Polynomial Identities and Enveloping Algebras for n-ary Structures

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Abstract

This thesis is devoted to studying the polynomial identities of alternating quaternary algebras structures, and the universal associative enveloping algebras of the $(n+1)$-dimensional n-Lie (Filippov) algebras, the 2-dimensional non-associative triple systems and the anti-Jordan triple system of $n \times n$ matrices.

Firstly, we determine the multiplicity of the irreducible representation $V(n)$ of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ as a direct summand of its fourth exterior power $\Lambda^4 V(n)$. The multiplicity is 1 (resp. 2) if and only if $n = 4, 6$ (resp. $n = 8, 10$). For these n we determine the multilinear polynomial identities of degree ≤ 7 satisfied by the $\mathfrak{sl}_2(\mathbb{C})$-invariant alternating quaternary algebra structures obtained from the projections $\Lambda^4 V(n) \to V(n)$.

Secondly, we study the universal associative enveloping algebras of n-Lie algebras. For n even and any $(n+1)$-dimensional n-Lie algebra, we construct a universal associative envelope and establish a generalization of the Poincaré-Birkhoff-Witt theorem for universal envelopes using noncommutative Gröbner bases. We provide computational evidence that the construction is much more difficult for n odd.

Thirdly, we construct universal associative envelopes for the non-associative triple systems arising from trilinear operations applied to the 2-dimensional simple associative triple system. We use noncommutative Gröbner bases to determine the monomial bases, the structure constants, and the centers of the universal envelopes. For the finite dimensional envelopes, we determine the Wedderburn decompositions and classify the irreducible representations.

Finally, we show that the universal associative envelope, of the simple anti-Jordan triple system of all $n \times n$ matrices $(n \geq 2)$ over an algebraically closed field of characteristic 0, is finite dimensional. We investigate the structure of the universal envelope and focus on the monomial basis, the structure constants, and the center. We explicitly determine the decomposition of the universal envelope into matrix algebras. We classify all finite dimensional irreducible representations of this simple anti-Jordan triple system, and show that the universal envelope is semisimple.

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I dedicate this work to my beloved mother and my sweety daughter Menna and my husband Zaki.

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List of Abbreviations

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{C}$	field of complex numbers
$\mathrm{LM}(f)$	leading monomial of f
$\mathrm{Nf}(f)$	normal form of f
$Z(A)$	center of A
$F\langle X\rangle$	free associative algebra over F
$M_{n\times n}(\mathbb{F})$	set of $n\times n$ matrices over $\mathbb{F}$
E_{ij}	matrix units
δ_{ij}	Kronecker delta
$\widehat{\delta}_{ij}$	$1-\delta_{i,j}$
$\Lambda^n V$	n-th exterior power of V
$U(A)$	universal associative enveloping algebra of A
$\mathfrak{sl}_2(\mathbb{C})$	Lie algebra of 2×2 complex matrices of trace 0
$\mathfrak{R}(A)$	radical of A
$A(\alpha,\beta,\gamma)$	down-up algebra
$GK(A)$	Gelfand Kirillov dimension
$\mathbb{Q}S_3$	group algebra over the rational field $\mathbb{Q}$
LCM	least common multiple
Tr	trace (of a matrix)
RCF	row canonical form

Chapter 1

Introduction

The present work is dedicated to studying the polynomial identities of alternating quaternary algebra structures and the universal associative enveloping algebras of non-associative n-ary structures. The thesis consists of six chapters. In the present introductory chapter, we introduce the suggested problems and the main results. The basic notations, definitions, and theorems that we use in the thesis are summarized in Chapter 2. Chapter 3 is devoted to the study of alternating quaternary algebra structures and polynomial identities. In Chapters 4 to 6 we focus on the universal associative envelopes of n-ary non-associative structures.

In Chapter 3, we use the representation theory of Lie algebras to construct new alternating algebra structures and to discover natural n-ary generalizations of Lie algebras. In particular, we study the exterior powers of an irreducible representation of a simple Lie algebra. If $\Lambda^n V$, the n-th exterior power of an irreducible representation V of a simple Lie algebra L, contains V itself as a direct summand, then the projection $\Lambda^n V \to V$ defines an alternating n-ary algebra structure on V which is L-invariant in the sense that the derivation algebra of this n-ary structure contains a subalgebra isomorphic to L. This approach was used by Bremner and Hentzel [8] in the case of the third exterior power of an irreducible representation of the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We extend this work to the fourth exterior power. In order to use this approach we first need to answer the following question: *How to determine the multiplicity of the irreducible representation $V(n)$ of the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ as a direct summand of its n-th exterior power:* $\dim \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\big(\Lambda^n V(n), V(n)\big)$*?* The famous Clebsch-Gordan Theorem provided the answer in the case of the second exterior power. In the case of the third exterior power, this problem has been solved by Bremner and Hentzel [9].

In the case of the fourth exterior power, we obtain a closed formula for the multiplicity using a general approach which applies to arbitrary exterior powers. We then determine the values of n for which the module $V(n)$ occurs in $\Lambda^4 V(n)$ with multiplicity 1 and 2. For these values of n, we obtain $\mathfrak{sl}_2(\mathbb{C})$-invariant alternating quaternary algebra structures from the projections $\Lambda^4 V(n) \to V(n)$. We determine the structure constants for these algebras and introduce the multilinear polynomial identities of degree ≤ 7 satisfied by the alternating quaternary algebra structures. We represent the polynomial identities as the nullspace of a large integer matrix and use computational linear algebra to find the canonical basis of the nullspace. (The results of Chapter 3 have been published [7].)

Chapter 4 is devoted to studying the universal associative enveloping algebras of the $(n+1)$-dimensional n-Lie (Filippov) algebras. Filippov [20] in 1985 introduced n-Lie algebras and classified the $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 0. Recall that an n-Lie algebra is a vector space L over a field F of characteristic $\neq 2$ with a multilinear operation $[x_1, x_2, \ldots, x_n]$ satisfying the alternating (or anticommutative) identity and the generalized Jacobi (or derivation) identity:

$$[x_1, x_2, \ldots, x_n] = \epsilon(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}] \quad (\sigma \in S_n),$$
$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

For $n = 2$ we obtain the definition of a Lie algebra.

The Poincaré-Birkhoff-Witt (PBW) theorem is an important tool in the representation theory of Lie algebras. The study of the representations of a Lie algebra L can be transformed into a problem of associative algebra by passage to the universal enveloping algebra $U(L)$ of L. From the Poincaré-Birkhoff-Witt theorem, L is embedded in $U(L)$. It also provides a basis for the universal associative enveloping algebra of any Lie algebra over any field, and allows us to make calculations in these noncommutative algebras. Bergman [4] in 1978 gave a new proof of the PBW theorem using noncommutative Gröbner bases in the free associative algebra. The central question in Chapter 4 is: *Whether any n-Lie (Filippov) algebra is isomorphic to a n-Lie subalgebra of an associative algebra?*

In Ling [31], it was proved that a simple finite-dimensional Filippov algebra over an algebraically closed field of characteristic 0 is unique up to isomorphism, and this is the simple algebra L_{n+1}, first described in Filippov (1985). Recall that L_{n+1} is the $(n+1)$-dimensional n-ary algebra with a basis $\{e_1, \ldots, e_{n+1}\}$ and with multiplication of basis elements given by

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] = (-1)^{n+1+i} e_i;$$

$\widehat{e_i}$ means that e_i is omitted. Recently, Cantarini and Kac [12] have extended Ling's results to a large class of infinite dimensional n-Lie algebras.

Pozhidaev [35] in 2003 considered the problem whether there exists an embedding of an arbitrary n-Lie algebra into an associative algebra. He showed that for $n \leq 5$ the simple finite-dimensional n-Lie algebra over an algebraically closed field of characteristic 0 can be embedded in an associative algebra, and made the following conjecture: *For any reductive finite-dimensional n-Lie algebra L over an algebraically closed field of characteristic 0 there exists an associative algebra A such that L is isomorphic to a subalgebra of A^-.*

By the work of Ling [31] it is known that any reductive finite-dimensional n-Lie algebra over an algebraically closed field of characteristic 0 decomposes into the direct sum of an abelian ideal and several copies of a simple ideal isomorphic to the simple $(n+1)$-dimensional n-Lie algebra L_{n+1}. Hence the main problem is to prove that L_{n+1} can be embedded into an associative algebra.

In Chapter 4, we study the universal associative enveloping algebras of n-Lie algebras and establish a generalization of the Poincaré-Birkhoff-Witt (PBW) theorem for $(n+1)$-dimensional n-Lie algebras when n is even. For n even, we prove Pozhidaev's conjecture on the existence of associative enveloping algebras for simple n-Lie (Filippov) algebras. More generally, for n even and any $(n+1)$-dimensional n-Lie algebra L, we construct a universal associative enveloping algebra $U(L)$ and show that the natural map $L \to U(L)$ is injective. We use noncommutative Gröbner bases to present $U(L)$ as a quotient of the free associative algebra on a basis of L and to obtain a monomial basis of $U(L)$. In the last section, we provide computational evidence that the construction of $U(L)$ is much more difficult for n odd. (The results of Chapter 4 have been published [16].)

Chapter 5 is concerned with the universal associative enveloping algebras of non-associative triple systems. By a multilinear n-ary operation we mean an element $\omega = \sum_{\sigma \in S_n} x_\sigma \sigma$ of the group algebra $\mathbb{Q}S_n$ of the symmetric group S_n over the rational field $\mathbb{Q}$. If A is an associative algebra over $\mathbb{Q}$, then ω defines a multilinear n-ary operation $\omega(a_1, \ldots, a_n)$ on the underlying vector space of A:

$$\omega(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)}.$$

In this way we obtain a nonassociative n-ary algebra which we denote by A^ω.

For $n = 2$, every bilinear operation is equivalent to either the zero operation, the associative operation ab, the Lie bracket $[a, b] = ab - ba$, or the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. The polynomial identities of degree ≤ 3 (≤ 4) satisfied by the Lie bracket (Jordan product) define Lie algebras (Jordan algebras), the most important varieties of non-associative algebras. For $n = 3$, Bremner and Peresi [10] found canonical representatives of the equivalence classes of trilinear operations, and identified 19 operations satisfying polynomial identities of degree 5 which do not follow from the identities of degree 3. These operations include the Lie, anti-Lie, Jordan, and anti-Jordan triple products.

In Chapter 5, we show how the theory of noncommutative Gröbner bases can be used to construct the universal associative enveloping algebra of any n-ary non-associative algebra. In particular we construct universal associative envelopes for the non-associative triple systems which arise from applying the non-associative trilinear operations classified by Bremner and Peresi [10] to the 2-dimensional simple associative triple system of the first kind in the sense of Lister [32], namely the space of 2×2 matrices $A = (a_{ij})$ with $a_{11} = a_{22} = 0$. We use noncommutative Gröbner bases to determine monomial bases, structure constants, and centers of the universal envelopes. We show that the infinite dimensional envelopes are closely related to the down-up algebras of Benkart and Roby [3]. For the finite dimensional envelopes, we determine the Wedderburn decompositions and classify the irreducible representations. (The results of Chapter 5 has been submitted [17].)

Finally in Chapter 6, we deal with the universal associative enveloping algebras and the representations of anti-Jordan triple systems. Anti-Jordan triple systems were introduced by Faulkner and Ferrar in [19]. The classification of finite-dimensional simple anti-Jordan triple systems over an algebraically closed field of characteristic 0 was given by Bashir [2]. In Chapter 5 we found an example of anti-Jordan triple system with infinite dimensional universal enveloping algebra. In general it is not hard to construct examples of anti-Jordan triple systems with infinite dimensional universal enveloping algebras by using noncommutative Gröbner bases. Recall that an anti-Jordan triple system is a vector space V over a field F endowed with a trilinear operation $V \times V \times V \to V$, $(a,b,c) \to \langle a,b,c\rangle$ satisfying the following identities for all $a,b,c,d,e \in V$:

$$\langle a,b,a\rangle = 0, \qquad \langle a,b,\langle c,d,e\rangle\rangle = \langle\langle a,b,c\rangle,d,e\rangle + \langle c,\langle b,a,d\rangle,e\rangle + \langle c,d,\langle a,b,e\rangle\rangle.$$

If A is an associative algebra, A defines an anti-Jordan triple system A_- relative to the product $\langle a,b,c\rangle = abc - cba$.

The goal of Chapter 6 is to provide a negative answer for the following question: *Whether the universal enveloping algebra of any anti-Jordan triple system is always infinite-dimensional?* We show that the universal associative enveloping algebra of the simple anti-Jordan triple system of all $n\times n$ matrices $(n \geq 2)$ over an algebraically closed field of characteristic 0 is finite dimensional. We investigate the structure of the universal envelope and focus on the monomial basis, the structure constants, and the center. We explicitly determine the decomposition of the universal envelope into matrix algebras. We classify all finite dimensional irreducible representations of the simple anti-Jordan triple system, and show that the universal envelope is semisimple. We also provide more examples to show that the universal associative envelopes of anti-Jordan triple systems are not necessary finite-dimensional. We conclude this chapter by giving a conjecture on the semisimplicity of the finite-dimensional universal associative envelopes of simple finite-dimensional anti-Jordan triple systems. (The results of Chapter 6 have been submitted [18].)

Chapter 2

Preliminaries

In this chapter we review some of the basic concepts that are used through this thesis. Sections 2.1–2.3 present some basic background information for Chapter 3. The algebraic background necessary for Chapters 4–6 is introduced in Sections 2.4–2.8.

In Section 2.1 we recall some standard facts about Lie algebras and representation theory. Section 2.2 reviews the basic representation theory of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In Section 2.3 we begin by introducing the relevant background material on the exterior powers. Then we explain briefly how the exterior powers of the irreducible representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ can be used to construct new algebra structures. In Section 2.4 we recall the basic facts about noncommutative Gröbner bases in free associative algebras. Section 2.5 presents some basic definitions and results on triple systems. In Section 2.6 we discuss the classification of the trilinear operations using the representation theory of the symmetric group. In Section 2.7 we outline the basic algorithms to compute the simple direct summands of the semisimple associative algebras. Finally, in the last section we provide some basic definitions and results about down-up algebras.

2.1 Basic concepts of Lie algebras

In this section, we review some of the basic concepts in Lie theory.

Definition 2.1.1. [24] A vector space L over a field $\mathbb{F}$, with operation $L \times L \to L$, denoted $(x, y) \to [x, y]$ and called the bracket or commutator of x and y, is called a **Lie algebra** over $\mathbb{F}$ if the following axioms are satisfied:

1. The bracket operation is bilinear.

2. $[x, x] = 0$ for all $x \in L$.

3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Remark 2.1.2. Every associative algebra A can be viewed as a Lie algebra under the natural Lie bracket $[x, y] = xy - yx$.

Definition 2.1.3. [24] A nonzero Lie algebra L is called **simple** if it has no ideals other than L or 0, and if moreover $[L, L] \neq 0$. We say a Lie algebra L is **semisimple** if it is a direct sum of non-trivial simple Lie algebras.

Definition 2.1.4. [24] Let L be a Lie algebra over the field $\mathbb{F}$. A **representation** of L on a vector space V is a homomorphism of Lie algebras $\rho : L \to \mathfrak{gl}(V)$, from L to the Lie algebra of endomorphisms on a vector space V.

If $\rho : L \to \mathfrak{gl}(V)$ is a representation of the Lie algebra L on the vector space V, then V is said to be an **L-module**. If from the context it is clear which representation ρ we mean, then we write $x \cdot v$ instead of $\rho(x)(v)$.

Definition 2.1.5. [24] Let V be an L-module and let W be a subspace of V that is stable under L, i.e., $x \cdot w \in W$ for all $x \in L$ and $w \in W$. Then W is called a **submodule** of V.

Definition 2.1.6. [24] The L-module V is called **irreducible** if it has no submodules other than 0 and V itself. It is called **completely reducible** if V is a direct sum of irreducible L-modules.

Definition 2.1.7. [24] Let $\rho : L \to \mathfrak{gl}(V)$ and $\rho' : L \to \mathfrak{gl}(W)$ be two representations of the Lie algebra L. A **homomorphism** of L-modules from V to W is a linear map $f : V \to W$ such that $\rho'(x)(f(v)) = f(\rho(x)(v))$ for all $v \in V$. If a homomorphism of L-modules $f : V \to W$ is bijective, then the representations ρ and ρ' are said to be **equivalent**.

Now we describe ways of making new L-modules from known ones. Let L be a Lie algebra over the field $\mathbb{F}$ and let V and W be two L-modules. Then the direct sum $V \oplus W$ becomes an L-module by setting

$$x \cdot (v + w) = x \cdot v + x \cdot w, \quad \text{for all } x \in L, v \in V \text{ and } w \in W.$$

Also the tensor product $V \otimes W$ can be made into an L-module by

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

Definition 2.1.8. [24] Let H be the Cartan subalgebra of a Lie algebra L and let $\lambda \in H^*$. Then λ is called a **weight** of the representation (ρ, V) if the space

$$V_\lambda = \{v \in V \mid \rho(h)v = \lambda(h)v \text{ for all } h \in H\}$$

is non-zero. In that case a nonzero $v \in V_\lambda$ is called a **weight vector** of weight λ.

Theorem 2.1.9. [22](Theorem of the Highest Weight) *Let L be a complex semisimple Lie algebra. Then*

1. *Every finite dimensional irreducible representation has a highest weight.*

2. *Two finite dimensional irreducible representations are equivalent if and only if they have the same highest weight.*

Definition 2.1.10. [22] A representation (ρ, V) of a complex semisimple Lie algebra L is said to be a **highest weight cyclic representation** with weight μ_0 if there exists $v \neq 0$ in V such that

1. v is a weight vector with weight μ_0

2. $\rho(X_\alpha)v = 0$ for all positive roots α

3. The smallest invariant subspace of V containing v is all of V.

The vector v is called a **cyclic vector** of ρ.

Proposition 2.1.11. [22] *Every finite dimensional irreducible representation of a complex semisimple Lie algebra L is a highest weight cyclic representation with a unique highest weight μ_0.*

2.2 The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its representations

Recall from [24] that the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has basis $\{H, E, F\}$ and commutation relations

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H.$$

All other brackets follow from bilinearity and anticommutativity. These relations are satisfied by the commutator $[X,Y] = XY - YX$ of 2×2 matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These three matrices form a basis of the vector space of all 2×2 matrices of trace 0.

2.2.1 The irreducible representation $V(n)$

For any nonnegative integer n, there is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ containing a nonzero vector v_n (called the highest weight vector) satisfying the conditions

$$H.v_n = nv_n, \qquad E.v_n = 0.$$

This representation is unique up to isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules. It is denoted $V(n)$ and called the **representation with highest weight** n. Its dimension is $n+1$; a basis of $V(n)$ consists of the $n+1$ vectors v_{n-2i} where

$$v_{n-2i} = \frac{1}{i!}F^i v_n, \quad 0 \le i \le n.$$

The action of $\mathfrak{sl}_2(\mathbb{C})$ on $V(n)$ is then as follows:

$$H.v_{n-2i} = (n-2i)v_{n-2i}, \tag{2.1}$$

$$E.v_n = 0, \qquad E.v_{n-2i} = (n-i+1)v_{n-2i+2}\ (i = 1, \cdots, n), \tag{2.2}$$

$$F.v_{n-2i} = (i+1)v_{n-2i-2}\ (i = 0, \cdots, n-1), \qquad F.v_{-n} = 0. \tag{2.3}$$

Theorem 2.2.1. [24] *Any finite dimensional irreducible representation of* $\mathfrak{sl}_2(\mathbb{C})$ *is isomorphic to* $V(n)$ *for some* n.

Theorem 2.2.2. [24] *Any finite dimensional representation of* $\mathfrak{sl}_2(\mathbb{C})$ *is isomorphic to a direct sum of irreducible representations.*

Theorem 2.2.3. [22](Clebsch-Gordan theorem) *For two non-negative integers* n *and* m*, consider* $V(n) \otimes V(m)$ *as a representation of* $\mathfrak{sl}_2(\mathbb{C})$*. Assume* $n \geq m$*. Then,*

$$V(n) \otimes V(m) \cong \bigoplus_{i=0}^{m} V(n+m-2i).$$

In the special case $n = m$ we obtain

$$V(n) \otimes V(n) \cong \bigoplus_{i=0}^{n} V(2n-2i).$$

2.3 Exterior powers

Fix a vector space V over a field $\mathbb{F}$. Let $T^0 V = \mathbb{F}$, $T^1 V = V$, $T^2 V = V \otimes V, \ldots, T^m V = V \otimes \cdots \otimes V$ (m copies). Define $T(V) = \oplus_{i=0}^{\infty} T^i V$, and introduce an associative product, defined on homogeneous generators of $T(V)$ by the obvious rule

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m \in T^{k+m} V.$$

This makes $T(V)$ an associative graded algebra with unity, which is generated by 1 along with any basis of V. We call it the tensor algebra on V.

Starting with the tensor algebra $T(V)$ of a vector space V, which is an associative algebra under the $\otimes$ operation, consider the two-sided ideal A generated by all elements of the form $x \otimes x$, where $x \in V$. The quotient algebra

$$E(V) = T(V)/A,$$

is an associative algebra with unity called the **exterior algebra** of the vector space V. It is customary to use the symbol $\wedge$ to denote the multiplication operation induced by $\otimes$ in the exterior algebra $E(V)$, so that we have

$$(t_1 + A) \wedge (t_2 + A) = (t_1 \otimes t_2) + A,$$

for the cosets of any tensors $t_1, t_2 \in T(V)$. If S_1 and S_2 are any two subspaces of the exterior algebra $E(V)$, then $S_1 \Lambda S_2$ denotes the subspace of all linear combinations of elements $s_1 \wedge s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$. Since $E(V)$ is an associative algebra, we have

$$S_1 \Lambda (S_2 \Lambda S_3) = (S_1 \Lambda S_2) \Lambda S_3,$$

for any three subspaces of the exterior algebra. The vector space V is naturally embedded in the exterior algebra $E(V)$, just as it is in the tensor algebra, and we may identify the vector $x \in V$ with the coset $x + A \in E(A)$. The exterior power

$$\Lambda^r V = V \Lambda \ldots \Lambda V \qquad (r \, \text{copies}),$$

is a subspace of the exterior algebra. The exterior square $\Lambda^2 V$ of a vector space V is a vector space generated by exterior products $x \wedge y$ of vectors x, y in V. If $x \in V$, then

$$x \wedge x = 0,$$

since $x \otimes x$ belongs to the ideal A. If x and y are in V, then

$$(x + y) \wedge (x + y) = 0,$$

and writing this out and using $x \wedge x = y \wedge y = 0$, we get

$$x \wedge y = -y \wedge x.$$

If V is an n-dimensional vector space with basis $e_1, \ldots, e_n$, then the exterior square $\Lambda^2 V$ is a vector space of dimension $\binom{n}{2} = n(n-1)/2$, with basis $e_i \wedge e_j$, where $1 \leq i < j \leq n$. In general, if V is an n-dimensional vector space with basis $e_1, \ldots, e_n$, the exterior power $\Lambda^r V$ has dimension

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{for } 0 \leq r \leq n,$$

and the elements

$$e_{j_1} \wedge \cdots \wedge e_{j_r}, \qquad j_1 < j_2 < \cdots < j_r,$$

form a basis for the exterior power space $\Lambda^r V$ since any change in the order of factors in a product at most changes its sign.

The exterior square $\Lambda^2 V$ may be identified with the subspace of $V \otimes V$ spanned by all the linear combinations of elements of the form

$$x \otimes y - y \otimes x,$$

where x and y are in V. More generally, the exterior power $\Lambda^k V$ is spanned by all linear combinations of elements of the form

$$\sum_{\sigma \in S_k} \epsilon(\sigma) \left(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \right),$$

where x_i are vectors in V, and the sum goes over all permutations σ of the indices $1, \ldots, k$. The sign factor $\epsilon(\sigma)$ here depends on the parity of the permutation. It is $+1$ for even permutations and -1 for odd permutations.

Now suppose that L is a Lie algebra and $\rho : L \to \mathfrak{gl}(V)$ is a representation. We may define a new representation $\wedge^2 \rho : L \to \mathfrak{gl}(\Lambda^2 V)$ by

$$(\wedge^2 \rho)(x)(v_i \wedge v_j) = \rho(x) v_i \wedge v_j + v_i \wedge \rho(x) v_j \quad \text{for } x \in L,$$

and extending it to linear combinations of basis elements. More generally, we can define a representation $\wedge^r \rho : L \to \mathfrak{gl}(\Lambda^r V)$ by

$$(\wedge^r \rho)(x)(v_{i_1} \wedge \cdots \wedge v_{i_r}) = \sum_{s=1}^{r} v_{i_1} \wedge \ldots \rho(x) v_{i_s} \cdots \wedge v_{i_r} \quad \text{for } x \in L.$$

We now show how the exterior powers can be used to construct new alternating algebra structures. Let $L = \mathfrak{sl}_2(\mathbb{C})$ and V be the irreducible representation $V(n)$ with highest weight n. Then, the multiplicity of $V(n)$ in its k-th exterior power $\Lambda^k V(n)$ is the dimension of the vector space

$$\mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\left(\Lambda^k V(n), V(n) \right)$$

of $\mathfrak{sl}_2(\mathbb{C})$-invariant linear maps $P \colon \Lambda^k V(n) \to V(n)$. If this multiplicity is positive then P defines an alternating k-ary algebra structure on $V(n)$,

$$[x_1, \cdots, x_k] = P(x_1 \wedge \cdots \wedge x_k), \quad (x_1, x_2, \ldots, x_k \in V(n))$$

which is $\mathfrak{sl}_2(\mathbb{C})$-invariant in the sense that the action of any $L \in \mathfrak{sl}_2(\mathbb{C})$ is a derivation of the k-ary multiplication: for any $x_1, \cdots, x_k \in V(n)$ we have

$$L.[x_1, \cdots, x_i, \cdots, x_k] = \sum_{i=1}^{k} [x_1, \cdots, L.x_i, \cdots, x_k].$$

This approach was used by Bremner and Hentzel [8, 9] in the case of the second and third exterior powers of an irreducible representation of the 3-dimensional simple Lie algebra and they got the following results:

Lemma 2.3.1. [8](Explicit version of Clebsch-Gordan theorem) *We have*

$$\dim \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\big(\Lambda^2 V(n), V(n)\big) = \begin{cases} 1 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $n = 2$ gives the 3-dimensional adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$, $n = 6$ gives the 7-dimensional simple non-Lie Malcev algebra, and $n = 10$ gives a new 11-dimensional anticommutative algebra satisfying a polynomial identity of degree 7.

Lemma 2.3.2. [9] *For* $n = 6q + r$ $(0 \le r \le 5)$ *we have*

$$\dim \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\big(\Lambda^3 V(n), V(n)\big) = \begin{cases} q & \text{if } r = 0, 1, 2, 4, \\ q+1 & \text{if } r = 3, 5. \end{cases}$$

The multiplicity is 1 for $n = 3, 5, 6, 7, 8, 10$; the corresponding $V(n)$ provide new examples of alternating ternary algebras.

2.4 Gröbner bases in free associative algebras

In this section, we recall the basic definitions and results in the theory of noncommutative Gröbner bases in free associative algebras following de Graaf [14, Chapter 6], which is based on the work of Bergman [4]. Further information can be found in [30].

Definition 2.4.1. [14] Let $X = \{x_1, \ldots, x_n\}$ be a set of symbols with the total order $x_i < x_j$ if and only if $i < j$. The **free monoid** generated by X is the set X^* of all (possibly empty) words $w = x_{i_1} \cdots x_{i_k}$ $(k \ge 0)$ with the (associative) operation of concatenation. For $w = x_{i_1} \cdots x_{i_k} \in X^*$ the **degree** is $\deg(w) = k$. The **free unital associative algebra** generated by X is the vector space $F\langle X \rangle$ with basis X^* and multiplication extended bilinearly from concatenation in X^*.

Definition 2.4.2. [14] Throughout this thesis we use the **degree-lexicographical** (**deglex**) order $<$ on X^* defined as follows: $u < v$ if and only if either (i) $\deg(u) < \deg(v)$ or (ii) $\deg(u) = \deg(v)$ and $u = wx_iu'$, $v = wx_jv'$ where $x_i < x_j$ $(w, u', v' \in X^*)$. We say that $u \in X^*$ is a **factor** of $v \in X^*$ if there exist $w_1, w_2 \in X^*$ such that $w_1uw_2 = v$. If w_1 (resp. w_2) is empty then u is a **left** (resp. **right**) factor of v.

Definition 2.4.3. [14] The **support** of a noncommutative polynomial $f \in F\langle X\rangle$ is the set of all monomials $w \in X^*$ that occur in f with nonzero coefficient. The **leading monomial** of $f \in F\langle X\rangle$, denoted $\mathrm{LM}(f)$, is the highest element of the support of f with respect to deglex order. If I is any ideal of $F\langle X\rangle$ then the set of **normal words** modulo I is defined by $N(I) = \{\, u \in X^* \mid u \neq \mathrm{LM}(f) \text{ for any } 0 \neq f \in I \,\}$. We write $C(I)$ for the subspace of $F\langle X\rangle$ spanned by $N(I)$.

Proposition 2.4.4. [14] *If $I \subset F\langle X\rangle$ is an ideal then $F\langle X\rangle = C(I) \oplus I$.*

Proof. de Graaf [14, Proposition 6.1.1]. □

If $f \in F\langle X\rangle$, then by Proposition 2.4.4 f has a unique expression as $f = v + p$ for $v \in C(I)$ and $p \in I$. The element $v \in C(I)$ is called the **normal form** of f modulo I. It is denoted by $\mathrm{Nf}_I(f)$, or if from the context it is clear which ideal we mean, by $\mathrm{Nf}(f)$.

Remark 2.4.5. Let $u, v \in C(I)$ and set $u * v = \mathrm{Nf}(uv)$. Then $C(I)$ together with $*$ becomes an associative algebra. It is immediate that this algebra is isomorphic to $F\langle X\rangle/I$. So if we have a method for computing normal forms, then we can conveniently calculate in $F\langle X\rangle/I$.

Definition 2.4.6. [14] Let $G \subset F\langle X\rangle$ be a subset generating an ideal $I \subset F\langle X\rangle$. A noncommutative polynomial $f \in F\langle X\rangle$ is in **normal form modulo** G if no monomial occurring in f has a factor of the form $\mathrm{LM}(g)$ for any $g \in G$.

Algorithm 2.4.7. [14, §6.1] **NormalForm**

Input: a generating set G of an ideal $I \subset F\langle X\rangle$, and an element $f \in F\langle X\rangle$.

Output: an element $\phi \in F\langle X\rangle$ that is in normal form modulo G and such that $f = \phi \mod I$.

- Step 1. Set $\phi := 0, h := f$.

- Step 2. If $h = 0$ then return ϕ. Otherwise set $u :=$ LM(h) and let λ be the coefficient of u in h.

- Let $g \in G$ be such that LM(g) is a factor of u. If there is no such g then set $h := h - \lambda u$, $\phi := \phi + \lambda u$ and return to Step 2.

- Let μ be the coefficient of LM(g) in g, and let $v, w \in X^*$ be such that $v\text{LM}(g)w = u$. Set $h := h - \frac{\lambda}{\mu} vgw$. Return to Step 2.

We can reformulate the algorithm NormalForm as follows. Let $h \in F\langle X\rangle$ and let $w \in X^*$ be an element of the support h such that LM(g) is a factor of w for some $g \in G$. Let $u, v \in X^*$ be such that $u\text{LM}(g)v = w$. Then we say that h reduces to h', where $h' = h - \frac{\lambda}{\mu} ugv$ and λ and μ are the coefficients of w and LM(g) in h and g respectively. More generally, if $h_1, \ldots, h_k$ are such that h_i reduces to h_{i+1} for $1 \leq i \leq k-1$, then we also say that h_1 reduces to h_k, and we write $h_1 \longrightarrow h_k$.

As the following example demonstrates, this algorithm may not return a normal form modulo the ideal I.

Example 2.4.8. [14] Let $X = \{x, y\}$ and $G = \{xy - x^2, x^3 - yx, y^3\}$ and let the order be deglex, with $x < y$. So LM$(G) = \{xy, x^3, y^3\}$. We consider calculating the normal form of $f = x^2y^2$. Since $f = x(xy)y$ we see that $f \longrightarrow x^3y$. Continuing like this we find that $x^3y \longrightarrow yxy \longrightarrow yx^2$. This last monomial is in normal form with respect to G. Hence we output it. However,

$$-(xy - x^2)y^2 + x(y^3) = x^2y^2.$$

So x^2y^2 lies in the ideal I generated by G, so that $\text{Nf}_I(f) = 0$. Also in the algorithm we often have a choice of $g \in G$ such that LM(g) is a factor of LM(h). In this particular case we could have made the following series

$$x^3y = x^2(xy) \longrightarrow x^4 = x(x^3) \longrightarrow xyx \longrightarrow x^3 \longrightarrow yx,$$

and the output is yx.

We see that the algorithm does not give a unique output. However, in the following we show that if the set G has the property of being a Gröbner basis, then the output of $NormalForm(G, f)$ is unique (and equals $\mathrm{Nf}(f)$).

Definition 2.4.9. [14] If $I \subset F\langle X \rangle$ is an ideal then a subset $G \subset I$ is a **Gröbner basis** of I if for all $f \in I$ there is a $g \in G$ such that $\mathrm{LM}(g)$ is a factor of $\mathrm{LM}(f)$.

Definition 2.4.10. A subset $G \subset F\langle X \rangle$ is **self-reduced** if every $g \in G$ is in normal form modulo $G \setminus \{g\}$ and every $g \in G$ is **monic**: the coefficient of $\mathrm{LM}(g)$ is 1. (This definition is stronger than [14, Definition 6.1.5].)

Definition 2.4.11. [14] Let $g, h \in F\langle X \rangle$ be two monic noncommutative polynomials. Assume that $\mathrm{LM}(g)$ is not a factor of $\mathrm{LM}(h)$ and that $\mathrm{LM}(h)$ is not a factor of $\mathrm{LM}(g)$. Let $u, v \in X^*$ be such that

(*i*) $\mathrm{LM}(g)\, u = v\, \mathrm{LM}(h)$,

(*ii*) u is a proper right factor of $\mathrm{LM}(h)$,

(*iii*) v is a proper left factor of $\mathrm{LM}(g)$.

In this case the element $gu - vh \in F\langle X \rangle$ is called a **composition** of g and h.

Lemma 2.4.12. [14](Diamond Lemma, (Bergman, 1978)) *If $I \subset F\langle X \rangle$ is an ideal generated by a self-reduced set G, then G is a Gröbner basis of I if and only if for all compositions f of the elements of G the normal form of f modulo G is zero.*

Remark 2.4.13. Lemma 2.4.12 yields a straightforward algorithm for computing a Gröbner basis of an ideal I generated by a finite set G.

Algorithm 2.4.14. (Gröbner basis)
Input: a finite generating set G of an ideal $I \subset F\langle X \rangle$.
Output: a Gröbner basis G for an ideal I.
While $H \neq \varnothing$ or $H \neq \{0\}$ do:

- Step 1. Compute all compositions of elements of G. Let H be the set of their normal forms modulo G.

- Step 2. Replace G by $G \cup H$, and self-reduce the new set G by replacing each element by its normal form modulo the other elements and return to Step 1.

Example 2.4.15. [14] Let $X = \{x, y\}$ and $G = \{g_1 = xy - x^2, g_2 = x^3 - yx, g_3 = y^3\}$ and let the order be deglex, with $x < y$. So $\mathrm{LM}(G) = \{xy, x^3, y^3\}$. First g_1 has no composition with itself, and neither with g_2; but it has a composition with g_3:

$$g_1 y^2 - x g_3 = -x^2 y^2.$$

As seen in Example 2.4.8 $NormalForm(G, f) = yx$; so we add $g_4 = yx$ to G. Now it is not difficult to check that all compositions of the elements $g_1, \dots, g_4$ reduce to zero modulo G. Hence G is a Gröbner basis.

Remark 2.4.16. The Gröbner basis might be infinite. To illustrate this let $X = \{x, y\}$ and I the ideal of $F\langle X\rangle$ generated by $f_1 = xyx - yx$. Here f_1 has a composition with itself

$$f_1 yx - xy f_1 = xy^2 x - yxyx$$

which modulo f_1 reduces to $xy^2x - y^2x$. Set $f_n = xy^n x - y^n x$, then the composition of f_k with f_l reduces to f_{k+l} (modulo f_k, f_l). Hence the Gröbner basis relative to deglex is equal to the set of all f_n for $n \geq 1$.

Definition 2.4.17. Let L be a Lie algebra over $\mathbb{F}$ and $U(L)$ an associative algebra. Let $i : L \to U(L)$ be a Lie homomorphism. The pair $(U(L), i)$ is called a **(universal) enveloping algebra** of L if for every associative algebra A and every Lie homomorphism $f : L \to A$ there is a unique algebra homomorphism $\bar{f} : U(L) \to A$ such that the map $\bar{f} \circ i = f$.

Theorem 2.4.18. (Poincaré-Birkhoff-Witt) *Let $\{x_j\}_{j \in \mathcal{J}}$ be a totally-ordered basis for L over $\mathbb{F}$. Then $U(L)$ has a basis consisting of PBW monomials, that is, monomials of the form*

$$x_{j_1}^{m_1} \dots x_{j_t}^{m_t},$$

where $j_1 < \dots < j_t$ are in $\mathcal{J}$ and t and each m_i are non-negative integers.

