UNIVERSITY OF SASKATCHEWAN

Geometry of the Hitchin Morphism and Spectral Correspondences

A dissertation submitted to the

College of Graduate and Postdoctoral Studies

in partial pulfillment of the requirements

for the degree of Doctor of Philosophy

in the Department of Mathematics

University of Saskatchewan

Saskatoon

By

Kuntal Banerjee

Permission to Use

Disclaimer

Abstract

We explore a strong categorical correspondence between isomorphism classes of sheaves of arbitrary rank on a given algebraic curve and twisted pairs on another algebraic curve, mostly from a linear-algebraic standpoint. We aim to generalize the language of classical spectral correspondence by the annihilating polynomials of pairs. In a particular application, we realize a generic elliptic curve as a spectral cover of the complex projective line and then construct examples of cyclic twisted pairs and co-Higgs bundles on the same curve. Afterwards, by appealing to a composite push-pull projection formula, we conjecture an iterated version of spectral correspondence. We prove this conjecture for a particular class of spectral covers of the complex projective line through Galois-theoretic arguments. The proof relies upon a classification of Galois groups into primitive and imprimitive types. In this context, we revisit a classical theorem of Ritt.

We move to examining the image of Hitchin morphism on algebraic varieties in general. In our context we work with rank 2 bundles on algebraic surfaces. Unlike the case of curves, Hitchin morphism on the space of twisted Higgs sheaves is not necessarily surjective though we study its properness, in case of twist by line bundles. We apply this idea to write down a proof of surjective property of Hitchin morphism. We also present illustrative examples in case of co-Higgs bundles following results by Rayan and Colmenares.

Acknowledgements

This thesis would not see daylight unless I had received generous help from many people and organizations. I owe my success to their support from depth of my heart. The following list contains most of them if not all.

First I thank my advisor Dr. Steven Rayan for his impact on my life for past five years and many years yet to come. I feel very lucky to work under his guidance. He protected me not only as a supervisor but also as a local guardian during my stay at Saskatoon. Dr. Rayan's enthusiasm attracted me just like many other students, since I first met him in 2019. My attraction towards his intellect and compassion led me to join University of Saskatchewan as a doctoral student in 2020. His patient guidance helped me to complete my curriculum remotely in my first year during the pandemic. Dr. Rayan's warm support helped me to live a decent life at Saskatoon and visit outside for the purpose of research. He successfully inculcated his mathematical excitement in me. I wish him a beautiful life ahead.

I thank Natural Sciences and Engineering Research Council of Canada College (NSERC) for the continual financial support they have provided for my doctoral research program via the Discovery Grant ("Hitchin Systems: Tame, Twisted, and Wild") of my advisor. I also acknowledge the College of Graduate and Postdoctoral Studies (CGPS) for further funding through the Graduate Teaching Fellowship program and for the smooth organization of my program of my study in general.

Teaching activities were an indispensable component of my research life for past three years. I thank Manuela, Amos for their guidance while I learnt significantly about formal teaching from Dr. Alwan, Dr. Chang and Dr. Patrick. I thank Kyla and Richard for their organizational and technical supports within the department.

My memory at Saskatoon would be enlightened by many friends and colleagues I met in Canada who are fantastic personalities themselves. I thank Ahmed, Christopher, Madeline, Mandana, Dat and Tayab for their wonderful company. I thank Ahmed, Eric and Evan for giving useful comments on my defense presentation. I would love to mention Yaindrila as a fantastic colleague and my elder sister of same ethnic origin. I am particularly fortunate to meet Mahmud, a sharp mathematician, a generous friend, a strong peer and my partner in sharing ethnic Bengali culture.

Here I take a chance to thank SURI program authorities at University of Saskatchewan and Indian Institute of Technology Gandhinagar, particularly the coordinator Dr. Raj Srinivasan for introducing me to the Department of Mathematics and Statistics.

I would like to thank Dr. Ajneet Dhillon, Department of Mathematics and Statistics, University of Western Ontario for hosting me for a short visit in Summer of 2023.

This section will remain incomplete without mentioning the role played in my life by the Indian Bengali community at Saskatoon whom I joined on many occasions including religious celebrations and social events. In particular, I thank Dr. Anindya Gangyly, Arpita Deb, Dr. Arkaprovo Ghosal, Madhurima Bandyopadhyay and Dr. Biswajit Saha for their invaluable company.

I would like to thank my comrades Moinak and Goutam with whom I have shared a significant part of my years during bachelor's and master's. Finally, I thank my parents Kartick Banerjee and Ruma Banerjee and my sister Debapriya Banerjee and all of the members of my family for their unconditional love and support over the years.

Declaration

The content of this thesis is, best to my knowledge, original except acknowledged to others within the bibliography.

The contents of chapters 2 and 3 have previously appeared as a joint arXiv preprint with my advisor Dr. Steven Rayan under the title "A generalized spectral correspondence" (arXiv:2310.02413, [BR23]). Our intellectual contribution to this jointly-authored manuscript are equal and I contributed significantly to every part of the manuscript and to the results in this paper.

Contents

1 Overview of thesis

1.1 Structure of the thesis

The thesis is divided into five chapters followed by an appendix. The first chapter gives an overview of the thesis. Parts of the second chapter are preliminary materials of the theory of twisted pairs on curves and the rest consists of original results followed by computed examples. The third chapter contains original results involving group theory and moduli theory of pairs. The fourth chapter contains excerpts of the theory of moduli of sheaves on arbitrary varieties. The fifth chapter is devoted to establishing results based on the material of the fourth section. Finally, the appendix summarizes some basic knowledge about invariants of varieties.

1.2 Preliminaries and motivation

Hitchin introduced the moduli space of stable Higgs bundles ([Hit87a], [Hit87b]) as part of studying the self-dual Yang-Mills equations on a Riemann surface, and the theory was taken up and further developed by Simpson, Beauville, Narasimhan, Ramanan, Nitsure, Biswas, and many others (cf. [BNR89], [Nit91], [Sim94a], [Sim94b], [BR94]). The moduli scheme of strongly equivalent semistable twisted pairs over curves was explored by Nitsure ([Nit91]) and more generally, sheaves of Λ-modules over varieties were studied by Simpson ([Sim94a], [Sim94b]). Hitchin morphism evaluates the characteristic coefficients of a coherent Λ-module. It is a proper morphism on moduli scheme of semistable pairsof fixed rank and fixed degree over curves ([Nit91]) and its generic fibers are given by coherent modules over the *spectral variety* by a sheaf theoretic correspondence (see [BNR89] and [Sim94b]). Hitchin morphism and Hitchin fibration played a crucial role in Ngô Bao Chau's proof of the *fundamental lemma* [Cha10].

Hitchin studied the characteristic polynomials of stable G-Higgs bundles (Higgs bundles with the canonical twist) in [Hit87b] for rank 2 and for a classical Lie group G established that the space of isomorphism classes of stable Higgs bundles with a smooth connected spectral curve is either a Jacobian of the spectral curve or a Prym variety. Thus it is a symplectic manifold and admits a structure of an algebraically completely integrable system. We extend this spectral correspondence to a nonabelian version in the following discussion.

1.3 A generalized spectral correspondence

In the first part of the thesis we expose a correspondence between twisted Higgs sheaves over a smooth curve and locally free sheaves over a spectral curve. Higgs bundles or Higgs sheaves are pairs $(E, \phi : E \to E \otimes V)$ over a variety X where E and V are vector bundles and ϕ is a bundle homomorphism with an integrability condition $\phi \wedge \phi = 0$. Over curves we usually choose a line bundle L in place of V, if not mentioned otherwise. This bestows E with a $\mathrm{Sym}(\mathcal{L}^{-1})$-module structure where $\mathcal{L}$ is the sheaf of sections of L. After defining stability, semistability and strong equivalence of pairs we define their annihilating polynomials with coefficients $s_1, ..., s_r$ in the affine base $\bigoplus_{i=1}^{r} H^0(X, L^i)$. The discussion of annihilating polynomials is finally reduced to an engineering of linear algebraic data over the function field over X. We see that the pairs with an irreducible annihilating polynomial p_s admits $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$-module structure which is the structure sheaf of the corresponding spectral curve $X_s = \mathrm{Spec}(\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}})$. This spectral curve X_s resides in the total space of L and the restriction of bundle projection π on this curve is finite degree covering map. Observe that the same spectral curve is also described as zero scheme of a section of $\pi^* L^r$ and a bundle M over X_s, also adjudged as a pair (M, η) is pushed down to an L-twisted pair on X. This assignment is a bijection which we frame as the following result.

Theorem 1.3.1. *Let X_s be a nonsingular, integral spectral curve over X with finite (so proper) covering map π. Then there is a one-to-one correspondence between isomorphism classes of vector bundles M of a finite rank over X_s and L-twisted Hitchin pairs (E, ϕ) over X annihilated by p_s. The correspondence is given by $(M, \eta) \mapsto (\pi_* M, \pi_* \eta)$ using projection formula*

$$\pi_*(M \otimes \pi^* L) \cong \pi_* M \otimes L.$$

This is an extension of *spectral correspondence* proved in [BNR89] and at the same time an algebro-geometric explanation of *nonabelianization* described in [HS14] to focus on annihilating polynomials of pairs replacing their characteristic polynomials. To mention in a nutshell, Hitchin and Schaposnik investigated G-Higgs bundles over curves for real subgroups G of complex Lie groups and exposed associated Prym varieties. In particular, we compute elementary, yet nontrivial numeric examples of semistable co-Higgs (Section 8 [BR23]) sheaves over projective line using the fact that a generic degree 2, 2-twisted spectral curve is elliptic and further explicit results for semistable bundles on an elliptic curve (see [Ati57b], [Tu93]).

1.4 An iterated spectral correspondence

Spectral covering maps on varieties play an important role in providing categorical constructions between a curve and its spectral curve, including the spectral correspondence that we have discussed so far. Our broad target is extending these constructions to their iterated versions via a *factorization* of the spectral covering map π into intermediate covers. Eventually, this problem reduces to investigation of Galois groups of the respective covers. In the thesis we unfold a minuscule piece of this large project. We exploit a very common spectral curve which we call a cyclic spectral curve. Here we showcase the discussion checking π is composition of two nonconstant holomorphic maps between curves over $\mathbb{C}$.

Let Z be a curve with a branched holomorphic covering map $\pi : Z \to X$ which defines a field extension $\pi^*\mathcal{M}(X) \subset \mathcal{M}(Z)$. Let Y be a curve and there exist maps $f : Z \to Y$ and $g : Y \to X$ such that π factors as $g \circ f$ and generic fibers of f and g contain more than one point. Then functorial properties produce $\pi^*\mathcal{M}(X) \subset f^*\mathcal{M}(Y) \subset \mathcal{M}(Z)$. We can prove a typical converse statement (see [BR23]) relating subfields of function fields. This reflects the categorical equivalence among function fields, algebraic curves and compact Riemann surfaces. We are inspired by techniques discussed in [BRSY21].

Proposition 1.4.1. *A nonconstant branched holomorphic map $\pi : Y \to X$ of compact connected Riemann surfaces is factorizable if and only if there exists a proper intermediate subfield $\pi^*\mathcal{M}(X) \subset K \subset \mathcal{M}(Y)$.*

Here we mention a century old theorem by J.F. Ritt ([Rit23]) that completely classifies the maps which admit non trivial factorizations. The subgroups of a permutation group S_r (where r is composite) are commonly divided into *primitive* and *imprimitive* categories. A subgroup G of S_r is imprimitive if there is a partition of r symbols into blocks of equal length so that action of G is a permutation on these blocks (see [LZ10]). We relate a covering map to finite permutation groups via its monodromy group. Joe Harris ([Har79]) proved that the monodromy group coincides with the Galois group of the covering map.

Theorem 1.4.2. *(Ritt) A nonconstant $r : 1$ holomorphic map between compact Riemann surfaces $\pi : Y \to X$ is factorizable if and only if Galois group of the branched covering* $\mathrm{Gal}(\mathcal{M}(Y)^{\mathrm{Gal}}/\pi^*\mathcal{M}(X))$ *is imprimitive.*

Though we do not use this theorem explicitly for our proofs still we need it for handling most general spectral covering maps over curves. We join two corners, field theory and permutation groups, to a third corner, that is, algebraic geometry of spectral covers. Observe that under a factorization $\pi = g \circ f$ as above, a π^*L-twisted pair on X_s is pushed down by f to a g^*L-twisted pair on Y via a *composite* version of bundle projection formula. We discuss its consequences in spectral correspondence and construction of invariant polynomials. We restrict our discussion to *cyclic* spectral covers.

Rayan and Sundbo studied the moduli space of holomorphic chains of twisted Higgs bundles on curves ([RS21]) in the context of twisted cyclic quiver varieties. A twisted cyclic pair is $(E, \phi : E \to E \otimes L)$ such that $E = \bigoplus_{i=1}^{n} E_i$ and $\phi = \bigoplus_{i=1}^{n} \phi_i$ while $\phi_i \in H^0(X, E_{i-1}^* \otimes E_i \otimes L)$ (for $i = 1, ..., n-1$) and $\phi_n \in H^0(X, E_n^* \otimes E_1 \otimes L)$. In case $n = \mathrm{rank}(E) = r$ the summand bundles $E_1, ..., E_n$ are line bundles. The Higgs bundles corresponding to underlying A-type quivers (that is, $\phi_n = 0$) are the fixed points of $\mathbb{C}^*$-action of stable Higgs bundles. These objects relate well to the objects explained in [LS12]. We generalize this discussion in the light of Galois theory.

We fix a line bundle L on a curve X and r be an integer. Let s be a holomorphic section of L^r with distinct zeros. We define an L-twisted pair (E, ϕ) as *cyclic* if it admits characteristic polynomial $\lambda^r - s = 0$ and this definition extends the definition in [RS21]. By a result in [BNR89] the underlying spectral curve X_s is smooth and integral and the spectral covering map π is a proper map. We can also interpret π as a nonconstant finite holomorphic map between compact Riemann surfaces. We investigate the map π through computation of its Galois group. We conjecture a feasible answer from the nomenclature immediately: Galois group of a cyclic spectral cover is cyclic. We verify this conjecture for $\mathbb{P}^1$ with a very elementary justification.

Recall that the global holomorphic sections of line bundles on $\mathbb{P}^1$ are given uniquely by polynomials (of maximal degree same as degree of the line bundle) and these polynomials define unique meromorphic functions on $\mathbb{P}^1$. The smooth connected spectral curves over $\mathbb{P}^1$ coincides with the curve defined by algebraic equation with meromorphic function coefficients. In particular, the cyclic spectral curves on projective line are merely classical cyclic covers once their definitions are unwrapped. This leads to the computation of Galois group as a cyclic group. Moreover, the cyclic spectral covers are Galois and we exploit the Fundamental theorem of Galois theory, in case $\deg(\pi) = r$ is a composite number, to obtain intermediate function fields and respective maps to factorize π.

Immediately using the spectral correspondence we identify twisted pairs on $\mathbb{P}^1$ with characteristic polynomial $\lambda^r - s$ as pushforward of vector bundles. We jot down this idea in form of the following theorem.

Theorem 1.4.3. *Let $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$ be a generic holomorphic section with $t \geq 2$ and r be a composite number.*

(A) Isomorphism classes of t-twisted Hitchin pairs (E, ϕ) of rank r on $\mathbb{P}^1$ satisfying the characteristic equation $\lambda^r = s$ (name it $\mathcal{N}''$) are in one-to-one correspondence with isomorphism classes of line bundles M on X_s. The correspondence is given with pushforward by the covering map π. In case we fix degree of E to be $d \in \mathbb{Z}$, we see that $\pi_* : \mathrm{Jac}^{d'}(X_s) \to \mathcal{M}_{\mathbb{P}^1}(r, d, t)$ is a one-to-one correspondence, while $d' = d + (r-1)(\frac{tr-2}{2} + 1)$.

(B) Given a factorization $r = mp$ with $p, m \geq 2$ there exists a compact Riemann surface X and nonconstant holomorphic maps $f : X_s \to X$ of degree m and $g : X \to \mathbb{P}^1$ of degree p such that $\pi = g \circ f$. If there is another compact Riemann surface $\tilde{X}$ and nonconstant holomorphic maps $\tilde{f} : X_s \to \tilde{X}$ of degree m and $\tilde{g} : \tilde{X} \to \mathbb{P}^1$ of degree p such that $\pi = \tilde{g} \circ \tilde{f}$ then $X \cong \tilde{X}$.

(C) Fix a chosen factorization of $r = mp$ and $\pi = g \circ f$. Let the space of isomorphism classes of stable $g^*\mathcal{O}(t)$-twisted pairs of rank m on X, be $\mathcal{N}$. Then $f_* : \mathrm{Pic}(X_s) \to \mathcal{N}$ is a well defined injective morphism with image $\mathcal{N}'$. There is a bijective correspondence $g_* : \mathcal{N}' \to \mathcal{N}''$. Given $\deg(E) = d$*, the pushforward morphism*

given by $f_* : \mathrm{Jac}^{d'}(X_s) \to \mathcal{M}'_X(m, d'', g^*\mathcal{O}(t))$ *is an injective morphism, wherein* $\mathcal{M}'_X(m, d'', g^*\mathcal{O}(t))$ *denotes isomorphism classes of stable* $g^*\mathcal{O}(t)$*-twisted Hitchin pairs of rank* m *and degree* $d'' = d + \frac{mtr(p-1)}{2}$ *on* X*. Let* $\mathcal{J}$ *denote image* $f_*(\mathrm{Jac}^{d'}(X_s))$*. Then* $g_* : \mathcal{J} \to H^{-1}(s)$ *is a bijective correspondence as* H *is the Hitchin morphism on* $\mathcal{M}_{\mathbb{P}^1}(r, d, \mathcal{O}(t))$*.*

A possible application of our results here is to integrable systems — specifically, to the interpretation of solutions of Lax pair equations. We know that we may linearize such equations on the Jacobian of the associated spectral curve (noting that time evolution in a Lax pair equation is isospectral, and so there is a unique spectral curve associated to the data of the problem) — see [Bea90, HSW99], for instance. Our results afford us the ability to interpret the linear flow along a moduli space of higher-rank vector bundles associated to an intermediate spectral curve, which may provide finer information about such systems, with some connection perhaps to integrable hierarchies. We do not pursue these ideas here.

1.5 Image of the Hitchin morphism over algebraic surfaces

The last component in this thesis (not available in [BR23]) aims to understand the image of twisted Hitchin morphism over surfaces. This question is derived from the properties of Hitchin morphism over curves. Hitchin used techniques from nonabelian Hodge theory to prove this is a proper morphism ([Hit87b]) for rank 2 bundles with canonical twist and Nitsure proved more general twist in [Nit91] using Langton's famous valuative criterion of properness ([Lan75]). It is a very important feature of Hitchin morphism that decides if Hitchin morphism is surjective. In [BNR89] Hitchin morphism $T^*\mathcal{N}(r, d) \to \bigoplus_{i=1}^r H^0(X, K_X^i)$ is dominant where $\mathcal{N}(r, d)$ denotes moduli space of stable bundles of rank r and degree d over a curve of genus > 1. So, proper (thus universally closed, so closed) Hitchin morphism is surjective. We want to emphasize a key argument that whenever image of Hitchin morphism contains a dense subset, surjectivity takes place immediately. We frame a more general question and its answer in 5.2.2. We set a hypothesis on a chosen line bundle L and an integer r such that the set of tuples s in $\bigoplus\limits_{i=1}^{r} H^0(X, L^i)$ with smooth integral spectral curve X_s forms a dense set. Further we choose a suitable line bundle on a spectral curve X_s and push it forward to X by spectral covering map. This will confirm existence of an L-twisted pair with fixed rank r and a fixed degree.

We intend to follow the same line of proof over algebraic surfaces though we should take cautious steps. Simpson gave moduli/ G.I.T construction for Gieseker semistable $\mathrm{Sym}(V^*)$-modules or V-twisted pure dimensional coherent sheaves, for a fixed polarization, in [Sim94a]. This semistability condition does not coincide with Mumford semistability generally. Moreover $\phi \wedge \phi = 0$ is a nontrivial condition that is essential $\mathrm{Sym}(V^*)$-structure on sheaves. This integrability condition holds trivially for a line bundle twist. We follow the above argument on curves by extending properness results which was proved by Simpson in [Sim94b]. In Simpson's construction semistable Higgs sheaves are identified with moduli space of coherent sheaves over projective completion of the twisting sheaf. This whole discussion leads us to Theorem 5.2.4 under specific hypotheses. We focus on rank 2 bundles all over and the classification results established by Rayan ([Ray14]) and Colmenares ([Col15]). Rayan observed existence co-Higgs bundles on the surfaces on the tail-end of Kodaira's classification of surfaces. However, in case V has rank 2, the question remains difficult to attack. A recent paper by Chen and Ngô ([CN20]) tackles this problem in the language of stacks discarding automorpisms. Their postulated image of Hitchin morphism is image of the closed subscheme defined by integrability condition $\phi \wedge \phi = 0$. We will follow a line of proof by Gallego, García Prada and Narasimhan [GGPN23] that uses Simpson's identification of Higgs sheaves as moduli spaces of sheaves on a projective variety.

Understanding the image of the Hitchin morphism for a fixed Hilbert polynomial is otherwise difficult due to hardness in computation of the first cohomology groups of twisted endomorphism bundles. Over $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ the classical Hitchin base $\bigoplus_{i=1}^{2} H^0(\mathrm{Sym}^i(\Omega^1))$ is trivial. so, we focus on their co-Higgs bundles. In case of line bundles we argue as we did for curves given L^r has an empty base locus. We can write down explicit information of some well known bundles and cohomology groups over projective 2-space and ruled surfaces. Unfortunately, the cohomology groups of twisted endomorphism bundles over surfaces with higher Kodaira dimensions are not yet addressed in existing literature. However, we need to prove that Hitchin morphism is proper for other twists over algebraic surfaces.

As a note to the reader, it is worth remarking that the subject of Higgs bundles has always been situated at the intersection of complex-analytic geometry and algebraic geometry, given the gauge-theoretic origins of the subject combined with the need for tools and perspectives in algebraic moduli theory. The present thesis is reflective of this reality; as such, we draw upon both viewpoints at once, including their respective notations and conventions, sometimes within the same development within the thesis. We hope this will not be too arduous or confusing for the reader.

Finally, it is also worth mentioning that a number of original lemmas and theorems in the thesis have a decidedly "classical" flavour to them, in some cases because of their intersection with foundational aspects of Galois theory. The new contributions in these cases are their geometric implications regarding spectral curves and moduli spaces. Lemmas and theorems with no qualifier beside their lemma or theorem number are original in the sense that they do not appear in the literature, at least to our knowledge at the time of writing. Results that appear elsewhere are qualified as such after the lemma or theorem number. Where possible, we attempt to give some additional context in the surrounding text as to the origins of results and their context in the literature.

1.6 Keywords

Spectral correspondence, spectral curve, twisted pair, Higgs bundle, co-Higgs bundle, moduli space, semistability, Hitchin fibration, projective line, elliptic curve, push-pull formula, cartographic group, Galois group, monodromy group, Hitchin morphism, algebraic surface, complex surface, spectral cover.

2 Generalized spectral correspondence

2.1 Introduction

The spectral correspondence for Higgs bundles was first identified by Hitchin in [Hit87b]. There, the correspondence plays a role in revealing the structure of the moduli space of semistable, finite-rank G-Higgs bundles over an algebraic curve X, where G is a reductive group and where the Higgs field is valued in the canonical line bundle K_X. At the level of an individual Higgs bundle, the correspondence produces a new curve $\widetilde{X}$ encoding the spectrum of the Higgs field as a finite-to-one branched cover of X as well as a rank-1 sheaf on $\widetilde{X}$ that encodes the eigenspaces of the Higgs field. When one pushes this data back to the original curve, the correspondence produces a representation of the original Higgs bundle in which the Higgs field is diagonalized almost everywhere (save for at the ramification points). In this way, the spectral correspondence is a globalization of familiar aspects of the linear algebra of operators on finite-dimensional vector spaces — in other words, of Higgs bundles over the point. The global spectral correspondence was subsequently expanded by Beauville, Narasimhan, and Ramanan in [BNR89] to the case of L-twisted Higgs fields, where L is now an arbitrary line bundle — later, it was extended to the case of vector-valued Higgs fields twisted by an arbitrary vector bundle of rank 2 as in [GGPN23]. Moduli spaces of semistable L-twisted Higgs bundles were constructed by Nitsure in [Nit91], who also gave a proof of properness of a morphism usually known in this context as the *Hitchin morphism.* The spectral correspondence leads to an explicit description of a generic fiber of the Hitchin morphism on this space in terms of Jacobians (or Prym varieties) of spectral curves. The spectral correspondence between twisted pairs and vector bundles of higher rank was examined by Hitchin and Schaposnik in [HS14] in the setting of Higgs bundles associated to real subgroups of complex Lie groups. They refer to this operation as a "nonabelianization" of Higgs bundles. The operation is nonabelian in two related ways, as the spectral bundle is no longer rank 1 and in the fact that the fiber of the analogous Hitchin map is no longer an abelian variety. The spectral curve, according to Hitchin and Schaposnik, is the underlying reduced curve defined by a non-reduced characteristic polynomial.

In this article, we first provide some exposition about the construction of spectral curves (Definition 2.11) and of the spectral correspondence for twisted Higgs bundles (interchangeably referred to as "twisted pairs") over a smooth algebraic curve (Theorem 2.6.4 and Corollary 2.6.5). This is done mostly in the language of linear algebra over unique factorization domains. We treat examples of cyclic Higgs bundles (Section 2.7) and co-Higgs bundles of higher ranks (Section 2.8) over $\mathbb{P}^1$ in this context. This setting also allows us to invoke

Galois-theoretic techniques (Proposition 3.4.1) to elicit a threefold avatar of spectral correspondence for cyclic pairs (3.5.2). The results here are algebro-geometric in nature but rely heavily on machinery afforded by classical abstract algebra.

2.2 Overview of the chapter

A source of inspiration for the algebraic construction of spectral curves in this manuscript is the approach taken in [Gal19]. Recall that a smooth curve X is a Noetherian scheme and each point $x \in X$ admits a Noetherian local ring (of stalks of regular functions). We replace annihilating polynomials (Definition 2.4.1) of twisted pairs with their counterparts on locally free stalks (Section 2.4) and explore maps on stalks linear over the function field (stalk of regular functions at the generic point) of X (by Theorem 2.6.2 and Theorem 2.6.3). These linear maps on stalks provide global characteristic polynomials of pairs and their invariant subbundles according to Remark 3. We package this discussion ultimately as a sheaf theoretic correspondence between X and a spectral curve X_s that is embedded in the total space of the twisting line bundle (Theorem 2.6.4). This further extends to a higher categorical correspondence between (semi)stable bundles on the spectral curve (semi)stable pairs on X (Corollary 2.6.5, Corollary 2.6.6, Proposition 2.6.7). In particular, spectral curves over $\mathbb{P}^1$ closely relate to complex affine algebraic curves in two variables (Lemma 2.7.1 and Lemma 2.7.3). We pay attention to cyclic pairs and numerical computations of pushforward sheaves from an elliptic spectral cover to $\mathbb{P}^1$ relies on splitting of holomorphic vector bundles ([Ati57b]) over curves (Lemma 2.8.4).

2.3 Preliminaries of L-twisted pairs on curves

Let X be an irreducible, nonsingular, projective algebraic curve over $\mathbb{C}$, equivalently, a smooth, compact, connected Riemann surface with genus $g_X \geq 0$. We will use 'curve'and 'Riemann surface'(or just 'surface') interchangeably to refer to such an object. Let L be a holomorphic line bundle on X with projection map $\pi : \mathrm{Tot}(L) \to X$ where $\mathrm{Tot}(L)$ is the total space of L. By an L-twisted *pair* or a *Hitchin pair* on X we understand a pair (E, ϕ) where E is a vector bundle over X of finite rank r and $\phi : E \to E \otimes L$ is a bundle morphism. The bundle morphism ϕ is alternatively referred to as an element of $H^0(X, \mathrm{End}(E) \otimes L)$. A *morphism* of L-twisted pairs (E, ϕ) and (E', ϕ') is a commutative diagram given as

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes L \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi \otimes 1 = \psi'} \\ E' & \xrightarrow{\phi'} & E' \otimes L \end{array} \tag{2.1}$$

in which $\psi : E \to E'$ is a bundle morphism and 1 denotes identity morphism on L. The pairs (E, ϕ) and (E', ϕ') are said to be *isomorphic* if there exists an isomorphism $\psi : E \to E'$ of bundles such that $\phi' = \psi' \circ \phi \circ \psi^{-1}$.

Definition 2.3.1. An L-twisted pair (E, ϕ) is said to be a *stable* (resp. *semistable*) pair if each nontrivial proper ϕ-invariant subbundle (also called *Hitchin subbundle*) F (i.e $\phi(F) \subseteq F \otimes L$) satisfies the slope inequality

$$\frac{\deg(F)}{\operatorname{rank}(F)} = \mu(F) < (\text{resp. } \leq) \mu(E) = \frac{\deg(E)}{\operatorname{rank}(E)}. \tag{2.2}$$

Remark 1. In the event that E is stable (respectively, semistable), any L-twisted pair with underlying bundle E is automatically stable (resp., semistable).

Hitchin first introduced moduli space of stable Higgs bundles i.e stable pairs twisted by K_X, the *canonical line bundle* on X, as solutions of reduced self-dual Yang-Mills equations in [Hit87a].

Proposition 2.3.2. *Let (E, ϕ) be be a semistable pair on X. Then there exists a finite filtration of ϕ-invariant subbundles of increasing ranks*

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E \tag{2.3}$$

such that for each $i = 1, ..., n$ we have $\mu(\frac{E_i}{E_{i-1}}) = \mu(E)$ and quotient pairs $(\frac{E_i}{E_{i-1}}, \phi_i : \frac{E_i}{E_{i-1}} \to \frac{E_i}{E_{i-1}} \otimes L)$ induced from ϕ are stable. The associated graded pair $\mathfrak{gr}(E, \phi) = \bigoplus_{i=1}^{n} (\frac{E_i}{E_{i-1}}, \phi_i)$ is unique up to isomorphism of (E, ϕ) and $\mathfrak{gr}(E, \phi)$ is also semistable.

Remark 2. The above filtration is called the *Jordan-Hölder filtration* of the pair (E, ϕ). We call L-twisted semistable pairs (E, ϕ) and (E', ϕ') to be *S-equivalent* if graded pairs $\mathfrak{gr}(E, \phi)$ and $\mathfrak{gr}(E', \phi')$ are isomorphic.

Nitsure [Nit91] established a moduli construction for the S-equivalence classes of L-twisted pairs on X as GIT quotients by actions of $GL(r, \mathbb{C})$ and $SL(r, \mathbb{C})$. By $\mathcal{M}(r, d, L)$ we denote the quasi-projective coarse moduli scheme of S-equivalence classes of L-twisted pairs for which underlying bundle E has rank r and degree d. This moduli scheme admits the scheme $\mathcal{M}'(r, d, L)$ of stable pairs as an open subscheme. Moreover, they established that the dimension of Zariski tangent space of $\mathcal{M}(r, d, L)$ satisfies the formula

$$\dim T_{(E,\phi)} = r^2 \deg(L) + 1 + \dim H^1(X, L) \tag{2.4}$$

in each of these following cases: $L \cong K_X$; $L^r \ncong K_X^r$ but $\deg(K_X) = \deg(L)$ and finally, $\deg(L) > \deg(K_X)$. Here on wards we will restrict to the twisting line bundles L with positive degrees.

2.4 Annihilating polynomials of pairs and the Hitchin morphism

In this context we aim to explore annihilating polynomials of a twisted bundle morphism, akin to their usual linear algebraic formulation. This includes the characteristic polynomial of a pair, which defines a morphism of varieties. We fix a line bundle L on X and let $s = (s_1, ..., s_n) \in \bigoplus_{i=1}^{n} H^0(X, L^i)$. For $\phi \in H^0(X, \operatorname{End}(E) \otimes L)$ conventionally, ϕ^0 is the identity morphism on E and inductively (denoting by 1 the identity morphism on L^i's, where i is an integer) as following. Consider $\phi \otimes 1 : E \otimes L^{i-1} \to (E \otimes L) \otimes L^{i-1} = E \otimes L^i$. Then $\phi^i : E \to E \otimes L^i$ is defined by $\phi^i := (\phi \otimes 1) \circ \phi^{i-1}$. This leads to construction of a global section of $\operatorname{End}(E) \otimes L^n$ defined by a polynomial given as $\phi^n + \sum_{i=1}^{n} s_i \otimes \phi^{n-i}$.

Definition 2.4.1. For a chosen tuple of global sections $s_1, ..., s_n$ of $L, ..., L^n$ the polynomial $p(\lambda) = \lambda^n + \sum_{i=1}^n s_i \lambda^{n-i}$ over the affine base space $\bigoplus_{i=1}^n H^0(X, L^i)$ is said to be an *annihilating* polynomial of ϕ if

$$\phi^n + \sum_{i=1}^{n} s_i \otimes \phi^{n-i} = 0. \tag{2.5}$$

We also say that ϕ satisfies p if p annihilates ϕ.

Let $\mathcal{O}_X$ denote the sheaf of regular functions on the algebraic curve X (alternatively, the sheaf of holomorphic functions on a compact Riemann surface X). We elicit a parallel set of constructions, via sheaf homomorphisms, over the Noetherian local ring of germs $\mathcal{O}_{X,x}$ at any point $x \in X$. Let $\mathcal{L}$ denote the sheaf of sections of L. Then $\mathcal{L}^i$ is locally free for any integer i and stalks are free of rank 1 at each point x. Let $\phi \in H^0(X, \mathrm{End}(E) \otimes L)$. For any open set U of X we are able to restrict ϕ on U. Now let U be a trivializing neighbourhood (connected, if necessary) of L^{-1} and Λ be a generator of restricted sheaf $\mathcal{O}(L^{-1})|_U$. That means, for each open subset V of U, we can generate sheaf of sections of L^{-1} by Λ on V. Moreover, we have $\phi \otimes \Lambda \in \mathcal{O}(\mathrm{End}(E) \otimes L)(V) \otimes \mathcal{O}(L^{-1})(V) \subset \mathcal{O}(\mathrm{End}(E))(V)$. Thus we obtain $\psi = \phi \otimes \Lambda$ as a local element in the sheaf $\mathcal{O}(\mathrm{End}(E))|_U$. We denote the $\mathcal{O}_X$-isomorphism $\mathcal{O}(\mathrm{End}(E)) \cong \mathrm{End}(\mathcal{O}(E))$ by (#). From (#) and definition of endomorphism sheaf, ψ is an $\mathcal{O}_X|_U$-endomorphism of $\mathcal{O}(E)|_U$ (taking restrictions on open subsets of U) for each trivializing neighbourhood U. For any section s_i of L^i we have $a_i = s_i \otimes \Lambda^i$ as sheaf elements in $\mathcal{O}_X|_U$. Taking the germs at $x \in U$, we have $\psi_x = \phi_x \otimes \Lambda_x$ and $a_{i,x} = s_{i,x} \otimes \Lambda_x^i$ taking tensor product over $\mathcal{O}_{X,x}$. Here $\phi_x \in \mathcal{O}(\mathrm{End}(E) \otimes L)_x$ and Λ_x denotes a generator of $\mathcal{O}_{X,x}$-free module $\mathcal{L}_x^{-1}$. Note that $a_{i,x}$, as an element of $\mathcal{O}_{X,x}$, is dependent on the choice of U. The transition maps play a role in change of choice of a trivializing neighbourhood. Owing to (#), we view ψ_x as an $\mathcal{O}_{X,x}$-linear endomorphism of the free module $\mathcal{O}(E)_x$.

A fixed tuple $(s_1, ..., s_n) \in \bigoplus_{i=1}^n H^0(X, L^i)$ determines a polynomial $\lambda^n + \sum_{i=1}^n a_i \lambda^{n-i}$ with coefficients in the restricted sheaf $\mathcal{O}_X|_U$ for any trivializing neighbourhood U of L^{-1}. Further we obtain the corresponding polynomial $\lambda^n + \sum_{i=1}^n a_{i,x} \lambda^{n-i}$ over $\mathcal{O}_{X,x}$. We casually refer to these polynomials as *associated polynomials* of $(s_1, ..., s_n)$ defined on U. We remark that a polynomial $p(\lambda) = \lambda^n + \sum_{i=1}^n s_i \lambda^{n-i}$ is an annihilating polynomial of ϕ if and only if the associated polynomial of p on each trivializing neighbourhood U of L^{-1} annihilates ψ if and only if at each point x, the associated polynomial $\lambda^n + \sum_{i=1}^n a_{i,x} \lambda^{n-i}$ over $\mathcal{O}_{X,x}$ annihilates ψ_x. For $\phi \in H^0(\mathrm{End}(E) \otimes L)$ the *characteristic coefficients* are defined as $s_i = (-1)^i \mathrm{tr}(\wedge^i \phi) \in H^0(X, L^i)$ where $\mathrm{tr}(\wedge^i \phi)$ is given by the following determinant computed at each point,

$$\mathrm{tr}(\wedge^i \phi) = \frac{1}{i!} \begin{vmatrix} \mathrm{tr}(\phi) & i-1 & 0 & \dots \\ \mathrm{tr}(\phi^2) & \mathrm{tr}(\phi) & i-2 & \dots \\ \vdots & \vdots & \vdots & \dots \\ \mathrm{tr}(\phi^{i-1}) & \mathrm{tr}(\phi^{i-2}) & \dots & 1 \\ \mathrm{tr}(\phi^i) & \mathrm{tr}(\phi^{i-1}) & \dots & \mathrm{tr}(\phi) \end{vmatrix}.$$

A twisted pair (E, ϕ) admits characteristic polynomial $\lambda^r + \sum_{i=1}^r s_i \lambda^{n-i}$. Replacing indeterminate λ with ϕ we obtain a global section of $\mathrm{End}(E) \otimes L^r$ given by $\phi^r + \sum_{i=1}^r s_i \otimes \phi^{r-i}$. By the Cayley-Hamilton theorem we conclude that

$$\phi^r + \sum_{i=1}^{r} s_i \otimes \phi^{r-i} = 0. \tag{2.6}$$

In particular, characteristic coefficients of the local sheaf homomorphism ψ (respectively, stalk wise homomorphism ψ_x) are given as

$$(-1)^i \mathrm{tr}(\wedge^i \psi) = (-1)^i \mathrm{tr}(\wedge^i \phi) \otimes \Lambda^i; \text{ (resp. } (-1)^i \mathrm{tr}(\wedge^i \psi_x) = (-1)^i \mathrm{tr}(\wedge^i \phi_x) \otimes \Lambda^i_x). \tag{2.7}$$

For a fixed rank r and a degree d we consider a morphism,

$$H : \mathcal{M}(r, d, L) \to \bigoplus_{i=1}^{r} H^0(X, L^i) \tag{2.8}$$

given by characteristic coefficients of a semistable pair (E, ϕ). In [Nit91], Nitsure proved that this *Hitchin morphism* H is a proper morphism, using valuative criterion. Spectral correspondence gives us a description of the fibers of Hitchin morphism. Note that given a choice of sections $s = (s_1, ..., s_r) \in \bigoplus_{i=1}^r H^0(X, L^i)$ we are able to frame an elementary example of a pair with the characteristic polynomial defined by s.

