Some Distributional Solutions of the CH, DP and CH2 Equations and the Lax Pair Formalism

A Thesis Submitted to the

College of Graduate Studies and Research

in Partial Fulfillment of the Requirements

for the degree of Doctor of Philosophy

in the Department of Mathematics and Statistics

University of Saskatchewan

Saskatoon

By

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Abstract

This dissertation deals with a class of nonlinear wave equations of the type discovered by R. Camassa and D. D. Holm which includes the Camassa-Holm, the Degasperis-Procesi, and the two component Camassa-Holm equations. All these equations admit certain non-smooth soliton-like solutions, called peakons as well as other non-smooth solutions like cuspons. We apply the techniques of the theory of distributions of L. Schwartz to study these solutions. In particular, every non-smooth traveling wave which is a distributional solution of the two component Camassa-Holm equation is a distributional solution of the Camassa-Holm equation if the set of points where the height of the wave equals its speed, is of measure zero. This includes peakon or cuspon traveling wave solutions.

We also develop a suitable modification of the classical Lax pair formalism to deal with singular solutions. We show that the Lax pair formalism can be extended to a distributional weak Lax pair which is appropriate for dealing with the peakon solutions of the Camassa-Holm equation.

Acknowledgements

I would like to thank professor Jacek Szmigielski for guiding me through the problems, patiently supervising the work and his tremendous help.

For financial support:

- The department of Mathematics and Statistics at the University of Saskatchewan for teaching assistantships and sessional lectureships.
- Prof. Jacek Szmigielski for financial assistance.

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Chapter 1

Introduction

In 1936, S. L. Sobolev introduced the concepts of generalized functions and derivatives to deal with non-smooth solutions of linear partial differential equations. In 1948, this theory was generalized by L. Schwartz [21], who introduced the concept of distributions. However, nonlinear operations cannot be easily performed on distributions. One of the problems is that the product of distributions cannot be defined in general. One can naturally define the product of a smooth function and a distribution but this product is not associative [19]. Schwartz [20] proved that if an associative differential algebra $(\mathcal{A}, \partial, \circ)$ contains the space $\mathcal{D}'(\Omega)$ of distributions over an open set $\Omega \subset \mathbb{R}^n$, then the operations $(\partial, \circ)$ in $\mathcal{A}$ cannot simultaneously be faithful extensions of the distributional derivatives and the product of continuous functions.

The purpose of this thesis is to study two classes of distributional solutions, namely, the traveling wave solutions and the peakon solutions of three nonlinear partial differential equations namely, Camassa-Holm (CH) [2] (also see [31]), Degasperis-Procesi (DP) [6] and the two-component Camassa-Holm (CH2) [12]. Given a fixed $\kappa \in \mathbb{R}$, the first two are the cases $b = 2$ and $b = 3$ of the following PDE respectively:

$$\begin{cases} u_t - u_{xxt} + (b+1)uu_x + \kappa u_x = bu_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

This equation can also be written as follows:

$$m_t + bmu_x + m_x u = 0, \tag{1.2}$$

where $m = u - u_{xx} + \frac{1}{b}\kappa$.

The Camassa-Holm and the Degasperis-Procesi equations admit a type of non-smooth solution (in the sense of distributions) that is called a *multipeakon*. A multipeakon is a train of finitely many peaked interacting waves that regain their original shape after interaction.

In general terms the main outcome of this dissertation is that the distributional solutions, at least in the narrow sense developed here, provide a wealth of mathematical connections with many areas of mathematics. We now present an outline of the content of this dissertation.

In Chapter 2, we present general information about the three equations mentioned above as well as the Korteweg de Vries equation

$$u_t + u_{xxx} + 6uu_x = 0, \tag{1.3}$$

which was historically the first equation of this type studied. In particular, we present a detailed derivation of the Camassa-Holm equation using variational methods applied to Euler's equation. We also discuss the so-called Lax formulation of these equations in terms of an overdetermined system of linear equations with a spectral parameter. For example, it is known that if $b = 2$ (the Camassa-Holm equation), then (1.2) is the compatibility condition $\psi_{xxt} = \psi_{txx}$, for the following system:

$$\begin{cases} \psi_{xx} = (\frac{1}{4} - zm)\psi, \\ \psi_t = -(\frac{1}{2z} + u)\psi_x + \frac{1}{2}u_x\psi. \end{cases} \tag{1.4}$$

A system of this type is called the Lax system.

In Chapter 3, we study the distributional traveling wave solutions of the two component Camassa-Holm equation. The two component Camassa-Holm equation was introduced recently by M. Chen, S. Liu and Y. Zhang [12] as a generalization of equation (1.2) for $b = 2$:

$$\begin{cases} m_t + 2mu_x + m_x u - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.5}$$

This equation has a potential application in image mapping. Indeed, D. Holm, A. Trouve and L. Younes rederive this equation in their manuscript [25] which contains a result that connects the process of metamorphosis in image matching to the physical concept of order parameter in the theory of complex fluids.

To obtain the traveling wave solutions we set $u = u(x - ct)$ and $\rho = \rho(x - ct)$ where c is the speed of the wave. Then, easy manipulations show that

$$\begin{cases} -2c(u' - u''') + 2\kappa u' + 3(u^2)' + ((u')^2)' - (u^2)''' = (\rho^2)', \\ -c\rho' + (\rho u)' = 0, \end{cases} \tag{1.6}$$

where u' is the derivative of u with respect to $x - ct$. These equations are valid in the sense of distributions if $u \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$. We generalize a result of J. Lenells [9] about the location of non-smooth points of functions that are distributional solutions of the Camassa-Holm equation. In our case, we show that if a non-smooth function u is a distributional solution of the two component Camassa-Holm equation, then its non-smooth points only appear when u is at the level of the speed c of the wave. Also, we prove that every non-smooth function u which is a distributional traveling wave solution to the two component Camassa-Holm equation is in fact a distributional solution to the

Camassa-Holm equation provided that the set $u^{-1}(c)$ is of measure zero. An example of a smooth solution of the two component Camassa-Holm equation which is not a solution to the Camassa-Holm equation is presented in this chapter. Also, we present an example of a non-smooth solution u to the two component Camassa-Holm equation, with $u^{-1}(c)$ of non-zero measure, which is not a solution of the Camassa-Holm equation.

In Chapter 4, we review basic facts about yet another type of distributional solution, namely, the peakon solutions of the equation (1.1). In order to produce multipeakon solutions to equations (1.1) or (1.2) we consider m as a discrete measure [3] defined by

$$m = \sum_{j=1}^{n} m_j(t)\delta_{x_j(t)},$$

where m_j and x_j are smooth functions of time. This leads to a possible solution of the form

$$u = -\frac{\kappa}{b} + \frac{1}{2}\sum_{j=1}^{n} m_j e^{|x-x_j|}. \tag{1.7}$$

We show that u given by (1.7) is a distributional solution to (1.1) if and only if m_j and x_j satisfy the following ODE:

$$\begin{cases} \dot{x}_j = -\frac{\kappa}{b} + \frac{1}{2}\sum_{i=1}^{n} m_i e^{-|x_j-x_i|}, \\ \dot{m}_j = \frac{1}{2}(b-1)\sum_{i=1}^{n} m_j m_i Sgn(x_j-x_i)e^{-|x_j-x_i|}. \end{cases} \tag{1.8}$$

Further in this chapter, we introduce a notion of the weak Lax pair for the Camassa-Holm equation, which we subsequently show is the right framework to study its multipeakon solutions. It is shown that, given certain conditions, the compatibility condition for the distributional system of equations

$$\begin{cases} (\frac{1}{4} - D_x^2)\psi = zm\psi, \\ D_t\psi = -(\frac{1}{2z} + u)\psi_x + \frac{1}{2}u_x\psi, \qquad z \in \mathbb{C}, \end{cases} \tag{1.9}$$

is equivalent to the system of ODEs given by (1.8) when $b = 2$ (the Camassa-Holm equation).

In Chapter 5, we review a surprising connection between the peakon solutions to the Camassa-Holm equation and continued fractions. R. Beals, D. H. Sattinger and J. Szmigielski [22] investigated the connection between the peakon solutions to Camassa-Holm equation and the Stieltjes continued fractions. They used classical results of Stieltjes to obtain explicit formulas for the peakon solutions of the Camassa-Holm equation. What follows is a brief review of their work [22].

The transformation $y = \tanh(\frac{x}{2})$, turns the first equation of (1.4) into the following equation:

$$\phi_{yy} = -zg(y)\phi, \tag{1.10}$$

where $\phi = (1-y^2)^{\frac{1}{2}}\psi$, and $g(y) = \frac{4m}{(1-y^2)^2}$. On the other hand, it is known that small vibrations $u(y,t)$ of a string with mass density $g(y)$ are described by $u_{yy} = g(y)u_{tt}$. Using the separation of variables $u(y,t) = \phi(y)\tau(t)$, we obtain the eigenvalue problem (1.10). Therefore, if $g(y)$ is a positive discrete measure, the eigenvalue problem (1.10), where $g(y) = \sum_{j=1}^{n} g_j\delta_{y_j}$, with the initial conditions

$$\begin{cases} \phi(-1,z) = 0, \\ \phi_y(-1^+,z) = 1, \end{cases} \tag{1.11}$$

describes a discrete string consisting of point-masses at y_j with masses g_j, tied at the left end [4]. Furthermore, if the distance between y_{j-1} and y_j is denoted by l_j, then the Weyl function of the discrete string problem that is defined by

$$W(z) = \frac{\phi_y(1^-,z)}{\phi(1,z)}, \tag{1.12}$$

has the following continued fractions representation:

$$W(z) = \cfrac{1}{l_{n+1} + \cfrac{1}{zy_n + \cfrac{1}{l_n + \cfrac{1}{\ddots + \cfrac{1}{zy_1 + \cfrac{1}{l_1}}}}}}. \tag{1.13}$$

In this chapter, we study the properties of the odd and even convergents of the Weyl function. Since $\frac{W(z)}{z}$ is analytic at ∞, for z large enough we can write

$$\frac{W(z)}{z} = \sum_{j=0}^{\infty} \frac{c_j}{z^{j+1}}. \tag{1.14}$$

Certain orthogonality conditions are reviewed and explicit formulas for the numerators and the denominators of the convergents in terms of the c_js are given. Given the Weyl function by (1.14) we show how to write $W(z)$ in the form given in equation (1.13) and thus how to recover the string data.

The dissertation concludes with two appendices:

- Appendix A (Functions of bounded variations, absolutely continuous functions)
- Appendix B (Basic statements about distributions)

All the results of Chapter 3 are new. The concept of the weak Lax form description of multipeakons of the Camassa-Holm equation and its subsequent distributional implementation presented in Chapter 4 are new. The presentation of Chapter 2 was in the literature. However, the statement and the proof of Theorem 2 is my own. The thesis is self-contained.

Chapter 2

A tale of three equations

2.1 Variational Derivation of the Camassa-Holm equation

2.1.1 Governing equations of water waves

The Camassa-Holm equation [2] (also see [31])

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{2.1}$$

is a model for the unidirectional propagation of shallow water waves over a flat bottom, with $u(x,t)$ representing the water's free surface, and $\kappa \in \mathbb{R}$ being a parameter related to the critical shallow water speed. In what follows, this equation is derived by the application of variational methods. We assume (see [8] and [15]) that water is moving in a domain with a free surface given by $z = h_0 + \eta(x,t)$ where $h_0 > 0$ represents the surface of stationary water. The two-dimensional velocity of water is $(u, 0, v)$. Thus, no motion takes place in the y direction. Furthermore, we suppose that the fluid is only affected by the acceleration of gravity g and ignore the effects of surface tension. Since water is incompressible we assume that the density ρ is a constant. Let us denote the pressure by p. Applying Euler's equation (see [16]), we have

$$\begin{cases} u_t + uu_x + vu_z = -\frac{1}{\rho}p_x, \\ v_t + uv_x + vv_z = -\frac{1}{\rho}p_z - g. \end{cases} \tag{2.2}$$

By the equation of mass conservation [16] we have

$$u_x + v_z = 0. \tag{2.3}$$

The kinematic boundary conditions [16] are

$$\begin{cases} v = \eta_t + u\eta_x \quad \text{on} \quad z = h_0 + \eta(x,t), \\ \\ v = 0 \quad \text{on} \quad z = 0. \end{cases} \tag{2.4}$$

The dynamic boundary condition [16] states that the pressure on the free surface is equal to the constant atmospheric pressure p_0. Thus, we have

$$p = p_0 \quad \text{on} \quad z = h_0 + \eta(x,t). \tag{2.5}$$

We non-dimensionalize these equations using the undisturbed depth of water h_0 as the vertical length scale, a typical wavelength λ as a horizontal scale and a typical amplitude of the wave a relative to the undisturbed surface of water. An appropriate scale for the horizontal velocity is $\sqrt{gh_0}$. Consequently, the time scale will be $\frac{\lambda}{\sqrt{gh_0}}$ and the scale for vertical velocity will be $\frac{h_0\sqrt{gh_0}}{\lambda}$. Thus, we have defined the non-dimensional variables

$$\begin{gathered} x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}}t, \\ u \mapsto \sqrt{gh_0}u, \quad v \mapsto \frac{h_0\sqrt{gh_0}}{\lambda}v. \end{gathered} \tag{2.6}$$

Note that the above notation simply means x is replaced by λx, so that afterwards, the symbol x represents the non-dimensional variable. Also, it follows that we have the non-dimensional variables

$$\begin{gathered} u_t \mapsto \frac{gh_0}{\lambda}u_t, \quad v_t \mapsto \frac{gh_0^2}{\lambda^2}v_t, \quad u_x \mapsto \frac{\sqrt{gh_0}}{\lambda}u_x, \\ v_x \mapsto \frac{h_0\sqrt{gh_0}}{\lambda^2}v_x, \quad u_z \mapsto \frac{\sqrt{gh_0}}{h_0}u_z, \quad v_z \mapsto \frac{\sqrt{gh_0}}{\lambda}v_z. \end{gathered} \tag{2.7}$$

Now, we define the non-dimensional pressure. Consider the stationary water, that is, $u \equiv v \equiv 0$. Then, the first equation of (2.2) implies that p only depends on z and consequently, using the second equation of (2.2) we obtain the hydrostatic pressure at the depth z as follows:

$$p = p_0 + \rho g h_0 - \rho g z. \tag{2.8}$$

So, the hydrostatic pressure in terms of the non-dimensional variable z (by abuse of notation) would be

$$p = p_0 + \rho g h_0 (1 - z). \tag{2.9}$$

We define the non-dimensional pressure by adding the hydrostatic pressure in (2.9) to the non-dimensional variable $p \mapsto \rho g h_0 p$. Thus, we have

$$p \mapsto p_0 + \rho g h_0 (1 - z) + \rho g h_0 p, \tag{2.10}$$

$$p_x \mapsto \frac{\rho g h_0}{\lambda} p_x, \quad p_z \mapsto -\rho g + \rho g p_z. \tag{2.11}$$

Note that, adding the hydrostatic pressure simplifies our equations more efficiently because it removes the term $-g$ from the second equation of (2.2). In fact, using the non-dimensional variables, we obtain

$$\begin{aligned} &u_t + u u_x + v u_z = -p_x, \\ &\delta^2 (v_t + u v_x + v v_z) = -p_z, \\ &u_x + v_z = 0, \end{aligned} \tag{2.12}$$

where $\delta = \frac{h_0}{\lambda}$. On the surface of water, we have

$$v = \epsilon(\eta_t + u \eta_x) \quad \text{and} \quad p = \epsilon \eta \quad \text{on} \quad z = 1 + \epsilon \eta(x, t), \tag{2.13}$$

where $\epsilon = \frac{a}{h_0}$, while on the bottom,

$$v = 0 \quad \text{on} \quad z = 0. \tag{2.14}$$

Now, using the re-scaling

$$p \mapsto \epsilon p, \qquad (u, v) \mapsto \epsilon (u, v), \tag{2.15}$$

we get

$$\begin{aligned}
&u_t + \epsilon(uu_x + vu_z) = -p_x,\\
&\delta^2\big(v_t + \epsilon(uv_x + vv_z)\big) = -p_z,\\
&u_x + v_z = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.16)\\
&v = \eta_t + \epsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \epsilon\eta(x,t),\\
&v = 0 \quad \text{on} \quad z = 0.
\end{aligned}$$

Furthermore, to remove δ, we introduce the variables

$$x \mapsto \frac{\delta}{\sqrt{\epsilon}}, \quad t \mapsto \frac{\delta}{\sqrt{\epsilon}}, \quad v \mapsto \frac{\sqrt{\epsilon}}{\delta}. \tag{2.17}$$

Thus, the second equation changes to

$$\epsilon\big(v_t + \epsilon(uv_x + vv_z)\big) = -p_z.$$

Thus, the form of the governing equations of water waves is:

$$\boxed{\begin{aligned}
&u_t + \epsilon(uu_x + vu_z) = -p_x,\\
&\epsilon\big(v_t + \epsilon(uv_x + vv_z)\big) = -p_z,\\
&u_x + v_z = 0,\\
&v = \eta_t + \epsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \epsilon\eta(x,t),\\
&v = 0 \quad \text{on} \quad z = 0.
\end{aligned}} \tag{2.18}$$

For waves of small amplitude, that is, when $\epsilon \to 0$, we have

$$\begin{aligned}
&u_t + p_x = 0,\\
&p_z = 0,\\
&u_x + v_z = 0,\\
&v = \eta_t \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1,\\
&v = 0 \quad \text{on} \quad z = 0.
\end{aligned} \tag{2.19}$$

Since, $p_z = 0$, then p does not depend on z. Consequently, since $p = \eta(x,t)$ on $z = 1$, then $p = \eta(x,t)$ for any $0 \le z \le 1$. Thus, using $u_t + p_x = 0$, we get

$$u = -\int \eta_x(x,t)dt + \mathcal{F}(x,z), \tag{2.20}$$

where $\mathcal{F}(x,z)$ is an arbitrary function which only depends on x and z. Hence, we have

$$u_x = -\int \eta_{xx}(x,t)dt + \mathcal{F}_x(x,z).$$

Therefore, by $u_x + v_z = 0$, and the boundary condition $v = 0$ on $z = 0$, we have

$$v = z\int \eta_{xx}(x,t)dt - \mathcal{G}(x,z) + \mathcal{G}(x,0), \tag{2.21}$$

where $\mathcal{G}_z(x,z) = \mathcal{F}_x(x,z)$. Now, since $v = \eta_t$ on $z = 1$, we have

$$\eta_t = \int \eta_{xx}(x,t)dt - \mathcal{G}(x,1) + \mathcal{G}(x,0).$$

Hence,

$$\eta_{tt} = \eta_{xx}. \tag{2.22}$$

Equation (2.22) is the well-known wave equation $\eta_{tt} = c^2\eta_{xx}$ with $c = 1$. So, the general solution of (2.22) is

$$\eta(x,t) = f(x-t) + g(x+t),$$

where f and g are in $C^2(\mathbb{R})$. Now, let us restrict the problem to the waves which propagate in only one direction $\eta(x,t) = f(x-t)$. Then, by (2.20), we have

$$u = \eta + \mathcal{F}(x,z),$$

and by (2.21), we get

$$v = -z\eta_x - \mathcal{G}(x,z) + \mathcal{G}(x,0).$$

Thus, the boundary condition $v = \eta_t$ on $z = 1$, implies that $\mathcal{G}(x,1) = \mathcal{G}(x,0)$. So, we see that the time evolution of u and v in the shallow water problem is entirely determined by the evolution of the function $\eta(x,t)$ which represents the displacement of the free surface from the undisturbed state.