Proof. de Graaf [14, Theorem 6.2.1]. □

2.5 Triple systems

In this section we recall some definitions and results about triple systems.

Definition 2.5.1. [34] Let T be a vector space over a field F. We say that T is a **triple system** if it is endowed with a trilinear map $\langle -,-,- \rangle$ from $T \times T \times T$ into T, called the triple product of T.

Definition 2.5.2. [34] A triple system T is called **associative** if the equalities

$$\langle\langle x,y,z\rangle,u,v\rangle = \langle x,\langle y,z,u\rangle,v\rangle = \langle x,y,\langle z,u,v\rangle\rangle$$

hold for all $x,y,z,u,v \in T$.

Definition 2.5.3. [34] A **subsystem** of $(T,\langle -,-,- \rangle)$ is a vector subspace U of T such that $\langle U,U,U\rangle \subset U$.

Definition 2.5.4. [34] Let $(T_1,\langle -,-,- \rangle_1)$ and $(T_2,\langle -,-,- \rangle_2)$ be triple systems. A linear map $\phi : T_1 \to T_2$ is called a **homomorphism**, if

$$\phi\langle x,y,z\rangle_1 = \langle \phi(x),\phi(y),\phi(z)\rangle_2,$$

for all $x,y,z \in T_1$.

Remark 2.5.5. The notations of epimorphism, monomorphism and isomorphism are the usual ones.

Definition 2.5.6. [34] A **right** (resp. **middle**, **left**) ternary **ideal** I of T is a linear subspace satisfying $\langle I,T,T\rangle \subset I$ (resp. $\langle T,I,T\rangle \subset I$, $\langle T,T,I\rangle \subset I$). A right, middle and left ternary ideal is simply called a **ternary ideal**.

Definition 2.5.7. [34] A vector space V over a field $\mathbb{F}$ endowed with a trilinear operation $V \times V \times V \to V$, $(x,y,z) \to \langle x,y,z\rangle$ is said to be a **Jordan triple system** if the following identities are satisfied:

$$\langle a,b,c\rangle = \langle c,b,a\rangle, \quad \langle a,b,\langle x,y,z\rangle\rangle = \langle\langle a,b,x\rangle,y,z\rangle - \langle x,\langle b,a,y\rangle,z\rangle + \langle x,y,\langle a,b,z\rangle\rangle,$$

for all $a,b,c,x,y,z \in V$.

Example 2.5.8. [27] Let D be an associative algebra over a field K. Set $V = Mat(p,q;D)$, the $(p \times q)$-matrices over D. This vector space is a Jordan triple system with respect to the product

$$\{x,y,z\} = xy^tz + zy^tx,$$

where y^t denotes the transpose matrix of y.

Definition 2.5.9. [19] A vector space V over a field $\mathbb{F}$ of characteristic $\neq 2$ endowed with a trilinear operation $V \times V \times V \to V$, $(x,y,z) \to \langle x,y,z\rangle$ is said to be an **anti-Jordan triple system** if the following identities are satisfied:

$$\langle a,c,b\rangle = -\langle b,c,a\rangle, \quad \langle a,b,\langle x,y,z\rangle\rangle = \langle\langle a,b,x\rangle,y,z\rangle + \langle x,\langle b,a,y\rangle,z\rangle + \langle x,y,\langle a,b,z\rangle\rangle,$$

for all $a,b,c,x,y,z \in V$.

Definition 2.5.10. [2] Let T be a triple system. Then T is called **polarized** if $T = T^+ \oplus T^-$ and the triple product satisfies $\langle T^\epsilon, T^\epsilon, T\rangle = 0 = \langle T, T^\epsilon, T^\epsilon\rangle$ and $\langle T^\epsilon, T^{-\epsilon}, T^\epsilon\rangle \subset T^\epsilon$, $\epsilon = \pm$.

Theorem 2.5.11. [2, Theorem 4] *Let F be an algebraically closed field of characteristic 0. The following are simple finite-dimensional anti-Jordan triple systems over F:*

(i) $T_1 = M_{nn}(F)$, *together with the trilinear map* $\langle -,-,-\rangle_1$ *defined by* $\langle x,y,z\rangle_1 = xyz - zyx$.

(ii) $T_2 = M_{mn}(F)$, *together with the trilinear map* $\langle -,-,-\rangle_2$ *defined by* $\langle x,y,z\rangle_2 = xy^taz - zy^tax$, *where* $a = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, *and* $2r = m$.

(iii) $T_3 = M_{mn}(F)$, *together with the trilinear map* $\langle -,-,-\rangle_3$ *defined by* $\langle x,y,z\rangle_3 = xby^tz - zby^tx$, *where* $b = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, *and* $2r = n$.

Remark 2.5.12. Theorem 2.5.11 is part of the classification of the simple anti-Jordan triple systems which was given by Bashir [2] (for the complete classification see [2, Theorem 6]).

Definition 2.5.13. [27] For $\delta = \pm 1$, a triple system $(a, b, c) \to [a, b, c]$, $a, b, c \in V$ is called **δ-Lie triple system**(δ-LTS) if the following three identities are satisfied:

$$[a, b, c] = -\delta[b, a, c],$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0,$$

$$[a, b, [x, y, z]] = [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]],$$

where $a, b, x, y, z \in V$. A 1-LTS is an **LTS** while a (-1)-LTS is an **anti-LTS**.

Example 2.5.14. [27] In any associative algebra $\mathfrak{A}$, the subspaces closed under $[[a, b], c] = abc - bac - cab + cba$ are L.T.S.

2.6 Classification of trilinear operations

In [10] Bremner and Peresi used the representation theory of the symmetric group to classify up to equivalence all trilinear operations. In this section we discuss briefly how the representation theory of the symmetric group was used.

Definition 2.6.1. [10] A **multilinear** operation on an n-ary associative algebra is a linear combination of permutations of the original operation:

$$[a_1, a_2, \ldots, a_n] = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)} \quad (x_\sigma \in \mathbb{Q}).$$

A multilinear operation can be identified with an element of the group algebra $\mathbb{Q}S_n$ of the symmetric group S_n.

A trilinear operation is an expression of the form

$$[a, b, c] = x_1 abc + x_2 acb + x_3 bac + x_4 bca + x_5 cab + x_6 cba \quad (x_i \in \mathbb{Q}).$$

Definition 2.6.2. [10] Two trilinear operations O_1 and O_2 are **equivalent** if and only if they generate the same left ideal in $\mathbb{Q}S_3$.

Remark 2.6.3. Equivalent operations do not always satisfy the same polynomial identities.

2.6.1 Irreducible representations of the symmetric group

We discuss the irreducible representations of the symmetric group. We recall the structure theory from [26]. The irreducible representations of S_n are in bijection with the partitions of n.

Definition 2.6.4. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n. Thus, $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r \geq 1$ and $\lambda_1 + \cdots + \lambda_r = n$. The **frame** of λ is an ordered set of boxes with λ_i boxes in row i. By placing the numbers 1 to n in any order into the n boxes, we obtain a **tableau**. Thus, there are $n!$ tableaux for a given frame. A tableau is called **standard** if the numbers increase in every row from left to right and in every column from top to bottom.

Theorem 2.6.5. [26] *For any partition λ of n, let d_λ be the number of standard tableaux. Then $\mathbb{Q}S_n$ can be represented as a direct sum over all λ of full matrix algebras of size $d_\lambda \times d_\lambda$:*

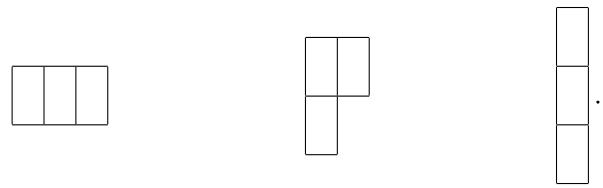
$$\phi : \mathbb{Q}S_n \xrightarrow{\cong} \bigoplus_\lambda M_{d_\lambda}(\mathbb{Q}). \tag{2.4}$$

An important problem is the following: Given a permutation $\pi \in S_n$ and a partition λ of n, compute the $d_\lambda \times d_\lambda$ matrix representing π, that is, compute the projection of π onto the summand M_{d_λ} in (2.6.5). A simple algorithm was found by Clifton [11]. Let $T_1, \ldots, T_d$ $(d = d_\lambda)$ be the standard tableaux for λ. Let E^λ_π be the matrix defined as follows: Apply π to the tableau T_j. If there exist two numbers that appear together in a column of T_i and a row of πT_j, then $(E^\lambda_\pi)_{ij} = 0$. If not, then $(E^\lambda_\pi)_{ij}$ equals the sign of the vertical permutation for T_i which leaves the columns of T_i fixed as sets and takes the numbers of T_i into the correct rows they occupy in πT_j.

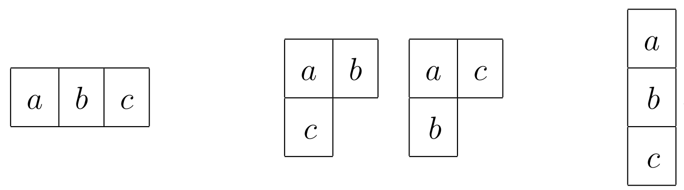
Remark 2.6.6. The matrix E^λ_{id} corresponding to the identity permutation is not necessarily the identity matrix, but it is always invertible.

Lemma 2.6.7. [11] *The matrix representing π in partition λ is equal to $(E^\lambda_{id})^{-1} E^\lambda_\pi$.*

Example 2.6.8. [10](Explicit decomposition of $\mathbb{Q}S_3$) Let the symmetric group S_3 act on $\{a, b, c\}$. There are three distinct partitions for $n = 3$: $\{3\}$, $\{2, 1\}$, and $\{1, 1, 1\}$. The corresponding frames are



The following are the corresponding standard tableaux:



Theorem 2.6.5 implies

$$\mathbb{Q}S_3 = \mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus \mathbb{Q}.$$

We have two bases of $\mathbb{Q}S_3$: the first (the permutation basis) consists of the words in lexicographical order,

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba;$$

the second consists of matrix units in standard order,

$$S, \quad E_{11}, \quad E_{12}, \quad E_{21}, \quad E_{22}, \quad A.$$

By Lemma 2.6.7, we get the complete isomorphism:

$$abc \longrightarrow 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus 1, \qquad acb \longrightarrow 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus -1,$$

$$bac \longrightarrow 1 \oplus \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \oplus -1, \qquad bca \longrightarrow 1 \oplus \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \oplus 1,$$

$$cab \longrightarrow 1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \oplus 1, \qquad cba \longrightarrow 1 \oplus \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \oplus -1.$$

We write M for the matrix whose columns express the matrix units as linear combinations of the permutations.

$$M = \frac{1}{6}\begin{bmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 & -1 \\ 1 & 2 & -2 & 0 & -2 & -1 \\ 1 & -2 & 2 & -2 & 0 & 1 \\ 1 & 0 & -2 & 2 & -2 & 1 \\ 1 & -2 & 0 & -2 & 2 & -1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

By Example 2.6.8 and Corollary 7 of [10], for representatives of the equivalence classes of trilinear operations we take ordered triples of matrices in row canonical form:

$$\left[y_1, \begin{bmatrix} y_2 & y_3 \\ y_4 & y_5 \end{bmatrix}, y_6 \right],$$

For the first and last components the possible row canonical forms are 0 and 1. For the middle component, the possible row canonical forms are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (q \in \mathbb{Q}).$$

We exclude the two trivial cases: the zero operation, and the original totally associative operation. This leaves ten cases: four families of operations with parameter q and six isolated operations.

Example 2.6.9. The following operations have the zero as the middle component.

$$\left[1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right] \longrightarrow \tfrac{1}{6}\left(abc + acb + bac + bca + cab + cba\right) \quad \text{(symmetric sum)},$$

$$\left[0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 \right] \longrightarrow \tfrac{1}{6}\left(abc - acb - bac + bca + cab - cba\right) \quad \text{(alternating sum)},$$

$$\left[1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 \right] \longrightarrow \tfrac{1}{3}\left(abc + bca + cab\right) \quad \text{(cyclic sum)}.$$

(see [10] for all the other trilinear operations).

2.7 Wedderburn decomposition of a finite dimensional associative algebra

A semisimple associative algebra is isomorphic to direct sum of full matrix algebras over possibly noncommutative fields. The most important problem is how to find an explicit isomorphism with such full matrix algebras. In this section we discuss briefly how to get this isomorphism.

Unless otherwise stated, we assume throughout this section that A is finite dimensional unital associative algebra over field $\mathbb{F}$ with basis $\{a_1, \ldots, a_n\}$ and structure constants $c_{ij}^k \in \mathbb{F}$ such that

$$a_i a_j = \sum_{k=1}^{n} c_{ij}^k a_k, \qquad 1 \le i, j, k \le n.$$

Definition 2.7.1. An element $x \in A$ is **nilpotent** if $x^m = 0$ for some positive integer m. An element x is **strongly nilpotent** if xy is nilpotent for every $y \in A$. The **radical** $\mathfrak{R}(A)$ is the set of strongly nilpotent elements of A.

Definition 2.7.2. The algebra A is **semisimple** if it contains no strongly nilpotent elements, i.e. $\mathfrak{R}(A) = 0$.

Theorem 2.7.3. [15](Wedderburn-Artin Theorem) *Every semisimple algebra Q has a unique decomposition $Q = Q_1 \oplus \cdots \oplus Q_c$ into the direct sum of simple ideals where $Q_i Q_j = 0$ for $i \neq j$. Every simple algebra is isomorphic to a full matrix algebra $M_n(D)$ for some division algebra D over $\mathbb{F}$.*

Theorem 2.7.4. [15](Dickson's Theorem) *If A is a finite dimensional algebra of matrices over a field $\mathbb{F}$ of characteristic 0 then x is in the radical $\mathfrak{R}(A)$ of A if and only if $trace(xy) = 0$ for every $y \in A$.*

Corollary 2.7.5. [6] *The radical $\mathfrak{R}(A)$ of A is the nullspace of the $n \times n$ matrix Δ such that*

$$\Delta_{ij} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{ji}^k c_{k\ell}^\ell.$$

Our next target is the Wedderburn decomposition of the radical-free part (semisimple part) $Q = A/\mathfrak{R}(A)$. By Wedderburn's Theorem there exists a decomposition

$$Q = Q_1 \oplus \cdots \oplus Q_c = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_c}(D_c),$$

where $Q_1, \ldots, Q_c$ are simple ideals. We give a method to find the bases for the ideals Q_i; this method was given by K. Friedl [21].

First we compute the center $Z(Q)$ of Q, where $Z(Q)$ is defined by

$$Z(Q) = \{x \in Q \mid xy = yx \text{ for every } y \in Q\}.$$

Corollary 2.7.6. [6] *Let $b_1, \ldots, b_r$ be a basis of $Q = A/\mathfrak{R}(A)$ with structure constants d_{ij}^k. The basis of the center $Z(Q)$ can be represented as a nullspace of the $r^2 \times r$ matrix as follows: the entry in row $(i-1)r + k$ and column j is $d_{ij}^k - d_{ji}^k$ for $1 \le i, j, k \le r$.*

It is not difficult to show that $Z(Q)$ is also a semisimple algebra. Moreover, the Wedderburn decomposition of $Z(Q)$ is inherited from the Wedderburn decomposition of Q in the following sense:

$$Z(Q) = Z(Q_1) \oplus \cdots \oplus Z(Q_c) = \mathbb{F}_1 \oplus \cdots \oplus \mathbb{F}_c,$$

where $\mathbb{F}_1, \ldots \mathbb{F}_c$ are extension fields of $\mathbb{F}$. From the Wedderburn decomposition of $Z(Q)$ we can easily recover the Wedderburn decomposition of Q. In fact, we have

$$Q_i = QZ(Q_i) = Q\mathbb{F}_i \quad \text{for } i = 1, \ldots, c. \tag{2.5}$$

This reduces the problem to the commutative case; if we can decompose $Z(Q)$ into direct sum of fields, we can decompose Q into direct sum of simple matrix algebras. Thus our next task is to decompose the commutative semisimple algebra $Z = Z(Q)$ into direct sum of fields. We need to find new basis $e_1, \ldots, e_c$ of orthogonal primitive idempotents: $e_i^2 = e_i$ and $e_i e_j = 0$ $(i \neq j)$. We use a recursive SPLIT Algorithm following [21, 6].

Algorithm 2.7.7. SPLIT(Z, I)

- Input: A semisimple commutative algebra Z and a nonzero ideal I of Z.

- Output: A decomposition of I into direct sum of fields.

(i) Choose a basis element v of I that is not a scalar multiple of the identity element of I.

(ii) Compute the the minimal polynomial f of v as an element of I, and factor f over $\mathbb{F}$. We have two cases:

1. If f is irreducible, then $\mathbb{F}(v)$ is a field. If $\mathbb{F}(v) = I$ then we are done: the ideal I is a field.
   If $\mathbb{F}(v) \neq I$ then we choose a basis element w of I with w is not in $\mathbb{F}(v)$ and compute the minimal polynomial of w over $\mathbb{F}(v)$. We repeat this process until we have either (a) constructed a proof that I is a field or (b) found an element of I whose minimal polynomial is reducible over $\mathbb{F}$.
2. If f is reducible, then $f = gh$ where $g, h \in \mathbb{F}$ are relatively prime. Hence there exist $s, t \in \mathbb{F}[x]$ for which $sg + th = 1$. It follows that the ideals J and K generated by $g(v)$ and $h(v)$ split I: that is, J and K are proper ideals of I such that $I = J \oplus K$ and $JK = \{0\}$. If J (resp. K) is not field then perform (i) and (ii) for $I = J$ (resp. $I = K$).

Remark 2.7.8. To get the decomposition of $Z = Z(Q)$ into direct sum of fields, we start Algorithm 2.7.7 by $I = Z$.

We now have a new basis $e_1, \ldots, e_c$ of orthogonal idempotents in $Z(Q)$. We compute a basis of Q_i by constructing $2r \times r$ matrix; in row j of the upper (resp. lower) $r \times r$ block we put the coefficients of $b_j e_i$ (resp. $e_i b_j$), with respect to $b_1, \ldots, b_r$. We compute the RCF; the nonzero rows form a basis of Q_i. Suppose that we have a basis $t^i_1, t^i_2, \ldots, t^i_{q^2}$ for Q_i. To construct an explicit isomorphism of Q_i to the full matrix algebra $M^i_q(\mathbb{F})$, we need to find new basis $E^i_{jk}(1 \leq j, k \leq q)$ satisfying the matrix unit relations $E^i_{jk} E^i_{ml} = \delta_{km} E^i_{jl}$. This is easy if we can find a basis for a minimal (q-dimensional) left ideal $I \subset Q_i$: we identify the basis elements of I with standard basis $U_1, \ldots, U_q \in \mathbb{F}^q$, and solve the the linear equations $E^i_{jk} U_m = \delta_{km} U_j$ to determine the elements E^i_{jk}. If $\mathbb{F}$ is finite then this can be done in polynomial time; but if $\mathbb{F} = \mathbb{Q}$, then the problem is more difficult. If we are lucky, one of the basis elements of Q generates a minimal left ideal.

2.8 Down-up algebras

G. Benkart and Roby [3] introduced the down-up algebras over complex numbers. We define it over an arbitrary field.

Definition 2.8.1. Let $\mathbb{F}$ be a field and $\alpha, \beta, \gamma \in \mathbb{F}$ be fixed arbitrary parameters. Then the **down-up algebra** $A(\alpha, \beta, \gamma)$ is the unital associative $\mathbb{F}$-algebra with generators a, b and relations,

$$b^2a = \alpha bab + \beta ab^2 + \gamma b, \quad ba^2 = \alpha aba + \beta a^2b + \gamma a.$$

Remark 2.8.2. We note that when $\gamma \neq 0$ the down-up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$ by the map $d \mapsto d'$, $u \mapsto \gamma u'$. Therefore for algebras with $\gamma \neq 0$ we may assume that $\gamma = 1$.

Example 2.8.3. Benkart and Roby [3] showed that the down-up algebra $A(2, -1, -2)$ is isomorphic to the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of 2×2 complex matrices of trace 0 with basis $\{h, e, f\}$ which satisfies: $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.

Indeed, in $U(\mathfrak{sl}_2(\mathbb{C}))$ the following relations hold:

$$e^2f = 2efe - fe^2 - 2e, \quad ef^2 = 2fef - f^2e - 2f.$$

Thus, $U(\mathfrak{sl}_2(\mathbb{C}))$ is a homomorphic image of $A(2, -1, 1)$ via the mapping $\phi : A(2, -1, 1) \to U(\mathfrak{sl}_2(\mathbb{C}))$, where $\phi(b) = e$, $\phi(a) = f$. Consider the mapping $\psi : \mathfrak{sl}_2(\mathbb{C}) \to A(2, -1, 1)$ defined by $\psi(e) = b$, $\psi(f) = a$ and $\psi(h) = ba - ab$. This mapping can be extended by the universal property of $U(\mathfrak{sl}_2(\mathbb{C}))$ to an algebra homomorphism $\psi : U(\mathfrak{sl}_2(\mathbb{C})) \to A(2, -1, 1)$. The two homomorphisms ϕ and ψ are inverses of each other.

Remark 2.8.4. [3] We can define an anti-automorphism η of the free associative algebra generated by a, b by $\eta(a) = b$ and $\eta(b) = a$. Then

$$\eta(b^2a) = ba^2 = \alpha aba + \beta a^2b + \gamma a = \eta(\alpha bab + \beta ab^2 + \gamma b),$$

and a similar relation holds for ba^2. Therefore, η induces an anti-automorphism on $A(\alpha, \beta, \gamma)$.

Definition 2.8.5. [29] Let A be a finitely generated (not necessarily associative) algebra over the field $\mathbb{F}$. Choose a finite dimensional $\mathbb{F}$-subspace V(containing 1_A) of A such that A is generated as an algebra over $\mathbb{F}$ by V. There is an ascending chain of subspaces

$$\mathbb{F} \subseteq V \subseteq V^2 \subseteq \cdots \subseteq V^n \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} V^n = A.$$

with $\dim_k(V^n) < \infty$, for each $n \in N$. The asymptotic behavior of the monotone increasing sequence $\dim_k(V^n)$ provides a useful invariant of the algebra A, known as the **growth** or **Gelfand-Kirillov dimension** of A, and defined by

$$GK\dim(A) = \limsup_{n\to\infty} \log_n(\dim_{\mathbb{F}}(V^n)).$$

It is known that the above definition does not depend on the choice of a particular finite dimensional vector space generating A (see [29]).

Definition 2.8.6. [29] A **grading** $\mathcal{A} = \{A_i\}_{i\in\mathbb{Z}}$ of the k-algebra A is a sequence of k-subspaces A_i of $\mathcal{A}$ such that

$$A = \bigoplus_{i\in\mathbb{Z}} A_i \quad \text{and} \quad A_iA_j \subseteq A_{i+j} \text{ for all } i, j \in \mathbb{Z}.$$

An algebra with a grading $\mathcal{A}$ is called $\mathcal{A}$-graded, or simply graded; it is finitely graded if each of the components A_i is a finite dimensional vector space. The elements of A_n are called homogenous of degree n. The component A_0 is a subalgebra of A which contains 1_A and hence the base field k.

Definition 2.8.7. [29] A $\mathbb{Z}$-filtration or simply a **filtration** of the k-algebra A is a sequence of k-subspaces

$$\cdots \subseteq A_{i-1} \subseteq A_i \subseteq A_{i+1} \subseteq \ldots, \; i \in \mathbb{Z},$$

such that

$$1 \in A_0, \; A_iA_j \subseteq A_{i+j} \text{ for all } i, j \in \mathbb{Z}, \text{ and } A = \bigcup_{i\in\mathbb{Z}} A_i.$$

The filtration is called finite if each A_i is finite dimensional, and it is called discrete if $A_i = 0$ for all $i < n_0$ for some integer $n_0 \leq 0$. The vector space

$$gr(A) = \bigoplus_{i \in \mathbb{Z}} A_i / A_{i-1},$$

equipped with a multiplication derived from the rule

$$(x + A_{i-1}).(y + A_{j-1}) = xy + A_{i+j-1},$$

is called the **associated graded algebra**.

Theorem 2.8.8. [[3], Theorem 3.1, Corollary 3.2] *Assume $A(\alpha, \beta, \gamma)$ is a down-up algebra. Then*

$$\mathfrak{B}_1 = \{a^i (ba)^j b^k \mid i, j, k \geq 0\},$$

is a basis of A. The Gelfand-Kirillov dimension of any down-up algebra is 3.

Lemma 2.8.9 ([38], Lemma 2.2)**.** *Assume $A(\alpha, \beta, \gamma)$ is a down-up algebra over $\mathbb{F}$, and $c_1, c_2 \in \mathbb{F}$ are arbitrary. Then*

$$\mathfrak{B}_2 = \{a^i (ba + c_1 ab + c_2)^j b^k \mid i, j, k \geq 0\},$$

is a basis of A.

Remark 2.8.10. E. Kirkman et [28] showed that

$$\mathfrak{B}_3 = \{(ab)^i (ba)^j b^k, (ab)^i (ba)^j a^{k+1} \mid i, j, k \geq 0\}.$$

is also a basis for $A(\alpha, \beta, \gamma)$ whenever $\beta \neq 0$.

Chapter 3

Alternating quaternary algebra structures on irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$

3.1 Introduction

The main focus of this chapter is to use the representation theory of Lie algebras to construct new alternating algebra structures and to discover natural n-ary generalizations of Lie algebras. In particular, we study the exterior powers of an irreducible representation of a simple Lie algebra. If $\Lambda^n V$, the n-th exterior power of an irreducible representation V of a simple Lie algebra L, contains V itself as a direct summand, then the projection $\Lambda^n V \to V$ defines an alternating n-ary algebra structure on V which is L-invariant in the sense that the derivation algebra of this n-ary structure contains a subalgebra isomorphic to L. This approach was used by Bremner and Hentzel [8] in the case of the third exterior power of an irreducible representation of the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In this chapter we extend this work to the fourth exterior power. We study alternating quaternary algebra structures on the irreducible representation $V(n)$ obtained from $\mathfrak{sl}_2(\mathbb{C})$-invariant linear maps $\Lambda^4 V(n) \to V(n)$. We first need to determine the multiplicity of the irreducible representation $V(n)$ of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ as a direct summand of its fourth exterior power $\Lambda^4 V(n)$; that is $\dim \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\big(\Lambda^4 V(n), V(n)\big)$. In Section 3.2 we obtain a closed formula for the multiplicity using a general approach which applies to arbitrary exterior powers. We determine the values of n for which the module

$V(n)$ occurs in $\Lambda^4 V(n)$ with multiplicity 1 and 2. For these values of n, we obtain $\mathfrak{sl}_2(\mathbb{C})$-invariant alternating quaternary algebra structures obtained from the projections $\Lambda^4 V(n) \to V(n)$. The structure constants for these algebras are determined in Section 3.3. In Section 3.4 we determine the multilinear polynomial identities of degree ≤ 7 satisfied by the alternating quaternary algebra structures.

The basic background information for this chapter is summarized in Sections 2.2-2.3.

3.2 Multiplicity formula

In this section we discuss how to determine the the multiplicity of the irreducible representation $V(n)$ in its k-th exterior power $\Lambda^k V(n)$. This problem had been solved in the cases of $k = 2$ and 3 (see Lemmas 2.3.1 & 2.3.2). For $k > 3$ the solution of this problem is not known. Our approach is to reduce the problem to a combinatorial question and apply the theory of Pólya enumeration. In this section we consider the case $k = 4$; since we are interested in studying alternating quaternary structures.

Lemma 3.2.1. [9, Lemma 5.1] *Let M be an $\mathfrak{sl}_2(\mathbb{C})$-module with $\dim M < \infty$. For $n \in \mathbb{Z}$ let $M_n = \{v \in M \mid H.v = nv\}$ be the subspace of all vectors of weight n together with 0. For $n \geq 0$ the multiplicity of $V(n)$ in the decomposition of M as a direct sum of simple $\mathfrak{sl}_2(\mathbb{C})$-modules is $\dim M_n - \dim M_{n+2}$.*

Lemma 3.2.2. [9, Lemma 5.2] *Let $M = \Lambda^k V(n)$ be the k-th exterior power of $V(n)$. If $w \in \mathbb{Z}$ with $kn \geq w \geq -kn$ and $w \equiv kn \,(\mathrm{mod}\, 2)$ then the dimension of the weight space M_w is the number of sequences $(w_1, w_2, \ldots, w_k) \in \mathbb{Z}^k$ satisfying*

$$n \geq w_1 > w_2 > \ldots > w_k \geq -n; \;\; w_1 + w_2 + \ldots + w_k = w; \;\; w_1, \cdots, w_k \equiv n \,(\mathrm{mod}\, 2).$$

We now specialize to $k = 4$ since we are interested in the fourth exterior power. To compute the multiplicity of $V(n)$ as a direct summand of $\Lambda^4 V(n)$ using Lemmas 3.2.1 and 3.2.2, we must determine the number of quadruples (p, q, r, s) satisfying

$$n \geq p > q > r > s \geq -n; \quad p + q + r + s = w; \quad p, q, r, s \equiv n \,(\mathrm{mod}\, 2) \tag{3.1}$$

for $w = n$ and $w = n+2$. Let n be a non-negative integer and let w be a weight of $\Lambda^4 V(n)$: thus w is an integer satisfying

$$4n \geq w \geq -4n, \qquad w \equiv 0 \,(\mathrm{mod}\, 2).$$

For integers p, q, r, s satisfying (3.1) we define

$$P' = p + n, \quad Q' = q + n, \quad R' = r + n, \quad S' = s + n.$$

Then (P', Q', R', S') is a quadruple of even integers satisfying

$$2n \geq P' > Q' > R' > S' \geq 0, \quad P' + Q' + R' + S' = W', \quad W' = w + 4n.$$

We need to count the number of partitions of W' into four distinct nonnegative even parts less than or equal to $2n$. We only need $W' = 5n$ and $W' = 5n+2$ corresponding to $w = n$ and $w = n+2$. It is clear that if n is odd then there are no solutions in both cases, so $V(n)$ does not occur as a summand of $\Lambda^4 V(n)$: the multiplicity is zero. Therefore we may assume that n is even and define

$$P = \frac{p+n}{2}, \quad Q = \frac{q+n}{2}, \quad R = \frac{r+n}{2}, \quad S = \frac{s+n}{2}, \quad W = \frac{w+4n}{2}.$$

Then (P, Q, R, S) is a quadruple of integers satisfying

$$n \geq P > Q > R > S \geq 0, \qquad P + Q + R + S = W.$$

Definition 3.2.3. [37, page 612] If G is a subgroup of the symmetric group S_n then the **cycle index** of G is the following polynomial in the indeterminates $x_1, x_2, \ldots, x_n$:

$$Z_G(x_1, x_2, \cdots, x_n) = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n};$$

here b_i is the number of cycles of length i in the disjoint cycle factorization of σ.

Lemma 3.2.4. [23, page 36] *The cycle index of the alternating group A_n is*

$$Z_{A_n}(x_1, x_2, \cdots, x_n) = Z_{S_n}(x_1, x_2, \cdots, x_n) + Z_{S_n}\left(x_1, -x_2, \cdots, (-1)^{n-1} x_n\right).$$

Proof. The definition of cycle index gives

$$Z_{A_n}(x_1, x_2, \cdots, x_n) = \frac{2}{n!}\left[\sum_{\sigma \in S_n} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} - \sum_{\sigma \in S_n \setminus A_n} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}\right].$$

Since $\sigma \in S_n \setminus A_n$ if and only if σ has an odd number of even length cycles, we get

$$Z_{A_n}(x_1, x_2, \cdots, x_n) =$$
$$\frac{2}{n!} \cdot \frac{1}{2}\left[\sum_{\sigma \in S_n} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} + \sum_{\sigma \in S_n} x_1^{b_1} (-x_2)^{b_2} \cdots \left((-1)^{n-1} x_n\right)^{b_n}\right].$$

This completes the proof. □

Definition 3.2.5. [36] Let A and B be two finite sets, R be a commutative ring with identity containing the field of rationals and let $w : B \to R$ assign a weight $w(b) \in R$ to each b in B. For $f : A \to B$, define

$$w(f) = \prod_{a \in A} w(f(a)) \in R;$$

here $w(f)$ is the weight of function f.

The next result is the special case $k = 4$ of Theorem 2 in Wu and Chao [37]; but note that we allow $0 \in S$.

Proposition 3.2.6. *If S is a set of non-negative integers then the number of partitions of an integer n into four distinct parts in S is the coefficient of x^n in*

$$Z_{A_4}\left(\sum_{i \in S} x^i, \sum_{i \in S} x^{2i}, \sum_{i \in S} x^{3i}, \sum_{i \in S} x^{4i}\right) - Z_{S_4}\left(\sum_{i \in S} x^i, \sum_{i \in S} x^{2i}, \sum_{i \in S} x^{3i}, \sum_{i \in S} x^{4i}\right).$$

Proof. Let $D_4 = \{1, 2, 3, 4\}$, let S be a set of non-negative integers and let S^{D_4} be the set of all functions from D_4 into S. Let the symmetric group S_4 act on D_4, and let the weight function $w : S \to \mathbb{Q}[x]$ be defined as $w(i) = x^i$ for all i in S. Also, let F be the subset of one-to-one functions in S^{D_4}, and for $f, g \in F$, define $f \sim g$ if and only if there exists $\sigma \in S_4$ such that $f(\sigma d) = g(d)$ for every $d \in D_4$. We claim that each equivalence class in F with weight n (i.e., every function in the equivalence class has weight n) determines a partition of n into S with 4 distinct parts. Let E be an equivalence class with weight n in F, and let f be function in E; then f has

4 distinct values in S such that $w(f) = x^n$. Let j_1, j_2, j_3, j_4 be the values of f in S; since the elements of S are nonnegative integers, we can arrange the 4 values of f, say $j_1 > j_2 > j_3 > j_4$. Since $w(f) = \prod_{i=1}^4 w(f(i)) = x^{j_1+j_2+j_3+j_4}$, then $j_1 + j_2 + j_3 + j_4 = n$. So, f corresponds to a partition of n with 4 distinct parts. Hence each equivalence class with weight n corresponds to a partition of n into S with 4 distinct parts.