Example 2.4.2. Let $E = \mathcal{O} \oplus L^{-1} \oplus ... \oplus L^{-(r-1)}$. A bundle map ϕ is defined by the companion matrix associated to $s = (s_1, ..., s_r) \in \bigoplus_{i=1}^r H^0(X, L^i)$ is,

$$\phi = \begin{bmatrix} 0 & 0 & ... & ... & -s_r \\ 1 & 0 & ... & ... & -s_{r-1} \\ 0 & 1 & 0 & ... & -s_{r-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & 1 & -s_1 \end{bmatrix} \tag{2.9}$$

where 1's denote sections of $H^0(X, L^{-i} \otimes L^i) = H^0(X, \mathcal{O}_X) = \mathbb{C}$ or, identity morphisms on L^{-i}. $\diamondsuit$

Proof of stability of such a pair in case $r = 2$ is sufficiently short. Let M be an invariant subbundle of E. Then the holomorphic projection $\pi_1 : M \to \mathcal{O}$ is a nonzero bundle map. Thus $\deg(M^* \otimes L^{-1}) \geq 0$ which implies slope of M is strictly less than slope of E. The general case of r is proved for $g_X \geq 2$ and $L = K_X$ in [HH21] Remark 3.8.

2.5 Construction of spectral curves

The following constructions are true for any algebraically closed field $\mathbb{k}$ of characteristic 0 according to [BNR89]. For a set of sections $s = (s_1, ..., s_n)$ of $L, ..., L^n$ as defined earlier, we obtain a 1-dimensional scheme X_s embedded in the total space of line bundle L. We can describe this scheme in two different ways interchangeably. The first definition is given in [BNR89], Section 3 which we modify into a convenient form. Recall that the tautological line bundle π^*L over L admits a tautological section η defined by $\eta(y) = (y, y) \in \pi^*L$ for a point y on any fiber of L. More rigorously, $\eta \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}^{-1}, \mathrm{Sym}(\mathcal{L}^{-1}))$ is the canonical morphism, equivalently a global section of π^*L by the adjunction formula of the pullback and the pushforward operations. The scheme X_s is defined as the space where the polynomial $\lambda^n + \sum_{i=1}^n s_i \lambda^{n-i}$ intersects η, that is,

$$X_s = \left\{ y \in L : \eta^n(y) + \sum_{i=1}^{n} (\pi^* s_i \otimes \eta^{n-i})(y) = 0 \right\}. \tag{2.10}$$

On the other hand, we give a more abstract algebro-geometric construction. As mentioned earlier, $\mathcal{L}$ be the sheaf of sections of L. For each $i = 1, ..., n$ we obtain a sheaf homomorphism defined by multiplication $s_i : \mathcal{O}(L^{-n}) = \mathcal{L}^{-n} \to \mathcal{L}^{-(n-i)} = \mathcal{O}(L^{-(n-i)})$ and for sake of completeness we mention s_0 to be the identity morphism on $\mathcal{L}^{-n}$. The sum of the maps defines a sheaf homomorphism $\bigoplus_{i=0}^n s_i : \mathcal{L}^{-n} \to \bigoplus_{i=0}^n \mathcal{L}^{-i} \subset \mathrm{Sym}(\mathcal{L}^{-1})$. We denote the ideal sheaf generated by the image of the sheaf morphism $\bigoplus_{i=0}^n s_i$ with $\mathcal{I}$. The sheaf $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$ over X is a quasi-coherent sheaf of algebra (which we will explicitly write out afterwards). We construct (see [BNR89] section 3)

$$X_s = \mathrm{Spec}\left(\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}\right). \tag{2.11}$$

The first definition in 2.10 describes the set of closed points of the scheme defined in 2.11.

Definition 2.5.1. The scheme X_s is defined to be the *spectral curve* associated to s. As X_s is embedded in total space of L we obtain a finite morphism $\pi : X_s \to X$ by restricting the bundle map, called the *spectral covering map.*

The space of all $s = (s_1, ..., s_n) \in \bigoplus_{i=1}^n H^0(X, L^i)$ such that X_s is smooth and integral is an open subset when L^n has no base points. Note that the set of spectral curves form a complete linear system of divisors of π^*L^n over L. By Bertini's theorem smooth, the integral divisors form a Zariski open subset in the projective completion of the linear system. On the other hand, the branch points on X of the finite morphism π are given by resultant of the polynomial $\lambda^n + \sum_{i=1}^n s_i\lambda^{n-i}$ and its derivative $n\lambda^{n-1} + \sum_{i=1}^{n-1}(n-i)s_i\lambda^{n-i-1}$ which is a global section of $L^{n(n-1)}$. A point on X is a branch point of π if and only if it is a zero of the resultant. Away from such points π is étale of degree n. The set of sections $(s_1, ..., s_n)$ whose resultants admit distinct zeros is Zariski open. The locus in the Hitchin base consisting of such spectral curves is called the *smooth locus of spectral curves.* We call a smooth, integral spectral curve a *generic spectral curve.*

We can compute the genus of X_s from this explanation. For a trivializing neighbourhood U of L^{-1}, we produce the sheaf of ideals restricted on U as $\mathcal{I}|_U =< \sum_{i=0}^n a_i\Lambda^{n-i} >$. We recall $\mathrm{Sym}(\mathcal{L}^{-1}) \cong \bigoplus_{i=0}^\infty \mathcal{L}^{-i}$ and on a trivializing neighbourhood U of L^{-1} we have

$$\mathrm{Sym}(\mathcal{L}^{-1})|_U = \mathcal{O}_X|_U[\Lambda] = \left\{ \sum_{i=0}^{k} f_i\Lambda^{k-i} : f_i \in \mathcal{O}_X|_U; k \geq 0 \right\}.$$

Consider any two trivializing neighbourhoods U and V with local generators Λ and μ of $\mathcal{L}^{-1}$. Then $\Lambda = g_{VU}\mu$ for some nonvanishing complex valued function g_{VU} on $U \cap V$. Restrictions of this equality also make sense on open subsets of $U \cap V$. This leads us to fix a transition between $\mathcal{I}|_U$ and $\mathcal{I}|_V$ (and between $\mathrm{Sym}(\mathcal{L}^{-1})|_U$ and $\mathrm{Sym}(\mathcal{L}^{-1})|_V$) such that division algorithm is a sheaf homomorphism (recall that $\mathcal{O}_X$ is a sheaf of commutative rings with identity and we are able to divide a polynomial by a monic polynomial). By gluing the local sheaf homomorphisms we obtain a global sheaf homomorphism.

$$\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{I} \cong \mathcal{O} \oplus \mathcal{L}^{-1} \oplus ... \oplus \mathcal{L}^{-(n-1)} \tag{2.12}$$

Let Λ_x be the germ of a generator of the stalk of the sheaf $\mathcal{L}^{-1}$ at x. Then we have

$$\mathrm{Sym}(\mathcal{L}^{-1})_x = \left\{ \sum_{i=0}^{k} f_i \Lambda_x^{k-i} : f_i \in \mathcal{O}_{X,x}; k \geq 0 \right\}.$$

In other words, $\mathrm{Sym}(\mathcal{L}^{-1})_x = \mathcal{O}_{X,x}[\Lambda_x]$. Likewise we have an explicit description of the germ of ideal $\mathcal{I}$ at x as a principal ideal

$$\mathcal{I}_x = \left\langle \sum_{i=0}^{n} a_{i,x} \Lambda_x^{n-i} \right\rangle. \tag{2.13}$$

As per the definition of X_s, the finite morphism π gives a sheaf isomorphism $\pi_* \mathcal{O}_{X_s} \cong \frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$ (see [Har77] page 128 Exercise 5.17) and the Euler characteristic of $\mathcal{O}_{X_s}$ is given as $\chi(X_s, \mathcal{O}_{X_s}) = \chi(X, \pi_* \mathcal{O}_{X_s}) = \chi\left(X, \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}\right)$. From Riemann-Roch theorem this leads to

$$\chi(X_s, \mathcal{O}_{X_s}) = -\frac{n(n-1)}{2} \deg(L) + n(1 - g_X). \tag{2.14}$$

We compute the genus of X_s by the formula (see [BNR89] section 3)

$$g_{X_s} = 1 - \chi(X_s, \mathcal{O}_{X_s}) = \deg(L) \frac{n(n-1)}{2} + n(g_X - 1) + 1. \tag{2.15}$$

2.6 Spectral correspondence for generic spectral covers

We have already obtained a local description of the sheaf $\mathcal{I} \subset \mathrm{Sym}(\mathcal{L}^{-1})$ as polynomials in local generator Λ. If V is a subset of a trivializing neighbourhood U then the ring $\mathcal{I}(V)$ is generated by a polynomial in $\Lambda(V)$. This polynomial should be irreducible over $\mathcal{O}_X(V)$ for the corresponding spectral curve to be integral. We consider a spectral curve X_s corresponding to s. We observe that X_s is integral if and only if for each open subset V of X the ring $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}(V)$ is an integral domain. This is possible if and only if $\mathcal{I}(V)$ is a prime ideal. An integral spectral cover admits a unique generic point $\mathrm{Spec}(K_{X_s})$ while K_{X_s} denotes the function field of X_s. This point can be identified with the spectrum of the zero ring associated to the empty subset of X. We package this conception in the following proposition.

Proposition 2.6.1. *The scheme X_s is integral if and only if the associated polynomial at each point of X, (for each trivializing neighbourhood) is irreducible.*

The Krull dimension of the integral scheme X_s is 1. This is given by computing Krull dimension of $\frac{\mathrm{Sym}(\mathcal{L}_x^{-1})}{\mathcal{I}_x}$ for each point $x \in X$. For a chosen trivializing neighbourhood U, with local generator Λ as before, the ring $\mathcal{O}_{X,x}[\Lambda_x]$ has Krull dimension 2 by Hilbert's basis theorem that a polynomial ring over a Noetherian ring is Noetherian. After taking quotient by the prime ideal $< \sum_{i=0}^{n} a_{i,x}\Lambda_x^{n-i} >$ the Krull dimension decreases by 1. Here we use Krull's height theorem which states that a principal prime ideal of a Noetherian ring has height 1.

We are in position to recall results on UFDs and their quotient fields which we will use in our explanations in spectral correspondence. Let R be a unique factorization domain and F be its field of fractions.

Theorem 2.6.2. *Let $f \in R[x]$ be a primitive polynomial and $g \in R[x]$. Then f divides g in $F[x]$ if and only if f divides g in $R[x]$.*

Theorem 2.6.3. *Let $f \in R[x]$ be a polynomial of degree $n \geq 1$. Then f is a product of two polynomials in $F[x]$ of degrees d and e respectively with $0 < d, e < n$ if and only if there exist polynomials $g, h \in R[x]$ of degrees d and e respectively with $0 < d, e < n$ such that $f = g \cdot h$.*

We employ these results on the UFD $\mathcal{O}_{X,x}$ to relate characteristic polynomials of twisted pairs to linear algebra over fields. Let (E, ϕ) be an L-twisted pair on X that admits $\lambda^r + \sum_{i=1}^{r} s_i\lambda^{r-i}$ as its characteristic polynomial, defined by s. As $\mathcal{O}(E)_x$ is a free module over $\mathcal{O}_{X,x}$ we can extend the module homomorphism on the K-vector space (of same rank) $V = K \otimes_{\mathcal{O}_{X,x}} \mathcal{O}(E)_x$ while K is the quotient field of $\mathcal{O}_{X,x}$. Here K is isomorphic to the function field of X because X is a non-singular curve. We can view characteristic polynomials of module homomorphisms of stalks same as the respective characteristic polynomials of respective K-endomorphisms on V. If there is a proper invariant subbundle then at each point x, characteristic polynomial of corresponding germ ψ_x is divisible by characteristic polynomial of the germ, say ψ'_x, contributed by the invariant subbundle, over quotient field K and divisibility over K descends to divisibility over UFD $\mathcal{O}_{X,x}$. This will be a polynomial over $\mathcal{O}_{X,x}$ with a strictly smaller degree. This contradicts the case that X_s is integral as in Proposition 2.6.1. Here we are able to pose a more vivid factorization of the characteristic polynomial as product of characteristic polynomials of $(F, \phi_{|F})$ and the induced quotient pair $(E/F, \phi_{E/F})$. Thus we have the following proposition.

Remark 3. If X_s is an integral scheme then a twisted pair (E, ϕ) with characteristic polynomial defined by s does not admit any nontrivial proper invariant subbundle $(F, \phi_{|F})$, so automatically stable.

We are now at the position to prove a categorical equivalence between the torsion-free sheaves over X_s and $\mathcal{O}_X$-locally free $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$-modules over X. This is also framed as a one-to-one correspondence between isomorphism classes of vector bundles over the reduced spectral curve and isomorphism classes of Hitchin pairs over X. Consider a smooth and integral spectral cover X_s. We denote the corresponding spectral polynomial by p_s, which we will simply refer to as p with choice of s understood. The associated polynomials over a trivializing neighbourhood are p', p'_x where the latter is irreducible over the UFD $\mathcal{O}_{X,x}$ and thus over function field K.

We begin with a locally free sheaf M of rank n with multiplication operation by the tautological section $\eta : M \to M \otimes \pi^* L$. This is pushed forward to a L-twisted Hitchin pair $(\pi_* M, \pi_* \eta)$ on X using projection formula. On the spectral curve, section $\eta^r + \sum_{i=1}^{r} \pi^* s_i \otimes \eta^{r-i}$ vanishes, thus the pair $(\pi_* M, \pi_* \eta)$ of rank nr satisfies p under pushforward operation π_* by adjoint operation of pulling back sections (see [HSW99] page 33 Proposition 4.2). This is same as giving $\pi_* \mathcal{O}_{X_s}$-structure on $\pi_* M$. So p is an annihilating polynomial of $\pi_* \eta$. This argument ensures a consistent definition of a correspondence in Theorem 2.6.4. Moreover, we obtain p^n as characteristic polynomial of the pair.

At a point x, polynomial ψ_x (defined over a chosen trivializing neighbourhood U) satisfies p'_x. Consider the minimal polynomial g of ψ_x over K, then g divides p'_x over K. If $\deg(g) < \deg(p'_x)$, then p'_x is reducible over K which contradicts the case. Thus, $\deg(g) = \deg(p'_x)$ and monic polynomial p'_x is the minimal polynomial over K. Recall that irreducible factors of the minimal polynomial and the characteristic polynomial of an endomorphism (on a finite dimensional vector space) are exactly same. So, we have p'^n_x as characteristic polynomial of K-linear map ψ_x. (Implicitly, we have p'^n_x is characteristic polynomial of ψ_x over $\mathcal{O}_{X,x}$.) Denoting coefficients of p^n by S_i's and coefficients of p'^n_x as A_i's, we have $A_i = (-1)^i \mathrm{tr}(\wedge^i \psi_x)$. Here the associated polynomial of p^n is p'^n. Thus $A_i = S_{i,x} \otimes \Lambda^i_x$. Assembling the formula in 2.7 we have $S_{i,x} = (-1)^i \mathrm{tr}(\wedge^i \phi_x)$ and this holds for any $x \in X$. This leads to the conclusion that $S_i = (-1)^i \mathrm{tr}(\wedge^i \phi)$ so p^n is the characteristic polynomial of ϕ.

On the other hand, let p to be an annihilating polynomial of (E, ϕ). Now the associated polynomial p'_x of p over $\mathcal{O}_{X,x}$ (over a trivializing neighbourhood U of L^{-1}) is an annihilating polynomial of the germ ψ_x. Treating ψ_x as a linear map over K-vector space V polynomial p'_x becomes an irreducible annihilating polynomial over K, thus the minimal polynomial over K. This leads to the fact that the rank of E is divisible by $\deg(p) = r$ say nr, and repeating the same argument once again, p^n is the characteristic polynomial of (E, ϕ). Henceforth, we will omit this stalk-wise explanation and directly work with spectral polynomial p. In particular, one can place an argument at the level of stalks to conclude that if (E, ϕ) is annihilated by p then a proper invariant subbundle $(F, \phi_{|F})$ (which is also annihilated by p) has characteristic polynomial p^k for some $k \leq \mathrm{rank}(E)$.

Here we consider a trivializing neighbourhood U of L^{-1} and an action on $\mathcal{O}(E)|_U$ by $\alpha|_U : \mathrm{Sym}(\mathcal{L}^{-1})|_U \to \mathcal{O}(\mathrm{End}(E))|_U \cong \mathrm{End}(\mathcal{O}(E))|_U$ given as $q \mapsto q(\psi)$. (Here, for each open subset V of U, we treat ψ as a sheaf homomorphism on $\mathcal{O}(E)|_V$.) We claim that $\ker(\alpha|_U) = \mathcal{I}|_U$. If $f \in \mathcal{I}|_U(V)$ then we have $f(\psi) = 0$. So $f \in \ker(\alpha|_U)(V)$. Now consider $f \in \ker(\alpha|_U)(V)$. Now let $x \in V$. Taking germs at $x \in V$ we have $f_x(\psi_x) = 0$. The minimal polynomial of ψ_x is p'_x as we regard ψ_x as a linear map over K. Then f_x is divisible by p'_x over K and over $\mathcal{O}_{X,x}$ because p'_x is monic, so primitive. Then we use division algorithm. We divide f by the restriction of p' defined on V, as elements of $\mathcal{O}_X(V)[\Lambda_V]$. There are unique elements g and h such that

$f = p'.g + h$ and $\deg(h) < \deg(p')$. But, for each $x \in V$ there is a neighbourhood W_x such that f is divided by the polynomial p' over W_x due to the divisibility of germs. From uniqueness of division algorithm over $\mathcal{O}_{X,x}$ we have the germ $h_x = 0$. This is true for each $x \in X$, thus $h(x) = 0$ for all $x \in V$. Thus we have $h = 0$ on V. Thus $f \in \mathcal{I}$ and $\ker(\alpha|_U) = \mathcal{I}|_U$ from the set theoretic equality on each open subset V of U. Finally, let $f_1 - f_2 \in \ker(\alpha|_U)(V) = \mathcal{I}|_U(V)$. The action of f_1 and f_2 on ψ is invariant. It confirms that there is an well-defined action $\alpha|_U$ by $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}|_U$ on $\mathrm{End}(\mathcal{O}(E))|_U$. The action is compatible over all the trivializing neighbourhoods of L^{-1} over X and defines a global action on $\mathcal{O}(E)$ by

$$\alpha : \frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}} \to \mathrm{End}(\mathcal{O}(E)).$$

Main ingredient of the spectral correspondence is the following categorical isomorphism via pushforward morphism π_*, due to [Har77] page 128 Exercise 5.17.

Remark 4. {Isomorphism classes of quasi-coherent sheaves of $\mathcal{O}_{X_s}$-modules}$\xrightarrow{\pi_*}$
{Isomorphism classes of $\mathcal{O}_X$-quasi-coherent sheaves of $\pi_*\mathcal{O}_{X_s}$-modules}.

So, $\mathcal{O}(E)$ can be written as pushforward of a locally free sheaf, that is sheaf of sections of a bundle M over X_s of rank n. Moreover $(E_1, \phi_1) \cong (E_2, \phi_2)$ on X, annihilated by p if and only if module structures defined by ϕ_1 and ϕ_2 induce isomorphic $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$-module structures. Injectivity of the correspondence that is inherent in 4 can be phrased: locally free sheaf M is obtained from $\pi_*\mathcal{O}_{X_s} = \frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$-module structure induced by $\pi_*\eta$ on π_*M. Keeping the same spirit we denote by M the associated unique vector bundle over X_s.

We conclude that $\pi_*(M, \eta) = (E, \phi)$. Let $\mathcal{F} = \mathcal{O}(\pi^*L)$. Then there is a multiplication map $-\eta : \mathcal{F}^{-1} \to \mathcal{O}_{X_s}$ and sheaf homomorphism $1 \oplus (-\eta) : \mathcal{F}^{-1} \to \mathcal{F}^{-1} \oplus \mathcal{O}_{X_s}$ defines an ideal $\mathcal{G} \subset \mathrm{Sym}(\mathcal{F}^{-1})$. By the division algorithm on restricted sheaf over trivializing neighbourhood we obtain $\frac{\mathrm{Sym}(\mathcal{F}^{-1})}{\mathcal{G}} \cong \mathcal{O}_{X_s}$. Hence, $\mathcal{O}_{X_s}$-locally free sheaf structure on M is isomorphic to the structure induced by $\frac{\mathrm{Sym}(\mathcal{F}^{-1})}{\mathcal{G}}$ via the algebra morphism $\frac{\mathrm{Sym}(\mathcal{F}^{-1})}{\mathcal{G}} \to \mathrm{End}(M)$ defined as $q \mapsto q(\eta.I_M)$. (Indeed any scalar multiple of the identity morphism is annihilated by a linear polynomial.) Here I_M denotes the identity morphism on M. That, $M \cong M'$ if and only if $(M, \eta) \cong (M', \eta)$, reflects the same fact. Also, η as a bundle morphism satisfies equation 2.10 defining X_s. So, $\frac{\mathrm{Sym}(\mathcal{L}^{-1})}{\mathcal{I}}$-structure on $\mathcal{O}(E)$ is obtained as pushforward of $\frac{\mathrm{Sym}(\mathcal{F}^{-1})}{\mathcal{G}}$-structure of M that is, $\pi_*(M, \eta) = (E, \phi)$. Keeping in mind that the spectral correspondence is written in terminology of bundles, we write the following statement for bundles.

Theorem 2.6.4. *Let X_s be a nonsingular, integral spectral curve over X with finite (so proper) covering map π. Then there is a one-to-one correspondence between isomorphism classes of vector bundles M of a finite rank over X_s and L-twisted Hitchin pairs (E, ϕ) over X annihilated by p_s. The correspondence is given by $(M, \eta) \mapsto (\pi_*M, \pi_*\eta)$ using projection formula*

$$\pi_*(M \otimes \pi^*L) \cong \pi_*M \otimes L.$$

If $F \subset E$ is an invariant subbundle then write $E = \pi_* M$ and $F = \pi_* N$ where $N \subset M$ as locally free subsheaf. To show that M/N is locally free we use $\mathcal{O}_X$-isomorphism $\pi_*(M/N) \cong \pi_* M/\pi_* N = E/F$. The latter being locally free, $\pi_*(M/N)$ is locally free so coherent sheaf M/N is indeed locally free due to the correspondence. We mention this fact in the following remark.

Remark 5. The correspondence preserves the subbundles of bundles over X_s and the invariant twisted subbundles over X.

Remark 6. In the classical case, $n = 1$ in Theorem 2.6.4, p is characteristic polynomial of underlying pairs on X.

As explored by Hitchin in [Hit87b] and Beauville, Narasimhan, Ramanan in [BNR89], as well as in further works by others, the generic fiber of Hitchin morphism 2.8 for a fixed rank and a fixed degree is the Jacobian of spectral curve X_s. In language of Hitchin, the sheaf M is defined as $\ker(\eta \cdot I - \pi^* \phi) \otimes \mathcal{L}'$ for a fixed invertible sheaf $\mathcal{L}'$ (see [HH21] Proposition 5.17). In case $n \geq 2$ we should not directly use the phrase 'fiber'(as there is no morphism to replace Hitchin morphism), rather we want to mention that the space of isomorphism classes of L-twisted stable pairs of given rank and degree which are annihilated by p, is a scheme for a generic choice of p, represented by, in case $g_{X_s} > 1$, the moduli space of the S-equivalence classes of semistable bundles on X_s for a fixed rank and a fixed degree. It has structure of an irreducible projective algebraic variety which contains the moduli space of isomorphism classes of stable bundles as an open smooth subvariety, but does not admit an abelian structure in general.

Corollary 2.6.5. *In Theorem 2.6.4, M is a stable (resp. semistable) bundle on X_s if and only if (E, ϕ) is a stable (resp. semistable) L-twisted pair on X.*

Proof. In this context we recall that degree of the pushforward bundle (more generally for pushforward of quasi-coherent sheaves) is given as

$$\deg(\pi_* M) = \deg(M) + \operatorname{rank}(M)(1 - g_{X_s}) - \deg(\pi)\operatorname{rank}(M)(1 - g_X). \tag{2.16}$$

A sample proof if this formula is modeled on [HSW99] Proposition 4.3. An argument that works for pushforward of a line bundle makes sense for vector bundles of arbitrary ranks. The rest follows from Corollary 5 and the obvious fact that a subbundle N of M obeys slope inequality if and only if the subbundle $f_* N$ of $f_* M$ obeys slope inequality 2.2. □

Corollary 2.6.6. *Let M be a semistable bundle on smooth integral spectral curve X_s of rank n. Consider a Jordan-Hölder filtration $0 = M_0 \subset M_1 \subset ... \subset M_k = M$ that is, $\mu(M) = \mu(M_i/M_{i-1})$ and M_i/M_{i-1} is stable for all i. Then a Jordan-Hölder filtration of $\pi_*(M, \eta)$ as defined in 2 is given by $0 = \pi_*(M_0, \eta) \subset \pi_*(M_1, \eta) \subset ... \subset \pi_*(M_k, \eta) = \pi_*(M, \eta)$. On the other hand, let (E, ϕ) be a semistable pair on X which is annihilated by p and a Jordan-Hölder filtration $0 = (E_0, \phi) \subset ... \subset (E_k, \phi) = (E, \phi)$ is obtained as pushforward of a filtration of M such that $\pi_*(M, \eta) = (E, \phi)$. Finally, $\mathfrak{gr}(E, \phi) \cong \pi_* \mathfrak{gr}(M, \eta)$.*

The proof is immediate: as π is a sufficiently nice morphism (in particular, finite), the pushforward operation commutes with quotients and the direct sum of bundles. We extend this equivalence for Harder-Narasimhan filtration of bundles and pairs.

Proposition 2.6.7. *(A) Let E be a vector bundle over a curve X. Then E has a unique increasing filtration by vector subbundles $0 = E_0 \subset E_1 \subset E_2 \subset ... \subset E_k = E$ such that $gr_i = E_i/E_{i-1}$ satisfies the following conditions: (i) quotient gr_i is semistable; (ii) $\mu(gr_i) > \mu(gr_{i+1})$ for $i = 1, ..., k-1$. (B) Likewise, let (E, ϕ) be an L-twisted pair. Then E has a unique increasing filtration by invariant subbundles $0 = E_0 \subset E_1 \subset E_2 \subset ... \subset E_k = E$ such that the quotient pair $gr_i = (E_i/E_{i-1}, \phi_i)$ satisfies following conditions: (i) the quotient gr_i is a semistable pair; (ii) $\mu(gr_i) > \mu(gr_{i+1})$ for $i = 1, ..., k-1$.*

We are able to write a proof of (B) which is identical to the one with vector bundle case in (A). We follow a proof given for vector bundles, available in [Dal07] Lemma 5.6, Proposition 5.7, Lemma 5.8 and Proposition 5.9. A detailed discussion on boundedness properties of bundles can be found in [Pot97].

Corollary 2.6.8. *The Harder-Narasimhan filtration of pairs (E, ϕ) on X are in one-to-one correspondence with the Harder-Narasimhan filtration of bundles M on smooth integral spectral cover X_s.*

2.7 Stable pairs and spectral curves on $\mathbb{P}^1$

We shift our focus to specific base curves for further exploration of the spectral correspondence. For the purpose of this article we explicitly write down the main objects on $\mathbb{P}^1$. Let $t \geq 2$. We denote by $\pi : \mathcal{O}(t) \to \mathbb{P}^1$ the unique line bundle of degree t admitting holomorphic transition data $z \mapsto z^t$ on the set of nonzero complex numbers. For a holomorphic vector bundle E of rank r on $\mathbb{P}^1$. Grothendieck splitting confirms that there are integers $m_1 \geq ... \geq m_r$ unique up to such that

$$E \cong \mathcal{O}(m_1) \oplus .. \oplus \mathcal{O}(m_r). \tag{2.17}$$

A holomorphic bundle map $\phi : E \to E \otimes \mathcal{O}(t)$ is said to be a t-twisted endomorphism on $\mathbb{P}^1$. Due to splitting of E in 2.17 we adopt a global representation of ϕ as an $r \times r$ matrix. The (i, j)-th entry $\phi_{i,j} \in H^0(\mathbb{P}^1, \mathcal{O}(m_i - m_j + t))$ reflecting the component wise maps between $\mathcal{O}(m_j) \to \mathcal{O}(m_i + t)$. Recalling that corresponding tautological line bundle $\pi^*\mathcal{O}(t)$ over $\text{Tot}(\mathcal{O}(t))$ admits a canonical section η the spectral curve defined by sections $s = (s_1, ..., s_r)$ of $\mathcal{O}(t), ..., \mathcal{O}(tr)$ respectively, is the curve

$$X_s = \left\{ y \in \mathcal{O}(t) : \eta^r(y) + s_1(\pi(y))\eta^{r-1}(y) + ... + s_r(\pi(y)) = 0 \right\}. \tag{2.18}$$

It is zero scheme of a global section of the line bundle $\pi^*\mathcal{O}(tr)$. A closer look at the space $\text{Tot}(\mathcal{O}(t))$ gives a clearer understanding of spectral curves. We first realize $\mathbb{P}^1$ as the complex space $(\mathbb{C} \bigsqcup \mathbb{C})/\Phi$ where Φ is a biholomorphism $\Phi : \mathbb{C}^* \to \mathbb{C}^*; \Phi(x) = \frac{1}{x}$. The space $\text{Tot}(\mathcal{O}(t))$ is then realized as $((\mathbb{C} \times \mathbb{C}) \bigsqcup (\mathbb{C} \times \mathbb{C}))/\Psi$ through identifying open subset $\mathbb{C}^* \times \mathbb{C}$ with itself by the biholomorphism $\Psi(x, y) = \left(\frac{1}{x}, \frac{y}{x^t}\right)$. (See for a reference [HSW99] page 39.) Indeed bundle map π takes $[(x, y)]$ to $[x] \in \mathbb{P}^1$. The space of global holomorphic sections of line bundles $\mathcal{O}(ti)$ is characterized by the complex polynomials of degree $\leq ti$. Thus we write a pair of complex affine curves simplifying definition of a spectral curve as following

$$\begin{cases} y^r + s_1(x)y^{r-1} + ... + s_r(x) = 0; \\ \tilde{y}^r + \tilde{s}_1(\tilde{x})\tilde{y}^{r-1} + ... + \tilde{s}_r(\tilde{x}) = 0 \end{cases} \tag{2.19}$$

We identify the disjoint union of curves coming from 2.19 by ψ to put down spectral curve X_s associated to s embedded in $\mathrm{Tot}(\mathcal{O}(t))$. We see that the projection map $\pi : X_s \to \mathbb{P}^1$ defined by $[(x, y)] \mapsto [x]$ is a finite morphism of complex analytic spaces allowing the curve to have singular points.

Lemma 2.7.1. *A spectral curve over $\mathbb{P}^1$ is integral if and only if one of two affine curves is irreducible (as a complex polynomial over two variables).*

Proof. If the spectral curve X_s is integral then the affine curves are irreducible together on the standard open neighbourhoods. If not then we would obtain a factorization of associated polynomials. On the other hand, let suppose that

$$y^r + s_1y^{r-1} + ... + s_r = (y^{r_1} + u_1y^{r_1-1} + ... + u_{r_1})(y^{r_2} + v_1y^{r_2-1} + ... + v_{r_2})$$

on one of two affine coordinate charts, meaning that all coefficients u_i's and v_j's are elements of $\mathbb{C}[x]$. Here $0 < r_1, r_2 < r$. It is enough to prove that $\deg(u_i) \leq ti$ and $\deg(v_j) \leq tj$ to show that polynomial on the other chart is reducible. Using the change of coordinates we obtain a factorization

$$\tilde{y}^r + \tilde{s}_1\tilde{y}^{r-1} + ... + \tilde{s}_r = (\tilde{y}^{r_1} + \tilde{u}_1\tilde{y}^{r_1-1} + ... + \tilde{u}_{r_1})(\tilde{y}^{r_2} + \tilde{v}_1\tilde{y}^{r_2-1} + ... + \tilde{v}_{r_2}).$$

Note that the left hand side is a monic polynomial with coefficients over UFD $\mathbb{C}[\tilde{x}]$ and the coefficients $\tilde{u}_i$'s and $\tilde{v}_j$'s are elements of $\mathbb{C}(\tilde{x})$. This is possible precisely when $\tilde{u}_i$'s and $\tilde{v}_j$'s are elements of $\mathbb{C}[\tilde{x}]$ that is, $\deg(u_i) \leq ti$ and $\deg(v_j) \leq tj$ for all i, j. The spectral curve X_s is both reduced and irreducible via the irreducible affine algebraic curves because in Zariski topology, ideal generated by each of the polynomials is an integral domain. This discussion indicates that all information of such a spectral curve is extracted from one single affine chart. □

We highlight a specific class of examples of non-generic points (in the sense that their discriminant sections do not necessarily admit distinct zeros) on the affine base. Let $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$ be a section with distinct zeros over $\mathbb{P}^1$. The set of such sections is a Zariski open subset of the affine space $H^0(\mathbb{P}^1, \mathcal{O}(tr))$ so we call such elements as *generic sections* of $\mathcal{O}(tr)$.

Definition 2.7.2. Let $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$. We call a t-twisted pair (E, ϕ) *cyclic* if it admits characteristic polynomial $\lambda^r - s$. In case s is a generic element that is, admits simple roots we call (E, ϕ) a *generic cyclic pair* and corresponding spectral curve a *generic cyclic spectral curve.*

Remark 7. A partial justification of above definition will be given in section 3.5.

The definition generalizes classical cyclic Higgs bundles associated to cyclic quivers. Observe that spectral polynomial $\lambda^r - s$ is not in smooth locus for $r > 2$. For $r = 2$ we have a spectral curve belonging to smooth locus. This is due to the fact that its resultant is given by (a nonzero constant multiple of) s^{r-1}. Smooth, integral cyclic spectral covers are characterized in [BNR89] Remark 3.1 and Remark 3.5. Generic cyclic spectral covers are integral. Choose a section s of the line bundle L^r such that s admits all distinct zeros. So we can not write this divisor of L^r as $m \cdot D$ for some divisor D on X with $m > 1$ dividing r. Moreover, this generic cyclic spectral covers are smooth due to Jacobian criterion of smoothness. We mention another algebraic proof in this context.

Lemma 2.7.3. *A generic cyclic spectral curve is integral on $\mathbb{P}^1$ so a generic cyclic pair is stable (for $r \geq 2$).*

Proof. All that we need to confirm is that the polynomial $y^r - s(x)$ on one affine coordinate is irreducible over $\mathbb{C}[x]$. This is a consequence of Eisenstein's Criterion of irreducibility over UFD $\mathbb{C}[x]$ because the roots of the polynomial defined by s are distinct. □

The next theorem restricts Grothendieck numbers (i.e. the degrees of the summand line bundles of the underlying bundle over $\mathbb{P}^1$) of semistable Hitchin pairs.

Theorem 2.7.4. *Let $E \cong \mathcal{O}(m_1) \oplus .... \oplus \mathcal{O}(m_r)$ be a vector bundle over $\mathbb{P}^1$ such that $m_1 \geq ... \geq m_r$. Let $t \geq 2$ be an integer. If E admits a t-twisted semistable Hitchin pair then, for $1 \leq i \leq r-1$,*

$$m_i \leq m_{i+1} + t. \tag{2.20}$$

Let suppose E follows inequality 2.20 and $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$. Then there exists a cyclic pair (E, ϕ_s) with characteristic coefficients $(0, .., 0, s)$.

