capabilities.

References

[1] Strogatz, S., 2019. *Nonlinear dynamics and chaos*. Boca Raton: CRC Press.

[2] Giordano, G. et al (2020). *Modelling the COVID-19 epidemic and implementation of population-wide interventions in Italy.* Nature Medicine, 26(6), 855-860.