Conversely, each partition $t_1 < t_2 < t_3 < t_4$ of n into S with 4 distinct parts determines an equivalence class with weight n in F. Clearly, $h(i) = t_i$ for $i = 1, \dots, 4$ is a function in F, and $w(h) = \prod_{i=1}^k x^{t_i} = x^{t_1+t_2+t_3+t_4} = x^n$. Thus, the partition of n into S with 4 parts determines the equivalence class containing h in S^{D_k}. The Pólya enumeration theorem for one-to-one functions (see p. 48 of [23]) completes the proof. □

Corollary 3.2.7. *If S is a set of non-negative integers then the number of partitions of a positive integer n into four distinct parts in S is the coefficient of x^n in*

$$Z_{S_4}\left(\sum_{i\in S} x^i, -\sum_{i\in S} x^{2i}, \sum_{i\in S} x^{3i}, -\sum_{i\in S} x^{4i}\right).$$

Proof. Take $n = 4$ in Lemma 3.2.4, set $x_j = \sum_{i\in S} x^{ji}$, and apply Proposition 3.2.6. □

Definition 3.2.8. For us $S = \{0, 1, \cdots, n\}$ so we define the following polynomials:

$$P_n(x) = Z_{S_4}\left(\sum_{i=0}^n x^i, -\sum_{i=0}^n x^{2i}, \sum_{i=0}^n x^{3i}, -\sum_{i=0}^n x^{4i}\right).$$

Lemma 3.2.9. *We have*

$$\left(\sum_{i=0}^n x^i\right)^t = \sum_{\ell=0}^{nt}\left[\sum_{k=0}^{\min(t, \lfloor \frac{\ell}{n+1} \rfloor)} (-1)^k \binom{t}{k}\binom{\ell-(n+1)k+t-1}{t-1}\right] x^\ell.$$

Proof. We use these three familiar identities:

$$\frac{1-x^{n+1}}{1-x} = \sum_{i=0}^n x^i, \quad \left(1-x^{n+1}\right)^t = \sum_{k=0}^t (-1)^k \binom{t}{k} x^{(n+1)k}, \quad \frac{1}{(1-x)^t} = \sum_{j=0}^\infty \binom{j+t-1}{t-1} x^j.$$

We obtain

$$\left(\sum_{i=0}^n x^i\right)^t = \frac{(1-x^{n+1})^t}{(1-x)^t} = \sum_{k=0}^t \sum_{j=0}^\infty (-1)^k \binom{t}{k}\binom{j+t-1}{t-1} x^{(n+1)k+j}. \tag{3.2}$$

We set $\ell = (n+1)k + j$ so that $\ell - j = (n+1)k$ and $\ell - j \equiv 0 \pmod{n+1}$. We also have $k = (\ell-j)/(n+1)$ and so $k \le \lfloor \ell/(n+1) \rfloor$. Substituting $j = \ell - (n+1)k$ in (3.2), and noting that nt is the largest power of x, we obtain the stated formula. □

Definition 3.2.10. We use the following notation:

$$\Delta_m^n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod m \\ 0 & \text{otherwise} \end{cases}, \qquad \Delta_{s,m}^n = \begin{cases} 1 & \text{if } n \equiv s \pmod m \\ 0 & \text{otherwise} \end{cases}.$$

Definition 3.2.11. We consider the following integer-valued functions of n:

$$\alpha(n) = \left\lceil \frac{n}{4} \right\rceil, \quad \beta(n) = \left\lceil \frac{3n}{4} \right\rceil, \quad \gamma(n) = \left\lfloor \frac{3n-2}{4} \right\rfloor, \quad \delta(n) = \left\lfloor \frac{5n}{6} \right\rfloor.$$

Proposition 3.2.12. *For even $n \in \mathbb{Z}$, the number of solutions $P, Q, R, S \in \mathbb{Z}$ to*

$$n \geq P > Q > R > S \geq 0, \qquad P + Q + R + S = \frac{5n}{2},$$

equals

$$\frac{23}{1152} n^3 - \frac{29}{96} n^2 + \frac{1}{288}\Big(-36\alpha(n) + 180\beta(n) + 36\gamma(n) + 27\Delta_4^n - 167\Big)n$$
$$+ \frac{1}{24}\Big(6\alpha(n)^2 - 6\beta(n)^2 - 6\gamma(n)^2 + 12\beta(n) - 12\gamma(n) + 8\delta(n) + 3\Delta_4^n - 6\Delta_8^n - 3\Big).$$

Proof. By Corollary 3.2.7, we need to find the coefficient of $x^{5n/2}$ in the polynomial $P_n(x)$ of Definition 3.2.8. The cycle index of S_4 is

$$Z_{S_4}(x_1, x_2, x_3, x_4) = \frac{1}{24}\big(x_1^4 + 6x_1^2 x_2 + 8x_1 x_3 + 3x_2^2 + 6x_4\big).$$

For the first four terms, we need to evaluate the following products:

$$A = \left(\sum_{i=0}^n x^i\right)^4, \; B = \left(\sum_{i=0}^n x^i\right)^2 \sum_{i=0}^n x^{2i}, \; C = \sum_{i=0}^n x^i \sum_{i=0}^n x^{3i}, \; D = \left(\sum_{i=0}^n x^{2i}\right)^2.$$

Lemma 3.2.9 gives

$$A = \sum_{\ell=0}^{4n} \sum_{k=0}^{\min(4, \lfloor \frac{\ell}{n+1} \rfloor)} (-1)^k \binom{4}{k} \binom{\ell-(n+1)k+3}{3} x^\ell.$$

Similarly,

$$B = \left(\sum_{\ell=0}^{2n} \sum_{k=0}^{\min(2, \lfloor \frac{\ell}{n+1} \rfloor)} (-1)^k \binom{2}{k} \big(\ell-(n+1)k+1\big) x^\ell \right) \sum_{i=0}^n x^{2i}.$$

The upper limit of k is 0 for $0 \leq \ell \leq n$, and 1 for $n+1 \leq \ell \leq 2n$. Hence

$$B = \left(\sum_{\ell=0}^{n} (\ell+1) x^\ell + \sum_{\ell=n+1}^{2n} \big[(\ell+1) - 2(\ell-n)\big] x^\ell \right) \sum_{i=0}^n x^{2i}$$

$$= \left(\sum_{\ell=0}^{n} (\ell+1)x^{\ell} + \sum_{\ell=n+1}^{2n} \big(2n-\ell+1\big)x^{\ell} \right) \sum_{i=0}^{n} x^{2i}$$
$$= \sum_{\ell=0}^{n} \sum_{m=0}^{n} (\ell+1)x^{\ell+2m} + \sum_{\ell=n+1}^{2n} \sum_{m=0}^{n} (2n-\ell+1)x^{\ell+2m}.$$

We now set $p = \ell+2m$, so that $\ell = p-2m$. For $0 \le \ell \le n$ we have $0 \le p-2m \le n$ and so $\frac{1}{2}(p-n) \le m \le \frac{1}{2}p$, but $m \in \mathbb{Z}$ so $\lceil \frac{1}{2}(p-n) \rceil \le m \le \lfloor \frac{1}{2}p \rfloor$; since also $0 \le m \le n$ we get $\max(0, \lceil \frac{1}{2}(p-n) \rceil) \le m \le \min(n, \lfloor \frac{1}{2}p \rfloor)$. Similarly, for $n+1 \le \ell \le 2n$ we obtain $\max\big(0, \lceil \frac{1}{2}(p-2n) \rceil\big) \le m \le \min\big(n, \lfloor \frac{1}{2}(p-(n+1)) \rfloor\big)$. Therefore

$$B = \sum_{p=0}^{3n} \sum_{m=\max\left(0,\lceil \frac{p-n}{2} \rceil\right)}^{\min\left(n,\lfloor \frac{p}{2} \rfloor\right)} (p-2m+1)\, x^p + \sum_{p=n+1}^{4n} \sum_{m=\max\left(0,\lceil \frac{p-2n}{2} \rceil\right)}^{\min\left(n,\left\lfloor \frac{p-(n+1)}{2} \right\rfloor\right)} (2n-(p-2m)+1)\, x^p.$$

Using a similar change of index we obtain

$$C = \sum_{i=0}^{n} \sum_{j=0}^{n} x^{i+3j} = \sum_{p=0}^{4n} \sum_{m=\max(0,\lceil \frac{p-n}{3} \rceil)}^{\min(n,\lfloor \frac{p}{3} \rfloor)} x^p = \sum_{p=0}^{4n} \left[\min\left(n, \left\lfloor \frac{p}{3} \right\rfloor\right) - \max\left(0, \left\lceil \frac{p-n}{3} \right\rceil\right) + 1 \right] x^p.$$

Replacing x by x^2 in Lemma 3.2.9 gives

$$D = \sum_{\ell=0}^{n} (\ell+1)x^{2\ell} + \sum_{\ell=n+1}^{2n} (2n-\ell+1)x^{2\ell}.$$

We now write

$$E = \sum_{\ell=0}^{n} x^{4\ell},$$

and obtain

$$\begin{aligned} &A - 6B + 8C + 3D - 6E \qquad (3.3)\\ &= \sum_{p=0}^{4n} \sum_{k=0}^{\min(4,\lfloor \frac{p}{n+1} \rfloor)} (-1)^k \binom{4}{k} \binom{p-(n+1)k+3}{3} x^p \\ &\quad - 6\left[\sum_{p=0}^{3n} \sum_{m=\max\left(0,\lceil \frac{p-n}{2} \rceil\right)}^{\min\left(n,\lfloor \frac{p}{2} \rfloor\right)} (p-2m+1)\, x^p + \sum_{p=n+1}^{4n} \sum_{m=\max\left(0,\lceil \frac{p-2n}{2} \rceil\right)}^{\min\left(n,\left\lfloor \frac{p-(n+1)}{2} \right\rfloor\right)} (2n-(p-2m)+1)\, x^p \right] \\ &\quad + 8\left[\sum_{p=0}^{4n} \left[\min\left(n, \left\lfloor \frac{p}{3} \right\rfloor\right) - \max\left(0, \left\lceil \frac{p-n}{3} \right\rceil\right) + 1 \right] x^p \right] \\ &\quad + 3\left[\sum_{\ell=0}^{n} (\ell+1)x^{2\ell} + \sum_{\ell=n+1}^{2n} (2n-\ell+1)x^{2\ell} \right] - 6 \sum_{\ell=0}^{n} x^{4\ell}. \end{aligned}$$

We need the coefficient T of $x^{5n/2}$ in the last equation:

$$\begin{aligned}
T = &\sum_{k=0}^{\left\lfloor \frac{5n}{2(n+1)} \right\rfloor} (-1)^k \binom{4}{k} \binom{\frac{5n}{2}-(n+1)k+3}{3} \\
&- 6\left[\sum_{m=\lceil \frac{3n}{4} \rceil}^{n} \left(\frac{5n}{2} - 2m + 1 \right) + \sum_{m=\lceil \frac{n}{4} \rceil}^{\lfloor \frac{3n-2}{4} \rfloor} \left(2m - \frac{n}{2} + 1 \right) \right] \\
&+ 8\left(\left\lfloor \frac{5n}{6} \right\rfloor - \frac{n}{2} + 1 \right) + 3\,\Delta_4^n \left(0 + \frac{3n}{4} + 1 \right) - 6\,\Delta_8^n.
\end{aligned}$$

For $n = 0$ and $n = 2$ we get $T = 0$; this is expected since the $\mathfrak{sl}_2(\mathbb{C})$-modules $V(0)$ and $V(2)$ have dimensions 1 and 3 respectively, so in both cases $\Lambda^4 V(n)$ is $\{0\}$. For $n \geq 4$ the upper limit of k is 2, and we use the formula

$$\sum_{m=a}^{b} m = \frac{1}{2}(b - a + 1)(b + a). \tag{3.4}$$

We obtain

$$\begin{aligned}
T = &\binom{\frac{5n}{2}+3}{3} - 4\binom{\frac{3n}{2}+2}{3} + 6\binom{\frac{n}{2}+1}{3} - 6\left(n - \left\lceil \frac{3n}{4} \right\rceil + 1 \right)\left(\frac{5n}{2} + 1 \right) \\
&+ 6\left(n - \left\lceil \frac{3n}{4} \right\rceil + 1 \right)\left(n + \left\lceil \frac{3n}{4} \right\rceil \right) - 6\left(\left\lfloor \frac{3n-2}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil + 1 \right)\left(-\frac{n}{2} + 1 \right) \\
&- 6\left(\left\lfloor \frac{3n-2}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil + 1 \right)\left(\left\lfloor \frac{3n-2}{4} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil \right) + 8\left\lfloor \frac{5n}{6} \right\rfloor - 4n + 8 \\
&+ 3\,\Delta_4^n \left(\frac{3n}{4} + 1 \right) - 6\,\Delta_8^n.
\end{aligned}$$

Expanding this and collecting terms with the same power of n gives

$$\begin{aligned}
&\frac{23}{48}n^3 - \frac{29}{4}n^2 + \frac{1}{12}\Big(-36\alpha(n) + 180\beta(n) + 36\gamma(n) + 27\Delta_4^n - 167 \Big) n \\
&+ \Big(6\alpha(n)^2 - 6\beta(n)^2 - 6\gamma(n)^2 + 12\beta(n) + 8\delta(n) - 12\gamma(n) + 3\Delta_4^n - 6\Delta_8^n - 3 \Big).
\end{aligned}$$

We check that this gives $T = 0$ for $n = 0$ and $n = 2$. Finally, we divide by 24. □

Corollary 3.2.13. *For even $n \in \mathbb{Z}$, write $n = 24q + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < 24$. The dimension of the weight space of weight n in the $\mathfrak{sl}_2(\mathbb{C})$-module $\Lambda^4 V(n)$ is*

$$\dim[\Lambda^4 V(n)]_n =$$

$$\frac{1}{1152}\begin{cases} 23n^3-42n^2+48n & (r=0) \\ 23n^3-42n^2-60n+104 & (r=2) \\ 23n^3-42n^2+48n+160 & (r=4) \\ 23n^3-42n^2-60n+360 & (r=6) \\ 23n^3-42n^2+48n-256 & (r=8) \\ 23n^3-42n^2-60n+232 & (r=10) \end{cases} \quad \begin{array}{|ll} 23n^3-42n^2+48n+288 & (r=12) \\ 23n^3-42n^2-60n+104 & (r=14) \\ 23n^3-42n^2+48n-128 & (r=16) \\ 23n^3-42n^2-60n+360 & (r=18) \\ 23n^3-42n^2+48n+32 & (r=20) \\ 23n^3-42n^2-60n+232 & (r=22) \end{array}$$

Proof. The dimension is given by the formula of Proposition 3.2.12. The LCM of the denominators of the functions $\alpha(n)$, $\beta(n)$, $\gamma(n)$, $\delta(n)$ and the periods of the functions Δ_4^n and Δ_8^n equals 24. Hence the dimension is given by a cubic polynomial in n which depends on the remainder of n modulo 24. □

Definition 3.2.14. We consider the following integer-valued functions of n:

$$\epsilon(n) = \left\lfloor \frac{3n}{4} \right\rfloor, \quad \zeta(n) = \left\lceil \frac{n+2}{4} \right\rceil, \quad \eta(n) = \left\lceil \frac{3n+2}{4} \right\rceil, \quad \theta(n) = \left\lfloor \frac{5n+2}{6} \right\rfloor.$$

Proposition 3.2.15. *For even $n \in \mathbb{Z}$, the number of solutions $P, Q, R, S \in \mathbb{Z}$ to*

$$n \geq P > Q > R > S \geq 0, \qquad P+Q+R+S = \frac{5n+2}{2},$$

equals

$$\frac{23}{1152}n^3 - \frac{21}{64}n^2 + \frac{1}{288}\Big(36\epsilon(n) - 36\zeta(n) + 180\eta(n) + 27\Delta_{4,2}^n - 254\Big)n$$
$$+ \frac{1}{48}\Big(-12\epsilon(n)^2 + 12\zeta(n)^2 - 12\eta(n)^2 - 12\epsilon(n) - 12\zeta(n) + 36\eta(n) + 16\theta(n)$$
$$+ 3\Delta_{4,2}^n - 12\Delta_{8,6}^n - 24\Big).$$

Proof. We need the coefficient T' of $x^{(5n+2)/2}$ in equation (3.3):

$$T' = \sum_{k=0}^{\left\lfloor \frac{5n+2}{2(n+1)} \right\rfloor} (-1)^k \binom{4}{k} \binom{\frac{5n+2}{2} - (n+1)k + 3}{3}$$
$$-6\left[\sum_{m=\left\lceil \frac{3n+2}{4} \right\rceil}^{n} \left(\frac{5n}{2} - 2m + 2 \right) + \sum_{m=\left\lceil \frac{n+2}{4} \right\rceil}^{\left\lfloor \frac{3n}{4} \right\rfloor} \left(2m - \frac{n}{2} \right) \right]$$
$$+ 8\left(\left\lfloor \frac{5n+2}{6} \right\rfloor - \frac{n}{2} \right) + 3\,\Delta_{4,2}^n \left(\frac{3n}{4} + \frac{1}{2} \right) - 6\,\Delta_{8,6}^n.$$

Using equation (3.4) gives

$$\begin{aligned}
T' = &\sum_{k=0}^{2}(-1)^k\binom{4}{k}\binom{\frac{5n+2}{2}-(n+1)k+3}{3}\\
&-6\Bigg[\left(\frac{5n}{2}+2\right)(n-\eta(n)+1)-(n-\eta(n)+1)(n+\eta(n))\\
&-\frac{n}{2}(\epsilon(n)+1-\zeta(n))+(\epsilon(n)-\zeta(n)+1)(\epsilon(n)+\zeta(n))\Bigg]\\
&+8\left(\theta(n)-\frac{n}{2}\right)+3\,\Delta^n_{4,2}\left(\frac{3n}{4}+\frac{1}{2}\right)-6\,\Delta^n_{8,6}.
\end{aligned}$$

Hence,

$$\begin{aligned}
T' = &\frac{23}{48}n^3+\tfrac{9}{8}n^2+\frac{5}{6}n-6\Bigg[\left(\frac{5n}{2}+2\right)(n-\eta(n)+1)-(n-\eta(n)+1)(n+\eta(n))\\
&-\frac{n}{2}(\epsilon(n)+1-\zeta(n))+(\epsilon(n)-\zeta(n)+1)(\epsilon(n)+\zeta(n))\Bigg]\\
&+8\left(\theta(n)-\frac{n}{2}\right)+3\,\Delta^n_{4,2}\left(\frac{3n}{4}+\frac{1}{2}\right)-6\,\Delta^n_{8,6}.
\end{aligned}$$

Collecting terms with the same power of n and dividing by 24 complete the proof. □

Corollary 3.2.16. *For even $n \in \mathbb{Z}$, write $n = 24q + r$ with $q, r \in \mathbb{Z}$ and $0 \le r < 24$. The dimension of the weight space of weight $n{+}2$ in the $\mathfrak{sl}_2(\mathbb{C})$-module $\Lambda^4 V(n)$ is*

$$\dim[\Lambda^4 V(n)]_{n+2} = \frac{1}{1152}\left\{\begin{array}{ll|ll}
23n^3-72n^2-48n & (r=0) & 23n^3-72n^2-48n & (r=12)\\
23n^3-72n^2+60n-16 & (r=2) & 23n^3-72n^2+60n-304 & (r=14)\\
23n^3-72n^2-48n-128 & (r=4) & 23n^3-72n^2-48n-128 & (r=16)\\
23n^3-72n^2+60n-432 & (r=6) & 23n^3-72n^2+60n-144 & (r=18)\\
23n^3-72n^2-48n+128 & (r=8) & 23n^3-72n^2-48n+128 & (r=20)\\
23n^3-72n^2+60n-272 & (r=10) & 23n^3-72n^2+60n-560 & (r=22)
\end{array}\right.$$

Proof. Similar to the proof of Corollary 3.2.13. □

Theorem 3.2.17. *If n is odd then $\dim \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\big(\Lambda^4 V(n), V(n)\big) = 0$. If n is even then $n = 24q + r$ with $0 \le r < 24$ (r even) and we have*

$$\dim \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\big(\Lambda^4 V(n), V(n)\big) =$$

$$\frac{1}{1152}\begin{cases} 30n^2+96n & \text{if } r=0,16 \\ 30n^2+96n+288 & \text{if } r=4,12 \\ 30n^2+96n-384 & \text{if } r=8 \\ 30n^2-120n+408 & \text{if } r=14 \end{cases} \left| \begin{array}{ll} 30n^2-120n+120 & \text{if } r=2 \\ 30n^2-120n+792 & \text{if } r=6,22 \\ 30n^2-120n+504 & \text{if } r=10,18 \\ 30n^2+96n-96 & \text{if } r=20 \end{array} \right.$$

Proof. apply Lemma 3.2.1 to Corollaries 3.2.13 and 3.2.16. □

Corollary 3.2.18. *The representation $V(n)$ of $\mathfrak{sl}_2(\mathbb{C})$ occurs in $\Lambda^4 V(n)$ with multiplicity 1 (resp. 2) if and only if $n=4$ or $n=6$ (resp. $n=8$ or $n=10$).*

Proof. The vertices of the parabolas in Theorem 3.2.17 occur at either $n=-8/5$ or $n=2$, so for each r the multiplicity is an increasing function of q. □

The next result shows that our method can also be applied to get the complete decomposition of the 4-th exterior power $\Lambda^4 V(n)$ into irreducible representations.

Lemma 3.2.19. *The decomposition of $\Lambda^4 V(n)$ for $n=4,6,8,10$ as a direct sum of irreducible representations is*

$$\begin{aligned} \Lambda^4 V(4) &\cong V(4), \\ \Lambda^4 V(6) &\cong V(12)\oplus V(8)\oplus V(6)\oplus V(4)\oplus V(0), \\ \Lambda^4 V(8) &\cong V(20)\oplus V(16)\oplus V(14)\oplus 2V(12)\oplus V(10)\oplus 2V(8)\oplus V(6) \\ &\quad \oplus 2V(4)\oplus V(0), \\ \Lambda^4 V(10) &\cong V(28)\oplus V(24)\oplus V(22)\oplus 2V(20)\oplus V(18)\oplus 3V(16)\oplus 2V(14) \\ &\quad \oplus 3V(12)\oplus 2V(10)\oplus 3V(8)\oplus V(6)\oplus 3V(4)\oplus V(0). \end{aligned}$$

Proof. For any n, the decomposition of $\Lambda^4 V(n)$ into irreducible modules can be computed by Corollary 3.2.7 and Lemma 3.2.1. The coefficient of $x^{(w+4n)/2}$ in the polynomial $P_n(x)$ (see Definition 3.2.8) is $\dim[\Lambda^4 V(n)]_w$ and Lemma 3.2.1 can be applied to find the multiplicity of $V(w)$ in $\Lambda^4 V(n)$. For $n=4,6,8,10$, we have

$$\begin{aligned} P_4(x) &= x^{10}+x^9+x^8+x^7+x^6, \\ P_6(x) &= x^{18}+x^{17}+2x^{16}+3x^{15}+4x^{14}+4x^{13}+5x^{12}+4x^{11}+4x^{10}+3x^9+2x^8 \\ &\quad +x^7+x^6, \end{aligned}$$

$$
\begin{aligned}
P_8(x) = x^{26} &+ x^{25} + 2x^{24} + 3x^{23} + 5x^{22} + 6x^{21} + 8x^{20} + 9x^{19} + 11x^{18} + 11x^{17} \\
&+ 12x^{16} + 11x^{15} + 11x^{14} + 9x^{13} + 8x^{12} + 6x^{11} + 5x^{10} + 3x^9 + 2x^8 + x^7 \\
&+ x^6, \\
P_{10}(x) = x^{34} &+ x^{33} + 2x^{32} + 3x^{31} + 5x^{30} + 6x^{29} + 9x^{28} + 11x^{27} + 14x^{26} + 16x^{25} \\
&+ 19x^{24} + 20x^{23} + 23x^{22} + 23x^{21} + 24x^{20} + 23x^{19} + 23x^{18} + 20x^{17} \\
&+ 19x^{16} + 16x^{15} + 14x^{14} + 11x^{13} + 9x^{12} + 6x^{11} + 5x^{10} + 3x^9 + 2x^8 \\
&+ x^7 + x^6.
\end{aligned}
$$

This completes the proof. □

In the rest of this chapter, for $n = 4, 6, 8, 10$ we use computational linear algebra to find all the multilinear polynomial identities of degree ≤ 7 satisfied by the resulting quaternary algebras.

3.3 Quaternary algebra structures

In this section, we explain how to compute explicitly the multiplication table for the alternating quaternary algebra structure on $V(n)$ obtained from a projection $\Lambda^4 V(n) \to V(n)$. Recall that $V(n)$ has the vector space basis $\{ v_{n-2i} \mid i = 0, 1, \cdots, n \}$ and that the subscript on v_{n-2i} is its weight: its eigenvalue for the action of $H \in \mathfrak{sl}_2(\mathbb{C})$.

Definition 3.3.1. The **tensor basis** of $\Lambda^4 V(n)$ consists of $\binom{n+1}{4}$ **quadruples**:

$$
v_p \wedge v_q \wedge v_r \wedge v_s = \sum_{\sigma \in S_4} \epsilon(\sigma) \big(v_{\sigma(p)} \otimes v_{\sigma(q)} \otimes v_{\sigma(r)} \otimes v_{\sigma(s)} \big),
$$

where $n \geq p > q > r > s \geq -n$ with $p, q, r, s \equiv n \pmod 2$ and $\epsilon \colon S_4 \to \{\pm 1\}$ is the sign homomorphism. We usually abbreviate $v_p \wedge v_q \wedge v_r \wedge v_s$ by $[p, q, r, s]$. The action of $L \in \mathfrak{sl}_2(\mathbb{C})$ satisfies the derivation property,

$$
\begin{aligned}
L.(v_p \wedge v_q \wedge v_r \wedge v_s) = L.v_p \wedge v_q \wedge v_r \wedge v_s + v_p \wedge L.v_q \wedge v_r \wedge v_s \\
+ v_p \wedge v_q \wedge L.v_r \wedge v_s + v_p \wedge v_q \wedge v_r \wedge L.v_s,
\end{aligned}
\tag{3.5}
$$

and hence the **weight** of the quadruple $T = [p, q, r, s]$ is $w(T) = p + q + r + s$. The **standard order** of the quadruples is given by decreasing weight, and within each weight by reverse lex order: $T = [p, q, r, s]$ precedes $T' = [p', q', r', s']$ if and only if either $w(T) > w(T')$, or $w(T) = w(T')$ and $t > t'$ where t, t' are the components of T, T' in the leftmost position where the components are not equal.

Example 3.3.2. The quadruples in the tensor basis of $\Lambda^4 V(n)$ for $n = 4, 6, 8$ are displayed in standard order in Tables A.2, A.4, A.9.

Remark 3.3.3. If we apply Lemma 3.2.1 to Tables A.2, A.4, A.9 then we obtain the decomposition of $\Lambda^4 V(n)$ for $n = 4, 6, 8$ as a direct sum of irreducible representations as given in Lemma 6.5.

The next step is to determine the highest weight vectors for the irreducible summands of $\Lambda^4 V(n)$ as linear combinations of the quadruples in the tensor basis.

Lemma 3.3.4. *The quadruple* $[n, n-2, n-4, n-6]$ *is the quadruple with highest weight in* $\Lambda^4 V(n)$ *and is a highest weight vector for the summand* $V(4n-12)$.

Proof. This follows directly from Eqs. (2.2) and (3.5). □

Example 3.3.5. For $n = 4$ we have $\Lambda^4 V(4) \cong V(4)$, and so the quadruple $[4, 2, 0, -2]$ is the only highest weight vector in $\Lambda^4 V(4)$. If we identify $[4, 2, 0, -2]$ with the highest weight vector v_4 of $V(4)$, and repeatedly apply F using equations (2.3) and (3.5), then we obtain the weight vectors of $\Lambda^4 V(4)$ corresponding to the basis vectors v_2, v_0, v_{-2}, v_{-4} of $V(4)$:

$$
\begin{aligned}
&v_4 = [4, 2, 0, -2] = v_4 \wedge v_2 \wedge v_0 \wedge v_{-2},\\
&v_2 = F.v_4 = F.(v_4 \wedge v_2 \wedge v_0 \wedge v_{-2})\\
&\ = F.v_4 \wedge v_2 \wedge v_0 \wedge v_{-2} + v_4 \wedge F.v_2 \wedge v_0 \wedge v_{-2} + v_4 \wedge v_2 \wedge F.v_0 \wedge v_{-2} + v_4 \wedge v_2 \wedge v_0 \wedge F.v_{-2}\\
&\ = v_4 \wedge v_2 \wedge v_0 \wedge 4v_{-4} = 4[4, 2, 0, -4],
\end{aligned}
$$

similarly,

$$
v_0 = \frac{1}{2!}F^2.v_4 = 6[4, 2, -2, -4], \quad v_{-2} = \frac{1}{3!}F^3.v_4 = 4[4, 0, -2, -4],
$$

$$v_{-4} = \frac{1}{4!}F^4.v_4 = [2, 0, -2, -4].$$

The matrix expressing the weight vectors in $V(4)$ in terms of the quadruples in $\Lambda^4 V(4)$ is $C = \mathrm{diag}(1, 4, 6, 4, 1)$. The matrix expressing the quadruples in $\Lambda^4 V(4)$ in terms of the weight vectors in $V(4)$ is $C^{-1} = \mathrm{diag}(1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1)$. We now have the structure constants for the $\mathfrak{sl}_2(\mathbb{C})$-invariant alternating quaternary algebra structure on $V(4)$, which we denote by $[v_p, v_q, v_r, v_s]$:

$$[v_4, v_2, v_0, v_{-2}] = v_4, \qquad [v_4, v_2, v_0, v_{-4}] = \tfrac{1}{4}v_2, \quad [v_4, v_2, v_{-2}, v_{-4}] = \tfrac{1}{6}v_0,$$
$$[v_4, v_0, v_{-2}, v_{-4}] = \tfrac{1}{4}v_{-2}, \quad [v_2, v_0, v_{-2}, v_{-4}] = v_{-4}.$$

The LCM of the denominators of the coefficients is 12. Taking $a = \sqrt[3]{12}$ and setting $v_t' = v_t/a$, we obtain integral structure constants (see Table A.3):

$$[v_4', v_2', v_0', v_{-2}'] = 12v_4', \quad [v_4', v_2', v_0', v_{-4}'] = 3v_2', \quad [v_4', v_2', v_{-2}', v_{-4}'] = 2v_0',$$
$$[v_4', v_0', v_{-2}', v_{-4}'] = 3v_{-2}', \quad [v_2', v_0', v_{-2}', v_{-4}'] = 12v_{-4}'.$$

In general, for all other weights $w < 4n-12$ we need to find a basis for the subspace of highest weight vectors of weight w in $\Lambda^4 V(n)$. The dimension of this subspace is the multiplicity of $V(w)$ as a summand of $\Lambda^4 V(n)$.

Definition 3.3.6. Suppose that $4n-14 \geq w \geq 0$ (w even). Let $d(w)$ be the dimension of the weight space of weight w in $\Lambda^4 V(n)$: the number of quadruples of weight w. We define the matrix $E_w^{(n)}$ of size $d(w+2) \times d(w)$ by setting the (i,j) entry equal to the coefficient of the i-th quadruple of weight $w+2$ in the expression for the action of $E \in \mathfrak{sl}_2(\mathbb{C})$ on the j-th quadruple of weight w. We call this the **E-action matrix** for weight w of $\Lambda^4 V(n)$; the nonzero vectors in its nullspace are the highest weight vectors of weight w in $\Lambda^4 V(n)$. We compute the row canonical form (RCF) and extract the canonical integral basis (CIB) by setting the free variables equal to the standard unit vectors, clearing denominators, and canceling common factors.

Example 3.3.7. For $\Lambda^4 V(6)$ we use the weight space basis of Table A.4 and obtain

$$E_8^{(6)} = \begin{bmatrix} 2 & 4 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & 2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$E_6^{(6)} = \begin{bmatrix} 1 & 4 & . \\ . & 2 & 5 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & -10 \\ . & 1 & 5/2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} 20 & -5 & 2 \end{bmatrix}$$

$$E_4^{(6)} = \begin{bmatrix} 4 & . & . & . \\ 1 & 3 & 5 & . \\ . & . & 2 & 6 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & -5 \\ . & . & 1 & 3 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} . & 5 & -3 & 1 \end{bmatrix}$$

$$E_0^{(6)} = \begin{bmatrix} 2 & 5 & . & . & . \\ . & 3 & . & 6 & . \\ . & 1 & 4 & . & 6 \\ . & . & . & 1 & 3 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . & 15 \\ . & 1 & . & . & -6 \\ . & . & 1 & . & 3 \\ . & . & . & 1 & 3 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -15 & 6 & -3 & -3 & 1 \end{bmatrix}$$

Example 3.3.8. For $\Lambda^4 V(8)$ we use the weight space basis of Table A.9 and obtain

$$E_{16}^{(8)} = \begin{bmatrix} 4 & 6 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & 3/2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -3 & 2 \end{bmatrix}$$

$$E_{14}^{(8)} = \begin{bmatrix} 3 & 6 & . \\ . & 4 & 7 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & -7/2 \\ . & 1 & 7/4 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} 14 & -7 & 4 \end{bmatrix}$$

$$E_{12}^{(8)} = \begin{bmatrix} 2 & 6 & . & . & . \\ . & 3 & 5 & 7 & . \\ . & . & . & 4 & 8 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & -5 & . & 14 \\ . & 1 & 5/3 & . & -14/3 \\ . & . & . & 1 & 2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} 15 & -5 & 3 & . & . \\ -42 & 14 & . & -6 & 3 \end{bmatrix}$$

$$E_{10}^{(8)} = \begin{bmatrix} 1 & 6 & . & . & . & . \\ . & 2 & 5 & 7 & . & . \\ . & . & 3 & . & 7 & . \\ . & . & . & 3 & 5 & 8 \\ . & . & . & . & . & 4 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . & 70 & . \\ . & 1 & . & . & -35/3 & . \\ . & . & 1 & . & 7/3 & . \\ . & . & . & 1 & 5/3 & . \\ . & . & . & . & . & 1 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -210 & 35 & -7 & -5 & 3 & . \end{bmatrix}$$

$$E_8^{(8)} = \begin{bmatrix} 6 & . & . & . & . & . & . & . \\ 1 & 5 & . & 7 & . & . & . & . \\ . & 2 & 4 & . & 7 & . & . & . \\ . & . & . & 2 & 5 & . & 8 & . \\ . & . & . & . & 3 & 6 & . & 8 \\ . & . & . & . & . & . & 3 & 5 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & 7 & . & 56/3 \\ . & . & 1 & . & . & -7 & . & -14 \\ . & . & . & 1 & . & -5 & . & -40/3 \\ . & . & . & . & 1 & 2 & . & 8/3 \\ . & . & . & . & . & . & 1 & 5/3 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} . & -7 & 7 & 5 & -2 & 1 & . & . \\ . & -56 & 42 & 40 & -8 & . & -5 & 3 \end{bmatrix}$$

$$E_6^{(8)} = \begin{bmatrix} 5 & . & 7 & . & . & . & . & . & . \\ 1 & 4 & . & 7 & . & . & . & . & . \\ . & 2 & . & . & 7 & . & . & . & . \\ . & . & 1 & 5 & . & . & 8 & . & . \\ . & . & . & 2 & 4 & 6 & . & 8 & . \\ . & . & . & . & . & 3 & . & . & 8 \\ . & . & . & . & . & . & 2 & 5 & . \\ . & . & . & . & . & . & . & 3 & 6 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . & . & . & . & . & -224/3 \\ . & 1 & . & . & . & . & . & . & 70/3 \\ . & . & 1 & . & . & . & . & . & 160/3 \\ . & . & . & 1 & . & . & . & . & -8/3 \\ . & . & . & . & 1 & . & . & . & -20/3 \\ . & . & . & . & . & 1 & . & . & 8/3 \\ . & . & . & . & . & . & 1 & . & -5 \\ . & . & . & . & . & . & . & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} 224 & -70 & -160 & 8 & 20 & -8 & 15 & -6 & 3 \end{bmatrix}$$

$$E_4^{(8)} = \begin{bmatrix} 4 & . & 7 & . & . & . & . & . & . & . & . \\ 1 & 3 & . & 7 & . & . & . & . & . & . & . \\ . & . & 5 & . & . & . & 8 & . & . & . & . \\ . & . & 1 & 4 & 6 & . & . & 8 & . & . & . \\ . & . & . & 2 & . & 6 & . & . & 8 & . & . \\ . & . & . & . & 2 & 4 & . & . & . & 8 & . \\ . & . & . & . & . & . & 1 & 5 & . & . & . \\ . & . & . & . & . & . & . & 2 & 4 & 6 & . \\ . & . & . & . & . & . & . & . & . & 3 & 7 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . & . & . & . & . & -28 & . & 98 \\ . & 1 & . & . & . & . & . & . & 14 & . & -245/3 \\ . & . & 1 & . & . & . & . & . & 16 & . & -56 \\ . & . & . & 1 & . & . & . & . & -2 & . & 21 \\ . & . & . & . & 1 & . & . & . & -4 & . & 14/3 \\ . & . & . & . & . & 1 & . & . & 2 & . & -7 \\ . & . & . & . & . & . & 1 & . & -10 & . & 35 \\ . & . & . & . & . & . & . & 1 & 2 & . & -7 \\ . & . & . & . & . & . & . & . & . & 1 & 7/3 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} 28 & -14 & -16 & 2 & 4 & -2 & 10 & -2 & 1 & . & . \\ -294 & 245 & 168 & -63 & -14 & 21 & -105 & 21 & . & -7 & 3 \end{bmatrix}$$

$$E_0^{(8)} = \begin{bmatrix} 2 & 7 & . & . & . & . & . & . & . & . & . & . \\ . & 3 & 6 & . & . & 8 & . & . & . & . & . & . \\ . & 1 & . & 6 & . & . & 8 & . & . & . & . & . \\ . & . & 4 & . & . & . & . & 8 & . & . & . & . \\ . & . & 1 & 3 & 5 & . & . & . & 8 & . & . & . \\ . & . & . & . & 2 & . & . & . & . & 8 & . & . \\ . & . & . & . & . & 4 & . & 6 & . & . & . & . \\ . & . & . & . & . & 1 & 3 & . & 6 & . & . & . \\ . & . & . & . & . & . & . & 1 & 4 & . & 7 & . \\ . & . & . & . & . & . & . & . & 2 & 5 & . & 7 \\ . & . & . & . & . & . & . & . & . & . & 2 & 4 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & . & 224 \\ . & 1 & . & . & . & . & . & . & . & . & . & -64 \\ . & . & 1 & . & . & . & . & . & . & . & . & 16 \\ . & . & . & 1 & . & . & . & . & . & . & . & 12 \\ . & . & . & . & 1 & . & . & . & . & . & . & -8 \\ . & . & . & . & . & 1 & . & . & . & . & . & 12 \\ . & . & . & . & . & . & 1 & . & . & . & . & -1 \\ . & . & . & . & . & . & . & 1 & . & . & . & -8 \\ . & . & . & . & . & . & . & . & 1 & . & . & -\frac{3}{2} \\ . & . & . & . & . & . & . & . & . & 1 & . & 2 \\ . & . & . & . & . & . & . & . & . & . & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} -448 & 128 & -32 & -24 & 16 & -24 & 2 & 16 & 3 & -4 & -4 & 2 \end{bmatrix}$$

Definition 3.3.9. We construct the **weight vector basis** of $\Lambda^4 V(n)$ as follows. We first determine the highest weight vectors for each summand of $\Lambda^4 V(n)$ as described in Definition 3.3.6 and Examples 3.3.7 and 3.3.8. For each highest weight vector X (of weight w, say) we apply $F \in \mathfrak{sl}_2(\mathbb{C})$ repeatedly w times to obtain weight vectors of weights $w-2, w-4, \cdots, -w$ forming a basis of the summand isomorphic to $V(w)$:

$$X, \quad F.X, \quad \frac{1}{2!}F^2.X, \quad \cdots, \quad \frac{1}{w!}F^w.X.$$

The set of all these weight vectors is the weight vector basis of $\Lambda^4 V(n)$. The **standard order** on this basis is as follows: We order the weight vectors first by decreasing weight of the corresponding highest weight vector and then by increasing power of F within each summand. (When there is more than one highest weight vector with the same weight, we order them as in the canonical integral basis.)