Proof. The proof of 2.20 appeared as ([Ray13] Theorem 6.1). We prove the rest of the statement of Theorem 2.7.4. Let E be a bundle over $\mathbb{P}^1$ such that $m_i \leq m_{i+1} + t$ for all $1 \leq i \leq r-1$ we have $H^0(\mathbb{P}^1, \mathcal{O}(m_{i+1} - m_i + t))$. See that Grothendieck numbers satisfy following identity $(m_2 - m_1 + t) + ... + (m_r - m_{r-1} + t) + (m_1 - m_r + t) = tr$ while $0 \leq m_2 - m_1 + t \leq t$, ..., $0 \leq m_r - m_{r-1} + t \leq t$. Thus $t \leq m_1 - m_r + t \leq tr$. Now choose a section $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$. We can represent s on an affine chart by a complex polynomial of degree $\leq tr$. By the fundamental theorem of algebra we exploit our liberty to distribute these roots, (it doesn't matter if they are distinct or not) wherever permitted. An explanation of the limiting cases suffices here. In the case that $m_2 - m_1 + t = ... = m_r - m_{r-1} + t = 0$ we have

$m_r - m_1 + t = tr$. This leads to construction of ϕ_s (adjusting signs as necessary) as

$$\phi_s = \begin{bmatrix} 0 & 0 & ... & ... & \pm s \\ 1 & 0 & ... & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}.$$

At the other extreme situation we have $m_2 - m_1 + t = ... = m_r - m_{r-1} + t = t$ and $m_r - m_1 + t = t$. We write $s = u_1 ... u_r$ each of degree $\leq t$ (perhaps some of the u_i's are 1) and construct ϕ_s (adjusting signs as necessary) as

$$\phi_s = \begin{bmatrix} 0 & 0 & ... & ... & \pm u_r \\ u_1 & 0 & ... & ... & 0 \\ 0 & u_2 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & u_{r-1} & 0 \end{bmatrix}.$$

To prove the statement in the other cases, we factorize a complex polynomial adjusting degrees of factors. Hence the result. □

The example of cyclic pairs we explicitly mentioned are associated to $(1, ..., 1)$-cyclic chains of t-twisted pairs. Topology and geometry of stable holomorphic chains appeared in [RS21]. We mention a group action by an $(r-1)$-dimensional compact group on such cyclic holomorphic chains following a stability criterion on cyclic holomorphic chains.

Let E be a bundle on $\mathbb{P}^1$ and $(u_1, ..., u_r)$ be sections on $\mathbb{P}^1$ as above. If $u_i = 0$ for some $1 \leq i \leq r-1$ then $E_i \cong \mathcal{O}(m_1) \oplus ... \oplus \mathcal{O}(m_i)$ is invariant and $\mu(E_i) \geq \mu(E)$. To confirm stability, we restrict $u_i \neq 0$ for $1 \leq i \leq r-1$. On the other hand, let $u_i \neq 0$ for $1 \leq i \leq r-1$ and $u_r \neq 0$. Then there is no nonzero proper invariant subbundle and stability is achieved at no cost. If $u_r = 0$ then a nonzero proper invariant subbundle is either of $\mathcal{O}(m_r); ...; \mathcal{O}(m_r) \oplus ... \oplus \mathcal{O}(m_1)$, confirming semistability of this pair. Finally, we restrict $\deg(E)$ and $\mathrm{rank}(E)$ to be mutually prime to confirm stability of each semistable cyclic pair.

The space of $(1,...,1)$-cyclic chains enjoy a group action by $\mathbb{S}^1 \times ... \times \mathbb{S}^1$ ($r-1$ times) as following:

$$(\lambda_1,...,\lambda_{r-1}).\begin{bmatrix} 0 & 0 & ... & ... & \phi_r \\ \phi_1 & 0 & ... & ... & 0 \\ 0 & \phi_2 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & \phi_{r-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & ... & ... & \lambda_1^{-1}...\lambda_{r-1}^{-1}\phi_r \\ \lambda_1\phi_1 & 0 & ... & ... & 0 \\ 0 & \lambda_2\phi_2 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_{r-1}\phi_{r-1} & 0 \end{bmatrix}.$$

If two such chains are equivalent under the group action they are isomorphic as pairs. Indeed we obtain $(E,(\lambda_1,...,\lambda_{r-1}).\phi) \cong (E,\phi)$ since $(\lambda_1,...,\lambda_{r-1}).\phi = \psi\phi\psi^{-1}$ where ψ denotes a diagonal matrix with i-th diagonal entry $\lambda_1...\lambda_{i-1}$. We denote orbit space on $\mathbb{P}^1$ by $\mathcal{M}(m_1,...,m_r,t)$ keeping in mind that $\sum m_i$ is co-prime to r and describe the moduli as quotient

$$\frac{\prod_{i=1}^{r-1} \mathbb{C}^{m_{i+1}-m_i+t+1}\backslash\{0\} \times \mathbb{C}^{m_1-m_r+t+1}}{(\mathbb{S}^1)^{r-1}}.$$

We have a proper group action by a compact Hausdorff group. Thus this quotient is a Hausdorff space and orbits are closed submanifolds of the parent space.

2.8 Numerical computation of co-Higgs sheaves

Over $\mathbb{P}^1$ an $\mathcal{O}(2)$-twisted pair is referred to as a *co-Higgs bundle*. A generic smooth spectral curve embedded in $\mathcal{O}(2)$ is an elliptic curve contributed by a co-Higgs bundle using formula in 2.15. The moduli space of semistable rank 2 co-Higgs bundles was explored in [Ray13] with greater detail including computation of topological invariants of moduli spaces. We explore the spectral correspondence for co-Higgs bundles of higher ranks. We exploit results in [Ati57b] at this stage to compute pushforward of vector bundles over an elliptic curve. Parts of them can be found in [Ray13]. We are able to extend our computations over an algebraically closed field of characteristic 0. All along we work with an elliptic curve X along with a $2:1$ branched covering map $f: X \to \mathbb{P}^1$.

Let E be an indecomposable bundle of rank n over X. **Case I:** Let $\deg(E) = 0$. Then degree of f_*E over $\mathbb{P}^1$ is given by, $\deg(f_*E) = -2n$. From Birkhoff-Grothendieck theorem, we write (as sum of line bundles) $f_*E \cong \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_{2n})$ and $a_1 + ... + a_{2n} = -2n$. Here $\dim H^0(X,E) = 0$ or 1, from [Ati57b] Lemma

15. (a) If $\dim H^0(X,E) = 0$ then $a_i < 0$, for all values of i. If not then it leads to $\dim H^0(X,E) \geq 1$. Thus, $a_i \leq -1$ holds $\forall i$ and $\sum_{i=1}^{2n} a_i \leq -2n$. In fact, equality holds if and only if $a_i = -1$, for all values of i. Thus,

$$f_*E \cong \mathcal{O}(-1) \oplus ... \oplus \mathcal{O}(-1). \tag{2.21}$$

(b) If $\dim H^0(X,E) = 1$ then $\exists i$ such that $a_i \geq 0$. If there are i,j with $a_i \geq 0$ and $a_j \geq 0$ then $\dim H^0(X,E) \geq 2$. So, $a_i \geq 0$ for only one value of i say $i = 1$. On the other hand, $a_1 > 0 \implies \dim H^0(X,E) > 1$, so $a_1 = 0$. Putting $a_1 = 0$, we have $a_2 + ... + a_{2n} = -2n$ with $a_2 < 0, ..., a_{2n} < 0$. From, $a_2 \leq -1, ..., a_{2n} \leq -1$ we manage $-2n+1$ and there is exactly one $i \geq 2$, say, $i = 2$ such that $a_2 = -2$. Thus,

$$f_*E \cong \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus ... \oplus \mathcal{O}(-1). \tag{2.22}$$

Case II: Let $\deg(E) = 1$. From [Ati57b] Lemma 15, $\dim H^0(X,E) = 1$. Let suppose $f_*E \cong \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_{2n})$. From degree computation of pushforward bundle, we have $\deg(f_*E) = 1 - 2n$. From earlier reasoning $a_1 = 0$ and thus $a_2 + ... + a_{2n} = 1 - 2n$. On the other hand, $a_2 \leq -1, ..., a_{2n} \leq -1$. This leads to, $a_2 = ... = a_{2n} = -1$. Finally,

$$f_*E \cong \mathcal{O} \oplus \mathcal{O}(-1) \oplus .. \oplus \mathcal{O}(-1). \tag{2.23}$$

Lemma 2.8.1. *Let E be an indecomposable vector bundle over an elliptic curve X such that $\deg(E) < 0$. Then $H^0(X,E)$ is trivial.*

Proof. Suppose that $\deg(E) = d < 0$. As E is indecomposable E^* is also indeomposable and so is $E^* \otimes K_X$. Now $\deg(E^* \otimes K_X) = \deg(E^*) = -d > 0$. From [Ati57b] Lemma 15, we have $\dim H^0(X, E^* \otimes K_X) = \deg(E^* \otimes K_X) = -d$. We have used that $\deg(K_X) = 0$. From Serre duality theorem, $\dim H^1(X,E) = -d$. Then from Riemann-Roch theorem, $\dim H^0(X,E) - \dim H^1(X,E) = \deg(E) = d \implies \dim H^0(X,E) = 0$. □

Case III: Let $\deg(E) = -1$. Then $\dim H^0(X,E) = 0$. On the other hand, $\deg(f_*E) = -1 - nr$. Writing $f_*E \cong \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_{2n})$ with $a_1 \leq -1, ..., a_{2n} \leq -1$. From the argument we have used in earlier cases, we have exactly unique i, say 1, such that $a_1 = -2$ and others $a_2, .., a_{2n}$ are -1. Thus we have

$$f_*E \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus ... \oplus \mathcal{O}(-1). \tag{2.24}$$

There is no immediate technique available if d is other than $0, 1, -1$. Let L be a line bundle over X of arbitrary degree. Then we can compute f_*L shifting its degree by projection formula [Ray13]. However, we finish this procedure in case of an indecomposable rank 2 bundle E over X which has degree 2. This suffices to put down pushforward of all semistable bundles of rank 2 over X. According to Appendix A (which relies upon [Tu93]), we confirm semistability of the indecomposable vector bundles over an elliptic curve. So we will be left with the decomposable semistable ones. A semistable bundle $E = L_1 \oplus L_2$ must have $\deg(L_1) = \deg(L_2) = m$ (say). Whatever be m we compute f_*E by the above computations $n = 1$ and the projection formula. On the other hand, if E is an indecomposable bundle of rank 2 using projection formula and it suffices to compute for $\deg(E) \in \{-1, 0, 1, 2\}$. Also decomposability remains invariant with multiplication by line bundles. To resolve $\deg(E) = 2$ case we introduce important definitions from [Ati57b].

Definition 2.8.2. Let X be a smooth projective algebraic curve and E be a vector bundle over X of rank r. A line subbundle L is said to be a *maximal line subbundle* if $\deg(L)$ is maximal among all line subbundles of E.

Definition 2.8.3. As per considerations in Definition 2.8.2, a collection of line bundles $(L_1, ..., L_r)$ is said to be a *maximal splitting* of E if (i) L_1 is a maximal line subbundle of E. (ii) $(L_2, ..., L_r)$ is a maximal splitting of E, inductively.

Lemma 2.8.4. ([Ati57b] Lemma 11) *Let E be an indecomposable vector bundle of rank r and degree r over an elliptic curve X. Then E has a maximal splitting $(L, ..., L)$ with* $\deg(L) = 1$.

Now consider an indecomposable vector bundle E of rank 2 and degree 2. It admits a subbundle L of rank 1 with $\deg(L) = 1$ and $L \cong E/L$. We recall that pushforward of bundles commutes with quotients. Also we mention from our previous computation that $f_*L \cong \mathcal{O} \oplus \mathcal{O}(-1)$ while L is a line bundle on elliptic curve X of degree 1. Using these we have $\mathcal{O} \oplus \mathcal{O}(-1) \cong f_*E/(\mathcal{O} \oplus \mathcal{O}(-1))$. Thus we can write the transition data of f_*E directly as

$$g_{\alpha\beta}(z) = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & \frac{1}{z} \end{matrix} & h_{\alpha\beta}(z) \\ \mathbf{0} & \begin{matrix} 1 & 0 \\ 0 & \frac{1}{z} \end{matrix} \end{bmatrix} \tag{2.25}$$

The function $h_{\alpha\beta}(z)$ uniquely correspond to an element in the cohomology group $H^1(\mathbb{P}^1, \mathrm{Hom}(E/F, F))$. In particular in 2.25 the latter space to be $H^1(\mathbb{P}^1, \mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1)))$. On the other hand, using formula of the endomorphism bundle, from Serre duality theorem we obtain that $\dim_{\mathbb{C}} H^1(\mathbb{P}^1, \mathrm{End}(\mathcal{O} \oplus \mathcal{O}(-1))) = 0$. The zero element in the vector space uniquely correspond to the splitting of E as $F \oplus E/F$ (see Proposition 2, [Ati57a]). Thus we can write 2.25 as

$$g_{\alpha\beta}(z) = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & \frac{1}{z} \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 1 & 0 \\ 0 & \frac{1}{z} \end{matrix} \end{bmatrix}$$

leading to our conclusion

$$f_*E \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1). \tag{2.26}$$

Remark 8. A careful observation leads to the understanding that given an indecomposable vector bundle E of rank r and degree r we have

$$f_*E \cong \underbrace{\mathcal{O} \oplus ... \oplus \mathcal{O}}_{r\text{-times}} \oplus \underbrace{\mathcal{O}(-1) \oplus ... \oplus \mathcal{O}(-1)}_{r\text{-times}}.$$

The proof follows from the application of the splitting principle and Serre duality theorem. We illustrate another example, $r = 3$ and use the principle of mathematical induction.

From the maximal splitting principle we have a degree 1 line bundle L such that

$$L \cong \frac{E/L}{L}.$$

Then taking pushforward by f and writing the transition data we obtain that

$$f_*(E/L) \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

By the same line of reasoning, we obtain

$$f_*E \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

However, if F is a subbundle of E then $H^1(X, \mathrm{Hom}(E/F, F))$ is not necessarily trivial. For example one can choose $E \cong \mathcal{O}(2) \oplus \mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O}(6)$ and $F \cong \mathcal{O}(4) \oplus \mathcal{O}(6)$.

The following lemma plays an important role in explicit construction of semistable co-Higgs bundles.

Lemma 2.8.5. *Let X be an algebraic curve. If M_1 and M_2 are line bundles with same degree over X then $M_1 \oplus M_2$ is semistable.*

Proof. Let us consider a line subbundle M of $M_1 \oplus M_2$. Then one of following bundle morphisms must be nonzero: $M \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi_1} M_1; M \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi_2} M_2$ where π_1 and π_2 are bundle projection maps on M_1 and M_2. This leads to one of $H^0(X, \mathrm{Hom}(M, M_1))$ or $H^0(X, \mathrm{Hom}(M, M_2))$ being non trivial. Thus $\deg(M) \leq \deg(M_1) = \deg(M_2)$. Thus $\mu(M) \leq \mu(M_1 \oplus M_2)$. Hence the result. □

Remark 9. We can extend this result for vector bundles of higher ranks also. Let M and N be semistable vector bundles of same slope μ on X. Then $M \oplus N$ is semistable.

Proof. See [Lan75] Example 2. On smooth curves every torsion-free sheaf is locally free. □

We implement Lemma 2.8.5 to construct semistable pairs of rank 4 over the projective line. The bundles which this construction yields (for $\deg(L_1) = \deg(L_2) = 0$ and $\deg(L_1) = \deg(L_2) = 1$) are $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2); \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2); \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. First we choose a smooth integral spectral elliptic curve defined by $s = (s_1, s_2) \in H^0(\mathcal{O}(2)) \oplus H^0(\mathcal{O}(4))$. Then define a Higgs bundles $(E, \phi) = (E_1, \phi_1) \oplus (E_2, \phi_2)$ such that $\deg(E_1) = \deg(E_2)$ and (E_i, ϕ_i) admits characteristic polynomial p_s. From the correspondence in Corollary 2.6.5, (E, ϕ) is semistable. Unfortunately, we are unable to explicitly construct semistable co-Higgs bundles while the underlying bundle has an odd degree. There exist stable co-Higgs bundles with odd degree obtained as pushforward of indecomposable bundles of rank 2 and odd degree over a spectral elliptic curve but we did not succeed in constructing one in matrix form. In any case, moduli description of the set of co-Higgs pairs for a fixed rank and a fixed degree is given by a theorem in [Tu93].

Theorem 2.8.6. ([Tu93] Theorem 1) *The moduli space $\mathcal{M}_{n,d}(C)$ of S-equivalence classes of semistable bundles of rank n, degree d over an elliptic curve curve C is isomorphic to the h-th symmetric product S^hC of the curve C where h is given as the greatest common divisor of n and d.*

Remark 10. The moduli space of stable bundles of rank n and degree d is isomorphic to C when n and d are relatively prime and empty otherwise.

We end this section with some moduli description of rank 4 co-Higgs sheaves. In the above construction a Jordan-Hölder series is obtained immediately. The pairs (E_i, ϕ_i) are stable for $i = 1, 2$ and each of them forms a Jordan-Hölder series of the parent pair. In [Ray13] the semistable (so stable) traceless co-Higgs bundles with underlying bundle $\mathcal{O} \oplus \mathcal{O}(-1)$ are characterized which will match in our present case of a generic cyclic spectral elliptic curve to a 1-complex dimensional space (that is, the spectral curve itself). On the other hand, only stable co-Higgs bundles with underlying bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ are *uniformizing* pairs on $\mathbb{P}^1$.

3 Iterated spectral correspondence

3.1 Iterated spectral correspondence

In this section we illustrate another side of the spectral correspondence discussed in section 2.6. We investigate a composite projection formula 3.2.3 of locally free sheaves under composition of finite morphisms. This begs an immediate question about the factorizability of a smooth spectral covering map. We arrive at an affirmative answer in a foundational, yet ultimately nontrivial, case of the complex projective line (Section 3.5). We focus on a typical class of non-generic spectral curve that we call a *cyclic spectral curve* due to the fact that their Galois groups are cyclic (Section 3.5). The Fundamental Theorem of Galois Theory and a categorical equivalence between function fields and algebraic curves (Proposition 3.4.1) are employed to complete the argument. We assemble the consequences of this so-called *iterated spectral correspondence* in Theorem 3.5.2. We close the article by revisiting a theorem of J.F. Ritt (Theorem 3.4.6) that inspired us during the course of this investigation. We also pose a conjecture 3.4.7 that anticipates a new direction of research.

3.2 A composite projection formula

In this section we illustrate another side of the spectral correspondence. As a result of the study above for $\mathbb{CP}^1$, we can embed Jacobian of a generic cyclic spectral curve into the open subscheme of stable Hitchin pairs over an intermediate curve that covers $\mathbb{CP}^1$ and admits the spectral cover as a branched cover. Here we warn the reader about shifting to the language of complex manifolds and Riemann surfaces and related reasoning. We will use statements of the spectral correspondence developed so far in this section. An important perspective of this investigation is giving an alternate description of solutions of Lax pair equations ([Bea90], [HSW99]). We first prove a composite version of the 'push-pull' projection formula. See [Gun67] for prerequisites. The following theorem 3.2.1 is well-known. We attach a proof for the sake of completeness.

Theorem 3.2.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of complex manifolds. Consider a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules and a sheaf $\mathcal{G}$ of $\mathcal{O}_Z$-modules. Then,* (i) *$(g \circ f)^*\mathcal{G} \cong f^*g^*\mathcal{G}$; and* (ii) *$(g \circ f)_*\mathcal{F} \cong g_*f_*\mathcal{F}$.*

Proof. To prove (i) we use sheaf isomorphisms of $\mathcal{O}_X$-modules given as $(g \circ f)^{-1}\mathcal{G} \cong f^{-1}g^{-1}\mathcal{G}$ and

$$f^{-1}(g^{-1}\mathcal{G} \bigotimes_{g^{-1}\mathcal{O}_Z} \mathcal{O}_Y) \cong f^{-1}g^{-1}\mathcal{G} \bigotimes_{f^{-1}g^{-1}\mathcal{O}_Z} f^{-1}\mathcal{O}_Y.$$

Recall that a given morphism of sheaves is an isomorphism of sheaves if and only if it boils down to an isomorphism of respective stalks at points. These sheaf isomorphisms can be verified by the isomorphisms of stalks at points. At each point $y \in Y$ $(g^{-1}\mathcal{G})_y \cong \mathcal{G}_{g(y)}$. Thus $(f^{-1}g^{-1}\mathcal{G})_x \cong (g^{-1}\mathcal{G})_{f(x)} \cong \mathcal{G}_{g(f(x))}$ at each point $x \in X$. Keeping bimodule structure of sheaves and stalks of sheaves in mind, we directly write sheaf isomorphisms

$$\begin{aligned} f^*g^*\mathcal{G} &= f^*(g^{-1}\mathcal{G} \bigotimes_{g^{-1}\mathcal{O}_Z} \mathcal{O}_Y) \\ &= f^{-1}(g^{-1}\mathcal{G} \bigotimes_{g^{-1}\mathcal{O}_Z} \mathcal{O}_Y) \bigotimes_{f^{-1}\mathcal{O}_Z} \mathcal{O}_X \\ &\cong (f^{-1}g^{-1}\mathcal{G} \bigotimes_{f^{-1}g^{-1}\mathcal{O}_Z} f^{-1}\mathcal{O}_Y) \bigotimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &\cong f^{-1}g^{-1}\mathcal{G} \bigotimes_{f^{-1}g^{-1}\mathcal{O}_Z} (f^{-1}\mathcal{O}_Y \bigotimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong f^{-1}g^{-1}\mathcal{G} \bigotimes_{f^{-1}g^{-1}\mathcal{O}_Z} \mathcal{O}_X \\ &\cong (g \circ f)^{-1}\mathcal{G} \bigotimes_{(g\circ f)^{-1}\mathcal{O}_Z} \mathcal{O}_X = (g \circ f)^*\mathcal{G}. \end{aligned}$$

Proof of (ii) follows from $(g \circ f)_*\mathcal{F}(U) = g^{-1}f^{-1}(U) = g^{-1}(f_*\mathcal{F}(U)) = (g_*f_*\mathcal{F})(U)$. □

Theorem 3.2.2. *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of complex analytic spaces and if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules and $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_Y$-modules of finite rank then there is a natural isomorphism of sheaves of $\mathcal{O}_Y$-modules $f_*(\mathcal{F} \bigotimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \bigotimes_{\mathcal{O}_Y} \mathcal{E}$.*

Corollary 3.2.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y); g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ and $h : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ be morphisms of complex analytic spaces satisfying $h = g \circ f$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules and $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_Z$ modules of finite rank then there is an isomorphism of $\mathcal{O}_Y$-modules $f_*(\mathcal{F} \bigotimes_{\mathcal{O}_X} h^*\mathcal{E}) \cong f_*\mathcal{F} \bigotimes_{\mathcal{O}_Y} g^*\mathcal{E}$.*

The isomorphism in corollary 3.2.3 is our *composite projection formula.* Next remark is the key information we will explore in rest of this article.

Remark 11. Let us consider assumptions in 3.2.3. Pushforward of a $f^*\mathcal{E}$-twisted pair (E, ϕ) on X, by f, is a $g^*\mathcal{E}$-twisted pair on Y.

In particular, we choose compact Riemann surfaces X, Y, Z with nonconstant holomorphic maps $f : X \to Y, g : Y \to Z$ and $h : X \to Z$ satisfying $h = g \circ f$. We fix holomorphic vector bundles F on X and E on Z. There is an isomorphism of vector bundles from corollary 3.2.3 $f_*(F \otimes h^*E) \cong f_*F \otimes g^*E$. On the other hand, denote by $\mathcal{O}(GL(r, \mathbb{C}))$ the (multiplicative) sheaf of holomorphic maps, on a compact Riemann surface, valued in nonsingular matrices of order r. Recall that a holomorphic vector bundle E on Z is an element $\{g_{\alpha\beta}\} \in H^1(Z, \mathcal{O}(GL(r, \mathbb{C})))$ where $r = \text{rank}(E)$. Thus, pullback bundles g^*E and f^*g^*E have representatives $\{(g_{\alpha\beta} \circ g)\} \in H^1(Y, \mathcal{O}(GL(r, \mathbb{C})))$ and $\{(g_{\alpha\beta} \circ g \circ f)\} \in H^1(X, \mathcal{O}(GL(r, \mathbb{C})))$ respectively.

Also pushforward of a holomorphic vector bundle is a holomorphic vector bundle. We can, in the same spirit, define twisted bundle pairs on compact Riemann surfaces and interpret Remark 11 in the context of holomorphic vector bundles: Let L be a line bundle on Z and $(E, \phi : E \to E \otimes h^*L)$ be a a bundle pair on X. Then $(f_*E, f_*\phi : f_*E \to f_*E \otimes g^*L)$ is a (g^*L-twisted) bundle pair on Y.

3.3 Imprimitive subgroups of a permutation group and monodromy groups

We briefly set the stage for a problem based on the modern aspect of computational group theory and enumerative algebraic geometry. Topological branched covers of oriented 2-manifolds is a research area that overlaps with the computational theory of permutation groups, graph theory, combinatorics, partition theory ([LZ10], [PP06]) and theory of dessins d'enfants ([JW18]). Classification of subgroups of finite permutation groups into *primitive* and *imprimitive* categories opens up an angle to study branched covers of surfaces. About 100 years ago, Ritt ([Rit23]) initiated techniques for studying the 2-sphere, which lead to modern study of *cartographic groups* or *monodromy groups* via *generalized constellations*. Ritt's theorem suggests that such groups are indispensable in illustrating the factorization of branched covers in to intermediate covers. A standard source for details of *oriented hypermaps* and cartographic groups is Chapter 1 of [LZ10]. A stronger form of Ritt's theorem plays an decisive role in the Hurwitz problem as well (constructing branched maps between oriented surfaces with prescribed branch data) ([PP06] Lemma 5.2 and Corollary 5.3). Interstingly, cartographic groups are closely related to Galois groups of holomorphic maps (cf. [Har79] page 689).

Let X and Z be compact connected Riemann surfaces. Given a nonconstant holomorphic map $\pi : X \to Z$ there is a unique degree d such that fiber of each point $z \in Z$ contains d points in X counting up to multiplicities. There is a finite subset $B \subset Z$ such that $\pi : X' = X \backslash \pi^{-1}(B) \to Z \backslash B$ is a topological covering map of degree d and is thus a local homeomorphism for each point $x \in X' = X \backslash \pi^{-1}(B)$. Fix a point $z_0 \in Z' = Z \backslash B$. Given a loop γ based at z_0, lifts of γ give d paths in Z permuting points in the fiber of z_0. Collection of all such permutations forms a transitive subgroup of S_d (recall that a subgroup H of permutation group of d symbols S_d is said to be a *transitive* subgroup if for each pair of symbols a_i, a_j there is an element $\sigma \in H$ such that $\sigma(a_i) = a_j$) which can be realized as image of a group homomorphism (cited as *monodromy representation*) $\rho : \pi_1(Z', z_0) \to S_d$. This transitive subgroup is defined to be the *cartographic group* or the *monodromy group* of π at z_0. It is important to note that a group homomorphism $\rho : \pi_1(Z', z_0) \to S_d$ with a transitive image gives a compact Riemann surface X with a nonconstant holomorphic map π which has branch points in B ([Mir95] page 91).

The problem of computing the monodromy groups is a complicated one in general. It necessitates application of several numerical techniques simultaneously. One can see this at play in, for example, sections 2.2, 3.1, and 3.2 of [DKPR22] . A closely related problem is determination of the Galois group of a branched holomorphic cover. A nonconstant branched holomorphic map $\pi : X \to Z$ is not necessarily a Galois cover in the sense that the field extension $\mathcal{M}(X)/\pi^*\mathcal{M}(Z)$ is not Galois. We consider the Galois closure $\mathcal{M}(X)^{\mathrm{Gal}}$ of $\mathcal{M}(X)$ and the Galois group $\mathcal{M}(X)^{\mathrm{Gal}}/\pi^*\mathcal{M}(Z)$. Now, a theorem by Harris in [Har79] states that fixing a generic base point $z_0 \in Z'$ we can embed the group $\mathrm{Gal}(\mathcal{M}(X)^{\mathrm{Gal}}/\pi^*\mathcal{M}(Z))$ in S_d and this image of the embedding is same as the monodromy group. This result opens up a direction of computation of the monodromy groups at a generic point — particularly when we can determine Galois groups from extensions defined by certain algebraic equations. We refer readers to Chapter 1 of [LZ10] and to [BRSY21] for further background.

Definition 3.3.1. Let $r \in \mathbb{N}$. A group of permutations G over r symbols $\{a_1, .., a_r\}$ is said to be *imprimitive* if there is a partition $\{B_1, ..., B_k\}$ of $\{a_1, .., a_r\}$, each of size l with $r > l > 1$, such that for each element $g \in G$ we have $g(B_i) = B_j$ that is, image of each block is some block.

The following examples are elementary.

Example 3.3.2. Any cyclic group generated by an r-cycle in permutation group of r symbols is transitive and in case r is composite, it is an imprimitive subgroup. ♢

Proof. Let $\{a_1, ..., a_r\}$ be r symbols. It suffices to prove the statement only for $< \sigma = (a_1....a_r) >$. Choose i, j, then $\sigma^{i-1}(a_1) = a_i$ and $\sigma^{j-1}(a_1) = a_j$. Thus $\sigma^{j-1} \circ \sigma^{1-i}(a_i) = a_j$. Let $u \geq 2$ be a divisor of r and $r = uv$. Organize v blocks $B_1 = [a_1, a_{u+1}, ..., a_{(v-1)u+1}], B_2 = [a_2, a_{u+2}, ..., a_{(v-1)u+2}], \dots, B_u = [a_p, a_{2p}, ..., a_r]$. It suffices to prove that for each B_i, $\sigma(B_i) = B_j$ for some j as $\sigma^2(B_i) = \sigma(B_j) = B_k$ and so on. Finally, the last statement is a direct consequence of $\sigma(B_i) = B_{i+1}$ for $i < u$ and $\sigma(B_u) = B_1$. This finishes our argument. □

Example 3.3.3. Let r be a composite number and a subgroup of order r of S_r be generated by an element $\sigma = \sigma_1\sigma_2$ where σ_1 and σ_2 are mutually disjoint. This is an imprimitive subgroup. ♢

Proof. It suffices to find blocks of same length that divides r. Let the lengths of σ_1 and σ_2 be l_1 and l_2 respectively. So, $l_1, l_2 \geq 2$. Let $q = \mathrm{g.c.d}(l_1, l_2) > 1$. Make blocks of length q out of σ_1 and σ_2 by the above procedure of 3.3.2. Among the remaining $r - (l_1 + l_2)$ elements we pick up identity blocks of length q because q divides $r - (l_1 + l_2)$. Otherwise, let $q = 1$. We assume without loss of generality that $l_1 > l_2$ because $l_1 = l_2$ is not an option. Also we have $l_1.l_2 = \mathrm{l.c.m}\{l_1, l_2\} = r$. As $l_2 \geq 2$ we have $r = l_1.l_2 \geq 2l_1$. Among the remaining $r - (l_1 + l_2)$ elements choose $l_1 - l_2$ elements (they are mapped to themselves by σ) and attach to σ_2 to make a block σ_2' of length l_1. Thus σ_1 gives a block of length l_1 say B_{σ_1}, σ_2' gives a block of length l_1, say B_{σ_2} and among the remaining $r - 2l_1$ elements, each mapping to itself by σ, we choose $\frac{r-2l_1}{l_1}$ many blocks each of length l_1, denoting each of them as B_i. Now the subgroup generated by σ is imprimitive because action σ on S_r preserve the blocks due to, $\sigma(B_{\sigma_1}) = B_{\sigma_1}$ and $\sigma(B_{\sigma_2'}) = B_{\sigma_2'}$ and finally $\sigma(B_i) = B_i$. □

Remark 12. If a cyclic subgroup H of order r in S_r is transitive then it is generated by an r-cycle.

Proof. Suppose H is generated by $\sigma = \sigma_1 ... \sigma_k$. Now choose symbols b_1 appearing in σ_1 and b_2 appearing in σ_2. Note that there is no element in H that maps b_1 to b_2 because elements of the cycles $\sigma_1, ..., \sigma_k$ are permuted within themselves. We conclude that if the monodromy group of an $r:1$ cover is cyclic of order r (taking r composite) then it is imprimitive, from Example 3.3.2. □

Definition 3.3.4. A map $\pi : Y \to X$ between compact Riemann surfaces is said to be *factorizable* if there exists a compact Riemann surface Z and nonconstant holomorphic maps $f : Y \to Z$ and $g : Z \to X$, both of degree > 1 such that $\pi = g \circ f$.

For completeness, we develop an abridged yet nonetheless explicit proof of Ritt's theorem which states that factorization of a branched cover of surfaces into two intermediate branched covers is possible precisely for imprimitive monodromy/Galois groups. A topological argument for Ritt's theorem in Theorem 1.7.6 of [LZ10] relies only on engineering of monodromy groups. A similar proof works for the holomorphic covers between compact Riemann surfaces.

Corollary 3.3.5. *If the monodromy/Galois group of an $r : 1$ holomorphic cover π of Riemann surfaces is a cyclic group of order r, then π is factorizable if and only if r is composite.*

The above corollary is stated in [DKPR22] Proposition 2.17 without enough clarification. We infer it from Example 3.3.2 and Remark 12. However, we explain an argument involving function fields over $\mathbb{C}$. The following result is an expected one though we do not find a reference that elaborates its proof.

3.4 Factorization through Galois groups of covers

A map $\pi : Y \to X$ of Riemann surfaces gives an inclusion $\pi^*\mathcal{M}(X) \subset \mathcal{M}(Y)$ where $\pi^*\mathcal{M}(X)$ denotes the meromorphic functions on Y of the form $\pi \circ f$ with $f \in \mathcal{M}(X)$. If π is factorizable there exists an intermediate subfield $\pi^*\mathcal{M}(X) \subset K \subset \mathcal{M}(Y)$. The converse statement is also true. This is a consequence of the categorical equivalence among compact connected Riemann surfaces, function fields over $\mathbb{C}$ and finally, smooth irreducible projective algebraic curves. The reader is advised to consult A.1.1 as a source of the background material.

Proposition 3.4.1. *A nonconstant branched holomorphic map $\pi : Y \to X$ of compact connected Riemann surfaces is factorizable if and only if there exists a proper intermediate subfield $\pi^*\mathcal{M}(X) \subset K \subset \mathcal{M}(Y)$.*

Let $\pi : Y \to X$ be a nonconstant holomorphic map of finite degree n. Then $\pi^*\mathcal{M}(X) \subset \mathcal{M}(Y)$ is a finite field extension of degree n. Let $\pi^*\mathcal{M}(X) \subseteq E \subseteq \mathcal{M}(Y)$ is an intermediate field, we will show there exist a compact Riemann surface Z, finite covering maps $f : Y \to Z$ and $g : Z \to X$ such that $g \circ f = \pi$ and $f^*\mathcal{M}(Z) = E$. We use following from [GGD12] page 64.

Theorem 3.4.2. *Let X_1 and X_2 be compact Riemann surfaces and Σ_1, Σ_2 be finite subsets of X_1, X_2 respectively. Assume that $X_1^* = X_1 \backslash \Sigma_1$ and $X_2^* = X_2 \backslash \Sigma_2$ are isomorphic. Then X_1 and X_2 are isomorphic.*

Theorem 3.4.3. *Let Y be a compact Riemann surface, $\Sigma \subset Y$ be a finite set. $f^* : X^* \to Y^*$ is an unramified holomorphic covering of finite degree. Then there exists a unique compact Riemann surface $X^* \subset X$ such that f^* extends a unique morphism $f : X \to Y$. Moreover $X \backslash X^*$ is a finite set.*

Corollary 3.4.4. *Let X, Y be compact Riemann surfaces and $\Sigma_1 \subset X, \Sigma_2 \subset Y$ be finite subsets. An unramified holomorphic covering of finite degree $f^* : X\backslash\Sigma_1 \to Y\backslash\Sigma_2$ extends to a morphism (i.e. a nonconstant holomorphic map) $f : X \to Y$.*

We discuss the proof of the Proposition 3.4.1 for $X = \mathbb{P}^1$ initially.

Proof. Let $\pi : Y \to \mathbb{P}^1$ be a given map with degree $n \geq 1$. Let us consider as in the proposition, a proper intermediate field E. So, $E = \pi^*\mathcal{M}(\mathbb{P}^1)(\alpha)$ and $\{1, \alpha, ..., \alpha^{r-1}\}$ is a basis of E over $\pi^*\mathcal{M}(\mathbb{P}^1)$, thus this field extension has degree r with $1 \leq r \leq n$. Here $E = \mathbb{C}(\pi, \alpha)$. We produce an irreducible polynomial $F(x, y) \in \mathbb{C}[x, y]$ such that $F(\pi, \alpha) = 0$ and a compact connected Riemann surface X^F compactifying the zero locus of F. Moreover $\mathcal{M}(X^F) = \mathbb{C}(\mathbf{x}, \mathbf{y})$ where $\mathbf{x}, \mathbf{y}$ denote the holomorphic projection of coordinates.

A standard construction of F available in [GGD12] page 68 and [JW18] pages 22-24. Let the irreducible minimal polynomial of α over $\mathcal{M}(\mathbb{P}^1)$ be $M(T) = T^r + \pi^*a_1T^{r-1} + ... + \pi^*a_r$. As α is annihilated by ψ, the minimal polynomial divides ψ over $\pi^*\mathcal{M}(\mathbb{P}^1)$. We clear denominators of $a_1, ..., a_r$ multiplying by the least common multiple to obtain a complex irreducible polynomial $F(x, y)$ out of M.

Remark 13. Here we make an observation that reflects an important connection between Riemann surfaces and roots of polynomials. Let S be the finite subset of $\mathbb{P}^1$ consisting of ∞, the branch points of π and the images under π of the poles (i.e. pre-images of ∞) of α. Then for each $q \in \mathbb{P}^1\backslash S$ there are n distinct points $y_1, ..., y_n \in Y$ such that $\pi(y_i) = q$. Now $\alpha(y_i) \in \mathbb{C}$ for all $1 \leq i \leq n$. Define the symmetric functions $S_1 = \sum_i \alpha(y_i); S_2 = \sum_{i<j} \alpha(y_i)\alpha(y_j); ...; S_n = \Pi_i\alpha(y_i)$. These are elementary symmetric (analytic) functions defined at q. We can define meromorphic functions $s_i = \pi^*S_i \in \mathbb{C}(\pi)$ by extension of S_i's as per the corollary 3.4.4. The polynomial $\psi(T) = T^n - s_1T^{n-1} + ... + (-1)^nT_n$ is satisfied by α as the point wise equation is satisfied by $y \in Y\backslash\pi^{-1}(S)$. The meromorphic functions $S_1, .., S_n$ are called the *elementary symmetric functions* of α with respect to π.