If the fluid is irrotational, the vorticity is zero, that is $u_z - v_x = 0$. If we non-dimensionalize this equation by (2.7), we obtain $u_z = \delta^2 v_x$. We re-scale again, using (2.17) to get $u_z = \epsilon v_x$. This shows that, for small amplitude waves, that is, $\epsilon \to 0$, we have $u_z = 0$. Hence, (2.20) and (2.21) will be replaced by

$$u = -\int \eta_x(x,t)dt + \mathcal{F}(x), \tag{2.23}$$

and

$$v = -zu_x = z\Big(\int \eta_{xx}(x,t)dt - \mathcal{F}'(x)\Big). \tag{2.24}$$

Note that, to get the second equation we have also used the boundary condition that $v = 0$ whenever $z = 0$. Now, since $v = \eta_t$ on $z = 1$, we have

$$\eta_t = \int \eta_{xx}(x,t)dt - \mathcal{F}'(x).$$

Thus, $\eta_{tt} = \eta_{xx}$. Again, we consider the traveling wave $\eta(x,t) = f(x-t)$ as a solution. Then, (2.23) and (2.24) imply that

$$u = \eta + \mathcal{F}(x), \qquad v = -z\big(\eta_x + \mathcal{F}'(x)\big).$$

Since, $v = \eta_t$ on $z = 1$ then $\mathcal{F}'(x) = 0$. Thus, $\mathcal{F}(x)$ is a constant. Hence, for the case of irrotational fluid we have the following solution:

$$\eta(x,t) = f(x-t), \qquad u = \eta + c_0, \qquad v = -z\eta_x, \tag{2.25}$$

where c_0 is a constant. The equation $u = \eta + c_0$ implies that if the surface of water is undisturbed then every particle of water moves at the speed of c_0. We are going to use this fact in the next section.

2.1.2 Variational computations

This section is primarily based on a paper of A. Constantin [8]. In Lagrangian formalism the motion of a fluid is described by a family of time-dependent diffeomorphisms $\gamma(t,\cdot)$ on the ambient space M (see [1]). Since $v = \eta_t$ on $z = 1$, every particle on the free surface of water does not leave the surface, so we can let M be a one-dimensional space. For the sake of presentation we assume $M = \mathbb{S}$, the unit circle. The material velocity is defined by $\gamma_t(t,x)$, while the spatial velocity is given by $w(t,X) = \gamma_t(x,t)$, where $X = \gamma(t,x)$, that is, $w(t,\cdot) = \gamma_t \circ \gamma^{-1}$. In terms of w, we have the Eulerian description, while in terms of γ_t we have the Lagrangian description of the motion. Let $\mathcal{D}$ be the Lie group of smooth orientation-preserving diffeomorphisms of $\mathbb{S}$ (see [30] for results specific to the Camassa-Holm equation). In Lagrangian description, the equation of motion is the equation satisfied by a critical point of a certain functional (called the action) defined on all paths $\{\gamma(t,\cdot), 0 \le t \le T\}$ in $\mathcal{D}$, having fixed endpoints. Applying equation (2.25) and using the fact that $v = -u_x$ on the surface of water where $z = 1$, we can approximately ($\epsilon \to 0$) compute the kinetic energy on the surface over one period as follows:

$$K = \frac{1}{2}\int_{\mathbb{S}}(u^2+v^2)dx \approx \frac{1}{2}\int_{\mathbb{S}}(u^2+u_x^2)dx \tag{2.26}$$

to this order of approximation. Note that if we replace $\gamma(t,\cdot)$ by $\gamma(t,\cdot)\circ\psi(\cdot)$, for a fixed time independent $\psi \in \mathcal{D}$, then the spatial velocity is unchanged. To see this we write

$$\big(\gamma(t,\cdot)\circ\psi(\cdot)\big)_t \circ \big(\gamma(t,\cdot)\circ\psi(\cdot)\big)^{-1} = \gamma_t(t,\cdot)\circ\psi(\cdot)\circ\psi^{-1}(\cdot)\circ\gamma^{-1}(t,\cdot) = \gamma_t\circ\gamma^{-1}. \tag{2.27}$$

For small surface elevations the potential energy is negligible, so K is transformed to a right-invariant Lagrangian. Hence, the action on a path $\gamma(t,\cdot) \in \mathcal{D}$ where $t \in [0,T]$, is given by

$$\mathfrak{a} = \frac{1}{2}\int_0^T \int_{\mathbb{S}} \{\big(\gamma_t\circ\gamma^{-1}\big)^2 + \big(\partial_x(\gamma_t\circ\gamma^{-1})\big)^2\} dxdt. \tag{2.28}$$

Suppose that a path $\gamma(t,\cdot)$, $t \in [0,T]$, parameterized by the arc length is a critical point of the action $\mathfrak{a}$ in the space of paths with fixed endpoints. Then we have

$$\frac{d}{d\epsilon}\mathfrak{a}(\gamma+\epsilon\phi)|_{\epsilon=0}=0, \tag{2.29}$$

for every path $\phi(t,\cdot)$, $t \in [0,T]$, with fixed endpoints at zero, that is, $\phi(0,\cdot)=0=\phi(T,\cdot)$. So, $\gamma+\epsilon\phi$ is a small variation of γ in $\mathcal{D}$. Thus, from (2.28) and (2.29), we have

$$\begin{aligned}\int_0^T\int_{\mathbb{S}}\{&(\gamma_t\circ\gamma^{-1})\frac{d}{d\epsilon}[(\gamma_t+\epsilon\phi_t)\circ(\gamma+\epsilon\phi)^{-1}]|_{\epsilon=0}\\&+\partial_x(\gamma_t\circ\gamma^{-1})\frac{d}{d\epsilon}[\partial_x\big((\gamma_t+\epsilon\phi_t)\circ(\gamma+\epsilon\phi)^{-1}\big)]|_{\epsilon=0}\}dxdt=0.\end{aligned} \tag{2.30}$$

Lemma 1.

$$\frac{d}{d\epsilon}[(\gamma+\epsilon\phi)^{-1}]|_{\epsilon=0}=-\frac{\phi\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}}. \tag{2.31}$$

Proof. Set $m(\epsilon)=(\gamma+\phi\epsilon)^{-1}$. We have

$$I=(\gamma+\epsilon\phi)\circ m(\epsilon)=\gamma\circ m(\epsilon)+\epsilon\phi\circ m(\epsilon).$$

Taking the derivative of this equation with respect to ϵ, we get

$$0=\big(\gamma_x\circ m(\epsilon)\big)\frac{d}{d\epsilon}m(\epsilon)+\phi\circ m(\epsilon)+\epsilon\big(\phi_x\circ m(\epsilon)\big)\frac{d}{d\epsilon}m(\epsilon).$$

Setting $\epsilon=0$, we have

$$(\gamma_x\circ\gamma^{-1})\frac{d}{d\epsilon}m(\epsilon)|_{\epsilon=0}+\phi\circ\gamma^{-1}=0.$$

Hence,

$$\frac{d}{d\epsilon}[(\gamma+\epsilon\phi)^{-1}]|_{\epsilon=0}=-\frac{\phi\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}}.$$

$\square$

Lemma 2.

$$\partial_x(\gamma_t\circ\gamma^{-1})=\frac{\gamma_{tx}\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}}. \tag{2.32}$$

Proof. We know that

$$\partial_x(\gamma^{-1}) = \frac{1}{\gamma_x \circ \gamma^{-1}}.$$

Therefore, we have

$$\partial_x(\gamma_t \circ \gamma^{-1}) = (\gamma_{tx} \circ \gamma^{-1})\partial_x(\gamma^{-1}) = \frac{\gamma_{tx} \circ \gamma^{-1}}{\gamma_x \circ \gamma^{-1}}.$$

□

Lemma 3.

$$\frac{d}{d\epsilon}[(\gamma_t + \epsilon\phi_t) \circ (\gamma + \epsilon\phi)^{-1}]|_{\epsilon=0} = \phi_t \circ \gamma^{-1} - (\phi \circ \gamma^{-1})\partial_x(\gamma_t \circ \gamma^{-1}). \tag{2.33}$$

Proof. We can write

$$\begin{aligned}
\frac{d}{d\epsilon}[(\gamma_t + \epsilon\phi_t) \circ (\gamma + \epsilon\phi)^{-1}]|_{\epsilon=0} &= \frac{d}{d\epsilon}[\gamma_t \circ (\gamma + \epsilon\phi)^{-1} + \epsilon\phi_t \circ (\gamma + \epsilon\phi)^{-1}]|_{\epsilon=0} \\
&= (\gamma_{tx} \circ \gamma^{-1})\frac{d}{d\epsilon}(\gamma + \epsilon\phi)^{-1}|_{\epsilon=0} + \phi_t \circ \gamma^{-1} \\
&= (\gamma_{tx} \circ \gamma^{-1})(-\frac{\phi \circ \gamma^{-1}}{\gamma_x \circ \gamma^{-1}}) + \phi_t \circ \gamma^{-1} \\
&= \phi_t \circ \gamma^{-1} - (\phi \circ \gamma^{-1})\partial_x(\gamma_t \circ \gamma^{-1}).
\end{aligned}$$

□

Lemma 4.

$$\begin{aligned}
\partial_t(\phi \circ \gamma^{-1}) &= \phi_t \circ \gamma^{-1} + (\phi_x \circ \gamma^{-1})\partial_t(\gamma^{-1}) \\
&= \phi_t \circ \gamma^{-1} - (\gamma_t \circ \gamma^{-1})\partial_x(\phi \circ \gamma^{-1}).
\end{aligned} \tag{2.34}$$

Proof. The first equation is obtained by the application of the chain rule to a function of the type $f\big(t, g(t)\big)$, that is,

$$\frac{d}{dt}f\big(t, g(t)\big) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial g}\frac{dg}{dt}.$$

To prove the second equation, we note that

$$\partial_x(\phi \circ \gamma^{-1}) = (\phi_x \circ \gamma^{-1})\partial_x(\gamma^{-1}) = \frac{\phi_x \circ \gamma^{-1}}{\gamma_x \circ \gamma^{-1}}.$$

On the other hand, taking the derivative of $\gamma \circ \gamma^{-1} = I$ with respect to t and applying the chain rule similar to the first equation, we obtain

$$\gamma_t \circ \gamma^{-1} + (\gamma_x \circ \gamma^{-1})\partial_t(\gamma^{-1}) = 0.$$

Thus, the Lemma follows. $\square$

Lemma 5.

$$\frac{d}{d\epsilon}[(\gamma_t + \epsilon\phi_t) \circ (\gamma + \epsilon\phi)^{-1}]|_{\epsilon=0}$$
$$= \partial_t(\phi \circ \gamma^{-1}) + (\gamma_t \circ \gamma^{-1})\partial_x(\phi \circ \gamma^{-1}) - (\phi \circ \gamma^{-1})\partial_x(\gamma_t \circ \gamma^{-1}). \quad (2.35)$$

Proof. Combining (2.33) and (2.34), we can write

$$\frac{d}{d\epsilon}[(\gamma_t + \epsilon\phi_t) \circ (\gamma + \epsilon\phi)^{-1}]|_{\epsilon=0} = \phi_t \circ \gamma^{-1} - (\phi \circ \gamma^{-1})\partial_x(\gamma_t \circ \gamma^{-1})$$
$$= \partial_t(\phi \circ \gamma^{-1}) + (\gamma_t \circ \gamma^{-1})\partial_x(\phi \circ \gamma^{-1}) - (\phi \circ \gamma^{-1})\partial_x(\gamma_t \circ \gamma^{-1}). \quad (2.36)$$

$\square$

Lemma 6.

$$\frac{d}{d\epsilon}[\partial_x\big((\gamma_t + \epsilon\phi_t) \circ (\gamma + \epsilon\phi)^{-1}\big)]|_{\epsilon=0}$$
$$= \partial_{tx}(\phi \circ \gamma^{-1}) + (\gamma_t \circ \gamma^{-1})\partial_x^2(\phi \circ \gamma^{-1}) - (\phi \circ \gamma^{-1})\partial_x^2(\gamma_t \circ \gamma^{-1}). \quad (2.37)$$

Proof. It follows from the previous Lemma. $\square$

Theorem 1. *In a periodic irrotational unidirectional shallow water flow, the motion of a particle is the critical point of the action given by* (2.28) *if and only if the horizontal velocity component* $u(x,t)$ *satisfies the Camassa-Holm equation* (2.1) *with* $\kappa = 0$.

Proof. We denote $\gamma_t \circ \gamma^{-1}$ by u. Thus, from (2.30), (2.35) and (2.37) it follows that

$$\int_0^T \int_{\mathbb{S}} \{u[\partial_t(\phi \circ \gamma^{-1}) + u\partial_x(\phi \circ \gamma^{-1}) - (\phi \circ \gamma^{-1})u_x]$$
$$+ u_x[\partial_{tx}(\phi \circ \gamma^{-1}) + u\partial_x^2(\phi \circ \gamma^{-1}) - (\phi \circ \gamma^{-1})u_{xx}]\}dxdt = 0. \quad (2.38)$$

Since ϕ has its endpoints at zero and u is a smooth periodic function, integration by parts yields

$$-\int_0^T \int_{\mathbb{S}} (\phi \circ \gamma^{-1})[u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx}]dxdt = 0. \quad (2.39)$$

This completes the proof. □

Theorem 2. *In a periodic irrotational unidirectional shallow water flow, the displacement of the free surface of water, denoted by $\eta(x,t)$, (x being measured in a coordinate system moving horizontally at the speed of c_0) satisfies the Camassa-Holm equation* (2.1) *with $\kappa = 2c_0$, where c_0 is the velocity of every particle when the water is undisturbed.*

Proof. To avoid any ambiguity we rename the variables x and t in the Camassa-Holm equation for the horizontal velocity. Thus, we can write

$$u_\tau - u_{\tau\chi\chi} + 3uu_\chi - 2u_\chi u_{\chi\chi} - uu_{\chi\chi\chi} = 0. \quad (2.40)$$

Now consider the Galilean transformation

$$x = \chi - c_0\tau, \qquad t = \tau. \quad (2.41)$$

Thus, we have

$$u_\tau = -c_0 u_x + u_t, \qquad u_\chi = u_x. \quad (2.42)$$

Now, substituting these into the Camassa-Holm equation, we obtain

$$-c_0 u_x + u_t + c_0 u_{xxx} - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \quad (2.43)$$

substituting $u = \eta + c_0$ from the equation (2.25) we obtain

$$\eta_t - \eta_{txx} + 2c_0\eta_x + 3\eta\eta_x - 2\eta_x\eta_{xx} - \eta\eta_{xxx} = 0.$$

$\square$

Remark. The first derivation of the Camassa-Holm equation in the group context was done by G. Misiolek [30] who used a one dimensional extension of the group of diffeomorphisms.

2.2 The Lax pair perspective

P. D. Lax [26] introduced a method of solving the KdV equation which is a generalization of the inverse scattering method and it can be applied to solve other partial differential equations. Consider the following system of equations:

$$\begin{aligned} L\phi &= z\phi, \\ \phi_t &= A\phi, \end{aligned} \tag{2.44}$$

where L is an operator, z is a spectral parameter and A describes the time evolution of the eigenfunction ϕ. Taking the partial derivative of the first equation in terms of t and applying the operator L to the second equations, we obtain

$$\begin{aligned} &L_t\phi + L\phi_t = z_t\phi + z\phi_t, \\ &L\phi_t = LA\phi. \end{aligned}$$

Also, we have

$$z\phi_t = zA\phi = Az\phi = AL\phi.$$

Thus, we have

$$(L_t + LA - AL)\phi = z_t\phi.$$

Hence, in order to get nontrivial eigenfunctions, we must have $L_t + LA - AL = 0$ if and only if the spectral parameter z is independent of time. The equation

$$L_t - [A, L] = 0, \tag{2.45}$$

where $[A, L] = AL - LA$, is called the Lax equation.

2.2.1 Formal pseudodifferential symbols and the KdV equation

In general, there is no method of finding a Lax pair for a given nonlinear partial differential equation. However, starting with the operator L, one can try to construct a class of operators like A such that the Lax equation produces a nonlinear partial differential equation for every operator A in the class. In what follows, we will try to find a method to calculate A with respect to a given L, so that the Lax pair formula can produce KdV and other similar equations [27]. First, we prove the following:

Lemma 7. *The KdV equation $u_t - 6uu_x + u_{xxx} = 0$ is equivalent to the Lax pair equation*

$$\frac{\partial}{\partial t} L = [A, L] = AL - LA, \tag{2.46}$$

where

$$\begin{aligned} L &= -D^2 + u, \\ A &= -4D^3 + 3(uD + Du), \end{aligned} \tag{2.47}$$

where $D = \partial/\partial x$.

Proof. We compute both AL and LA as follows:

$$\begin{aligned} AL &= \big(-4D^3 + 3(uD + Du)\big)(-D^2 + u) \\ &= 4D^5 - 4(u_{xxx} + 3u_{xx}D + 3u_xD^2 + uD^3) \\ &- 3uD^3 + 3u(u_x + uD) - 3(u_xD^2 + uD^3) + 3(2uu_x + u^2D) \\ &= 4D^5 - 4u_{xxx} - 12u_{xx}D - 15u_xD^2 - 10uD^3 + 9uu_x + 6u^2D. \end{aligned} \tag{2.48}$$

$$
\begin{aligned}
LA &= (-D^2+u)\big(-4D^3+3(uD+Du)\big) \\
&= 4D^5 - 3(u_{xx}D + 2u_xD^2 + uD^3) - 3(u_{xxx} + 3u_{xx}D + 3u_xD^2 + uD^3) \\
&- 4uD^3 + 3(u^2D + uu_x + u^2D) \\
&= 4D^5 - 12u_{xx}D - 15u_xD^2 - 10uD^3 - 3u_{xxx} + 6u^2D + 3uu_x.
\end{aligned} \tag{2.49}
$$

Thus, the assertion follows. □

We call the highest power of D in every operator, the order of the operator. Since $ord(L) = 2$ and $ord(\frac{\partial}{\partial t}L)$=0, we should look for a polynomial operator A such that $ord(([A, L]) = 0$.

Definition 1. A pseudodifferential symbol M is given by

$$M = \sum_{i=-\infty}^{n} a_i(x)D^i, \tag{2.50}$$

where each $a_i : \mathbb{R} \to \mathbb{C}$ is a smooth function of x and t, and n is an integer.

The product of pseudodifferential symbols can be naturally defined using the combination of the usual Leibniz Rule and

$$D^{-1}f = \sum_{k=0}^{\infty}(-1)^k f^{(k)} D^{-1+k}.$$

Let M_+ denote the part of M which has no negative power of D. Also, set

$$M_- = M - M_+$$

.

Lemma 8. *If a pseudodifferential symbol M commutes with L, then*

$$ord([M_+, L]) = 0.$$

Moreover $[M_+, L] = a(x)$, where $a(x)$ is a smooth function.

Proof. Suppose $ord(M) \geq 0$ and $[M, L] = 0$. Then we have $[M_+, L] = -[M_-, L]$. Therefore,

$$ord([M_+, L]) \leq ord(M_-) + ord(L) - 1 \leq 0.$$

On the other hand, since M_+ has no terms with a negative power of D, then

$$ord([M_+, L]) \geq 0.$$

Hence, $ord([M_+, L]) = 0$. □

Lemma 9. *Set $L = -D^2 + u$ and suppose u is a smooth function. There exists a unique pseudodifferential symbol K, such that $K \cdot K = L$.*

Proof. It is obvious that $ord(K) = 1$ and the highest order term is of the form iD, where $i^2 = -1$. Thus, we can write

$$K = \sum_{n=-\infty}^{0} a_n(x) D^n + iD.$$

We proceed by induction on the order of terms. So, by induction hypothesis we assume that the coefficients $a_n(x)$ for $-k \leq n \leq 0$ are all determined uniquely. Set

$$P_{-k} = \sum_{n=-k}^{1} a_n(x) D^n.$$

We write

$$\left(\sum_{n=-\infty}^{-k-2} a_n(x)D^n + a_{-k-1}(x)D^{-k-1} + P_{-k}\right)\left(\sum_{n=-\infty}^{-k-2} a_n(x)D^n + a_{-k-1}(x)D^{-k-1} + P_{-k}\right) = L.$$

we observe that in the above product, the only contributions to a term of order $-k$ can appear in the following products

$$a_{-k-1}(x)D^{-k-1}P_{-k}, \qquad P_{-k}a_{-k-1}D^{-k-1}, \qquad P_{-k}P_{-k}.$$

Since P_{-k} is known by induction hypothesis, then we can uniquely determine the coefficient a_{-k-1}. □

Proposition 1. *For every positive integer k, $[L_+^{k/2}, L]$ is a smooth function.*

Proof. For every positive integer k we have $[L^{k/2}, L] = 0$. Therefore, by Lemma 8, $ord([L_+^{k/2}, L]) = 0$. $\square$

Now, let us try $k = 1$. We have

$$[L_+^{1/2}, L] = -iD^3 + i(u_x + uD) + iD^3 - iuD = iu_x.$$

Thus, the Lax pair formula becomes

$$u_t = u_x.$$

For $k = 3$, we have

$$L^{3/2} = L \cdot L^{1/2} = i(-D^2 + u)(D + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \cdots).$$

So in order to compute $L_+^{3/2}$, we only need to know a_0, a_{-1} and a_{-2}. We write

$$(D + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \cdots)(D + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \cdots) = -L.$$

The contribution to the term of order 1 only appears in the product

$$Da_0 + a_0 D = a_0' + 2a_0 D.$$

Thus, $a_0 = 0$. Therefore, the contribution to the term of order 0 appears only in the product

$$Da_{-1}D^{-1} + a_{-1}D^{-1}D = a_{-1}'D^{-1} + 2a_{-1}.$$

So, $a_{-2} = -\frac{u}{2}$. Finally, for the contributions to the term of order -1 we need to consider the term $ia_{-1}'D^{-1}$ and the product

$$Da_{-2}D^{-2} + a_{-2}D_{-2}D^{-2}D = a_{-2}'D^{-2} + 2a_{-2}D^{-1}.$$

Thus, we must have $2a_{-2} + a'_{-1} = 0$. Therefore, $a_{-2} = \frac{u'}{4}$. Hence, we write

$$
\begin{aligned}
L^{3/2} = L \cdot L^{1/2} &= i(-D^2 + u)(D - \frac{u}{2}D^{-1} + \frac{u'}{4}D^{-2} + \cdots) \\
&= i(-D^3 + \frac{u}{2}D + u' + \frac{u''}{2}D^{-1} - \frac{u'}{4} - \frac{u''}{2}D^{-1} - \frac{u'''}{4}D^{-2} \\
&+ uD - \frac{u^2}{2}D^{-1} + \frac{uu'}{4}D^{-2} + \cdots).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L_+^{3/2} &= i(-D^3 + \frac{3}{2}uD + \frac{3}{4}u') \\
&= \frac{i}{4}\big(-4D^3 + 3(uD + Du)\big).
\end{aligned}
\tag{2.51}
$$

Thus, the Lax pair formula with $A = -4iL_+^{3/2}$ becomes the KdV equation.