Example 3.3.10. The weight vector basis of $\Lambda^4 V(6)$ is displayed in Table A.5. The weight vector basis of $\Lambda^4 V(8)$ is displayed in Tables A.10–A.13.

Definition 3.3.11. The **weight vector matrix** C is the $\binom{n+1}{4} \times \binom{n+1}{4}$ matrix which expresses the weight vector basis in terms of the tensor basis: the (i, j) entry is the coefficient of the i-th quadruple in the j-th element of the weight vector basis.

Definition 3.3.12. The **alternating quaternary algebra structure** on $V(n)$ is defined in terms of structure constants as follows. The inverse C^{-1} of the weight vector matrix expresses the tensor basis in terms of the weight vector basis. Let $[p, q, r, s]$ be the j-th quadruple in the tensor basis. Column j of C^{-1} expresses $[p, q, r, s]$ as a linear combination of the elements of the weight vector basis. Suppose that the highest weight vector for the summand of $\Lambda^4 V(n)$ isomorphic to $V(n)$ is the k-th element of the weight vector basis. The entries of C^{-1} in column j and rows $i = k, \cdots, k+n$ are the coefficients of the projection of $[p, q, r, s]$ onto the summand isomorphic to $V(n)$. Let $P\colon \Lambda^4 V(n) \to V(n)$ be this surjective homomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules. The quadruple $[p, q, r, s]$ has weight $p+q+r+s$, and the summand isomorphic to $V(n)$ has (at most) one basis vector of this weight. Hence there is at most one nonzero entry in C^{-1} in column j and rows $i = k, \cdots, k+n$. If all these entries are zero then $P(v_p \wedge v_q \wedge v_r \wedge v_s) = 0$. If there is a nonzero entry, say in row ℓ, then $P(v_p \wedge v_q \wedge v_r \wedge v_s) = (C^{-1})_{i\ell}\, v_{p+q+r+s}$. The resulting alternating quaternary algebra structure on $V(n)$ is denoted by $[v_p, v_q, v_r, v_s]$ and defined by $[v_p, v_q, v_r, v_s] = P(v_p \wedge v_q \wedge v_r \wedge v_s)$.

Example 3.3.13. For $n = 6$, the weight vector matrix is given in Table A.6. From rows 23 to 29 of the weight matrix inverse (see Table A.7) we obtain the structure constants for the alternating quaternary algebra structure on $V(6)$. We ignore the equations for which $|p+q+r+s| > 6$ since in these cases the result is obviously zero: there is no vector of the given weight in $V(6)$. The LCM of the denominators of the coefficients is 120, so we can scale the basis vectors of $V(n)$ by setting $v'_t = v_t / \sqrt[3]{120}$ to obtain integral structure constants. We use the shorthand notation $[p,q,r,s] = c$ for the equation $[v'_p, v'_q, v'_r, v'_s] = cv'_{p+q+r+s}$. After making these changes we obtain the structure constants in Table A.8.

Remark 3.3.14. In this chapter we also study the alternating quaternary algebra structures on $V(10)$ but in this case the tables of the tensor basis and the weight vector basis are too large to include.

3.4 Polynomial identities and computational methods

Definition 3.4.1. The nonassociative polynomial $I(x_1,\ldots,x_n)$ is a **polynomial identity** for the algebra A if $I(x_1,\cdots,x_n) = 0$ for all $x_1,\ldots,x_n \in A$.

We are concerned with multilinear polynomial identities of degree n for an alternating quaternary algebra. This means that each term consists of a coefficient and a monomial, where the monomial is some permutation of n distinct variables x_1, x_2, $\cdots$, x_n together with some association type, by which we mean some placement of brackets representing the quaternary operation. For any k-ary operation, the degree of a monomial has the form $n = 1 + \ell(k-1)$ where ℓ is the number of occurrences of the operation in the monomial. Thus for a quaternary operation the degree of a monomial is congruent to 1 modulo 3.

In degree 4, we have only the single association type $[-,-,-,-]$; the alternating property implies that we have only the single monomial $[x_1,x_2,x_3,x_4]$. In degree 7, the alternating property implies that we have only one association type

$[[-,-,-,-],-,-,-]$ and only $\binom{7}{4} = 35$ distinct multilinear monomials,

$$[[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}], x_{\sigma(5)}, x_{\sigma(6)}, x_{\sigma(7)}],$$

where $\sigma \in S_7$ is a $(4,3)$-shuffle; that is, $1 \leq \sigma(1) < \sigma(2) < \sigma(3) < \sigma(4) \leq 7$ and $1 \leq \sigma(5) < \sigma(6) < \sigma(7) \leq 7$. In degree 10, the alternating property implies that we have two association types,

$$[[[-,-,-,-],-,-,-],-,-,-], \qquad [[-,-,-,-],[-,-,-,-],-,-],$$

and that the corresponding numbers of distinct multilinear monomials are

$$\binom{10}{4,3,3} + \frac{1}{2}\binom{10}{4,4,2} = 4200 + 1575 = 5775.$$

3.4.1 Fill-and-reduce algorithm

The basic references for this algorithm and the next one are [8, 9].

Suppose we wish to find all the multilinear polynomial identities of degree n satisfied by an algebra A of dimension d. We assume that we have chosen a basis of A and that we know the structure constants with respect to this basis. We write m for the number of distinct multilinear monomials of degree n, and we assume that these monomials are ordered in some way. We create a matrix X of size $(m+d) \times m$ and initialize it to zero; the columns of M correspond bijectively to the monomials. We choose two parameters p and s: we generate pseudorandom integers in the range 0 to $p-1$.

We perform the following "fill-and-reduce" algorithm until the rank of the matrix X has remained stable for s iterations:

1. Generate n pseudorandom elements $a_1, \ldots, a_n$ of A: vectors of length d in which the components are integers in the range 0 to $p-1$.

2. For j from 1 to m do:

   (a) Evaluate monomial j by setting the variable x_k equal to the vector a_k for $k = 1, \ldots, n$ and using the structure constants for A, obtaining another vector b of length d.

(b) Store b as a column vector in column j of X in rows $m+1$ to $m+d$.

3. Compute the row canonical form of X; the last d rows are now zero.

After this process has terminated, if the nullspace of X is not zero then it contains candidates for polynomial identities satisfied by A. We usually find that $s = 10$ is a sufficient number of iterations after the rank has stabilized, but we use $s = 100$ to increase our confidence in the results. We now compute the canonical integral basis of the nullspace.

3.4.2 Module generators algorithm

We assume that we have the canonical integral basis of the nullspace of the matrix X used in the fill-and-reduce algorithm. Let r be the number of these basis vectors; these are the coefficient vectors of polynomial identities satisfied by the algebra A. These identities are linearly independent over $\mathbb{Q}$ but they are not necessarily independent as generators of the S_n-module of identities. We want to find a minimal set of module generators. We start by sorting the basis vectors by increasing Euclidean norm. We create a new matrix M of size $(m+n!) \times m$ and initialize it to zero.

We perform the following algorithm for k from 1 to r:

1. For i from 1 to $n!$ apply permutation i of the variables $\{x_1, \ldots, x_n\}$ to basis identity k and store the result in row $m+i$ of M. More precisely, for each nonzero coefficient c of the identity, apply permutation i to the corresponding monomial, use the alternating property to straighten the monomial, obtain a standard basis monomial (with index j say) and store $\pm c$ in position $(m+i, j)$ of M (straightening may introduce a sign change).

2. Compute the row canonical form of M. If the rank has increased from the previous iteration, then we record basis identity k as a new generator.

3.5 Multiplicity 1: representation $V(4)$

In this section and the next we describe computer searches for polynomial identities satisfied by the two irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ which admit an alternating quaternary structure which is unique up to a scalar multiple; we determine all their identities of degree 7. For all our calculations we use the computer algebra system `Maple`, especially the packages `LinearAlgebra` and `LinearAlgebra[Modular]`. The proof that follow depend on computational results; in some cases, we are able to give independent theoretical proofs.

Theorem 3.5.1. *The vector space of multilinear polynomial identities in degree* 7 *for the alternating quaternary structure on* $V(4)$ *has dimension 21.*

Proof. We use the fill-and-reduce algorithm with $n = 7$, $d = 5$, $m = 35$, $p = 10$ and $s = 100$. We create a matrix X of size 40×35 consisting of an upper block of size 35×35 and a lower block of size 5×35; the columns are labeled by the ordered basis of multilinear monomials in degree 7 for an alternating quaternary operation. We generate seven random elements of $V(4)$ and evaluate the 35 monomials on these seven elements. We put the 35 resulting elements of $V(4)$ as column vectors into the lower block of the matrix. Each of the last five rows of the matrix now contains a linear relation that must be satisfied by the coefficients of any identity for the alternating quaternary structure on $V(4)$. We repeat the fill-and-reduce process until the rank of the matrix stabilizes. The rank reached 14 and did not increase further. Therefore the nullspace of the matrix has dimension 21. □

Theorem 3.5.2. *Every multilinear polynomial identity in degree 7 for the alternating quaternary structure on* $V(4)$ *is a consequence of the alternating property in degree 4 together with the quaternary derivation identity in degree 7:*

$$[a,b,c,[d,e,f,g]] =$$
$$[[a,b,c,d],e,f,g] + [d,[a,b,c,e],f,g] + [d,e,[a,b,c,f],g] + [d,e,f,[a,b,c,g]]$$

Proof. We use the module generators algorithm, slightly modified to use less memory. We create a matrix of size 59×35 with an upper block of size 35×35 and a lower block of size 24×35. We generate all 5040 permutations of seven letters and divide them into 210 groups of 24 permutations. For each of the 21 basis identities, we perform the following computation. For each group of permutations, we apply the corresponding 24 permutations to the identity, obtain 24 new identities which we store in the lower block of the matrix, and compute the row canonical form of the matrix. After all 210 groups of permutations have been processed, the rank of the matrix is equal to the dimension of the module generated by all the identities up to and including the current identity. After the first identity has been processed, the rank of the matrix is 21, which is the same as the entire nullspace; the rank does not increase as we process the remaining identities. Therefore every identity is a consequence of the first identity, which has the form

$$[[a,b,c,d],e,f,g] - [[a,b,c,e],d,f,g] + [[a,b,c,f],d,e,g] - [[a,b,c,g],d,e,f] \\ + [[d,e,f,g],a,b,c]$$

Applying the alternating property of the quaternary product, we see that this identity can be written in the stated form. □

Remark 3.5.3. The alternating property in degree 4 and the quaternary derivation identity together define the case $n = 4$ of the variety of n-Lie algebras introduced by Filippov [20]. Thus the isomorphism $\Lambda^4 V(4) \cong V(4)$ makes $V(4)$ into an alternating quaternary algebra isomorphic to simple 5-dimensional 4-Lie algebras in Filippov's classification of $(n+1)$-dimensional n-Lie algebras.

3.6 Multiplicity 1: representation $V(6)$

Theorem 3.6.1. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure on $V(6)$ has dimension 1.*

Proof. We use the fill-and-reduce algorithm with $n = 7$, $d = 7$, $m = 35$, $p = 10$ and $s = 100$. The details of the computations are similar to those described in the proof of Theorem 3.5.1. The rank reached 34 and did not increase further. Therefore the nullspace of the matrix has dimension 1. □

Theorem 3.6.2. *Every multilinear polynomial identity in degree* 7 *for the alternating quaternary structure on* $V(6)$ *is a consequence of the alternating property in degree 4 together with the quaternary alternating sum identity in degree 7:*

$$\sum_{\sigma \in S_7} \epsilon(\sigma)\,[[a^\sigma, b^\sigma, c^\sigma, d^\sigma], e^\sigma, f^\sigma, g^\sigma] = 0.$$

Proof. Since the nullspace has dimension 1, this is an immediate corollary of Theorem 3.6.1; we do not need to apply the module generators algorithm. □

Remark 3.6.3. Theorem 3.6.2 can be proved as follows: The quaternary alternating sum identity is an alternating multilinear function of 7 variables. Evaluating this function on the 7-dimensional space $V(6)$ gives a map $\alpha\colon \Lambda^7 V(6) \to V(6)$. But $\Lambda^7 V(6)$ is 1-dimensional (it is isomorphic to $V(0)$ as an $\mathfrak{sl}_2(\mathbb{C})$-module), and α is invariant under the action of $\mathfrak{sl}_2(\mathbb{C})$. Hence the image of α is an $\mathfrak{sl}_2(\mathbb{C})$-submodule which has dimension 0 or 1. Since $V(6)$ is irreducible, it has no submodule of dimension 1, and so α must be the zero map.

Remark 3.6.4. It is shown in [5] (Theorems 3 and 4) that the quaternary alternating sum identity in degree 7 is satisfied by the following multilinear operation (the alternating quaternary sum) in every totally associative quadruple system,

$$[x_1, x_2, x_3, x_4] = \sum_{\pi \in S_4} \epsilon(\pi)\, x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} x_{\pi(4)},$$

and that the quaternary alternating sum identity of Theorem 3.6.2 is a consequence of the quaternary derivation identity of Theorem 3.5.1.

3.7 Multiplicity 2: representation $V(8)$

In this section and the next we describe computer searches for polynomial identities satisfied by the two irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ which admit a two-dimensional space of alternating quaternary structures; we determine all their identities of degree 7.

Any $\mathfrak{sl}_2(\mathbb{C})$-module homomorphism $\Lambda^4 V(8) \to V(8)$ is a linear combination of the structures f and g from Tables A.14 and A.15. Up to a scalar multiple, we need to consider only the single structure g and the one-parameter family of structures $f + xg$ for $x \in \mathbb{C}$. For g our methods are similar to those used for $V(4)$ and $V(6)$. For $f + xg$ we need to use the Smith normal form to determine the values of the parameter x which produce a nonzero nullspace. For this we use the Maple command `linalg[smith]` instead of `LinearAlgebra[SmithForm]` since the former is much more efficient than the latter.

Theorem 3.7.1. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure g on $V(8)$ has dimension 1 and is spanned by the quaternary alternating sum identity.*

Proof. Similar to the proofs of Theorems 3.6.1 and 3.6.2. □

Theorem 3.7.2. *For any $x \in \mathbb{C}$, the vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure $f + xg$ on $V(8)$ has dimension 1 and is spanned by the quaternary alternating sum identity.*

Proof. In order to determine how the space of identities depends on the parameter x, we use the Smith normal form of a matrix over the polynomial algebra $\mathbb{C}[x]$. Since the computation of the Smith form performs not only row operations but also column operations, we must fill the matrix using a suitable number of trials, and then compute the Smith form once. In the general case, we create a matrix of size $t(n{+}1) \times m$ where n is the highest weight (recall that $V(n)$ has dimension $n{+}1$) and m is the number of multilinear monomials in degree d; the matrix consists of t blocks of size $(n{+}1) \times m$. We choose t so that $t(n{+}1) > m$ in order to guarantee that we have enough nonzero rows in the matrix to eliminate false nullspace vectors. We perform the following algorithm:

1. For b from 1 to t do:

    (a) Generate d pseudorandom vectors of length $n{+}1$ representing elements of $V(n)$.

    (b) For j from 1 to m do:

   i. Evaluate the j-th alternating quaternary monomial on the d pseudorandom vectors to obtain another vector of length $n+1$ with components which are polynomials in the parameter x.
   ii. Put the resulting vector into column j of block t.

2. Compute the Smith normal form of the matrix.

For $n = 8$ and $d = 7$ we have $m = 35$ and we choose $t = 4$. The entries of the resulting 36×35 matrix are quadratic polynomials in the parameter x since each monomial involves two occurrences of the quaternary operation. In the Smith normal form of the matrix, the diagonal entries are 1 (34 times) and 0 (once). It follows that the matrix has a one-dimensional nullspace for every value of x. In [5] (Proposition 3, page 85) it is shown that there is a unique 1-dimensional S_7-submodule of the 35-dimensional module with basis consisting of the alternating quaternary monomials in degree 7, and this submodule is spanned by the quaternary alternating sum identity. Hence the nullspace basis does not depend on the value of the parameter x, and this completes the proof. We checked this result independently by evaluating the quaternary alternating sum identity on pseudorandom vectors for the product $f + xg$ with indeterminate x and verifying that the result was zero. □

Remark 3.7.3. It is an open problem to determine whether the alternating quaternary structures on $V(8)$ are isomorphic for all values of the parameter x.

3.8 Multiplicity 2: representation $V(10)$

As in the previous section, any $\mathfrak{sl}_2(\mathbb{C})$-module homomorphism $\Lambda^4 V(10) \to V(10)$ is a linear combination of two structures f and g from Tables A.16-A.19, and we consider separately the single structure g and the one-parameter family of structures $f + xg$ for $x \in \mathbb{C}$.

Theorem 3.8.1. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure g on $V(10)$ has dimension 0: every identity is a consequence of the alternating properties in degree 4.*

Proof. Similar to the proofs of Theorem 3.7.1 except that the matrix achieves the full rank of 35. □

Theorem 3.8.2. *For $x = \frac{5}{4}$, the vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure $f + xg$ on $V(10)$ has dimension 1 and is spanned by the quaternary alternating sum identity. For all other $x \in \mathbb{C}$, the vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure $f + xg$ on $V(10)$ has dimension 0.*

Proof. Similar to the proof of Theorem 3.7.2 except that now $n = 10$. As before, the entries of the resulting 44×35 matrix are quadratic polynomials in the parameter x. In the Smith normal form of this matrix, the diagonal entries are 1 (28 times) and $x - \frac{5}{4}$ (7 times). It follows that the matrix has zero nullspace except in the case $x = \frac{5}{4}$. We now specialize to this value of x and consider the structure $f + \frac{5}{4}g$; the rest of the proof is similar to that of Theorems 3.6.1 and 3.6.2. □

Chapter 4

Universal associative envelopes of $(n+1)$-dimensional n-Lie algebras

4.1 Introduction

Filippov [20] in 1985 introduced n-Lie (Filippov) algebras and classified the $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 0. For a very recent comprehensive survey on the physical applications of n-ary algebras, see de Azcárraga and Izquierdo [13].

Ling [31] in 1993 showed that for each $n \geq 3$ there exists up to isomorphism a unique simple finite-dimensional n-Lie algebra over an algebraically closed field of characteristic 0. Pozhidaev [35] in 2003 showed that for $n \leq 5$ the simple finite-dimensional n-Lie algebra over an algebraically closed field of characteristic 0 can be embedded in an associative algebra, and made the conjecture that such associative enveloping algebras exist for all n. The aim of this chapter is to study the universal associative enveloping algebras of n-Lie algebras and to establish a generalization of the Poincaré-Birkhoff-Witt (PBW) theorem for $(n+1)$-dimensional n-Lie algebras when n is even. Our approach uses noncommutative Gröbner bases to construct universal associative enveloping algebras. In Section 4.2 we recall basic facts about n-Lie algebras. In Section 4.3 we prove a theorem on the normal form of a composition of ideal generators for universal associative enveloping algebras of $(n+1)$-dimensional alternating n-ary algebras. For n even, this allows us to construct a basis for $U(L)$ where L is any $(n+1)$-dimensional n-Lie algebra. In Section 4.4 we establish Pozhidaev's conjecture for the simple n-Lie algebra when n is even. In Section 4.5 we establish analogous results for the non-simple $(n+1)$-dimensional n-Lie algebras. Finally, in Section 4.6 we describe some calculations with the computer algebra system Maple which suggest that extending these results to n odd may be difficult.

Unless otherwise stated, we assume throughout that all vector spaces are over an algebraically closed field F of characteristic 0.

4.2 n-Lie or Filippov algebras

Definition 4.2.1. [20] An **n-Lie algebra** is a vector space L over a field F of characteristic $\neq 2$ with a multilinear operation $[x_1, x_2, \ldots, x_n]$ satisfying the **alternating (or anticommutative) identity** and the **generalized Jacobi (or derivation) identity**:

$$[x_1, x_2, \ldots, x_n] = \epsilon(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}] \quad (\sigma \in S_n),$$
$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Remark 4.2.2. For $n = 2$ we obtain the definition of a Lie algebra, but for $n \geq 3$ the structure of n-Lie algebras is quite different.

Definition 4.2.3. [20] If a subspace S of an n-Lie algebra L satisfying $[x_1, ..., x_n] \in S$ for any $x_1, ..., x_n \in S$, then S is called a **subalgebra** of L. The subalgebra $L^1 = [L, L, ..., L]$ is called the **derived algebra** of L. If $L^1 = 0$, then L is called an **Abelian** n-Lie algebra.

Definition 4.2.4. [20] An **ideal** I of an n-Lie algebra L is a subspace of L such that $[I, L, ..., L] \subseteq I$. If $L^1 \neq 0$ and L has no ideals except 0 and itself, then L is said to be **simple** n-Lie algebra. The subset $Z(L) = \{x \in A \mid [x, y_1, ..., y_{n-1}] = 0 \text{ for all } y_1, ..., y_{n-1} \in L\}$ is called the **center** of L. The **radical** $\mathfrak{R}(L)$ of an n-Lie algebra L is the maximal solvable ideal of L. An n-Lie algebra L is called **reductive** when $\mathfrak{R}(L) = Z(L)$.

We will give an example of $(n+1)$-dimensional n-Lie algebra which is an analogue of the three-dimensional Lie algebra with the cross product as multiplication.

Example 4.2.5. [20] Let L be the $(n+1)$-dimensional (real) Euclidean space. We denote by $[x_1, \ldots, x_n]$ the vector product of the vectors $x_1, \ldots, x_n \in L$. If $e_1, \ldots, e_{n+1}$ form an orthonormal basis of the space L, then the vector product is equal to the determinant:

$$[x_1, \ldots, x_n] = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} & e_1 \\ x_{21} & x_{22} & \ldots & x_{2n} & e_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n+11} & x_{n+12} & \ldots & x_{n+1n} & e_{n+1} \end{vmatrix}, \tag{4.1}$$

where $(x_{1i}, \ldots, x_{n+1i})$ are the coordinates of the vectors x_i, $i = 1, \ldots, n$. From (4.1) we obtain the following multiplication table of the basis vectors:

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] = (-1)^{n+i+1} e_i, \quad (1 \le i \le n+1); \tag{4.2}$$

$\widehat{e_i}$ means that e_i is omitted. The remaining products of the basis vectors are either equal to zero or obtained from (4.2) and anticommutativity. If the n-ary operation in L is defined as the vector product, then L becomes an $(n+1)$-dimensional n-Lie algebra (see Proposition 1 of [20]).

Definition 4.2.6. [20] Let $n \ge 3$ and let F be a field of characteristic $\ne 2$. Let L_{n+1} be the $(n+1)$-dimensional n-Lie algebra over F with basis $e_1, \ldots, e_{n+1}$ such that

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{n+i+1} e_i, \quad (1 \le i \le n+1);$$

$\widehat{e_i}$ means that e_i is omitted. Filippov [20, Theorem 4] shows that L_{n+1} is simple.

Remarks 4.2.7. In Ling [31], it was proved that for each $n \ge 3$ a simple finite-dimensional n-Lie algebra over an algebraically closed field of characteristic 0 is unique up to isomorphism, and this is the simple algebra L_{n+1}.

We first recall Filippov's classification of $(n+1)$-dimensional n-Lie algebras. If L is an n-Lie algebra then L^1 is its derived algebra and $Z(L)$ is its center.

Theorem 4.2.8. (Classification Theorem) [20] *Let $n \ge 3$ and let L be an $(n+1)$-dimensional n-Lie algebra with basis $e_1, e_2, \ldots, e_{n+1}$ over F. Up to isomorphism, exactly one of the following cases holds; omitted brackets are assumed to be zero:*

(0) If $\dim L^1 = 0$ *then* L *is the Abelian* n*-Lie algebra.*

(1) If $\dim L^1 = 1$ *then we write* $L^1 = Fe_1$ *and we have two cases:*

- *(a) If* $L^1 \subseteq Z(L)$ *then* $[e_2, \dots, e_{n+1}] = e_1$.
- *(b) If* $L^1 \nsubseteq Z(L)$ *then* $[e_1, \dots, e_n] = e_1$.

(2) If $\dim L^1 = 2$ *then we write* $L^1 = Fe_1 \oplus Fe_2$ *and we have two cases:*

- *(a)* $[e_2, \dots, e_{n+1}] = e_1$ *and* $[e_1, e_3, \dots, e_{n+1}] = e_2$.
- *(b)* $[e_2, \dots, e_{n+1}] = e_1 + \beta e_2$ *for* $\beta \in F \smallsetminus \{0\}$ *and* $[e_1, e_3, \dots, e_{n+1}] = e_2$.

(r) If $\dim L^1 = r$ *for* $3 \le r \le n+1$ *then we write* $L^1 = Fe_1 \oplus \cdots \oplus Fe_r$ *and we have* $[e_1, \dots, \widehat{e_i}, \dots, e_{n+1}] = e_i$ *for* $1 \le i \le r$.

Proof. This is Filippov's classification [20, Section 3] of $(n+1)$-dimensional n-Lie algebras in the simplified version of Bai and Song [1, Theorem 3.1]. □

4.3 Universal associative envelopes of alternating n-ary algebras

Let L be an $(n+1)$-dimensional n-ary algebra with an alternating product which does not necessarily satisfy the generalized Jacobi identity. We are primarily interested in n-Lie algebras but in this section we consider a more general situation.

Definition 4.3.1. Let A be an associative algebra. On the underlying vector space of A we define a new operation, the n-ary **alternating sum**:

$$\operatorname{alt}(x_1, x_2, \dots, x_n) = \sum_{\sigma \in S_n} \epsilon(\sigma)\, x_{\sigma(1)}\, x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

We write A^- for the **minus algebra**: the alternating n-ary algebra obtained by replacing the associative product by the alternating sum.

Definition 4.3.2. A **universal associative envelope** of the alternating n-ary algebra L consists of a unital associative algebra U and a linear map $i: L \to U$ satisfying

$$i([x_1, x_2, \dots, x_n]) = \operatorname{alt}\big(i(x_1), i(x_2), \dots, i(x_n)\big) \quad (x_1, \dots, x_n \in L),$$

such that for any unital associative algebra A and linear map $j: L \to A$ satisfying the same equation with j in place of i, there is a unique homomorphism of unital associative algebras $\psi: U \to A$ such that $\psi \circ i = j$.

Notation 4.3.3. *Let $B = \{e_1, e_2, \dots, e_{n+1}\}$ be an ordered basis of L. Consider the bijection $\phi: B \to X = \{x_1, x_2, \dots, x_{n+1}\}$ defined by $\phi(e_i) = x_i$. We extend ϕ to a linear map $\phi: L \to F\langle X\rangle$ and write $y_i = \phi([e_1, \dots, \widehat{e_i}, \dots, e_{n+1}])$.*

We refer to Section 2.4 for the basic notations and definitions about Gröbner bases.

Definition 4.3.4. Consider the following elements of $F\langle X\rangle$ for $1 \le i \le n+1$:

$$G_i = (-1)^{\lfloor n/2 \rfloor}\big(\operatorname{alt}(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}) - y_i\big).$$

The factor $(-1)^{\lfloor n/2 \rfloor}$ ensures that $\operatorname{LM}(G_i) = x_{n+1} \cdots \widehat{x_i} \cdots x_1$ has coefficient 1.

Notation 4.3.5. *Let $I \subset F\langle X\rangle$ be the ideal generated by $G_1, \dots, G_{n+1}$. We write $U = F\langle X\rangle / I$ with surjection $\pi: F\langle X\rangle \to U$ sending f to $f + I$, and $i = \pi \circ \phi$ for the natural map $i: L \to U$.*

Lemma 4.3.6. *The unital associative algebra U and the linear map i form the universal associative envelope of the alternating n-ary algebra L.*

Proof. Similar to the case $n = 2$; see Humphreys [24, §17.2]. □

Lemma 4.3.7. *There is only one overlap among $\operatorname{LM}(G_1), \dots, \operatorname{LM}(G_{n+1})$, namely $\operatorname{LM}(G_1) = x_{n+1} \cdots x_2$ and $\operatorname{LM}(G_{n+1}) = x_n \cdots x_1$ have the common factor $x_n \cdots x_2$. Hence there is only one composition among the generators: $G_1 x_1 - x_{n+1} G_{n+1}$.*

Proof. The subscripts in $\operatorname{LM}(G_i)$ are the sequence $n{+}1 > \cdots > \widehat{i} > \cdots > 1$. □

Theorem 4.3.8. *The normal form of the composition* $G_1 x_1 - x_{n+1} G_{n+1}$ *is*

$$N = (-1)^n \sum_{i=1}^{n+1} (-1)^i \big(x_i G_i - (-1)^n G_i x_i \big) = (-1)^{n+1} \sum_{i=1}^{n+1} (-1)^i \big(x_i y_i - (-1)^n y_i x_i \big).$$

Proof. We first observe that N can be rewritten as follows:

$$(-1)^n \sum_{i=1}^{n+1} (-1)^i \big(x_i G_i - (-1)^n G_i x_i \big) = G_1 x_1 - x_{n+1} G_{n+1} + S + T,$$

where

$$S = (-1)^n \sum_{i=2}^{n} (-1)^i \big(x_i G_i - (-1)^n G_i x_i \big), \quad T = -(-1)^n x_1 G_1 + (-1)^n G_{n+1} x_{n+1}.$$

To compute the normal form of $G_1 x_1 - x_{n+1} G_{n+1}$ using noncommutative division with remainder, we perform two steps. First, we eliminate occurrences of $\mathrm{LM}(G_1)$, …, $\mathrm{LM}(G_{n+1})$ as factors in the monomials; this corresponds to the sum S, and introduces new occurrences of $\mathrm{LM}(G_1)$ and $\mathrm{LM}(G_{n+1})$. Second, we eliminate these last two occurrences; this corresponds to the sum T. This shows that N can be obtained from $G_1 x_1 - x_{n+1} G_{n+1}$ by a sequence of reductions modulo the generators $G_1, \dots, G_{n+1}$ of the ideal I. Thus, in order to prove that N is the normal form of the composition, it remains to show that no monomial occurring in N has a factor equal to $\mathrm{LM}(G_i)$ for any $i = 1, \dots, n+1$.

For the following calculations, it is convenient to write

$$G_i = (-1)^{\lfloor n/2 \rfloor} \Big(\sum_{\sigma \in S_n^{(i)}} \epsilon(\sigma) x_{\sigma(1)} \cdots \widehat{x}_{\sigma(i)} \cdots x_{\sigma(n+1)} - y_i \Big),$$

where $S_n^{(i)} \cong S_n$ is the symmetric group on $\{1, \dots, \widehat{i}, \dots, n+1\}$. To simplify the signs, we factor out $(-1)^n (-1)^{\lfloor n/2 \rfloor}$ from the entire calculation. Thus we consider the following simplified versions of N and the ideal generators G_i:

$$N = \sum_{i=1}^{n+1} (-1)^i \big(x_i G_i - (-1)^n G_i x_i \big), \quad G_i = \sum_{\sigma \in S_n^{(i)}} \epsilon(\sigma) x_{\sigma(1)} \cdots \widehat{x}_{\sigma(i)} \cdots x_{\sigma(n+1)} - y_i.$$

We rewrite $x_i G_i$ and $G_i x_i$ as follows:

$$x_i G_i = \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = i}} (-1)^{i-1} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)} - x_i y_i, \tag{4.3}$$

$$G_i x_i = \sum_{\substack{\tau \in S_{n+1} \\ \tau(n+1) = i}} (-1)^{n+1-i} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i - y_i x_i. \tag{4.4}$$

In $x_i G_i$, the symbol x_i has moved left past $i{-}1$ symbols, so $\epsilon(\sigma) = (-1)^{i-1}\epsilon(\tau)$. In $G_i x_i$, the symbol x_i has moved right past $n{+}1{-}i$ symbols, so $\epsilon(\sigma) = (-1)^{n+1-i}\epsilon(\tau)$. Therefore

$$x_i G_i - (-1)^n G_i x_i = -\big(x_i y_i - (-1)^n y_i x_i\big) \\ + \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = i}} (-1)^{i-1} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)} - \sum_{\substack{\tau \in S_{n+1} \\ \tau(n+1) = i}} (-1)^{i-1} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i.$$

From this we obtain

$$(-1)^i \big(x_i G_i - (-1)^n G_i x_i\big) = -(-1)^i \big(x_i y_i - (-1)^n y_i x_i\big) \\ - \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = i}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)} + \sum_{\substack{\tau \in S_{n+1} \\ \tau(n+1) = i}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i.$$

Summing over $i = 1, \ldots, n{+}1$ gives

$$\sum_{i=1}^{n+1} (-1)^i \big(x_i G_i - (-1)^n G_i x_i\big) = Q + R, \quad Q = -\sum_{i=1}^{n+1} (-1)^i \big(x_i y_i - (-1)^n y_i x_i\big), \\ R = -\sum_{i=1}^{n+1} \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = i}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)} + \sum_{i=1}^{n+1} \sum_{\substack{\tau \in S_{n+1} \\ \tau(n+1) = i}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i.$$

It remains to show that $R = 0$. In the first (respectively second) double sum, we separate terms according to the last (respectively first) symbol in each monomial:

$$R = -\sum_{i=1}^{n+1} \sum_{\substack{j = 1 \\ j \neq i}}^{n+1} \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = i \\ \tau(n+1) = j}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n)} x_j \\ + \sum_{i=1}^{n+1} \sum_{\substack{j = 1 \\ j \neq i}}^{n+1} \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = j \\ \tau(n+1) = i}} \epsilon(\tau) x_j x_{\tau(2)} \cdots x_{\tau(n)} x_i = 0,$$

since both sums are over all pairs (i, j) with $1 \leq i \neq j \leq n{+}1$. □

Remark 4.3.9. For n even (respectively odd) the terms of N can be written as Lie brackets (respectively Jordan products):

$$x_i G_i - (-1)^n G_i x_i = [x_i, G_i] \;\; (n \text{ even}), \quad x_i G_i - (-1)^n G_i x_i = x_i \circ G_i \;\; (n \text{ odd}).$$

4.4 Proof of Pozhidaev's conjecture for simple n-Lie algebras (n even)

In the rest of this chapter, we assume that L is an n-Lie algebra. Pozhidaev [35] considered the problem whether there exists an embedding of an arbitrary n-Lie algebra into an associative algebra, and made the following conjecture:

Conjecture 4.4.1. *For any reductive finite-dimensional n-Lie algebra L over an algebraically closed field of characteristic 0 there exists an associative algebra A such that L is isomorphic to a subalgebra of A^-.*

By the work of Ling [31] it is known that any reductive finite-dimensional n-Lie algebra over an algebraically closed field of characteristic 0 decomposes into the direct sum of an Abelian ideal and several copies of a simple ideal isomorphic to the simple $(n+1)$-dimensional n-Lie algebra L_{n+1}. Hence the main problem is to prove that L_{n+1} can be embedded into an associative algebra.

Theorem 4.4.2. *Let $n \geq 3$ and let F be a field of characteristic $\neq 2$. Let L be the simple n-Lie algebra L_{n+1} over F. The generators $\{G_1, \dots, G_{n+1}\}$ of Definition 4.3.4 form a Gröbner basis for the ideal $I = \langle G_1, \dots, G_{n+1} \rangle$ in the free associative algebra $F\langle x_1, \dots, x_{n+1} \rangle$ if and only if n is even.*

Proof. The structure constants for L_{n+1} give $y_i = (-1)^{n+i+1} x_i$ and so

$$G_i = (-1)^{\lfloor n/2 \rfloor} \Big(\mathrm{alt}(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}) + (-1)^{n+i} x_i \Big).$$

By Theorem 4.3.8 the normal form of the single composition of these generators is

$$N = \sum_{i=1}^{n+1} \big(1 - (-1)^n\big) x_i^2 = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 \sum_{i=1}^{n+1} x_i^2 & \text{if } n \text{ is odd.} \end{cases}$$

Since $\operatorname{char} F \neq 2$ we have $N = 0$ if and only if n is even, and by Theorem 2.4.12 this is equivalent to $\{G_1, \dots, G_{n+1}\}$ being a Gröbner basis. □

Corollary 4.4.3. *Let $n \geq 4$ be even and let F be a field of characteristic $\neq 2$. Let L be the simple n-Lie algebra L_{n+1} over F. The universal associative enveloping algebra $U(L)$ is infinite-dimensional, and a basis consists of the monomials which do not contain any factor of the form $x_{i_1} \cdots x_{i_n}$ with $i_1 > \cdots > i_n$.*

Proof. Since $\{G_1, \dots, G_{n+1}\}$ is Gröbner basis, Proposition 2.4.4 shows that the normal words of $F\langle X \rangle$ modulo I, or equivalently the coset representatives for $U(L) = F\langle X \rangle / I$, are those that do not contain any $\mathrm{LM}(G_i)$ as a factor. □

Corollary 4.4.4. *Let $n \geq 4$ be even and let F be a field of characteristic $\neq 2$. For the simple n-Lie algebra L_{n+1} the natural map $i \colon L_{n+1} \to U(L_{n+1})$ is injective.*

Proof. The intersection of $I = \langle G_1, \dots, G_{n+1} \rangle$ with $\mathrm{span}(x_1, \dots, x_{n+1})$ is 0, and hence the cosets of the x_i are linearly independent in $U(L_{n+1})$. □

We now obtain a proof of Pozhidaev's conjecture [35] in the case of n even.