The importance of dragging ψ on stage is that for a well chosen complex number x_0 the equation $F(x_0, y) = 0$ has solution $\alpha(y_0)$ such that $\pi(y_0) = x_0$. Let η be the polynomial of two variables clearing the denominators of coefficients of ψ (multiplying by the least common multiple of the denominator). Indeed a root of $F(x_0, y)$ is a root of $\eta(x_0, y)$ and roots of $\eta(x_0, y)$ are of the form $\alpha(y_0)$ such that $\pi(y_0) = x_0$. Counting up to multiplicity there are r roots of $F(x_0, y) = 0$ for generic x_0 (of the form $\alpha(y_1), ..., \alpha(y_r)$ such that $\pi(y_1) = ... = \pi(y_r) = x_0$).

Let $\phi : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ given as $\phi(y) = (\pi(y), \alpha(y))$ which is a holomorphic map. We restrict the map to image $\phi(Y)$, a compact connected analytic variety. Moreover ϕ is proper (that is, inverse image of a compact set is compact). Writing $F(x,y) = p_0(x)y^r + p_1(x)y^{r-1} + ... + p_r(x)$ with $p_0(x)$ a nonzero polynomial in X, we define a connected smooth Riemann surface (charts are given by Implicit Function Theorem) $C^x := \{(x_0, y_0) \in Z(F) : F_x(x_0, y_0) \neq 0; p_0(x_0) \neq 0\}$ and a holomorphic map $\mathbf{x} : C^x \to \mathbb{P}^1$ (see [GGD12] page 69). We complete C^x to X^F by adding finitely many points. We remove a finite set of points which are *not* valued in $\mathbb{C}^2$ so that the image lies inside $Z(F)$. Moreover irreducible polynomial F intersects F_x at finitely many points and common solutions of p_0 and F are finite. Thus we remove finitely many points from $Z(F)$ and the pre-image of these points (under ϕ) and map $\phi : Y'' \to C^x$. This map is proper and holomorphic. So it has a finite degree. Finally we see that ϕ has finitely many ramification points in Y'' because π has finitely many ramification points in Y. We remove finitely many branch points from C^x and ramification points from Y'' to get a unbranched map $\phi : Y' \to C^{x'}$. The restricted map of ϕ rewritten $f' : Y' \to X'$ is an unramified holomorphic covering of finite degree. This leads to a holomorphic map $f : Y \to X^F$ from Corollary 3.4.4. From the fact that $\mathcal{M}(X^F) = \mathbb{C}(\mathbf{x}, \mathbf{y})$ (see [GGD12] page 74 Corollary 1.93) we obtain $f^*\mathbf{x} = \pi$ and $f^*\mathbf{y} = \alpha$ (using identity theorem). Thus $E = f^*\mathcal{M}(X^F)$. The map $g : X^F \to \mathbb{P}^1$ is the map $\mathbf{x}$. We obtain that $g(f(y)) = \pi(y)$ on Y punctured at finitely many points, thus $\pi = g \circ f$. We have $[E : \pi^*\mathcal{M}(\mathbb{P}^1)] = \deg(g) = r$ and $[\mathcal{M}(Y) : E] = \deg(f) = n/r = m$.

Remark 14. Here we mention a fact which has an elementary proof. Let $a_1(x), ..., a_r(x), b_1(x), ..., b_r(x)$ are complex polynomials. We denote the least common multiple of $b_1, ..., b_r$ by b and assume that a_i and b_i are mutually prime polynomials for all i. The polynomial $y^r + \frac{a_1}{b_1}y^{r-1} + ... + \frac{a_r}{b_r}$ is irreducible over $\mathbb{C}(x)$ if and only if $b \cdot (y^r + \frac{a_1}{b_1}y^{r-1} + ... + \frac{a_r}{b_r})$ is an irreducible polynomial in $\mathbb{C}[x, y]$. This statement confirms that F in above explanation is an irreducible polynomial.

We generalize above argument. We simply choose the compact Riemann surface defined by an irreducible polynomial over function field of X. This is equivalent to taking the analytic continuation of germs of local solutions of an algebraic equation. See in [For81] page 53 Theorem 8.9. (The notation and symbols of the following statement are independent of ones in the proof of Proposition 3.4.1.)

Theorem 3.4.5. *Suppose X is a compact Riemann surface and $P(T) = T^r + c_1T^{r-1} + ... + c_r \in \mathcal{M}(X)[T]$ is an irreducible polynomial of degree r. Then there exists a compact Riemann surface Y, a branched holomorphic r-sheeted covering $\pi : Y \to X$ and a meromorphic function $F \in \mathcal{M}(Y)$ such that $(\pi^*P)(F) = 0$. The triple (Y, π, F) is uniquely determined: If (Z, τ, G) has the corresponding properties, then there exists exactly one biholomorphic mapping $\zeta : Z \to Y$ such that $G = \zeta^*F$.*

□

Theorem 3.4.6. *(Ritt) A nonconstant $r : 1$ holomorphic map between compact Riemann surfaces $\pi : Y \to X$ is factorizable if and only if Galois group of the branched covering* $\mathrm{Gal}(\mathcal{M}(Y)^{\mathrm{Gal}}/\pi^*\mathcal{M}(X))$ *is imprimitive.*

Proof. A topological proof using the monodromy groups without involving the Galois groups is found in [LZ10] page 65; Theorem 1.7.6 with construction of 2-surfaces out of the generalized constellations. However, we discuss a proof available in [BRSY21] Proposition 1. One direction is simple. We identify the Galois group with the cartographic group at a generic point due to the argument of Harris, then prove the cartographic group imprimitive. To prove the converse, we return to the actual Galois group itself. Let K be the Galois closure of the field extension $\mathcal{M}(Y)/\pi^*\mathcal{M}(X)$. (Our aim is to set up a proper subfield between $\mathcal{M}(Y)$ and $\pi^*\mathcal{M}(X)$.) Let us recall the Fundamental Theorem of Galois Theory briefly. Let K/L be a finite Galois extension of fields. Then there is an inclusion-reversing bijective correspondence between (i) fixed subfield K^H intermediate between K/L corresponding to a subgroup H of $\mathrm{Gal}(K/L)$ and (ii) the automorphism group $\mathrm{Aut}(J/L)$ for an intermediate field J between K/L. Finally, the degree of extension, $[K^H : L]$ is same as the group index $[\mathrm{Gal}(K/L) : H]$. So, subgroups of the Galois group $G = \mathrm{Gal}(K/\pi^*\mathcal{M}(X))$ are in one-to-one (inclusion reversing) correspondence with the intermediate subfields. We choose a nontrivial block B_1. Its stabilizer subgroup L i.e. group elements σ in the Galois group such that $\sigma(B_1) = B_1$, forms a subgroup H'. We denote the subgroup H of G such that the fixed field of H written as $K^H = \mathcal{M}(Y)$. Then H' is a proper subgroup of G properly containing H. We consider a compact Riemann surface $\tilde{X}$ associated to $K^{H'}$. We can come up with intermediate covering maps of π defined by inclusion $\pi^*\mathcal{M}(X) \subset K^{H'} \subset \mathcal{M}(Y)$ by Proposition 3.4.1. □

According to Theorem 3.4.6, deciding if a generic spectral covering map on a smooth spectral curve is a composition of two intermediate maps is a question of computing the Galois group. We are in a position now to pose a research question that is in our knowledge, unexplored. The question is appealing as it bridges pure aspects of geometry with computational group theory. Let X be a smooth irreducible projective algebraic curve. Given a holomorphic line bundle L over X we recall that smoothness of a spectral curve defined by $s = (s_1, ..., s_r) \in \bigoplus_{i=1}^r H^0(X, L^i)$ is an open condition. That is, given a suitable s_0, we have smooth spectral curves X_s for each s near s_0. The spectral curves will be decidedly different as will their spectral covering maps (that is, they will be non-isomorphic) for any infinitesimal change of the spectral coefficients; that said, they still retain with the same genus (according to 2.15). An immediate question is: are Galois groups invariant under small perturbation of spectral coefficients? (However, a spectral covering map is not necessarily Galois and we should consider Galois closure of field extensions.) It is an open question if Galois groups are still imprimitive for any s chosen generically. It is interesting to observe that in case of $\mathbb{P}^1$ the smooth compact connected spectral covers are same as the one defined by respective meromorphic functions on $\mathbb{P}^1$. We will explore this further. The following is a stepping stone towards solution of this question.

Conjecture 3.4.7. *Let X be a compact Riemann surface of genus g_X and $L \to X$ be a holomorphic line bundle on X with a positive degree. We fix a generic section $s \in H^0(X, L^r)$ with distinct zeros over X. Consider extension of function fields defined by smooth integral cyclic spectral cover $\pi : X_s \to X$ is Galois and the Galois group of cover π is a cyclic group of order r.*

We verify this conjecture for $g_X = 0$[1]. From the classical spectral correspondence given in [BNR89] we

[1] We were informed in the meanwhile that the conjecture holds for higher genus also. But we are not aware of any definite reference of its proof.

can embed the Jacobian of a spectral curve inside a quasi-projective variety of S-equivalent semistable pairs over an intermediate spectral curve.

3.5 Galois theory of cyclic spectral covers of $\mathbb{P}^1$

Here we mention another point of view of the generic spectral covers. The sections s_i are indeed polynomials over the affine coordinate charts of $\mathbb{P}^1$ which we can assume as meromorphic functions on $\mathbb{P}^1$. This understanding leads to an irreducible algebraic polynomial equation over the function field of $\mathbb{P}^1$. Theorem 3.4.5 addresses an analytic approach to prove that there is a root of such an equation in a finite extension K of $\mathcal{M}(\mathbb{P}^1)$ and there is a unique compact Riemann surface Y such that K is $\mathbb{C}$-algebra isomorphic to $\mathcal{M}(Y)$. This construction of X^F in Theorem 3.4.1 is construction of that unique compact Riemann surface, as we have said. This time we intend to explore a particular case of the same construction where the constructed compact Riemann surface is away from the smooth locus over $\mathbb{P}^1$.

This next discussion is adopted from [Mir95] comparing constructions for the hyperelliptic curves. Let $t \geq 2$ and $r \geq 2$. Choose a generic section s of $\mathcal{O}(tr)$ which has tr distinct zeros $B = \{z_1, ..., z_{tr}\} \subset \mathbb{P}^1$. It can be represented by a complex polynomial s with distinct zeros of degree tr or $tr-1$. Making a small change in notation, the spectral curve X_s corresponding to $\lambda^r - s$ is given by

$$\begin{cases} y^r - s(x) = 0 \\ \tilde{y}^r - \tilde{s}(\tilde{x}) = 0 \end{cases} \tag{3.1}$$

with identification given in equations 2.19. Strictly speaking, this following construction only makes sense in case section s has distinct zeros. If we allow s with repeated zeros we obtain a singular spectral curve. We are able to desingularize a singular spectral curve but there is no guarantee of spectral correspondence to hold for that desingularized spectral curve.

We first expose inherent properties of the above spectral cover as a compact Riemann surface and see that zeros of section s are only branch points of spectral covering map, each having singleton fiber. There are exactly tr ramification points each with multiplicity r in X_s. We recall definition of the polynomial $\tilde{s}(z) = z^{tr}s(\frac{1}{z})$. Note that, factorizing the polynomial s in its distinct roots, we see $\tilde{s}$ admits all distinct roots too and degree tr or $tr-1$. We have already presented the associated affine plane curves in $\mathbb{C}^2$ as

$$C_1 = \{(x, y) \in \mathbb{C}^2 : y^r = s(x)\}; C_2 = \{(\tilde{x}, \tilde{y}) \in \mathbb{C}^2 : \tilde{y}^r = \tilde{s}(\tilde{x})\}.$$

Here C_1, C_2 are noncompact smooth connected Riemann surfaces due to the fact that s and $\tilde{s}$ admit distinct roots. To establish X_s a compact Riemann surface we adopt following steps. Consider open subsets U, V of X, Y respectively:

$$U := \{(x, y) \in \mathbb{C}^2 : y^r = s(x); x \neq 0\}; V := \{(\tilde{x}, \tilde{y}) \in \mathbb{C}^2 : \tilde{y}^r = \tilde{s}(\tilde{x}); \tilde{x} \neq 0\}.$$

We consider a holomorphic map $\phi : U \to V$ by

$$\phi(x, y) = \left(\frac{1}{x}, \frac{y}{x^t}\right).$$

It is apparent that ϕ is an isomorphism. There are only finitely-many points in $C_1 \backslash U$ and $C_2 \backslash V$. As per the definition of the spectral curve X_s we take disjoint union of $C_1 \sqcup C_2$ along ϕ. That is, we identify each point in $C_1 \backslash U$ to itself, each point in $C_2 \backslash V$ to itself and each point $u \in U$ to itself or to $\phi(u)$. The 'disjoint-union' topology of $C_1 \sqcup C_2$ descends to quotient topology of $C_1 \sqcup C_2/\phi$. Finally, the space $C_1 \sqcup C_2/\phi$ which is nothing but X_s is a compact (restriction on closed unit discs) connected (due to non-empty intersection of connected components) Hausdorff second countable topological space. The holomorphic charts of C_1 and C_2 produce charts of points in X_s via inclusion maps on C_1 and C_2. (See [Mir95] page: 60.) Thus X_s is a compact Riemann surface. Observe that we can embed C_1 and C_2 into X_s and $X_s \backslash C_1$ and $X_s \backslash C_2$ are finite sets. Indeed, X_s is compact completion of both C_1 and C_2. We want to find the genus of X_s in an alternative way. To do this computation we want to obtain X_s as a finite branched cover of $\mathbb{P}^1$. Observe that it is enough to understand the calculus over C_1 because C_2 contributes only finitely many points to X_s.

We have the first holomorphic projection coordinate map $\pi' : C_1 \to \mathbb{C}$ as a holomorphic surjective finite branched map with zeros of s as branch points. (We can explore $\mathbb{P}^1$ as $\mathbb{C} \sqcup \mathbb{C}/\phi$ while $\phi : \mathbb{C}^* \to \mathbb{C}^*$ is defined as $\phi(z) = \frac{1}{z}$. Here $0 \in \mathbb{P}^1$ is denoted by 0 in first summand $\mathbb{C}$ and ∞ is denoted by second summand $\mathbb{C}$.) This is restriction of the bundle map $\mathcal{O}(t) \to \mathbb{P}^1$. The pre-image of each branch point is singleton i.e. each of ramification points of π' has multiplicity r. The map is extended to a holomorphic branched covering map π to $\mathbb{P}^1$ of finite degree r. In case s has degree tr we observe 0 is not a complex root of $\tilde{s}$ i.e. zeros of s are the only branch points of π. It is same thing as saying that $\infty \in \mathbb{P}^1$ is not a branch point of π. On the other hand, in case s has degree $tr - 1$, we have 0 as a root of $\tilde{s}$ and there is a ramification point in X_s which has multiplicity r and π maps that point to $\infty \in \mathbb{P}^1$. Thus in each case we have tr ramification points in X_s each with multiplicity r. We are in position to apply Hurwitz formula: $2(g_{X_s} - 1) = b_\pi + 2r(g_{\mathbb{P}^1} - 1) = tr(r-1) - 2r \implies g_{X_s} = \frac{(tr-2)(r-1)}{2}$.

Remark 15. The above computation of genus is consistent with the genus of a spectral curve in equation 2.15.

We have access to another meromorphic function on X_s. This is given by projecting y-coordinate. Denote this as f_y. The y-coordinate map $C_1 \to \mathbb{C}$ is extended to C by mapping the points $(0, \tilde{y})$ (which are contributed only by other affine component of the spectral curve) to $\infty \in \mathbb{P}^1$. Here is a specific indication. The polynomial s is a meromorphic function on $\mathbb{P}^1$ defining $\infty \mapsto \infty$. From the explanation of the holomorphic map π we realize that $f_y^r = \pi^* s$ on C_1, so on all over X_s. The degree of s as a meromorphic function on $\mathbb{P}^1$ of is polynomial degree tr or $tr - 1$ whichever applicable. Thus $\pi^* s$ has degree $= r.\deg(s)$ and $\deg(f_y) = \deg(s)$.

Remark 16. In Theorem 3.4.5 if we plug $c_1 = ... = c_{r-1} = 0$ and $c_r = -s$ (as a meromorphic function on the projective line) then (X_s, π, f_y) is the unique solution of the corresponding irreducible polynomial.

This explanation gives a complete understanding of the function field of X_s. We claim that $\mathcal{M}(X_s) = \mathbb{C}(\pi, f_y)$ or $\pi^*\mathcal{M}(\mathbb{P}^1)(f_y)$. First see that $\pi^*\mathcal{M}(\mathbb{P}^1) \subset \pi^*\mathcal{M}(\mathbb{P}^1)(f_y) \subseteq \mathcal{M}(X_s)$ and both the field extensions $\pi^*\mathcal{M}(\mathbb{P}^1) \subset \mathcal{M}(X_s)$ and $\pi^*\mathcal{M}(\mathbb{P}^1) \subset \pi^*\mathcal{M}(\mathbb{P}^1)(f_y)$ are $r:1$ extensions. So we reach to our conclusion. We obtain that the field of meromorphic functions is given as $\mathcal{M}(X_s) = \{\sum_{j=0}^{r-1} \pi^* r_j f_y^j : r_j \in \mathcal{M}(\mathbb{P}^1)\}$. We also obtain that $T^r - \pi^* s$ is a separable polynomial that is, all of its roots are distinct, given as $\{\xi^i f_y : i = 0, ..., r-1\}$ where ξ is an imprimitive r-th root of 1 and $\mathcal{M}(X_s)$ is the splitting field of $T^r - \pi^* s$. Thus this extension is a Galois extension because $\xi^i f_y$ lies in $\mathcal{M}(X_s)$ for all i.

We are in position to compute Galois group of extension $\pi^*\mathcal{M}(\mathbb{P}^1) \subset \mathcal{M}(X_s)$. The equation $T^r - \pi^* s = 0$ over $\pi^*\mathcal{M}(\mathbb{P}^1)$ admits all roots in the function field of X_s. The set of roots is given by (for the time being, $f_y = \sqrt[r]{\pi^* s}$) the set $\{\sqrt[r]{\pi^* s}, \xi\sqrt[r]{\pi^* s}, ..., \xi^{r-1}\sqrt[r]{\pi^* s}\}$ and it is enough to define a Galois group by writing action on $\sqrt[r]{\pi^* s}$. Galois group is cyclic generated by an element σ while $\sigma(\sqrt[r]{\pi^* s}) = \xi\sqrt[r]{\pi^* s}$ while ξ denotes a primitive r-th root of unity $e^{\frac{2\pi i}{r}}$. On the other hand, $\sigma(\sqrt[r]{\pi^* s})$ is a root of the equation $T^r - \pi^* s = 0$.

The deck transformation group ([For81] page 57 Theorem 8.12) will be isomorphic to the Galois group. In [Mir95] page 74 we find an independent computation of the deck transformation group of π by the following way. Observe that

$$(x, y) \mapsto (x, \xi^i y)$$

is a deck transformation on X_s for $i = 0, ..., r-1$. Indeed these are the only deck transformations. See that for any $\sigma \in \mathrm{Deck}(\pi)$ we have $\sigma^r(y) = s(x) = y^r$. Thus we have for each point, $\sigma^r(y) = \xi^i y$ for some i. By continuity of automorphism σ we have i will be same everywhere. Hence we classify the group of all the deck transformations as a cyclic one generated by $\sigma_1 = (x, y) \mapsto (x, \xi y)$. We remark that spectral cover defined by 2.19 is appropriately named as a *cyclic cover* in [Mir95] page 73 because Galois group along with deck transformation groups is cyclic.

Let $r \geq 4$ is a composite number. We denote the respective Galois group by G and its subgroups by H. Here we see that for each divisor m of r there is a unique subgroup of order m. Index of such a subgroup is p where $r = m.p$. Now a subgroup $0 \subset H \subset G$ of order m uniquely associates to a finite extension $\mathcal{M}(X_s) \supset K \supset \pi^*\mathcal{M}(\mathbb{P}^1)$ by the Fundamental Theorem of Galois theory. From Proposition 3.4.1 we obtain that there exists a holomorphic map $f : X_s \to X$ such that $K = f^*\mathcal{M}(X)$. Thus $\deg(f) = [\mathcal{M}(X_s) : K] = m$ and $\pi^*\mathcal{M}(\mathbb{P}^1) \subset K = f^*\mathcal{M}(X)$ such that there is a holomorphic map $g : X \to \mathbb{P}^1$ such that $\pi = g \circ f$.

Moreover, X is unique up to isomorphism in this particular case of spectral covers, based on degrees of f and g. Let suppose there are maps $\tilde{f} : X_s \to \tilde{X}$ and $\tilde{g} : \tilde{X} \to \mathbb{P}^1$ such that $\pi = \tilde{g} \circ \tilde{f}$ and $\deg(f) = \deg(\tilde{f}) = m$ (thus $\deg(g) = \deg(\tilde{g})$) then $f^*\mathcal{M}(X)$ and $\tilde{f}^*\mathcal{M}(\tilde{X})$ are subfields of $\mathcal{M}(X_s)$ with same degree of extension. Thus they must be same as there is unique subfield of index m. Thus X and $\tilde{X}$ are isomorphic as they have

$\mathbb{C}$-algebra isomorphic function fields. (See [GGD12], [Mir95] for further reference on function fields.) We write $\pi = g \circ f$ where $\deg(g) = p \geq 2$ and $\deg(f) = m \geq 2$ and $r = mp$.

The ramification points of g along with their multiplicities are obvious, hence genus of X. Consider tr many distinct points

$$f(z_1), ..., f(z_{tr}) \in X.$$

Now $\text{mult}_\pi(z_i) = \text{mult}_f(z_i).\text{mult}_g(f(z_i)) \leq m.p = r$. Equality takes place if and only if $\text{mult}_f(z_i) = m$ and $\text{mult}_g(f(z_i)) = p$. Thus each of these points is a ramification point under g. Moreover, these are only ramification points of g. They are all distinct. If $x_0 \in X$ is a ramification point, choose z in (non-empty) fiber of x_0 under f. Then $\text{mult}_\pi(z) = \text{mult}_f(z).\text{mult}_g(x_0) \geq 2$ i.e. z is a ramification point of π. Thus $z = z_i$ for some i and $x_0 = f(z_i)$. Thus we obtain that $f(z_1), ..., f(z_{tr})$ are only ramification points of g each of multiplicity p. Thus $b_g = tr(p-1)$. From Hurwitz formula we have genus of X.

Example 3.5.1. $g_X = \frac{tr(p-1)}{2} + 1 - p.$ ♢

In case $t = 2$ and $r = 4$, we obtain a hyper-elliptic curve X which has genus 3. As we have seen beforehand this is a unique curve.

From the formula in 2.16 we understand stability conditions of Hitchin pairs continuing above argument. Let us consider a $g^*(\mathcal{O}(t))$-twisted pair $(E', \phi' : E' \to E' \otimes g^*\mathcal{O}(t))$ on X such that $(g_*E', g_*\phi')$ is a stable Hitchin pair. Let suppose if possible that (E', ϕ') admits a nontrivial proper ϕ'-invariant subbundle F' such that $\mu_{F'} \geq \mu_{E'}$. Then g_*F is a nontrivial proper $g_*\phi'$-invariant subbundle of g_*E' such that $\mu_{g_*F'} \geq \mu_{g_*E'}$. This is contradictory. So (E', ϕ') must be stable. Same thing holds with semistability. This is observed in the following diagram.

$$\begin{array}{ccccccc} F' & \xrightarrow{\phi'} & F' \otimes g^*\mathcal{O}(t) & & g_*F' & \xrightarrow{g_*\phi'} & g_*F' \otimes \mathcal{O}(t) \\ \big\downarrow i & & \big\downarrow i\otimes id & \xrightarrow{g_*} & \big\downarrow i & & \big\downarrow i\otimes id \\ E' & \xrightarrow{\phi'} & E' \otimes g^*\mathcal{O}(t) & & g_*E' & \xrightarrow{g_*\phi'} & g_*E' \otimes \mathcal{O}(t) \end{array} \tag{3.2}$$

Let $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$ be a generic section and (E, ϕ) is a generic pair with characteristic polynomial $\lambda^r - s$ as in 2.11. From the factorization of $\pi = g \circ f$ there is a pair on X namely $(f_*M, f_*\eta)$ and $(E, \phi) \cong g_*f_*(M, \eta) \cong g_*(f_*M, f_*\eta)$. Further we have $[(M, \eta)] \mapsto [(f_*M, f_*\eta)]$ is an injective morphism into space of isomorphism classes of $g^*\mathcal{O}(t)$-twisted pairs on X. Indeed $(f_*M, f_*\eta) \cong (f_*M', f_*\eta) \implies g_*(f_*M, f_*\eta) \cong g_*(f_*M', f_*\eta) \implies \pi_*(M, \eta) \cong \pi_*(M', \eta)$. From the spectral correspondence, $M \cong M'$.

Let the space of isomorphism classes of stable $g^*\mathcal{O}(t)$-twisted pairs of rank m on X, be denoted by $\mathcal{N}$ and $f_*\text{Pic}(X_s) = \mathcal{N}'$ and isomorphism classes of t-twisted Hitchin pairs (E, ϕ) of rank r on $\mathbb{P}^1$ satisfying characteristic equation $\lambda^r = s$, name it $\mathcal{N}''$. Then the pushforward morphism g_* is immediately injective and surjective. The pair $(f_*M, f_*\eta)$ satisfies an equation $\lambda^r + g^*s_1\lambda^{r-1} + ... + g^*s_r$. In case of cyclic covers, we have $s_1 = s_2 = ... = s_{r-1} = 0$ over $\bigoplus_{i=1}^{r} H^0(\mathbb{P}^1, g^*\mathcal{O}(ti))$. However, this is certainly not the characteristic polynomial of this pair on X; rather it is an annihilating polynomial.

Remark 17. Let $\deg(E) = d$, then degree of shifted Jacobian on X_s as degree of a line bundle M as $d' = d + (g_{X_s} - 1) + r = d + (r-1)(\frac{tr-2}{2} + 1)$ while $H^{-1}(s) \cong \mathrm{Jac}^{d'}(X_s)$. From $\pi = g \circ f$ with $\deg(f) = m$ and $\deg(g) = p$ immediately, $\deg(f_* M) = d + \frac{mtr(p-1)}{2}$.

Though existence of iterated spectral covers is established we lack direct control over them. We obtain a lower bound on Nitsure's dimension of the Zariski tangent space on an intermediate (or iterated) spectral cover X in case of twisting by $g^*\mathcal{O}(t)$ by smooth immersion f_*: it is at least the genus g_{X_s} of the spectral curve X_s due to containment of $\mathcal{J}$. From $\deg(g^*\mathcal{O}(t)) = tp$ we see that $2(g_X - 1) - \deg(g^*\mathcal{O}(t)) = p(t(m(p-1)-1)-2)$. This number is generally positive and 0 at the base case $t = 2, r = 4$. So, there is lack of information to compute Nitsure's dimension in any of these cases $t \geq 2$ and $r \geq 4$. Also, we are unable to comment if $\mathcal{J}$ is the full space $\mathcal{M}'_X(m, d'', g^*\mathcal{O}(t))$. It is also difficult to decide whether X can be embedded within $\mathrm{Tot}(\mathcal{O}(t))$.

We can omit twisting endomorphism $f_*\eta$ and isomorphism class of E' with rank m, degree d'' (which are of the form f_*M) can be identified as the Jacobian of X_s. We can refer to this object as an "iterated Hitchin fiber". The spectral correspondence of line bundles and pairs thus extends to a threefold correspondence in following theorem. Part (A) follows from the classical spectral correspondence applied to the case where the spectral curve is a cyclic cover. Part (B) is a direct consequence of Galois theory applied to a cyclic extension of a function field. Finally, Part (C) follows from the iterated spectral correspondence applied on intermediate spectral curve.

Theorem 3.5.2. *Let $s \in H^0(\mathbb{P}^1, \mathcal{O}(tr))$ be a generic holomorphic section with $t \geq 2$ and r be a composite number.*

(A) Isomorphism classes of t-twisted Hitchin pairs (E, ϕ) of rank r on $\mathbb{P}^1$ satisfying the characteristic equation $\lambda^r = s$ (name it $\mathcal{N}''$) are in one-to-one correspondence with isomorphism classes of line bundles M on X_s. The correspondence is given with pushforward by the covering map π. In case we fix degree of E to be $d \in \mathbb{Z}$, we see that $\pi_* : \mathrm{Jac}^{d'}(X_s) \to \mathcal{M}_{\mathbb{P}^1}(r, d, t)$ is a one-to-one correspondence, while $d' = d + (r-1)(\frac{tr-2}{2} + 1)$.

(B) Given a factorization $r = mp$ with $p, m \geq 2$ there exists a compact Riemann surface X and nonconstant holomorphic maps $f : X_s \to X$ of degree m and $g : X \to \mathbb{P}^1$ of degree p such that $\pi = g \circ f$. If there is another compact Riemann surface $\tilde{X}$ and nonconstant holomorphic maps $\tilde{f} : X_s \to \tilde{X}$ of degree m and $\tilde{g} : \tilde{X} \to \mathbb{P}^1$ of degree p such that $\pi = \tilde{g} \circ \tilde{f}$ then $X \cong \tilde{X}$.

(C) Fix a chosen factorization of $r = mp$ and $\pi = g \circ f$. Let the space of isomorphism classes of stable $g^*\mathcal{O}(t)$-twisted pairs of rank m on X, be $\mathcal{N}$. Then $f_* : \mathrm{Pic}(X_s) \to \mathcal{N}$ is a well defined injective morphism with

image $\mathcal{N}'$. There is a bijective correspondence $g_* : \mathcal{N}' \to \mathcal{N}''$. Given $\deg(E) = d$*, the pushforward morphism given by* $f_* : \mathrm{Jac}^{d'}(X_s) \to \mathcal{M}'_X(m, d'', g^*\mathcal{O}(t))$ *is an injective morphism, wherein* $\mathcal{M}'_X(m, d'', g^*\mathcal{O}(t))$ *denotes isomorphism classes of stable $g^*\mathcal{O}(t)$-twisted Hitchin pairs of rank m and degree* $d'' = d + \frac{mtr(p-1)}{2}$ *on X. Let $\mathcal{J}$ denote image* $f_*(\mathrm{Jac}^{d'}(X_s))$*. Then $g_* : \mathcal{J} \to H^{-1}(s)$ is a bijective correspondence as H is the Hitchin morphism on $\mathcal{M}_{\mathbb{P}^1}(r, d, \mathcal{O}(t))$.*

Remark 18. Repeating our argument for a series of subgroups $0 \subset H_1 \subset H_2 \subset ... \subset H_k \subset G$, we decompose covering map π on X_s in a polygonal series of iterated covers $X_1, .., X_k$ and an intermediate series of twisted Hitchin pairs using the composite projection formula.

3.6 Further directions of research

3.6.1 Galois group and involutions on Jacobian of spectral curves

A prominent direction of research is the group of involutions as Hitchin explained in [Hit87b] the role of Prym varieties in moduli problem regarding Higgs bundles. Unfortunately we have not yet tackled the general problem iterated spectral covers for K_X. For $G = \mathrm{Sp}(m, \mathbb{C})$. A point $T^*\mathcal{N}(r, d)$ consists of a stable vector bundle of rank $2m$ with a symplectic form together with a holomorphic section $\phi \in H^0(M, \mathrm{End}(V) \otimes K)$ which satisfies $< \phi(v), w >= - < v, \phi(w) >$. The characteristic polynomial is of the form $\det(A - \lambda \cdot I) = \lambda^{2m} + a_2 x^{2m-2} + ... + a_{2m}$ and the polynomials $a_2, ..., a_{2m}$ (only even coefficients survive) form a basis for the invariant polynomials on the Lie algebra $\mathfrak{sp}(m)$. The spectral curve possesses an involution (that is, the curve admits an automorphism σ with $\sigma^2 = Id$). We may attempt to look into more general involution groups on Jacobian which is uniquely specified by the spectral curve itself by Torelli's theorem and relate it to the Galois group (that is, the automorphism group of field extension to Galois closure) of the corresponding smooth spectral cover. We have not come across any material yet that has talked about this delicate question.

In the same line of thoughts we can raise questions about the effects of Galois groups on the real structures of moduli space of Higgs bundles. In the works of Baraglia and Schaposnik ([BS13]) effects of involution groups on moduli spaces of G-Higgs bundles. So far, we have not discussed possible role of Galois groups for curves with genus > 0. This is because the spectral curves are very much abstract for higher genus curves. However, a good stepping stone in this scenario will be the general spectral curves on $\mathbb{P}^1$ which are interpreted merely as complex algebraic plane curves.

4 Moduli scheme of coherent Sym(V^*)-modules

4.1 Geometric invariant theory

Invariant theory was first used by Seshadri to establish existence of moduli scheme of stable bundles. This is more generalized by Simpson ([Sim94a], [Sim94b]) for coherent modules. We give a quick review of the main results here. We compute all invariants with respect to a very ample line H_X. If H_X is an ample line bundle on X then H_X^m is very ample for some integer m and we can continue computation of invriants to check boundedness, with respect to ample polarization. Moreover, Serre's vanishing theorem holds for ample twist too (see [Har77] page:229 Theorem 5.3). Here we will discuss the computations with respect to a very ample polarization. All the material of this section is collected from [Sim94a] and [Sim94b]. In the context of [Sim94a] and [Sim94b] we define rank of Hilbert polynomial $P(n)$ as the leading coefficient of $\frac{n^d}{d!}$. This depends upon choice of a polarization. The explanation given in this section is algebro-geometric in nature.

Let X be a smooth irreducible projective algebraic variety proper over an algebraically closed field $\mathbb{k}$ of characteristic 0. We will consider the functor $M^*(\mathcal{O}_X, P)$ which associates to the set of semistable sheaves $\mathcal{E}$ on X of pure dimension d and with Hilbert polynomial P. We will construct a moduli space $M(\mathcal{O}_X, P)$ that universally corepresents this functor. We recall definitions of stable and semistable coherent sheaves and Jordan-Hölder filtration A.1.47 and A.1.48. Two semistable coherent sheaves E and F are said to be S-equivalent if $\mathfrak{gr}(E) \cong \mathfrak{gr}(F)$.

We state results from [Sim94a] for boundedness (according to Mumford-Castelnuovo regularity) of sheaves.

Theorem 4.1.1. *Fix P such that $\deg(P) = d$ and b. The set of sheaves $\mathcal{E}$ on X woth Hilbert polynomial P, such that $\mathcal{E}$ has pure dimension d, and for any subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $\mu(\mathcal{F}) \leq b$, is bounded. In particular, the set of μ-semistable sheaves with Hilbert polynomial P is bounded.*

The following lemma from [Sim94a] is useful for boundedness results.

Lemma 4.1.2. *Suppose F is a k-bounded family of coherent sheaves of pure dimension d and rank r on $\mathbb{P}^n$ with $k \leq d-1$. Then there is an integer B such that for all $\mathcal{E}$ in F and all m, we have*

$$h^0(\mathbb{P}^n, \mathcal{E}(m)) \leq \begin{cases} 0; \ m+B \leq 0 \\ r(m+B)^d/d!; \ m+B \geq 0. \end{cases}$$

Hilbert schemes are generalized version of Grassmannian variety of finite dimensional vector spaces. We will describe the theory as explained in [Sim94a] in relative cases. We let X to be a scheme over a separated scheme S and H is a very ample line bundle on X. Fix a polynomial $P(n)$. Suppose $\mathcal{W}$ is a coherent sheaf on X. The Hilbert scheme represents a functor parametrizing quotients $\mathcal{W} \to \mathcal{F} \to 0$ with Hilbert polynomial P. For any connected scheme $\sigma : S' \to S$, the S'-valued points of $\mathrm{Hilb}(\mathcal{W}, P)$ are the isomorphism classes of quotients on $X \times_S S'$ given as $\sigma^*(\mathcal{W}) \to \mathcal{F} \to 0$ where $\mathcal{F} \to 0$ where $\mathcal{F}$ is flat over S' and $P(\mathcal{F}, n) = P(n)$. The fiber of $\mathrm{Hilb}(\mathcal{W}, P)$ over a closed point $s \in S$ is $\mathrm{Hilb}(\mathcal{W}_s, P)$. For large values of m there are closed projective embedding ψ_m and $\mathcal{L}_m$ denotes corresponding very ample line bundle.

Lemma 4.1.3. *There exists M such that for $m \geq M$, the following holds. Suppose $V \otimes \mathcal{W} \to \mathcal{F} \to 0$ is a point in $\mathrm{Hilb}(V \otimes \mathcal{W}, P) \to \mathcal{F} \to 0$. For any subspace $W \subset V$, let $\mathcal{G}$ denote the subsheaf of $\mathcal{F}$ generated by $W \otimes \mathcal{W}$. Suppose that $P(\mathcal{G}, m) > 0$ and $\frac{\dim(W)}{P(\mathcal{G},m)} \leq \frac{\dim(V)}{P(m)}$ for all nonzero proper subspaces W. Then the point is semistable with respect to the linearized invertible sheaf $\mathcal{L}_m$ and the action of $SL(V)$.*

Lemma 4.1.4. *There exists M such that for $m \geq M$, if $\mathcal{F}$ is the quotient sheaf represented by a point of $\mathrm{Hilb}(V \otimes \mathcal{W}, P)$ which is semistable with respect to $\mathcal{L}_m$ and the action of $SL(V)$, then the following property holds. For any nonzero subspace $W \subset V$, let $\mathcal{G} \subset \mathcal{F}$ be the subsheaf generated by $W \otimes \mathcal{W}$. Then $r(\mathcal{G}) > 0$ and $\frac{\dim(W)}{r(\mathcal{G})} \leq \frac{\dim(V)}{r(\mathcal{F})}$.*

Suppose S is a scheme of finite type over $\mathrm{Spec}(\Bbbk)$ and suppose $X \to S$ is projective. We will consider the functor $M^*(\mathcal{O}_X, P)$ which associates to any S-scheme S' the set of semistable sheaves $\mathcal{E}$ on $X \times_S S'/S'$, of pure dimension d, with Hilbert polynomial P. Fix a large number N. We already have a very ample locally free sheaf of rank 1 given as $\mathcal{O}_X(1) = f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let $W = \mathcal{O}_X(-N)$ and and $V = \Bbbk^{P(N)}$. For an S-scheme S' the set of S'-valued points in $\mathrm{Hilb}(V \otimes W, P)$ may be described as the set of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E}$ is a coherent sheaf on $X' = X \times_S S'$ flat over S' with Hilbert polynomial P and $\alpha : V \otimes \mathcal{O}_{S'} \to H^0(X'/S', \mathcal{E}(N))$ is a morphism such that the sections in the image of α generate $\mathcal{E}(N)$.