2.2.2 Lax pair representations of the Camassa-Holm, the two component Camassa-Holm and the Degasperis-Procesi equations

Suppose u and m are smooth functions of x and t. Consider the operators

$$
L(z) = D^2 + zm - \frac{1}{4}, \qquad A(z) = -(\frac{1}{2z} + u)D + \frac{u_x}{2}, \quad z \in \mathbb{C}.
\tag{2.52}
$$

Proposition 2. *$u(x,t)$ satisfies the Camassa-Holm equation (2.1) with $\kappa = 0$ if and only if*

$$
\frac{\partial}{\partial t}L(z) = [A(z), L(z)] - 2u_x L(z),
\tag{2.53}
$$

for at least two distinct values of z.

Proof. We compute AL and LA.

$$
\begin{aligned}
AL = &-(\frac{1}{2z} + u)D^3 - (\frac{1}{2} + zu)(m_x + mD) + \frac{1}{4}(\frac{1}{2z} + u)D \\
&+ \frac{u_x}{2}D^2 + \frac{zu_x m}{2} - \frac{u_x}{8}.
\end{aligned}
\tag{2.54}
$$

$$
\begin{aligned}
LA = &-\big(u_{xx}D + 2u_xD^2 + (\frac{1}{2z} + u)D^3\big) + \frac{1}{2}(u_{xxx} + 2u_{xx}D + u_xD^2) \\
&- zm(\frac{1}{2z} + u)D + \frac{zmu_x}{2} + \frac{1}{4}(\frac{1}{2z} + u)D - \frac{u_x}{8}.
\end{aligned}
\tag{2.55}
$$

Thus, the equation (2.53) is equivalent to

$$z(m_t + um_x + 2u_x m) + \frac{1}{2}(m_x - u_x + u_{xxx}) = 0. \tag{2.56}$$

$\square$

To obtain the two component Camassa-Holm equation, we need to modify the operator L. Thus, we set

$$L_1(z) = L(z) - z^2\rho^2, \quad z \in \mathbb{C}, \tag{2.57}$$

where ρ is a smooth function of x and t.

Proposition 3. *$u(x,t)$ and $\rho(x,t)$ satisfy the two component Camassa-Holm equation*

$$\begin{cases} m_t + 2mu_x + m_x u - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \\ m_x = u_x - u_{xxx}, \end{cases} \tag{2.58}$$

if and only if

$$\frac{\partial}{\partial t} L_1(z) = [A, L_1(z)] - 2u_x L_1(z), \tag{2.59}$$

for at least three distinct values of z.

Proof. From equation (2.59) we have

$$L_t - z^2(\rho^2)_t = [A, L] - 2u_x L + \frac{z}{2}(\rho^2)_x + z^2 u(\rho^2)_x + 2z^2 u_x \rho^2.$$

Hence,

$$-z^2\big((\rho^2)_t + u(\rho^2)_x + 2u_x\rho^2\big) + z\big(m_t + um_x + 2u_x m - \frac{1}{2}(\rho^2)_x\big) + \frac{1}{2}(m_x - u_x + u_{xxx}) = 0.$$

$\square$

The Lax pair operators for the Degasperis-Procesi equation are given by (see [17])

$$L(z) = D - D^3 - zm, \qquad A(z) = -\frac{1}{z}D^2 - uD + (u_x + \frac{1}{z}). \tag{2.60}$$

Proposition 4. *$u(x,t)$ satisfies the Degasperis-Procesi equation*

$$\begin{cases} m_t + 3mu_x + m_x u = 0, \\ \\ m_x = u_x - u_{xxx}, \end{cases} \tag{2.61}$$

if and only if

$$\frac{\partial}{\partial t}L(z) = [A(z), L(z)] - 3u_x L(z), \tag{2.62}$$

for at least one nonzero value of z.

Proof. We have

$$\begin{aligned} AL = &-\frac{1}{z}D^3 + \frac{1}{z}D^5 + m_{xx} + 2m_x D + mD^2 \\ &- uD^2 + uD^4 + zu(m_x + mD) \\ &+ (u_x + \frac{1}{z})D - (u_x + \frac{1}{z})D^3 - m(zu_x + 1), \end{aligned}$$

$$\begin{aligned} LA = &-\frac{1}{z}D^3 - (u_x D + uD^2) + u_{xx} + (u_x + \frac{1}{z})D \\ &+ \frac{1}{z}D^5 + (u_{xxx}D + 3u_{xx}D^2 + 3u_x D^3 + uD^4) \\ &- \Big(u_{xxxx} + 3u_{xxx}D + 3u_{xx}D^2 + (u_x + \frac{1}{z})D^3\Big) \\ &+ mD^2 + zumD - m(zu_x + 1). \end{aligned}$$

Thus,

$$[A, L] - 3u_x L = 2(m_x - u_x + u_{xxx})D + m_{xx} - u_{xx} + u_{xxxx} + z(um_x + 3u_x m).$$

Hence, the assertion follows. $\square$

Chapter 3

Traveling wave solutions to the two component Camassa-Holm equation

R. Camassa and D. Holm [2] discovered that the equation (2.1) has non-smooth solitary waves that retain their individual characteristics after the interaction and eventually emerge with their original shapes and speeds. They called these solutions multipeakons. The simplest of them has the form of a traveling wave. The traveling wave solutions of the Camassa-Holm equation have been subsequently classified by J. Lenells [9]. An alternative, and useful for generalizations form of this equation is

$$m_t + um_x + 2mu_x = 0, \tag{3.1}$$

where $m = u - u_{xx} + \frac{1}{2}\kappa$.

One such generalization has been introduced by M. Chen, S. Liu and Y. Zhang [12]:

$$\begin{cases} m_t + um_x + 2mu_x - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{3.2}$$

The traveling wave solutions are obtained by setting $u = u(x - ct)$ and $\rho = \rho(x - ct)$. In this case, easy manipulations show that (3.2) can be written as follows

$$\begin{cases} -2c(u' - u''') + 2\kappa u' + 3(u^2)' + ((u')^2)' - (u^2)''' = (\rho^2)', \\ -c\rho' + (\rho u)' = 0. \end{cases} \tag{3.3}$$

These equations are valid in the sense of distributions, if $u \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$. Indeed, for a given function ρ, if $(\rho^2)' \in \mathcal{D}'(\mathbb{R})$, then $\rho \in L^2_{loc}(\mathbb{R})$.

Since every distribution has a primitive which is a distribution (see [18]), we can integrate and then rewrite

$$\begin{cases} (v^2)'' = (v')^2 + p(v) - \rho^2, \\ \\ \rho v = B_1. \end{cases} \tag{3.4}$$

where $v = u - c$ and $p(v) = 3v^2 + (2\kappa + 4c)v + K$ for some constants K and B_1.

Definition 2. A pair of functions (u, ρ) where $u \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$, is called a traveling wave solution for (3.2) if u and ρ satisfy (3.4) in the sense of distributions.

The following Lemma is due to J. Lenells [9].

Lemma 10. *Let $p(v)$ be a polynomial with real coefficient. Assume that $v \in H^1_{loc}(\mathbb{R})$ satisfies*

$$(v^2)'' = (v')^2 + p(v) \quad \text{in} \ \ \mathcal{D}'(\mathbb{R}). \tag{3.5}$$

Then

$$v^k \in C^j(\mathbb{R}) \quad \text{for} \ \ k \geq 2^j. \tag{3.6}$$

In our case, we have the following generalization:

Lemma 11. *Let $p(v)$ be a polynomial with real coefficients. Assume that $v \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$ satisfy the following system in $\mathcal{D}'\big(\mathbb{R}\big)$:*

$$\begin{cases} (v^2)'' = (v')^2 + p(v) - \rho^2, \\ \\ \rho v = B_1. \end{cases} \tag{3.7}$$

Then

$$v^k \in C^j\big(\mathbb{R}\big) \quad \text{for} \ \ k \geq 2^j \quad \text{and} \ \ j \geq 0. \tag{3.8}$$

Proof. Since $v \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$, (3.7) implies that $(v^2)'' \in L^1_{loc}(\mathbb{R})$. Therefore, $(v^2)'$ is absolutely continuous (see Appendix A) and $v^2 \in C^1(\mathbb{R})$. Also, since $v \in H^1_{loc}(\mathbb{R})$, then v is absolutely continuous and we can claim

$$(v^k)' = \frac{k}{2}\big(v^{k-2}(v^2)'\big) \quad \text{for} \quad k \geq 3.$$

To see why the claim is true, we first note that in fact, it is obviously true if k is an even number. Also, note that since the first derivative of an absolutely continuous function exists almost everywhere, in taking the first derivative of the product of two absolutely continuous functions we can use the Leibniz Rule almost everywhere. Now, if k is an odd number, let us say $k = 2n+1$, then we can write

$$\begin{aligned}(v^k)' &= (v^{2n}v)' = v(v^{2n})' + v'v^{2n}\\ &= v(nv^{2(n-1)})(v^2)' + \frac{1}{2}(v^2)'v^{2n-1}\\ &= \frac{k}{2}v^{k-2}(v^2)'.\end{aligned}$$

Thus, we have

$$\begin{aligned}(v^k)'' &= \frac{k}{2}\big(v^{k-2}(v^2)'\big)'\\ &= \frac{k}{2}\big((v^{k-2})'(v^2)' + v^{k-2}(v^2)''\big)\\ &= k(k-2)v^{k-2}(v')^2 + \frac{k}{2}v^{k-2}(v^2)'' \quad \text{for} \quad k \geq 3.\end{aligned}$$

Substituting from (3.7) we have

$$\begin{aligned}(v^k)'' &= k(k-2)v^{k-2}(v')^2 + \frac{k}{2}v^{k-2}\big((v')^2 + p(v) - \rho^2\big)\\ &= k(k-\frac{3}{2})v^{k-2}(v')^2 + \frac{k}{2}v^{k-2}p(v) - \frac{k}{2}B_1v^{k-3}\rho.\end{aligned} \tag{3.9}$$

For $k \geq 3$ the right hand side of the above equation belongs to $L^1_{loc}(\mathbb{R})$. Therefore

$$v^k \in C^1\big(\mathbb{R}\big) \quad \text{for} \quad k \geq 2. \tag{3.10}$$

Thus, the assertion holds for $j = 1$. We proceed by induction on j. Suppose

$$v^k \in C^{j-1}\big(\mathbb{R}\big) \quad \text{for} \quad k \geq 2^{j-1} \quad \text{and} \quad j \geq 2.$$

Then for $k \geq 2^j$ we have

$$\begin{aligned} v^{k-2}(v')^2 &= \frac{1}{2^{j-1}}(2^{j-1}v^{2^{j-1}-1}v')\frac{1}{k-2^{j-1}}((k-2^{j-1})v^{k-2^{j-1}-1}v') \\ &= \frac{1}{2^{j-1}(k-2^{j-1})}(v^{2^{j-1}})'(v^{k-2^{j-1}})' \in C^{j-2}(\mathbb{R}). \end{aligned} \tag{3.11}$$

Also, we have $v^{k-2}p(v) \in C^{j-1}(\mathbb{R})$ and $v^{k-3}\rho = B_1 v^{k-4} \in C^{j-2}(\mathbb{R})$. Therefore the right hand side of equation (3.9) belongs to $C^{j-2}(\mathbb{R})$. Hence,

$$v^k \in C^j\big(\mathbb{R}\big) \quad \text{for} \quad k \geq 2^j.$$

$\square$

Remark. Lemma (11) implies that v' is possibly discontinuous only at points where $v = 0$. In fact, a much stronger result is true:

Corollary 1. If $v \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$ satisfy (3.7) in $\mathcal{D}'\big(\mathbb{R}\big)$, then

$$v \in C^\infty\big(\mathbb{R} \setminus v^{-1}(0)\big)$$

and

$$\rho \in C^\infty\big(\mathbb{R} \setminus v^{-1}(0)\big).$$

Proof. Suppose $k \geq 2$. Then $v^k \in C^1(\mathbb{R})$. Therefore

$$kv^{k-1}v' = (v^k)' \in C(\mathbb{R}).$$

This implies that $v' \in C\big(\mathbb{R} \setminus v^{-1}(0)\big)$. Thus, $v \in C^1\big(\mathbb{R} \setminus v^{-1}(0)\big)$.

Now, assume that $v \in C^j\big(\mathbb{R} \setminus v^{-1}(0)\big)$ for $j \geq 1$. For $k \geq 2^{j+1}$, we have $v^k \in C^{j+1}(\mathbb{R})$. Therefore

$$kv^{k-1}v' = (v^k)' \in C^j(\mathbb{R}).$$

This shows that $v' \in C^j\big(\mathbb{R} \setminus v^{-1}(0)\big)$. Hence, $v \in C^{j+1}\big(\mathbb{R} \setminus v^{-1}(0)\big)$. Thus, u is in the desired space. Now the statement for ρ follows from the second equation of (3.4). $\square$

Remark. Since $v = u - c$, Corollary (1) shows that $u \in C^\infty\big(\mathbb{R} \setminus u^{-1}(c)\big)$.

Since $\mathbb{R} \setminus u^{-1}(c)$ is an open set, we have

$$\mathbb{R} \setminus u^{-1}(c) = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

So, u is smooth in every interval (a_i, b_i) where the following Lemma holds (below $(a_i, b_i) = (a, b)$):

Lemma 12. *Let (u, ρ) be a traveling wave solution to (3.2). Suppose u is smooth in the interval (a, b). Then in the interval (a, b), u satisfies the following equation:*

$$(u - c)^2 u'^2 = P(u), \tag{3.12}$$

where

$$P(u) = (u^2 + \kappa u + A)(u - c)^2 + C(u - c) + B, \tag{3.13}$$

and A, B and C are some constants.

Proof. Since both u and ρ are smooth in (a, b) we use standard calculus rules. By the first equation of (3.4), we have

$$2(v')^2 + 2vv'' = (v')^2 + p(v) - \rho^2.$$

Therefore,

$$(v')^2 + 2vv'' = p(v) - \rho^2.$$

Multiplying by v' we have

$$(v')^3 + v\big((v')^2\big)' = v'p(v) - v'\rho^2.$$

Thus,

$$\big(v(v')^2\big)' = v'p(v) - v'\rho^2.$$

Hence,

$$\big(v(v')^2\big)' = (3v^2 + (2\kappa + 4c)v + K)v' - \frac{Bv'}{v^2},$$

where $B = B_1^2$.

Integration yields

$$v(v')^2 = v^3 + (\kappa + 2c)v^2 + Kv + \frac{B}{v} + C.$$

Now, multiplying this equation by v we get

$$v^2(v')^2 = \big(v^2 + (\kappa + 2c)v + K\big)v^2 + Cv + B.$$

Substituting $v = u - c$ and simplifying, we have

$$(u-c)^2(u')^2 = (u^2 + \kappa u + A)(u-c)^2 + C(u-c) + B,$$

for some constant A. $\square$

Theorem 3. *Suppose (u, ρ) is a non-smooth traveling wave solution to (3.2). If $u^{-1}(c)$ is a set of measure zero, then u is a solution to the Camassa-Holm equation.*

Proof. Suppose $\xi \in \mathbb{R} \setminus u^{-1}(c)$. Since, $u^{-1}(c) \neq \varnothing$, there exists an $\eta \in u^{-1}(c)$ such that either $\xi > \eta$ or $\xi < \eta$. Without loss of generality, assume that $\xi < \eta$. Let $\eta_0 = \inf\{\eta \in u^{-1}(c) : \eta > \xi\}$. Since $u^{-1}(c)$ is a closed set, $\eta_0 \in u^{-1}(c)$. So, $(\xi, \eta_0) \subseteq \mathbb{R} \setminus u^{-1}(c)$. Thus, we have proved that there exists an $\eta \in u^{-1}(c)$ such that either $(\xi, \eta) \subseteq \mathbb{R} \setminus u^{-1}(c)$ or $(\eta, \xi) \subseteq \mathbb{R} \setminus u^{-1}(c)$. Now, consider the equation (3.12) and set $F(u) = \frac{P(u)}{(u-c)^2}$. We claim that B in (3.12) equals 0. Suppose $B \neq 0$. Since $B = B_1^2$, we have $B > 0$. Then (3.13) implies that

$$\frac{1}{\sqrt{F(u)}} = \frac{1}{\sqrt{B}}|u - c| + \mathcal{O}\big((u-c)^2\big) \qquad u \to c.$$

On the other hand, we have

$$\frac{d\xi}{du} = \pm\frac{1}{\sqrt{F(u)}}.$$

Since $u \in C(\mathbb{R})$, for ξ close enough to η, integration yields

$$|\xi - \eta| = \frac{1}{2\sqrt{B}}(u-c)^2 + \mathcal{O}\big((u-c)^3\big) \qquad u \to c. \tag{3.14}$$

Therefore,

$$|\xi - \eta| = \frac{1}{2\sqrt{B}}(u-c)^2\big(1 + \mathcal{O}\big(u-c\big)\big) \qquad u \to c.$$

So,

$$|\xi - \eta|^{\frac{1}{2}} = \frac{1}{\sqrt{2\sqrt{B}}}|u-c|\sqrt{\big(1 + \mathcal{O}\big(u-c\big)\big)} \qquad u \to c.$$

Thus,

$$|\xi - \eta|^{\frac{1}{2}} = \frac{1}{\sqrt{2\sqrt{B}}}|u-c|\big(1 + \mathcal{O}\big(u-c\big)\big) \qquad u \to c.$$

Hence,

$$|\xi - \eta|^{\frac{1}{2}} = \frac{1}{\sqrt{2\sqrt{B}}}|u-c| + \mathcal{O}\big((u-c)^2\big) \qquad u \to c.$$

This implies that

$$(u-c) = \mathcal{O}\big((\xi-\eta)^{\frac{1}{2}}\big) \qquad \xi \to \eta.$$

Therefore,

$$(u-c)^2 = \mathcal{O}(\xi-\eta) \qquad \xi \to \eta.$$

Thus, we have

$$|u-c| = \sqrt{2\sqrt{B}}|\xi-\eta|^{\frac{1}{2}} + \mathcal{O}(\xi-\eta) \qquad \xi \to \eta. \tag{3.15}$$

Hence,

$$\begin{aligned} |\xi-\eta|^{-\frac{1}{2}} - \sqrt{2\sqrt{B}}|u-c|^{-1} &= \mathcal{O}\big(\frac{|\xi-\eta|^{\frac{1}{2}}}{u-c}\big) \\ &= \mathcal{O}(1) \qquad \xi \to \eta. \end{aligned}$$

So,

$$|u-c|^{-1} = \frac{1}{\sqrt{2\sqrt{B}}}|\xi-\eta|^{-\frac{1}{2}} + \mathcal{O}(1) \qquad \xi \to \eta. \tag{3.16}$$

On the other hand, from (3.13) we have

$$|u'| = \sqrt{B}(u-c)^{-1} + \mathcal{O}(1) \qquad \xi \to \eta. \tag{3.17}$$

Now combining (3.17) and (3.16), we have

$$|u'| = \frac{\sqrt[4]{B}}{\sqrt{2}}|\xi - \eta|^{-\frac{1}{2}} + \mathcal{O}(1) \qquad \xi \to \eta. \tag{3.18}$$

Hence, $u' \notin L^2_{loc}(\mathbb{R})$. This contradiction shows that $B = 0$. Therefore, the second equation of (3.4) implies that $\rho = 0$ almost everywhere. $\square$

Now, we provide an example of a smooth solution of (3.2) that is not a solution of Camassa-Holm equation.