Corollary 4.4.5. *Let $n \geq 4$ be even and let F be a field of characteristic $\neq 2$. There exists an associative algebra A such that the simple n-Lie algebra L_{n+1} is isomorphic to a subalgebra of A^-.*

Proof. Take $A = U(L)$ and apply Corollary 4.4.4. □

We also obtain the following new proof of Pozhidaev's Corollary 2.1 [35].

Corollary 4.4.6. *Let $n \geq 3$ be odd, let F be a field of characteristic $\neq 2$, and let L be the simple n-Lie algebra L_{n+1}. If A is an associative algebra and $j \colon L \to A^-$ is a homomorphism of alternating n-ary algebras, then $j(e_1)^2 + \cdots + j(e_{n+1})^2 = 0$.*

Proof. The proof of Theorem 4.4.2 shows that $x_1^2 + \cdots + x_{n+1}^2 = 0$ in $U(L)$, and so the claim follows from the universal property of $U(L)$. □

For n odd, finding a Gröbner basis of $I = \langle G_1, \dots, G_{n+1} \rangle$ for the simple n-Lie algebra L_{n+1} seems to be much more difficult; see the calculations in Section 4.6.

4.5 The non-simple n-Lie algebras (n even)

We now consider the other $(n+1)$-dimensional n-Lie algebras in the classification of Theorem 4.2.8. We divide these non-simple algebras into three cases depending on the complexity of the resulting Gröbner basis.

4.5.1 Case 1

This includes cases (0), (1a), (2a) and (r) of Theorem 4.2.8.

Theorem 4.5.1. *Let $n \geq 4$ be even, let F be an algebraically closed field of characteristic 0, and let L be an $(n+1)$-dimensional n-Lie algebra from Theorem 4.2.8. In the following four cases, the original ideal generators $\{G_1, \ldots, G_{n+1}\}$ of Definition 4.3.4 are a Gröbner basis for the ideal $I = \langle G_1, \ldots, G_{n+1} \rangle \subseteq F\langle X \rangle$:*

(0) $L^1 = \{0\}$*: L is the Abelian n-Lie algebra.*

(1a) $L^1 = Fe_1$ *where* $[e_2, \ldots, e_{n+1}] = e_1$.

(2a) $L^1 = Fe_1 \oplus Fe_2$ *where* $[e_2, \ldots, e_{n+1}] = e_1$ *and* $[e_1, e_3, \ldots, e_{n+1}] = e_2$.

(r) $L^1 = Fe_1 \oplus \cdots \oplus Fe_r$ $(3 \leq r \leq n)$ *where* $[e_1, \ldots, \widehat{e}_i, \ldots, e_{n+1}] = e_i$ *for* $1 \leq i \leq r$.

Proof. In each case we verify that the normal form N of the unique composition of the original ideal generators is equal to 0. This is trivial in case (0). In case (1)(a), Theorem 4.3.8 gives $N = x_1^2 - x_1^2 = 0$. In case (2)(a), we get $N = (x_1^2 - x_1^2) - (x_2^2 - x_2^2) = 0$. In case (r), we get $N = -\sum_{i=1}^{r} (-1)^i (x_i^2 - x_i^2) = 0$. We note that in all these cases, either $y_i = x_i$ or $y_i = 0$ for $i = 1, \ldots, n+1$. □

4.5.2 Case 2

This is case (1b) of Theorem 4.2.8: $L^1 = Fe_1$ where $[e_1, \ldots, e_n] = e_1$. The original ideal generators are

$$G_i = (-1)^{\lfloor n/2 \rfloor} \mathrm{alt}(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}) \quad (1 \leq i \leq n),$$
$$G_{n+1} = (-1)^{\lfloor n/2 \rfloor} \big(\mathrm{alt}(x_1, \ldots, x_n) - x_1 \big).$$

Lemma 4.5.2. *The composition $G_1 x_1 - x_{n+1} G_{n+1}$ has normal form*

$$N = x_{n+1} x_1 - x_1 x_{n+1}.$$

Proof. This follows directly from Theorem 4.3.8. □

We must include N as a new generator and modify the original generators by replacing them by their normal forms modulo N.

Notation 4.5.3. *For $i = 2, \ldots, n$ we write $T_n^{(i)}$ for the set of all permutations of $\{1, \ldots, \widehat{i}, \ldots, n{+}1\}$ in which 1 and $n{+}1$ do not appear consecutively. We consider the following corresponding elements of $F\langle X\rangle$:*

$$H_i = (-1)^{\lfloor n/2 \rfloor} \sum_{\sigma \in T_n^{(i)}} \epsilon(\sigma)\, x_{\sigma(1)}\, x_{\sigma(2)} \cdots x_{\sigma(n)} \quad (2 \le i \le n).$$

Theorem 4.5.4. *Let $n \ge 4$ be even and let F be any field. Let L be the $(n{+}1)$-dimensional n-Lie algebra with structure constants $[e_1, \ldots, e_n] = e_1$. A Gröbner basis for the ideal $I = \langle G_1, \ldots, G_{n+1} \rangle \subseteq F\langle X \rangle$ consists of the elements*

$$\{\, G_1,\, H_2,\, \ldots,\, H_n,\, G_{n+1},\, N \,\}.$$

Proof. We have $\mathrm{LM}(N) = x_{n+1} x_1$ and obviously this never occurs as a factor of any monomial in G_1 or G_{n+1}. If $x_{n+1} x_1$ is a factor of a term $\epsilon w = \pm u x_{n+1} x_1 v$ occurring in G_i for some $i = 2, \ldots, n$, then we reduce w using N. This simply means that we replace ϵw by $\epsilon w' = \pm u x_1 x_{n+1} v$. But since G_i is an alternating sum, the term $-\epsilon w'$ appears in G_i, and the terms $\epsilon w'$ and $-\epsilon w'$ cancel. The remaining terms in G_i correspond to the permutations in $T_n^{(i)}$ and so we obtain the new generators $H_2, \ldots, H_n$. No further reductions are possible in the set of generators: the set $\{G_1, H_2, \ldots, H_n, G_{n+1}, N\}$ is self-reduced. The leading monomials of the generators $G_1, H_2, \ldots, H_n, G_{n+1}$ have strictly decreasing subscripts, and hence never have x_1 as the first symbol or x_{n+1} as the last symbol; it follows that no further compositions with N are possible. Hence we now have a Gröbner basis for the ideal I. □

4.5.3 Case 3

This is case (2b) of Theorem 4.2.8: $L^1 = Fe_1 \oplus Fe_2$ where

$$[e_2, \ldots, e_{n+1}] = e_1 + \beta e_2 \ (\beta \neq 0), \qquad [e_1, e_3, \ldots, e_{n+1}] = e_2.$$

The original ideal generators are

$$\begin{aligned} G_1 &= (-1)^{\lfloor n/2 \rfloor}\big(\mathrm{alt}(x_2, \ldots, x_{n+1}) - (x_1 + \beta x_2)\big), \\ G_2 &= (-1)^{\lfloor n/2 \rfloor}\big(\mathrm{alt}(x_1, x_3, \ldots, x_{n+1}) - x_2\big), \\ G_i &= (-1)^{\lfloor n/2 \rfloor}\mathrm{alt}(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}) \quad (3 \leq i \leq n{+}1). \end{aligned}$$

Lemma 4.5.5. *The composition $G_1 x_1 - x_{n+1} G_{n+1}$ has normal form*

$$N = x_2 x_1 - x_1 x_2.$$

Proof. This follows directly from Theorem 4.3.8 since $\beta \neq 0$. □

We must include N as a new generator and modify the original generators by replacing them by their normal forms modulo N.

Notation 4.5.6. *We write V_{n+1} for the subset of S_{n+1} in which 1 and 2 do not appear consecutively. We write $V_n^{(i)}$ ($i \neq 1, 2$) for the set of permutations of $\{1, \ldots, \widehat{i}, \ldots, n{+}1\}$ in which 1 and 2 do not appear consecutively. We consider the corresponding elements of $F\langle X \rangle$:*

$$K_i = -(-1)^{\lfloor n/2 \rfloor} \sum_{\sigma \in V_n^{(i)}} \epsilon(\sigma)\, x_{\sigma(1)}\, x_{\sigma(2)} \cdots x_{\sigma(n)} \quad (3 \leq i \leq n{+}1).$$

The extra minus sign appears because $\mathrm{LM}(K_i)$ differs by a transposition from $\mathrm{LM}(G_i)$: the leading monomial of K_i is

$$x_{n+1} \cdots x_5 x_2 x_4 x_1 \ (i = 3), \qquad x_{n+1} \cdots \widehat{x}_i \cdots x_4 x_2 x_3 x_1 \ (i \geq 4).$$

Theorem 4.5.7. *Let $n \geq 4$ be even and let F be any field. Let L be the $(n{+}1)$-dimensional n-Lie algebra with structure constants*

$$[e_2, \ldots, e_{n+1}] = e_1 + \beta e_2 \ (\beta \neq 0), \qquad [e_1, e_3, \ldots, e_{n+1}] = e_2.$$

A Gröbner basis for the ideal $I = \langle G_1, \dots, G_{n+1} \rangle \subseteq F\langle X \rangle$ *consists of the elements*

$$\{\, G_1, G_2, K_3, \dots, K_{n+1}, N \,\}.$$

Proof. We first use N to reduce the original generators $G_1, \dots, G_{n+1}$. Clearly G_1 and G_2 do not change, since G_1 (resp. G_2) does not contain x_1 (resp. x_2). The monomials in $G_3, \dots, G_{n+1}$ of the form $\cdots x_2 x_1 \cdots$ reduce to $\cdots x_1 x_2 \cdots$; hence all the monomials containing $x_2 x_1$ and $x_1 x_2$ cancel, and $G_3, \dots, G_{n+1}$ reduce to $K_3, \dots, K_{n+1}$. It is easy to check that G_1, G_2, K_3, $\dots$, K_{n+1}, N have only one overlap among their leading monomials: $\mathrm{LM}(G_1) = x_{n+1} \cdots x_2$, $\mathrm{LM}(N) = x_2 x_1$. Hence there is a single new composition,

$$P = G_1 x_1 - x_{n+1} x_n \dots x_3 N.$$

To complete the proof, it suffices to show that the normal form of P is 0.

Following the proof of Theorem 4.3.8, we first eliminate from P all occurrences of the leading monomials of G_2, K_3, $\dots$, K_{n+1}, N. This gives $P + Q$ where

$$Q = -G_2 x_2 + x_2 G_2 + \sum_{i=3}^{n+1} (-1)^{i+1} x_i K_i + (-1)^{n/2} \Bigg[\beta N - \sum_{\substack{\tau \in S_{n+1} \\ \tau(n) = 2 \\ \tau(n+1) = 1}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n-1)} N \Bigg].$$

We next eliminate from $P + Q$ all occurrences of the leading monomials of G_1, K_3, $\dots$, K_{n+1}, N. This gives $P + Q + R$ where

$$R = -x_1 G_1 - \sum_{i=3}^{n+1} (-1)^{i+1} K_i x_i + (-1)^{n/2} \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = 2 \\ \tau(2) = 1}} \epsilon(\tau) N x_{\tau(3)} \cdots x_{\tau(n+1)}.$$

This shows that P reduces to $M = P + Q + R$. It remains to show that $M = 0$.

Combining the terms in P, Q, R we obtain $M = A + B + C$ where

$$A = \sum_{i=1}^{2} (-1)^{i+1} (G_i x_i - x_i G_i), \qquad B = \sum_{i=3}^{n+1} (-1)^{i+1} (x_i K_i - K_i x_i),$$

$$C = (-1)^{n/2} \Bigg[\beta N + \sum_{\substack{\tau \in S_{n+1} \\ \tau(1) = 2 \\ \tau(2) = 1}} \epsilon(\tau) N x_{\tau(3)} \cdots x_{\tau(n+1)} - \sum_{\substack{\tau \in S_{n+1} \\ \tau(n) = 2 \\ \tau(n+1) = 1}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n-1)} N \Bigg].$$

We factor out $(-1)^{n/2}$ from the following calculation to simplify the signs.

Using the definitions of the ideal generators, we rewrite A as follows:

$$A = \sum_{i=1}^{2}\left[\sum_{\substack{\tau \in S_{n+1} \\ \tau(n+1)=i}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i - \sum_{\substack{\tau \in S_{n+1} \\ \tau(1)=i}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)}\right] - \beta(x_2x_1 - x_1x_2).$$

The signs $(-1)^{i+1}$ cancel using equations (4.3) and (4.4). We now separate the monomials which either begin or end with either x_1x_2 or x_2x_1:

$$\begin{aligned} A = &\sum_{i=1}^{2}\left[\sum_{\substack{\tau \in V_{n+1} \\ \tau(n+1)=i}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i - \sum_{\substack{\tau \in V_{n+1} \\ \tau(1)=i}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)}\right] \\ &+ \sum_{\substack{\tau \in S_{n+1} \\ \tau(n)=2 \\ \tau(n+1)=1}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n-1)} (x_2x_1 - x_1x_2) \\ &- \sum_{\substack{\tau \in S_{n+1} \\ \tau(1)=2 \\ \tau(2)=1}} \epsilon(\tau)(x_2x_1 - x_1x_2) x_{\tau(3)} \cdots x_{\tau(n+1)} - \beta(x_2x_1 - x_1x_2). \end{aligned}$$

Similarly, we obtain

$$B = \sum_{i=3}^{n+1}\left[\sum_{\substack{\tau \in V_{n+1} \\ \tau(n+1)=i}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i - \sum_{\substack{\tau \in V_{n+1} \\ \tau(1)=i}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)}\right],$$

using the same relation between $\epsilon(\sigma)$ and $\epsilon(\tau)$ as in equations (4.3) and (4.4).

Since $N = x_2x_1 - x_1x_2$ we obtain

$$\begin{aligned} C = &\sum_{\substack{\tau \in S_{n+1} \\ \tau(1)=2 \\ \tau(2)=1}} \epsilon(\tau)(x_2x_1 - x_1x_2) x_{\tau(3)} \cdots x_{\tau(n+1)} \\ &- \sum_{\substack{\tau \in S_{n+1} \\ \tau(n)=2 \\ \tau(n+1)=1}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n-1)} (x_2x_1 - x_1x_2) + \beta(x_2x_1 - x_1x_2). \end{aligned}$$

Adding the last three expressions for A, B and C gives

$$M = \sum_{i=1}^{n+1} \sum_{\substack{\tau \in V_{n+1} \\ \tau(n+1)=i}} \epsilon(\tau) x_{\tau(1)} \cdots x_{\tau(n)} x_i - \sum_{i=1}^{n+1} \sum_{\substack{\tau \in V_{n+1} \\ \tau(1)=i}} \epsilon(\tau) x_i x_{\tau(2)} \cdots x_{\tau(n+1)}.$$

The argument at the end of the proof of Theorem 4.3.8 now shows that $M = 0$. □

Corollary 4.5.8. *Let $n \geq 4$ be even and let L be any non-simple $(n+1)$-dimensional n-Lie algebra over F. In cases (0), (1a), (2a) and (r) of Theorem 4.2.8, a basis of the universal associative envelope $U(L)$ consists of the monomials which do not contain any factor of the form*

$$x_{i_1}x_{i_2}\cdots x_{i_n} \ (i_1 > i_2 > \cdots > i_n).$$

In case (1b) of Theorem 4.2.8, a basis of the universal associative envelope $U(L)$ consists of the monomials which do not contain any factor of the form

$$x_{n+1}x_1 \qquad or \qquad x_{i_1}x_{i_2}\cdots x_{i_n} \ (i_1 > i_2 > \cdots > i_n).$$

In case (2b) of Theorem 4.2.8, a basis of the universal associative envelope $U(L)$ consists of the monomials which do not contain any factor of the form

$$x_2x_1, \qquad x_{n+1}x_n \ldots x_2, \qquad x_{n+1}x_n \ldots x_3x_1,$$
$$x_{n+1}\cdots x_5x_2x_4x_1 \qquad or \qquad x_{n+1}\cdots\widehat{x}_i\cdots x_4x_2x_3x_1 \ (i \geq 4).$$

Hence in every case $U(L)$ is infinite-dimensional.

Corollary 4.5.9. *Let $n \geq 4$ be even and let F be a field of characteristic $\neq 2$. For any non-simple $(n+1)$-dimensional n-Lie algebra L the natural map $i: L \to U(L)$ is injective.*

4.6 Computational results for n odd

In this section we present computational results to illustrate the complexity of finding a Gröbner basis for the ideal $I = \langle G_1, \ldots, G_{n+1} \rangle \subseteq F\langle x_1, \ldots, x_{n+1} \rangle$ when n is odd. These computations were done with the computer algebra system `Maple`.

We consider the 4-dimensional simple 3-Lie algebra L_4; to clarify the notation in this special case, we write a, b, c, d in place of x_1, x_2, x_3, x_4 for the basis elements. The structure constants are then as follows:

$$[a,b,c] = d, \qquad [a,b,d] = -c, \qquad [a,c,d] = b, \qquad [b,c,d] = -a.$$

The original set of ideal generators, which is already self-reduced, is as follows:

$$
\begin{aligned}
G_1 &= dcb - dbc - cdb + cbd + bdc - bcd - a,\\
G_2 &= dca - dac - cda + cad + adc - acd + b,\\
G_3 &= dba - dab - bda + bad + adb - abd - c,\\
G_4 &= cba - cab - bca + bac + acb - abc + d.
\end{aligned}
$$

Noncommutative polynomials will made monic and their terms will be listed in reverse deglex order so that their leading monomials occur first; sets of polynomials will be listed in reverse deglex order of their leading monomials.

- First iteration:

  Lemma 4.3.7 shows that there is only one composition among $G_1, \ldots, G_4$ and Theorem 4.3.8 gives its normal form:

  $$G_1 a - dG_4 \xrightarrow{\text{nf}} N = d^2 + c^2 + b^2 + a^2.$$

  Since $N \neq 0$, we must add N to the set of ideal generators and repeat the process. The new set of generators, which is already self-reduced, is

  $$\{G_1, G_2, G_3, G_4, N\}.$$

- Second iteration:

  We obtain three new compositions and compute their normal forms:

  $$
  \begin{aligned}
  Ncb - dG_1 \xrightarrow{\text{nf}} P_1 &= dcdb - dbdc - cdbd + c^3b - c^2bc - cbc^2 - cb^3 - caba + bdcd\\
  &\quad + bc^3 + bcb^2 + b^2cb - b^3c + baca + acab - abac + 2da - 2ad,\\
  Nca - dG_2 \xrightarrow{\text{nf}} P_2 &= dcda - dadc - cdad + c^3a - c^2ac - cac^2 - cab^2 - ca^3 + b^2ca\\
  &\quad - b^2ac + adcd + ac^3 + acb^2 + aca^2 + a^2ca - a^3c - db + bd,\\
  Ncb - dG_3 \xrightarrow{\text{nf}} P_3 &= dbda - dadb + cbca - cacb - bdad - bcac + b^3a - b^2ab - bab^2\\
  &\quad - ba^3 + adbd + acbc + ab^3 + aba^2 + a^2ba - a^3b + 2dc - 2cd.
  \end{aligned}
  $$

  The new set of generators, which is already self-reduced, is

  $$\{\, P_1,\, P_2,\, P_3,\, G_1,\, G_2,\, G_3,\, G_4,\, N \,\}.$$

- Third iteration:

  We obtain five new compositions:

$$P_1 da - dcP_3 \xrightarrow{\text{nf}} Q_1, \qquad Ncdb - dP_1 \xrightarrow{\text{nf}} Q_2, \qquad Ncda - dP_2 \xrightarrow{\text{nf}} Q_3,$$
$$Nbda - dP_3 \xrightarrow{\text{nf}} Q_4, \qquad P_1 a - dcG_3 \xrightarrow{\text{nf}} Q_5.$$

  These compositions have 34, 20, 20, 20, 23 terms respectively; their normal forms have 178, 35, 33, 56, 6 terms respectively. The simplest new generator is

$$Q_5 = dc^2 + db^2 + da^2 - c^2 d - b^2 d - a^2 d.$$

  The leading monomials of the others are

$$\text{LM}(Q_1) = dc^2bca, \quad \text{LM}(Q_2) = dc^3b, \quad \text{LM}(Q_3) = dc^3a, \quad \text{LM}(Q_4) = dbc^2a.$$

  We add these new noncommutative polynomials to the set of generators and obtain

$$\{\, Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3, Q_5, G_1, G_2, G_3, G_4, N \,\}.$$

  However, this set of the generators is not self-reduced: the leading monomials of some generators are factors of monomials occurring in other generators. After performing self-reduction, we find that Q_2 and Q_3 become 0, and Q_1 and Q_4 respectively become R_1 and R_2 with 187 and 58 terms and leading monomials dbc^3a and dbc^2a. The new self-reduced set of generators is

$$\{\, R_1, R_2, P_1, P_2, P_3, Q_5, G_1, G_2, G_3, G_4, N \,\}.$$

- Fourth iteration:

  We obtain six new compositions:

$$P_1c^3a - dcR_1, \quad P_1c^2a - dcR_2, \quad Nbc^3a - dR_1, \quad Nbc^2a - dR_2, \quad Nc^2 - dQ_5.$$

  These compositions have 203, 74, 189, 60, 8 terms respectively. This suggests that the algorithm may not terminate and that the Gröbner basis obtained by this process from the original set of ideal generators may in fact be infinite.

Chapter 5

Associative enveloping algebras for non-associative triple systems

5.1 Introduction

In this chapter we use noncommutative Gröbner bases to construct the universal associative enveloping algebras of the nonassociative triple systems which arise from applying the non-associative trilinear operations classified by Bremner and Peresi [10] to the 2-dimensional simple associative triple system of the first kind in the sense of Lister [32], namely the space of 2×2 matrices $A = (a_{ij})$ with $a_{11} = a_{22} = 0$.

By a multilinear n-ary operation we mean an element ω of the group algebra $\mathbb{Q}S_n$ of the symmetric group S_n over the rational field $\mathbb{Q}$. Following [10] we say that two operations ω_1, ω_2 are *equivalent* if they generate the same left ideal: each can be expressed as a linear combination of permutation of the other operation. If A is an associative algebra over $\mathbb{Q}$, then ω defines a multilinear n-ary operation $\omega(a_1, \ldots, a_n)$ on the underlying vector space of A:

$$\omega = \sum_{\sigma \in S_n} x_\sigma \sigma \qquad \Longrightarrow \qquad \omega(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)}.$$

In this way we obtain a non-associative n-ary algebra which we denote by A^ω.

For $n = 2$, every bilinear operation is equivalent to either the zero operation, the associative operation ab, the Lie bracket $[a, b] = ab - ba$, or the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. The polynomial identities of degree ≤ 3 (≤ 4) satisfied by the Lie bracket (Jordan product) define Lie algebras (Jordan algebras), the most important varieties of nonassociative algebras. For $n = 3$, Bremner and Peresi [10] found canonical representatives of the equivalence classes of trilinear operations, and identified 19 operations satisfying polynomial identities of degree 5 which do not follow from the identities of degree 3. These operations include the Lie, anti-Lie, Jordan, and anti-Jordan triple products (see Section 2.5 for definitions).

In Section 5.2 we find simpler operations equivalent to those of [10]; our operations have coefficients $\pm 1, \pm 2$ and most have coefficients ± 1. We augment this list with the symmetric, alternating and cyclic sums; see Table 5.1.

In Section 5.3 we explain the construction of the universal associative envelopes $U(A^\omega)$ of nonassociative n-ary algebras A^ω (defined by multilinear operations ω) by means of noncommutative Gröbner bases for ideals in free associative algebras.

In Section 5.4 we recall the down-up algebras of Benkart and Roby [3]. We then consider the cases in which $U(A^\omega)$ is infinite dimensional: we determine monomial bases and structure constants, identify the center, and determine the Gelfand-Kirillov dimension. In every case, $U(A^\omega)$ is either a free associative algebra, a down-up algebra, or a quotient of a down-up algebra.

In Section 5.5 we consider the cases in which $U(A^\omega)$ is finite dimensional. We use an algorithmic version of the structure theory for finite dimensional associative algebras to determine the Wedderburn decompositions and classify the irreducible representations. In most cases we obtain only the trivial 1-dimensional representation and the natural 2-dimensional representation.

The results of Sections 5.4 and 5.5 are summarized in Table 5.2. We distinguish trilinear operations of "Lie type" for which $U(A^\omega)$ is infinite dimensional, and those of "Jordan type" for which $U(A^\omega)$ is finite dimensional. Recall that for a finite dimensional Lie algebra L, the universal associative envelope $U(L)$ is infinite dimensional, and the map $L \to U(L)$ is injective; whereas for a finite dimensional Jordan algebra J, the universal associative envelope $U(J)$ is finite dimensional, and the map $J \to U(J)$ is injective if and only if J is special.

We assume throughout that the base field $\mathbb{F}$ has characteristic 0; if necessary, we assume that $\mathbb{F}$ is algebraically closed. The basic background information for this chapter is summarized in Sections 2.4-2.7.

5.2 The twenty-two trilinear operations

A natural basis for $\mathbb{Q}S_3$ consists of the six permutations in lexicographical order: $\{abc, acb, bac, bca, cab, cba\}$. Another natural basis consists of the matrix units $\{S, E_{11}, E_{12}, E_{21}, E_{22}, A\}$ for the decomposition as a direct sum $\mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus \mathbb{Q}$ of simple ideals corresponding to the partitions $3 = 2 + 1 = 1 + 1 + 1$ which label the irreducible representations of S_3. Recall the matrix M whose columns express the matrix units as linear combinations of the permutations; see Example 2.6.8:

$$M = \frac{1}{6}\begin{bmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 & -1 \\ 1 & 2 & -2 & 0 & -2 & -1 \\ 1 & -2 & 2 & -2 & 0 & 1 \\ 1 & 0 & -2 & 2 & -2 & 1 \\ 1 & -2 & 0 & -2 & 2 & -1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Given a trilinear operation $\omega = x_1 abc + x_2 acb + x_3 bac + x_4 bca + x_5 cab + x_6 cba$ with coefficient vector $X = [x_1, x_2, x_3, x_4, x_5, x_6]^t$, one obtains its matrix form,

$$Y = \left[y_1, \begin{bmatrix} y_2 & y_3 \\ y_4 & y_5 \end{bmatrix}, y_6 \right],$$

by $Y = M^{-1}X$. Two operations are equivalent if and only if their matrix forms are row-equivalent; hence canonical representatives of the equivalence classes are the operations for which each component matrix is in row canonical form (see Section 2.5).

To find the simplest representative of each equivalence class, we first consider the $3^5 = 243$ operations whose coefficients in the permutation basis are $[1, x_2, \ldots, x_6]$ where $x_2, \ldots, x_6 \in \{1, 0, -1\}$; for each operation we compute its row canonical matrix form. We record those operations whose row canonical matrix forms equal to one of the triples in the second column of Table 5.1: this gives 20 of the 22 operations; the only ones missing are Jordan $q = \frac{1}{2}$ and anti-Jordan $q = \infty$. We next consider the $5^5 = 3125$ operations whose coefficients in the permutation basis are $[1, x_2, \ldots, x_6]$

where $x_2, \ldots, x_6 \in \{2, 1, 0, -1, -2\}$; this gives all 22 operations, and also produces alternative forms of the last two operations. In more than half of the cases, the simplified operations of Table 5.1 are more natural than the original operations appearing in [10], since they can be easily expressed in terms of the Lie bracket and the Jordan product.

5.3 Universal associative envelopes of n-ary non-associative algebras

In this section we show how the theory of non-commutative Gröbner bases can be applied to the construction of universal associative envelopes.

Given an associative algebra A, and a multilinear n-ary operation ω, we obtain a non-associative n-ary algebra A^ω.

Definition 5.3.1. A **universal associative envelope** of A^ω consists of a unital associative algebra $U(A^\omega)$ and a linear map $i: A^\omega \to U(A^\omega)$ satisfying

$$i(\omega(x_1, x_2, \ldots, x_n)) = \omega\left(i(x_1), i(x_2), \ldots, i(x_n)\right),$$

for all $x_1, \ldots, x_n \in A^\omega$, such that for any unital associative algebra $\mathbb{A}$ and linear map $j: A^\omega \to \mathbb{A}$ satisfying the same equation with j in place of i, there is a unique homomorphism of unital associative algebras $\psi: U(A^\omega) \to \mathbb{A}$ such that $\psi \circ i = j$.

Notation 5.3.2. *Let $B = \{e_1, e_2, \ldots, e_m\}$ be an ordered basis of A^ω, and let $\phi: B \to X = \{x_1, x_2, \ldots x_m\}$ be the bijection $\phi(e_i) = x_i$. We extend ϕ to a linear map, denoted by the same symbol, $\phi: A^\omega \to F\langle X\rangle$.*

Definition 5.3.3. Consider the following elements of $F\langle X\rangle$:

$$G_{i_1,\ldots,i_n} = \omega(x_{i_1}, \ldots, x_{i_n}) - \phi(\omega(e_{i_1}, \ldots, e_{i_n})), \qquad 1 \le i_1, \ldots, i_n \le m.$$

Let I be the ideal generated by the set of all $G_{i_1,\ldots,i_n}$, and define $U(A^\omega) = F\langle X\rangle / I$. We have the natural surjection $\pi: F\langle X\rangle \to U(A^\omega)$ sending f to $f + I$, and the composition $i = \pi \circ \phi: A^\omega \to U(A^\omega)$.

Lemma 5.3.4. *The algebra $U(A^\omega)$ and the map $i\colon A^\omega \to U(A^\omega)$ form the universal associative envelope of the non-associative n-ary algebra A^ω.*

Proof. Similar to Lemma 4.3.6. □

To obtain the elements $G_{i_1,\dots,i_n}$, we use the structure constants of A^ω. We then use Theorem 2.4.12 to compute a Gröbner basis of the ideal I, and Proposition 2.4.4 to determine a basis of $U(A^\omega)$.

Notation 5.3.5. *We write $\delta_{i,j}$ for the Kronecker delta, and $\widehat{\delta}_{i,j} = 1 - \delta_{i,j}$.*

5.4 Infinite dimensional envelopes

In this section, we consider the trilinear operations of "Lie type".

Definition 5.4.1. [3] Let $\mathbb{F}$ be a field and let $\alpha, \beta, \gamma \in \mathbb{F}$ be parameters. The **down-up algebra** $A(\alpha,\beta,\gamma)$ is the unital associative algebra with generators a, b and relations

$$b^2a = \alpha bab + \beta ab^2 + \gamma b, \qquad ba^2 = \alpha aba + \beta a^2b + \gamma a.$$

Theorem 5.4.2. [3, Theorem 3.1, Corollary 3.2] *The down-up algebra $A(\alpha,\beta,\gamma)$ has basis*

$$\mathfrak{B}_1 = \{a^i(ba)^jb^k \mid i,j,k \geq 0\},$$

and its Gelfand-Kirillov dimension is 3.

Lemma 5.4.3. [38, Lemma 2.2] *For any $c_1, c_2 \in \mathbb{F}$, the down-up algebra $A(\alpha,\beta,\gamma)$ has basis*

$$\mathfrak{B}_2 = \{a^i(ba + c_1ab + c_2)^jb^k \mid i,j,k \geq 0\}.$$

For the rest of this chapter, A is the associative triple system with the basis

$$e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

and the operation of ordinary matrix multiplication. We make the underlying vector space of A into a non-associative triple system A^ω in different ways corresponding to the trilinear operations ω of Table 5.1. Let $X = \{a, b\}$ with deglex $<$ where $a < b$, and define $\phi\colon A^\omega \to F\langle X\rangle$ by $\phi(e_1) = a$, $\phi(e_2) = b$. By definition the ideal I is generated by a set of these eight elements (see Definition):

$$G_{1,1,1} = \omega(a,a,a) - \phi(\omega(e_1,e_1,e_1)), \qquad G_{1,2,1} = \omega(a,b,a) - \phi(\omega(e_1,e_2,e_1)),$$
$$G_{2,2,1} = \omega(b,b,a) - \phi(\omega(e_2,e_2,e_1)), \qquad G_{1,2,2} = \omega(a,b,b) - \phi(\omega(e_1,e_2,e_2)),$$
$$G_{1,1,2} = \omega(a,a,b) - \phi(\omega(e_1,e_1,e_2)), \qquad G_{2,1,1} = \omega(b,a,a) - \phi(\omega(e_2,e_1,e_1)),$$
$$G_{2,1,2} = \omega(b,a,b) - \phi(\omega(e_2,e_1,e_2)), \qquad G_{2,2,2} = \omega(b,b,b) - \phi(\omega(e_2,e_2,e_2)).$$

5.4.1 The symmetric sum

The structure constants for A^ω are determined by

$$[e_1,e_1,e_1] = [e_2,e_2,e_2] = 0, \qquad [e_2,e_1,e_1] = 2e_1, \qquad [e_1,e_2,e_2] = 2e_2.$$

Lemma 5.4.4. *A basis for $U(A^\omega)$ is the set $\{\, a^i(ba)^j b^k \mid 0 \le i,k \le 2, j \ge 0 \,\}$.*

Proof. We have $U(A^\omega) = F\langle a,b\rangle/I$ where I is generated by $G = \{G_1, G_2, G_3, G_4\}$:

$$G_1 = b^3, \quad G_2 = b^2a + bab + ab^2 - b, \quad G_3 = ba^2 + aba + a^2b - a, \quad G_4 = a^3.$$

We show that the set G is a Gröbner basis of I. There are seven compositions:

$$S_1 = G_1a - bG_2, \quad S_2 = G_1ba - b^2G_2, \quad S_3 = G_1a^2 - b^2G_3, \quad S_4 = G_2a^2 - b^2G_4,$$
$$S_5 = G_2a - bG_3, \quad S_6 = G_3a^2 - baG_4, \quad S_7 = G_3a - bG_4.$$

We eliminate from $S_1, \dots, S_7$ all occurrences of the leading monomials of G_1, G_2, G_3, G_4; we write $\equiv$ to indicate congruence modulo G:

$$S_1 = -b^2ab - bab^2 + b^2 \equiv -\left(-bab - ab^2 + b\right)b - bab^2 + b^2 \equiv 0,$$
$$S_2 = -b^2\left(bab + ab^2 - b\right) \equiv -\left(-bab - ab^2 + b\right)b^2 \equiv 0,$$
$$S_3 = -b^2aba - b^2a^2b + b^2a = -b^2a\left(ba + ab - 1\right)$$
$$\equiv -\left(-bab - ab^2 + b\right)\left(ba + ab - 1\right)$$
$$\equiv ba\left(-bab - ab^2 + b\right) + babab - bab + a\left(-bab - ab^2 + b\right)b - ab^2$$

$$+ bab + ab^2 - b - bab + b = -ba^2b^2 + a\left(-bab - ab^2 + b\right)b$$
$$\equiv \left(aba + ab^2 - a\right)b^2 - abab^2 + ab^2 \equiv 0,$$
$$S_4 = baba^2 + ab^2a^2 - ba^2 \equiv ba\left(-aba - a^2b\right) + a\left(-bab - ab^2 + b\right)a$$
$$\equiv -\left(-aba - a^2b + a\right)ba + a\left(-bab - ab^2 + b\right)a \equiv 0,$$
$$S_5 = ab^2a - ba^2b \equiv -a(bab + ab^2 - b) + (aba + a^2b - a)b = 0,$$
$$S_6 = (aba + a^2b - a)a^2 \equiv -a^2(aba + a^2b - a) \equiv 0,$$
$$S_7 = aba^2 + a^2ba - a^2 \equiv a(-aba - a^2b + a) + a^2ba - a^2 \equiv 0.$$

Hence by Lemma 2.4.12, G is a Gröbner basis of I. From this and Proposition 2.4.4 we obtain a basis for the universal envelope $U(A^\omega)$: the cosets of the monomials which do not contain the leading monomial of any element of the Gröbner basis. Since a^3 and b^3 do not occur then any monomial basis has the form

$$\ldots a^{\epsilon_1}\, b^{\epsilon_2}\, a^{\epsilon_3}\, b^{\epsilon_4} \ldots a^{\epsilon_n}\, b^{\epsilon_{n+1}} \ldots$$

where $0 \leq \epsilon_1, \ldots, \epsilon_{n+1} \leq 2$. Since b^2a (resp. ba^2) does not occur, then b^2 can only occur at the end (resp. a^2 at the beginning). Thus any monomial basis has the form

$$a^{\epsilon_1}\, b^{\epsilon_2}\, a^{\epsilon_3}\, b^{\epsilon_4} \ldots a^{\epsilon_n}\, b^{\epsilon_{n+1}},$$

where $0 \leq \epsilon_1, \epsilon_{n+1} \leq 2$, $\epsilon_2 = \cdots = \epsilon_n = 0$ or 1. This completes the proof. □

Corollary 5.4.5. *In $U(A^\omega)$, we have the relations*

$$a^3 = b^3 = 0, \qquad b^2a = -bab - ab^2 + b, \qquad ba^2 = -aba - a^2b + a.$$

Hence $U(A^\omega)$ is the quotient of $A(-1,-1,1)$ by the ideal generated by a^3 and b^3.

Definition 5.4.6. Consider the anti-automorphism $\zeta\colon F\langle a,b\rangle \to F\langle a,b\rangle$ defined by $\zeta(a) = b$ and $\zeta(b) = a$. Since $\zeta(G_4) = G_1$, $\zeta(G_1) = G_4$, $\zeta(G_2) = G_3$, $\zeta(G_3) = G_2$, we see that ζ induces an anti-automorphism on $U(A^\omega)$, also denoted ζ.