Let $Q_1 \subset \mathrm{Hilb}(V \otimes \mathcal{W}, P)$ denote the open set where the sheaf $\mathcal{E}$ has pure dimension d and is Gieseker semistable. The set of Gieseker semistable sheaves on the fibers with Hilbert polynomial P is bounded. We assume that N is chosen large enough so that every Gieseker-semistable sheaf with Hilbert polynomial P appears as a quotient corresponding to a point in Q_1 and for any Gieseker semistable sheaf with Hilbert polynomial P, the space $H^0(X'/S', \mathcal{E}(N))$ is locally free over S' of rank $P(N)$. Set Q_2 as the open set in Q_1 where α is an isomorphism. then Q_2 represents a functor that associates to an S-scheme S' the set of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E}$ is a Gieseker semistable sheaf on $X' =$ with Hilbert polynomial P and $\alpha : V \otimes \mathcal{O}_{S'} \cong H^0(X'/S', \mathcal{E}(N))$. Let Q_2 denote the open subset of Q_1 where α is an isomorphism. Here Q_2 represents the functor that associates to an S-scheme S' the set of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E}$ is a Gieseker semistable coherent sheaf on $X \times_S S' = X'$ with Hilbert polynomial and α is an isomorphism.

We also fix M large and and consider $m \geq M$. We may assume that for each such m there is an embedding ψ_m of $\mathrm{Hilb}(V \otimes W, P)$ in a Grasmannian corresponding to a very ample line bundle $\mathcal{L}_m$. The open subset Q_2 is preserved by this action.

Let Q_2^* and $SL(V)^*$ denote the functors represented by Q_2 and and $SL(V)$ respectively. We write the quotient functor $Q_2^*/SL(V)^*$ associating to an S-scheme S' the quotient set $Q_2(S')/SL(V)(S')$. There is a natural morphism of functors $Q_2 \to M^*(\mathcal{O}_X, P)$ invariant under the group action giving a natural morphism $Q_2^*/SL(V)^* \to M^*(\mathcal{O}_X, P)$. To construct a scheme $M(\mathcal{O}_X, P)$ universally corepresenting the functor $M^*(\mathcal{O}_X, P)$ it suffices to construct a scheme universally corepresenting the quotient $Q_2^*/SL(V)^*$ that is to say a universal categorical quotient Q_2 by action of $SL(V)$. We will construct the moduli space as a good quotient $M(\mathcal{O}_X, P) = Q_2/SL(V)$. We will discuss a criterion for semistability of points in the Hilbert scheme with respect to the action of $SL(V)$ and linearlized invertible sheaf $\mathcal{L}_m$ for large m and $m \geq M(N)$. Let d be the degree of the polynomial P. Define $\mathrm{Hilb}(V \otimes W, P, d)$ to be the closure in $\mathrm{Hilb}(V \otimes W, P)$ of the set of points such that the quotient sheaf $\mathcal{E}$ is of pure dimension d. (We want to confirm that the quotient sheaves by subsheaves are pure too.) We have $Q_2 \subset \mathrm{Hilb}(V \otimes W, P, d)$.

Lemma 4.1.5. *If $\mathcal{E}$ is the quotient sheaf represented by a point of* $\mathrm{Hilb}(V \otimes W, P, d)$*, let $\mathcal{L}$ denote the coherent subsheaf of sections supported in dimension $\leq d-1$, then there is a sheaf $\mathcal{E}'$ of pure dimension d, with Hilbert polynomial P and an inclusion $0 \to \mathcal{E}/\mathcal{L} \to \mathcal{E}'$.*

Lemma 4.1.6. *There exists N_0 such that for all $N \geq N_0$ the following statement is true. Suppose $\mathcal{E}$ is a Gieseker semistable sheaf on a fiber X_s with Hilbert polynomial P. then for all subsheaves $\mathcal{F} \subset \mathcal{E}$, we have*

$$\frac{h^0(\mathcal{F}(N))}{r(\mathcal{F})} \leq \frac{P(N)}{r(\mathcal{E})}$$

and if equality holds then

$$\frac{p(\mathcal{F}, m)}{r(\mathcal{F})} = \frac{P(m)}{r(\mathcal{E})}$$

for all m.

Theorem 4.1.7. *Fix a polynomial P of degree d. There exist M and N such that for $m \geq M$, the following statement is true. A point $\mathcal{E}$ in* $\mathrm{Hilb}(V \otimes W, P, d)$ *is semistable (respectively, stable) for the action of $SL(V)$ with respect to the embedding determined by m, if and only if the quotient $\mathcal{E}$ is a Gieseker semistable (resp. Gieseker stable) coherent sheaf of pure dimension d and the map $V \to H^0(\mathcal{E})$ is an isomorphism.*

Corollary 4.1.8. *The scheme Q_2 is equal to the set of semistable points of* $\mathrm{Hilb}(V \otimes \mathcal{W}, P, d)$ *under the action of $SL(V)$. The open subset Q_2^s parametrizing Gieseker stable sheaves is equal to the set of stable points under the action of $SL(V)$.*

Theorem 4.1.9. *Let $M(\mathcal{O}_X, P) := Q_2/SL(P(N))$ be the good quotient applied to the group action on* $\mathrm{Hilb}(V \otimes \mathcal{W}, P, d)$*. (i) There exists a natural transformation $\Phi : M^*(\mathcal{O}_X, P) \to M(\mathcal{O}_X, P)$ such that*

$M(\mathcal{O}_X, P)$ universally corepresents $M^*(\mathcal{O}_X, P)$. (ii) The scheme $M(\mathcal{O}_X, P)$ is a projective scheme. (iii) The points of $M(\mathcal{O}_X, P)$ represent the equivalence classes of semistable sheaves under isomorphic grading associated to Jordan-Hölder. (iv) There is an open subset $M^s(\mathcal{O}_X, P)$ with inverse image equal to Q_2^s whose points represent Gieseker stable sheaves. (v) If $x \in M^s(\mathcal{O}_X, P)$ is a point such that Q_2^s is smooth at the inverse image of x, then $M^s(\mathcal{O}_X, P)$ is smooth at x.

The proof of this theorem is available in [Sim94a] Theorem 1.21.

4.2 A parameterizing scheme for Gieseker semistable Λ-modules

4.2.1 Sheaves of rings of differential operators

Suppose S is a Noetherian scheme over $\mathbb{C}$ and $f : X \to S$ is a scheme of finite type over S. A sheaf of rings of differential operators on X over S is a sheaf of $\mathcal{O}_X$-algebras Λ over X with a filtration $\Lambda_0 \subset \Lambda_1 \subset ...$ which satisfies the following properties. (i) $\lambda = \bigcup\limits_{i=0}^{\infty} \Lambda_i$ and $\Lambda_i \Lambda_j \subset \Lambda_{i+j}$. (ii) The image of the morphism $\mathcal{O}_X \to \Lambda$ is equal to Λ_0. (iii) The image of $f^{-1}(\mathcal{O}_S)$ in $\mathcal{O}_X$ is contained in the center of Λ. (So Λ needs not be commutative.) (iv) The left and right $\mathcal{O}_X$-module structures on $Gr_i(\Lambda) := \Lambda_i/\Lambda_{i-1}$ are equal.(v) The sheaves of $\mathcal{O}_X$-modules $Gr_i(\Lambda)$ are coherent.(vi) The sheaf of graded $\mathcal{O}_X$-algebras $Gr(\Lambda) := \bigoplus\limits_{i=0}^{\infty} Gr_i(\Lambda)$ is generated by $Gr_1(\Lambda)$ in the sense that the morphism of sheaves

$$Gr_1(\Lambda) \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} Gr_1(\Lambda) \to Gr_i(\Lambda)$$

is surjective.

Let V be a locally free sheaf on X of rank n. Then $Gr_i(V) = \mathrm{Sym}^i(V)$. We explain coherent $\mathrm{Sym}(V^*)$-modules as Λ-modules. Recall that $\mathrm{Sym}^0(V^*) = \mathcal{O}_X$ and for any n the n-th symmetric product is defined as the locally free sheaf $\mathrm{Sym}^n(V^*) = (V^* \otimes ... \otimes V^*)/ < x \otimes y - y \otimes x >$. Finally, $\mathrm{Sym}(V^*) = \bigoplus\limits_{i=0}^{\infty} \mathrm{Sym}^i(V^*)$. Here we explain an integrability condition. Let $\phi : E \to E \otimes V$ and $\gamma' : V \to \wedge^2 V \otimes V^*$ be canonically written as $u \mapsto \sum\limits_{i=1}^{n} (u \wedge v_i) \otimes v_i^*$ on trivializing neighbourhood with a local basis $\{u_1, ..., u_n\}$. Further, $\gamma : E \otimes V \to E \otimes \wedge^2 V \otimes \mathcal{V}^*$ is modified by $Id \otimes \gamma'$. On the other hand, $\phi' : E \otimes V^* \to E$ from ϕ according to isomorphism $\mathrm{Hom}(E, E \otimes V) = \mathrm{Hom}(V^* \otimes E, E)$. We call ϕ integrable if $\phi' \gamma \phi = 0$. If $n = 1$ we have $\gamma = 0$. On a coherent sheaf of modules E, a module homomorphism $\phi : E \to E \otimes V$ defines a $\mathrm{Sym}(V^*)$-module structure if it satisfies $\phi \wedge \phi = 0$. Recall that E is coherent if and only if each point $x \in X$ admits an open neighbourhood U and a finite number of sections of $E_{|U}$ such that for each $y \in U$ corresponding $\mathcal{O}_{X,y}$-module E_y is generated by these sections. From corresponding morphism $\phi' : V^* \otimes E \to E$ we have a global action on E via $V^* \to \mathrm{End}(E)$ given as $v^* \mapsto (g : e \mapsto \phi'(v^* \otimes e))$. On the other hand, an action of $\mathrm{Sym}(V^*)$ on V defines a global morphism $V^* \otimes E \to E$, thus a global morphism $\phi : E \to E \otimes V$. We will work on coherent $\mathrm{Sym}(V^*)$-modules in this thesis.

4.2.2 Invariant theory for Λ-modules

We explain the moduli space of $\mathrm{Sym}(V^*)$-modules under Gieseker (semi)stability.

Definition 4.2.1. A $\mathrm{Sym}(V^*)$-module $\mathcal{E}$ is a on X over $\mathrm{Spec}(\Bbbk)$ with pure dimension d, is said to be Gieseker (semi)stable if for each sub $\mathrm{Sym}(V^*)$-module $\mathcal{F}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$ there exists a large integer N such that

$$\frac{P(\mathcal{F}, n)}{\mathrm{rank}(\mathcal{F})} (\leq) < \frac{P(\mathcal{E}, n)}{\mathrm{rank}(\mathcal{E})}$$

holds for all $n \geq N$. Slope (semi)stability is defined likewise for invariant coherent subsheaves.

Lemma 4.2.2. *Suppose $\mathcal{E}$ is a coherent $\mathrm{Sym}(V^*)$-module on X over $\mathrm{Spec}(\Bbbk)$. There is a unique filtration of sub $\mathrm{Sym}(V^*)$-module $0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_k = \mathcal{E}$ such that the quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are Gieseker semistable $\mathrm{Sym}(V^*)$-modules pure of dimension d with strictly decreasing normalized Hilbert polynomials.*

Remark 19. Suppose that $\mathcal{E}$ is a Gieseker semistable coherent $\mathrm{Sym}(V^*)$-module. Then there exists a unique filtration by subsheaves with torsion-free quotients preserved by action of $\mathrm{Sym}(V^*)$ such that the quotients are direct sums of Gieseker stable Λ-modules with the same normalized Hilbert polynomials. Two such Gieseker semistable modules are isomorphic if their graded modules (that is, direct sums of the quotients) are isomorphic.

Remark 20. Slope stability of $\mathrm{Sym}(V^*)$-modules, Jordan-Hölder filtration and Harder-Narasimhan filtration are defined as in usual way we do for coherent sheaves in A.1.43 and A.1.44. The same implications of μ-(semi)stability and Gieseker (semi)stability.

We mention results from [Sim94a] about boundedness properties. We mention them for $\mathrm{Sym}(V^*)$-modules because we are working on such sheaves of differential operators.

Lemma 4.2.3. *Let m denote a number such that $V \otimes \mathcal{O}_X(m)$ is generated by global sections. Then for any slope semistable coherent $\mathrm{Sym}(V^*)$-module $\mathcal{E}$ of dimension d and rank r and any subsheaf $\mathcal{F} \subset \mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E}) + mr$. In other words, $\mu(\mathcal{F}) - \mu(\mathcal{E})$ is bounded above depending on r and V.*

Corollary 4.2.4. *The set of slope semistable $\mathrm{Sym}(V^*)$-modules on X with a given Hilbert polynomial P is bounded.*

We recall following theorem from [Sim94a] to explain parametrizing scheme of $\mathrm{Sym}(V^*)$-modules.

Theorem 4.2.5. *Fix a polynomial P, and let $N \geq N_0$ where N_0 is a well chosen integer. The unctor which associates to each S-scheme S' the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$, where $\mathcal{E}$ is a Gieseker semistable $\mathrm{Sym}(V^*)$-module with Hilbert polynomial P on $X' = X \times_S S'$ and $\alpha : (\mathcal{O}'_S)^{P(N)} \to H^0(X'/S', \mathcal{E}(N))$ is an isomorphism, is representable by a quasiprojective scheme Q over S.*

Lemma 4.2.6. *There is an integer B depending on Λ, r and d such that if $\mathcal{E}$ is a slope semistable Λ-module with pure dimension d and rank r on a fiber X_s, then*

$$h^0(X_s, \mathcal{E}(k)) \leq \begin{cases} 0; \ \mu(\mathcal{E}) + k + B \leq 0 \\ r(\mu(\mathcal{E}) + k + B)^d/d!; \ \mu(\mathcal{E}) + k + B \geq 0 \end{cases}$$

for any k.

Lemma 4.2.7. *There exists N_0 depending upon Λ and P such that for all $N \geq N_0$ the following is true. Suppose $\mathcal{E}$ is Gieseker semistable Λ-module with Hilbert polynomial P on a fiber X_s. Then for all submodules $\mathcal{F} \subset \mathcal{E}$, we have*

$$\frac{h^0(\mathcal{F}(N))}{r(\mathcal{F})} \leq \frac{P(N)}{r(\mathcal{E})}$$

and if equality holds then

$$\frac{p(\mathcal{F}, m)}{r(\mathcal{F})} = \frac{P(m)}{r(\mathcal{E})}$$

for all m.

We refer to the basic features of Hilbert schemes of semistable Λ-modules as in [Sim94a] Section:4 similar to ordinary coherent sheaves. We mention the properties of quotients here. Let $X \to S$ be a projetive morphism and $\Lambda = \mathrm{Sym}(V^*)$ a sheaf of rings of differential operators on X over S. We can construct moduli spaces for Gieseker semistable $\mathrm{Sym}(V^*)$-modules of rank $\leq r$ and normalized Hilbert polynomial P/r, on fibers X_s. Let $Q \to S$ be the parameter scheme already mentioned. Consider the sheaf $\mathcal{W} = \mathrm{Sym}^r(V^*) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)$ and $V = \Bbbk^{P(N)}$, then there is a locally closed embedding $Q \subset \mathrm{Hilb}(\mathcal{W} \otimes V, P)$. Let $M^*(\Lambda, P)$ (keeping in mind that $\Lambda = \mathrm{Sym}(V^*)$) denote the functor of schemes over S which associates to $S' \to S$ the set of isomorphism classes of Gieseker semistable $\mathrm{Sym}(V^*)'$ modules on $X' = X \times_S S'$ over S' with Hilbert polynomial P. The following theorem is [Sim94b] Theorem 4.7. We refer to [Sim94a] Section 4 for its proof.

Theorem 4.2.8. *Let $M(\Lambda, P) = Q/SL(V)$ be the good quotient. There is a morphism of functors $\phi : M^*(\Lambda, P) \to M(\Lambda, P)$ such that $(M(\Lambda, P), \phi)$ universally corepresents the functor $M^*(\Lambda, P)$. The following properties are satisfied.(i) $M(\Lambda, P)$ is a quasi-projective variety. (ii) The geometric points of $M(\Lambda, P)$ represent the equivalence classes of Gieseker semistable* $\mathrm{Sym}(V^*)$*-modules with Hilbert polynomial P on fibers X_s, under S-equivalence. The equivalence class of a Λ-module $\mathcal{F}$ on X_s corresponds to the point $\phi(\mathcal{F})$ in the fiber of $M(\mathrm{Sym}(V^*), P)$ over $s \in S$. (iii) There is an open subset $M^s(\Lambda, P) \subset M(\Lambda, P)$ whose points represent isomorphism classes of Gieseker stable Λ-modules. Locally in the etale topology on $M^s(\Lambda, P)$ there is a universal Λ-module $\mathcal{F}^{univ}$ such that if $\mathcal{F}$ is an element of $M^*(\Lambda, P)(S')$ whose fibers $\mathcal{F}_s$ are stable, then the pullback of $\mathcal{F}^{univ}$ via $S' \to M^s(\Lambda, P)$ is isomorphic to $\mathcal{F}$ after tensoring with the line bundle on S'.*

5 Hitchin morphism on smooth varieties

5.1 Overview of the problems

An important aspect of Hitchin morphism, as studied by Hitchin (see [Hit87b]) for the case of curves, is giving a concrete characterization of the image space. This is entirely a question of algebraic geometry though Hitchin's proof was analytic. Nitsure's arguments paved a direction of exploring the possible image spaces of Hitchin morphism in case of L-twisted pairs, not only on smooth curves but also on smooth varieties. Recall for $L = K_X$ that Beauville, Narasimhan and Ramanan argued for $g_X > 1$ that the characteristic polynomial morphism $T^*\mathcal{N}(r,d) \to \mathbb{A}(r,\Omega_X)$ is dominant ([BNR89] Theorem 1). Thus the image of $T^*\mathcal{N}(r,d)$ is open dense in affine Hitchin base for any choice of rank r and degree d. As L-twisted Hitchin morphism proper (thus universally closed) thus closed and the image of Hitchin morphism is closed and it contains the closure of image of $T^*\mathcal{N}(r,d)$ which is the full affine base space. So, Hitchin morphism is surjective here.

Statement of Questions:
(1) Let X be a smooth irreducible projective curve and L be a line bundle on X. Given integers r, d, is Nitsure's Hitchin morphism $H : \mathcal{M}_X(r,d,L) \to \bigoplus_{i=1}^{r} H^0(X, L^i)$ surjective?
(2) Let X be a smooth irreducible projective variety and L be a line bundle on X. We denote by $\mathcal{M}_{X,H_X}(P,L)$ Simpson's moduli space of semistable $\mathrm{Sym}(L^{-1})$-modules on X. For a given integer r and a Hilbert polynomial P and a well-chosen ample polarization H_X, is Simpson's Hitchin morphism (defined in [Sim94b]) $\mathcal{H} : \mathcal{M}_{X,H_X}(P,L) \to \bigoplus_{i=1}^{r} H^0(X, L^i)$ surjective?

We observe that if image of a proper morphism contains a dense subset it is surjective. For this purpose, we set a hypothesis on line bundle L such that we access to a dense set of smooth integral spectral covers. The proof then relies on a proof of properness on varieties. This is based on the fact that the moduli space of Giesker semistable torsion-free coherent sheaves on a smooth projective variety X is separated (see [Mar77]). We overall follow the lines of a proof available in [GGPN23]. In this context, for ample twisting line bundle, we mention from Bertini's theorem ([Har77] Theorem 8.18) an explicit description of spectral varieties via an immersion to a projective space.

However, this strategy does not apply universally to any twisting sheaf due to the integrability condition on the Higgs field. Integrable Higgs fields are equivalent to $\mathrm{Sym}(V^*)$-module structures on coherent sheaves. In the work of Chen and Ngô (see [CN20]), a closed subscheme of the base $\bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i(T^*X))$ is postulated as the image of the Hitchin morphism on a surface ([CN20] Conjecture 5.2). They also provided an updated version of the spectral correspondence. From the spectral correspondence, this postulate image or postulated Hitchin base admits an open dense subspace of points that admit non empty Hitchin fiber (see [CN20] Proposition 6.3). We do not elaborate on this approach but derive an inspiration. On the other hand, Gallego, García-Prada, Narasimhan ([GGPN23]) studied pairs on a curve twisted by a vector bundle and corresponding spectral covers and the spectral correspondence. The commonality between these two articles is that both of these feature the Hitchin morphism in case of a twist by a vector bundle. This becomes another useful source of inspiration for us. In the present context we will explore determinant morphism on trace-less co-Higgs fields exploiting a set of examples studied by Rayan ([Ray11]) and Colmenares ([Col15]). We have not included any deep study of spectral varieties for surfaces.

5.2 Outline of proofs from the spectral view point

5.2.1 Immersion of spectral varieties for an ample line bundle twist

Let X be a smooth integral projective algebraic variety proper over algebraically closed field $\Bbbk$ of characteristic 0. Let $\pi : L \to X$ be a line bundle and M be a line bundle on X. We will prove π^*M is ample too. Note that X is Noetherian over $\Bbbk$. The total space L is Noetherian also because π is affine. Let F be coherent sheaf on L, then π_*F is coherent on X. There is an integer n_0 such that for all $n \geq n_0$ the sheaf $\pi_*F \otimes M^n$ is finitely generated. Equivalently, there is an index set I such that there is a surjection $\mathcal{O}_X^{\oplus I} \to \pi_*F \otimes M^n$ for all $n \geq n_0$. Now taking pullback of such surjection we have $\mathcal{O}_L^{\oplus I} \cong \pi^*\mathcal{O}_X^{\oplus I} \to \pi^*(\pi_*F \otimes M^n) \cong \pi^*\pi_*F \otimes \pi^*M^n$. There is a canonical morphism $\pi^*\pi_*F \to F$ which is a surjective morphism reduced to the following surjective module homomorphism. Let $\psi : A \to B$ be a ring homomorphism with $\psi(1_A) = 1_B$, we denote a B-module N as an A-module denoted by N_A. The map $N_A \otimes_A B \to N_A$ defined as $n \otimes b \mapsto n \cdot b$ is surjective. Thus we have a surjective morphism $\mathcal{O}_L^{\oplus I} \to F \otimes \pi^*M^n$ for $n \geq n_0$. Thus π^*M is ample on L. Recall that M^r is very ample for some large integer r. We highlight the role of very ample line bundles to express embedding of varieties into projective spaces. A very ample line bundle L on X is base point free. Let $\dim_{\Bbbk} H^0(X, L) = N + 1$ and $s_0, ..., s_N$ be a basis of L. Then $f : X \to \mathbb{P}^N_{\Bbbk}$ defined by $x \mapsto [s_0(x) : ... : s_N(x)]$ is an immersion meaning that its image is a quasi-projective variety. We will observe that specific spectral schemes admit immersion to $\mathbb{P}^n_{\Bbbk}$ as quasi-projective spaces. Let $r \geq 2$ be an integer such that π^*L^r is very ample meaning that L can be embedded as a quasi-projective variety by a basis of global sections of π^*L^r. We analyze the immersion $L \subset Z = \mathbb{P}(H^0(L, \pi^*L^r))$ as a locally closed variety. Observe that Z is an N-dimensional projective space where $N = \sum_{i=1}^{r} h^0(X, L^i)$ is the dimension of L-twisted Hitchin base. From the bundle projection formula with respect to the morphism π we obtain $\Bbbk$-isomorphism of vector spaces $\bigoplus_{i=0}^{r} H^0(X, L^i) \cong H^0(L, \pi^*L^r)$ while the isomorphism is given by

$$s = (s_0, ..., s_r) \mapsto p_s = \sum_{i=0}^{r} \pi^* s_i \otimes \eta^{r-i}.$$

This is due to $\pi_* \mathcal{O}_L \cong \bigoplus_{i=0}^{\infty} L^{-i}$ and the fact that negative powers of an ample line bundle admit only trivial global section. The elements in Z are represented by the elements of the form $\sum_{i=0}^{r} \pi^* s_i \otimes \eta^{r-i}$ such that all of $s_0, ..., s_r$ do not vanish simultaneously. In particular, the L-twisted Hitchin base giving spectral polynomials are represented by fixing $s_0 = 1$, forming an open dense subset in the projective space Z. We view spectral curves as the intersection of hyperplanes in projective space with L. Let $s_0, ..., s_N$ be a basis of $\bigoplus_{i=1}^{r} H^0(X, L^i)$. Then an immersion of L is given as

$$y \mapsto [\pi^* s_0(y) : ... : \pi^* s_N(y) \eta^r(y)].$$

Now a spectral polynomial is written as $c_0 \pi^* s_0 + ... + c_N \pi^* s_N \eta^r$ by these basis elements and its corresponding hyperplane H is given as

$$c_0 z_0 + ... + c_N z_N = 0$$

and

$$H \cap L = \{y \in L : c_0 \pi^* s_0(y) + ... + c_N \pi^* s_N \eta^r(y) = 0\}$$

and this description coincides with the definition of a spectral variety. We can view spectral varieties as divisors corresponding to the line bundle $\pi^* L^r$ on L. Total space of a line bundle is quasi-projective and spectral curves admit a quasi-projective embedding in $\mathbb{P}^N_{\Bbbk}$. By Bertini's theorem smooth spectral varieties form an open dense subset.

5.2.2 Construction of a proof for curves (Question (1))

Hypothesis: Let L be a line bundle on a smooth projective curve X and r be an integer such that the set of elements $s \in \bigoplus_{i=1}^{r} H^0(X, L^i)$ with smooth integral spectral curve X_s.

Under the above hypothesis consider Nitsure's $H : \mathcal{M}_X(r, d, L) \to \bigoplus_{i=1}^{r} H^0(X, L^i)$ proper Hitchin morphism. We complete a proof that H is surjective for such an integer r. We already used the formula for (geometric) genus of a smooth integral spectral curve for fixed s that

$$g_{X_s} = \deg(L) \frac{r(r-1)}{2} + r(g-1) + 1$$

and define d' by the equation $d = d' + (1 - g_{X_s}) - r \cdot (1 - g)$. Consider a line bundle M on X_s with $\deg(M) = d'$ then $(E, \phi) = \pi_{s*}(M, \eta)$ is a pair on X which admits p_s as its characteristic polynomial. Such an L-twisted pair is stable because p_s is irreducible. Thus image of $H : \mathcal{M}_X(r, d, L) \to \bigoplus_{i=1}^{r} H^0(X, L^i)$ contains a dense subset of the present Hitchin base. Finally, H is surjective due to properness. □

5.2.3 Twisted Higgs sheaves on smooth surfaces

We will generalize the argument attaching more assumption, for algebraic surfaces for simplicity. We assume that X is a smooth irreducible projective surface. Let $H_X := \mathcal{O}_X(1)$ be an ample line bundle on X. As we have seen, the moduli space of semistable pure coherent sheaves E of pure dimension 2 (thus torsion free) under polarization H_X form a projective scheme. Moreover the moduli space of semistable pure coherent sheaves of $\mathrm{Sym}(F^*)$-modules dimension 2 forms a quasi-projective variety.

Here we take a chance to mention Simpson's second construction as mentioned in [Sim94b]. We assume in our context that F is a vector bundle of rank 2 and Z is a projective completion (we can think it as $\mathbb{P}(F \oplus \mathcal{O})$) such that the bundle projection $\pi : F \to X$ can be extended to an affie morphism on Z. We define the hyperplane divisor $D = Z \backslash F$. If we assume F a line bundle the explanations will be simpler as we have seen in Chapter 3 We consider the corresponding $\Lambda = \mathrm{Sym}(\mathcal{F}^*)$ where $\mathcal{F}$ denotes the sheaf of sections of F. On a trivializing neighbourhood U of F we can explicitly write $\pi_*\mathcal{O}_F|_U \cong \mathcal{O}_X[\lambda, \mu]$ where λ, μ are generators of local sections of F on U. Let E be a coherent sheaf of pure dimension 2 and V be an open dense subset of X such that E is locally free of rank r, say, on V. Consider a sheaf homomorphism $\phi : E \to E \otimes F$ such that $\phi \wedge \phi = 0$ meaning that $\phi_{|_V} \wedge \phi_{|_V} = 0$. To be precise, if E is locally free then $\phi \wedge \phi$ is locally $[\theta_1, \theta_2]\mu \wedge \lambda$ where $\phi = \theta_1\mu + \theta_2\lambda$. More generally, let E be a locally free sheaf and $\theta, \psi : E \to E \otimes F$ be morphisms of locally free sheaves. We can explain a global morphism $\theta \wedge \psi : E \to E \otimes \wedge^2 F$ with a local description $\sum_{i<j}(\theta_i\psi_j - \psi_j\theta_i) \otimes u_i \wedge u_j$. Here u_i's form a local basis of V. These elements $(\theta_i\psi_j - \psi_j\theta_i)$ via transition data. Coming to the original case, we need $[\theta_1, \theta_2] = 0$ for integrability. Here action of ϕ on E is determined by action of a polynomial $f(\theta_1, \theta_2)$ for a polynomial $f(\mu, \lambda) \in \mathcal{O}_X[\lambda, \mu]$. This explanation holds trivially for a twisting invertible sheaf L. We refer to the following lemma from [Sim94b] for twisting locally free sheaf F of rank ≤ 2. The statement is proved using the same argument from [Har77] page: 128 Exercise 5.17 for affine morphism π. We define an ample sheaf $\mathcal{O}_Z(1) := \pi^*\mathcal{O}_X(m) \otimes \mathcal{O}_Z(D)$ for some large positive integer m and the ample sheaf $\pi^*\mathcal{O}_X(m)$ on F. In the following statement stability is defined with respect to these ample invertible sheaves.

Lemma 5.2.1. *There is an one-to-one correspondence between (i) isomorphism classes of pure coherent torsion-free sheaves $\mathcal{E}$ of dimension 2 on Z such that $\mathrm{Support}(\mathcal{E}) \cap D$ is empty and (ii) isomorphism classes of pure coherent torsion-free $\mathrm{Sym}(\mathcal{F}^*)$-modules on X. This equivalence is extended to categorical correspondences for Gieseker (semi)stable sheaves $\mathcal{E}$ and Gieseker (semi)stable $\mathrm{Sym}(\mathcal{F}^*)$-modules E.*

5.2.4 Spectral correspondence on a smooth surface (Question (2))

We shift our attention to line bundle twists for surfaces. can explain spectral correspondence for a line bundle twist following the argument given for curves. Let $\pi : L \to X$ be a line bundle and η is the tautological section $\pi^* L$. The automatic definition of a spectral surface is given as

$$X_s := \left\{ y \in L : \eta^r(y) + \sum_{i=1}^{r} (\pi^* s_i \otimes \eta^{r-i})(y) = 0 \right\}.$$

An obvious description as a scheme can also be given. This is a Noetherian scheme of dimension 2 as per the reasoning we gave for spectral curves. We mention the spectral correspondence likewise.

Theorem 5.2.2. *Let X_s be an integral spectral cover of surface X with finite $r:1$ covering map π. Then there is an equivalence of categories between torsion-free sheaves of rank 1 on X_s and torsion-free L-twisted Higgs bundles of rank r on X. This equivalence reduces to a one-to-one correspondence of isomorphism classes of corresponding sheaves.*

Hypothesis: (a) We choose a line bundle L and we consider $r \geq 2$ such that the set of (tuples of) sections s in affine base space $\bigoplus_{i=1}^{r} H^0(X, L^i)$ with integral spectral surface X_s is a dense subset.

By pushforward operation via the finite branched covering map π of degree r is there is a one-to-one correspondence between isomorphism classes of pure torsion-free coherent sheaves of rank n over X_s and pure coherent torsion-free twisted Higgs sheaves of rank nr over X generalizing theorem 5.2.2. This leads to correspondence for slope (respectively, Gieseker)(semi)stability of torsion-free sheaves on X_s and slope (respectively, Gieseker) (semi)stability of torsion-free coherent sheaves on X. Here we want to fix the ample line bundle $\mathcal{O}_X(1)$. We utilize the ample sheaf $\pi^*\mathcal{O}_X(m)$ on X_s. We fix $\mathcal{O}_{X_s}(1) = \pi^*\mathcal{O}_X(m)$. Here we can allow common m for all spectral surfaces X_s. This is because spectral surfaces are closed subvarieties inside the total space of L. With respect to this ample line bundle we relate Hilbert polynomials of two sheaves $P(\mathcal{E}, n) = P(\pi_*\mathcal{E}, mn)$ by projection formula of sheaves.

To explain properness results on a surface we must specify a Hilbert polynomial P which has degree 2 and $\alpha_2 = \alpha_2(\mathcal{O}_X) \cdot r$ (computed with respect to ample sheaf $\mathcal{O}_X(1)$ on X) while r is specified above for ampleness of $\pi^* L$. The pure coherent Higgs sheaves E with $\alpha_2(E) = \alpha_2(\mathcal{O}_X) \cdot r$ have rank r and we work with the moduli space of L-twisted Gieseker semistable Higgs sheaves with fixed Hilbert polynomial P denoted as $\mathcal{M}_X(P, L)$. Here we modify our ample sheaf on X from $H_X := \mathcal{O}_X(1)$ to $H_X^m = \mathcal{O}_X(m)$ and with respect to this modified ample sheaf we obtain for all $n \in \mathbb{N}$ that $P(\mathcal{E}, \mathcal{O}_{X_s}(1), n) = P(\pi_*\mathcal{E}, H_X^m, n)$. Note that Gieseker semistable Higgs sheaves on X with Hilbert polynomial $m^*P(n \mapsto P(mn))$ with respect to H_X are same as Gieseker semistable Higgs sheaves with Hilbert polynomial P with respect to ample sheaf H_X^m.

Hypothesis: (b) Let P be a Hilbert polynomial such that for each integral subscheme Y of Z of codimension 1 admits a torsion-free sheaf of rank 1 with Hilbert polynomial P with respect to ample line bundle $\mathcal{O}_Y(1) := \pi_Y^*\mathcal{O}_X(1)$.

With respect to this changed ample sheaf on X we consider Hitchin's characteristic polynomial morphism

$$\mathcal{H} : \mathcal{M}_{X,H_X^m}(P, L) \to \bigoplus_{i=1}^{r} H^0(X, L^i).$$

There is an open dense subset inside $\bigoplus_{i=1}^{r} H^0(X, L^i)$ such that spectral surface X_s is smooth, integral and proper over k. Consider torsion-free sheaves of rank 1 on such a spectral variety X_s and pushforward via the map π confirms existence of an open dense subset in the image of $\mathcal{H}$. Now we aim to show properness of $\mathcal{H}$.

We explain Maruyama's result of separated moduli schemes. This is described in a relative case. Let $f : X \to S$ be a smooth projective geometrically integral morphism of locally Noetherian schemes with an f-very ample invertible sheaf $\mathcal{O}_X(1)$. Maruyama ([Mar77]) explained a functor $\sum_{X/S}^{P}(T)$ on a scheme T locally of finite type on S. The functor has a coarse moduli scheme ([Mar77] Theorem 5.6).

Theorem 5.2.3. *The functor $\sum_{X/S}^{P}(T)$ has a coarse moduli scheme $M_{X/S}(P)$ in the category (Sch/S) of schemes on S. Moreover, $M_{X/S}(P)$ is separated and locally of finite type over S.*

This result explains separatedness of moduli scheme of coherent sheaves on a projective variety. We will use it in further reasoning. We mention a proof of properness using the line of arguments [GGPN23] Proposition 3.9. This is an abridged version of Simpson's original proof in [Sim94b].

Let R be a discrete valuation ring and $\Delta = \mathrm{Spec}(R)$ denotes the spectrum consisting of only two points. one of them is a closed point p and another one is the generic point $\Delta' = \mathrm{Spec}(K)$ where K is the quotient field of R. Consider a map $g' : \mathrm{Spec}(K) = \Delta' \to \mathcal{M}_{X,H_x^m}(P, L)$ such that $h \circ g$ extends to a map $f : \Delta = \mathrm{Spec}(R) \to \bigoplus_{i=1}^{r} H^0(X, L^i)$. We extend g' to $g : \Delta \to \bigoplus_{i=1}^{r} H^0(X, L^i)$.

As we have explained earlier that $\mathcal{M}_{X,H_X^m}(P, L)$ is represented by the moduli scheme of sheaves $\mathcal{E}$ inside $\mathcal{M}(\mathcal{O}_Z, P)$ which vanish on the hyperplane divisor D. Thus we obtain a morphism $g'' : \Delta' \to \mathcal{M}(\mathcal{O}_Z, P)$. This morphism g'' extends to a morphism (extending to closure) $g : \Delta \to \mathcal{M}(\mathcal{O}_Z, P)$ because the latter space is projective (that is, same as its projective closure). We aim to confirm that g takes p to $\mathcal{M}_{X,H_X^m}(P, L)$. Apart from separatedness, we confirm existence of a morphism.