Example. Let $P(u)$ be as in the previous Theorem. Observe that $P(u) = (u-G)^2(u-L)^2$ if and only if

$$\begin{cases} \kappa = 2\big(c - (L+G)\big), \\ A = 2c\kappa - c^2 + (L+G)^2 + 2LG, \\ C = 2cA - \kappa c^2 - 2LG(L+G), \\ B = Cc - Ac^2 + L^2G^2. \end{cases} \tag{3.19}$$

Suppose $|u| < 1$ and $c > 1$. Therefore, if $G = -1$ and $L = 1$, integration yields

$$(1-u)^{1-c}(1+u)^{1+c} = e^{2(\xi - \xi_0)}. \tag{3.20}$$

Let us say $c = 2$ and $\xi_0 = 0$. We observe that the equation

$$\frac{(1+u)^3}{1-u} = e^{2\xi},$$

provides a smooth solution of (3.2) which is not a solution of Camassa-Holm equation. See figure 3.1.

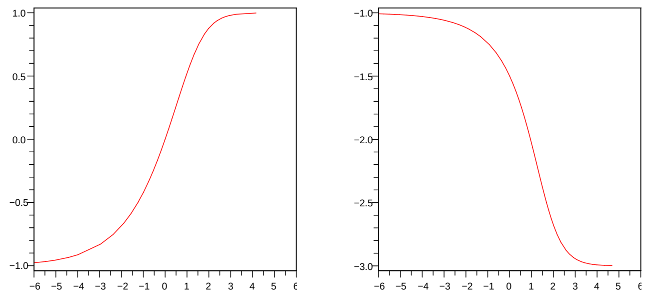

Figure 3.1: (u on the left and ρ on the right) A smooth solution of (3.2) which is not a solution of Camassa-Holm equation.

The following Lemma provides necessary and sufficient conditions for a piecewise smooth function to be a distributional solution to (3.2).

Lemma 13. *Suppose u is a piecewise smooth function. The pair (u, ρ) is a distributional solution to* (3.2) *in the sense of definition 2 if and only if all of the following conditions hold:*

1. $u \in H^1_{loc}(\mathbb{R})$ and $\rho \in L^2_{loc}(\mathbb{R})$.

2. $(u-c)^2 \in W^{2,1}_{loc}(\mathbb{R})$.

3. u and ρ satisfy the equation (3.4) *locally with the same constant K on every interval where u is smooth.*

Proof. The part ($\Rightarrow$) is easy. For the converse ($\Leftarrow$), we note that since $(u-c)^2 \in W^{2,1}_{loc}(\mathbb{R})$, then $((u-c)^2)'$ is absolutely continuous and has no jumps. Therefore, $((u-c)^2)''$ defines a regular distribution [18] .Thus, every term in the equation (3.4) can be represented by an integral that defines a distribution on the space of test functions and we are allowed to write each integral as a finite sum of integrals over local intervals and use condition 3 to prove that u and ρ satisfy (3.4) in the sense of distributions. □

Remark. We note that if the measure of $u^{-1}(c)$ is not zero, then the equation (3.4) implies that $\rho^2 = K$ on $u^{-1}(c)$. However in the Camassa-Holm equation if the measure of $u^{-1}(c)$ is not zero, then $K = 0$ because $\rho = 0$. This implies that solutions of the form given in the following example cannot arise from the Camassa-Holm equation.

Example. Set $\kappa = 0$. The pair of functions (u, ρ) given by

$$u(x) = \begin{cases} ce^{1-|x|} & \text{if } |x| > 1, \\ c & \text{if } |x| < 1, \end{cases}$$

$$\rho(x) = \begin{cases} c & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

is a solution to (3.2) but u is not a solution of Camassa-Holm equation. To see this, observe that the left hand side derivative of u at -1 and the right hand side derivative of u at 1 are non-zero and finite in contrast with the Camassa-Holm equation for which Lenells [9] showed that if the measure of $u^{-1}(c)$ is not zero, then these limits cannot be finite. See figure 3.2.

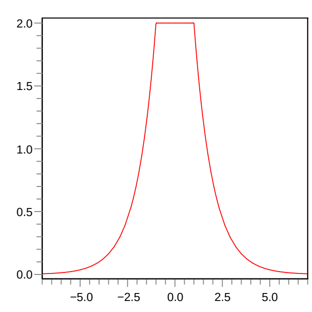

Figure 3.2: $u(x)$ is a solution of (3.2) but it is not a solution of Camassa-Holm equation.

Definition 3. Suppose f is a continuous function on $\mathbb{R}$.

1. We say f has a peak at x if f is smooth locally on both sides of x and

$$0 \neq \lim_{y \to x^+} f'(y) = -\lim_{y \to x^-} f'(y) \neq \pm\infty.$$

Traveling wave solutions of (3.2) with peaks are called peakons.

2. We say f has a cusp at x if f is smooth locally on both sides of x and

$$\lim_{y \to x^+} f'(y) = -\lim_{y \to x^-} f'(y) = \pm\infty.$$

Traveling wave solutions of (3.2) with cusps are called cuspons.

3. We say that f has a stump if there is an interval $[a, b]$ on which f is a constant and f is smooth locally to the left of a and to the right of b and

$$0 \neq \lim_{x \to a^-} f'(x) = -\lim_{x \to b^+} f'(x).$$

Traveling wave solutions of (3.2) with stumps are called stumpons. Note that, in the definition of a stump the limits can be either finite or infinite.

Theorem 3 limits the existence of new distributional peakon or cuspon solutions to the (3.2).

Corollary 2. Every peakon or cuspon traveling wave solution to (3.2) is a traveling wave solution to the Camassa-Holm equation.

Finally we would like to comment on the peaked solution reported in [12]. For reasons explained below, that solution is not a distributional solution. First, we note that by Corollary (1) the non-smooth points of a distributional solution u can only appear when

$u = c$. Also, Lemma (11) shows that if (u, ρ) is a traveling wave solution to (3.2), then $(u-c)^2 \in C^1(\mathbb{R})$. Now, consider the peaked function (see [12])

$$u = \chi + \sqrt{\chi^2 - c^2}, \quad \rho = \sqrt{-cK_1}\left(1 + \sqrt{\frac{\chi + c}{\chi - c}}\right), \quad \chi = -(c + K_1)\cosh(x - ct) + K_1,$$

where $K_1 = -\frac{1}{4}\kappa$, $K_1 < 0$ and $c > |K_1| > 0$. Away from it's non-smooth point, u is a solution to (3.2). However, it is clear that u is not smooth at $\xi = 0$ and $u(0) = -c$. Furthermore, $(u-c)^2 \notin C^1(\mathbb{R})$ because

$$\lim_{\xi \to 0^+} \left((u-c)^2\right)' - \lim_{\xi \to 0^-} \left((u-c)^2\right)' = -8c\sqrt{c(c + K_1)},$$

where $\xi = x - ct$. Therefore, u is not a distributional solution to (3.2) even though it superficially looks like a peakon solution (see figure 3.3).

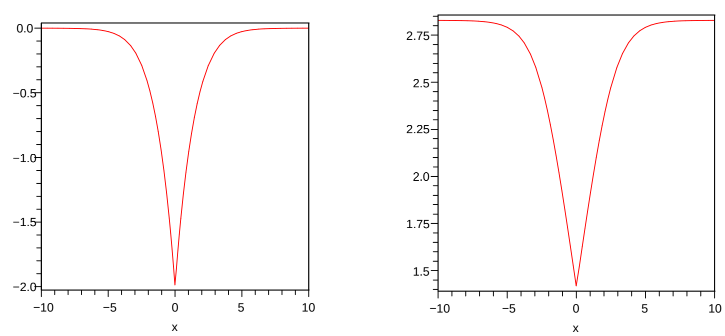

Figure 3.3: (u on the left and ρ on the right) This pair is not a distributional traveling wave solution of (3.2). $c = 2$ and $K_1 = -1$.

Chapter 4

Peakons and the Lax pair

4.1 Multipeakons

Recall the equation

$$\begin{cases} u_t - u_{xxt} + (b+1)uu_x + \kappa u_x = bu_x u_{xx} + uu_{xxx} & t > 0 \quad x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{4.1}$$

This is the Camassa-Holm equation if $b = 2$, and the Degasperis-Procesi equation if $b = 3$. We can rewrite (4.1) in the following form:

$$m_t + bmu_x + m_x u = 0, \tag{4.2}$$

where $m = u - u_{xx} + \frac{1}{b}\kappa$. The equation (4.1) can also be written in the following form:

$$2(u_t - u_{xxt}) + (b+1)(u^2)_x + 2\kappa u_x = (u^2)_{xxx} + (b-3)(u_x^2)_x. \tag{4.3}$$

For what follows, the equation (4.3) allows us to define distributional solutions to (4.1). Suppose $u(t,x)$ is a t-dependent family of functions in $H^1_{loc}(\mathbb{R})$ where $t \in [0,T]$. Then for every t in $[0,T]$, u, u_x, u^2 and u_x^2 are distributions. Also, observe that if $b = 3$, one only needs to have $u(t,x)$ in $L^2_{loc}(\mathbb{R})$.

Definition 4. Consider the map $(0,T) \xrightarrow{f} \mathcal{D}'(\mathbb{R})$ (see Appendix B for notation). The derivative $D_t f$ (if it exists) is defined as follows:

$$\langle D_t f, \phi \rangle = \lim_{h \to 0} \frac{1}{h} \langle f(t+h) - f(t), \phi \rangle, \tag{4.4}$$

for every $\phi \in \mathcal{D}(\mathbb{R})$.

Using the theorem of completeness of distributions (see Appendix B), we see that $D_t f$ defined above, is indeed a distribution in $\mathcal{D}'(\mathbb{R})$. Now, we rewrite equation (4.3) in the following form:

$$2D_t(1 - D_x^2)u + (b+1)D_x(u^2) + 2\kappa D_x u = D_x^3(u^2) + (b-3)D_x(D_x u)^2. \tag{4.5}$$

where D_x is the distributional derivative of u, D_t is the derivative in the sense of equation (4.4) and u_x is the weak derivative of u with respect to x. Now, we are ready to present the definition of a distributional solution to (4.1).

Definition 5. Suppose $u \in C^1\big(0, T; H^1_{loc}(\mathbb{R})\big)$. Then u is called a distributional solution of (4.1) if it satisfies (4.5) in the sense of distributions.

Lemma 14. *D_t in the sense of equation (4.4) commutes with D_x in the sense of distributions.*

Proof. Consider the map $(0, T) \xrightarrow{f} \mathcal{D}'(\mathbb{R})$ and let ϕ be a compactly supported C^∞ function. By the definition of the derivative of a distribution, we have

$$\langle D_x D_t f, \phi \rangle = -\langle D_t f, \phi_x \rangle.$$

On the other hand, using the equation (4.4), we have

$$\begin{aligned} \langle D_t D_x f, \phi \rangle &= \lim_{h \to 0} \frac{1}{h} \langle D_x f(t+h) - D_x f(t), \phi(x) \rangle \\ &= -\lim_{h \to 0} \frac{1}{h} \langle f(t+h) - f(t), \phi_x \rangle \\ &= -\langle D_t f, \phi_x \rangle. \end{aligned}$$

□

It is well known that (4.1) admits multipeakon solutions. A multipeakon is a train of interacting peakons that retain their original shapes after interaction. In order to obtain multipeakon solutions to (4.1) we consider m as a discrete measure given by

$$m = \frac{\kappa}{b} + (1 - D_x^2)u = \sum_{j=1}^{n} m_j(t)\delta_{x_j(t)}, \tag{4.6}$$

where $m_j \in C^\infty(0,T)$. This prompts a possible solution of the form

$$u = -\frac{\kappa}{b} + \frac{1}{2}\sum_{j=1}^{n} m_j e^{-|x-x_j|}. \tag{4.7}$$

This is a piecewise smooth and continuous function having its sharp edges at $x_j(t)$. For convenience let us assume that $x_j(t) \in C^\infty(0,T)$. In order to characterize such a solution we need the following lemma:

Lemma 15. *Suppose $g(t) \in C^\infty(\mathbb{R})$. Then*

$$D_t(g\delta_{x_j}) = \dot{g}\delta_{x_j} - g\dot{x_j}\delta'_{x_j}. \tag{4.8}$$

Proof. For every $\phi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}
\langle D_t(g\delta_{x_j}), \phi\rangle &= \lim_{h\to 0}\frac{1}{h}\langle g(t+h)\delta_{x_j(t+h)} - g(t)\delta_{x_j(t)}, \phi\rangle \\
&= \lim_{h\to 0}\frac{1}{h}\big(g(t+h)\phi(x_j(t+h)) - g(t)\phi(x_j(t))\big) \\
&= \frac{d}{dt}\big(g(t)\phi(x_j(t))\big) \\
&= \dot{g}\phi(x_j) + g\dot{x_j}\phi_x(x_j) \\
&= \langle \dot{g}\delta_{x_j} - g\dot{x_j}\delta'_{x_j}, \phi\rangle.
\end{aligned}$$

□

Corollary 3. Suppose u is a piecewise smooth and continuous function which satisfies the equation (4.6). Then we have

$$D_t(1 - D_x^2)u = \sum_{j=1}^{n}(\dot{m_j}\delta_{x_j} - m_j\dot{x_j}\delta'_{x_j}). \tag{4.9}$$

Proof. Immediately follows from the previous lemma. $\square$

Definition 6. Suppose $u(x,t)$ is a piecewise continuous function with non-smooth edges located at $x_j(t)$, $j = 1, \ldots, n$. We define the distribution $\partial_x^m u$ as follows:

$$\langle \partial_x^m u, \phi \rangle = \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} \frac{\partial^m u}{\partial x^m} \phi \, dx \tag{4.10}$$

for every $\phi \in \mathcal{D}(\mathbb{R})$. Note that the integrals can be improper. We denote $\partial_x u$, $\partial_x^2 u$ and $\partial_x^3 u$ by u_x, u_{xx} and u_{xxx} respectively.

Definition 7. Let $f(x)$ be a continuous function for $x \in [a,b] \setminus \{x_0\}$, where $a < x_0 < b$. Suppose both $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist. We define the jump of f at x_0 as follows,

$$[f](x_0) = \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x).$$

We also define

$$\langle f \rangle(x_0) = \frac{1}{2}\big(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x) \big).$$

Lemma 16. *Suppose $u(x,t)$ is a piecewise smooth function with non-smooth edges located at $x_j(t)$, $j = 1, \ldots, n$. Then the following equality holds in the sense of distributions:*

$$D_x^m u = \partial_x^m u + \sum_{j=1}^{n} \sum_{k=0}^{m-1} [\partial_x^k u](x_j) \delta_{x_j}^{(m-k-1)}, \tag{4.11}$$

where $\delta_{x_j}^{(r)} = D_x^r \delta_{x_j}$.

Proof. We prove this by induction on m.

$$\begin{aligned}
\langle D_x u, \phi \rangle &= -\langle u, \phi_x \rangle = -\int_{-\infty}^{\infty} u \phi_x \, dx = -\sum_{j=0}^{n} \int_{x_j^+}^{x_{j+1}^-} u \phi_x \, dx \\
&= \sum_{j=0}^{n} -\Big(\big(u(x_{j+1}^-) - u(x_j^+)\big)\phi(x_j) - \int_{x_j^+}^{x_{j+1}^-} u_x \phi \, dx \Big) \\
&= \sum_{j=1}^{n} [u](x_j)\phi(x_j) + \sum_{j=0}^{n} \int_{x_j^+}^{x_{j+1}^-} u_x \phi \, dx \\
&= \langle u_x + \sum_{j=1}^{n} [u](x_j) \delta_{x_j}, \phi \rangle.
\end{aligned}$$

Now, suppose

$$D_x^{m-1}u = \partial_x^{m-1}u + \sum_{j=1}^{n}\sum_{k=0}^{m-2}[\partial_x^k u](x_j)\delta_{x_j}^{(m-k-2)}.$$

Then the proof we have just stated above, shows that

$$\begin{aligned} D_x^m u &= \partial_x^m u + \sum_{j=1}^{n}[\partial_x^{m-1}u](x_j)\delta_{x_j} + \sum_{j=1}^{n}\sum_{k=0}^{m-2}[\partial_x^k u](x_j)\delta_{x_j}^{(m-k-1)} \\ &= \partial_x^m u + \sum_{j=1}^{n}\sum_{k=0}^{m-1}[\partial_x^k u](x_j)\delta_{x_j}^{(m-k-1)}. \end{aligned}$$

□

Proposition 5. *If X is an open subset of $\mathbb{R}^n$ and K is a compact subset of X, then there exist a compactly supported C^∞ function ϕ on X, such that $0 \le \phi \le 1$ and $\phi = 1$ on a neighborhood of K.*

Proof. See [28]. □

Corollary 4. Suppose $u(x,t)$ and $v(x,t)$ are piecewise smooth functions with possible non-smooth edges located at $x_j(t)$, $j = 1,\dots,n$. If

$$D_x^m u = v + \sum_{j=1}^{n}\sum_{k=0}^{m-1} a_{jk}\delta_{x_j}^{(m-k-1)},$$

then $v = \partial_x^m u$ and $a_{jk} = [\partial_x^k u](x_j)$, for every $j = 1,\dots,n$, $k = 0,\dots,m-1$.

Proof. Using Lemma (16), we have

$$v - \partial_x^m u + \sum_{j=1}^{n}\sum_{k=0}^{m-1}\big(a_{jk} - [\partial_x^k u](x_j)\big)\delta_{x_j}^{(m-k-1)} = 0.$$

So, for every ϕ with $\text{Support}(\phi) \cap \{x_1,\dots,x_n\} = \emptyset$, we have

$$v - \partial_x^m u = 0.$$

Therefore, away from the jumps $v = \partial_x^m u$. Now, choose a test function ϕ such that $x_j \in \text{Support}(\phi)$, $\phi(x) = 1$ on a neighborhood of x_j and zero elsewhere. Also $x_i \notin \text{Support}(\phi)$, for $i \neq j$. This implies that $a_{j,m-1} = [u](x_j)$. Choosing the test function $(x-x_j)^{m-1-k}\phi(x)$, we can show that for every $k = 0, \dots, m-1$, $a_{jk} = [\partial_x^k u](x_j)$. Since j was arbitrary, the proof is complete. $\square$

Theorem 4. *Suppose that $u(x,t)$ is given by*

$$u = -\frac{\kappa}{b} + \frac{1}{2}\sum_{j=1}^{n} m_j e^{-|x-x_j|}$$

where $m_j,\ x_j \in C^\infty(0,T)$ and for every $t \in (0,T)$ we have

(a) $x_1(t) < x_2(t), \dots, < x_n(t)$,

(b) $m_j(t) \neq 0$.