A filtration $\{0\} \subseteq V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq \bigcup_n V^{(n)} = U(A^\omega)$ is defined by letting $V^{(n)}$ be the subspace with basis consisting of all $a^i(ba)^jb^k$ where $0 \leq i,k \leq 2$, $j \geq 0$, and $i + 2j + k \leq n$. The associated graded algebra is

$$\mathrm{gr}(U(A^\omega)) = \bigoplus_{i\geq 0} \mathcal{G}^i(U(A^\omega)), \qquad \mathcal{G}^i(U(A^\omega)) = V^{(i)}/V^{(i-1)}, \qquad V^{-1} = \{0\}.$$

Corollary 5.4.7. *The dimension of $\mathcal{G}^n(U(A^\omega))$ is 1 if $n = 0$, 2 if $n = 1$, 4 if $n = 2$ or $n \geq 3$ (odd), and 5 if $n \geq 4$ (even).*

Proof. For $n = 0$, there is one monomial: $(i, j, k) = (0, 0, 0)$. For $n = 1$, there are two: $(1, 0, 0)$, $(0, 0, 1)$. For $n = 2$, there are four: $(0, 1, 0)$, $(0, 0, 2)$, $(1, 0, 1)$, $(2, 0, 0)$. For $n \geq 3$ and odd, four: $(0, \frac{n-1}{2}, 1)$, $(1, \frac{n-1}{2}, 0)$, $(2, \frac{n-3}{2}, 1)$, $(1, \frac{n-3}{2}, 2)$. For $n \geq 4$ and even, five: $(0, \frac{n}{2}, 0)$, $(1, \frac{n-2}{2}, 1)$, $(2, \frac{n-2}{2}, 0)$, $(0, \frac{n-2}{2}, 2)$, $(2, \frac{n-4}{2}, 2)$. □

Corollary 5.4.8. *The Gelfand-Kirillov dimension of $U(A^\omega)$ is 1.*

Proof. We have

$$GK \dim U(A^\omega) = \limsup_{n\to\infty} \log_n \dim V^{(n)} = \lim_{n\to\infty} \frac{\ln \dim V^{(n)}}{\ln n} = 1,$$

since Corollary 5.4.7 implies that $\dim V^{(n)}$ is a polynomial of degree 1: $\dim V^{(n)} = \sum_{k=0}^{n} \dim \mathcal{G}^k(U(A^\omega))$. □

Corollary 5.4.9. *A $\mathbb{Z}$-grading of $U(A^\omega)$ is given by*

$$U(A^\omega) = U(A^\omega)_{-2} \oplus U(A^\omega)_{-1} \oplus U(A^\omega)_0 \oplus U(A^\omega)_1 \oplus U(A^\omega)_2,$$

where $U(A^\omega)_n = \operatorname{span}\{\, a^i(ba)^j b^k \mid j \geq 0,\ 0 \leq i, k \leq 2,\ i - k = n \,\}$.

Proof. Recall that $U(A^\omega)$ is the quotient $F\langle a, b\rangle / I$, where I is the two sided ideal with four generators: b^3, $b^2a + bab + ab^2 - b$, $ba^2 + aba + a^2b - a$, a^3 by the proof of Lemma 5.4.4. Clearly $F\langle a, b\rangle$ can be graded by defining $\deg(b) = -1$, $\deg(a) = 1$ and $\deg(ab) = \deg(a) + \deg(b)$. Then $U(A^\omega)$ inherits the grading; since the ideal I is homogenous: $\deg(b^3) = -3$, $\deg(a^3) = 3$, $\deg(b^2a) = \deg(bab) = \deg(ab^2) = \deg(b) = -1$ and $\deg(ba^2) = \deg(aba) = \deg(a^2b) = \deg(a) = 1$. □

Our next goal is to compute the structure constants of $U(A^\omega)$.

Definition 5.4.10. For $j, \ell, r, m \geq 0$, we define the following polynomials:

$$L^m_{j,\ell,r} = \sum_{t=0}^{j} (-1)^{j+t} \binom{j}{t} a^\ell (ba)^{j+m-t} b^r.$$

Lemma 5.4.11. *We have*

$$-L^{j-1}_{m+1,\ell,r} + L^{j-1}_{m,\ell,r} = L^{j}_{m,\ell,r} \ (j > 0).$$

Proof. Let $T = -L^{j-1}_{m+1,\ell,r} + L^{j-1}_{m,\ell,r}$, hence

$$T = \sum_{t=0}^{m+1} (-1)^{t+m} \binom{m+1}{t} a^{\ell}(ba)^{j+m-t} b^r + \sum_{t=0}^{m} (-1)^{t+m} \binom{m}{t} a^{\ell}(ba)^{j+m-t-1} b^r.$$

Changing the index in the second sum, we obtain

$$\begin{aligned} T = {} & (-1)^m a^{\ell}(ba)^{j+m} b^r + \sum_{t=1}^{m+1} (-1)^{t+m} \binom{m+1}{t} a^{\ell}(ba)^{j+m-t} b^r \\ & - \sum_{t=1}^{m} (-1)^{t+m} \binom{m}{t-1} a^{\ell}(ba)^{j+m-t} b^r - (-1)^{2m+1} a^{\ell}(ba)^{j-1} b^r. \end{aligned}$$

Using Pascal's formula,

$$\binom{m}{t} = \binom{m-1}{t} + \binom{m-1}{t-1}, \tag{5.1}$$

for the second sum we get

$$\begin{aligned} T = {} & (-1)^m a^{\ell}(ba)^{j+m} b^r + \sum_{t=1}^{m+1} (-1)^{t+m} \binom{m+1}{t} a^{\ell}(ba)^{j+m-t} b^r \\ & - \sum_{t=1}^{m} (-1)^{t+m} \binom{m+1}{t} a^{\ell}(ba)^{j+m-t} b^r + \sum_{t=1}^{m} (-1)^{t+m} \binom{m}{t} a^{\ell}(ba)^{j+m-t} b^r \\ & - (-1)^{2m+1} a^{\ell}(ba)^{j-1} b^r \\ = {} & \sum_{t=0}^{m} (-1)^{t+m} \binom{m}{t} a^{\ell}(ba)^{j+m-t} b^r = L^{j}_{m,\ell,r}. \end{aligned}$$

This completes the proof. □

Lemma 5.4.12. *If $j, m \geq 0$, then in $U(A^{\omega})$ we have*

$$(ba)^j \cdot a(ba)^m = -\widehat{\delta}_{j,0}\, L^{j-1}_{m,2,1} + L^{m}_{j,1,0}, \tag{5.2}$$

$$(ba)^j b \cdot (ba)^m = -\widehat{\delta}_{m,0}\, L^{m-1}_{j,1,2} + L^{j}_{m,0,1}, \tag{5.3}$$

$$(ba)^j b^2 \cdot a^2 = -L^{0}_{j+1,1,1} + (ba)^{j+1} - (ba)^{j+2} + a^2(ba)^j b^2. \tag{5.4}$$

Proof. For (5.2), we use induction on j. Clearly the claim is true for $j = 0$. To prove it for $j = 1$, we use induction on m. For $m = 0$, Lemma 5.4.5 implies

$$(ba)a = ba^2 = -aba - a^2 b + a = -L^{0}_{0,2,1} + L^{0}_{1,1,0}.$$

By the inductive hypothesis, we have

$$(ba)a(ba)^m = (ba)a(ba)^{m-1}ba = \left(-L^0_{m-1,2,1} + L^{m-1}_{1,1,0}\right) ba$$
$$= (-1)^m \sum_{t=0}^{m-1} (-1)^t \binom{m-1}{t} a^2(ba)^{m-t-1}b^2a - a(ba)^{m+1} + a(ba)^m.$$

Use the second relation of Lemma 5.4.5:

$$(ba)a(ba)^m$$
$$= (-1)^m \sum_{t=0}^{m-1} (-1)^{t+1} \binom{m-1}{t} a^2(ba)^{m-t}b + (-1)^m \sum_{t=0}^{m-1} (-1)^{t+1} \binom{m-1}{t} a^2(ba)^{m-t-1}ab^2$$
$$+ (-1)^m \sum_{t=0}^{m-1} (-1)^t \binom{m-1}{t} a^2(ba)^{m-t-1}b - a(ba)^{m+1} + a(ba)^m.$$

Use Pascal's formula in the first sum and change index in the third sum:

$$(ba)a(ba)^m$$
$$= (-1)^{m+1}\left(a^2(ba)^m b + \sum_{t=1}^{m-1} (-1)^t \binom{m}{t} a^2(ba)^{m-t}b\right) + (-1)^m \sum_{t=1}^{m-1} (-1)^t \binom{m-1}{t-1} a^2(ba)^{m-t}b$$
$$+ (-1)^m \sum_{t=0}^{m-1} (-1)^{t+1} \binom{m-1}{t} a^2(ba)^{m-t-1}ab^2 + (-1)^m \sum_{t=1}^{m} (-1)^{t-1} \binom{m-1}{t-1} a^2(ba)^{m-t}b$$
$$- a(ba)^{m+1} + a(ba)^m.$$

The first $m-1$ terms of the second and fourth sums cancel:

$$(ba)a(ba)^m = -L^0_{m,2,1} + (-1)^m \sum_{t=0}^{m-1} (-1)^{t+1} \binom{m-1}{t} a^2(ba)^{m-t-1}ab^2 + L^m_{1,1,0}.$$

To complete the proof for $j = 1$, it suffices to show that the second sum is 0. But this holds since $a^2(ba)^\ell a = 0$ for $\ell \geq 0$. To show this, we use induction on ℓ; the claim is true for $\ell = 0$ by the first equation of Lemma 5.4.5. For $\ell = 1$, Lemma 5.4.5 implies $a^2(ba)a = a^2(-aba - a^2b + a) = 0$. For $\ell \geq 1$, Lemma 5.4.5 and the inductive hypothesis give $a^2(ba)^\ell a = a^2(ba)^{\ell-1}(-aba - a^2b + a) = 0$.

We now consider the case $j \geq 1$. Using the inductive hypothesis and Lemma 5.4.5, we obtain

$$(ba)^{j+1}a(ba)^m = ba(ba)^j a(ba)^m = ba\left(- L^{j-1}_{m,2,1} + L^m_{j,1,0}\right)$$
$$= (-1)^j \sum_{t=0}^{j} (-1)^t \binom{j}{t} (ba)a(ba)^{j+m-t}$$

$$= (-1)^{m+1}\sum_{t=0}^{j}\sum_{s=0}^{j+m-t}(-1)^s\binom{j}{t}\binom{j+m-t}{s}a^2(ba)^{j+m-t-s}b$$
$$+ (-1)^{j+1}\sum_{t=0}^{j}(-1)^t\binom{j}{t}a(ba)^{j+m-t+1} + (-1)^j\sum_{t=0}^{j}(-1)^t\binom{j}{t}a(ba)^{j+m-t}.$$

Use Pascal's formula in the second sum and change index in the third sum:

$$(ba)^{j+1}a(ba)^m = (-1)^{m+1}\sum_{t=0}^{j}\sum_{s=0}^{j+m-t}(-1)^s\binom{j}{t}\binom{j+m-t}{s}a^2(ba)^{j+m-t-s}b$$
$$+ (-1)^{j+1}\sum_{t=0}^{j}(-1)^t\binom{j+1}{t}a(ba)^{j+m-t+1}$$
$$- (-1)^{j+1}\sum_{t=1}^{j}(-1)^t\binom{j}{t-1}a(ba)^{j+m-t+1}$$
$$- (-1)^{j}\sum_{t=1}^{j}(-1)^t\binom{j}{t-1}a(ba)^{j+m-t+1} + a(ba)^m.$$

The last two sums cancel and the previous expression simplifies to

$$(-1)^{m+1}\sum_{t=0}^{j}\sum_{s=0}^{j+m-t}(-1)^s\binom{j}{t}\binom{j+m-t}{s}a^2(ba)^{j+m-t-s}b + L^m_{j+1,1,0}.$$

To complete the proof, it suffices to show that

$$\sum_{t=0}^{j}\sum_{s=0}^{j+m-t}(-1)^s\binom{j}{t}\binom{j+m-t}{s}a^2(ba)^{j+m-t-s}b = \sum_{t=0}^{m}(-1)^t\binom{m}{t}a^2(ba)^{j+m-t}b.$$

Let C_r denote the coefficient of $a^2(ba)^{j+m-r}b$ in the left side:

$$C_r = \sum_{t=0}^{j}\sum_{s=0}^{j+m-t}\delta_{s+t,r}(-1)^s\binom{j}{t}\binom{j+m-t}{s} = \sum_{t=0}^{j}\binom{j}{t}(-1)^{r-t}\binom{j+m-t}{r-t}$$
$$= (-1)^r\sum_{t=0}^{j}(-1)^t\binom{j}{t}\binom{j+m-t}{r-t} = (-1)^r\binom{m}{r}.$$

For the last equality, see [36, Example 10.3]. This completes the proof of (5.2). The proof of (5.3) is obvious by using the anti-automorphism ζ:

$$(ba)^jb(ba)^m = \zeta((ba)^ma(ba)^j) = -\widehat{\delta}_{m,0}\,L^{m-1}_{j,1,2} + L^j_{m,0,1}.$$

For (5.4), we use Lemma 5.4.5 and get

$$(ba)^jb^2a^2 = (ba)^j(-bab - ab^2 + b)a = -(ba)^{j+2} - (ba)^jab^2a + (ba)^{j+1}. \tag{5.5}$$

Write $T = -(ba)^jab^2a$. Lemma 5.4.5 implies

$$T = -(ba)^ja(-bab - ab^2 + b) = (ba)^ja(ba)b + \delta_{j,0}\,a^2b^2 - (ba)^jab.$$

Using (5.2) and Lemma 5.4.11 we get

$$\begin{aligned}T &= \big(-\widehat{\delta}_{j,0}\, L^{j-1}_{1,2,1} + L^{1}_{j,1,0} + \widehat{\delta}_{j,0}\, L^{j-1}_{0,2,1} - L^{0}_{j,1,0}\big)b + \delta_{j,0}\, a^2 b^2 \\ &= \widehat{\delta}_{j,0}\, L^{j}_{0,2,1}\, b - L^{0}_{j+1,1,0}\, b + \delta_{j,0}\, a^2 b^2 = a^2(ba)^j b^2 - L^{0}_{j+1,1,1}.\end{aligned}$$

Using T in (5.5) completes the proof of (5.4). □

Theorem 5.4.13. *The structure constants of $U(A^\omega)$ are*

$$a^i(ba)^j b^k \cdot a^\ell (ba)^m b^n = a^i(ba)^{j+k+\ell+m} b^n,\ \text{if } (k,\ell) = (0,0) \text{ or } (k,\ell) = (1,1), \tag{5.6}$$

$$a^i(ba)^j \cdot a(ba)^m b^n = -\delta_{i,0}\widehat{\delta}_{n,2}\widehat{\delta}_{j,0}\, L^{j-1}_{m,2,n+1} + \widehat{\delta}_{i,2} L^{m}_{j,i+1,n}, \tag{5.7}$$

$$a^i(ba)^j b \cdot (ba)^m b^n = -\delta_{n,0}\widehat{\delta}_{i,2}\widehat{\delta}_{m,0}\, L^{m-1}_{j,i+1,2} + \widehat{\delta}_{n,2}\, L^{j}_{m,i,n+1}, \tag{5.8}$$

$$a^i(ba)^j b^2 \cdot a(ba)^m b^n = -\delta_{n,0}\widehat{\delta}_{i,2}\, L^{m}_{j,i+1,2} + \widehat{\delta}_{n,2}\, L^{j}_{m+1,i,n+1}, \tag{5.9}$$

$$a^i(ba)^j b \cdot a^2(ba)^m b^n = -\delta_{i,0}\widehat{\delta}_{n,2} L^{j}_{m,2,n+1} + \widehat{\delta}_{i,2}\, L^{m}_{j+1,i+1,n}, \tag{5.10}$$

$$a^i(ba)^j b^2 \cdot (ba)^m b^n = \delta_{m,0}\delta_{n,0}\, a^i(ba)^j b^2, \tag{5.11}$$

$$a^i(ba)^j \cdot a^2(ba)^m b^n = \delta_{i,0}\delta_{j,0}\, a^2(ba)^m b^n, \tag{5.12}$$

together with

$$\begin{aligned}&a^i(ba)^j b^2 \cdot a^2(ba)^m b^n \\ &= \sum_{k=0}^{j+1}(-1)^{k+j}\binom{j+1}{k}\big[-\delta_{n,0}\delta_{i,0}\widehat{\delta}_{m,0}\, L^{m-1}_{j-k+1,2,2} + \widehat{\delta}_{n,2}\widehat{\delta}_{i,2}\, L^{j-k+1}_{m,i+1,n+1}\big] \\ &\quad + a^i(ba)^{j+m+1}b^n - a^i(ba)^{j+m+2}b^n + \delta_{i,0}\delta_{m,0}\delta_{n,0}\, a^2(ba)^j b^2.\end{aligned} \tag{5.13}$$

Proof. For (5.6), use the associativity of $U(A^\omega)$. For (5.7) and (5.8) use Lemma 5.4.5 and equations (5.2) and (5.3) of Lemma 5.4.12. For (5.9), Lemma 5.4.5 implies

$$\begin{aligned}a^i(ba)^j b^2 a(ba)^m b^n &= a^i(ba)^j\left(-bab - ab^2 + b\right)(ba)^m b^n \\ &= -a^i(ba)^{j+1} b(ba)^m b^n - \delta_{m,0}\delta_{n,0} a^i(ba)^j ab^2 + a^i(ba)^j b(ba)^m b^n.\end{aligned}$$

Using (5.7) and (5.8) and Lemma 5.4.11 we obtain (5.9):

$$\begin{aligned}a^i(ba)^j b^2 a(ba)^m b^n = {}& \delta_{n,0}\widehat{\delta}_{i,2}\widehat{\delta}_{m,0} L^{m-1}_{j+1,i+1,2} - \widehat{\delta}_{n,2} L^{j+1}_{m,i,n+1} - \delta_{m,0}\delta_{n,0}\widehat{\delta}_{i,2} L^{0}_{j,i+1,2} \\ &- \delta_{n,0}\widehat{\delta}_{i,2}\widehat{\delta}_{m,0} L^{m-1}_{j,i+1,2} + \widehat{\delta}_{n,2} L^{j}_{m,i,n+1}\end{aligned}$$

$$= -\delta_{n,0}\widehat{\delta}_{i,2}\widehat{\delta}_{m,0}\, L^m_{j,i+1,2} + \widehat{\delta}_{n,2}\, L^j_{m+1,i,n+1} - \delta_{m,0}\widehat{\delta}_{n,0}\widehat{\delta}_{i,2}\, L^0_{j,i+1,2}.$$

For (5.10) use (5.9) and the anti-automorphism ζ. The proofs of (5.11) and (5.12) are obvious by Lemma 5.4.5. For (5.13), we use (5.4) of Lemma 5.4.12 and obtain

$$\begin{aligned} a^i(ba)^j b^2 a^2 (ba)^m b^n &= -a^i L^0_{j+1,1,1}(ba)^m b^n + a^i (ba)^{j+1}(ba)^m b^n - a^i (ba)^{j+2}(ba)^m b^n \\ &\quad + a^{i+2}(ba)^j b^2 (ba)^m b^n. \end{aligned}$$

Using Lemma 5.4.5, we get

$$\begin{aligned} a^i(ba)^j b^2 a^2 (ba)^m b^n &= -\widehat{\delta}_{i,2} L^0_{j+1,i+1,1}(ba)^m b^n + a^i (ba)^{j+1+m} b^n - a^i (ba)^{j+m+2} b^n \\ &\quad + \delta_{i,0}\delta_{m,0}\delta_{n,0}\, a^2 (ba)^j b^2. \end{aligned} \tag{5.14}$$

Write $A = L^0_{j+1,i+1,1}(ba)^m b^n$ and use (5.8) to obtain

$$\begin{aligned} A &= \sum_{k=0}^{j+1} (-1)^{k+j+1} \binom{j+1}{k} a^{i+1}(ba)^{j-k+1} b (ba)^m b^n \\ &= \sum_{k=0}^{j+1} (-1)^{k+j+1} \binom{j+1}{k} \left[-\delta_{n,0}\delta_{i,0}\widehat{\delta}_{m,0}\, L^{m-1}_{j-k+1,2,2} + \widehat{\delta}_{n,2}\widehat{\delta}_{i,2}\, L^{j-k+1}_{m,i+1,n+1} \right]. \end{aligned}$$

Using A in (5.14) completes the proof of (5.13). □

Our next goal is to describe the center $Z(U(A^\omega))$ of $U(A^\omega)$.

Notation 5.4.14. *We consider the following functions:*

	$\gamma_1(m)$	$\gamma_2(m)$	$\gamma_3(m)$	$\gamma_4(m)$
m even	$m+1$	-3	0	$-m+2$
m odd	$-(m-3)$	-1	-2	m

Definition 5.4.15. We consider the following elements:

$$\begin{aligned} \mathcal{Z}(m) &= \widehat{\delta}_{m,2} \sum_{j=1}^{m-2} (-1)^{j+1} \binom{m-1}{j-1} \big((ba)^j - a(ba)^{j-1} b \big) + \gamma_1(m)(ba)^{m-1} + \gamma_2(m)(ba)^m \\ &\quad + \gamma_3(m) a (ba)^{m-1} b + \gamma_4(m) a (ba)^{m-2} b + 3a^2 (ba)^{m-2} b^2. \end{aligned}$$

Theorem 5.4.16. *The center of $U(A^\omega)$ is the polynomial algebra in $\mathcal{Z}(m)$, $m \geq 2$:*

$$Z(U(A^\omega)) = \mathbb{F}[\, \mathcal{Z}(m) \mid m \geq 2 \,].$$

Proof. By Corollary 5.4.9 we know $Z(U(A^\omega))$ is graded. Thus if z is central and $z = z_{-2} + z_{-1} + z_0 + z_1 + z_2$ is its decomposition into homogenous components, then each z_i is itself central. We now show that $z \in U(A^\omega)_0$. First assume

$$0 \neq z_{-2} = \sum_{j \geq 0} s_j (ba)^j b^2 \in Z(U(A^\omega)), \quad s_j \in \mathbb{F}.$$

It follows that

$$0 = z_{-2}\, a - a\, z_{-2} = \sum_{j\geq 0} s_j (ba)^j b^2 a - \sum_{j \geq 0} s_j a (ba)^j b^2.$$

Using (5.9) of Theorem 5.4.13, we obtain

$$0 = \sum_{j\geq 0}\sum_{t=0}^{j} (-1)^{j+t+1} \binom{j}{t} s_j a (ba)^{j-t} b^2 - \sum_{j\geq 0} s_j (ba)^{j+1} b + \sum_{j \geq 0} s_j (ba)^j b - \sum_{j\geq 0} s_j a (ba)^j b^2.$$

Comparing the coefficients on both sides gives $s_j = 0$ for all j. Now assume

$$0 \neq z_2 = \sum_{j\geq 0} s_j a^2 (ba)^j \in Z(U(A^\omega)), \quad s_j \in \mathbb{F}.$$

It follows that

$$0 = b\, z_2 - z_2\, b = \sum_{j \geq 0} s_j b a^2 (ba)^j - \sum_{j\geq 0} s_j a^2 (ba)^j b.$$

Applying the anti-automorphism ζ to both sides gives

$$\sum_{j\geq 0} s_j (ba)^j b^2 a = \sum_{j \geq 0} s_j a (ba)^j b^2.$$

Hence $[\, \sum_{j\geq 0} s_j (ba)^j b^2,\, a\,] = 0$, contradicting the previous case. Assume

$$0 \neq z_1 = \sum_{j \geq 0} s_j\, a (ba)^j + \sum_{\ell \geq 0} t_\ell\, a^2 (ba)^\ell b \in Z(U(A^\omega)), \quad s_j, t_\ell \in \mathbb{F}.$$

It follows that

$$0 = b z_1 - z_1 b = \sum_{j\geq 0} s_j\, (ba)^{j+1} + \sum_{\ell \geq 0} t_\ell\, b a^2 (ba)^\ell b - \sum_{j \geq 0} s_j\, a (ba)^j b - \sum_{\ell\geq 0} t_\ell\, a^2 (ba)^\ell b^2.$$

Using (5.10) of Theorem 5.4.13 gives

$$0 = \sum_{j\geq 0} s_j\, (ba)^{j+1} - \sum_{\ell\geq 0}\sum_{t=0}^{\ell} (-1)^{\ell+t} \binom{\ell}{t} t_\ell\, a^2 (ba)^{\ell - t} b^2 - \sum_{\ell \geq 0} t_\ell\, a (ba)^{\ell+1} b + \sum_{\ell\geq 0} t_\ell\, a (ba)^\ell b$$
$$- \sum_{j \geq 0} s_j\, a (ba)^j b - \sum_{\ell\geq 0} t_\ell\, a^2 (ba)^\ell b^2.$$

Comparing the coefficients on both sides gives $s_j = 0 = t_\ell$ for all j, ℓ. Next assume

$$0 \neq z_{-1} = \sum_{j\geq 0} s_j\,(ba)^j b + \sum_{\ell\geq 0} t_\ell\, a(ba)^\ell b^2 \in Z(U(A^\omega)), \quad s_j, t_\ell \in \mathbb{F}.$$

It follows that

$$0 = az_{-1} - z_{-1}a = \sum_{j\geq 0} s_j\, a(ba)^j b + \sum_{\ell\geq 0} t_\ell a^2(ba)^\ell b^2 - \sum_{j\geq 0} s_j\,(ba)^j ba - \sum_{\ell\geq 0} t_\ell\, a(ba)^\ell b^2 a.$$

Applying the anti-automorphism ζ to both sides gives

$$0 = \sum_{j\geq 0} s_j\, a(ba)^j b + \sum_{\ell\geq 0} t_\ell\, a^2(ba)^\ell b^2 - \sum_{j\geq 0} s_j\,(ba)^{j+1} - \sum_{\ell\geq 0} t_\ell\, ba^2(ba)^\ell b.$$

Hence $[\sum_{j\geq 0} s_j\, a(ba)^j + \sum_{\ell\geq 0} t_\ell\, a^2(ba)^\ell b, b] = 0$, contradicting the previous case. Therefore $z \in U(A^\omega)_0$.

Now $U(A^\omega)_0$ is a commutative subalgebra: any element in $U(A^\omega)_0$ is a linear combination of $(ba)^j$, $a(ba)^k b$, $a^2(ba)^\ell b^2$ for $j, k, \ell \geq 0$, and these elements commute by Theorem 5.4.13. Using the anti-automorphism ζ, we see that an element in $U(A^\omega)_0$ commutes with a if and only if it commutes with b. So it suffices to determine the elements that commute with a. We choose

$$z = \sum_{j=0}^{m}\sum_{i=0}^{2} s_{i,j}\, a^i(ba)^j b^i \in U(A^\omega)_0, \quad m \geq 0,\ s_{i,j} \in \mathbb{F}.$$

By the relations of Lemma 5.4.5 we have

$$az = \sum_{j=0}^{m} s_{0,j}\, a(ba)^j + \sum_{j=0}^{m} s_{1,j}\, a^2(ba)^j b. \tag{5.15}$$

On the other hand,

$$za = \sum_{j=0}^{m} s_{0,j}\,(ba)^j a + \sum_{j=0}^{m} s_{1,j}\, a(ba)^{j+1} + \sum_{j=0}^{m} s_{2,j}\, a^2(ba)^j b^2 a.$$

Using (5.7) and (5.9) of Theorem 5.4.13 we obtain

$$\begin{aligned} za = {} & s_{0,0}\, a + \sum_{j=1}^{m} s_{0,j}\left(-a^2(ba)^{j-1}b + \sum_{t=0}^{j}(-1)^{j+t}\binom{j}{t} a(ba)^{j-t}\right) + \sum_{j=0}^{m} s_{1,j}\, a(ba)^{j+1} \\ & - \sum_{j=0}^{m} s_{2,j}\, a^2(ba)^{j+1}b + \sum_{j=0}^{m} s_{2,j}\, a^2(ba)^j b \\ = {} & s_{0,0}\, a - \sum_{j=1}^{m} s_{0,j} a^2(ba)^{j-1}b + A - \sum_{j=0}^{m} s_{2,j}\, a^2(ba)^{j+1}b + \sum_{j=0}^{m} s_{2,j}\, a^2(ba)^j b, \end{aligned} \tag{5.16}$$

where

$$A=\sum_{j=1}^{m}\sum_{t=0}^{j}(-1)^{j+t}\binom{j}{t}s_{0,j}\,a(ba)^{j-t}+\sum_{j=0}^{m}s_{1,j}\,a(ba)^{j+1}.$$

We rewrite A and obtain

$$\begin{aligned}A&=\sum_{r=0}^{m}\left(\sum_{j=1}^{m}(-1)^{r}\binom{j}{r}s_{0,j}\right)a(ba)^{r}+\sum_{j=1}^{m+1}s_{1,j-1}\,a(ba)^{j}\\&=\sum_{j=1}^{m}s_{0,j}\,a+\sum_{r=1}^{m}\left(\sum_{j=1}^{m}(-1)^{r}\binom{j}{r}s_{0,j}+s_{1,r-1}\right)a(ba)^{r}+s_{1,m}\,a(ba)^{m+1}.\end{aligned}\tag{5.17}$$

Using (5.17) in (5.16) gives

$$\begin{aligned}za=&\left(s_{0,0}+\sum_{j=1}^{m}s_{0,j}\right)a-\sum_{j=1}^{m}s_{0,j}\,a^{2}(ba)^{j-1}b+\sum_{r=1}^{m}\left(\sum_{j=1}^{m}s_{0,j}\,(-1)^{r}\binom{j}{r}+s_{1,r-1}\right)a(ba)^{r}\\&+s_{1,m}\,a(ba)^{m+1}-\sum_{j=0}^{m}s_{2,j}\,a^{2}(ba)^{j+1}b+\sum_{j=0}^{m}s_{2,j}\,a^{2}(ba)^{j}b.\end{aligned}$$

Changing index in the second and fourth sums and combining coefficients gives

$$\begin{aligned}za=&\left(s_{0,0}+\sum_{j=1}^{m}s_{0,j}\right)a+\left(-s_{0,1}+s_{2,0}\right)a^{2}b+\sum_{j=1}^{m-1}\left(-s_{0,j+1}-s_{2,j-1}+s_{2,j}\right)a^{2}(ba)^{j}b\\&+\sum_{r=1}^{m}\left(\sum_{j=1}^{m}s_{0,j}(-1)^{r}\binom{j}{r}+s_{1,r-1}\right)a(ba)^{r}+s_{1,m}\,a(ba)^{m+1}\\&+(-s_{2,m-1}+s_{2,m})\,a^{2}(ba)^{m}b-s_{2,m}\,a^{2}(ba)^{m+1}b.\end{aligned}$$

Comparing the coefficients in this expression with (5.15), we get this linear system:

$$\begin{cases}\sum_{j=1}^{m}s_{0,j}=0,\\-s_{0,1}+s_{2,0}-s_{1,0}=0,\\-s_{0,j+1}-s_{2,j-1}+s_{2,j}-s_{1,j}=0\quad(1\le j\le m-1),\\(-1)^{r}\sum_{j=1}^{m}\binom{j}{r}s_{0,j}+s_{1,r-1}-s_{0,r}=0\quad(1\le r\le m),\\s_{2,m}=s_{1,m}=s_{2,m-1}=0.\end{cases}\tag{$\mathcal{T}$}$$

For $m<2$, the only solution is trivial. For $m\ge 2$, a calculation (see Lemma B.0.3 of Appendix B) shows that $(\mathcal{T})$ has $m{-}1$ linearly independent solutions. For each m, we have the following solution:

$$s_{2,m-2}=1,\quad s_{2,j}=0\ (j\ne m-2),\qquad s_{0,j}=\frac{1}{3}(-1)^{j+1}\binom{m-1}{j-1}\ (1\le j\le m-2),$$

$$s_{0,m-1} = \begin{cases} (m+1)/3 & \text{if } m \text{ is even} \\ (3-m)/3 & \text{if } m \text{ is odd} \end{cases}, \qquad s_{0,m} = \begin{cases} -1 & \text{if } m \text{ is even} \\ -1/3 & \text{if } m \text{ is odd} \end{cases},$$

$$s_{1,i-1} = -s_{0,i} \ (1 \le i \le m-2), \qquad s_{1,m-2} = -s_{0,m-1} + 1,$$

$$s_{1,m-1} = \begin{cases} 0 & \text{if } m \text{ is even} \\ -2/3 & \text{if } m \text{ is odd} \end{cases}, \qquad s_{1,m} = 0.$$

Using these coefficients in z, and observing that any solution for $m-1$ is also a solution for m, we obtain a complete list of linearly independent solutions for $(\mathcal{T})$. □

5.4.2 The alternating sum

The structure constants for A^{ω} are 0, the set of ideal generators is empty, and hence $U(A^{\omega})$ is the free associative algebra on a and b. The Gelfand-Kirillov dimension of $U(A^{\omega})$ is ∞ (see [29]).

5.4.3 The cyclic sum

The results are identical to those for the symmetric sum, since the structure constants are

$$[e_2, e_1, e_1] = e_1, \qquad [e_1, e_2, e_2] = e_2, \qquad [e_1, e_1, e_1] = [e_2, e_2, e_2] = 0.$$

5.4.4 The Lie family, $q = \infty$

The structure constants for $A^{\omega_L^{\infty}}$ are determined by

$$[e_1, e_2, e_1] = 2e_1, \qquad [e_2, e_2, e_1] = -2e_2.$$

Lemma 5.4.17. *The universal associative envelope $U(A^{\omega_L^{\infty}})$ is isomorphic to the down-up algebra $A(2,-1,-2)$.*

Proof. We have $U\left(A^{\omega_L^{\infty}}\right) = F\langle a, b\rangle / I$, where I is the ideal generated by these two elements, which form a Gröbner basis: $b^2a - 2bab + ab^2 + 2b$, $ba^2 - 2aba + a^2b + 2a$. □

Remark 5.4.18. If we replace $\omega_L^\infty(a,b,c) = [a,[b,c]]$ by the equivalent operation $\omega'(a,b,c) = [[a,b],c]$ then we get the 2-dimensional simple Lie triple system $A^{\omega'}$ with relations $[e_1,e_2,e_1] = 2e_1$, $[e_1,e_2,e_2] = -2e_2$; the results for $U(A^{\omega'})$ are identical to those for ω_L^∞.

Recall that Benkart and Roby [3] showed that the down-up algebra $A(2,-1,-2)$ is isomorphic to the universal associative envelope $U(\mathfrak{sl}_2(\mathbb{C}))$ of the simple Lie algebra of 2×2 matrices of trace 0 with basis $\{h,e,f\}$ and relations $[e,f] = h$, $[h,e] = 2e$, and $[h,f] = -2f$ (see Example 2.8.3). In $U(\mathfrak{sl}_2(\mathbb{C}))$ we have $ef - fe = h$, $he - eh = 2e$, $hf - fh = -2f$.