The good quotient $Q_2 \to \mathcal{M}(\mathcal{O}_Z, P)$ is a surjective map of finite type. Thus we obtain an integral extension $R \subset \tilde{R}$, where $\tilde{R}$ is another discrete valuation ring over $\Bbbk$ with quotient field $\tilde{K}$ so that, if we denote $\tilde{\Delta} = \mathrm{Spec}(\tilde{R})$ and $\tilde{\Delta}' = \mathrm{Spec}(\tilde{K})$ and $\alpha : \tilde{\Delta} \to \Delta$ the map induced by the extension then we have the following commutative diagram.

$$\begin{array}{ccccc} \tilde{\Delta}' & \longrightarrow & \tilde{\Delta} & \longrightarrow & Q_2 \\ \downarrow & & \downarrow \alpha & & \downarrow \\ \Delta' & \longrightarrow & \Delta & \xrightarrow{g} & M(Z,P) \end{array} \tag{5.1}$$

The map $\tilde{\Delta} \to Q_2$ indicates a sheaf $\mathcal{E}$ on $Z_{\tilde{\Delta}}$ and for $s \in \tilde{\Delta}'$ we obtain that $g \circ \alpha(s) \in \mathcal{M}_{X,H_X^m}(P,L)$. Thus $\mathcal{E}|_{\tilde{\Delta}'}$ is supported inside of $L_{\tilde{\Delta}'}$ and it corresponds to a family of L-twisted Gieseker semistable Higgs sheaves E' on X parametrized by $\tilde{\Delta}'$ with $E'_s = g \circ \alpha(s)$. Therefore the map $\tilde{\Delta}' \to \bigoplus_{i=1}^{r} H^0(X, L^i)$ given by $s \mapsto \mathcal{H}(E'_s)$ extends to a map $b = f \circ \alpha : \tilde{\Delta} \to \bigoplus_{i=1}^{r} H^0(X, L^i)$.

From the spectral correspondence $\mathcal{E}$ is supported on a closed subscheme W inside $L_{\tilde{\Delta}}$. Hence $\mathcal{E}$ corresponds to a family E of semistable L-twisted torsion-free sheaves parametrized by $\tilde{\Delta}$ with $E'_{|\tilde{\Delta}'}$. That is, we have a map $\tilde{\Delta} \to \mathcal{M}(\mathcal{O}_Z, P)$ extending $g' \circ \alpha$. As $\mathcal{M}(\mathcal{O}_Z, P)$ is separated there is a unique such map and it is $g \circ \alpha$. So, if s_0 is the closed point of $\tilde{\Delta}$ we have $g(p) \in \mathcal{M}_{X,H_X^m}(P,L)$. □

We package our discussion in the following theorem under the assumptions mentioned so far.

Theorem 5.2.4. *The Hitchin morphism* $\mathcal{H} : \mathcal{M}_{X,H_X^m}(P,L) \to \bigoplus_{i=1}^{r} H^0(X, L^i)$ *is surjective.*

5.3 Hitchin morphism with specified underlying vector bundles

In the approach of Chen and Ngô in [CN20] the moduli stack of Higgs bundles was given more importance than the moduli scheme of Λ-modules itself. We are unable to extract enough information about image of Hitchin morphism from this approach. We investigate spectral data of specific algebraic surfaces over $\mathbb{C}$ in a holomorphic setting. As Rayan proved existence of co-Higgs bundles on the surfaces along the tail-end of Kodaira dimensions we highlight these cases here for investigating the image space of Hitchin morphism. The following discussion relies heavily on characterization of stable bundles on algebraic surfaces as explained by Rayan ([Ray14]) and Colmenares ([Col15]).

We explain existence of nonempty semistable moduli spaces as necessary. On $\mathbb{P}^2$ all the necessary information about the tangent bundle and the cotangent bundle are inherent to the following spectral sequences.

$$0 \to \mathcal{O} \to \bigoplus_{i=1}^{3} \mathcal{O}(1) \to T_{\mathbb{P}^1} \to 0 \tag{5.2}$$

and the dual sequence

$$0 \to T^*_{\mathbb{P}^1} \to \bigoplus_{i=1}^{3} \mathcal{O}(-1) \to \mathcal{O} \to 0. \tag{5.3}$$

The surface $\mathbb{P}^1 \times \mathbb{P}^1$ is the 0-th Hirzebruch surface where we have explicit descriptions for the tangent and the cotangent bundles

$$T_{\mathbb{P}^1 \times \mathbb{P}^1} \cong \pi_1^* T_{\mathbb{P}^1} \oplus \pi_2^* T_{\mathbb{P}^1}$$

and

$$T^*_{\mathbb{P}^1\times\mathbb{P}^1} \cong \pi_1^* T^*_{\mathbb{P}^1} \oplus \pi_2^* T^*_{\mathbb{P}^1}.$$

Discussions by Chen and Ngô on Hitchin morphism are concentrated on the classical case of cotangent sheaves and spectral correspondence by the Cohen-Macaulay sheaves on spectral varieties. We observe that cotangent sheaves on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ admit trivial Hitchin bases.

Remark 21. On $\mathbb{P}^2$ there is an exact sequence $0 \to S^2(T^*_{\mathbb{P}^2}) \to \bigoplus_{i=1}^{6} \mathcal{O}(-2) \to \bigoplus_{i=1}^{3} \mathcal{O}(-1)$. Thus we have $H^0(\mathbb{P}^2, T^*_{\mathbb{P}^2}) = 0$ and $H^0(\mathbb{P}^2, S^2(T^*_{\mathbb{P}^2})) = 0$.

Remark 22. We have direct computations on $\mathbb{P}^1 \times \mathbb{P}^1$. Observe that

$$S^2(\pi_1^*\mathcal{O}(-2) \oplus \pi_2^*\mathcal{O}(-2)) \cong \pi_1^*\mathcal{O}(-4) \oplus \pi_1^*\mathcal{O}(-2) \otimes \pi_2^*\mathcal{O}(-2) \oplus \pi_2^*\mathcal{O}(-4).$$

This gives trivial base $H^0(\mathbb{P}^1 \times \mathbb{P}^1, T^*_{\mathbb{P}^1\times\mathbb{P}^1}) \oplus H^0(\mathbb{P}^1 \times \mathbb{P}^1, S^2(T^*_{\mathbb{P}^1\times\mathbb{P}^1}))$. In other words, we have nilpotent cone consisting of the whole moduli scheme of semistable classical Higgs sheaves. Here the formula we are using is given

$$S^n(E_1 \oplus E_2) \cong \bigoplus_{i=0}^{n} S^i(E_1) \otimes S^{n-i}(E_2).$$

Here is another point of view from Nonabelian Hodge correspondence that $\mathbb{P}^1 \times \mathbb{P}^1$ has trivial fundamental group, so only trivial representations.

5.4 Cohomology group computations on $\mathbb{P}^2$

Here we refer to A.2.1 for background material. Moreover we have stable tangent and cotangent bundles on $\mathbb{P}^2$. A proof is available in [Lan75] that involves an argument that every torsion-free subsheaf of degree 0 of $\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ is trivial. We use this information to compute some cohomological invariants. We will use following Lemma 5.4.1 simultaneously for our computations. These elementary computations partly appeared in [Ray11].

Lemma 5.4.1. *(Kodaira-Nakano) $H^1(\mathbb{P}^1, \mathcal{O}(d))$ is trivial for $d \in \mathbb{Z}$.*

From 5.3 we write the corresponding long exact sequence and obtain

Proposition 5.4.2. *(i)* $h^0(\mathcal{O}_{\mathbb{P}^2}(d)) = \begin{cases} \frac{(d+1)(d+2)}{2}; d \geq 0 \\ 0; d < 0 \end{cases}$ *and (ii)* $h^0(T(d)) = \begin{cases} (d+2)(d+4); d \geq 0 \\ 3; d = -1 \\ 0; d < -1. \end{cases}$

From 5.2 we write the corresponding long exact sequence and obtain that

Proposition 5.4.3. $h^0(T^*(d)) = \begin{cases} 0; d < 1 \\ d^2 - 1 + h^1(T^*(d)); d \geq 1. \end{cases}$.

Moreover, we can compute from a truncated exact sequence

$$H^1(\mathcal{O}(d)) = 0 \to \bigoplus_{i=1}^{3} H^1(\mathcal{O}(d+1)) = 0 \to H^1(T(d)) \to H^2(\mathcal{O}(d)) \to \bigoplus_{i=1}^{3} H^2(\mathcal{O}(d+1)) \to H^2(T(d)) \to 0$$

to compute the following invariants.

Proposition 5.4.4. $h^1(T(d)) = \begin{cases} 0; d > -3 \\ h^2(T(d)) - (d+3); d \leq -3. \end{cases}$

Similarly, we have from truncated spectral sequence associated to 5.2, the following invariants.

Proposition 5.4.5. $h^1(T^*(d)) = \begin{cases} 0; d < 0 \\ 1; d = 0 \\ h^0(T^*(d)) - (d^2 - 1); d \geq 1. \end{cases}$

Remark 23. We extend our computations for $H^0(\text{End}(T) \otimes \mathcal{O}(-d))$. For $d = 0$ we have from stability that $h^0(\text{End}(T)) = 1$. For $d > 0$ we have an exact sequence $0 \to T^*(-d) \to \bigoplus_{i=1}^{3} T^*(1-d) \to \text{End}(T) \otimes \mathcal{O}(-d) \to 0$ and corresponding long exact sequence is given by

$$0 \to H^0(T^*(-d)) \to \bigoplus_{i=1}^{3} H^0(T^*(1-d)) \to H^0(\text{End}(T) \otimes \mathcal{O}(-d)) \to H^1(T^*(-d)) \to ....$$

Here the spaces $H^0(T^*(-d)), H^1(T^*(d)), H^0(T^*(1-d))$ are trivial and thus $H^0(\text{End}(T) \otimes \mathcal{O}(-d))$ is trivial.

Example 5.4.6. We will calculate the dimensions $h^0(\text{End}(T) \otimes \mathcal{O}(d))$ for $d > 0$ and then $h^0(\text{End}(T) \otimes T)$ is computed. We first consider the short exact sequence $0 \to T^*(d) \to \bigoplus_{i=1}^{3} T^*(d+1) \to \text{End}(T) \otimes \mathcal{O}(d) \to 0$ and then the long exact sequence

$$0 \to H^0(T^*(d)) \to \bigoplus_{i=1}^{3} H^0(T^*(d+1)) \to H^0(\text{End}(T) \otimes \mathcal{O}(d)) \to H^1(T^*(d)) \to \bigoplus_{i=1}^{3} H^1(T^*(d+1)) \to H^2(T^*(d)).$$

Observe that $h^2(T^*(d)) = h^0(T(-d-3)) = 0$ and $h^0(\text{End}T(d)) = 3(h^0(T^*(d+1)) - h^1(T^*(d+1))) - (h^0(T^*(d)) - h^1(T^*(d))) = 3 \cdot ((d+1)^2 - 1) - (d^2 - 1) = 2d^2 + 6d + 1.$

♢

Example 5.4.7. We check that $h^1(\text{End}(T)) = 0$. From the short exact sequence $0 \to T^* \to \bigoplus_{i=1}^{3} T^*(1) \to \text{End}(T) \to 0$ we have the long exact sequence

$$0 \to H^0(T^*) = 0 \to \bigoplus_{i=1}^{3} H^0(T^*(1)) \to H^0(\text{End}(T)) = \mathbb{C} \to H^1(T^*) = H^1(T(-3)) = 0$$
$$\to \bigoplus_{i=1}^{3} H^1(T^*(1)) \to H^1(\text{End}(T)) \to H^2(T^*) = H^0(T(-3)).$$

Thus we have $h^1(\text{End}(T)) = 3 \cdot (h^1(T^*(1)) - h^0(T^*(1)) + 1 - 1 = 0.$

♢

Example 5.4.8. The short exact sequence $0 \to \mathrm{End}(T) \to \bigoplus_{i=1}^{3} \mathrm{End}(T)(1) \to \mathrm{End}(T) \otimes T \to 0$ defines a long exact sequence $0 \to H^0(\mathrm{End}(T)) = \mathbb{C} \to \bigoplus_{i=1}^{3} H^0(\mathrm{End}(T)(1)) \to H^0(\mathrm{End}(T) \otimes T) \to H^1(\mathrm{End}(T)) = 0 \to \bigoplus_{i=1}^{3} H^1(\mathrm{End}(T)(1)) \to H^2(\mathrm{End}(T)) = H^0(\mathrm{End}(T)(-3)) = 0$. This leads to $h^1(\mathrm{End}(T)(1)) = 0$ and most importantly, $h^0(\mathrm{End}(T) \otimes T) = 26$. ♢

Remark 24. From the exact sequence $0 \to \bigoplus_{i=1}^{3} \mathcal{O}(1) \to \bigoplus_{i=1}^{6} \mathcal{O}(2) \to \mathrm{Sym}^2(T)$ we have $h^0(\mathrm{Sym}^2(T)) = 27$. Here we have involved the computational information from the above discussions.

We can not characterize the image space or postulated image of Hitchin morphism completely from a moduli point of view. Rather we explain some examples with specific bundles. Our approach bases on examples of twisted Higgs bundles explored by other mathematicians.

5.4.1 Examples on co-Higgs bundles

As Rayan proved in [Ray11] we can bound degrees of the semistable co-Higgs bundles for decomposable underlying bundles. This is an important result that is used for the characterization of co-Higgs bundles on $\mathbb{P}^2$.

Theorem 5.4.9. *Let $m_1 \geq ... \geq m_r$ be integers and $V = \bigoplus_{i=1}^{r} \mathcal{O}(m_i)$ be a bundle on $\mathbb{P}^2$ and $\phi : V \to V \otimes T$ is semistable. Then $m_i - m_{i+1} \leq 1$ hold for all i.*

Remark 25. This theorem is a version of 2.7.4 for the case twisting sheaf tangent bundle. This can be generalized for any projective space $\mathbb{P}^n$.

Proof. This was proved in [Ray14] Theorem 5.1. Without loss of generality we consider only the case where all these integers $m_1, ..., m_r$ are distinct. If there is i such that $m_i - m_{i+1} > 1$ then $m_i > m_j + 1$ for all $j > i$. More over for all $s \leq i$ we have $m_s \geq m_j + 1$ for all $j > i$. This leads to an invariant destabilizing subbundle $\mathcal{O}_1 \oplus ... \oplus \mathcal{O}_i$ due to the fact that $T(m_j - m_s + 1)$ has only trivial global section for $s \leq i$ and $j > i$. □

We explicitly put down image of determinant morphism of rank 2 trace-free twisted Higgs bundles because trace is a surjective morphism and a twisted Higgs field ϕ is written as a pair $(\lambda, \phi_0 = \phi - \frac{1}{2}\lambda \cdot \phi)$ where λ is trace of ϕ and its second component is trace-free. The only surviving spectral coefficient the trace-free part is its determinant which is yet to be settled. Spectral coefficients of ϕ are uniquely determined by λ and $\det(\phi_0)$. Trace is surjective and it is enough to investigate the image space of determinant morphism on the trace-free part. Also we observe that a subsheaf or subbundle is invariant under ϕ if and only if invariant under trace-free part ϕ_0. We aim to present parametrizing spaces of trace-free determinant sections under specified vector bundles. We only investigate the case $r = 2$. Let $E = \mathcal{O} \oplus \mathcal{O}(-1)$.

Theorem 5.4.10. *Let $\mathcal{M}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1), T)$ denote the space of trace-free Gieseker semistable co-Higgs bundles on $\mathcal{O} \oplus \mathcal{O}(-1)$. Then points image of determinant morphism is in one-to-one correspondence with points in $\frac{H^0(\mathcal{O}(2))^* \times H^0(T(-1))^*}{(q,C) \sim (\frac{1}{\lambda^2} \cdot q', \lambda \cdot C)}$ or 0.*

Proof. Here the underlying bundle admits co-prime rank and degree, so Gieseker semistability of a co-Higgs bundle coincides with Gieseker stability. Any Gieseker stable (E, ϕ) prevents $\mathcal{O}$ to be invariant. Then the stable trace-free co-Higgs bundles with underlying bundle V are of the form

$$\phi = \begin{bmatrix} \lambda & \mu \\ 1 & -\lambda \end{bmatrix} \otimes C$$

where $C \in H^0(T(-1)) \backslash \{0\}$ and $\lambda \in \mathcal{O}(1), \mu \in \mathcal{O}(2)$. This is justified by Rayan ([Ray11]). Any such ϕ is written as

$$\phi = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}$$

where $A \in H^0(T), B \in H^0(T(1))$ and $C \in H^0(T(-1))$. For integrability we need $\phi \wedge \phi = 0$ that is $A \wedge B = 0, A \wedge C = 0, B \wedge C = 0$. Now the second Chern class of $T(-1)$ is 1 which confirms that zero scheme of a nonzero section is a point. Combining with simultaneous equations of integrability and Hartog's theorem on $\mathbb{P}^2$ we obtain this particular form of ϕ. We keep $C \neq 0$ to prevent destabilization by subbundle $\mathcal{O}$. On the other hand, any Gieseker stable co-Higgs sheaf is of this form (due to slope $\frac{-1}{2}$ and A.1.45 confirms we should prevent $\mathcal{O}$ from destabilizing) and these are only possible semistable co-Higgs bundles due to odd degree.

We claim that any section $s \in H^0(\mathcal{O}(2))$ is written as $\lambda^2 + \mu$. Our argument will consist of a nested sequence of cases. On a trivializing neighbourhood $\mathbb{C} \times \mathbb{C}$ with coordinates (z, w) we can write s uniquely as a complex polynomial of two variables $az^2 + bzw + cw^2 + dz + ew + f$. In case $a \neq 0$ we can write $az^2 + bzw + cw^2 + dz + ew + f = (\sqrt{a}z + \frac{b}{2\sqrt{a}}w + \frac{d}{2\sqrt{a}})^2 + (c - \frac{b^2}{4a})w^2 + (e - \frac{bd}{2a})w + (f - \frac{d^2}{4a})$. For $a = 0$ we first consider $c \neq 0$ then $s = cw^2 + bzw + dz + ew + f = (\sqrt{c}w + \frac{b}{2\sqrt{c}}z + \frac{e}{2\sqrt{c}})^2 + (-\frac{b^2}{4c}z^2 + (d - \frac{be}{2c})z + (f - \frac{e^2}{4c}))$. If $c = 0$, in case $b \neq 0$, we write $s = bzw + dz + ew + f$. We apply a rotation $z = z' + w'$ and $w = z' - w'$ on the chart and $s = bzw + dz + ew + f = (\sqrt{b}z' + \frac{d+e}{2\sqrt{b}})^2 - bw'^2 + (d - e)w' + (f - \frac{(d+e)^2}{4b})$. If a, b, c are all zero then s is a linear polynomial in z, w. Out of this explicit computation we construct ϕ going in a reverse direction.

Finally, let q, q' be nonzero sections of $H^0(\mathcal{O}(2))$ and C, C' be nonzero sections of $H^0(T(-1))$ such that $q \otimes C \otimes C = q' \otimes C' \otimes C'$. Then using local coordinates we can say that $C \wedge C' = 0$. Using Hartogs' theorem we obtain $C' = \lambda \cdot C$ where λ is a nonzero complex constant and $q' = \frac{1}{\lambda^2} \cdot q$. Obviously, λ or C is zero precisely if we have determinant 0. Thus $q' = \frac{1}{\lambda^2} \cdot q$. We can identify a parametrizing scheme for the image space of nonzero determinants as

$$\frac{H^0(\mathcal{O}(2))^* \times H^0(T(-1))^*}{\sim}$$

where $\lambda \cdot (q, C) = (\frac{1}{\lambda^2} \cdot q, \lambda \cdot C)$. The total image space of determinants $\mathcal{D}$ is obtained by attaching an extra point denoting 0 section of $S^2(T)$.

$\square$

We consider another situation with underlying vector bundle $E = \mathcal{O} \oplus \mathcal{O}$. We can check straight forward that this is a Gieseker semistable bundle by considering rank 1 subsheaves. Thus any pair (E, ϕ) is Gieseker semistable.

Theorem 5.4.11. *Let $\mathcal{M}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}, T)$ denote the space of trace-free Gieseker semistable co-Higgs bundles on $\mathcal{O} \oplus \mathcal{O}$. Then points in the image of determinant morphism are in one-to-one correspondence with points in $\frac{H^0(\mathcal{O}(2))^* \times H^0(T(-1))^*}{(q,C)\sim(\frac{1}{\lambda^2}\cdot q', \lambda\cdot C)}$ or $\frac{H^0(T)^*}{\{\pm 1\}}$ or 0.*

Proof. Consider the cases as following. First we write

$$\phi = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}$$

and $C \neq 0$. Then any trace-free co-Higgs bundle ϕ, using the same argument on vanishing wedge product, can be written as

$$\phi = \begin{bmatrix} \lambda & \mu \\ \mu' & -\lambda \end{bmatrix} \otimes C'$$

while λ, μ, μ' are sections of $\mathcal{O}(1)$ and C' is a nonzero section of $H^0(T(-1))$.

The other case is $C = 0$. Then

$$\phi = \begin{bmatrix} A & B \\ 0 & -A \end{bmatrix}$$

with $A, B \in H^0(T)$. We consider $B \neq 0$. From integrability we have

$$\phi = \begin{bmatrix} \lambda & \mu \\ 0 & -\lambda \end{bmatrix} \otimes B'$$

where B' is a nonzero section of $H^0(T(-1))$. In case $B = 0$ we have $\phi = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$. Here $\phi \wedge \phi = 0$ automatically. Determinant of a co-Higgs field is of the form $q \otimes C \otimes C$ in the first two cases and in the final case, $A \otimes A$ where A is a global section of $H^0(T)$. We can identify a parametrizing scheme of image of determinant morphism say $\mathcal{D}$ as disjoint union of three spaces. We denote $\mathcal{D}_1 = \frac{H^0(\mathcal{O}(2))^* \times H^0(T(-1))^*}{\sim}$ and $\mathcal{D}_2 = \frac{H^0(T)^*}{\{\pm 1\}}$. From representation of sections on trivializing neighbourhoods we have $A \mapsto A \otimes A$ an injective map up to a change of sign. So we can identify a point in the image uniquely as these determinants with $\mathcal{D}_2$. We finally add a single point for the zero determinant. □

We observe another case while $E = T$. The trivializing neighbourhoods of tangent bundle on $\mathbb{P}^2$ are given as $\phi_1 : U_1(= [x : y : 1]) \to \mathbb{C}^2; \phi_2 : U_2(= [x : 1 : y] \to \mathbb{C}^2; \phi_3 : U_3(= [1 : x : y] \to \mathbb{C}^2)$ with $\phi_1(x : y : 1) = \phi_2(x : 1 : y) = \phi_3(1 : x : y) = (x, y)$. The transition data are given as

$$g'_{12} = \begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & -\frac{1}{y^2} \end{bmatrix}$$

and

$$g'_{23} = \begin{bmatrix} -\frac{1}{x^2} & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

and

$$g'_{31} = \begin{bmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ -\frac{1}{x^2} & 0 \end{bmatrix}.$$

We can check that

$$g'_{31}(g_{12}(g_{23}(x,y))) \cdot g'_{12}(g_{23}(x,y)) \cdot g'_{23}(x,y) = I_2$$

where $g_{12}(x,y) = (\frac{x}{y}, \frac{1}{y}); g_{23}(x,y) = (\frac{1}{x}, \frac{y}{x}); g_{31}(x,y) = (\frac{y}{x}, \frac{1}{x})$. The power series technique to find global sections of $T_{\mathbb{P}^2}$ by the formula

$$s_j(g_{12}(x,y)) = g'_{ji}(x,y) \cdot s_i(x,y).$$

Also we find global sections of $T(d)$ just by modifying the transition data and writing the power series expansion of holomorphic functions of two variables, around $(0,0)$. We move to some other computations.

The determinant morphism $\det : H^0(\mathrm{End}_0 T(1)) \to H^0(\mathcal{O}(2))$ is surjective. The matrix equation has the following form.

$$\begin{bmatrix} F(x,y) & G(x,y) \\ H(x,y) & -F(x,y) \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & -\frac{1}{y^2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{x}{y} \\ 0 & -\frac{1}{y} \end{bmatrix} \cdot \begin{bmatrix} f(\frac{x}{y}, \frac{1}{y}) & g(\frac{x}{y}, \frac{1}{y}) \\ h(\frac{x}{y}, \frac{1}{y}) & -f(\frac{x}{y}, \frac{1}{y}) \end{bmatrix}.$$

Here F, G, H and f, g, h are represented by power series of complex coefficients. On one side we have symbols x, y and on the other side we have symbols $\frac{x}{y}, \frac{1}{y}$. By comparing coefficients H and h are constants.

Theorem 5.4.12. *Let $\mathcal{M}_{\mathbb{P}^2}(T,T)$ denote the space of trace-free Gieseker semistable co-Higgs bundles on $T_{\mathbb{P}^2}$. The points in the image space of determinant morphism are in one-to-one correspondence with the spaces $\frac{H^0(\mathcal{O}(2))^* \times H^0(T(-1))^*}{(q,C)\sim(\frac{1}{\lambda^2}\cdot q', \lambda\cdot C)}$ or 0.*

Proof. Any trace-free co-Higgs field is of the form $\phi \otimes C$ where $\phi \in H^0(\mathrm{End}_0 T(1))$ and $C \in H^0(T(-1))$. This is due to an isomorphism $H^0(\mathrm{End}_0 T(1)) \otimes H^0(T(-1)) \cong H^0(\mathrm{End}_0 T \otimes T)$. Locally on a trivializing neighbourhood, ϕ is written as

$$\phi = \begin{bmatrix} \lambda & \mu \\ 1 & -\lambda \end{bmatrix}$$

such that λ is a bivariate polynomial of degree ≤ 1, μ is a bivariate polynomial of degree ≤ 2. This local representation gives a global surjective determinant morphism to $H^0(\mathcal{O}(2))$. Thus determinant of a co-Higgs bundle is of the form $q \otimes C \otimes C$ and conversely. Just as the previous example we can characterize the image space of determinant morphism $\mathcal{D}_1 = \frac{H^0(\mathcal{O}(2))^* \times H^0(T(-1))^*}{\sim}$ added a point for zero determinant section. □

5.4.2 Co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$

The following discussion is based on results in [Col15]. The moduli space of $\mathcal{M}(-F, 0)$ of rank 2 stable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with first Chern class $-F$ and second Chern class 0 is a 6-dimensional smooth variety isomorphic to the moduli space $\mathcal{M}_{\mathbb{P}^1}(-1)$ of rank 2 stable co-Higgs bundles on $\mathbb{P}^1$. We follow conventional symbols as used in [Col15]. Let $\mathbb{F}_n$ denote the n-th Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ and F and C_0 denote the two standard generating divisor of $\text{Pic}(\mathbb{F}_n)$. The ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained as $\mathbb{F}_0$. We fix the polarization $H = C_0 + F$ and second chern class $c_2 = 0$ on a rank 2 vector bundle E. By $\mathcal{O}(a, b)$ we denote the line bundle $\pi_1^*\mathcal{O}(a) \otimes \pi_2^*\mathcal{O}(b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 5.4.13. Suppose that $c_1(E) = -F$. If (E, ϕ) is a stable trace-free co-Higgs bundle then $E = \mathcal{O} \oplus \mathcal{O}(-1, 0)$ and ϕ is of the form

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{bmatrix}$$

while $A \in H^0(\mathcal{O}(2, 0)), B \in H^0(\mathcal{O}(3, 0)), C \in H^0(\mathcal{O}(1, 0))$. Here we can identify A, B, C as respective sections of $\mathcal{O}(2), \mathcal{O}(3)$ and $\mathcal{O}(1)$ on $\mathbb{P}^1$. As $C \neq 0$ we claim that determinant sections form whole $\mathcal{O}(4)$. A section s of $\mathcal{O}(4)$ is polynomial of degree ≤ 4. Then factorize s as product of a polynomial of degree ≤ 1 (for C) and a polynomial of degree ≤ 3 (for B). Here we fix $A = 0$ and finish the proof that the image space of determinant is $H^0(\mathcal{O}(4))$. We want to highlight Kunneth formula in this context

$$H^k(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m, n)) = \bigoplus_{i+j=k} H^i(\mathbb{P}^1, \mathcal{O}(m)) \otimes_{\mathbb{C}} H^j(\mathbb{P}^1, \mathcal{O}(n)).$$

$\diamondsuit$

Example 5.4.14. Suppose that $c_1(E) = 0$ and $c_2(E) = 0$. If (E, ϕ) is a semistable co-Higgs bundle then $E = \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(0, 1) \oplus \mathcal{O}(0, -1)$ or $\mathcal{O}(1, 0) \oplus \mathcal{O}(-1, 0)$. We in particular consider $\mathcal{O} \oplus \mathcal{O}$ which is strictly semistable. Any element of $H^0(\text{End}_0 E \otimes T)$ is of the form

$$\phi = \begin{bmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{bmatrix} + \begin{bmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{bmatrix}$$

with $A_1, B_1, C_1 \in H^0(\mathcal{O}(2, 0))$ and $A_2, B_2, C_2 \in H^0(\mathcal{O}(0, 2))$. The strictly semistable trace-free Higgs bundles admit S-equivalence classes given as

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$$

while $A \in H^0(T)$. Two such matrices given by A and B are equivalent if and only if $A = \pm B$. So determinants take the form $A \otimes A$ and $A \to A \otimes A$ being an injective assignment up to a change of sign, we have the points of image of determinant are in one-to-one correspondence with $(H^0(T) \backslash \{0\})/\{\pm 1\} \cup \{0\}$. $\diamondsuit$

Example 5.4.15. Now we consider the case of strictly stable co-Higgs bundles in this situation and these are characterized as

$$\begin{bmatrix} 0 & B \\ 1 & 0 \end{bmatrix}$$

where $B \in H^0(\mathcal{O}(4))$. Obviously we have the image of the determinant map $H^0(\mathcal{O}(4))$. $\diamondsuit$

5.4.3 The Higgs sheaves twisted by line bundles

Next we consider semistable d-twisted (or $\mathcal{O}(d)$-twisted) semistable Higgs bundle with $d > 0$. We know that $E \cong \mathcal{O}(m) \oplus \mathcal{O}(n)$ admits semistable d-twisted Higgs fields if and only if $|m - n| \leq d$. But an explicit description of image of determinant morphism is not so easy. This is due to failure of factorization of polynomials of two variables. However, in certain cases, we can consider generic sections s of $\mathcal{O}(2d)$ such that $\lambda^2 - s$ is irreducible and our postulated image will be the whole space.

We restrict ourselves to $|m - n| \leq 1$ such that identifying the semistable Higgs fields is easy. We start with $E \cong \mathcal{O} \oplus \mathcal{O}(-1)$. The semistable Higgs bundles are only the stable ones. See that the antidiagonal entries of a Higgs field ϕ are sections of $H^0(\mathcal{O}(d+1))$ and $H^0(\mathcal{O}(d-1))$. The entry which takes care of the component $\mathcal{O} \to \mathcal{O}(-1)$ must be nonzero to avoid destabilization. On the other hand if such a pair (E, ϕ) is unstable then there is a destabilizing invariant sub-line bundle (as a subsheaf, not necessarily as a subbundle) which is necessarily $\mathcal{O}$ due to the slope inequality and 1-dimensional global sections of $\mathcal{O} \to \mathcal{O}(-1)$. To avoid that we want the bottom antidiagonal term to be nonzero. There is an open set in $H^0(\mathcal{O}(2d))$ (sections with nonvanishing coefficients of z^{2d}) on which we are able to write the determinant section in $\mathcal{O}(2d)$ as $\lambda^2 + \mu \cdot \mu'$ where $\mu \cdot \mu'$ is a section of $\mathcal{O}(d)$. This is not a fruitful description in general but case $d = 1$ we can explicitly write down the image. We will take $\mu' = 1$ and μ. So determinant is a surjective map.

Example 5.4.16. In case $E \cong \mathcal{O} \oplus \mathcal{O}$ then

$$\phi = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where $a, b, c \in H^0(\mathcal{O}(d))$. Let $s \in H^0(\mathcal{O}(2d))$. Then we write $s = -a^2 - b$ for $a \in H^0(\mathcal{O}(d))$ and $b \in H^0(\mathcal{O}(2d))$ and we put ϕ in the form

$$\phi = \begin{bmatrix} a & b'' \\ b' & -a \end{bmatrix}$$

For $d = 1$ we consider a section s of $\mathcal{O}(2)$. Our argument will consist of a nested sequence of cases. On a trivializing neighbourhood $\mathbb{C} \times \mathbb{C}$ with coordinates (z, w) we can write s uniquely as a complex polynomial of two variables $az^2 + bzw + cw^2 + dz + ew + f$. In case $a \neq 0$ we can write $az^2 + bzw + cw^2 + dz + ew + f = (\sqrt{a}z + \frac{b}{2\sqrt{a}}w + \frac{d}{2\sqrt{a}})^2 + (c - \frac{b^2}{4a})w^2 + (e - \frac{bd}{2a})w + (f - \frac{d^2}{4a})$. For $a = 0$ we first consider $c \neq 0$ then $s = cw^2 + bzw + dz + ew + f = (\sqrt{c}w + \frac{b}{2\sqrt{c}}z + \frac{e}{2\sqrt{c}})^2 + (-\frac{b^2}{4c}z^2 + (d - \frac{be}{2c})z + (f - \frac{e^2}{4c}))$. If $c = 0$, in case $b \neq 0$, we write $s = bzw + dz + ew + f$. We apply a rotation $z = z' + w'$ and $w = z' - w'$ and $s = bzw + dz + ew + f = (\sqrt{b}z' + \frac{d+e}{2\sqrt{b}})^2 - bw'^2 + (d-e)w' + (f - \frac{(d+e)^2}{4b})$. If a, b, c are all zero then s is a linear polynomial. Otherwise we take the liberty to factorize the remainder that is a degree 2 polynomial $b = b' \cdot b''$(of a single variable).

$\diamondsuit$

Example 5.4.17. In case $E \cong \mathcal{O} \oplus \mathcal{O}(1)$ we have the stable Higgs fields are of the form $\phi = \begin{bmatrix} a & b \\ 1 & -a \end{bmatrix}$. Here the determinant morphism is surjective as we have already discussed. $\diamondsuit$

5.5 Further questions

Our motivation behind discussing these examples is verifying a conjecture about the postulated Hitchin base available in [CN20]. The set of examples we have posed here are not complete. The research surrounding twisted Higgs bundles over algebraic surfaces is not yet over. Recent works by Boulter and Moraru [BM23] demonstrate existence of stable co-Higgs bundles on certain elliptic surfaces. A new question is explaining differential geometric properties (structures as completely integrable systems) of Hitchin base over surfaces.

Appendix A

Sheaves over curves and surfaces

A.1 Finiteness theorems on complex manifolds

In this section we introduce the geometry of compact Riemann surfaces equivalently, smooth projective algebraic curves (constructed by as curves of two variables.) However, we will introduce the finiteness theorems wherever applicable for projective algebraic surfaces and other distinguished varieties. Throughout the write up all the varieties are connected.

A.1.1 Branched covers of Riemann surfaces

Definition A.1.1. Let X be a two-dimensional real manifold i.e. a Hausdorff, second countable topological space and for each point of X there is a neighbourhood U_α homeomorphic to an open subset V_α of $\mathbb{C}$ via a homeomorphism ϕ_α. Here X is said to be a *Riemann surface* if $\phi_\alpha \circ \phi_\beta^{-1}$ is a biholomorphic map $\forall \alpha, \beta$. This equivalence of charts is called holomorphic compatibility. The collection $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ is called an atlas of X.

Furthermore, If X is a $2n$-dimensional real manifold with the coordinate transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are biholomorphic maps between open sets of $\mathbb{C}^n$ we call it an n-dimensional complex manifold. We will explore particular two-dimensional complex manifolds in later sections.

The most common examples of compact Riemann surfaces are the complex projective line and the elliptic curves.

Example A.1.2. (i) The Riemann sphere or the complex projective line denoted by $\mathbb{P}^1$ which is obtained by adjoining an external point ∞ to $\mathbb{C}$. Here define $\phi_1 : \mathbb{C} \to \mathbb{C}$ the identity map and $\phi_2 : \mathbb{C}^* \cup \{\infty\} \to \mathbb{C}$ as $\phi_2(z) = \frac{1}{z}$ for z in $\mathbb{C}^*$ and $\phi_2(z) = 0$ for $z = \infty$.

(ii) Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where ω_1 and ω_2 be complex numbers, linearly independent over $\mathbb{R}$. Two complex numbers z_1 and z_2 are said to be *equivalent* iff $z_1 - z_2 \in \Gamma$. A torus $\mathbb{C}/\Gamma$ where Γ is a lattice, is a Riemann surface. A complex atlas is given by an open mapping. Let $V \subseteq \mathbb{C}$ be an open set such that no two points of V are equivalent. Then consider the map $(\pi_{|V})^{-1}$ which is a bijective continuous open mapping to produce the chart for this Riemann surface.
By the mapping $[\lambda\omega_1 + \mu\omega_2] \mapsto (e^{2\pi i\lambda}, e^{2\pi i\mu})$ while $\lambda, \mu \in \mathbb{R}$ any torus can be shown to be homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. For a complex number $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ the action of $\mathbb{Z}$ on $\mathbb{C}^*$ given by $z \mapsto \lambda^n z$ gives rise to the

quotient space $\mathbb{C}^*/\mathbb{Z}$. Topologically this is a torus having a complex structure. In fact this is another model of the example described in 2.4. $\diamondsuit$

Let X be a Riemann surface and $a \in X$. A function $f : X \to \mathbb{C}$ is said to be *holomorphic* if $\exists$ a chart $\phi : U \to V$ of a such that $f \circ \phi^{-1} : V \to \mathbb{C}$ is holomorphic at $\phi(a)$. Moreover, f is said to be *holomorphic* if it is holomorphic at each point of its domain. Let X and Y be two Riemann surfaces and a continuous mapping $f : X \to Y$ is said to be holomorphic at $a \in X$ if there is a chart $\phi : U_1 \to V_1$ of a and $\psi : U_2 \to V_2$ of $f(a)$ with $f(U_1) \subseteq U_2$ such that $\psi \circ f \circ \phi^{-1} : V_1 \to V_2$ is holomorphic at $\phi(a)$. A map $f : X \to Y$ is said to be holomorphic if it is holomorphic at each point.