Then u is a solution to (4.1) if and only if the following system of ODEs holds:

$$\begin{cases} \dot{x}_j = -\frac{\kappa}{b} + \frac{1}{2}\sum_{i=1}^{n} m_i e^{-|x_j-x_i|}, \\ \dot{m}_j = \frac{1}{2}(b-1)\sum_{i=1}^{n} m_j m_i Sgn(x_j-x_i) e^{-|x_j-x_i|}. \end{cases} \tag{4.12}$$

Proof. We compute every term of the equation (4.5).

$$2D_t(1-D_x^2)u = 2\sum_{j=1}^{n}(\dot{m}_j\delta_{x_j} - m_j\dot{x}_j\delta'_{x_j}).$$

$$(b+1)D_x(u^2) = (b+1)(u^2)_x.$$

$$2\kappa D_x(u) = 2\kappa u_x.$$

$$D_x^3(u^2) = (u^2)_{xxx} + \sum_{j=1}^{n}\big([(u^2)_x](x_j)\delta'_{x_j} + [(u^2)_{xx}](x_j)\delta_{x_j}\big).$$

$$(b-3)D_x(u_x^2) = (b-3)(u_x^2)_x + (b-3)\sum_{j=1}^{n}[u_x^2](x_j)\delta_{x_j}.$$

On the other hand, we have

$$[(u^2)_x](x_j) = [2uu_x](x_j) = 2u(x_j)[u_x](x_j) = \frac{2\kappa}{b}m_j - \sum_{i=1}^{n} m_j m_i e^{-|x_j-x_i|}.$$

Also, since $[u_{xx}](x_j) = 0$, we have

$$[(u^2)_{xx}](x_j) = 2[u_x^2 + uu_{xx}](x_j) = 2[u_x^2](x_j).$$

Since for a given function f,

$$[f^2](x) = 2\langle f\rangle(x)[f](x),$$

we have

$$[u_x^2](x_j) = 2\langle u_x\rangle(x_j)[u_x](x_j) = \sum_{i=1}^{n} m_j m_i Sgn(x_j - x_i)e^{-|x_j - x_i|}.$$

Hence,

$$\dot{x_j} = -\frac{\kappa}{b} + \frac{1}{2}\sum_{i=1}^{n} m_i e^{-|x_j - x_i|},$$

and

$$\dot{m_j} = \frac{1}{2}(b-1)\sum_{i=1}^{n} m_j m_i Sgn(x_j - x_i)e^{-|x_j - x_i|}.$$

$\square$

4.2 Multipeakons and the Lax pair

Recall the Lax pair representation of Camassa-Holm given in (2.52),

$$L(z) = D^2 + zm - \frac{1}{4}, \qquad A(z) = -(\frac{1}{2z} + u)D + \frac{u_x}{2}, \quad z \in \mathbb{C}.$$

Now, we consider

$$m = \sum_{j=1}^{n} m_j \delta_{x_j},$$

where $m_j(t),\ x_j(t) \in C^\infty(0,T)$. Suppose $\psi(x,t)$ satisfies $L(z)\psi = 0$ in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, in other words:

$$(\frac{1}{4} - D_x^2)\psi = z\sum_{j=1}^{n} m_j \psi(x_j)\delta_{x_j}, \tag{4.13}$$

Now, using the distributional ODE

$$(\frac{1}{4} - D_x^2)E = \delta_0,$$

which has the general solution

$$E = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{1}{2}x} + e^{-\frac{1}{2}|x|},$$

we have,

$$\psi = A(t)e^{-\frac{1}{2}x} + B(t)e^{\frac{1}{2}x} + z\sum_{j=1}^{n} m_j \psi(x_j) e^{-\frac{1}{2}|x-x_j|}, \tag{4.14}$$

where $t \in (0,T)$. Therefore, for every $t \in (0,T)$, $\psi(x)$ is a piecewise smooth and absolutely continuous function with sharp edges at $x_j(t)$.

Lemma 17. *There exists a solution $\psi(x,t)$ of the equation* (4.13) *such that*

$$\psi(x,t) = A_j(t)e^{-\frac{1}{2}x} + B_j(t)e^{\frac{1}{2}x}$$

for every $x \in (x_j, x_{j+1})$, where $A_j, B_j \in C^\infty(0,T)$ for every $j = 1, \ldots, n$. Furthermore, $\psi\big(t, x_j(t)\big) \in C^\infty(0,T)$ for every $j = 1, \ldots, n$.

Proof. By equation (4.14), on (x_j, x_{j+1}) we have

$$\psi = A_j e^{-\frac{1}{2}x} + B_j e^{\frac{1}{2}x},$$

where $x_0 = -\infty$, $x_{n+1} = \infty$ and

$$A_j = A(t) + z\sum_{i=1}^{j} m_i \psi(x_i) e^{\frac{1}{2}x_i}, \qquad B_j = B(t) + z\sum_{i=j+1}^{n} m_i \psi(x_i) e^{-\frac{1}{2}x_i}. \tag{4.15}$$

Suppose that $\psi(x) = A_0(t)e^{-\frac{1}{2}x} + B_0(t)e^{\frac{1}{2}x}$ for every $x \in (-\infty, x_1)$ where $A_0, B_0 \in C^\infty(0,T)$. Then by the continuity of ψ we have

$$\psi(x_1) = A_0 e^{-\frac{1}{2}x_1} + B_0 e^{\frac{1}{2}x_1}.$$

Hence $\psi\big(t, x_1(t)\big) \in C^\infty(0,T)$. Equation (4.13) implies that, for every $j = 1, \ldots, n$,

$$[\psi](x_j) = \frac{1}{2}\big((B_j - B_{j-1})e^{\frac{x_j}{2}} - (A_j - A_{j-1})e^{-\frac{x_j}{2}}\big) = zm_j\psi(x_j).$$

Again by the continuity of ψ we have

$$\psi(x_j) = A_{j-1}e^{-\frac{x_j}{2}} + B_{j-1}e^{\frac{x_j}{2}} = A_j e^{-\frac{x_j}{2}} + B_j e^{\frac{x_j}{2}}. \tag{4.16}$$

Therefore,

$$(B_j - B_{j-1})e^{\frac{x_j}{2}} = (A_{j-1} - A_j)e^{-\frac{x_j}{2}}.$$

Thus,

$$(A_j - A_{j-1}) = -zm_j\psi(x_j)e^{\frac{1}{2}x_j}, \qquad (B_j - B_{j-1}) = zm_j\psi(x_j)e^{-\frac{1}{2}x_j}.$$

Hence,

$$A_j = A_0 - z\sum_{i=1}^{j} m_i\psi(x_i)e^{\frac{1}{2}x_i}, \qquad B_j = B_0 + z\sum_{i=1}^{j} m_i\psi(x_i)e^{-\frac{1}{2}x_i}. \tag{4.17}$$

This implies that $A_1(t), B_1(t) \in C^\infty(0,T)$. Now, suppose that

$$A_{j-1}(t), \quad B_{j-1}(t), \quad \psi\big(t, x_{j-1}(t)\big) \in C^\infty(0,T).$$

Then, by equation (4.16), $\psi\big(t, x_j(t)\big) \in C^\infty(0,T)$. Hence, by equations (4.17),

$$A_j(t), \quad B_j(t) \in C^\infty(0,T).$$

Thus, by the mathematical induction the proof is complete. □

Remark. Equations (4.15) show that the solution constructed in the proof of Lemma 17 satisfies (4.14) with $A(t),\ B(t) \in C^\infty(0,T)$. Thus, from now on, we assume that ψ is given as in Lemma 17.

Lemma 18. *$D_t\psi$ and $D_t(\frac{1}{4} - D_x^2)\psi$ exist in the sense of definition* (4.4) *and the following relations hold:*

$$\langle D_t\psi, \phi\rangle = \sum_{j=0}^{n}\int_{x_j(t)}^{x_{j+1}(t)} \frac{\partial}{\partial t}\psi(x,t)\phi(x)dx. \tag{4.18}$$

$$D_t(\frac{1}{4} - D_x^2)\psi = \sum_{j=1}^{n} \big(zm_j\psi(x_j)\big)_t \delta_{x_j} - z\sum_{j=1}^{n} m_j\psi(x_j)\dot{x_j}\delta'_{x_j}. \tag{4.19}$$

Proof. Suppose $\phi \in \mathcal{D}(\mathbb{R})$.

$$\begin{aligned}\langle D_t\psi, \phi\rangle &= \lim_{h\to 0}\frac{1}{h}\langle \psi(x, t+h) - \psi(x,t), \phi(x)\rangle,\\ &= \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}} \big(\psi(x,t+h) - \psi(x,t)\big)\phi(x)dx.\end{aligned}$$

Suppose $\epsilon > 0$ is given. Since $\psi(x)$ is a piecewise smooth function we can choose $\delta > 0$ such that for every $x \in (x_j, x_{j+1}) \cap \text{Support}(\phi)$, $j = 0, 1, \ldots, n$, $x_0 = -\infty$, $x_{n+1} = \infty$, if $|h| < \delta$, then

$$\left|\frac{1}{h}\big(\psi(x,t+h) - \psi(x,t)\big) - \frac{\partial}{\partial t}\psi(x,t)\right| < \epsilon.$$

So, we can write

$$\begin{aligned}&\left|\frac{1}{h}\int_{\mathbb{R}}\big(\psi(x,t+h) - \psi(x,t)\big)\phi(x)dx - \sum_{j=0}^{n}\int_{x_j(t)}^{x_{j+1}(t)}\frac{\partial}{\partial t}\psi(x,t)\phi(x)dx\right|\\ &< \sum_{j=0}^{n}\int_{x_j(t)}^{x_{j+1}(t)}\left|\left(\frac{1}{h}\big(\psi(x,t+h) - \psi(x,t)\big) - \frac{\partial}{\partial t}\psi(x,t)\right)\phi(x)\right|dx\\ &< \epsilon\int_{\mathbb{R}}\big|\phi(x)\big|dx.\end{aligned}$$

Therefore,

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\big(\psi(x,t+h) - \psi(x,t)\big)\phi(x)dx = \sum_{j=0}^{n}\int_{x_j(t)}^{x_{j+1}(t)}\frac{\partial}{\partial t}\psi(x,t)\phi(x)dx.$$

Hence,

$$\langle D_t\psi, \phi\rangle = \sum_{j=0}^{n}\int_{x_j(t)}^{x_{j+1}(t)}\frac{\partial}{\partial t}\psi(x,t)\phi(x)dx.$$

The existence of $D_t(\frac{1}{4} - D_x^2)\psi$ follows from equations (4.13) and (4.8) and we have

$$\begin{aligned}(\frac{1}{4}D_t - D_tD_x^2)\psi &= D_t\left(z\sum_{j=1}^{n} m_j\psi(x_j)\delta_{x_j}\right)\\ &= \sum_{j=1}^{n}\big(zm_j\psi(x_j)\big)_t\delta_{x_j} - z\sum_{j=1}^{n} m_j\psi(x_j)\dot{x_j}\delta'_{x_j}.\end{aligned}$$

$\square$

Definition 8. Let $u(x,t) \in C^1\big((0,T); W^{1,1}_{loc}(\mathbb{R})\big)$ and suppose $(1-D_x^2)u + \frac{\kappa}{2} = \sum_{i=1}^n m_i \delta_{x_i}$.

The system

$$\begin{cases} (\frac{1}{4} - D_x^2)\psi = z \sum_{j=1}^n m_j \psi(x_j)\delta_{x_j}, & z \in \mathbb{C} \\ D_t\psi = -(\frac{1}{2z} + u)\psi_x + (\frac{1}{2}u_x + \alpha)\psi, & z \in \mathbb{C} \end{cases} \tag{4.20}$$

where $\alpha = \frac{1}{2}(\frac{1}{2z} - \frac{\kappa}{2})$ and every term is considered as a distribution in $\mathcal{D}'(\mathbb{R})$, is called a peakon weak Lax pair.

Remark. It makes sense to consider a more general weak Lax pair,

$$\begin{cases} (\frac{1}{4} - D_x^2)\psi = zm\psi, & z \in \mathbb{C}, \\ D_t\psi = -(\frac{1}{2z} + u)\psi_x + (\frac{1}{2}u_x + \alpha)\psi, & z \in \mathbb{C}, \end{cases} \tag{4.21}$$

where m is an arbitrary measure, that is, $m = D_x M$, with M being a function of bounded variation (see Appendix A).

We are going to show that, *if ψ satisfies the peakon weak Lax pair* (4.20) *with*

$$u(x,t) = -\frac{\kappa}{2} + \frac{1}{2}\sum_{j=1}^n m_j e^{-|x-x_j|},$$

then the following system of ODEs hold:

$$\begin{cases} \dot{x_j} = u(x_j), \\ \dot{m_j} = -\big(\langle u_x\rangle(x_j)\big)m_j. \end{cases}$$

Applying D_t to the first equation of (4.20) and using Lemma 18, we get

$$(\frac{1}{4}D_t - D_tD_x^2)\psi = z\sum_{j=1}^n \Big(\big(m_j\psi(x_j)\big)_t \delta_{x_j} - m_j\psi(x_j)\dot{x_j}\delta'_{x_j}\Big). \tag{4.22}$$

Applying $\frac{1}{4} - D_x^2$ to the second equation of (4.20) and using Lemma 16, we obtain

$$
\begin{aligned}
(\frac{1}{4}D_t - D_x^2 D_t)\psi = &-\frac{1}{4}(\frac{1}{2z} + u)\psi_x + \frac{1}{8}u_x\psi + \frac{1}{4}\alpha\psi \\
&+ u_{xx}\psi_x + 2u_x\psi_{xx} + (\frac{1}{2z} + u)\psi_{xxx} \\
&+ \sum_{j=1}^{n}\big([u_x\psi_x](x_j) + (\frac{1}{2z} + u(x_j))[\psi_{xx}](x_j)\big)\delta_{x_j} \\
&+ \sum_{j=1}^{n}(\frac{1}{2z} + u(x_j))[\psi_x](x_j)\delta'_{x_j} \\
&- \frac{1}{2}u_{xxx}\psi - u_{xx}\psi_x - (\frac{1}{2}u_x + \alpha)\psi_{xx} \\
&- \frac{1}{2}\sum_{j=1}^{n}\big(\psi(x_j)[u_{xx}](x_j) + [(u_x + 2\alpha)\psi_x](x_j)\big)\delta_{x_j} \\
&- \frac{1}{2}\sum_{j=1}^{n}\psi(x_j)[u_x](x_j)\delta'_{x_j}.
\end{aligned}
$$

The first equation of (4.20) implies that, away from the jumps, $\frac{1}{4}\psi = \psi_{xx}$. So,

$$
\frac{1}{4}(\frac{1}{2z} + u)\psi_x = (\frac{1}{2z} + u)\psi_{xxx}
$$

and $\frac{3}{2}u_x\psi_{xx} = \frac{3}{8}u_x\psi$. Thus we can write

$$
\begin{aligned}
(\frac{1}{4}D_t - D_x^2 D_t)\psi = &\frac{1}{2}(u_x - u_{xxx})\psi \\
&+ \frac{1}{2}\sum_{j=1}^{n}\big([u_x\psi_x](x_j) - 2\alpha[\psi_x](x_j) - \psi(x_j)[u_{xx}](x_j)\big)\delta_{x_j} \\
&+ \sum_{j=1}^{n}\big((\frac{1}{2z} + u(x_j))[\psi_x](x_j) - \frac{1}{2}\psi(x_j)[u_x](x_j)\big)\delta'_{x_j}.
\end{aligned}
\tag{4.23}
$$

Now, by Lemma 14, $\psi(\frac{1}{4}D_t - D_t D_x^2) = (\frac{1}{4}D_t - D_x^2 D_t)\psi$. Thus, comparing (4.22) and (4.23), we obtain the following:

1.

$$
zm_j\psi(x_j)\dot{x_j} = \frac{1}{2}\psi(x_j)[u_x](x_j) - (\frac{1}{2z} + u(x_j))[\psi_x](x_j).
$$

Note that the first equation of (4.20) implies $[\psi_x](x_j) = -zm_j\psi(x_j)$. Therefore we have

$$
\dot{x_j} = \frac{1}{2z}\big(\frac{[u_x](x_j)}{m_j} + 1\big) + u(x_j), \quad \text{for every } z \in \mathbb{C}. \tag{4.24}
$$

Thus

$$[u_x](x_j) = -m_j, \qquad \dot{x_j} = u(x_j). \tag{4.25}$$

2.

$$2\big(zm_j\psi(x_j)\big)_t = [u_x\psi_x](x_j) - 2\alpha[\psi_x](x_j) - \psi(x_j)[u_{xx}](x_j). \tag{4.26}$$

Lemma 19.

$$[\dot{x_j}\psi_x + D_t\psi](x_j) = 0.$$

Proof. By (4.22) and Lemma 14 we have

$$D_x^2(D_t\psi) = \frac{1}{4}D_t\psi - z\sum_{j=1}^{n}\Big(\big(m_j\psi(x_j)\big)_t\delta_{x_j} - m_j\psi(x_j)\dot{x_j}\delta'_{x_j}\Big).$$

Therefore, Corollary 4 implies that $[D_t\psi](x_j) = z\dot{x_j}m_j\psi(x_j)$. Thus the assertion follows.

□

Corollary 5.

$$\lim_{x\to x_j}\big(\dot{x_j}\psi_x(x) + D_t\psi(x)\big) = \dot{x_j}\langle\psi_x\rangle(x_j) + \langle D_t\psi\rangle(x_j).$$

Proof. Let

$$a^+ = \lim_{x\to x_j^+}\dot{x_j}\psi_x(x), \qquad a^- = \lim_{x\to x_j^-}\dot{x_j}\psi_x(x),$$

and

$$b^+ = \lim_{x\to x_j^+}D_t\psi(x), \qquad b^- = \lim_{x\to x_j^-}D_t\psi(x).$$

By Lemma 19 we have $a^+ - a^- = b^- - b^+$. Hence,

$$\frac{a^+ + a^-}{2} + \frac{b^+ + b^-}{2} = a^+ + b^+ = a^- + b^-.$$

□

Proposition 6.

$$\big(\psi(x_j)\big)_t = \dot{x_j}\langle\psi_x\rangle(x_j) + \langle D_t\psi\rangle(x_j).$$

Proof. The remark after Lemma 17 implies that $\psi = \psi_1 + \psi_2$ where

$$\psi_1 = z \sum_{i=1}^{n} m_i \psi(x_i) e^{-\frac{1}{2}|x-x_i|} \tag{4.27}$$

and $\psi_2(x,t)$ is a C^∞ function with respect to t and x. So, ψ_2 obviously satisfies the assertion. For ψ_1 we proceed as follows:

$$\langle D_t \psi_1 \rangle(x_j) = z \sum_{i=1}^{n} \Big((m_i \psi(x_i))_t e^{-\frac{1}{2}|x_j - x_i|} + \frac{1}{2} \dot{x}_i m_i \psi(x_i) e^{-\frac{1}{2}|x_j - x_i|} Sgn(x_j - x_i) \Big),$$

$$\langle \psi_{1x} \rangle(x_j) = z \sum_{i=1}^{n} -\frac{1}{2} m_i \psi(x_i) e^{-\frac{1}{2}|x_j - x_i|} Sgn(x_j - x_i),$$

and

$$\begin{aligned}\big(\psi_1(x_j)\big)_t &= z \sum_{i=1}^{n} \Big((m_i \psi(x_i))_t e^{-\frac{1}{2}|x_j - x_i|} - \frac{1}{2} (\dot{x}_j - \dot{x}_i) Sgn(x_j - x_i) e^{-\frac{1}{2}|x_j - x_i|} \Big) \\ &= \dot{x}_j \langle \psi_{1x} \rangle(x_j) + \langle D_t \psi_1 \rangle(x_j).\end{aligned}$$

□

Now, we are ready to prove the main theorem of this section.

Theorem 5. *Suppose that*

$$u(x,t) = -\frac{\kappa}{2} + \frac{1}{2} \sum_{j=1}^{n} m_j e^{-|x-x_j|}.$$

Then (4.20) *implies the following system of ODEs:*

$$\begin{cases} \dot{x}_j = u(x_j), \\ \dot{m}_j = -\big(\langle u_x \rangle(x_j)\big) m_j. \end{cases} \tag{4.28}$$

Proof. The first equation of the system follows from (4.24)).