Lemma 5.4.19. *If $\ell,k,m,j \geq 0$ then in $U(\mathfrak{sl}_2(\mathbb{C}))$ we have*

$$e^\ell \cdot h^k = \sum_{q=0}^{k} (-1)^q 2^q \binom{k}{q} \ell^q h^{k-q} e^\ell, \tag{5.18}$$

$$h^k \cdot f^m = \sum_{q=0}^{k} (-1)^q 2^q \binom{k}{q} m^q f^m h^{k-q}, \tag{5.19}$$

$$e^\ell \cdot f^j = \ell! j! \sum_{r=0}^{\min(j,\ell)} \frac{f^{j-r}}{(j-r)!(\ell-r)!} \binom{h-j-\ell+2r}{r} e^{\ell-r}. \tag{5.20}$$

Proof. For (5.18), we use induction on k. The claim is clear for $k = 0$. To prove the claim for $k = 1$, we use induction on ℓ. For $\ell = 1$, the claim holds since $eh = he - 2e$. Assume that $\ell \geq 1$. By the inductive hypothesis we have

$$e^{\ell+1} h = ee^\ell h = \left(ehe^\ell - 2\ell e^{\ell+1}\right) = he^{\ell+1} - 2e^{\ell+1} - 2\ell e^{\ell+1},$$

so the claim is true for $k = 1$. For $k \geq 1$, the inductive hypothesis implies

$$\begin{aligned} e^\ell h^{k+1} &= \sum_{q=0}^{k} (-1)^q 2^q \binom{k}{q} \ell^q h^{k-q} e^\ell h = \sum_{q=0}^{k} (-1)^q 2^q \binom{k}{q} \ell^q h^{k-q} \left(he^\ell - 2\ell e^\ell\right) \\ &= \sum_{q=0}^{k} (-1)^q 2^q \binom{k}{q} \ell^q h^{k-q+1} e^\ell + \sum_{q=1}^{k+1} (-1)^q 2^q \binom{k}{q-1} \ell^q h^{k-q+1} e^\ell \\ &= h^{k+1} e^\ell + \sum_{q=1}^{k} (-1)^q 2^q \left[\binom{k}{q} + \binom{k}{q-1}\right] \ell^q h^{k+1-q} e^\ell + (-1)^{k+1} 2^{k+1} \ell^{k+1} e^\ell. \end{aligned}$$

Using Pascal's formula for binomial coefficients we obtain

$$e^\ell h^{k+1} = h^{k+1} e^\ell + \sum_{q=1}^{k} (-1)^q 2^q \binom{k+1}{q} \ell^q h^{k+1-q} e^\ell + (-1)^{k+1} 2^{k+1} \ell^{k+1} e^\ell.$$

This proves (5.18), and (5.19) is similar; for (5.20), see Humphreys [24, Lemma 26.2]. □

Theorem 5.4.20. *The structure constants of* $U(\mathfrak{sl}_2(\mathbb{C}))$ *are*

$$(f^i h^j e^k) \cdot (f^\ell h^m e^n) = k!\,\ell! \sum_{r=0}^{\min(\ell,k)} \sum_{q=0}^{j} \sum_{i=0}^{m} (-1)^{q+i} 2^{q+i} \binom{j}{q} \binom{m}{i} \frac{(\ell-r)^q}{(\ell-r)!} \frac{(k-r)^i}{(k-r)!} \times$$
$$f^{\ell-r+i} h^{j-q+m-i} \binom{h-k-\ell+2r}{r} e^{k-r+n}.$$

Proof. Using the third equation of Lemma 5.4.19, we get

$$\begin{aligned}(f^i h^j e^k)(f^\ell h^m e^n) &= f^i h^j k!\,\ell! \left(\sum_{r=0}^{\min(\ell,k)} \frac{f^{\ell-r}}{(\ell-r)!} \binom{h-k-\ell+2r}{r} \frac{e^{k-r}}{(k-r)!} \right) h^m e^n \\ &= k!\,\ell!\, f^i \left(\sum_{r=0}^{\min(\ell,k)} \frac{h^j f^{\ell-r}}{(\ell-r)!} \binom{h-k-\ell+2r}{r} \frac{e^{k-r} h^m}{(k-r)!} \right) e^n.\end{aligned}$$

Using the first and the second equations of Lemma 5.4.19 completes the proof. □

Remark 5.4.21. Using Theorem 5.4.20 and the homomorphism $\psi\colon \mathfrak{sl}_2(\mathbb{C}) \to A(2,-1,1)$ (see Example 2.8.3), we obtain the structure constants of $U\big(A^{\omega_L^\infty}\big)$ with respect to the basis $\mathfrak{B}_2$ with $(c_1, c_2) = (-1, 0)$ (see Lemma 5.4.3).

5.4.5 The Lie family, $q = \frac{1}{2}$

The structure constants for $A^{\omega_L^{1/2}}$ are zero.

Lemma 5.4.22. *The universal associative envelope* $U(A^{\omega_L^{1/2}})$ *is isomorphic to the down-up algebra* $A(0,1,0)$.

Proof. We have $U(A^{\omega_L^{1/2}}) = F\langle a, b\rangle / I$, where I is the ideal generated by these two elements, which form a Gröbner basis: $b^2 a - a b^2$, $b a^2 - a^2 b$. □

Remark 5.4.23. If we replace $\omega_L^{1/2}(a,b,c) = [a, b \circ c]$ by the equivalent operation $\omega''(a,b,c) = [a \circ b, c]$ then we get an anti-Lie triple system $A^{\omega''}$, and the results for $U(A^{\omega''})$ are the same as those for $A^{\omega_L^{1/2}}$.

Lemma 5.4.24. *If $i, j \geq 0$, then in $U(A^{\omega_L^{1/2}})$ we have*

$$b^i \cdot a^j = \begin{cases} a^{j-1}(ba)b^{i-1} & \text{if } i,j \text{ are both odd,} \\ a^j b^i & \text{otherwise.} \end{cases} \tag{5.21}$$

$$(ba)^j \cdot a^i = \begin{cases} a^{i+1}(ba)^{j-1}b & \text{if } i \text{ is odd, } j \neq 0, \\ a^i(ba)^j & \text{otherwise.} \end{cases} \tag{5.22}$$

$$b^i \cdot (ba)^j = \begin{cases} a(ba)^{j-1}b^{i+1} & \text{if } i \text{ is odd, } j \neq 0, \\ (ba)^j b^i & \text{otherwise.} \end{cases} \tag{5.23}$$

$$(ba)^i \cdot a(ba)^j = \begin{cases} a^{2j+2}(ba)^{i-j-1}b^{2j+1} & \text{if } i > j, \\ a^{2i+1}(ba)^{j-i}b^{2i} & \text{otherwise.} \end{cases} \tag{5.24}$$

Proof. For (5.21), we use induction on i. The claim is clear for $i = 0$. To prove the claim for $i = 1$ we use induction on j. For $j = 0$ or 1, the claim is obvious. For $j = 2$ the claim holds since $ba^2 = a^2b$. We now prove the claim for $j \geq 2$. By the inductive hypothesis, we have

$$ba^j = \begin{cases} a^{j-2}baa & \text{if } j-1 \text{ is odd} \\ a^{j-1}ba & \text{otherwise} \end{cases} = \begin{cases} a^{j-1}ba & \text{if } j \text{ is odd} \\ a^{j-2}baa & \text{otherwise} \end{cases} = \begin{cases} a^{j-1}ba & \text{if } j \text{ is odd} \\ a^j b & \text{otherwise} \end{cases}.$$

So the claim holds for $i = 1$. We now consider the case $i \geq 1$. If $i + 1$ is odd, then the inductive hypothesis implies

$$b^{i+1}a^j = bb^i a^j = ba^j b^i = \begin{cases} a^{j-1}(ba)b^i & \text{if } j \text{ is odd} \\ a^j b^{i+1} & \text{otherwise} \end{cases}.$$

If $i + 1$ is even, then the inductive hypothesis gives

$$b^{i+1}a^j = bb^i a^j = \begin{cases} ba^{j-1}(ba)b^{i-1} & \text{if } j \text{ is odd} \\ ba^j b^i & \text{otherwise} \end{cases} = \begin{cases} a^{j-1}b(ba)b^{i-1} & \text{if } j \text{ is odd} \\ a^j b^{i+1} & \text{otherwise} \end{cases}.$$

Using $b^2a = ab^2$ we get $b^{i+1}a^j = a^{j-1}ab^2b^{i-1}$ if j is odd, $a^j b^{i+1}$ otherwise; in both cases the result is $a^j b^{i+1}$. This completes the proof of (5.21).

For (5.22) we use induction on i. The claim is obvious for $i = 0$. To prove the claim for $i = 1$, we use induction on j. For $j = 0$, the claim is obvious. For $j = 1$, the claim holds by using $ba^2 = a^2b$. We now consider the case of general j. By the inductive hypothesis, we have $(ba)^j a = ba(ba)^{j-1}a = baa^2(ba)^{j-2}b = a^2ba(ba)^{j-2} = a^2(ba)^{j-1}b$. So the claim is true for $i = 1$. We now consider the case $i \geq 1$. The claim is obvious for $j = 0$, so we assume that $j \neq 0$. If $i+1$ is odd, then the inductive hypothesis implies $(ba)^j a^{i+1} = a^i(ba)^j a = a^{i+2}(ba)^{j-1}b$. If $i+1$ is even, then $(ba)^j a^{i+1} = a^{i+1}(ba)^{j-1}ba = a^{i+1}(ba)^j$. This completes the proof of (5.22). The proof of (5.23) follows by using the anti-automorphism η (see Remark 2.8.4) of the down-up algebra $A(0,1,0)$.

For (5.24), we use induction on i. The claim is obvious for $i = 0$. To prove the claim for $i = 1$, we use induction on j. The claim holds for $j = 0$ by using $(ba)a = a^2b$. For $j \geq 1$, $ba^2 = a^2b$ and (5.23) imply $baa(ba)^j = a^2b(ba)^j = a^3(ba)^{j-1}b^2$. So the claim is true for $i = 1$. We now consider the case of $i > 1$. By the inductive hypothesis, we have

$$(ba)^{i+1}a(ba)^j = ba(ba)^i a(ba)^j = \begin{cases} baa^{2j+2}(ba)^{i-j-1}b^{2j+1} & \text{if } i > j \\ baa^{2i+1}(ba)^{j-i}b^{2i} & \text{if } i \leq j \end{cases}.$$

Therefore

$$(ba)^{i+1}a(ba)^j = \begin{cases} ba^{2j+3}(ba)^{i-j-1}b^{2j+1} & \text{if } i > j \\ ba^{2i+2}(ba)^{j-i}b^{2i} & \text{if } i \leq j \end{cases}.$$

Two cases need to be considered. (I) If $i+1 > j$, then $i = j$ or $i > j$. Hence,

$$(ba)^{i+1}a(ba)^j = \begin{cases} ba^{2j+3}(ba)^{i-j-1}b^{2j+1} & \text{if } i > j \\ ba^{2i+2}b^{2i} & \text{if } i = j \end{cases}.$$

Using (5.21) we obtain

$$(ba)^{i+1}a(ba)^j = \begin{cases} a^{2j}ba^3(ba)^{i-j-1}b^{2j+1} = a^{2j+2}(ba)^{i-j}b^{2j+1} & \text{if } i > j \\ a^{2i+2}b^{2i+1} & \text{if } i = j \end{cases}.$$

Therefore, $(ba)^{i+1}a(ba)^j = a^{2j+2}(ba)^{i-j}b^{2j+1}$. (II) If $i+1 \leq j$, then $i < j$. Hence $(ba)^{i+1}a(ba)^j = ba^{2i+2}(ba)^{j-i}b^{2i}$. Using (5.21) and (5.23) we obtain $(ba)^{i+1}a(ba)^j =$

$a^{2i+2}b(ba)^{j-i}b^{2i} = a^{2i+3}(ba)^{j-i-1}b^{2(i+1)}$. Combining the results of (I) and (II) completes the proof of (5.24). □

Theorem 5.4.25. *The structure constants of* $U(A^{\omega_L^{1/2}})$ *are*

$$a^i(ba)^jb^k \cdot a^\ell(ba)^mb^n$$
$$= \begin{cases} a^{i+\ell-1}(ba)^{j+m+1}b^{k-1+n} & \text{if } k, \ell \text{ are both odd,} \\ a^{i+\ell}(ba)^{j+m}b^{k+n} & \text{if } k, \ell \text{ are both even,} \\ \chi_{j,m}\, a^{2m+i+\ell+1}(ba)^{j-m-1}b^{2m+k+n+1} & \\ \quad + (1-\chi_{j,m})\, a^{2j+i+\ell}(ba)^{m-j}b^{2j+k+n} & \text{if } k \text{ is even, } \ell \text{ is odd,} \\ \chi_{j,m-1}\, a^{2m+i+\ell}(ba)^{j-m}b^{2m+k+n} & \\ \quad + (1-\chi_{j,m-1})\, a^{2j+i+\ell+1}(ba)^{m-j-1}b^{2j+k+n+1} & \text{if } k \text{ is odd, } \ell \text{ is even,} \end{cases}$$

where $\chi_{\ell,t} = 1$ *if* $\ell > t$ *and 0 otherwise.*

Proof. We use equations (5.21), (5.22) and (5.23). If k and ℓ are odd, then

$$a^i(ba)^jb^k \cdot a^\ell(ba)^mb^n = a^i(ba)^ja^{\ell-1}(ba)b^{k-1}(ba)^mb^n = a^{i+\ell-1}(ba)^{j+1+m}b^{k-1+n}.$$

If k and ℓ are even, then

$$a^i(ba)^jb^k \cdot a^\ell(ba)^mb^n = a^i(ba)^ja^\ell b^k(ba)^mb^n = a^{i+\ell}(ba)^{j+m}b^{k+n}.$$

If k is even and ℓ is odd, then

$$\begin{aligned} a^i(ba)^jb^k \cdot a^\ell(ba)^mb^n &= a^i(ba)^ja^\ell b^k(ba)^mb^n \\ &= \widehat{\delta}_{j,0}\, a^{i+\ell+1}(ba)^{j-1}(b^{k+1}(ba)^m)b^n + \delta_{j,0}\, a^{i+\ell}b^k(ba)^mb^n \\ &= \widehat{\delta}_{j,0}\left[\widehat{\delta}_{m,0}\, a^{i+\ell+1}(ba)^{j-1}a(ba)^{m-1}b^{k+2+n} + \delta_{m,0}\, a^{i+\ell+1}(ba)^{j-1}b^{k+n+1}\right] \\ &\quad + \delta_{j,0}\, a^{i+\ell}(ba)^mb^{k+n}. \end{aligned}$$

Using (5.24) completes the proof. If k is odd and ℓ is even, then

$$\begin{aligned} a^i(ba)^jb^ka^\ell(ba)^mb^n &= a^i(ba)^ja^\ell b^k(ba)^mb^n = a^{i+\ell}(ba)^jb^k(ba)^mb^n \\ &= \widehat{\delta}_{m,0}\, a^{i+\ell}(ba)^ja(ba)^{m-1}b^{k+n+1} + \delta_{m,0}\, a^{i+\ell}(ba)^jb^{k+n}. \end{aligned}$$

Using (5.24) again completes the proof. □

5.4.6 The anti-Jordan family, $q = \infty$

The structure constants for $A^{\omega_{AJ}^{\infty}}$ are

$$[e_1, e_1, e_2] = -2e_1, \quad [e_2, e_1, e_1] = 2e_1, \quad [e_1, e_2, e_2] = 2e_2, \quad [e_2, e_2, e_1] = -2e_2.$$

Proposition 5.4.26. *The universal associative envelope $U(A^{\omega_{AJ}^{\infty}})$ is isomorphic to the down-up algebra $A(2, -1, -2)$, so we have $U(A^{\omega_{AJ}^{\infty}}) \cong U(A^{\omega_{L}^{\infty}})$.*

Proof. Similar to the proof of Lemma 5.4.17. □

5.4.7 The anti-Jordan family, $q = \frac{1}{2}$

The structure constants for $A^{\omega_{AJ}^{1/2}}$ are zero.

Proposition 5.4.27. *The universal associative envelope $U\big(A^{\omega_{AJ}^{1/2}}\big)$ is isomorphic to the down-up algebra $A(0, 1, 0)$, so we have $U\big(A^{\omega_{AJ}^{1/2}}\big) \cong U\big(A^{\omega_{L}^{1/2}}\big)$.*

Proof. Similar to the proof of Lemma 5.4.22. □

5.5 Finite dimensional envelopes

In this section, we consider the trilinear operations of "Jordan type".

5.5.1 The Jordan family, $q = \infty$

The structure constants for $A^{\omega_{J}^{\infty}}$ are

$$[e_1, e_2, e_1] = 2e_1, \qquad [e_2, e_1, e_2] = 2e_2.$$

Theorem 5.5.1. *A basis for $U(A^{\omega_{J}^{\infty}})$ consists of the elements 1, a, b, ab, ba. The structure constants are $a \cdot b = ab$, $a \cdot ba = a$, $b \cdot a = ba$, $b \cdot ab = b$, $ab \cdot a = a$, $ab \cdot ab = ab$, $ba \cdot b = b$, $ba \cdot ba = ba$. The Wedderburn decomposition is $U(A^{\omega_{J}^{\infty}}) = \mathbb{Q} \oplus M_{2\times 2}$. The only finite dimensional irreducible representations are the trivial 1-dimensional representation and the natural 2-dimensional representation.*

Proof. We have $U = U(A^{\omega_{\mathcal{J}}^{\infty}}) = F\langle a,b\rangle / I$ where I is generated by b^3, $b^2a + ab^2$, $bab - b$, $ba^2 + a^2b$, $aba - a$, a^3. We compute a Gröbner basis of I. There are four compositions with normal forms ab^2, a^2b, b^2, a^2. Including these with the original generators and self-reducing the resulting set produces the four generators $bab - b$, $aba - a$, b^2, a^2. All compositions of these elements reduce to 0, and so we have a Gröbner basis. A basis for the quotient algebra consists of the cosets of the monomials which are not divisible by the leading monomial of any element of the Gröbner basis. This gives the stated basis for U. It follows that U satisfies $a^2 = 0$, $b^2 = 0$, $aba = a$, $bab = b$ and these give the stated structure constants.

To decompose U we follow the algorithms in Section 2.7. Using Corollary 2.7.5 we verify that the radical is zero, and hence U is semisimple. By Corollary 2.7.6 the center $Z(U)$ has dimension 2, basis $z_1 = 1$, $z_2 = ab + ba$, and structure constants $z_1 \cdot z_1 = z_1$, $z_1 \cdot z_2 = z_2$, $z_2 \cdot z_2 = z_2$. Since $z_2^2 = z_2$, the minimal polynomial of z_2 is $t^2 - t$. Thus $Z(U)$ splits in two 1-dimensional ideals with bases $z_2 - z_1$ and z_2. Scaling these basis elements to obtain idempotents gives $e_1 = -z_2 + z_1$, $e_2 = z_2$. The corresponding elements in U are $e_1 = -ab - ba + 1$, $e_2 = ab + ba$. The ideals in U generated by e_1 and e_2 have dimensions 1 and 4 respectively, and this gives the Wedderburn decomposition. The 4-dimensional ideal generated by e_2 has basis $\alpha = a$, $\beta = b$, $\gamma = ab$, $\delta = ba$. Each of the left ideals generated by these elements has dimension 2. In particular, α generates a 2-dimensional left ideal with basis $U_1 = a$, $U_2 = ba$. We identify U_1, U_2 with $(1,0)$, $(0,1) \in \mathbb{Q}^2$ and solve for the matrix units; we obtain $E_{11} = ab$, $E_{12} = a$, $E_{21} = b$, $E_{22} = ba$. We now have two bases for U, and so we can express all the elements of U in terms of matrix units:

$$1 \to [1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; a \to [0], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \; b \to [0], \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \; ab \to [0], \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \; ba \to [0], \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We obtain the 1-dimensional trivial representation and the 2-dimensional natural representation. □

5.5.2 The Jordan family, $q = 0$

The structure constants for $A^{\omega_J^0}$ are

$$[e_1, e_2, e_1] = [e_2, e_1, e_1] = e_1, \qquad [e_2, e_1, e_2] = [e_1, e_2, e_2] = e_2.$$

Theorem 5.5.2. *A basis for $U(A^{\omega_J^0})$ consists of the elements 1, a, b, a^2, ab, ba, b^2, aba, ab^2. The structure constants are $a \cdot a = a^2$, $a \cdot b = ab$, $a \cdot ba = aba$, $a \cdot b^2 = ab^2$, $b \cdot a = ba$, $b \cdot b = b^2$, $b \cdot a^2 = a - aba$, $b \cdot ab = b - ab^2$, $b \cdot aba = ba$, $b \cdot ab^2 = b^2$, $ab \cdot a = aba$, $ab \cdot b = ab^2$, $ab \cdot a^2 = a^2$, $ab \cdot ab = ab$, $ab \cdot aba = aba$, $ab \cdot ab^2 = ab^2$, $ba \cdot a = a - aba$, $ba \cdot b = b - ab^2$, $ba \cdot ba = ba$, $ba \cdot b^2 = b^2$, $aba \cdot a = a^2$, $aba \cdot b = ab$, $aba \cdot ba = aba$, $aba \cdot b^2 = ab^2$. The Wedderburn decomposition is $U(A^{\omega_J^0}) = \mathfrak{R} \oplus \mathbb{Q} \oplus M_{2\times2}$ where $\mathfrak{R}$ is the radical of dimension 4. There are only two finite dimensional irreducible representations.*

Proof. We have $U = U(A^{\omega_J^0}) = F\langle a, b\rangle / I$ where I is generated by b^3, b^2a, $bab + ab^2 - b$, $ba^2 + aba - a$, a^2b, a^3. This set is a Gröbner basis for I. Hence U is finite dimensional and has the stated basis. The following relations hold in U: $b^3 = 0$, $b^2a = 0$, $bab = -ab^2 + b$, $ba^2 = -aba + a$, $a^2b = 0$, $a^3 = 0$. These imply the stated structure constants. Using Corollary 2.7.5, a basis of the radical $\mathfrak{R} = \mathfrak{R}(U)$ consists of the elements $\xi_1 = a - aba$, $\xi_2 = a^2$, $\xi_3 = b^2$, $\xi_4 = ab^2$. Hence we have these relations in $Q = U/\mathfrak{R}$: $a = aba$, $a^2 = b^2 = ab^2 = 0$. The semisimple quotient Q has dimension 5, and a basis consists of the cosets of $\eta_1 = 1$, $\eta_2 = b$, $\eta_3 = ab$, $\eta_4 = ba$, $\eta_5 = aba$. The center $Z(Q)$ has dimension 2, basis $z_1 = \eta_1$, $z_2 = \eta_3 + \eta_4$, and structure constants $z_1 \cdot z_1 = z_1$, $z_1 \cdot z_2 = z_2 \cdot z_1 = z_2$, $z_2 \cdot z_2 = z_2$. Since $z_2^2 = z_2$, the minimal polynomial of z_2 is $t^2 - t$. Thus $Z(Q) = J \oplus K$ where $J = \langle z_2 - z_1\rangle$ and $K = \langle z_2\rangle$ and both ideals are 1-dimensional. Scaling the basis elements to obtain idempotents gives $e_1 = z_1 - z_2$, $e_2 = z_2$. The corresponding elements in Q are $e_1 = \eta_1 - \eta_3 - \eta_4$, $e_2 = \eta_3 + \eta_4$. The ideals in Q generated by e_1 and e_2 have dimensions 1 and 4 respectively, hence $Q \simeq \mathbb{Q} \oplus M_{2\times2}$.

The 4-dimensional ideal generated by e_2 has basis: η_2, η_3, η_4, η_5. The left ideal generated by η_2, η_3, η_4 or η_5 has dimension 2. In particular, η_2 generates a 2-dimensional left ideal with basis $U_1 = \eta_2$ and $U_2 = \eta_3$. Identify U_1, U_2 with $(1,0), (0,1) \in \mathbb{Q}^2$, and solve for the matrix units and obtain $E_{11} = \eta_4$, $E_{12} = \eta_2$, $E_{21} = \eta_5$, $E_{22} = \eta_3$.

We now have two basis for Q the old basis is: $\eta_1,\ \eta_2,\ \eta_3, \eta_4,\ \eta_5$ and the new basis is: $e_1,\ E_{11},\ E_{12},\ E_{21},\ E_{22}$. Using the matrix of the change the basis, we can express the basis elements of Q in terms of the matrix units. Since the other basis elements of $U(A^{\omega_J^0})$ are congruent modulo $\mathfrak{R}$ to elements of Q, then we can express the nine basis elements in terms of matrix units:

$$\eta_1 \to [1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \eta_2 \to [0], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \eta_3 \to [0], \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \eta_4 \to [0], \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \eta_5 \to [0], \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We obtain the 1-dimensional trivial representation and the 2-dimensional natural representation. □

5.5.3 The Jordan family, $q = \frac{1}{2}$

The structure constants for $A^{\omega_J^{1/2}}$ are

$$[e_1, e_1, e_2] = [e_1, e_2, e_1] = [e_2, e_1, e_1] = 2e_1,$$
$$[e_2, e_2, e_1] = [e_2, e_1, e_2] = [e_1, e_2, e_2] = 2e_2.$$

Theorem 5.5.3. *We have the isomorphism* $U(A^{\omega_J^{1/2}}) \cong U(A^{\omega_J^{\infty}})$.

Proof. We have $U = U(A^{\omega_J^{1/2}}) = F\langle a, b\rangle / I$ where I is generated by b^3, $b^2a + \frac{1}{2}bab - \frac{1}{2}b$, $b^2a + 2bab + 3ab^2 - 2b$, $ba^2 + \frac{2}{3}aba + \frac{1}{3}a^2b - \frac{2}{3}a$, $aba + 2a^2b - a$, a^3. The first iteration of the Gröbner basis algorithm produces the seven compositions $bab^2 - 2b^2$, $bab^2 - \frac{1}{2}b^2$, $bab - b$, $bab + 2ab^2 - b$, $bab + \frac{3}{2}ab^2 - b$, a^2b, a^2. Including these with the original generators, and self-reducing the resulting set, produces the same ideal generators as for $q = \infty$; hence the two quotient algebras are isomorphic. □

5.5.4 The Jordan family, $q = 1$

The structure constants for $A^{\omega_J^1}$ are

$$[e_1, e_1, e_2] = [e_1, e_2, e_1] = e_1, \qquad [e_2, e_2, e_1] = [e_2, e_1, e_2] = e_2.$$

Theorem 5.5.4. *A basis for* $U(A^{\omega_J^1})$ *consists of the elements* 1, a, b, a^2, ab, ba, b^2, a^2b, bab. *The structure constants of* $U(A^{\omega_J^1})$ *are* $a \cdot a = a^2$, $a \cdot b = ab$, $a \cdot ab = a^2b$,

$a \cdot ba = a - a^2b,\ a \cdot bab = ab,\ b \cdot a = ba,\ b \cdot b = b^2,\ b \cdot ab = bab,\ b \cdot ba = b - bab,\ b \cdot bab = b^2,$ $a^2 \cdot b = a^2b,\ a^2 \cdot ba = a^2,\ a^2 \cdot bab = a^2b,\ ab \cdot a = a - a^2b,\ ab \cdot ab = ab,\ ba \cdot b = bab,\ ba \cdot ba = ba,$ $ba \cdot bab = bab,\ b^2 \cdot a = b - bab,\ b^2 \cdot ab = b^2,\ a^2b \cdot a = a^2,\ a^2b \cdot ab = a^2b,\ bab \cdot a = ba,\ bab \cdot ab = bab.$ *The Wedderburn decomposition is* $U(A^{\omega_J^1}) = \mathfrak{R} \oplus \mathbb{Q} \oplus M_{2\times 2}$ *where* $\mathfrak{R}$ *is the radical of dimension 4. There are two finite dimensional irreducible representations.*

Proof. The original set of generators of the ideal I is a Gröbner basis and consists of the six elements b^3, $b^2a + bab - b$, ba^2, ab^2, $aba + a^2b - a$, a^3. Hence $U = U(A^{\omega_J^1})$ is finite dimensional with the stated basis. The following relations hold in U: $b^3 = 0$, $b^2a = -bab + b$, $ba^2 = 0$, $ab^2 = 0$, $aba = -a^2b + a$, $a^3 = 0$. These give the stated structure constants. A basis of the radical $\mathfrak{R} = \mathfrak{R}(U)$ consists of the elements $\xi_1 = b - bab$, $\xi_2 = a^2$, $\xi_3 = b^2$, $\xi_4 = a^2b$ which give these relations in $Q = U/\mathfrak{R}$: $b = bab$, $a^2 = b^2 = a^2b = 0$. The semisimple quotient Q has dimension 5 and a basis consists of the cosets of $\eta_1 = 1$, $\eta_2 = a$, $\eta_3 = ab$, $\eta_4 = ba$, $\eta_5 = bab$. The center $Z(Q)$ has dimension 2 with basis $z_1 = \eta_1$, $z_2 = \eta_3 + \eta_4$ and structure constants $z_1 \cdot z_1 = z_1$, $z_1 \cdot z_2 = z_2 \cdot z_1 = z_2$, $z_2 \cdot z_2 = z_2$. Since $z_2^2 = z_2$, the minimal polynomial of z_2 is $t^2 - t$. Thus $Z(Q) = J \oplus K$, where $J = \langle z_2 - z_1 \rangle$ and $K = \langle z_2 \rangle$; both ideals are 1-dimensional. Scaling these basis elements to obtain idempotents gives $e_1 = z_1 - z_2$, $e_2 = z_2$. The corresponding elements of Q are $e_1 = \eta_1 - \eta_3 - \eta_4$, $e_2 = \eta_3 + \eta_4$. The ideals in Q generated by e_1 and e_2 have dimensions 1 and 4 respectively. Thus $Q \simeq \mathbb{Q} \oplus M_{2\times 2}$. The 4-dimensional ideal generated by e_2 has basis: η_2, η_3, η_4, η_5. The left ideal generated by η_2, η_3, η_4 or η_5 has dimension 2. In particular, η_2 generates a 2-dimensional left ideal with basis $U_1 = \eta_2$ and $U_2 = \eta_3$. We identify U_1, U_2 with $(1,0), (0,1) \in \mathbb{Q}^2$, and solve for the matrix units; we obtain $E_{11} = \eta_3$, $E_{12} = \eta_2$, $E_{21} = \eta_5$, $E_{22} = \eta_4$. We now have two basis for Q the old basis is: η_1, η_2, η_3, η_4, η_5 and the new basis is e_1, E_{11}, E_{12}, E_{21}, E_{22}. By using the matrix of the change the basis, we can express the basis elements of Q in terms of the matrix units. Since the other basis elements of $U(A^{\omega_J^1})$ are congruent modulo $\mathfrak{R}$ to elements of Q, then we can express the nine basis elements in terms of matrix units:

$$\eta_1 \to [1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \eta_2 \to [0], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \eta_3 \to [0], \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \eta_4 \to [0], \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \eta_5 \to [0], \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We obtain the 1-dimensional trivial representation and the 2-dimensional natural representation. □

5.5.5 The anti-Jordan family, $q = -1$

The structure constants for $A^{\omega_{AJ}^{-1}}$ are

$$[e_1, e_1, e_2] = -e_1, \qquad [e_1, e_2, e_1] = e_1, \qquad [e_2, e_1, e_2] = e_2, \qquad [e_2, e_2, e_1] = -e_2.$$

Theorem 5.5.5. *We have the isomorphisms $U\big(A^{\omega_{AJ}^{-1}}\big) \cong U\big(A^{\omega_J^{1/2}}\big) \cong U\big(A^{\omega_J^{\infty}}\big)$.*

Proof. We have $U = U\big(A^{\omega_{AJ}^{-1}}\big) = F\langle a, b\rangle / I$ where I is generated by $b^2a - bab + b$ and $aba - a^2b - a$. The first iteration of the Gröbner basis algorithm produces one composition, $bab - b$. Including this element with the original generators, and self-reducing the resulting set, produces a new set of three generators: b^2a, $bab - b$, $aba - a^2b - a$. The second iteration produces three compositions: ba^2b, a^2b^2, b^2. Including these elements with the previous generators, and self-reducing the resulting set, produces a new set of four generators: ba^2b, $bab - b$, $aba - a^2b - a$, b^2. The third iteration produces two compositions: $ba^3b + ba^2$, a^2b. Including these elements with the previous generators, and self-reducing the resulting set, produces a new set of five generators: $bab - b$, ba^2, $aba - a$, a^2b, b^2. The fourth iteration produces one composition, a^2. Including this element with the previous generators, and self-reducing the resulting set, produces a new set of four generators: $bab - b$, $aba - a$, b^2, a^2. This is a Gröbner basis for the ideal, and is the same Gröbner basis as for the Jordan cases $q = \infty$, $q = \frac{1}{2}$; hence the quotient algebras are isomorphic. □

5.5.6 The anti-Jordan family, $q = 2$

The structure constants for $A^{\omega_{AJ}^{2}}$ are

$$[e_1, e_2, e_1] = e_1, \qquad [e_2, e_1, e_1] = -e_1, \qquad [e_2, e_1, e_2] = e_2, \qquad [e_1, e_2, e_2] = -e_2.$$

Proposition 5.5.6. *We have $U\big(A^{\omega_{AJ}^{2}}\big) \cong U(A^{\omega_{AJ}^{-1}})$.*

Proof. Similar to the proof of Theorem 5.5.5. □

5.5.7 The last nine operations

We first consider the fourth family with $q = \infty$. The structure constants for $A^{\omega_F^\infty}$ are

$$[e_1, e_1, e_2] = [e_2, e_1, e_1] = -e_1, \qquad [e_1, e_2, e_1] = e_1,$$
$$[e_2, e_2, e_1] = [e_1, e_2, e_2] = -e_2, \qquad [e_2, e_1, e_2] = e_2.$$

Proposition 5.5.7. *We have* $U(A^{\omega_F^\infty}) \cong U(A^{\omega_J^\infty})$.

Proof. We have $U\big(A^{\omega_F^\infty}\big) = F\langle a, b\rangle / J$ where J is generated by b^3, $b^2a - bab + ab^2 + b$, $bab - b$, $ba^2 - aba + a^2b + a$, $aba - a$, a^3. Self-reducing this set of generators gives the set of generators for the Jordan case, $q = \infty$ (see the proof of Theorem 5.5.1). □

For the fourth family with $q = 0$, the structure constants for $A^{\omega_F^0}$ are

$$[e_1, e_2, e_1] = e_1, \qquad [e_2, e_1, e_2] = e_2.$$

Proposition 5.5.8. *We have* $U(A^{\omega_F^0}) \cong U(A^{\omega_J^0})$.

Proof. We have $U(A^{\omega_F^0}) = F\langle a, b\rangle / J$ where J is generated by b^3, $b^2a - bab - ab^2 + b$, b^2a, $ba^2 + aba - a^2b - a$, a^2b, a^3. Self-reducing this set of generators gives the set of generators in the Jordan case $q = 0$ (see the proof of Theorem 5.5.2). □

For the fourth family with $q = 1$, the structure constants for $A^{\omega_F^1}$ are

$$[e_1, e_2, e_1] = e_1, \qquad [e_2, e_1, e_2] = e_2.$$

Proposition 5.5.9. *We have* $U(A^{\omega_F^1}) \cong U(A^{\omega_J^1})$.

Proof. We have $U(A^{\omega_F^1}) = F\langle a, b\rangle / J$ where J is generated by b^3, $b^2a + bab - ab^2 - b$, $ba^2 - aba - a^2b + a$, ba^2, ab^2, a^3. This set generates the ideal I in the Jordan case $q = 1$ (see the proof of Theorem 5.5.4). □

We consider the last six operations together.

Fourth family, $q = -1$: The structure constants for $A^{\omega_F^{-1}}$ are

$$[e_1, e_2, e_1] = e_1, \qquad [e_2, e_1, e_1] = 2e_1, \qquad [e_2, e_1, e_2] = e_2, \qquad [e_1, e_2, e_2] = 2e_2.$$

Fourth family, $q = 2$: The structure constants for $A^{\omega_F^2}$ are

$$[e_1, e_1, e_2] = 2e_1, \qquad [e_1, e_2, e_1] = e_1, \qquad [e_2, e_2, e_1] = 2e_2, \qquad [e_2, e_1, e_2] = e_2.$$

Fourth family, $q = \frac{1}{2}$: The structure constants for $A^{\omega_F^{1/2}}$ are

$$[e_1, e_1, e_2] = [e_1, e_2, e_1] = [e_2, e_1, e_1] = e_1,$$
$$[e_2, e_2, e_1] = [e_2, e_1, e_2] = [e_1, e_2, e_2] = e_2.$$

Cyclic commutator: The structure constants for $A^{\omega_{cc}}$ are

$$[e_1, e_1, e_2] = -e_1, \qquad [e_1, e_2, e_1] = e_1, \qquad [e_2, e_2, e_1] = -e_2, \qquad [e_2, e_1, e_2] = e_2.$$

Weakly commutative operation (we consider the second form of Table 5.1): The structure constants for $A^{\omega_{wc}}$ are

$$[e_1, e_1, e_2] = -e_1, \qquad [e_1, e_2, e_1] = e_1, \qquad [e_2, e_1, e_1] = 2e_1,$$
$$[e_2, e_2, e_1] = -e_2, \qquad [e_2, e_1, e_2] = e_2, \qquad [e_1, e_2, e_2] = 2e_2.$$

Weakly anti-commutative operation (we consider the second form of Table 5.1): The structure constants for $A^{\omega_{wa}}$ are

$$[e_1, e_2, e_1] = [e_1, e_1, e_2] = e_1, \qquad [e_2, e_1, e_1] = -2e_1,$$
$$[e_2, e_1, e_2] = [e_2, e_2, e_1] = e_2, \qquad [e_1, e_2, e_2] = -2e_2.$$

Proposition 5.5.10. *We have the following isomorphisms:*

$$U(A^{\omega_F^{-1}}) \cong U(A^{\omega_F^{1/2}}) \cong U(A^{\omega_F^2}) \cong U(A^{\omega_{cc}}) \cong U(A^{\omega_{wc}}) \cong U(A^{\omega_{wa}}) \cong U(A^{\omega_J^{\infty}}).$$

Proof. We have $U(A^{\omega_F^{-1}}) = F\langle a, b\rangle / I$ where I is generated by b^3, $b^2a + bab + ab^2 - b$, $b^2a + \frac{1}{2}ab^2$, $bab + \frac{1}{2}ab^2 - b$, $ba^2 + 2aba - 2a$, $ba^2 + aba + a^2b - a$, $ba^2 + 2a^2b$, a^3. We compute a Gröbner basis for I. The first iteration produces eight compositions with the normal forms a^2b^2, $a^2ba - a^2$, a^2ba, $aba - a^2b - a$, ab^2, a^2b, b^2, a^2. Including these elements with the original generators, and then self-reducing the resulting set, produces a new set of four ideal generators: $bab - b$, $aba - a$, a^2, b^2. This is a Gröbner basis for I. In fact, this is the same Gröbner basis as in the Jordan case, $q = \infty$ (see the proof of Theorem 5.5.1).