Example A.1.3. (i) As the chart on a torus $\mathbb{C}/\Gamma$ is given by the open continuous projection map π, the map $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is itself a holomorphic map.

(ii) Note that $\mathbb{P}^1\backslash\{\infty\} = \mathbb{C}$ and any holomorphic function $\phi : \mathbb{P}^1\backslash\{0\} \to \mathbb{C}$ is given by $\phi(z) = f(\frac{1}{z})$ while $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic map. $\diamondsuit$

These definitions extend to more general complex manifolds as well. There are instances of holomorphic maps for algebraic varieties higher dimensions. For example, let $\mathbb{P}^2$ denote the space of lines in $\mathbb{C}$ passing through the origin. There is a holomorphic map

$$\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$$

defined by

$$\rho(x, y) = (xy, \frac{y}{x}).$$

Let us state few theorems which are going to be useful in studying the holomorphic maps on Riemann surfaces. We denote the ring of holomorphic functions $f : U \to \mathbb{C}$ with $\mathcal{O}(U)$.

Definition A.1.4. Let X be a Riemann surface. By a *meromorphic function* on X we mean a holomorphic function $f : Y \to \mathbb{C}$ where Y is an open subset of X and the following holds:
(i) $X\backslash Y$ contains isolated points only, called the *poles*.
(ii) For every point, $p \in X\backslash Y$, $\lim_{x \to p} \mid f(x) \mid = \infty$.

The collection of all meromorphic functions on any open set U is denoted by $\mathcal{M}(U)$. The set of meromorphic functions on U forms a field under usual addition and multiplication. Let (U, z) be a coordinate chart of a point p then a meromorphic function has a Laurent series expansion at p.

Any compact Riemann surface X admits a global meromorphic function i.e. $\mathcal{M}(X)$ is non-empty. See [Mir95] for more details. This result helps us to prove that each compact Riemann surface is projective algebraic.

For $n \geq 1$, $F(z) = z^n + c_1 z^{n-1} + .. + c_n$ be a polynomial and $F(\infty) = \infty$ defines a meromorphic function on $\mathbb{P}^1$ with only pole at ∞. A more no-trivial example of a meromorphic function is Weierstrass ρ-function on an elliptic curve (which is assumed to be a complex torus according to A.1.2.)

$$\rho(z) = \frac{1}{z^2} + \sum_{\lambda \in \Gamma \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}. \tag{A.1}$$

The derivative of ρ-function denoted as ρ' is another important meromorphic function on the elliptic curves. We will see their importance further.

Now we state an important theorem on local behaviour of holomorphic mappings which will be useful to prove the analogous theorems of complex analysis. The rigorous proofs can be found in [For81].

Theorem A.1.5. *Let $f : X \to Y$ be a nonconstant holomorphic mapping between two Riemann surfaces. For $a \in X$ and $b = f(a) \in Y$ there exists $k \in \mathbb{N}$ and charts $\phi : U \to V$ of X and $\psi : U' \to V'$ on Y such that*
(i) $a \in U$, $b \in U'$, $\phi(a) = \psi(b) = 0$.
(ii) $f(U) \subseteq U'$.
(iii) $F := \psi \circ f \circ \phi^{-1} : V \to V'$ is given by $F(z) = z^k$.

An important corollary is that any holomorphic function on a compact Riemann surface X is constant and immediately concluded as $\mathbb{C}$ is noncompact.

Remark 26. The integer k will play an important role in understanding the maps between compact Riemann surfaces. If $k \geq 2$ then the point a is a *ramification point* i.e. there is no neighbourhood U of a such that f is injective on U. The unique integer k is defined to be the multiplicity of a for taking value b. Here the point b is called a *branch point* of f.

Here we skip some more details on algebraic topology of Riemann surfaces as described in chapter 1 in [For81] and jump to an important conclusion about the branch points of a nonconstant holomorphic map and mention an important theorem for a rough counting of branch points and their multiplicities.

Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces then we have,

Remark 27. (i) The set of ramification points of f are closed and discrete. Moreover, the set of branch points in Y is a closed and discrete set. (ii) The map restricted on the non-ramification points is a topological covering map of finite number of sheets and (iii) the number of sheets of f is extended to at a branch point f by counting the multiplicities of the ramification points. We often call f a *branched holomorphic covering.*

Some common examples are the following.

Example A.1.6. (i) Any holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function i.e. a quotient of two polynomials. The field of quotients of complex polynomials is denoted as $\mathbb{C}(z)$.

(ii) The ρ-function in A.1 is a branched holomorphic covering of the complex projective line $\mathbb{P}^1$ by an elliptic curve X, with 2-sheets. branched over four distinct points in $\mathbb{P}^1$. Indeed there is an invariant (see [JW18]) based on the four points that determine the elliptic curve explicitly.

An important way of viewing the theory of compact Riemann surfaces is the field theory of function fields given by the fields of meromorphic functions. As a result, we can explicitly write the field of meromorphic functions on an elliptic curve $X = \mathbb{C}/\Gamma$ as following.

$$\mathcal{M}(\mathbb{C}/\Gamma) = \{f(\rho) + \rho' g(\rho) : f, g \in \mathbb{C}(z)\}. \tag{A.2}$$

$\diamondsuit$

Let us conclude this section with the main theorem of counting for the holomorphic branched covers between compact Riemann surfaces. Here we mention an important invariant of a compact Riemann surface which we further define rigorously namely, the *genus* denoted by g_X. For any compact Riemann surface genus is a non-negative integer. For example, the genus of the complex projective line is 0 while an elliptic curve has genus 1.

Theorem A.1.7. *(Riemann-Hurwitz) Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces of r-sheets. At $x \in X$ we denote the multiplicity of f by $v(f, x)$. The total multiplicity of f, denoted by $b(f)$ is the integer $\sum_{x \in X} (v(f, x) - 1)$. We have,*

$$g_X - 1 = \frac{b(f)}{2} + r(g_Y - 1) \tag{A.3}$$

In this context we mention an important theorem named after Russian mathematician Belyi, on classification of smooth projective algebraic curves that cover $\mathbb{P}^1$ over three branch points.

A.1.2 The Čech cohomology groups of compact Riemann surfaces.

The Čech cohomology groups of the algebraic varieties are important objects for understanding certain invariants of the sheaves and specifically the vector bundles. The relevant finiteness theorems that we discuss here hold a strong place in algebraic geometry. Sheaves have been discussed extensively for more general objects called *schemes* in [Har77]. our present discussion is found in [For81].

Definition A.1.8. Let X be a topological space with topology τ and let $(\mathcal{F}, \rho)$ be a family of abelian groups and group homomorphisms such that the following hold:

(i) each $U \in \tau$ is attached with an abelian group $\mathcal{F}(U)$.

(ii) for $V \subseteq U$, $V \in \tau$, $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is a group homomorphism which follows $\rho_U^U = Id_U$ and $\rho_W^V \circ \rho_V^U = \rho_W^U$ while $W \subseteq V \subseteq U$.

(iii) Every open subset U of X and a cover $(U_i)_i$ of U, if $f, g \in \mathcal{F}(U)$ are elements such that $f|U_i = g|U_i$, for all i then $f = g$.

(iv) Given elements $f_i \in \mathcal{F}(U_i)$, such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all i, j then there exists an element $f \in \mathcal{F}(U)$ such that $f|U_i = f_i$ for all i.

Example A.1.9. (i) The symbol $\mathcal{O}$ denotes the sheaf of holomorphic functions on a Riemann surface M and $\mathcal{O}(U)$ denotes the space associated to U.
(ii) The symbol $\mathcal{O}^*$ denotes the sheaf of multiplicative group of nonvanishing holomorphic functions.
(iii) We often describe the sheaf of locally constant functions taking values in $\mathbb{C}$ and $\mathbb{Z}$ directly by $\mathbb{C}$ and $\mathbb{Z}$. ♢

Remark 28. For smooth varieties we define the sheaf of sections of vector bundles and construct the cohomology groups. We will further see that vector bundles are not only a class of geometric objects but also a class of sheaves.

In the context of sheaves, Stalks are as important as sheaves themselves while discuss the morphisms of sheaves. See [For81] for more details. For a sheaf $\mathcal{F}$ and $x \in X$, we define an equivalence relation on $\sqcup_{x \in U} \mathcal{F}(U)$ by $(U, f) \sim (V, g)$ if there is an open subset $(a \in) W \subseteq U \cap V$ such that $\rho_W^U(f) = \rho_W^V(g)$. The quotient is called the stalk of x, denoted by $\mathcal{F}_x$.

Now we introduce the cohomology groups for a sheaf.

Definition A.1.10. Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. Let us consider a family of open sets of X viz, $\mathcal{U} = \{U_i\}_i$ such that $\bigcup_i U_i = X$. For $q = 0, 1, 2...$ define the q-th cochain group of $\mathcal{F}$ as $C^q(\mathcal{U}, \mathcal{F}) = \Pi_{(i_0, i_1..i_q)} \mathcal{F}(U_{i_0} \cap ... \cap U_{i_q})$. The elements of this group are called the q-cochains.

Remark 29. We can replace the sheaf of abelian groups by the sheaf of commutative rings, modules and vector spaces.

Definition A.1.11. Let $\delta_0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ and $\delta_1 : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ be defined as $\delta_0((f_i)_i) = (f_j - f_i)_{i,j}$ and $\delta_1((f_{ij})_{i,j}) = (f_{jk} - f_{ik} + f_{ij})_{i,j,k}$.

Now we define the *first cohomology group*.

Definition A.1.12. Let $Z^1(\mathcal{U}, \mathcal{F}) = \ker(\delta_1)$ and $B^1(\mathcal{U}, \mathcal{F}) = Im(\delta_0)$. It is elementary to check that $B^1(\mathcal{U}, \mathcal{F}) \subseteq Z^1(\mathcal{U}, \mathcal{F})$. The quotient group

$$H^1(\mathcal{U}, \mathcal{F}) = Z^1(\mathcal{U}, \mathcal{F}) / B^1(\mathcal{U}, \mathcal{F})$$

is defined to be the first cohomology group with respect to $\mathcal{U}$.

The cohomology group of higher orders is given by general definition of the operator as follows: $\delta_p : C^p \to C^{p+1}$, $(\delta f)_{i_0 \ldots i_{p+1}} = \sum_j (-1)^j f_{i_0 \ldots i_{j-1} i_{j+1} \ldots i_{p+1}} | U_{i_0} \cap \ldots \cap U_{i_{p+1}}$. Thus $H^p(\mathcal{U}, \mathcal{F}) := \ker(\delta_p)/Im(\delta_{p-1})$. Now we define the cohomology group which depends up on X only.

An open covering $\mathcal{B} = (V_k)_{k \in K}$ of X is said to be *finer* than the open covering $\mathcal{U} = (U_i)_{i \in I}$ (often denoted as $\mathcal{B} < \mathcal{U}$) if there is a mapping $\tau : K \to I$ such that $V_k \subseteq U_{\tau(k)}$ for all k.

Remark 30. The mapping τ induces a mapping $t^{\mathcal{U}}_{\mathcal{B}} : Z^1(\mathcal{U}, \mathcal{F}) \to Z^1(\mathcal{B}, \mathcal{F})$ given by $t^{\mathcal{U}}_{\mathcal{B}}((f_{ij})) = (g_{kl})$ where $g_{kl} = f_{\tau(k), \tau(l)} | V_k \cap V_l$.

The mapping $t^{\mathcal{U}}_{\mathcal{B}}$ induces a group homomorphism $H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{B}, \mathcal{F})$ which is independent of the choice of the map τ. Moreover, this map is an injective group homomorphism.

If one has three open coverings $\mathcal{W} < \mathcal{B} < \mathcal{U}$ then $t^{\mathcal{B}}_{\mathcal{W}} \circ t^{\mathcal{U}}_{\mathcal{B}} = t^{\mathcal{U}}_{\mathcal{W}}$. Let us define an equivalence relation $\sim$ on the disjoint union of $H^1(\mathcal{U}, \mathcal{F})$ for all open coverings $\mathcal{U}$ of X. Let $\xi \in H^1(\mathcal{U}, \mathcal{F})$ and $\eta \in H^1(\mathcal{U}', \mathcal{F})$. Then $\xi \sim \eta$ iff $\exists$ an open covering $\mathcal{B}$ with $\mathcal{B} < \mathcal{U}$ and $\mathcal{B} < \mathcal{U}'$ such that $t^{\mathcal{U}}_{\mathcal{B}}(\xi) = t^{\mathcal{U}'}_{\mathcal{B}}(\eta)$.

Definition A.1.13. $H^1(X, \mathcal{F}) = (\bigsqcup_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}))/\sim$.

Remark 31. We can extend the above definition for higher cohomology groups also. The cohomology spaces will be extremely important while we will discuss the exact sequences. As soon as we work on the line bundles the dimensions of the cohomology spaces will be important. We will see that the previously mentioned invariant in A.1.7, the genus of a compact Riemann surface, as dimension of $H^1(M, \mathcal{O})$.

A.1.3 Serre duality and Hirzebruch-Riemann-Roch theorem

The cohomology groups of sheaves of sections of vector bundles over the smooth projective algebraic varieties are normed-linear spaces of finite dimensions. The most finiteness theorems are Serre duality theorem and Hirzebruch-Riemann-Roch theorem are proved for complex projective algebraic manifolds. We discuss the material as found in [For81], [HSW99], [ROW13].

Definition A.1.14. Let X be a compact Riemann surface (more generally a smooth complex projective algebraic manifold) and $\pi : L \to X$ be a map such that

(i) For each $x \in X$ the set $L_x = \pi^{-1}\{x\}$ is a one dimensional vector space over $\mathbb{C}$.

(ii) For each $x \in X$, $\exists$ an open set $x \in U \subseteq X$ and a homeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ such that $\pi_1 \circ \phi_U = \pi$ for the map $\pi_1 : U \times \mathbb{C} \to U$, $\pi_1(p, z) = p$. Thus the following diagram commutes.

$$\begin{array}{ccc} L_U & \xrightarrow{\phi_U} & U \times \mathbb{C} \\ \Big\downarrow \pi & & \Big\downarrow \pi_1 \\ U & \xrightarrow{Id} & U \end{array} \tag{A.4}$$

(iii) The map $\phi_U : L_p \to \{p\} \times \mathbb{C}$ is a vector space isomorphism over $\mathbb{C}$ for each $p \in U$.

The triple (π, L, X) is said to be a *line bundle* over X.

Remark 32. (i) The map $\phi_V \circ \phi_U^{-1} : \{x\} \times \mathbb{C} \to \{x\} \times \mathbb{C}$ is a vector space isomorphism so must be associated to an element $\mathbb{C}^*$. We have a mapping $g_{VU} : U \cap V \to \mathbb{C}^*$ such that $\phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{C} \to (U \cap V) \times \mathbb{C}$ is given by

$$\phi_V \circ \phi_U^{-1}(m, w) = (m, g_{VU}(m)w).$$

(ii) For the covering $\mathcal{U}$ of local trivialization maps on x with $g_{UU} = 1$ and $g_{UV} g_{VW} g_{WU} = g_{UU} = 1$.

(iii) If g_{VU} is holomorphic for all U, V, they give rise to an element $\{g_{VU}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ from (ii) and we call L a *holomorphic line bundle* on X.

We further study the holomorphic line bundles over compact Riemann surfaces with greater details.

Remark 33. We have not talked of the complex manifold structure on L. Let X be a compact Riemann surface. For $x \in X$ we have an open set U_x and a homeomorphism $\psi : U_x \to V_x$(open in $\mathbb{C}$). Now taking the intersection $W = U \cap U_x$ we find a homeomorphism $\Psi_W = (\psi_W, Id) \circ \phi_W$ between $\pi^{-1}(W)$ and $\psi(W)$. Moreover, the maps $\Psi_{W_1} \circ \Psi_{W_2}^{-1}$ are holomorphic due to the fact that the maps $\phi_{W_2} \circ \phi_{W_1}^{-1}$ and $\psi_{W_2} \circ \psi_{W_1}^{-1}$ are holomorphic. Thus the total space L has a 2-dimensional complex manifold structure.

Let $\pi : L \to X$ be a holomorphic line bundle then the bundle map π is a holomorphic map between the complex manifolds L and X.

Let X be a complex projective algebraic manifold. A line bundle gives rise to an element (g_{ij}) in the group $H^1(X, \mathcal{O}^*)$ for which there is a unique holomorphic line bundle up to isomorphism. So it is enough to define the transition maps to define a holomorphic line bundle. The proof is elementary and available in [HSW99].

Observe that holomorphically isomorphic line bundles L and L' admitting transition functions,$\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ on a common collection of trivializing open sets are related by $\frac{g'_{\alpha\beta}}{g_{\alpha\beta}} = \frac{h_\alpha}{h_\beta}$ while $h_\alpha : U_\alpha \to \mathbb{C}^*$ are holomorphic maps. Thus these two sets of transition data will produce the same element in $H^1(X, \mathcal{O}^*)$.

Let us consider an element in $H^1(X, \mathcal{O}^*)$. Then its representations with two sets of transition functions say, $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are related by the following: $\frac{g'_{\alpha\beta}}{g_{\alpha\beta}} = \frac{h_\alpha}{h_\beta}$ while $h_\alpha : U_\alpha \to \mathbb{C}^*$ are holomorphic maps. Each of these two sets of transition functions produce holomorphic line bundles as we mentioned earlier and they are isomorphic.

The group $H^1(X, \mathcal{O}^*)$ is defined to be the *Picard group* of X having multiplication of tensor product of line bundles, which we define further. The Picrd group itself has a complex manifold structure. However, taking X to be a compact Riemann surface we survey some examples of holomorphic line bundles. Almost all of these line bundles can be defined to higher dimensional projective algebraic complex manifolds and projective varieties.

Example A.1.15. (i) The trivial bundle is $\pi : X \times \mathbb{C} \to X$, $\pi(x, z) = x$. This line bundle has transition map

$$g_{VU} = 1, \forall U, V.$$

We will see that this bundle will play the role of a multiplicative indentity in the Picard group.

(ii) A bundle L_p is defined with help of a local coordinate chart at p. Let (U_0, z) be a coordinate map at p such that $z(p) = 0$. Let $U_1 = X \backslash \{p\}$ and $g_{01} = z$ on $U_0 \cap U_1$. Naturally the transition map is a nonvanishing holomorphic function. It is obvious that change of such coordinate map z will produce more such line bundles. In the language of divisors on compact Riemann surfaces, this line bundle associates to the divisor $D = p$ on X. We generalize it further. See [For81] and [Mir95] for the correspondence of the line bundles and the divisors.

(iii) The dual bundle of a line bundle L is defined by the transition maps $g^*_{VU} = \frac{1}{g_{VU}}$ having the total space $\sqcup_{x \in M} L^*_x$.

(iv) For two holomorphic line bundles L_1 and L_2 the tensor product bundle is given by the disjoint union of all $L_{1,x} \otimes L_{2,x}$ and the transition functions are given by $h_{VU} = g_{VU} g'_{VU}$.

(v) The holomorphic tangent bundle $(\pi, T^{(1,0)}X, X)$ is defined by the complex coordinate maps on the Riemann surface X. For two complex coordinate maps (U, z) and $(V, \tilde{z})$ the transition map $g_{VU} = \frac{d\tilde{z}}{dz}$. (In fact, the fibre of this line bundle is the holomorphic summand of the complexified real tangent space.)

(v) The holomorphic cotangent bundle or the canonical line bundle in case of a compact Riemann surface, $(\pi^*, T^{*(1,0)}X, X)$ is also defined by the complex coordinate maps. It is the dual bundle of the tangent bundle with transition functions $g_{VU} = \frac{dz}{d\tilde{z}}$, while $z : U \to \mathbb{C}$ and $\tilde{z} : V \to \mathbb{C}$ are two holomorphic coordinate maps.

(In fact, the fibre of this line bundle is the holomorphic summand of the complexified real cotangent space.)

(vi)(*The pullback bundle*) Let us consider a holomorphic map $f : Y \to X$ where X, Y are two compact Riemann surfaces. For a holomorphic line bundle (π, E, X) let us consider the bundle $f^*\pi$ on Y with the total space $\{(x, y) : f(y) = \pi(x)\}$ and $f^*\pi(x, y) = y$. Actually, here we have the total space to be $\sqcup_{y\in Y} E_{f(y)}$ and the transition functions are given by $g_{VU} \circ f$. This is called the pullback bundle.

(vii) The homomorphism bundle $\text{Hom}(L, \tilde{L})$ with fibre $\text{Hom}(L_x, \tilde{L}_x)$ is isomorphic to the line bundle $L^* \otimes \tilde{L}$. In fact, an isomorphism can be given, defining for the element

$$\phi \otimes w \mapsto (v \mapsto \phi(v)w).$$

(Here make note that we pick up $\phi \in L^*_x$ and $w \in \tilde{L}_x$. Now for $v \in L_x$, $\phi(x) \in \mathbb{C}$. Thus the mapping makes sense.) Moreover, the bundle $\text{Hom}(L, L)$ is nothing but the trivial bundle as the linear maps on L_x, the fiber of $x \in X$ are characterized by merely the complex numbers. Thus $L^* \otimes L$ is isomorphic to the trivial bundle and $L^* := L^{-1}$. ♢

Now we move to the holomorphic sections of a holomorphic line bundle. Again, the definitions can be extended for higher dimensional projective algebraic manifolds.

Definition A.1.16. A holomorphic map $s : X \to L$ is said to be a holomorphic section of line bundle (π, L, M) if $s(m) \in \pi^{-1}\{m\}$ for all $m \in M$.

Note that a holomorphic section s can be produced by a collection of holomorphic maps $s_U : U \to \mathbb{C}$ for which $\phi_U \circ s = (Id, s_U)$ where the ϕ_U's are the local trivialization maps. Moreover, these maps s_U's satisfy $s_V = g_{VU}s_U$ on $U \cap V$ and two sets of maps $\{s_U\}$ and $\{t_U\}$ produces a global meromorphic map $\frac{s}{t}$ patching the maps $\frac{s_U}{t_U}$ (accepting for the time being that the zeros of t_U's are discrete). This is a well defined map as $\frac{s_U}{t_U} = \frac{s_V}{t_V}$ on $U \cap V$.

We have a natural vector space structure on the set of all global sections viz. $H^0(X, L)$. For $s, t \in H^0(M, L)$ we define $s + t \in H^0(X, L)$ as $(s + t)(m) := s(m) + t(m)$ and for $c \in \mathbb{C}$ we define the scalar multiplication cs as $(cs)(m) := cs(m)$. In fact, the holomorphic section $s + t$ is defined by the holomorphic functions $s_U + t_U$ and the holomorphic section cs is defined by the holomorphic functions cs_U.

The sheaf of local sections is denoted by $\mathcal{O}(L)(U)$ for all open set U of Riemann surface X.

Theorem A.1.17. *If X is a compact Riemann surface then the vector space $H^0(X, L)$ is a finite dimensional vector space over $\mathbb{C}$.*

Proof. A detailed proof can be found in [For81]. □

Here we present few important examples which will be extremely important in proving important theorems.

Example A.1.18. (i) The line bundle L_p has a section s_p given by the maps maps $s_{U_0}(m) = z(m)$ and $s_{U_1}(m) = 1$. Also $s_{U_0} = g_{01}s_{U_1}$.

(ii) The holomorphic cotangent bundle has the space of global holomorphic sections isomorphic to the space of holomorphic 1-forms. In fact, a holomorphic 1-form ω which has representations $f_i dz_i$ on coordinate neighborhood U_i, while $f_i : U_i \to \mathbb{C}$ is a holomorphic function, produces a global section s defined by $s_{U_i} = f_i$. The construction of holomorphic 1-form out of a section is also obvious from this construction. It produces a vector space isomorphism too.

(iii) For $X = \mathbb{P}^1$ the bundle $\mathcal{O}(n)$ is given by the map $g_{01}(z) = z^n$. Note that an holomorphic function f on $\mathbb{P}^1 \backslash \{\infty\} = \mathbb{C}$ has a representation as $f(z) = \Sigma_{m=0}^{\infty} a_m z^m$. Moreover, note that any holomorphic function g on $\mathbb{P}^1 \backslash \{0\}$ has a unique representation $g(z) = \Sigma_{m=0}^{\infty} \frac{b_m}{z^m}$. If s_{U_0} and s_{U_1} represent the same section of line bundle $\mathcal{O}(n)$ over $\mathbb{P}^1$. Then we have

$$s_{U_0}(z) = \Sigma_{m=0}^{\infty} a_m z^m = z^n s_{U_1}(z) = z^n (\Sigma_{m=0}^{\infty} \frac{b_m}{z^m}). \tag{A.5}$$

The above equality gives $b_0 = a_n, ..., b_n = a_0$ and $b_m = a_m = 0$, for all $n > m$. Thus $(n + 1)$ complex numbers uniquely determine a section of $\mathcal{O}(n)$ and conversely. Also this correspondence respects linearity. From this information we can show that $H^0(\mathbb{P}^1, \mathcal{O}(n)) \cong \mathbb{C}^{(n+1)}$ thus has dimension $n + 1$. However, this is true for $n \geq 0$. For $n \leq -1$ we will see that $H^0(\mathbb{P}^1, \mathcal{O}(n)) = 0$. $\diamondsuit$

The canonical bundle K_X which is same as the holomorphic cotangent bundle in case of a compact Riemann surface, is given by the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^1$. We use this line bundle to produce the following definition.

Definition A.1.19. For a compact Riemann surface X the dimension of $H^0(X, K)$ is said to be the *genus* of X and denoted by g.

Now on wards we will consider the Riemann surfaces to be compact, if not mentioned otherwise. Here we calculate the genus of two Riemann surfaces, the Riemann sphere and a torus.

Remark 34. (i) On $X = \mathbb{P}^1$ the holomorphic 1-forms are the sections and the holomorphic 1-form will be of the form $f(z)dz$ on U_0 and $\tilde{f}(\tilde{z})d\tilde{z}$ on U_1 while f and $\tilde{f}$ are holomorphic functions on $U_0 = \mathbb{C}$. Thus we have

$$\tilde{z} = \frac{1}{z} \text{ and } d\tilde{z} = -\frac{1}{z^2}dz.$$

Now on $U_0 \cap U_1$, these two 1-forms must be same. So we write $f(z)dz = -\frac{1}{z^2}\tilde{f}(\frac{1}{z})dz$ and we see that the surviving powers in the left hand side are non negative and the powers in the right hand side are all negative. Thus both f_0 and f_1 must be zero i.e. the only holomorphic section is the zero section.

(ii) We note that $\frac{1}{z}dz$ is holomorphic 1-form on $\mathbb{C}^*$. Note that the holomorphic map $z \mapsto az$ for $a \neq 0$ keeps the form invariant. Thus the action $(n, z) \mapsto \lambda^n z$ keeps the holomorphic 1-form invariant and defines a nonvanishing holomorphic 1-form on the torus $\mathbb{C}^*/\mathbb{Z}$ that is, a global holomorphic section is given as $1.d[z]$.

Here we observe an important result. If a line bundle L has a nonvanishing section then it is isomorphic to the trivial bundle. In fact, we consider a nonvanishing section s of L and $(m, w) \mapsto ws(m)$ is an holomorphic isomorphism from the trivial bundle $X \times \mathbb{C}$ to L.

Here we have a nonvanishing section of the canonical bundle K_X over a torus or an elliptic curve X. Thus K_x is the trivial bundle and we note that the sections of the trivial bundle are the holomorphic functions. Now a torus being compact the only holomorphic maps are the constants. Thus $H^0(X, K_X) \cong \mathbb{C}$ and the genus of a torus is 1.

Now we discuss the general holomorphic vector bundle for arbitrary rank over a complex projective algebraic manifold X.

Definition A.1.20. Let $\pi : E \to X$ be a continuous map such that

(i) For each $x \in X$ the set $E_x = \pi^{-1}\{x\}$ is a r dimensional vector space over $\mathbb{C}$.

(ii) For each $x \in X$, $\exists$ an open set $x \in U \subseteq X$ and a homeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that $\pi_1 \circ \phi_U = \pi$ for the map $\pi_1 : U \times \mathbb{C}^r \to U$, $\pi_1(p, z) = p$. That is, the following diagram commutes,

$$\begin{array}{ccc} E_U & \xrightarrow{\phi_U} & U \times \mathbb{C}^r \\ \big\downarrow \pi & & \big\downarrow \pi_1 \\ U & \xrightarrow{Id} & U \end{array} \tag{A.6}$$

(iii) The map $\phi_U : E_p \to \{p\} \times \mathbb{C}^r$ is a vector space isomorphism over $\mathbb{C}$ for each $p \in U$. The triple (π, E, X) is said to be a vector bundle over M.

We see that there is a complex manifold structure on E of dimension $r+1$ in case X is a compact Riemann surface. The vector bundle is called holomorphic vector bundle if the map π is holomorphic.

Remark 35. (i) The map $\phi_V \circ \phi_U^{-1} : \{x\} \times \mathbb{C}^r \to \{x\} \times \mathbb{C}^r$ is a vector space isomorphism so must be associated to an element $GL(r, \mathbb{C})$. We have a mapping $g_{VU} : U \cap V \to GL(r, \mathbb{C})$ such that $\phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{C} \to$

$(U \cap V) \times \mathbb{C}$ is given by

$$\phi_V \circ \phi_U^{-1}(m, w) = (m, g_{VU}(m)w).$$

(ii) For the covering $\mathcal{U}$ of local trivialization maps on M such that $g_{UU} = I_r$ and $g_{UV}g_{VW}g_{WU} = g_{UU} = I_r$.

(iii) If g_{VU} is holomorphic for all U, V, they give rise to an element $\{g_{VU}\} \in Z^1(\mathcal{U}, \mathcal{F})$ from (ii) while $\mathcal{F}$ denotes the sheaf of holomorphic maps to $GL(r, \mathbb{C})$. Though the this group is not an abelian group it is harmless to define the cohomology groups $H^0(M, \mathcal{F})$ and $H^1(M, \mathcal{F})$.

(iv) Like the line bundles we can construct a unique holomorphic vector bundle up to holomorphic isomorphism, of rank r if an element $\{g_{VU}\}$ in $Z^1(\mathcal{U}, \mathcal{F})$ is given.

We pose a number of examples of vector bundles like the line bundles.

Example A.1.21. (i) The dual bundle of a vector bundle E is E^* given by the transition data $g'_{UV} = (g_{UV}^{-1})^T$. The total space being the disjoint union of the dual spaces of fibres of the vector bundle.

(ii) The sum bundle is given by the transition map the matrix having the diagonal blocks g_{UV} and g'_{UV} where they are the transition data of two vector bundles L and L'. Clearly the rank of the direct sum bundle is the sum of the ranks of the vector bundles.

(iii) The tensor product bundle of the bundles E and E' viz. $E \otimes E'$ is given by the transition map $g_{UV} \otimes g'_{UV}$. Clearly, rank of $E \otimes E'$ is the product of product of ranks of L and L'.

(iv) For a general vector bundle E, we can talk of a holomorphic sub bundle. A holomorphic bundle E' is a holomorphic subbundle of E if there is an injective holomorphic bundle homomorphism $f : E' \to E$. If E' is a holomorphic subbundle of E then we construct the quotient bundle with total space $\sqcup_{x \in M} E_x/E'_x$ which is a vector bundle of rank $rank(E) - rank(E')$.

$\diamondsuit$

Remark 36. The above one is not the actual definition of the holomorphic subbundle. Let (E, π, X) be a holomorphic vector bundle and let F be a holomorphic sub manifold of E and $\pi_{|F \to X}$ is a holomorphic vector bundle by its own right. Then $(F, \tilde{\pi}, X)$ is said to be a holomorphic subbundle of (E, π, X). In above case, we can show that the image of an injective bundle morphism is indeed a subbundle.

The sections of holomorphic vector bundles have useful properties. Like the line bundles, the holomorphic sections of vector bundles are given by holomorphic maps which take value in $\mathbb{C}^r$. However, the vector space of holomorphic sections of a vector bundle is a finite dimensional vector space over $\mathbb{C}$.

Remark 37. (i) The sum bundle $E \oplus E'$ has sections of the form $s \oplus s'$ defined by the holomorphic maps $s_U \oplus s'_U$.

(ii) The tensor product bundle $E \otimes E'$ has the sections of the form $s \otimes s'$ defined by the holomorphic maps $s_U \otimes s'_U$.

(iii) We recall the pullback bundle. For a holomorphic map $f : \tilde{X} \to X$ and a vector bundle E of rank r the pullback of E is given by the transition maps $f^* g_{UV} := g_{UV} \circ f$. The total space of the pull back bundle is given by $\sqcup_{x' \in \tilde{X}} E_{f(x')}$. Thus the rank of the pullback bundle is rank(E). Observe that, any section s of pullback bundle is given by a holomorphic map $h : \tilde{M} \to E$ such that $\pi \circ h = f$.

Here we mention an important categorical equivalence between the sheaves and the vector bundles. Let $\mathcal{R}$ be a sheaf of commutative rings over a topological space X.

Definition A.1.22. Define $\mathcal{R}^p$ for $p \geq 0$ by the pre sheaf $U \mapsto \mathcal{R}^p(U) := \mathcal{R}(U) \oplus .. \oplus \mathcal{R}(U)$($p$-times).

(i) If $\mathcal{M}$ is sheaf of $\mathcal{R}$-modules such that $\mathcal{M}(U) \simeq \mathcal{R}^p(U)$ for some p, for all U then $\mathcal{M}$ is said to be a free sheaf. (ii) Let $\mathcal{M}|U$ denote the restriction sheaf of an open set U. A sheaf $\mathcal{M}$ of $\mathcal{R}$-modules is said to be *locally free* if for each point x in X has a neighbourhood U such that $\mathcal{M}|U$ is free.

Theorem A.1.23. *Let X be a complex manifold. Then there is a one-to-one correspondence (categorical isomorphism) between isomorphism classes of holomorphic bundles over X and locally free sheaves of modules over the sheaf of holomorphic functions $\mathcal{O}_X$ on X.*

Proof. For an open set $U \subseteq X$, we denote $\mathcal{S}(U, E) := \mathcal{S}(U, E|U)$. The sheaf defined by the holomorphic sections is a sheaf of modules over the ring of holomorphic functions $\mathcal{O}$ on X. Note that $\mathcal{S}(E)$ is a locally free sheaf of $\mathcal{O}$ modules because the local triviality property of vector bundles. Conversely, let $\mathcal{L}$ be a locally free sheaf of $\mathcal{O}_X$ modules while $\mathcal{O}_X$ denotes the ring of holomorphic functions. We find an open covering $\{U_\alpha\}$ of X such that $g_\alpha : \mathcal{L}|_{U_\alpha} \to \mathcal{S}^r|_{U_\alpha}$ is an isomorphism. Let us define $g_{\alpha\beta} = g_\alpha \circ g_\beta^{-1}$. Then g takes each point x of $U_\alpha \cap U_\beta$ to an invertible matrix in $GL(r, \mathbb{C})$ taht is, we obtain an element in $H^1(X, GL(r, \mathbb{C}))$ thus we obtain a unique vector bundle on X. □

Here we mention an important theorem on vanishing of cohomology spaces which will be utilized later on.

Theorem A.1.24. *Let X be a Riemann surface. If $\mathcal{S} = \mathcal{O}(L)$ is the sheaf of holomorphic sections of the line bundle L, then $H^p(X, \mathcal{S}) = 0$ for $p > 1$ and $\mathcal{S} = \mathbb{C}$ or $\mathbb{Z}$, then $H^p(X, \mathcal{S}) = 0$ for $p > 2$.*

The next theorem deals with an interconnection between zeroth and first cohomology spaces for sheaf of sections.

Theorem A.1.25. *(Serre duality Theorem) For a vector bundle E over a compact Riemann surface X,*

$$H^1(X, E) \cong H^0(X, K \otimes E^*)^*. \tag{A.7}$$

Proof. See [ROW13] for a detailed proof. ☐

Remark 38. The duality theorem of Serre is given for arbitrary compact complex manifolds. Let X be a compact complex manifold of complex dimension n and its holomorphic cotangent bundle is given as $T^*(X)$. The line bundle $\Omega^p_X = \wedge^p T^*(X)$ is the space of global holomorphic n-forms. Let E be a holomorphic vector bundle on X and We are at the position of taking the bundle $\Omega^p_X \otimes E := \Omega^p(E)$ and its sheaf of holomorphic sections. Then we have the duality theorem in general form. See [ROW13] for a detailed proof.

Theorem A.1.26.

$$H^r(X, \Omega^p(E)) \cong (H^{n-r}(X, \Omega^{n-p}(E^*)))^*. \tag{A.8}$$

Now we are position to explore the short exact sequences of sheaves and the corresponding long exact sequence of the cohomology groups.

Let us consider a sequence of sheaf group homomorphisms

$$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$$

such that the homomorphisms of the stalks are giving an exact sequence i.e.

$$0 \to \mathcal{S}_x \to \mathcal{T}_x \to \mathcal{U}_x \to 0$$

is exact for all $x \in X$. Then this sequence of sheaves is called an exact sequence of sheaves.

This sequence generates an exact sequence of the cohomology groups,

$$0 \to H^0(X, \mathcal{S}) \to H^0(X, \mathcal{T}) \to H^0(X, \mathcal{U}) \to H^1(X, \mathcal{S}) \to H^1(X, \mathcal{T}) \to H^1(X, \mathcal{U}) \to H^2(X, \mathcal{S}) \to .... \tag{A.9}$$

We describe the maps between cohomology groups as following. Let $\{u_i\}_i \in H^0(X, \mathcal{U})$. It will satisfy the relation $u_i = u_j$ on $U_i \cap U_j$. There exists $\{t_i\}_i$ such that $t_i \mapsto u_i$. Thus $\{t_i - t_j\}_{i,j} \mapsto u_i - u_j = 0$. By the exactness of the sheaves, there is unique $s = \{s_{ij}\}_{i,j}$ such that $s_{ij} \mapsto t_i - t_j$. Then $s \mapsto 0$ and $s \in H^1(X, \mathcal{S})$. Then we define $\delta_0(u) = s$.