Now consider the equation (4.26). We have

$$[u_x \psi_x](x_j) = [u_x](x_j) \langle \psi_x \rangle(x_j) + [\psi_x](x_j) \langle u_x \rangle(x_j).$$

Therefore, applying Proposition 6 and using $[u_x](x_j) = -m_j$ and $[\psi_x](x_j) = -zm_j\psi(x_j)$, we obtain

$$\begin{aligned} &2z\big(\dot{m_j}\psi(x_j) + m_j\dot{x_j}\langle\psi_x\rangle(x_j) + m_j\langle\psi_t\rangle(x_j)\big) \\ &= -m_j\langle\psi_x\rangle(x_j) - zm_j\psi(x_j)\langle u_x\rangle(x_j) + 2\alpha z m_j\psi(x_j) - \psi(x_j)[u_{xx}](x_j). \end{aligned} \tag{4.29}$$

Now, from the second equation in (4.20) we get,

$$\langle\psi_t\rangle(x_j) = -(\frac{1}{2z} + u(x_j))\langle\psi_x\rangle(x_j) + \big(\frac{1}{2}\langle u_x\rangle(x_j) + \alpha\big)\psi(x_j). \tag{4.30}$$

Substituting (4.30) for $\langle\psi_t\rangle(x_j)$ in the equation (4.29), using $\dot{x_j} = u(x_j)$ and simplifying the equation we obtain

$$\dot{m_j} = -\big(\langle u_x\rangle(x_j)\big)m_j.$$

□

Theorem 6. *Suppose that*

$$u = -\frac{\kappa}{2} + \frac{1}{2}\sum_{i=1}^{n} m_i e^{-|x-x_i|}.$$

Then there exists a ψ that satisfies (4.20), $\psi(x,t;z) = e^{\frac{1}{2}x}$ *on* $(-\infty, x_1)$ *and*

$$\psi(x,t;z) = A_n(t;z)e^{-\frac{1}{2}x} + B_n(t;z)e^{\frac{1}{2}x}$$

on (x_n, ∞)*, where B_n is independent of time.*

Proof. Suppose that $x < x_1$. We can write

$$u = -\frac{\kappa}{2} + \frac{1}{2}M^{-}e^{\frac{1}{2}x},$$

where $M^{-} = \sum_{i=1}^{n} m_i e^{-x_i}$. Therefore, by the second equation of (4.20) we have

$$0 = -(\frac{1}{2z} - \frac{\kappa}{2} + \frac{1}{2}M^{-}e^{x})(\frac{1}{2}e^{\frac{1}{2}x}) + (\frac{1}{4}M^{-}e^{x} + \alpha)e^{\frac{1}{2}x}.$$

Thus,

$$0 = \big(\alpha - \frac{1}{2}(\frac{1}{2z} - \frac{\kappa}{2})\big)e^{\frac{1}{2}x}.$$

This is true because $\alpha = \frac{1}{2}(\frac{1}{2z} - \frac{\kappa}{2})$. The proof of Lemma 17 implies that, on (x_j, x_{j+1}),

$$\psi = A_j e^{-\frac{1}{2}x} + B_j e^{\frac{1}{2}x},$$

where $A_0 = 0$, $B_0 = 1$ and

$$B_j = 1 + z\sum_{i=1}^{j} m_i\psi(x_i)e^{-\frac{1}{2}x_i}, \tag{4.31}$$

$$A_j = -z\sum_{i=1}^{j} m_i\psi(x_i)e^{\frac{1}{2}x_i}. \tag{4.32}$$

Now suppose that $x > x_n$ and let

$$u = \frac{\kappa}{2} + \frac{1}{2}M^+e^{-x},$$

where $M^+ = \sum_{i=1}^{n} m_i e^{x_i}$. By the second equation of (4.20) we have

$$\dot{A}_n e^{-\frac{1}{2}x} + \dot{B}_n e^{\frac{1}{2}x} = -(\frac{1}{2z} - \frac{\kappa}{2} + \frac{M^+}{2}e^{-x})(-\frac{1}{2}A_n e^{-\frac{1}{2}x} + \frac{1}{2}B_n e^{\frac{1}{2}x})$$
$$(-\frac{M^+}{4}e^{-x} + \alpha)(A_n e^{-\frac{1}{2}x} + B_n e^{\frac{1}{2}x}).$$

Comparing the coefficients of $e^{-\frac{1}{2}x}$ and $e^{\frac{1}{2}x}$ in both sides of the equation, we obtain

$$\begin{aligned} \dot{B}_n &= \big(\alpha - \frac{1}{2}(\frac{1}{2z} - \frac{\kappa}{2})\big)B_n, \\ \dot{A}_n &= \big(\alpha + \frac{1}{2}(\frac{1}{2z} - \frac{\kappa}{2})\big)A_n - \frac{1}{2}M^+B_n. \end{aligned} \tag{4.33}$$

Since $\alpha = \frac{1}{2}(\frac{1}{2z} - \frac{\kappa}{2})$, then B_n must be independent of time. □

Corollary 6. Suppose that $\psi(x, t; z)$ is given as in Theorem 6. Then we have

$$\begin{aligned} &\dot{B}_n(z) = 0, \quad \text{for every } z \in \mathbb{C}, \\ &\dot{A}_n(z) = 2\alpha(z)A_n(z) - \frac{1}{2}M^+B_n(z), \quad \text{for every } z \in \mathbb{C}. \end{aligned} \tag{4.34}$$

Furthermore, if z_j is a root of polynomial $B_n(z)$, then

$$A_n(t; z_j) = A_n(0; z_j)e^{2\alpha(z_j)t}.$$

Proof. Equations (4.34) are obtained in the proof of Theorem 6. If z_j is a root of $B_n(z)$, then by the second equation of (4.34) we have

$$\dot{A}_n(z_j) = 2\alpha(z_j)A_n(z_j). \tag{4.35}$$

Hence, $A_n(t; z_j) = A_n(0; z_j)e^{2\alpha(z_j)t}$. $\square$

In Chapter 5 we show how this corollary is applied to solve peakon equations (4.28).

Chapter 5

Peakons and the Continued Fractions

5.1 The Camassa-Holm equation and the string problem

Let us look at the Camassa-Holm equation from a different point of view. This is based on the work of R. Beals, D. H. Sattinger and J. Szmigielski [22]. Suppose u is a solution to the equation

$$\begin{cases} u_t - u_{xxt} + \kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

As we have seen before, we can rewrite equation (5.1) in the following form:

$$m_t + 2mu_x + m_x u = 0, \tag{5.2}$$

where $m = u - u_{xx} + \frac{1}{2}\kappa$. In Chapter 2, we observed that equation (5.2) is the compatibility condition $\psi_{xxt} = \psi_{txx}$, for the system

$$\begin{cases} \psi_{xx} = \frac{1}{4}\psi - zm\psi, \\ \psi_t = -(\frac{1}{2z} + u)\psi_x + \frac{1}{2}u_x\psi. \end{cases} \tag{5.3}$$

Now consider the change of variable $y = \tanh(\frac{x}{2})$. We have

$$\frac{dy}{dx} = \frac{1}{2}sech^2(\frac{x}{2}) = \frac{1}{2}(1-y^2).$$

and

$$\psi_x = \frac{1}{2}(1-y^2)\psi_y.$$

$$\psi_{xx} = \frac{1}{2}(1-y^2)\big(-y\psi_y + \frac{1}{2}(1-y^2)\psi yy\big).$$

Therefore

$$\begin{aligned} \psi_{xx} - \frac{1}{4}\psi &= -\frac{1}{4}\psi - \frac{1}{2}y(1-y^2)\psi_y + \frac{1}{4}(1-y^2)^2\psi_{yy} \\ &= \frac{1}{4}(1-y^2)^{\frac{3}{2}}\bigg(-(1-y^2)^{-\frac{3}{2}}\psi - 2y(1-y^2)^{-\frac{1}{2}}\psi_y + (1-y^2)^{\frac{1}{2}}\psi_{yy}\bigg). \end{aligned} \tag{5.4}$$

Now let $\phi = (1-y^2)^{\frac{1}{2}}\psi$. Then we have

$$\begin{aligned} \phi_{yy} &= \big((1-y^2)^{\frac{1}{2}}\big)_{yy}\psi + 2\big((1-y^2)^{\frac{1}{2}}\big)_y\psi_y + (1-y^2)^{\frac{1}{2}}\psi_{yy} \\ &= -(1-y^2)^{-\frac{3}{2}}\psi - 2y(1-y^2)^{-\frac{1}{2}}\psi_y + (1-y^2)^{\frac{1}{2}}\psi_{yy}. \end{aligned} \tag{5.5}$$

Comparing (5.4) and (5.5) we have

$$\psi_{xx} - \frac{1}{4}\psi = \frac{1}{4}(1-y^2)^{\frac{3}{2}}\phi_{yy}.$$

Therefore, from the first equation of (5.3) we get

$$\frac{1}{4}(1-y^2)^{\frac{3}{2}}\phi_{yy} = -zm(1-y^2)^{-\frac{1}{2}}\phi.$$

Hence

$$\phi_{yy} = \frac{-4zm}{(1-y^2)^2}\phi.$$

In other words we can write

$$\phi_{yy} = -zg(y)\phi, \qquad -1 < y < 1,$$

where $g(y) = \frac{4m}{(1-y^2)^2}$.

Proposition 7. *If $m(x) = \sum_{i=1}^n m_i\delta_{x_i}$, is a discrete measure then the above transformation transforms $m(x)$ to $g(y) = \sum_{i=1}^n g_i\delta_{y_i}$ where $g_i = \frac{2m_i}{1-y_i^2}$.*

Proof. We can write

$$
\begin{aligned}
g(y) &= \big(\frac{2}{1-y^2}\big)^2 m(x) \\
&= \sum_{i=1}^{n} \big(\frac{2}{1-y_i^2}\big)^2 m_i \delta_{x_i} \\
&= \sum_{i=1}^{n} \big(\frac{2}{1-y_i^2}\big)^2 m_i \frac{dy}{dx}(y_i) \delta_{y_i} \\
&= \sum_{i=1}^{n} \big(\frac{2}{1-y_i^2}\big)^2 m_i \frac{1-y_i^2}{2} \delta_{y_i} \\
&= \sum_{i=1}^{n} \frac{2}{1-y_i^2} m_i \delta_{y_i}
\end{aligned}
$$

□

5.2 A discrete string problem

Consider the inhomogeneous string equation

$$
\begin{aligned}
&u_{xx} = m(x) u_{tt}, \\
&u(-1,t) = u(1,t) = 0,
\end{aligned}
\tag{5.6}
$$

where $m(x) = \sum_{j=1}^{n} m_j \delta_{x_j}$ is a positive discrete measure, and use the method of separation of variables. Let $u(x,t) = \phi(x) g(t)$. We will have $\phi_{xx} g(t) = m(x)\phi g_{tt}$ or $\phi_{xx} = \frac{g_{tt}}{g} m(x)\phi$. Setting $\lambda = \frac{g_{tt}}{g}$, we obtain the following eigenvalue problem:

$$
\phi_{xx} = \lambda m(x) \phi, \tag{5.7}
$$

with the initial conditions $\phi(-1,\lambda) = 0,\ \phi_x(-1^+,\lambda) = 1$, first considered by M. G. Krein [5]. If the distance between x_{j-1} and x_j is denoted by l_j, we have the following relations:

$$
\begin{aligned}
&\phi(x_1,\lambda) = l_1, \\
&\phi_x(x_1^+,\lambda) - \phi_x(x_1^-,\lambda) = \lambda m_1 \phi(x_1,\lambda), \\
&\phi_x(x_1^-,\lambda) = \phi_x(-1^+,\lambda),
\end{aligned}
$$

and more generally

$$
\begin{aligned}
&\phi(x_j,\lambda)-\phi(x_{j-1},\lambda)=l_j\phi_x(x_{j-1}^+,\lambda),\\
&\phi_x(x_j^+,\lambda)-\phi_x(x_j^-,\lambda)=\lambda m_j\phi(x_j,\lambda),\\
&\phi_x(x_j^-,\lambda)=\phi_x(x_{j-1}^+,\lambda),
\end{aligned}
\tag{5.8}
$$

where $j=1,\ldots,n+1$, $x_0=-1$ and $x_{n+1}=1$. Equations (5.8) describe a discrete string consisting of point masses at x_j with masses m_j, tied at the left end. Now let $q_j=\phi(x_j,\lambda)$, $p_j=\phi_x(x_j^+,\lambda)$ for $j=0,\ldots,n+1$. Then we have

$$
\begin{aligned}
&q_j-q_{j-1}=l_jp_{j-1},\\
&p_j-p_{j-1}=\lambda m_jq_j,
\end{aligned}
\tag{5.9}
$$

for $j=1,\ldots,n+1$ and the initial conditions $q_0=0$, $p_0=1$. Let's write

$$
\begin{aligned}
&q_j=l_jp_{j-1}+q_{j-1},\\
&p_j=\lambda m_jq_j+p_{j-1}.
\end{aligned}
\tag{5.10}
$$

Then we have

$$\frac{p_{j-1}}{q_j}=\cfrac{1}{l_j+\cfrac{q_{j-1}}{p_{j-1}}}=\cfrac{1}{l_j+\cfrac{1}{\lambda m_{j-1}+\cfrac{p_{j-2}}{q_{j-1}}}}.$$

In particular we have

$$\frac{p_1}{q_2}=\cfrac{1}{l_2+\cfrac{q_1}{p_1}}=\cfrac{1}{l_2+\cfrac{1}{\lambda m_1+\cfrac{1}{l_1}}}.$$

Hence, by induction we can prove that

$$\frac{p_n}{q_{n+1}} = \cfrac{1}{l_{n+1} + \cfrac{1}{\lambda m_n + \cfrac{1}{l_n + \cfrac{1}{\ddots + \cfrac{1}{\lambda m_1 + \cfrac{1}{l_1}}}}}}. \tag{5.11}$$

Definition 9. The Weyl function of the discrete string problem is defined by

$$W(\lambda) = \frac{\phi_x(1^-,\lambda)}{\phi(1,\lambda)}.$$

Definition 10. The continued fractions

$$f_{2k}(\lambda) = \cfrac{1}{l_{n+1} + \cfrac{1}{\lambda m_n + \cfrac{1}{l_n + \cfrac{1}{\ddots + \cfrac{1}{l_{n-k+2} + \cfrac{1}{\lambda m_{n-k+1}}}}}}}, \tag{5.12}$$

$k = 1, \ldots, n,$

and

$$f_{2k+1}(\lambda) = \cfrac{1}{l_{n+1} + \cfrac{1}{\lambda m_n + \cfrac{1}{l_n + \cfrac{1}{\ddots + \cfrac{1}{\lambda m_{n-k+1} + \cfrac{1}{l_{n-k+1}}}}}}}, \tag{5.13}$$

$k = 0, \ldots, n,$

are called the $2k$th and the $2k+1$th convergents of the Weyl function respectively. These convergents were originally introduced by Stieltjes [29].

Now let

$$M_j = \begin{pmatrix} 1 & 0 \\ \lambda m_j & 1 \end{pmatrix}, \quad L_j = \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix}. \tag{5.14}$$

Therefore, (5.10) implies

$$\begin{pmatrix} q_j \\ p_{j-1} \end{pmatrix} = L_j \begin{pmatrix} q_{j-1} \\ p_{j-1} \end{pmatrix}, \quad \begin{pmatrix} q_j \\ p_j \end{pmatrix} = M_j \begin{pmatrix} q_j \\ p_{j-1} \end{pmatrix}. \tag{5.15}$$

Hence, applying the initial conditions $q_0 = 0,\ p_0 = 1$, we have

$$\begin{pmatrix} q_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} \phi(1,\lambda) \\ \phi_x(1^-,\lambda) \end{pmatrix} = L_{n+1} M_n L_n \cdots L_2 M_1 L_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.16}$$

Let

$$L_{n+1} = \begin{pmatrix} Q_0 & Q_1 \\ P_0 & P_1 \end{pmatrix}.$$

Since multiplication by M_j from right leaves the second column invariant, and multiplication by L_j from right leaves the first column invariant, we can write

$$L_{n+1}M_n = \begin{pmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{pmatrix},$$

and

$$L_{n+1}M_nL_n = \begin{pmatrix} Q_2 & Q_3 \\ P_2 & P_3 \end{pmatrix}.$$

We observe that

$$\frac{P_0}{Q_0} = \frac{0}{1},\ \frac{P_1}{Q_1} = \frac{1}{l_{n+1}},\ \frac{P_2}{Q_2} = \frac{\lambda m_n}{1+\lambda m_n l_{n+1}},\ \frac{P_3}{Q_3} = \frac{\lambda m_n l_n + 1}{\lambda m_n l_n l_{n+1} + l_n + l_{n+1}}$$

are the convergents of the Weyl function defined above. By induction we can define P_i and Q_i as follows:

$$\begin{pmatrix} Q_{2k} & Q_{2k-1} \\ P_{2k} & P_{2k-1} \end{pmatrix} = \begin{pmatrix} Q_{2k-2} & Q_{2k-1} \\ P_{2k-2} & P_{2k-1} \end{pmatrix} M_{n-k+1}, \quad k = 1, \ldots, n, \tag{5.17}$$

and

$$\begin{pmatrix} Q_{2k} & Q_{2k+1} \\ P_{2k} & P_{2k+1} \end{pmatrix} = \begin{pmatrix} Q_{2k} & Q_{2k-1} \\ P_{2k} & P_{2k-1} \end{pmatrix} L_{n-k+1}, \quad k = 1, \ldots, n, \tag{5.18}$$

or equivalently,

$$\begin{aligned} &P_{2k} = \lambda m_{n-k+1} P_{2k-1} + P_{2k-2}, \\ &Q_{2k} = \lambda m_{n-k+1} Q_{2k-1} + Q_{2k-2}, \\ &k = 1, \ldots, n, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} &P_{2k+1} = l_{n-k+1} P_{2k} + P_{2k-1}, \\ &Q_{2k+1} = l_{n-k+1} Q_{2k} + Q_{2k-1}, \\ &k = 1, \ldots, n. \end{aligned} \tag{5.20}$$

Theorem 7. *The leading coefficients of $Q_{2k}(\lambda)$ and $Q_{2k+1}(\lambda)$ are equal to*

$$l_{n+1}m_nl_nm_{n-1}\cdots l_{n-k+2}m_{n-k+1}$$

and

$$l_{n+1}m_nl_n\cdots m_{n-k+1}l_{n-k+1}$$

respectively.

Proof. Use (5.19) and (5.20) and apply induction. □

Theorem 8. *The following statements hold:*

a) $\frac{P_i}{Q_i}$ *is the ith convergent of* $W(\lambda)$*. In particular,* $W(\lambda) = \frac{P_{2n+1}}{Q_{2n+1}}$.

b) P_{2k+1}*,* Q_{2k+1}*,* P_{2k} *and* Q_{2k} *are of degree* k*.*

c) $P_{2k}(0) = 0$*,* $Q_{2k}(0) = 1$*,* $P_{2k+1}(0) = 1$ *and* $Q_{2k+1}(0) = \sum_{j=0}^{k} l_{n-j+1}$.

$k = 0, \ldots, n$.

Proof. (a) By induction we can show that equations (5.19) and (5.20) characterize the numerators and the denominators of the convergents of $W(\lambda)$.

(b) Use (5.19) and (5.20) and apply induction.

(c) It is proved by setting $\lambda = 0$ in (5.17) and (5.18) and applying induction. □

Proposition 8. *The eigenvalues of the boundary value problem*

$$\phi_{xx} = \lambda m\phi,$$

$$\phi(-1, \lambda) = \phi(1, \lambda) = 0,$$

where $m = \sum_{j=1}^{n} m_j\delta_{x_j}$*, are all real and negative.*

Proof. We have $\bar{\phi}_x(x_j^+, \lambda) - \bar{\phi}_x(x_j^-, z) = \bar{\lambda}m_j\bar{\phi}(x_j, \lambda)$, where $\bar{\lambda}$ is the complex conjugate of the number λ. So we can write

$$(\bar{\phi}_x(x_j^+,\lambda) - \bar{\phi}_x(x_j^-,\lambda))\phi(x_j) = \bar{\lambda} m_j \bar{\phi}(x_j)\phi(x_j),$$

$$(\phi_x(x_j^+,\lambda) - \phi_x(x_j^-,\lambda))\bar{\phi}(x_j) = \lambda m_j \phi(x_j)\bar{\phi}(x_j).$$

Therefore,

$$(\phi_x(x_j^+,\lambda) - \phi_x(x_j^-,\lambda))\bar{\phi}(x_j) - (\bar{\phi}_x(x_j^+,\lambda) - \bar{\phi}_x(x_j^-,\lambda))\phi(x_j) = (\lambda - \bar{\lambda}) m_j |\phi(x_j)|^2.$$

In other words,

$$\begin{vmatrix} \phi(x_j) & \bar{\phi}(x_j) \\ \phi_x(x_j^-) & \bar{\phi}_x(x_j^-) \end{vmatrix} - \begin{vmatrix} \phi(x_j) & \bar{\phi}(x_j) \\ \phi_x(x_j^+) & \bar{\phi}_x(x_j^+) \end{vmatrix} = (\lambda - \bar{\lambda}) m_j |\phi(x_j)|^2. \tag{5.21}$$

Now using $\phi_x(x_j^-,\lambda) = \phi_x(x_{j-1}^+,\lambda)$ and $\phi(x_j,\lambda) - \phi(x_{j-1},\lambda) = l_j \phi_x(x_{j-1}^+,\lambda)$, we see that

$$\begin{vmatrix} \phi(x_{j-1}) & \bar{\phi}(x_{j-1}) \\ \phi_x(x_{j-1}^+) & \bar{\phi}_x(x_{j-1}^+) \end{vmatrix} = \begin{vmatrix} \phi(x_j) & \bar{\phi}(x_j) \\ \phi_x(x_j^-) & \bar{\phi}_x(x_j^-) \end{vmatrix}.$$

Hence,

$$0 = \begin{vmatrix} \phi(-1,\lambda) & \bar{\phi}(-1,\lambda) \\ \phi_x(-1^-,\lambda) & \bar{\phi}_x(-1^-,\lambda) \end{vmatrix} - \begin{vmatrix} \phi(1,\lambda) & \bar{\phi}(1,\lambda) \\ \phi_x(1^+,\lambda) & \bar{\phi}_x(1^+,\lambda) \end{vmatrix} = (\lambda - \bar{\lambda}) \sum_{j=0}^{n+1} m_j |\phi(x_j)|^2.$$

Thus λ is real. To prove λ is negative we note that we can write

$$(\phi_x(x_j^+,\lambda) - \phi_x(x_j^-,\lambda))\phi(x_j,\lambda) = \lambda m_j \phi^2(x_j,\lambda).$$

Therefore

$$\phi_x(1^+,\lambda)\phi(1,\lambda) - \phi_x(-1^-,\lambda)\phi(-1,\lambda) + \sum_{j=0}^{n} -l_{j+1}\phi_x^2(x_j^+,\lambda) = \lambda \sum_{j=0}^{n+1} m_j \phi^2(x_j,\lambda).$$

So, using the boundary conditions we have

$$\sum_{j=0}^{n} -l_{j+1}\phi_x^2(x_j^+,\lambda) = \lambda \sum_{j=1}^{n} m_j \phi^2(x_j,\lambda).$$

Hence λ is negative. $\square$

The following proposition was presented in [23].

Proposition 9. *The problem* (5.9) *with* $q_{n+1} = 0$ *implies the following matrix problem:*

$$Lq = \lambda Mq, \tag{5.22}$$

where

$$L = \begin{pmatrix} -(\frac{1}{l_1} + \frac{1}{l_2}) & \frac{1}{l_2} & 0 & 0 & \dots & 0 \\ \frac{1}{l_2} & -(\frac{1}{l_2} + \frac{1}{l_3}) & \frac{1}{l_3} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \frac{1}{l_{n-1}} & -(\frac{1}{l_{n-1}} + \frac{1}{l_n}) & \frac{1}{l_n} \\ 0 & \dots & 0 & 0 & \frac{1}{l_n} & -(\frac{1}{l_n} + \frac{1}{l_{n+1}}) \end{pmatrix},$$

$$q = (q_1, q_2, \dots, q_n)^T, \quad M = diag\{m_1, \dots, m_n\}.$$

Proof. Suppose (5.9) holds. In the first equation of (5.9) we solve for p_{j-1} and substitute that into the second equation. We get

$$\frac{1}{l_j} q_{j-1} - (\frac{1}{l_j} + \frac{1}{l_{j+1}}) q_j + \frac{1}{l_{j+1}} q_{j+1} = \lambda m_j q_j, \quad j = 1, \dots, n. \tag{5.23}$$

Now (5.23) and $q_0 = q_{n+1} = 0$, imply (5.22). $\square$

Proposition 10. *The problem* (5.22) *implies the Jacobi spectral problem*

$$JU = \lambda U, \tag{5.24}$$

where

$$U = M^{1/2} q, \quad J = M^{-1/2} L M^{-1/2}. \tag{5.25}$$

We see that

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & \\ a_1 & b_2 & a_2 & \ddots & & \vdots \\ 0 & a_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{n-1} & a_{n-1} \\ 0 & \dots & \dots & 0 & a_{n-1} & b_n \end{pmatrix}, \tag{5.26}$$

where

$$b_j = -\frac{1}{m_j}(\frac{1}{l_j} + \frac{1}{l_{j+1}}),\ j = 1, \dots, n, \tag{5.27}$$

$$a_j = \frac{1}{l_{j+1}\sqrt{m_j m_{j+1}}},\ j = 1, \dots, n-1. \tag{5.28}$$

Theorem 9. *The zeros of $\phi(1,\lambda)$ are all simple.*

Proof. Every zero of $\phi(1,\lambda)$ is an eigenvalue of the problem (5.24). Now, looking at the problem (5.9) we observe that for every λ there is a unique solution up to a constant multiple depending on p_0. Therefore, the geometric multiplicity of those eigenvalues of the problem (5.24) that are zeros of $\phi(1,\lambda)$ is equal to 1. Since J is a symmetric matrix, it is diagonalizable. So, the geometric multiplicity of every eigenvalue of J is equal to its algebraic multiplicity. Hence, every zero of $\phi(1,\lambda)$ is simple. □

Corollary 7. The set of zeros of $\phi(1,\lambda)$ is equal to the set of eigenvalues of the problem (5.24).

Proof. Every zero of $\phi(1,\lambda)$ is a zero of $|J-\lambda I|$. Since, both $\phi(1,\lambda)$ and $|J-\lambda I|$ are of degree n and every zero of $\phi(1,\lambda)$ is simple, the assertion follows. □

Corollary 8.

$$\phi(1,\lambda) = 2\prod_{j=1}^{n}(1 - \frac{\lambda}{\lambda_j}). \tag{5.29}$$

Proof. If $\lambda = 0$, (5.8) implies that $\phi(x, 0) = x + 1$. Therefore, $\phi(1, 0) = 2$. So the constant term of $\phi(1, \lambda)$ is equal to 2. $\square$

Theorem 10. *The residue of* $\frac{W(\lambda)}{\lambda}$ *at any eigenvalue* λ_j *is positive.*

Proof. Suppose that λ is a zero of $\phi(1, \lambda)$. Let $p_j'(\lambda)$ denote the derivative of the polynomial $p_j(\lambda)$ with respect to λ. From (5.9) we have

$$(p_j'(\lambda) - p_{j-1}'(\lambda))q_j(\lambda) = m_j q_j^2(\lambda) + \lambda m_j q_j(\lambda) q_j'(\lambda),$$

and

$$(p_j(\lambda) - p_{j-1}(\lambda))q_j'(\lambda) = \lambda m_j q_j(\lambda) q_j'(\lambda).$$

Therefore

$$(p_j'(\lambda) - p_{j-1}'(\lambda))q_j(\lambda) - (p_j(\lambda) - p_{j-1}(\lambda))q_j'(\lambda) = m_j q_j^2(\lambda).$$

Thus, using the fact that $q_{n+1}(\lambda) = 0$ implies $p_{n+1}(\lambda) = p_n(\lambda)$, we have

$$-p_n(\lambda) q_{n+1}'(\lambda) = \sum_{j=1}^{n+1} m_j q_j^2(\lambda).$$

So, $p_n(\lambda)$ and $q_{n+1}'(\lambda)$ have opposite signs. Since λ_j is negative and it is a simple zero of $q_{n+1}(\lambda)$, we have

$$Res_{\lambda=\lambda_j} \frac{W(\lambda)}{\lambda} = \frac{p_n(\lambda_j)}{\lambda_j q_{n+1}'(\lambda_j)} > 0.$$

$\square$

Corollary 9.

$$\frac{W(\lambda)}{\lambda} = \frac{1}{2\lambda} + \sum_{j=1}^{n} \frac{r_j}{\lambda - \lambda_j}, \qquad r_j > 0, \tag{5.30}$$

where $r_j = Res_{\lambda=\lambda_j} \frac{W(\lambda)}{\lambda}$.

Corollary 10.

$$\frac{W(\lambda)}{\lambda} = \int \frac{d\mu(x)}{\lambda - x}, \tag{5.31}$$

where $d\mu = \sum_{j=0}^{n} r_j \delta_{\lambda_j}$, $\lambda_0 = 0$ and $r_0 = 1/2$.

Now we compute the coefficients of the Laurent expansion of $\frac{W(\lambda)}{\lambda}$ (See [29]). Since $\frac{W(\lambda)}{\lambda}$ is analytic at ∞, for λ large enough we can write

$$\frac{W(\lambda)}{\lambda} = \sum_{j=0}^{\infty} \frac{c_j}{\lambda^{j+1}}, \tag{5.32}$$

where $c_j = \int x^j d\mu(x)$. So, we have $c_j = \sum_{k=0}^{n} r_k \lambda_k^j$, for $j = 0, 1, 2, \ldots$.

We observe that if j is even, $c_j > 0$ and if j is odd, $c_j < 0$. We can write

$$\frac{W(\lambda)}{\lambda} = \sum_{j=0}^{\infty} \frac{(-1)^j A_j}{\lambda^{j+1}}, \tag{5.33}$$

where $A_j = (-1)^j c_j$. So, $A_j = (-1)^j \int x^j d\mu(x) = \int (-x)^j d\mu(x)$.

The following approximations are due to Stieltjes [29].

Proposition 11.

$$W(\lambda) - \frac{P_{2k}(\lambda)}{Q_{2k}(\lambda)} = \mathcal{O}(\frac{1}{\lambda^{2k}}), \quad \lambda \to \infty, \quad k = 0, 1, \ldots, n \tag{5.34}$$

and

$$W(\lambda) - \frac{P_{2k+1}(\lambda)}{Q_{2k+1}(\lambda)} = \mathcal{O}(\frac{1}{\lambda^{2k+1}}), \quad \lambda \to \infty, \quad k = 0, 1, \ldots, n. \tag{5.35}$$

Proof. Using the fact that

$$\begin{gathered} det \begin{pmatrix} Q_{2k} & Q_{2k+1} \\ P_{2k} & P_{2k+1} \end{pmatrix} = 1, \quad det \begin{pmatrix} Q_{2k} & Q_{2k-1} \\ P_{2k} & P_{2k-1} \end{pmatrix} = 1 \\ k = 0, 1, \ldots, n, \qquad\qquad k = 1, \ldots, n, \end{gathered} \tag{5.36}$$

we have

$$\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} = \frac{1}{Q_{2n}Q_{2n+1}} = \mathcal{O}(\frac{1}{\lambda^{2n}}), \quad \lambda \to \infty, \tag{5.37}$$

and

$$\frac{P_{2n}}{Q_{2n}} - \frac{P_{2n-1}}{Q_{2n-1}} = \frac{-1}{Q_{2n-1}Q_{2n}} = \mathcal{O}(\frac{1}{\lambda^{2n-1}}), \quad \lambda \to \infty. \tag{5.38}$$

Now adding (5.37) and (5.38) we see that

$$\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n-1}}{Q_{2n-1}} = \mathcal{O}(\frac{1}{\lambda^{2n-1}}), \quad \lambda \to \infty.$$

Hence, induction completes the proof. $\square$

Proposition 12. *(orthogonality)*

$$\int x^j Q_{2k}(x) d\mu(x) = 0, \quad j = 0, \dots, k-1, \tag{5.39}$$

$$\int x^j Q_{2k+1}(x) d\mu(x) = 0, \quad j = 1, \dots, k, \tag{5.40}$$

$$\int Q_{2k+1}(x) d\mu(x) = 1. \tag{5.41}$$

Proof. Using equation (5.34) we can write

$$W(\lambda)Q_{2k}(\lambda) - P_{2k}(\lambda) = \mathcal{O}(\frac{1}{\lambda^k}), \quad \lambda \to \infty, \quad k = 0, \dots, n. \tag{5.42}$$

Also,

$$\lambda^j (W(\lambda)Q_{2k}(\lambda) - P_{2k}(\lambda)) = \mathcal{O}(\frac{1}{\lambda^{k-j}}), \quad \lambda \to \infty, \quad k = 0, \dots, n, \quad j = 0, \dots, k-2.$$

Now, let γ be a circle of large radius containing the support of μ in its interior. Then using equation (5.31) we have,

$$\frac{1}{2\pi i} \int_\gamma \int \frac{\lambda^{j+1} Q_{2k}(\lambda)}{\lambda - x} d\mu(x) d\lambda = 0, \quad j = 0, \dots, k-2.$$

Thus, applying Fubini's theorem and Cauchy's residue theorem we get

$$\int x^j Q_{2k}(x) d\mu(x) = 0, \quad j = 1, \dots, k-1.$$

The case $j = 0$ follows from the fact that $P_{2k}(0) = 0$ and dividing equation (5.42) by λ. Similarly, (5.35) implies that

$$W(\lambda)Q_{2k+1}(\lambda) - P_{2k+1}(\lambda) = \mathcal{O}(\frac{1}{\lambda^{k+1}}), \quad \lambda \to \infty, \quad k = 0, \dots, n. \tag{5.43}$$

Therefore,

$$\lambda^j(W(\lambda)Q_{2k+1}(\lambda) - P_{2k+1}(\lambda)) = \mathcal{O}(\frac{1}{\lambda^{k+1-j}}),\ \lambda \to \infty,\ k = 0, \dots, n, \quad j = 0, \dots, k-1.$$

So we have

$$\frac{1}{2\pi i} \int_\gamma \int \frac{\lambda^{j+1} Q_{2k+1}(\lambda)}{\lambda - x} d\mu(x) d\lambda = 0, \quad j = 0, \dots, k-1.$$

Again applying Fubini's theorem and Cauchy's residue theorem we obtain

$$\int x^j Q_{2k+1}(x) d\mu(x) = 0, \quad j = 1, \dots, k.$$

Now (5.41) follows from the fact that $P_{2k+1}(0) = 1$ and dividing equation (5.43) by λ. $\square$

We are going to construct the approximants $\frac{P_i}{Q_i}$ according to the asymptotic behaviour of $W(\lambda)$. The formulas are due to Stieltjes [29].

Theorem 11.

$$Q_{2k}(\lambda) = \frac{1}{\Delta_k^1} \begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^k \\ c_0 & c_1 & c_2 & \dots & c_k \\ c_1 & c_2 & c_3 & \dots & c_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{k-1} & c_k & c_{k+1} & \dots & c_{2k-1} \end{vmatrix}, \quad k = 0, 1, \dots, n, \tag{5.44}$$

$$Q_{2k+1}(\lambda) = \frac{1}{\Delta_{k+1}^0} \begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^k \\ c_1 & c_2 & c_3 & \dots & c_{k+1} \\ c_2 & c_3 & c_4 & \dots & c_{k+2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & c_{k+2} & \dots & c_{2k} \end{vmatrix}, \quad k = 0, 1, \dots, n, \tag{5.45}$$

where

$$\Delta_k^1 = \begin{vmatrix} c_1 & c_2 & \dots & c_k \\ c_2 & c_3 & \dots & c_{k+1} \\ \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & \dots & c_{2k-1} \end{vmatrix}, \quad \Delta_k^0 = \begin{vmatrix} c_0 & c_1 & \dots & c_{k-1} \\ c_1 & c_2 & \dots & c_k \\ \vdots & \vdots & & \vdots \\ c_{k-1} & c_k & \dots & c_{2k-2} \end{vmatrix}, \quad k \in \mathbb{N}, \tag{5.46}$$

and by convention

$$\Delta_0^1 = \Delta_0^0 = 1. \tag{5.47}$$

Proof. We write $Q_{2k}(\lambda) = \sum_{i=0}^{k} q_i \lambda^i$. Now, using orthogonality condition (5.39) and the fact that $Q_{2k}(0) = 1$ we have

$$\int x^j (1 + \sum_{i=1}^{k} q_i x^i) d\mu(x) = 0, \quad j = 0, 1, \dots, k-1.$$

Since the jth moment is given by $c_j = \int x^j d\mu(x)$ we have

$$\sum_{i=1}^{k} c_{i+j} q_i = -c_j, \quad j = 0, 1, \dots, k-1.$$

Hence we obtain the system

$$Bq = -c,$$

where

$$B = \begin{pmatrix} c_1 & c_2 & \dots & c_k \\ c_2 & c_3 & \dots & c_{k+1} \\ \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & \dots & c_{2k-1} \end{pmatrix}, \quad q = (q_1, q_2 \dots, q_k)^T, \quad c = (c_0, c_1, \dots, c_{k-1})^T.$$

Thus, Cramer's rule implies that

$$Q_{2k}(\lambda) = \frac{1}{\Delta_k^1} \begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^k \\ c_0 & c_1 & c_2 & \dots & c_k \\ c_1 & c_2 & c_3 & \dots & c_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{k-1} & c_k & c_{k+1} & \dots & c_{2k-1} \end{vmatrix}.$$

Similarly, using (5.40) and (5.41) we obtain the following system

$$Cq = e_1,$$

where

$$C = \begin{pmatrix} c_0 & c_1 & \dots & c_k \\ c_1 & c_2 & \dots & c_{k+1} \\ \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & \dots & c_{2k} \end{pmatrix}, \quad q = (q_0, q_1 \dots, q_k)^T, \quad e_1 = (1, 0, \dots, 0)^T.$$

Therefore, we have $Q_{2k+1}(\lambda) = \frac{1}{\Delta_{k+1}^0} \sum_{j=0}^k (-1)^j C_j \lambda^j$, where C_j is obtained by eliminating the first row and the $j+1$th column of C. Thus

$$Q_{2k+1}(\lambda) = \frac{1}{\Delta_{k+1}^0} \begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^k \\ c_1 & c_2 & c_3 & \dots & c_{k+1} \\ c_2 & c_3 & c_4 & \dots & c_{k+2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & c_{k+2} & \dots & c_{2k} \end{vmatrix}.$$

$\square$

Theorem 12. *Let l_j and m_j be as in* (5.8). *Then*

$$m_{n-k+1} = \frac{(\Delta_k^0)^2}{\Delta_{k-1}^1 \Delta_k^1}, \quad k = 1, \dots, n, \tag{5.48}$$

$$l_{n-k+1} = \frac{(\Delta_k^1)^2}{\Delta_k^0 \Delta_{k+1}^0}, \quad k = 0, \dots, n, \tag{5.49}$$

where Δ_k^0 *and* Δ_k^1 *are given by* (5.46) *and* (5.47)*.*

Proof. Using equations (5.44) and (5.45) and Theorem 7 we have

$$\frac{(-1)^k \Delta_k^0}{\Delta_k^1} = l_{n+1} m_n l_n m_{n-1} \cdots l_{n-k+2} m_{n-k+1}$$

and

$$\frac{(-1)^k \Delta_k^1}{\Delta_{k+1}^0} = l_{n+1} m_n l_n \cdots m_{n-k+1} l_{n-k+1}$$

Hence, the assertion follows. □

5.3 Time Evolution of Peakons

In this section, we use the results obtained for the string problem to solve the peakon equations (4.28) [24]. We consider the peakon weak Lax pair (4.20) with $\psi(x, t; z)$ as in Theorem 6 and the boundary value problem

$$\begin{aligned} &\phi_{yy} = \lambda g \phi, \\ &\phi(-1, \lambda) = \phi(1, \lambda) = 0, \end{aligned} \tag{5.50}$$

where $\lambda = -z$, $\phi = (1 - y^2)^{\frac{1}{2}} \psi$ and g is given by Proposition 7. Now, we are going to see how the residues of $\frac{W(\lambda)}{\lambda}$ evolve in time. In the following theorem we assume that A_n and B_n are given as in Theorem 6.