It is customary to mention that if the sheaves are of rings or vector spaces then the corresponding cohomology groups will be rings and vector spaces respectively.

Example A.1.27. We give an example that is necessary for many computations of bundles. First we define a new sheaf. Let us fix $p \in X$ then line bundle L_p has a section s_p which vanishes at p. Let us define a sheaf $\mathcal{O}_p(L)$ by

$$\mathcal{O}_p(L)(U) = \begin{cases} \pi^{-1}\{p\}, & p \in U \\ 0, & p \notin U \end{cases} \tag{A.10}$$

From the short exact sequence

$$0 \to \mathcal{O}(LL_p^{-1}) \to \mathcal{O}(L) \to \mathcal{O}_p(L) \to 0,$$

of vector spaces we produce exact sequence of cohomology vector spaces,

$$0 \to H^0(X, LL_p^{-1}) \xrightarrow{\delta_1} H^0(X, L) \xrightarrow{\delta_2} H^0(M, \mathcal{O}_p(L)) \cong \mathbb{C} \xrightarrow{\delta} H^1(X, LL_p^{-1}) \to H^1(X, L) \to H^1(X, \mathcal{O}_p(L)) \to ..... \tag{A.11}$$

Here $H^0(X, \mathcal{O}_p(L)) \cong \pi^{-1}\{p\} \cong \mathbb{C}$. Now let us consider the map $\delta : H^0(M, \mathcal{O}_p(L)) \cong \mathbb{C} \to H^1(X, LL_p^{-1})$. If δ is a nonzero map then the $\ker(\delta)$ being a subspace of $\mathbb{C}$ is 0.

The sequence of vector spaces being exact, the map $\delta_2 : H^0(X, L) \to \mathbb{C}$ is the zero map as its image is 0. Again by exactness we conclude that the linear map $\delta_1 : H^0(X, LL_p^{-1}) \to H^0(X, L)$ is surjective and as it is injective already. Thus, we have $H^0(X, LL_p^{-1}) \cong H^0(X, L)$ while the isomorphism is given by $s \mapsto ss_p$ and any holomorphic section of L can be written in this form. Thus all holomorphic sections of L vanish at p. ♢

We use A.9 to define the degree of a line bundle L over a compact Riemann surface X. Now we look at another sheaf exact sequence and corresponding sequence of cohomology groups.

Let us consider the short exact sequence of sheaves on a compact Riemann surface X as the following:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1.$$

Here the sheaf $\mathbb{Z}$ is the sheaf of locally constant functions taking integer values and the map $\mathcal{O} \to \mathcal{O}^*$ is given by

$$f \mapsto e^{2\pi i f}.$$

Now it is easy to see that $H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ as groups and from the fact that the holomorphic functions over a compact Riemann surface are constants we conclude: $H^0(X, \mathcal{O}) \cong \mathbb{C}$ and $H^0(X, \mathcal{O}^*) \cong \mathbb{C}^*$.
Thus the exact sequence is given by:

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}). \tag{A.12}$$

Now the map $\mathbb{C} \to \mathbb{C}^*$ is given by the exponential map which is surjective. Thus by exactness, the map

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O})$$

is injective and we naturally have an exact sequence involving the quotient group. Using as a fact $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ and from previous theorem we have $H^2(X, \mathcal{O})$ is trivial. It produces the following exact sequence:

$$0 \to \frac{H^1(X, \mathcal{O})}{H^1(X, \mathcal{O}^*)} \to H^1(X, \mathcal{O}^*) \to \mathbb{Z} \to 0. \tag{A.13}$$

Definition A.1.28. The map $\delta : H^1(X, \mathcal{O}^*) \to \mathbb{Z}$ is called the *degree* map which takes each isomorphism class $[L]$ of holomorphic line bundles to an integer $\delta([L])$. Without any hesitation we can also write it as $\deg(L)$.

Remark 39. (i) As described in [Har77] and [Fri98] we can consider the degree of a line bundle over a smooth projective variety X as the first Chern class $c_1(L)$ an element of the Chow ring or an element in $H^2(X, \mathbb{Z})$ as image of a group homomorphism from the Chow ring. However, over the higher dimensional varieties the degrees of line bundles are not necessarily integers (e.g. ruled surfaces).

(ii) The exact sequence in A.13 is indeed valid for compact complex manifolds of higher dimensions. See [Fri98] for a complete discussion. However, both the definitions agree for line bundles.

Remark 40. (i) From the group homomorphism property we have:

$$\deg(L \otimes \tilde{L}) = \deg(L) + \deg(\tilde{L}). \tag{A.14}$$

(ii) As an important information we note that $\deg(L_p) = 1$. However, this can be verified from an alternate definition of degree. The line bundle L_p has a nonzero holomorphic section s_p that has a simple zero at p. So (the number of zeros)$-$(the number of poles) $= \deg(L_p) = 1$. (See that, for two nonzero meromorphic sections s and t we can provide with a global meromorphic function f over X which must have same number of zeros and poles counted up to multiplicities.This fact leads to the fact that (the number of zeros of s) - (the number of poles of s) = (the number of zeros of t) - (the number of poles of t) i.e. the number is independent of choice of nonzero meromorphic section.) (iii) For a vector bundle E of rank m, the determinant bundle $\det(E)$ is the line bundle having the fibre space $\Lambda^m(E_x)$ over the field $\mathbb{C}$ and transition data $g'_{\alpha\beta} = \det(g_{\alpha\beta})$.

We define

$$\deg(E) := \deg(\det(E)). \tag{A.15}$$

Now we mention two theorems involving degree of a line bundle and holomorphic sections. See [HSW99] for proofs.

Theorem A.1.29. *If $s \in H^0(X, L)$ vanishes at points $p_1, p_2, ..., p_n$ with multiplicities $m_1, ..., m_n$ respectively then*

$$\deg(L) = \Sigma_{i=1}^n m_i. \tag{A.16}$$

We must notice that the holomorphic section s is given by holomorphic functions s_U and $p_1, .., p_n$ are zeros of these locally defined holomorphic functions with respective multiplicities.

Remark 41. For a compact Riemann surface X, the line bundle L_p^{-1} has trivial space of holomorphic sections as the section s_p has a zero at p. Rather, s_p^{-1} is a meromorphic section having a simple pole at p.

Corollary A.1.30. *If* $\deg(L) < 0$ *then* L *has only the trivial holomorphic section.*

Now we look in to the space of holomorphic line bundles which have fixed degree. We often fix the degree to be 0 and the corresponding space is called the Jacobian denoted by $Jac(X)$ or the Picard variety of X. The construction for complex projective algebraic manifolds the construction of Picard variety is similar. Here we see that the Jacobian of a compact riemann surface X is a complex torus of g-dimensions embedded in the Picard group.

For the trivial line bundle L we have $L^* \cong L$ and $\mathcal{O}(L) = \mathcal{O}$. Thus applying the Serre duality theorem, we write $H^1(X, \mathcal{O}) \cong H^0(X, K_X)^*$.

Given the genus g of the compact Riemann surface X, we have $H^1(X, \mathcal{O})$ is a g dimensional vector space over $\mathbb{C}$.

Now we use a short exact sequence of sheaves as following:

$$0 \to \mathbb{C} \to \mathcal{O} \to \mathcal{O}(K) \to 0.$$

Here the map

$$\mathcal{O} \to \mathcal{O}(K)$$

is given by $f \mapsto df$.

As we have discussed earlier, we mention the following information.
$H^0(X, \mathbb{C}) \cong \mathbb{C}$ as it is the space of all constant functions and we have $H^0(X, \mathcal{O}) \cong \mathbb{C}$ as all holomorphic functions on a compact Riemann surfaces are constants. Finally we recall that all cohomology groups of order > 2 vanish.

Thus the corresponding long exact sequence of vector spaces will be

$$0 \to \mathbb{C} \to \mathbb{C} \to H^0(X, \mathcal{O}(K_X)) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}(K_X)) \to H^2(X, \mathbb{C}). \quad (A.17)$$

Now we apply the Serre duality theorem on the canonical bundle $\mathcal{O}(K_X)$. As per we described the trivial bundle case we see that

$$\begin{cases} H^1(X,\mathcal{O}(K_X)) \cong \mathbb{C} \\ H^0(X,\mathcal{O}^*) \cong \mathbb{C}^* \\ H^2(X,\mathbb{C}) \cong \mathbb{C} \end{cases} \tag{A.18}$$

Applying repeatedly the exactness of the sequence of cohomology spaces and using the rank-nullity theorem we conclude that

$$\begin{cases} \dim(H^1(X,\mathbb{C})) = 2g \\ H^1(X,\mathbb{C}) \cong \mathbb{C}^{2g} \end{cases} \tag{A.19}$$

The same technique proves that $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ as $\mathbb{Z}$ modules. Here one fact is quite important: The module $H^1(X,\mathbb{Z})$ is torsion-free over the principal ideal domain $\mathbb{Z}$ thus free over $\mathbb{Z}$. In that case, we will start with the sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}(K_X) \to 0.$$

However. we have, involving a quotient group, the exact sequence that proves our claim.

$$0 \to \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \to H^1(X,\mathcal{O}^*) \to \mathbb{Z} \to 0 \tag{A.20}$$

Remark 42. As a general case we have

(i)

$$H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^{2g}, & i = 1 \\ \mathbb{Z}, & i = 2 \\ 0 & i \geq 3 \end{cases}$$

(ii)

$$H^i(X,\mathcal{O}) = \begin{cases} \mathbb{C}, & i = 0 \\ \mathbb{C}^g, & i = 1 \\ 0, & i \geq 2 \end{cases}$$

Here X is a compact Riemann surface of genus g. We refer to [Har77] for a very good view on these results.

Remark 43. A similar exact sequence is available for compact complex projective algebraic manifold X. The Picard torus or the Picard variety $\mathrm{Pic}^0(X)$ is a complex torus of dimension $\dim H^1(X,\mathcal{O}_X) := h^{0,1}(X)$. We obtain it from the exponential cohomology sequence

$$H^1(X,\mathbb{Z}) \xrightarrow{i} H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}). \tag{A.21}$$

by

$$\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/i(H^1(X, \mathbb{Z})). \tag{A.22}$$

Here we emphasize on the assumption that X is projective algebraic because we need a 'Kähler' manifold that we discuss further. We refer to [Fri98] for detailed workout.

Before moving to the Riemann-Roch theorem we mention an important result and a remark.

Theorem A.1.31. *Let $\mathcal{O}(E)$ be the sheaf of sections of a vector bundle E over a compact Riemann surface X then $H^p(X, \mathcal{O}(E))$ is trivial $\forall p > 1$.*

Theorem A.1.32. *(The Riemann Roch theorem) A compact Riemann surface X of genus g satisfies the equality*

$$\dim H^0(X, E) - \dim H^1(X, E) = \deg(E) + rank(E)(1 - g). \tag{A.23}$$

The theorem of Riemann-Roch is extended to higher dimensional compact complex manifolds. We have mentioned in A.24 the version for compact complex projective algebraic non-singular surfaces.

A.1.4 Vector bundles over the complex projective line and an elliptic curve

Now we move to the classification of vector bundles over the complex projective line $\mathbb{P}^1$: the Birkhoff-Grothendieck theorem and see the consequences. However, this classification is valid over any algebraically closed field k of characteristic zero. Further we will revisit the main classification theorems of indeomposable vector bundles over an elliptic curve as proved in [Ati57b].

We recall the following exact sequence of groups,

$$0 \to \frac{H^1(X, \mathcal{O})}{H^1(X, \mathbb{Z})} \cong \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \to H^1(X, \mathcal{O}^*) \to \mathbb{Z} \to 0.$$

Now for the Riemann sphere $\mathbb{P}^1$ the genus $g = 0$. Thus the exact sequence turns into $0 \to H^1(X, \mathcal{O}^*) \to \mathbb{Z} \to 0$. This converts the degree map to a group isomorphism i.e. each integer d is uniquely associated to an isomorphic class of line bundles. This is not only true for the $\mathbb{P}^1$ but also for $\mathbb{P}^n$ for all $n \in \mathbb{N}$.

We note that $\mathcal{O}(1)$ has degree 1 as the $\mathcal{O}(1)$ is isomorphic to L_0. Thus from the group homomorphism property, $\mathcal{O}(d)$ has degree d. Thus it is the only line bundle of degree d up to isomorphism i.e. the line bundles over $\mathbb{P}^1$ are given by the isomorphism classes of $\mathcal{O}(d)$. In algebraic geometry terms $\mathcal{O}(1)$ is called the *hyperplane bundle*. See [Har77] for more details. We mention the hyperplane bundle in case of projective spaces $\mathbb{P}^n$ further.

Remark 44. Over $\mathbb{P}^1$, $K_X \cong \mathcal{O}(-2)$. Now we see that the bundle defined by $g_{01}(z) = -z$ is a L_0 bundle so has degree 1. Thus this is $\mathcal{O}(1)$. Thus the line bundle having transition function $-z^2$ has degree 2 and the line bundle having transition map $\frac{-1}{z^2}$ has degree -2. Again the line bundle $\mathcal{O}(-2)$ has degree -2. Thus K and $\mathcal{O}(-2)$ are in the same isomorphism class.

Now we prove that the vector bundles over $\mathbb{P}^1$ can be written as a direct sum of line bundles $\mathcal{O}(d)$.

Theorem A.1.33. *(Birkhoff- Grothendieck) If E is a vector bundle of rank r then $E \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) .... \oplus \mathcal{O}(a_m)$ for some $a_1, .., a_m \in \mathbb{Z}$. The integers $a_1, ..., a_m$ are unique up to a permutation.*

A.1.5 Coherent and torsion-free sheaves on smooth projective varieties

In this section we adapt the language of sheaves with more generality to present more coherent sheaves and their geometry. We use the books of Huybrechts and Lehn and Friedmann for these details.

Definition A.1.34. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules such that

(i) $\mathcal{F}$ is of finite type over $\mathcal{O}_X$ meaning that every point $x \in X$ admits an open neighbourhood U such that there exists an integer n and there exists a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$.

(ii) For any open set $U \subseteq X$ and any morphism $\phi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$ for any natural number n, admits kernel tht is of finite type.

Let X be a Noetherian scheme (for example smooth vaieties over algebraically closed fields). By Coh(X) we denote the category of coherent sheaves on X. A coherent sheaf E on a smooth projective integral scheme admits a global resolution of locally sheaves $0 \to E_0 \to ... \to E_n \to E$ and homological invariants of E are determined by this global resolution. Also, direct sum, tensor product of coherent sheaves are coherent and kernel, cokernel, image of a homomorphism of coherent sheaves are coherent again. We define the rank of E as the rank of $E(x) = E \otimes_{\mathcal{O}_X} k(x)$ where $k(x)$ denotes the function field of X.

Definition A.1.35. The support of a coherent sheaf E is the closed set $\mathrm{Supp}(E) = \{x \in X : E_x \neq 0\}$. its dimension is called the dimension of E or $\dim(E)$.

Definition A.1.36. A coherent sheaf E is said to be pure of dimension d if $\dim(F) = d$ for all subsheaves $F \subseteq E$. A coherent sheaf is torsion-free if and only if it is pure of dimension $d = \dim(X)$.

We see that for a torsion-free coherent sheaf E, its double dual E^{**} is locally free. We omit a few details of sheaves and mention further main results as following. Let X be a smooth integral projective variety over an algebraically closed field k of characteristic 0. We fix an ample line bundle $\mathcal{O}_X(1)$ on X.

Definition A.1.37. For a coherent sheaf E on X the Hilbert polynomial of E is defined to be $m \mapsto \chi(E \otimes \mathcal{O}_X(m))$.

There are unique integers α_i for $i = 0, ..., \dim(E)$ such that

$$P(E, m) = \sum_{i=0}^{\dim(E)} \alpha_i(E) \frac{m^i}{i!}.$$

Definition A.1.38. If E is a coherent sheaf of dimension $d = \dim(X)$ then the rank of E is defined to be $\text{rank}(E) = \frac{\alpha_d(E)}{\alpha_d \mathcal{O}_X)}$.

On X a d-dimensional sheaf E admits an open dense subset U such that $E|_U$ is locally free. Then the rank of E is the rank of corresponding locally free sheaf. Now onwards we will only consider pure coherent sheaves of dimension $\dim(X)$ and assume this in all the definitions. We note that such sheaves are always torsion-free sheaves. Recall that a coherent sheaf E is torsion-free if and only if for each $x \in X$ and $s \in \mathcal{O}_{X,x} \backslash \{0\}$ multiplication by s is an injective morphism. This is equivalent to say that it is pure of dimension $\dim(X)$.

Definition A.1.39. The reduced Hilbert polynomial $p(E)$ of a coherent sheaf E is defined as $p(E, m) = \frac{P(E,m)}{\alpha_d(E)}$. A coherent sheaf E is said to be Gieseker semistable if for each coherent subsheaf F with $0 < \text{rank}(F) < \text{rank}(E)$ the inequality $p(E, m) \geq p(F, m)$ holds for all large integers m. We can write the strict inequality to define stability.

Corollary A.1.40. *If E is a stable sheaf then $H^0(\text{End}(E)) \cong k$. We call such sheaves simple.*

We have the classical definition of slope stability due to Mumford. We can define this in two ways. We can define $\deg(E) = c_1(E) \cdot c_1(\mathcal{O}_X(d-1))$ where c_1 denotes the first Chern class of a torsion-free coherent sheaf. We can manipulate this definition of degree to the expression $\deg(E) = \alpha_{d-1}(E) - \text{rank}(E) \cdot \alpha_{d-1}(\mathcal{O}_X)$. Then $\mu(E) = \frac{\deg(E)}{\text{rank}(E)}$.

Definition A.1.41. A coherent sheaf E is said to be semistable or slope semistable if for all coherent subsheaves F with $0 < \text{rank}(F) < \text{rank}(E)$ the inequality $\mu(E) \geq \mu(F)$ holds true. We define stability if the inequality is strict.

Proposition A.1.42. *Let E be a coherent sheaf then we have a chain of implications: E is slope stable $\implies$ Gieseker stable $\implies$ Gieseker semistable $\implies$ slope semistable.*

Theorem A.1.43. *Let V be a semistable torsion-free coherent sheaf with $\mu(V) = \mu$. There is a filtration of coherent subsheaves $0 \subset F_1 \subset ... \subset F_k = V$ such that F_i/F_{i-1} is torsion-free and stable for each i and $\mu(F_i/F_{i-1}) = \mu$ for all i. This is called a Jordan-Hölder filtration.*

Theorem A.1.44. *If V is a μ-semistable torsion-free sheaf with $\mu(V) = \mu$. There is a filtration of coherent subsheaves $0 \subset F_1 \subset ... \subset F_k = V$ such that F_i/F_{i-1} is torsion-free and μ-semistable for every i such that $\mu(F_{i+1}/F_i) < \mu(F_i/F_{i-1})$ for every i. This is called the Harder-Narasimhan filtration of V.*

On an algebraic surface X, we for a vector bundle E we have, using Riemann-Roch theorem that $p(E, m) = \frac{P(E,m)}{r} = \frac{c_1(H)^2 m^2}{2} + [\frac{c_1(V) \cdot c_1(H)}{r} - \frac{c_1(K_X) \cdot c_1(H)}{2}]m + \frac{1}{r}[\frac{c_1(V)^2 - c_1(V) \cdot c_1(K_X)}{2} - c_2(V)] + \chi(\mathcal{O}_X) = \frac{c_1(H)^2 m^2}{2} + [\frac{c_1(V) \cdot c_1(H)}{r} - \frac{c_1(K_X) \cdot c_1(H)}{2}]m + \frac{\chi(V)}{r}$. It follows that if V is Gieseker stable if and only if for all rank s subsheaves W of V with $0 < s < r$, either $\mu(W) < \mu(V)$ or $\mu(W) = \mu(V)$ and $\chi(W)/s < \chi(V)/r$.

We change the definition to choice of an ample line bundle. Let $f : X \to \mathbb{P}^n$ be a closed immersion. We define $H = \mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}^n}(1)$ which is a very ample line bundle on X. We will consider a torsion-free sheaf E on X that is a pure coherent sheaf of dimension 2. Note that the *rank* of E is the rank of the locally free sheaf on an open dense subset U. We can write the Hilbert polynomial $P(E, m) = \chi(E \otimes \mathcal{O}_X(m))$. We will consider moduli of pure coherent Λ-modules of pure dimension $\dim(X)$. Such sheaves are always torsion-free. They are connected to pure coherent sheaves of dimension $\dim(X)$ on the projective completion of the total space of vector bundles.

We explain some important results on slope-unstable (that is, not semistable) vector bundles over algebraic surfaces.

Proposition A.1.45. *Suppose that V is a rank 2 bundle on a smooth connected projective surface X. Then there exists a unique sub-line bundle (a line bundle whose sheaf of sections is a subsheaf) F of V with torsion-free quotient such that $\mu(F) > \mu(V)$. If L is a sub-line bundle of V such that $\mu(L) \geq \mu(V)$, then L is a subsheaf of F ad $\mu(L) \leq \mu(F)$. Here equality takes place if and only if $L = F$.*

Proposition A.1.46. *Let V be a semistable but not stable bundle of rank 2 on a smooth onnected projective surface X. Then exactly one of the following holds:*

(i) There is a unique sub-line bundle F of V with $\mu(F) = \mu(V)$. The quotient V/F is necessarily torsion-free, and V is given canonically as an extension

$$0 \to F \to V \to F' \otimes I_Z \to 0$$

where I_Z is the ideal sheaf defined by a closed subscheme Z of X.

(ii) There are exactly two distinct sub-line bundles F and G of V with $\mu(F) = \mu(G) = \mu(V)$. In this case $V = F \oplus G$.

(iii) $V = F \oplus F$ and there are infinitely many sub-line bundles with normalized degree $\mu(V)$, exactly corresponding to the choice of a line in $H^0(V \otimes F^{-1})$.

To discuss coherent sheaves in depth we define and explain existence of a Harder-Narasimhan filtration and grading of semistable coherent sheaves.

Proposition A.1.47. *A torsion-free coherent sheaf E on X admits a unique filtration*

$$0 = E_0 \subset E_1 \subset ... \subset E_l = E$$

such that the quotients E_i/E_{i-1} are Gieseker semistable of (pure dimension d) and normalized Hilbert polynomials $P(E_i/E_{i-1})/\mathrm{rank}(E_i/E_{i-1})$ is strictly decreasing for all large values of n.

Proposition A.1.48. *If E is a Gieseker semistable coherent sheaf then there is a filtration*

$$0 = E_0 \subset E_1 \subset ... \subset E_l = E$$

such that the quotients E_i/E_{i-1} are Gieseker stable of pure dimension d, with the same normalized Hilbert polynomials $P(E_i/E_{i-1})/\mathrm{rank}(E_i/E_{i-1})$. The coherent sheaf $gr(E) = \bigoplus_{i=1}^{l} E_i/E_{i-1}$ is unique up to isomorphism on E.

A.2 Sheaf theoretic results on algebraic surfaces

A.2.1 The moduli of bundles of rank 2

Let V be a vector bundle on a quasi-projective variety X. Then we can define the Chern classes of V as follows.

Let $L = \det(V)$. Then $c_1(V) := c_1(L) \in H^2(X, \mathbb{Z})$. To define $c_i(V)$ in general we first define it on direct sum of line bundles

$$V = L_1 \oplus ... \oplus L_n.$$

by comparing the coefficients of the polynomials

$$\sum_{i=0}^{n} c_i(V)t^i = (1 + c_1(V)t)...(1 + c_n(V)t).$$

See that $c_i(V) \in H^{2i}(X, \mathbb{Z})$ and by the usual cup product formula we obtain $c_i(V).c_j(V) \in H^{2(i+j)}(X, \mathbb{Z})$. We further extend the definition in case of filtration of subbundles V_i's of V such that $V_i/V_{i-1} \cong L_i$ is a line bundle. Now to define the chern classes for general V we construct a variety Y and a morphism $\pi : Y \to X$ such that $\pi^* : H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ is injective for all values of i and such that π^*V has a filtration as mentioned above. Then we can set $c_i(V)$ to be the unique class such that

$$\pi^* c_i(V) = c_i(\pi^* V).$$

Finally the *total Chern class* is defined as

$$c(V) = \sum_{i=0}^{r} c_i(V).$$

We list below the properties of the Chern classes.

Remark 45. (i) For an exact sequence of bundles

$$0 \to V' \to V \to V'' \to 0$$

we have

$$c(V) = c(V')c(V'').$$

(ii) If the dual bundle of V is denoted by V^* then $c(V^*) = c(V)^*$, in other words

$$c(V^*) = \sum_{i=0}^{r} (-1)^i c_i(V).$$

(iii) If L is a line bundle and V is a vector bundle of rank r then

$$c_1(V \otimes L) = c_1(V) + r.c_1(L)$$

and

$$c_2(V \otimes L) = c_2(V) + (r-1)c_1(V).c_1(L) + \binom{r}{2} c_1(L)^2.$$

(iv) Finally, for a vector bundle V of rank r on a smooth surface X, we produce the Riemann-Roch formula

$$\chi(V) = \frac{c_1(V).(c_1(V) - c_1(K_X))}{2} - c_2(V) + r\chi(\mathcal{O}_X) \tag{A.24}$$

where

$$\chi(V) = h^0(X, V) - h^1(X, V) + h^2(X, V).$$

We will derive more formulas regarding the chern classes as necessary. For more details see [Fri98]. We list out more facts about Chern classes of vector bundles on surfaces which we can prove by computing in the case of decomposable vector bundles.

1. $c_2(L) = 0$ for any line bundle L.

2. For a vector bundle E we have $c_1(E^*) = -c_1(E)$ and in case E has rank 2 we have $c_2(E^*) = c_2(E)$.

3. $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$.

4. If E and F are vector bundles of rank 2 then for decomposable case $E = L_1 \oplus L_2$ and $F = M \oplus N$ we obtain

   $$c_1(E \otimes F) = 2(c_1(E) + c_1(F))$$

   and

   $$c_2(E \otimes F) = 2(c_2(E) + c_2(F)) + c_1(E)^2 + c_1(F)^2 + 3 \cdot c_1(E) \cdot c_1(F).$$

   This can be generalized easily for indecomposable cases also.

5. $c_1(T_{\mathbb{P}^2}) = 3$ and $c_2(T_{\mathbb{P}^2}) = 3$. This obtained from the fact that c_1 and c_2 are given by the coefficients of the polynomial $(1+t)^3$.

Here we compute an invariant of a vector bundle V of rank 2 which may be important in case of studying the moduli space of holomorphic bundles over an algebraic surface. Here we want to find the dimension of the moduli space of μ-stable vector bundles V with rank 2 and the first Chern class c_1, we should perhaps see how the second Chern class c_2 affects the moduli construction. Let us use a result which we can prove using Riemann-Roch theorem and the splitting principle of vector bundles of rank 2.

$$\chi(\mathrm{End}(V)) = c_1^2 - 4c_2 + \chi(\mathcal{O}_X). \tag{A.25}$$

Proof. It is enough for us to prove the result for a bundle V such that $V \cong L_1 \oplus L_2$. We write the Chern polynomial of V,

$$1 + c_1(V)t + c_2(V)t^2 = 1 + (c_1(L_1) + c_1(L_2))t + (c_1(L_1).c_1(L_2))t^2 \tag{A.26}$$

which leads to

$$\begin{cases} c_1(V) = c_1(L_1) + c_1(L_2); \\ c_2(V) = c_1(L_1).c_1(L_2) \end{cases} \tag{A.27}$$

Now we realize the endomorphism bundle $\mathrm{End}(V)$ using L_1 and L_2 from A.27 as following,

$$\mathrm{End}(V) \cong \mathcal{O}_X \oplus \mathcal{O}_X \oplus (L_1 \otimes L_2^*) \oplus (L_2 \otimes L_1^*). \tag{A.28}$$

Further we use the Chern polynomial of $\mathrm{End}(V)$ with the above splitting and the fact $c_1(\mathcal{O}_X) = 0$ to obtain

$$\begin{cases} c_1(\mathrm{End}(V)) = 0; \\ c_2(\mathrm{End}(V)) = -(c_1(L_1) - c_1(L_2))^2 = 4c_2(V) - c_1^2(V). \end{cases} \tag{A.29}$$

Due to the vanishing of the first Chern class of $End(V)$ we have, from Riemann-Roch formula in A.24 we produce

$$\chi(\mathrm{End}(V)) = -c_2(\mathrm{End}(V)) + 4.\chi(\mathcal{O}_X).$$

Finally from A.29 we conclude the result. □

Now if V is slope stable we have $H^0(X, \mathrm{End}(V))$ to be 1-dimensional. On the other hand we using Serre duality theorem obtain that $H^2(X, \mathrm{End}(V)) \cong H^0(X, \mathrm{End}(V) \otimes K_X)^*$. Now V being μ-stable we have $H^0(X, \mathrm{End}(V) \otimes K_X) \cong H^0(X, K_X) = 0$ if $\deg(K_X) < 0$. Finally, we obtain the dimension of the space $H^1(X, \mathrm{End}(V))$ using A.24.

$$h^1(X, \mathrm{End}(V)) = 1 + h^0(X, K_X) - (\chi(\mathcal{O}_X) + c_1^2 - 4c_2). \tag{A.30}$$

Next we find $\chi(\mathcal{O}_X)$. From the definition of χ we have $\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X)$. From the fact that $h^0(X, \mathcal{O}_X) = 1$ and $h^2(X, \mathcal{O}_X) = h^0(X, K_X)$ and denoting $h^1(X, \mathcal{O}_X) = q$ we obtain $\chi(\mathcal{O}_X) = 1 - q + h^0(X, K_X)$. Going back to A.30 we conclude

$$h^1(X, \mathrm{End}(V)) = q - c_1^2 + 4c_2. \tag{A.31}$$

Even if E is a μ-stable bundle we don't know yet the dimension of the space $H^0(X, \mathrm{End}(V) \otimes \Omega_X^1)$. But we already know the dimension of $H^0(X, \mathrm{End}(V) \otimes \Omega_X^2)$ which is $h^0(K)$. This piece of information gives us some hints to understand the space of pairs $(E, \phi : E \to E \otimes \Omega_X^2)$. In [HL10] the moduli spaces of the torsion-free sheaves have been thoroughly discussed. Finding the dimension of the space $\mathcal{M}^s \backslash T^*\mathcal{N}^s$ is another question. Here $\mathcal{N}^s$ is the space of stable bundles in case it attains a moduli structure. Here we see that smoothness of moduli space of torsion sheaves for higher dimensional smooth varieties is more complicated than the curve case. Indeed in many cases we obtain local smoothness rather than global smoothness.

References

[Ati57a] M. F. Atiyah. *Complex Analytic Connections in Fibre Bundles. Transactions of the American Mathematical Society*, 85(1):181–207, 1957.

[Ati57b] M. F. Atiyah. *Vector bundles over an elliptic curve. Proc. London Math. Soc.*, 7(1):414–452, 1957.

[Bea90] Arnaud Beauville. *Jacobiennes des courbes spectrales et systèmes hamiltoniens complètement intégrables. Acta Mathematica*, 164:211–235, 1990.

[BM23] Eric Boulter and Ruxandra Moraru. *(Co)-Higgs bundles on non-Kähler elliptic surfaces.* (arXiv: 2210.09839), 2023.

[BNR89] A. Beauville, M. S. Narasimhan, and S. Ramanan. *Spectral curves and the generalized theta divisor. Journal für die reine und angewandte Mathematik*, 398:169–179, 1989.

[BR94] I. Biswas and S. Ramanan. *An infinitesimal study of the moduli of Hitchin pairs. Journal of the London Mathematical Society*, 49(2):219–231, 1994.

[BR23] Kuntal Banerjee and Steven Rayan. *A generalized spectral correspondence.* (arXiv:2310.02413), 2023.

[BRSY21] Taylor Brysiewicz, Jose Israel Rodriguez, Frank Sottile, and Thomas Yahl. *Solving decomposable sparse systems. Numerical Algorithms*, 88(1):453–474, 2021.

[BS13] David Baraglia and Laura Schaposnik. Real structures on moduli spaces of higgs bundles. *Advances in Theoretical and Mathematical Physics*, 20, 2013.

[Cha10] Ngô Bao Chau. Le lemme fondamental pour les algèbres de lie. *Publications mathématiques de l'IHÉS*, 111:1618–1913, 2010.

[CN20] T. Chen and B. Ngô. *On the Hitchin morphism for higher-dimensional varieties. Duke Mathematical Journal*, 169:1971–2004, 06 2020.

[Col15] Alejandra Vicente Colmenares. *Semistable rank 2 co-Higgs bundles over Hirzebruch surfaces.* PhD thesis, University of Waterloo, 2015.

[Dal07] Ciara Daly. *Rank two vector bundles on elliptic curves. Irish Math. Soc. Bulletin*, 60:113–135, 2007.

[DKPR22] Timothy Duff, Viktor Korotynskiy, Tomas Pajdla;, and Margaret H. Regan. *Galois/monodromy groups for decomposing minimal problems in 3D reconstruction. SIAM Journal on Applied Algebra and Geometry*, 6(4):740–772, 2022.

[For81] Otto Forster. *Lectures on Riemann Surfaces.* Graduate Texts in Mathematics. Springer Newyork, NY, 1981.

[Fri98] Robert Friedman. *Algebraic surfaces and holomorphic vector bundles.* https://www.springer.com/series/223. Springer New York, NY, 1998.

[Gal19] Guillermo Gallego. *Higgs bundles twisted by a vector bundle.* Master's thesis, Universidad Complutense de Madrid, 9 2019.

[GGD12] Ernesto Girondo and Gabino González-Diez. *Introduction to compact Riemann surfaces and Dessins d'Enfants.* Number 79 in London Mathematical Society Student Texts. Cambidge University Press, 2012.

[GGPN23] Guillermo Gallego, Oscar Garcia-Prada, and M. S. Narasimhan. *Higgs bundles twisted by a vector bundle.* (arXiv:2105.05543), 2023.

[Gun67] R. C. Gunning. *Lectures on Vector Bundles over Riemann Surfaces*, volume 6 of *Mathematical Notes.* Princeton University Press and the University of Tokyo Press, 1967.

[Har77] Robin Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer New York, NY, 1977.

[Har79] Joe Harris. *Galois groups of enumerative problems. Duke Mathematical Journal*, 46(4):685–724, 1979.

[HH21] Tamás Hausel and Nigel James Hitchin. *Very stable Higgs bundles, equivariant multiplicity and mirror symmetry. Inventiones mathematicae*, 228:893 – 989, 2021.

[Hit87a] N. J. Hitchin. *The self-duality equations on a Riemann surface. Proceedings of the London Mathematical Society*, 55(1):91–114, 1987.

[Hit87b] Nigel Hitchin. *Stable bundles and integrable systems. Duke Mathematical Journal*, 54(1):91–114, 1987.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves, second edition.* Cambridge University Press, 2010.

[HS14] Nigel Hitchin and Laura P. Schaposnik. *Nonabelianization of Higgs bundles. Journal of Differential Geometry*, 97(1):79–89, 2014.

[HSW99] N.J. Hitchin, G.B. Segal, and R.S. Ward. *Integrable Systems: Twistors, Loop Groups and Riemann Surfaces*, volume 4 of *Oxford Graduate Texts in Mathematics*. Clarendon Press, 1999.

[JW18] Gareth A. Jones and Jürgen Wolfart. *Dessins d'Enfants on Riemann Surfaces.* Springer Cham, 2018.

[Lan75] Stacy G. Langton. *Valuative criteria for families of vector bundles on algebraic varieties. Annals of Mathematics*, 101(1):88–110, 1975.

[LS12] Andreas Laudin and Alexander Schmitt. *Recent results on quiver sheaves. Central European Journal of Mathematics*, 10:1246–1279, 2012.

[LZ10] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on Surfaces and Their Applications.* https://www.springer.com/series/855. Springer Berlin, Heidelberg, 2010.

[Mar77] Masaki Maruyama. *Moduli of stable sheaves, I. Journal of Mathematics of Kyoto University*, 17(1):91 – 126, 1977.

[Mir95] Rick Miranda. *Algebraic Curves and Riemann Surfaces*, volume 5 of *Graduate Studies in Mathematics.* American Mathematical Society, 1995.

[Nit91] Nitin Nitsure. *Moduli space of semistable pairs on a curve. Proc. London Math. Soc.*, (3)(62):275–300, 1991.

[Pot97] J. Le Potier. *Lectures on Vector Bundles*. Cambridge studies in advanced mathematics. Cambridge University Press, 1997.

[PP06] Ekaterina Pervova and Carlo Petronio. *On the existence of branched coverings between surfaces with prescribed branch data, I. Algebraic & Geometric Topology*, 6(4):1957–1985, 2006.

[Ray11] Steven Rayan. *Geometry of co-Higgs bundles*. PhD thesis, University of Oxford, 2011.

[Ray13] Steven Rayan. *Co-Higgs bundles on* $\mathbb{P}^1$. *New York Journal of Mathematics*, 19:925–945, 2013.

[Ray14] Steven Rayan. *Constructing co-Higgs bundles on* $\mathbb{CP}^2$. *The Quarterly Journal of Mathematics*, 65(4):1437–1460, 2014.

[Rit23] J.F. Ritt. *Permutable rational functions. Transactions of the American Mathematical Society*, 25(3):399–448, 1923.

[ROW13] Jr R. O. Wells. *Differential analysis on complex manifolds*. https://www.springer.com/series/136. Springer New York, NY, 2013.

[RS21] Steven Rayan and Evan Sundbo. *Twisted cyclic quiver varieties on curves. Eur. J. Math*, 7(1):205–225, 2021.

[Sim94a] Carlos T. Simpson. *Moduli of representations of the fundamental group of a smooth projective variety. I. Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 79:47–129, 1994.

[Sim94b] Carlos T. Simpson. *Moduli of representations of the fundamental group of a smooth projective variety. II. Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 80:5–79, 1994.

[Tu93] Loring W. Tu. *Semistable bundles over an elliptic curve. Advances in Mathematics*, 98:1–26, 1993.