Theorem 13. *The following hold:*

1. *The boundary value problem* (5.50) *is iso-spectral, that is its eigenvalues* λ_j *are time independent.*

2. $B_n(-\lambda_j) = 0.$

3. If r_j is the residue of $\frac{W(\lambda)}{\lambda}$ at $\lambda = \lambda_j$, then

$$r_j = r_j(0)e^{-\frac{1}{2}(\frac{1}{\lambda_j}+\kappa)t}. \tag{5.51}$$

where

$$r_j(0) = \frac{A_n(0; -\lambda_j)}{2\prod_{k\neq j}(1-\frac{\lambda_k}{\lambda_j})}$$

Proof. From Theorem 6 we have

$$\psi(x,t;z) = A_n(t;z)e^{-\frac{1}{2}x} + B_n(z)e^{\frac{1}{2}x}, \qquad x > x_n.$$

Using the transformation $y = \tanh(\frac{x}{2})$, which was introduced in section 5.1 and $z = -\lambda$, we obtain the following

$$\psi(x,t;z) = (1-y^2)^{-\frac{1}{2}}\phi(y,t;\lambda) = (\frac{1-y}{1+y})^{\frac{1}{2}}A_n(t;-\lambda) + (\frac{1+y}{1-y})^{\frac{1}{2}}B_n(-\lambda).$$

Thus,

$$\phi(y,t;\lambda) = (1-y)A_n(t;-\lambda) + (1+y)B_n(-\lambda), \qquad y_n < y < 1. \tag{5.52}$$

The spectrum of the boundary value problem (5.50) is determined by $\phi(1,t;\lambda) = 0$. Thus, if λ_j is an eigenvalue, $B_n(-\lambda_j) = 0$. Therefore by Corollary 6, λ_j is iso-spectral. Since $W(\lambda) = \frac{\phi_y(1^-,\lambda)}{\phi(1,\lambda)}$ and $\phi(1,\lambda_j) = 0$ we have

$$Rez_{\lambda=\lambda_j}\frac{W(\lambda)}{\lambda} = \frac{\phi_y(1^-,\lambda_j)}{\lambda_j\phi_\lambda(1,\lambda_j)}.$$

By (5.52) and $B_n(-\lambda_j) = 0$, we get

$$\phi_y(1^-,\lambda_j) = -A_n(t;-\lambda_j), \quad \phi_\lambda(1,\lambda_j) = -2B_n'(-\lambda_j).$$

Also, by (5.29) we have

$$\phi_\lambda(1,\lambda_j) = -\frac{2}{\lambda_j}\prod_{k\neq j}(1-\frac{\lambda_k}{\lambda_j}).$$

Hence, by Corollary 6 we have

$$r_j = \frac{A_n(0;-\lambda_j)}{2\prod_{k\neq j}(1-\frac{\lambda_k}{\lambda_j})}e^{-\frac{1}{2}(\frac{1}{\lambda_j}+\kappa)t}.$$

□

Now we are ready to say something about the time evolution of $x_j(t)$. In fact, by equation (5.49) we have

$$l_j = \frac{(\Delta^1_{n-j+1})^2}{\Delta^0_{n-j+1}\Delta^0_{n-j+2}}, \quad j = 1,\ldots,n. \tag{5.53}$$

This implies that

$$y_j(t) = -1 + \sum_{i=1}^{j} \frac{(\Delta^1_{n-i+1})^2}{\Delta^0_{n-i+1}\Delta^0_{n-i+2}}. \tag{5.54}$$

This gives the time evolution of $y_j(t)$ because $c_k(t) = \sum_{i=1}^n r_i(t)\lambda_i^k$. Hence, using the transformation $y = \tanh(\frac{x}{2})$, one can obtain the time evolution of multi-peakons introduced in Chapter 4, as follows:

$$x_j(t) = \ln\left(\frac{S_j}{2-S_j}\right), \tag{5.55}$$

where $S_j = \sum_{i=1}^{j} \frac{(\Delta^1_{n-i+1})^2}{\Delta^0_{n-i+1}\Delta^0_{n-i+2}}$.

Chapter 6

Summary and open problems

This Chapter contains a brief summary, open problems and some possible directions for future research.

Summary: The following results have been obtained in this work:

It was shown that every non-smooth traveling wave solution to the two component Camassa-Holm equation is a solution of the Camassa-Holm equation provided that the set of points where the height of the wave is equal to its speed, is of measure zero. This includes all cuspon and peakon traveling wave solutions. A weak form of the Lax pair that is appropriate for dealing with the peakon solutions of the Camassa-Holm equation was obtained thus extending the original Lax pair formalism which only deals with smooth solutions of a PDE. It was shown that one can work with the peakon solutions of the Camassa-Holm equation in the framework of an interpretation of the Lax pair in the sense of distributions.

Open problems:

1. Can one use a general weak Lax pair (4.21) to construct solutions to the Camassa-Holm equation when $m \in \mathcal{M}$?

2. In the case of Degasperis-Procesi equation ($b = 3$), it is known that there exists a class of non-smooth solutions more general than peakons, called shockpeakons [11],

$$u(x,t) = \sum_{j=1}^{n} m_j(t) e^{-|x-x_j|} - \sum_{i=1}^{n} s_j(t) Sgn(x - x_j) e^{-|x-x_j|}. \tag{6.1}$$

   Can one formulate the correct version of the weak Lax pair for this case?

3. Investigate the existence of multipeakon solutions to the two component Camassa-Holm equation. More generally, investigate the existence of piecewise smooth distributional solutions to the two component Camassa-Holm equation where the discontinuities are located at $x_1(t), \ldots, x_n(t)$.

Appendix A

Functions of Bounded Variation

This Appendix is based on [13] and [14].

Definition 11. The set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, is called a partition of the interval $[a, b]$. The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition 12. Consider the function $f : [a, b] \to \mathbb{R}$ and suppose $P \in \mathcal{P}[a, b]$. Let $\Delta f_k = f(x_k) - f(x_{k-1})$. The sum $\sum_{k=1}^{n} |\Delta f_k|$ is called a variation of f on $[a, b]$. If there exists a number $M > 0$ such that for every $P \in \mathcal{P}[a, b]$,

$$\sum_{k=1}^{n} |\Delta f_k| < M,$$

then f said to be a function of bounded variation on $[a, b]$. The supremum of all the variations of f on $[a, b]$ is called the total variation of f on $[a, b]$ and is denoted by $V_f[a, b]$.

Example 1. If $\alpha : [a, b] \to \mathbb{R}$ is monotonic then it is a function of bounded variation on $[a, b]$.

Proof. Suppose α is an increasing function. Let $P \in \mathcal{P}[a, b]$. Then we have

$$\sum_{k=1}^{n} |\Delta \alpha_k| = \sum_{k=1}^{n} \Delta \alpha_k = \alpha(b) - \alpha(a).$$

$\square$

Example 2. The function f defined by

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0, \\ x \cos(\frac{\pi}{2x}) & \text{if } x \neq 0, \end{cases}$$

is not a function of bounded variation on $[0, 1]$.

Proof. Suppose $P = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{2}, 1\}$. Thus, we have

$$\sum_{k=1}^{n} |\Delta \alpha_k| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

$\square$

Proposition 13. *Let f be a continuous function on $[a, b]$. Suppose that f' exists on (a, b) and it is bounded. Then f is of bounded variation on $[a, b]$.*

Proof. Let us say there exist a number C such that $|f'(x)| \leq C$ for every $x \in (a,b)$. Suppose that $P \in \mathcal{P}[a,b]$. By the mean value theorem there exists a $c_k \in (x_{k-1}, x_k)$ such that $\Delta f_k = f'(c_k)\Delta x_k$. Thus, we have

$$\sum_{k=1}^{n} |\Delta f_k| \leq C(b-a).$$

□

Example 3. The function f defined by $f(x) = x^2\cos(\frac{1}{x})$, is a function of bounded variation on $[0,1]$, because for every $x \neq 0$, $f'(x) = \sin(\frac{1}{x}) + 2x\cos(\frac{1}{x})$ which is bounded on $(0,1)$.

Boundedness of f' is not a necessary condition for f to be of bounded variation.

Example 4. Let $f(x) = x^{1/3}$. This functions is of bounded variation on every closed interval because it is monotonic. However, $f'(x) \to +\infty$ as $x \to 0$.

Proposition 14. *Suppose that f is of bounded variation on $[a,b]$. Then f is bounded on $[a,b]$.*

Proof. Suppose that $x \in (a,b)$ and consider the partition $P = \{a, x, b\}$. For some number M we have

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M.$$

So, $|f(x) - f(a)| \leq M$ and consequently $|f(x)| \leq |f(a)| + M$. □

Proposition 15. *If f and g are of bounded variation on $[a,b]$, then so are $f \pm g$ and fg. Also, we have*

$$V_{f\pm g} \leq V_f + V_g, \qquad V_{fg} \leq AV_f + BV_g, \tag{A.1}$$

where

$$A = \sup_{a\leq x\leq b} |g(x)|, \qquad B = \sup_{a\leq x\leq b} |f(x)|.$$

Proof. Let $P \in \mathcal{P}[a,b]$. To prove the first inequality we can write

$$|f(x_k) \pm g(x_k) - [f(x_{k-1}) \pm g(x_{k-1})]| \leq |\Delta f_k| + |\Delta g_k| \leq V_f + V_g.$$

The second inequality follows from the observation

$$\begin{aligned} &|f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\ &\quad = |[f(x_k)g(x_k) - f(x_{k-1})g(x_k)] + [f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})]| \\ &\quad\quad \leq A|\Delta f_k| + B|\Delta g_k| \leq AV_f + BV_g. \end{aligned}$$

□

Proposition 16. *Suppose that f is a function of bounded variation on $[a,b]$ and assume that there exists a number m such that $0 < m \leq |f(x)|$ for every $x \in [a,b]$. Then $\frac{1}{f}$ is also a function of bounded variation on $[a,b]$. Moreover, $V_{1/f} \leq \frac{V_f}{m^2}$.*

Proof. We have

$$\left|\frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})}\right| = \left|\frac{\Delta f_k}{f(x_k)f(x_{k-1})}\right| \leq \frac{|\Delta f_k|}{m^2}.$$

□

Proposition 17. *Suppose that f is a function of bounded variation on $[a, b]$ and $c \in [a, b]$. Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have*

$$V_f[a, b] = V_f[a, c] + V_f[c, b]. \tag{A.2}$$

Proof. Let $P_1 \in \mathcal{P}[a, c]$ and $P_2 \in \mathcal{P}[c, b]$. Then $P_1 \cup P_2 \in \mathcal{P}[a, b]$ and we have

$$\sum_{x_i \in P_1} |\Delta f_i| + \sum_{x_j \in P_2} |\Delta f_j| = \sum_{x_k \in P_1 \cup P_2} |\Delta f_k| \leq V_f[a, b].$$

Therefore, f is of bounded variation on both $[a, c]$ and $[c, b]$. Also it follows that

$$V_f[a, c] + V_f[c, b] \leq V_f[a, b].$$

To prove the reverse inequality, suppose that $P \in \mathcal{P}[a, b]$ and $c \in [x_{j-1}, x_j]$ and let $P_c = P \cup \{c\}$. Then $P_c \cap [a, c] \in \mathcal{P}[a, c]$ and $P_c \cap [c, b] \in \mathcal{P}[c, b]$ and we have

$$|\Delta f_j| \leq |f(c) - f(x_{j-1})| + |f(x_j) - f(c)|.$$

Thus, we can write

$$\sum_{x_k \in P} |\Delta f_k| \leq \sum_{x_k \in P_c \cap [a,c]} |\Delta f_k| \; + \sum_{x_k \in P_c \cap [c,b]} |\Delta f_k| \leq V_f[a, c] + V_f[c, b].$$

This completes the proof. □

Proposition 18. *Suppose that f is of bounded variation on $[a, b]$ and let V be the function given by $V(x) = V_f[a, x]$. Then we have*
a) V is increasing on $[a, b]$.
b) $V - f$ is increasing on $[a, b]$.

Proof. Assume that $a \leq x < y \leq b$. We have

$$V(y) = V_f[a, y] = V_f[a, x] + V_f[x, y].$$

Therefore

$$V(y) - V(x) = V_f[x, y] \geq 0.$$

To verify (b), we write

$$V(y) - f(y) - [V(x) - f(x)] = V_f[x, y] - [f(y) - f(x)]$$

Since $|f(y) - f(x)| \leq V_f[x, y]$, the proof is complete. □

Theorem 14. *A function f is of bounded variation on $[a, b]$ if, and only if, it can be expressed as the difference of two increasing functions.*

Proof. Suppose that f is on bounded variation on $[a, b]$. Let V and $V - f$ be as in the previous proposition. Thus f is the difference of two increasing functions. To see the converse, we remember that every increasing function on $[a, b]$ is of bounded variation. Thus, their difference is also of bounded variation on $[a, b]$. □

Theorem 15. *Suppose that f is of bounded variation on $[a, b]$ and $V(x) = V_f[a, x]$ for every $x \in [a, b]$. Then every point of continuity of f is a point of continuity of V and vice versa.*

Proof. Suppose that $a < x < y \le b$. Since V is an increasing function, the one-sided limits $V(x+)$ and $V(x-)$ exists. Also, since f is the difference of two increasing functions then $f(x+)$ and $f(x-)$ exist. We have

$$|f(x) - f(y)| \le V_f[x, y] = V(y) - V(x).$$

If we let $y \to x$, we obtain

$$|f(x) - f(x+)| \le V(x+) - V(x).$$

Similarly, we can show that

$$|f(x) - f(x-)| \le V(x) - V(x-).$$

This proves that every point of continuity of V is a point of continuity of f. To see the converse, let f be continuous at $c \in (a, b)$. So for every $\epsilon > 0$ there exists a $\delta > 0$ such that, $|f(x) - f(c)| < \frac{\epsilon}{2}$ if $|x - c| < \delta$. Also there exists a partition $P \in \mathcal{P}[c, b]$ given by

$$P = \{c = x_0, x_1, \ldots, x_n = b\},$$

such that

$$V_f[c, b] - \frac{\epsilon}{2} < \sum_{k=1}^{n} \Delta f_k.$$

Since adding more points to the partition can only make the same on the right hand side bigger, we can assume that $x - x_1 < \delta$. Thus, we have

$$V_f[c, b] - \frac{\epsilon}{2} < \frac{\epsilon}{2} + \sum_{k=2}^{n} \Delta f_k \le \frac{\epsilon}{2} + V_f[x_1, b].$$

Hence,

$$V(x_1) - V(c) = V_f[c, x_1] = V_f[c, b] - V_f[x_1, b] < \epsilon.$$

Thus, $V(c+) = V(c)$. Similarly, we can see that $V(c-) = V(c)$. So, V is continuous at c. The argument for the endpoints of the interval is similar. □

The following useful corollary follows from Theorem 14 and Theorem 15:

Theorem 16. *Suppose that f is continuous on $[a, b]$. Then f is of bounded variation on $[a, b]$ if, and only if, f can be expressed as the difference of two increasing continuous functions.*

Definition 13. A real valued function f defined on $[a, b]$ is said to be absolutely continuous on $[a, b]$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

for every finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Corollary 11. Every absolutely continuous function is continuous.

Lemma 20. *Let f be a nonnegative function which is integrable over a set E. Then given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$ we have*

$$\int_A f < \epsilon.$$

Corollary 12. Every function which is an indefinite integral of another function is absolutely continuous.

Lemma 21. *If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.*

Corollary 13. If f is absolutely continuous, then f has a derivative almost everywhere.

Lemma 22. *If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ almost everywhere, then f is constant.*

Theorem 17. *A function F is an indefinite integral if and only if it is absolutely continuous.*

Corollary 14. Every absolutely continuous function is the indefinite integral of its derivative.

Appendix B

Distributions

This Appendix is based on [19], [18] and [28].

Definition 14. The vector space of all compactly supported C^∞ functions $\phi : \mathbb{R} \to \mathbb{C}$, is called the space of test functions and it is denoted by $\mathcal{D}$. The vector space of all compactly supported functions $\phi : \mathbb{R} \to \mathbb{C}$ having continuous derivatives up to order m, is denoted by $\mathcal{D}^m$.

The following Lemma can be used to show that the space of test functions is nontrivial.

Lemma 23. *Let $P(x)$ be a polynomial. Then the function f given by*

$$f(x) = \begin{cases} P(1/x)e^{-1/x} & \textit{if} \quad x > 0, \\ 0 & \textit{if} \quad x \leq 0, \end{cases} \tag{B.1}$$

belongs to $C^\infty(\mathbb{R})$.

Proof. Since, $f'(x) = \frac{P(1/x)-P'(1/x)}{x^2}e^{-1/x}$ for $x \neq 0$, it suffices to prove that

$$\lim_{x \downarrow 0} x^{-n} e^{-1/x} = 0.$$

To see this, we write

$$\lim_{x \downarrow 0} x^{-n} e^{-1/x} = \lim_{t \uparrow +\infty} \frac{t^n}{e^t} = 0.$$

□

Lemma 24. *There exists a non-negative function $\phi \in \mathcal{D}$ such that $\phi(0) > 0$.*

Proof. By lemma 23, the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases} \tag{B.2}$$

belongs to $C^\infty(\mathbb{R})$. Hence, $\phi(x) = f(1 - x^2)$ is the desired function. □

Proposition 19. *If X is an open subset of $\mathbb{R}^n$ and K is a compact subset of X, then there exist a compactly supported C^∞ function ϕ on X, such that $0 \leq \phi \leq 1$ and $\phi = 1$ on a neighborhood of K.*

Proof. See [28]. □

Definition 15. A distribution u in $\mathbb{R}$ is a linear functional on $\mathcal{D}$ such that for every compact set $K \subset \mathbb{R}$, there exist constants C and k such that

$$|u(\phi)| \leq C \sum_{\alpha \leq k} \sup|\partial^\alpha \phi|, \tag{B.3}$$

for every $\phi \in \mathcal{D}$ with Support$(\phi) \subset K$. The set of all distributions in $\mathbb{R}$ is denoted by $\mathcal{D}'$.

If the same integer can be used in (B.3) for every compact subset K, we say that $order(u) \leq k$. The set of all distributions u with $order(u) \leq k$ is denoted by $\mathcal{D}'^k$.

Definition 16. A Radon measure μ in $\mathbb{R}$ is a linear functional on $\mathcal{D}^0$ such that for every compact set $K \subset \mathbb{R}$, there exists a constant C such that

$$|\mu(\phi)| \leq C \sup|\phi|, \tag{B.4}$$

for every $\phi \in \mathcal{D}^0$ with Support$(\phi) \subset K$. The set of all Radon measures in $\mathbb{R}$ is denoted by $\mathcal{M}$.

Example 5. If $x_0 \in \mathbb{R}$, then $\delta_{x_0}^{(n)}$ which is defined by

$$\delta_{x_0}^{(n)}(\phi) = (-1)^n \phi^{(n)}(x_0), \quad \text{for every } \phi \in \mathcal{D},$$

is a distribution of order n.

Proof. It is clear that $\delta_{x_0}^{(n)}$ satisfies (B.3) with $k = n$. So, it suffices to show that the order is not smaller than n. To see this, we choose $\psi \in \mathcal{D}$ with $\psi(0) = 1$ and define

$$\phi_\epsilon(x) = (x - x_0)^n \psi(\frac{x - x_0}{\epsilon}).$$

Then, $\delta_{x_0}^{(n)}(\phi_\epsilon) = (-1)^n n!$. But, if $m < n$, then

$$\sup|\partial^m \phi_\epsilon| \leq C\epsilon^{n-m}.$$

Thus, if $\epsilon \to 0$ then $\sup|\partial^m \phi_\epsilon| \to 0$. ☐

Example 6. If $x_j \in \mathbb{R}$ is a sequence of numbers with no limit point in $\mathbb{R}$, and if n_j is a sequence of non-negative integers, then

$$u = \sum_j \delta_{x_j}^{(n_j)},$$

defines a distribution in $\mathbb{R}$, because a compact subset can only contain finitely many x_j. By the previous example, u is of finite order if and only if the sequence n_j is bounded. In this case, the order is $\max(n_j)$.

The following theorem states that the continuity condition (B.3) is equivalent to the sequential continuity.

Theorem 18. *A linear functional u on $\mathcal{D}$ is a distribution if and only if $u(\phi_j) \to 0$ when $j \to 0$ for every sequence $\phi_j \in \mathcal{D}$ converging to 0 in the sense that* $\sup|\partial^\alpha \phi_j| \to 0$ *for every fixed α and* Support$(\phi_j) \subset K$ *for all j and some fixed compact set $k \subset \mathbb{R}$.*

Proof. See [28]. ☐

Theorem 19 (**Completeness Property**)**.** *If u_j is a sequence in $\mathcal{D}'$ and*

$$u(\phi) = \lim_{j\to\infty} u_j(\phi),$$

exists for every $\phi \in \mathcal{D}$, then $u \in \mathcal{D}'$.

Proof. See [28]. ☐

Proposition 20. *Every function $f \in L^1_{loc}(\mathbb{R})$ defines a distribution.*

Proposition 21. *Every distribution of order 0 is a Radon measure.*

Proposition 22. *Every Radon measure on $\mathbb{R}$ is the first derivative of a function of (locally) bounded variation.*

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