For $U(A^{\omega_F^{1/2}})$, we have $U(A^{\omega_F^{1/2}}) = F\langle a,b\rangle/J$ where J is generated by b^3, $b^2a+bab+ab^2-b$, $b^2a+\frac{1}{2}bab-\frac{1}{2}b$, $bab+2ab^2-b$, $ba^2+\frac{1}{2}aba-\frac{1}{2}a$, $ba^2+aba+a^2b-a$, $aba+2a^2b-a$, a^3. We compute a Gröbner basis for J. The first iteration of the Gröbner basis algorithm produces four compositions with the normal forms: ab^2, a^2b, b^2, a^2. Including these elements with the original generators, and then self-reducing the resulting set of twelve elements, produces a new set of four ideal generators which is a Gröbner basis of the ideal J: $bab-b$, $aba-a$, b^2, a^2. Similarly we can show that $U(A^{\omega_F^2}) \cong U(A^{\omega_J^\infty})$.

For $U(A^{\omega_{cc}})$, we have $U(A^{\omega_{cc}}) = F\langle a,b\rangle/K$ where K is generated by b^2a-ab^2, $b^2a-bab+b$, $bab-ab^2-b$, ba^2-a^2b, $ba^2-aba+a$, $aba-a^2b-a$. The first iteration of the Gröbner basis algorithm produces four compositions with the normal forms: ab^2, a^2b, b^2, a^2. Including these elements with the original generators, and then self-reducing the resulting set of twelve elements, produces a new set of four ideal generators which a Gröbner basis of the ideal K: $bab-b$, $aba-a$, b^2, a^2.

For $U(A^{\omega_{wc}})$, we have $U(A^{\omega_{wc}}) = F\langle a,b\rangle/L$, where L is generated by b^3, $b^2a-\frac{1}{3}bab+\frac{1}{3}b$, $b^2a-bab-2ab^2+b$, $bab-b$, $ba^2+\frac{1}{2}aba-\frac{1}{2}a^2b-\frac{1}{2}a$, $aba-a$, $aba-3a^2b-a$, a^3. The first iteration of the Gröbner basis algorithm produces five compositions with the normal forms: a^2b^2, ab^2, a^2b, b^2, a^2. Including these elements with the original generators, and then self-reducing the resulting set of thirteen elements, produces a new set of four ideal generators which is a Gröbner basis for the ideal L: $bab-b$, $aba-a$, b^2, a^2.

Finally, we have $U(A^{\omega_{wa}}) = F\langle a,b\rangle/M$ where M is generated by $b^2a+bab-2ab^2-b$, $b^2a-bab+b$, $bab-ab^2-b$, $ba^2-aba+a$, $ba^2-\frac{1}{2}aba-\frac{1}{2}a^2b+\frac{1}{2}a$, $aba-a^2b-a$. The first iteration of the Gröbner basis algorithm produces four compositions with the normal forms: ab^2, a^2b, b^2, a^2. Including these elements with the original generators, and then self-reducing the resulting set of ten elements, produces a new set of four ideal generators, which is a Gröbner basis of the ideal M: $bab-b$, $aba-a$, b^2, a^2. □

symmetric sum	$\left(1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0\right)$	$a(b \circ c) + b(c \circ a) + c(a \circ b)$
alternating sum	$\left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1\right)$	$a[b,c] + b[c,a] + c[a,b]$
cyclic sum	$\left(1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc + bca + cab$
Lie $q = \infty$	$\left(0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\right)$	$[a,[b,c]]$
Lie $q = \frac{1}{2}$	$\left(0, \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, 0\right)$	$[a, b \circ c]$
Jordan $q = \infty$	$\left(1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\right)$	$abc + cba$
Jordan $q = 0$	$\left(1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0\right)$	$(a \circ b)c$
Jordan $q = \frac{1}{2}$	$\left(1, \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, 0\right)$	$a(b \circ c) + c(a \circ b) + (c \circ a)b$
Jordan $q = 1$	$\left(1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, 0\right)$	$a(b \circ c)$
anti-Jordan $q = \infty$	$\left(0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 1\right)$	$a[b,c] + c[a,b] + [c,a]b$
anti-Jordan $q = -1$	$\left(0, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, 1\right)$	$a[b,c]$
anti-Jordan $q = \frac{1}{2}$	$\left(0, \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc - cba$
anti-Jordan $q = 2$	$\left(0, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, 1\right)$	$[a,b]c$
fourth family $q = \infty$	$\left(1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc - acb - bac$
fourth family $q = 0$	$\left(1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc - acb + bca$
fourth family $q = 1$	$\left(1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc - bac + cab$
fourth family $q = -1$	$\left(1, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc + bac + cab$
fourth family $q = 2$	$\left(1, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc + acb + bca$
fourth family $q = \frac{1}{2}$	$\left(1, \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, 1\right)$	$abc + acb + bac$
cyclic commutator	$\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0\right)$	$abc - bca$
weakly commutative	$\left(1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0\right)$	$\begin{cases} abc + acb + bac - cba, \\ abc - acb + 2bac \end{cases}$
weakly anticommutative	$\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1\right)$	$\begin{cases} abc + acb - bac - cba, \\ abc + acb - 2bac \end{cases}$

Table 5.1: The twenty-two basic trilinear operations

Operations of Lie type

	$\dim U(A^\omega)$	GK-dim	$U(A^\omega)$
symmetric sum; cyclic sum	∞	1	$A(-1,-1,1)/\langle a^3, b^3\rangle$
alternating sum	∞	∞	$F\langle a, b\rangle$
Lie $q = \infty$; anti-Jordan $q = \infty$	∞	3	$A(2,-1,-2)$
Lie $q = \frac{1}{2}$; anti-Jordan $q = \frac{1}{2}$	∞	3	$A(0,1,0)$

Operations of Jordan type

	$\dim U(A^\omega)$	GK-dim	$U(A^\omega)$
Jordan $q = \infty$; Jordan $q = \frac{1}{2}$; anti-Jordan $q = -1$; anti-Jordan $q = 2$; fourth family $q = \infty$; fourth family $q = -1$; fourth family $q = 2$; fourth family $q = \frac{1}{2}$; cyclic commutator; weakly commutative; weakly anticommutative	5	0	$\mathbb{Q} \oplus M_{2\times 2}$
Jordan $q = 0$; fourth family $q = 0$	9	0	$\mathfrak{R} \oplus \mathbb{Q} \oplus M_{2\times 2}$
Jordan $q = 1$; fourth family $q = 1$	9	0	$\mathfrak{R} \oplus \mathbb{Q} \oplus M_{2\times 2}$

Table 5.2: Structure of the universal associative envelopes

Chapter 6

The universal associative envelope of the anti-Jordan triple system of $n \times n$ matrices

6.1 Introduction

Anti-Jordan triple systems were introduced by Faulkner and Ferrar in [19]. The classification of finite-dimensional simple anti-Jordan triple systems over an algebraically closed field of characteristic 0 was given by Bashir [2, Theorem 6].

Definition 6.1.1. [2] A vector space $\mathfrak{J}$ over a field F endowed with a trilinear operation $\mathfrak{J} \times \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$, $(a, b, c) \to \langle a, b, c\rangle$ is said to be an **anti-Jordan triple system** if the following conditions are fulfilled for all $a, b, c, d, e \in V$:

$$\langle a, b, a\rangle = 0, \qquad \langle a, b, \langle c, d, e\rangle\rangle = \langle\langle a, b, c\rangle, d, e\rangle + \langle c, \langle b, a, d\rangle, e\rangle + \langle c, d, \langle a, b, e\rangle\rangle.$$

Any associative algebra A defines an anti-Jordan triple system A_- relative to the product $\langle a, b, c\rangle = abc - cba$.

Definition 6.1.2. A **representation** of an anti-Jordan triple system $\mathfrak{J}$ is a homomorphism $\rho: \mathfrak{J} \to (\operatorname{End} V)_-$ from $\mathfrak{J}$ to the anti-Jordan triple system of endomorphisms of a vector space V. In other words, ρ is a linear mapping that satisfies

$$\rho(\langle a, b, c\rangle) = \rho(a)\rho(b)\rho(c) - \rho(c)\rho(b)\rho(a),$$

for all $a, b, c \in \mathfrak{J}$. Two representations ρ_1 and ρ_2 of an anti-Jordan triple system $\mathfrak{J}$ on the same vector space V are **equivalent** if there exists an invertible endomorphism T such that $\rho_2(a) = T^{-1}\rho_1(a)T$ for all $a \in \mathfrak{J}$.

This chapter is structured as follows. In Section 6.2, we prove that the universal associative envelope of the simple anti-Jordan triple system of $n \times n$ matrices over an algebraically closed field is finite-dimensional using noncommutative Gröbner bases in free associative algebras. In Section 6.3, we determine the structure constants of the universal enveloping algebra. In Section 6.4, we determine the center of the universal enveloping algebra. In Section 6.5, we explicitly determine the complete decomposition of the universal enveloping algebra into a direct sum of matrix algebras. In the last section, we provide examples to show that the universal envelopes of anti-Jordan triple systems are not necessary finite-dimensional.

Unless otherwise stated, we assume throughout that all vector spaces are over an algebraically closed field F of characteristic 0.

6.2 The universal associative enveloping algebra

Definition 6.2.1. Let $\mathfrak{J}$ be an anti-Jordan triple system over a field F and let $\mathfrak{A}$ be an associative algebra. Let $i : \mathfrak{J} \to \mathfrak{A}$ be an anti-Jordan homomorphism. The pair $(\mathfrak{A}, i)$ is called a **universal enveloping algebra** of $\mathfrak{J}$ if for any associative algebra A and every anti-Jordan homomorphism $f : \mathfrak{J} \to A_-$ there exists a unique algebra homomorphism $\bar{f} : \mathfrak{A} \to A$ such that the map $\bar{f} \circ i = f$.

For the rest of this chapter we assume that $\mathfrak{J}$ is the anti-Jordan triple system of all $n \times n$ matrices over an algebraically closed field F of characteristic 0 with the triple product $\langle a, b, c \rangle = abc - cba$.

Notation 6.2.2. *We write $\delta_{i,j}$ for the Kronecker delta, and $\widehat{\delta}_{i,j} = 1 - \delta_{i,j}$.*

Definition 6.2.3. Let $\Omega = \{1, 2, \dots, n\}$ be a finite index set. Let $B = \{E_{i,j}\}_{i,j \in \Omega}$ be a basis of $\mathfrak{J}$, where $E_{i,j}$ is the matrix with a single 1 in the ith row and jth column, and zeros elsewhere. The structure constants for $\mathfrak{J}$ are

$$\langle E_{i,j}, E_{k,\ell}, E_{m,t} \rangle = \delta_{j,k}\delta_{\ell,m}E_{i,t} - \delta_{t,k}\delta_{\ell,i}E_{m,j}, \quad \text{for all } \ i, j, k, \ell, m, t \in \Omega.$$

Consider the bijection $\phi : B \to X = \{e_{i,j}\}_{i,j \in \Omega}$ defined by $\phi(E_{i,j}) = e_{i,j}$. We extend ϕ to a linear map $\phi : \mathfrak{J} \to F\langle X \rangle$.

Throughout this chapter we use the deglex order $<$ where $e_{i,j} < e_{k,\ell}$ if either $i < k$, or $i = k$ and $j < \ell$.

Definition 6.2.4. Let $G \subset F\langle X\rangle$ consist of these elements $(i,j,k,r,s,t \in \Omega)$:

$$\begin{aligned}
\mathcal{R}_1^{(i,j,k,t)} &= e_{i,j}e_{j,k}e_{k,t} - e_{k,t}e_{j,k}e_{i,j} - e_{i,t} \quad (k < i),\\
\mathcal{R}_2^{(i,j,t)} &= e_{i,j}e_{j,i}e_{i,t} - e_{i,t}e_{j,i}e_{i,j} - e_{i,t} \quad (t < j),\\
\mathcal{R}_3^{(i,j,k,t)} &= e_{i,j}e_{k,i}e_{t,k} - e_{t,k}e_{k,i}e_{i,j} + e_{t,j} \quad (t < i),\\
\mathcal{R}_4^{(i,j,k)} &= e_{i,j}e_{k,i}e_{i,k} - e_{i,k}e_{k,i}e_{i,j} + e_{i,j} \quad (k < j),\\
\mathcal{R}_5^{(i,j,k,t,r,s)} &= e_{i,j}e_{k,t}e_{r,s} - e_{r,s}e_{k,t}e_{i,j} \quad (r < i,\ j \neq k \text{ or } t \neq r,\ s \neq k \text{ or } t \neq i)\,,\\
\mathcal{R}_6^{(i,j,k,t,s)} &= e_{i,j}e_{k,t}e_{i,s} - e_{i,s}e_{k,t}e_{i,j} \quad (s < j,\ j \neq k \text{ or } t \neq i,\ s \neq k \text{ or } t \neq i)\,.
\end{aligned}$$

Let $I \subset F\langle X\rangle$ be the ideal generated by G. We write $\mathfrak{A} = F\langle X\rangle/I$ with surjection $\pi\colon F\langle X\rangle \to \mathfrak{A}$ sending f to $f + I$, and $i = \pi \circ \phi$ for the natural map $i\colon \mathfrak{J} \to \mathfrak{A}$.

Lemma 6.2.5. *The unital associative algebra $\mathfrak{A}$ and the linear map i form the universal associative envelope of the anti-Jordan triple system $\mathfrak{J}$.*

Proof. Similar to the construction of the universal enveloping algebra of a Lie or Jordan algebra (see [24, §17.2]). □

6.2.1 Normal forms of compositions of the ideal generators

Our goal is to complete the set G to a Gröbner basis for the ideal I. This will be achieved by repeatedly adding normal forms of compositions of generators to the set G of the original generators. In this subsection we obtain normal forms of some compositions. In the next subsection we show that adding these normal forms to the set G gives a Gröbner basis for the ideal I.

Lemma 6.2.6. *The set of all normal forms modulo G of nontrivial compositions among elements of G includes the set G_1 which consists of these elements:*

$$\begin{aligned}
\mathcal{G}_1^{(r,t,m)} &= e_{r,t}e_{t,m} - e_{r,1}e_{1,m} \quad (m \neq r,\ t \neq 1),\\
\mathcal{G}_2^{(i,t,\ell)} &= e_{i,t}e_{\ell,i} - e_{1,t}e_{\ell,1} \quad (t \neq \ell,\ i \neq 1),\\
\mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j}e_{k,\ell} \quad (i \neq \ell,\ j \neq k).
\end{aligned}$$

Proof. For all $s < t$, we consider the set of the following compositions:

$$S^{(r,t,s,m)} = \mathcal{R}_2^{(r,t,s)} e_{s,m} - e_{r,t} \mathcal{R}_1^{(t,r,s,m)}.$$

We eliminate from $S^{(r,t,s,m)}$ all occurrences of the leading monomials of elements of G as factors in the monomials; we write $\equiv$ to indicate congruence modulo G:

$$\begin{aligned} S^{(r,t,s,m)} &= -e_{r,s} e_{t,r} e_{r,t} e_{s,m} - e_{r,s} e_{s,m} + e_{r,t} e_{s,m} e_{r,s} e_{t,r} + e_{r,t} e_{t,m} \\ &\equiv -e_{r,s} \left(e_{s,m} e_{r,t} e_{t,r} - \delta_{m,r} e_{s,r} \right) - e_{r,s} e_{s,m} + \left(e_{r,s} e_{s,m} e_{r,t} - \delta_{m,r} e_{r,t} \right) e_{t,r} + e_{r,t} e_{t,m} \\ &= \delta_{m,r} e_{r,s} e_{s,r} - e_{r,s} e_{s,m} - \delta_{m,r} e_{r,t} e_{t,r} + e_{r,t} e_{t,m}. \end{aligned}$$

Clearly, if $m = r$ then $S^{(r,t,s,r)} \equiv 0$. If $m \neq r$ then we obtain the set L of nonzero normal forms modulo G:

$$L = \left\{ \mathcal{N}^{(r,t,m,s)} = e_{r,t} e_{t,m} - e_{r,s} e_{s,m} \mid s < t,\ m \neq r \right\}.$$

The set L is not self-reduced. Therefore, for all $1 < s < t \leq n$, we eliminate from the element $\mathcal{N}^{(r,t,m,s)}$ the leading monomial of $\mathcal{N}^{(r,s,m,1)}$ and obtain a self-reduced set consisting of the elements $\mathcal{G}_1^{(r,t,m)}$.

For all $(r,\ell) < (i,k)$, we consider the set of the following compositions:

$$S_1^{(i,k,r,t,\ell,s)} = \mathcal{R}_1^{(i,k,r,t)} e_{\ell,s} - e_{i,k} \mathcal{R}_5^{(k,r,r,t,\ell,s)} \quad (t \neq \ell, \text{ and } s \neq r \text{ or } t \neq k).$$

We eliminate from $S_1^{(i,k,r,t,\ell,s)}$ all occurrences of the leading monomials of elements of G:

$$\begin{aligned} S_1^{(i,k,r,t,\ell,s)} &= -e_{r,t} e_{k,r} e_{i,k} e_{\ell,s} - e_{i,t} e_{\ell,s} + e_{i,k} e_{\ell,s} e_{r,t} e_{k,r} \\ &\equiv -e_{r,t} \big(e_{\ell,s} e_{i,k} e_{k,r} - \delta_{s,i} e_{\ell,r} \big) - e_{i,t} e_{\ell,s} + e_{r,t} e_{\ell,s} e_{i,k} e_{k,r} \\ &= \delta_{s,i} e_{r,t} e_{\ell,r} - e_{i,t} e_{\ell,s}. \end{aligned}$$

Hence, for all $(r,\ell) < (i,k)$, the (monic) normal form of $S^{(i,k,r,t,\ell,s)}$ is

$$e_{i,t} e_{\ell,s} \ (\text{if } i \neq s), \qquad e_{i,t} e_{\ell,i} - e_{r,t} e_{\ell,r} \ (\text{if } i = s). \tag{6.1}$$

For all $(r,t) < (i,k)$, we consider the set of the following compositions:

$$S_2^{(i,j,k,\ell,r,s,t,m)} = \mathcal{R}_5^{(i,j,k,\ell,r,s)} e_{t,m} - e_{i,j} \mathcal{R}_5^{(k,\ell,r,s,t,m)}$$

$$(s \neq k \text{ or } \ell \neq i, j \neq k \text{ or } \ell \neq r, m \neq r \text{ or } s \neq k) \text{ and } (\ell \neq r \text{ or } s \neq t).$$

We eliminate from $S_2^{(i,j,k,\ell,r,s,t,m)}$ all occurrences of the leading monomials of elements of G:

$$\begin{aligned} S_2^{(i,j,k,\ell,r,s,t,m)} &= -e_{r,s}e_{k,\ell}e_{i,j}e_{t,m} + e_{i,j}e_{t,m}e_{r,s}e_{k,\ell} \\ &\equiv -e_{r,s}\left(e_{t,m}e_{i,j}e_{k,\ell} + \delta_{\ell,i}\delta_{j,t}e_{k,m} - \delta_{j,k}\delta_{m,i}e_{t,\ell}\right) \\ &\quad + \left(e_{r,s}e_{t,m}e_{i,j} + \delta_{j,t}\delta_{m,r}e_{i,s} - \delta_{s,t}\delta_{m,i}e_{r,j}\right)e_{k,\ell} \\ &= \delta_{j,t}\left(-\delta_{\ell,i}e_{r,s}e_{k,m} + \delta_{m,r}e_{i,s}e_{k,\ell}\right) - \delta_{m,i}\left(-\delta_{j,k}e_{r,s}e_{t,\ell} + \delta_{s,t}e_{r,j}e_{k,\ell}\right). \end{aligned}$$

We first note that if $(m,j,s) = (i,k,t)$ then the (monic) normal form of $S_2^{(i,k,k,\ell,r,t,t,i)}$ modulo G coincides with the element $\mathcal{N}^{(r,k,\ell,t)}$, so we ignore this case. For all $(r,t) < (i,k)$, the possible non-zero (monic) normal forms of $S_2^{(i,j,k,\ell,r,s,t,m)}$ modulo G are

$$\begin{array}{ll} e_{r,s}e_{k,m}\ \ (m \neq r, s \neq k), & e_{i,s}e_{k,\ell}\ \ (\ell \neq i, s \neq k), \\ e_{r,s}e_{t,\ell}\ \ (r \neq \ell, s \neq t), & e_{r,j}e_{k,\ell}\ \ (\ell \neq r, j \neq k), \\ e_{r,t}e_{k,i}\ \ (t \neq k,\ r < i), & e_{i,s}e_{k,i} - e_{r,s}e_{k,r}\quad (s \neq k). \end{array} \tag{6.2}$$

Combining (6.1) and (6.2) gives the following normal forms:

$$\begin{aligned} \mathcal{L}^{(i,s,k,r)} &= e_{i,s}e_{k,i} - e_{r,s}e_{k,r} \quad (r < i,\ s \neq k), \\ \mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j}e_{k,\ell} \quad (i \neq k,\ j \neq \ell). \end{aligned}$$

We observe that the set $\left\{\mathcal{L}^{(i,s,k,r)} \mid \text{for all } r < i,\ s \neq k\right\}$ is not self-reduced. Therefore, for all $1 < r < i \leq n$, we eliminate from $\mathcal{L}^{(i,s,k,r)}$ the leading monomial of $\mathcal{L}^{(r,s,k,1)}$ and obtain a self-reduced set consisting of the elements $\mathcal{G}_2^{(i,s,k)}$.

For $n = 2$, we cannot obtain $\mathcal{G}_3^{(1,2,1,2)}$, $\mathcal{G}_3^{(2,2,1,1)}$ and $\mathcal{G}_3^{(2,1,2,1)}$ from $S_1^{(i,k,r,t,\ell,s)}$ or $S_2^{(i,j,k,\ell,r,s,t,m)}$. Thus, we consider three more compositions in this case:

$$\begin{aligned} &S_3 = \mathcal{R}_6^{(1,2,1,2,1)}e_{1,1} - e_{1,2}\mathcal{R}_4^{(1,2,1)}, \qquad S_4 = \mathcal{R}_1^{(2,2,1,2)}e_{1,1} - e_{2,2}\mathcal{R}_3^{(2,1,1,1)}, \\ &S_5 = \mathcal{R}_2^{(2,2,1)}e_{2,1} - e_{2,2}\mathcal{R}_6^{(2,2,2,1,1)}. \end{aligned}$$

We eliminate from S_3 all occurrences of the leading monomials of elements of G:

$$S_3 = -e_{1,1}e_{1,2}^2e_{1,1} + e_{1,2}e_{1,1}^2e_{1,2} - e_{1,2}^2$$

$$\equiv -e_{1,1}\left(e_{1,1}e_{1,2}^2\right) + \left(e_{1,1}^2 e_{1,2} - e_{1,2}\right)e_{1,2} - e_{1,2}e_{1,2} = -2e_{1,2}^2.$$

Similarly, we can show that $S_4 \equiv -2e_{2,2}e_{1,1}$ and $S_5 \equiv -2e_{2,1}e_{2,1}$. The monic forms of the last three elements give the required elements. This completes the proof. □

Lemma 6.2.7. *The set of all normal forms modulo $G \cup G_1$ of nontrivial compositions among elements of $G \cup G_1$ includes the set G_2 which consists of these elements:*

$$\mathcal{G}_4^{(r,i)} = e_{r,i}e_{i,r} - e_{r,1}e_{1,r} + e_{1,1}^2 - e_{1,i}e_{i,1} \quad (r, i \in \Omega \setminus \{1\}).$$

Proof. For all $(s,t) < (r,i)$, we consider the set of the following compositions:

$$S^{(r,i,s,t,m)} = \mathcal{R}_1^{(r,i,s,t)} e_{t,m} - e_{r,i}\mathcal{R}_1^{(i,s,t,m)}.$$

We eliminate from $S^{(r,i,s,t,m)}$ all occurrences of the leading monomials of elements of G:

$$\begin{aligned} S^{(r,i,s,t,m)} &= -e_{s,t}e_{i,s}e_{r,i}e_{t,m} - e_{r,t}e_{t,m} + e_{r,i}e_{t,m}e_{s,t}e_{i,s} + e_{r,i}e_{i,m} \\ &\equiv -e_{s,t}\left(e_{t,m}e_{r,i}e_{i,s} - \delta_{m,r}e_{t,s}\right) - e_{r,t}e_{t,m} + \left(e_{s,t}e_{t,m}e_{r,i} - \delta_{m,r}e_{s,i}\right)e_{i,s} + e_{r,i}e_{i,m} \\ &= \delta_{m,r}e_{s,t}e_{t,s} - e_{r,t}e_{t,m} - \delta_{m,r}e_{s,i}e_{i,s} + e_{r,i}e_{i,m}. \end{aligned}$$

We now eliminate from $S^{(r,i,s,t,m)}$ all occurrences of the leading monomials of elements of G_1. Clearly, if $m \neq r$ then $S^{(r,i,s,t,m)} \equiv 0 \mod G_1$, using the relations $\mathcal{G}_1^{(r,t,m)}$(if $t \neq 1$) and $\mathcal{G}_1^{(r,i,m)}$. If $m = r$ then we obtain the set $\mathcal{N}$ of nonzero normal forms of $S^{(r,i,s,t,r)}$ modulo $G \cup G_1$:

$$\mathcal{N} = \left\{\mathcal{N}^{(r,i,t,s)} = e_{r,i}e_{i,r} - e_{r,t}e_{t,r} - e_{s,i}e_{i,s} + e_{s,t}e_{t,s} \mid \text{for all } (s,t) < (r,i)\right\}.$$

We observe that the set $\mathcal{N}$ is not self-reduced and the element $\mathcal{N}^{(r,i,1,1)}$ coincides with $\mathcal{G}_4^{(r,i)}$ for all $r, i \neq 1$. Assume now that $s, t \neq 1$. For all $(s,t) < (r,i)$, we eliminate from $\mathcal{N}^{(r,i,t,s)}$ the leading monomials of $\mathcal{N}^{(r,t,1,1)}$, $\mathcal{N}^{(s,i,1,1)}$ and $\mathcal{N}^{(s,t,1,1)}$ and again obtain $\mathcal{G}_4^{(r,i)}$. A similar argument can be used if $s \neq 1$ or $t \neq 1$. The result is a self-reduced set consisting of the elements $\mathcal{G}_4^{(r,i)}$. □

Lemma 6.2.8. *The set of all normal forms modulo $G \cup G_1 \cup G_2$ of nontrivial compositions among elements of $G \cup G_1 \cup G_2$ includes the set G_3 which consists of these elements:*

$$\begin{aligned}
\mathcal{G}_5^{(r,i)} &= e_{r,1}e_{1,i}e_{i,1} - e_{1,1}^2 e_{r,1} - e_{r,1} \quad (r < i;\ i, r \in \Omega \setminus \{1\}),\\
\mathcal{G}_6^{(i,r)} &= e_{i,1}e_{1,i}e_{r,1} - e_{1,1}^2 e_{r,1} \quad (i < r;\ i, r \in \Omega \setminus \{1\}),\\
\mathcal{G}_7^{(t,\ell)} &= e_{1,t}e_{t,1}e_{1,\ell} - e_{1,1}^2 e_{1,\ell} \quad (t < \ell;\ t, \ell \in \Omega \setminus \{1\}),\\
\mathcal{G}_8^{(\ell,t)} &= e_{1,\ell}e_{t,1}e_{1,t} - e_{1,1}^2 e_{1,\ell} + e_{1,\ell} \quad (\ell < t; \ell, t \in \Omega \setminus \{1\}),\\
\mathcal{G}_9^{(r)} &= e_{r,1}e_{1,r}e_{r,1} - 2e_{1,1}^2 e_{r,1} - e_{r,1} \quad (r \in \Omega \setminus \{1\}),\\
\mathcal{G}_{10}^{(r)} &= e_{1,r}e_{r,1}e_{1,r} - 2e_{1,1}^2 e_{1,r} + e_{1,r} \quad (r \in \Omega \setminus \{1\}),\\
\mathcal{G}_{11}^{(r,i,\ell)} &= e_{r,1}e_{1,i}e_{\ell,1} \quad (\ell, r \neq i; \ell, i, r \in \Omega),\\
\mathcal{G}_{12}^{(\ell,i,r)} &= e_{1,\ell}e_{i,1}e_{1,r} \quad (\ell, r \neq i; \ell, i, r \in \Omega),\\
\mathcal{G}_{13}^{(i)} &= e_{1,1}e_{1,i}e_{i,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{14}^{(i)} &= e_{1,1}e_{i,1}e_{1,i} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}).
\end{aligned}$$

Proof. For all $r, t, i, \ell, k \in \Omega$, we consider the set of the following compositions:

$$\begin{aligned}
S_1^{(r,t,i,\ell)} &= \mathcal{G}_1^{(r,t,i)} e_{i,\ell} - e_{r,t}\mathcal{G}_1^{(t,i,\ell)} \quad (1 \neq i \neq r,\ \ell \neq t \neq 1),\\
S_2^{(r,i,t)} &= \mathcal{G}_4^{(r,i)} e_{r,t} - e_{r,i}\mathcal{G}_1^{(i,r,t)} \quad (i, r \neq 1,\ i \neq t),\\
S_3^{(i,r,t)} &= \mathcal{G}_4^{(i,r)} e_{t,r} - e_{i,r}\mathcal{G}_2^{(r,i,t)} \quad (i, r \neq 1,\ i \neq t),\\
S_4^{(r,t,i,\ell,k)} &= \mathcal{G}_1^{(r,t,i)} e_{\ell,k} - e_{r,t}\mathcal{G}_3^{(t,i,\ell,k)} \quad (r \neq i \neq \ell, k \neq t \neq 1),\\
S_5^{(t,\ell,i,k,r)} &= \mathcal{G}_2^{(t,\ell,i)} e_{k,r} - e_{t,\ell}\mathcal{G}_3^{(i,t,k,r)} \quad (\ell \neq i \neq r, k \neq t \neq 1),\\
S_6^{(r,i,t,\ell)} &= \mathcal{G}_4^{(r,i)} e_{t,\ell} - e_{r,i}\mathcal{G}_3^{(i,r,t,\ell)} \quad (i, r \neq 1, \ell \neq i, r \neq t).
\end{aligned}$$

We eliminate from these compositions all occurrences of the leading monomials of elements of $G \cup G_1 \cup G_2$. For the composition $S_1^{(r,t,i,\ell)}$, we have

$$\begin{aligned}
S_1^{(r,t,i,\ell)} &= -e_{r,1}e_{1,i}e_{i,\ell} + e_{r,t}e_{t,1}e_{1,\ell}\\
&\equiv -\delta_{\ell,1}e_{r,1}e_{1,i}e_{i,1} - \widehat{\delta}_{\ell,1}e_{r,1}e_{1,1}e_{1,\ell} + \delta_{r,1}e_{1,t}e_{t,1}e_{1,\ell} + \widehat{\delta}_{r,1}e_{r,1}e_{1,1}e_{1,\ell} \bmod G_1.
\end{aligned}$$

We note first that if $\ell, r \neq 1$ then $S_1^{(r,t,i,\ell)} \equiv 0 \bmod G_1$. Three cases need to be considered. Case I. If $(\ell, r) = (1,1)$ then

$$S_1^{(1,t,i,1)} \equiv -e_{1,1}e_{1,i}e_{i,1} + e_{1,t}e_{t,1}e_{1,1} \bmod G_1$$
$$\equiv -e_{1,1}e_{1,i}e_{i,1} + e_{1,1}e_{t,1}e_{1,t} + e_{1,1} \bmod G,$$

since by definition $t \neq 1$. Hence the (monic) normal form of $S_1^{(1,t,i,1)}$ in this case is

$$\mathcal{G}'^{(t,i)} = e_{1,1}e_{t,1}e_{1,t} - e_{1,1}e_{1,i}e_{i,1} + e_{1,1} \quad (t, i \in \Omega \setminus \{1\}). \tag{6.3}$$

Case II. If $\ell = 1$ and $r \neq 1$ then

$$S_1^{(r,t,i,1)} \equiv -e_{r,1}e_{1,i}e_{i,1} + e_{r,1}e_{1,1}^2 \bmod G_1$$
$$\equiv -e_{r,1}e_{1,i}e_{i,1} + e_{1,1}^2 e_{r,1} + e_{r,1} \bmod G.$$

Clearly, if $r < i$ then the monic form of the last equation coincides with $\mathcal{G}_5^{(r,i)}$. If $i < r$ then the element $e_{r,1}e_{1,i}e_{i,1}$ of the last equation can be reduced further modulo G: $e_{r,1}e_{1,i}e_{i,1} \equiv e_{i,1}e_{1,i}e_{r,1} + e_{r,1} \bmod G$. Using this in the last equation gives $\mathcal{G}_6^{(i,r)}$.
Case III. If $\ell \neq 1$ and $r = 1$ then

$$S_1^{(1,t,i,\ell)} \equiv -e_{1,1}^2 e_{1,\ell} + e_{1,t}e_{t,1}e_{1,\ell} \bmod G_1.$$

Clearly, if $t < \ell$ then the normal form of $S_1^{(1,t,i,\ell)}$ in this case coincides with $\mathcal{G}_7^{(t,\ell)}$. If $\ell < t$ then the element $e_{1,t}e_{t,1}e_{1,\ell}$ of the last equation can be reduced further modulo G: $e_{1,t}e_{t,1}e_{1,\ell} \equiv e_{1,\ell}e_{t,1}e_{1,t} + e_{1,\ell} \bmod G$. Using this in the last equation gives $\mathcal{G}_8^{(\ell,t)}$.
For the composition $S_2^{(r,i,t)}$, we have

$$S_2^{(r,i,t)} = -e_{r,1}e_{1,r}e_{r,t} + e_{1,1}^2 e_{r,t} - e_{1,i}e_{i,1}e_{r,t} + e_{r,i}e_{i,1}e_{1,t}$$
$$\equiv -\delta_{t,1}e_{r,1}e_{1,r}e_{r,1} - \widehat{\delta}_{t,1}e_{r,1}e_{1,1}e_{1,t} + \delta_{t,1}e_{1,1}^2 e_{r,1} + e_{r,1}e_{1,1}e_{1,t} \bmod G_1$$
$$\equiv -\delta_{t,1}e_{r,1}e_{1,r}e_{r,1} + \delta_{t,1}\left(2e_{1,1}^2 e_{r,1} + e_{r,1}\right) \bmod G.$$

Hence, for $t = 1$ the (monic) normal form of $S_2^{(r,i,1)}$ coincides with $\mathcal{G}_9^{(r)}$. For the composition $S_3^{(i,r,t)}$, we have

$$S_3^{(i,r,t)} = -e_{i,1}e_{1,i}e_{t,r} + e_{1,1}^2 e_{t,r} - e_{1,r}e_{r,1}e_{t,r} + e_{i,r}e_{1,i}e_{t,1}$$
$$\equiv \delta_{t,1}e_{1,1}^2 e_{1,r} - \widehat{\delta}_{t,1}e_{1,r}e_{1,1}e_{t,1} - \delta_{t,1}e_{1,r}e_{r,1}e_{1,r} + e_{1,r}e_{1,1}e_{t,1} \bmod G_1$$

$$\equiv \delta_{t,1}\left(e_{1,1}^2 e_{1,r} - e_{1,r}e_{r,1}e_{1,r} + e_{1,1}^2 e_{1,r} - e_{1,r}\right) \bmod G.$$

Hence, for $t = 1$ the (monic) normal form of $S_3^{(i,r,1)}$ coincides with $\mathcal{G}_{10}^{(r)}$. Next, we consider the composition $S_4^{(r,t,i,\ell,k)}$:

$$S_4^{(r,t,i,\ell,k)} = -e_{r,1}e_{1,i}e_{\ell,k} \equiv -\delta_{k,1}e_{r,1}e_{1,i}e_{\ell,1} \bmod G_1.$$

Obviously, for $k = 1$ the (monic) normal form of $S_4^{(r,t,i,\ell,1)}$ coincides with $\mathcal{G}_{11}^{(r,i,\ell)}$. Similarly, we can show that for $k = 1$, the (monic) normal form of $S_5^{(t,\ell,i,k,r)}$ coincides with $\mathcal{G}_{12}^{(\ell,i,r)}$. Finally, for the composition $S_6^{(r,i,t,\ell)}$, we have

$$\begin{aligned} S_6^{(r,i,t,\ell)} &= -e_{r,1}e_{1,r}e_{t,\ell} + e_{1,1}^2 e_{t,\ell} - e_{1,i}e_{i,1}e_{t,\ell} \\ &\equiv -\delta_{\ell,1}e_{r,1}e_{1,r}e_{t,1} + \delta_{t,1}e_{1,1}^2 e_{1,\ell} + \widehat{\delta}_{t,1}\delta_{\ell,1}e_{1,1}^2 e_{t,1} - \delta_{t,1}e_{1,i}e_{i,1}e_{1,\ell} \bmod G_1. \end{aligned}$$

Clearly, if $\ell = 1$ and $t \neq 1$ then the (monic) normal form of $S_6^{(r,i,t,\ell)}$ coincides with $\mathcal{G}_6^{(r,t)}$ (if $r < t$) and $\mathcal{G}_5^{(t,r)}$ (if $t < r$). If $\ell \neq 1$ and $t = 1$ then the (monic) normal form of $S_6^{(r,i,t,\ell)}$ coincides with $\mathcal{G}_7^{(i,\ell)}$ (if $i < \ell$) and $\mathcal{G}_8^{(\ell,i)}$ (if $\ell < i$). If $(\ell, t) = (1,1)$ then

$$\begin{aligned} S_6^{(r,i,1,1)} &\equiv -e_{r,1}e_{1,r}e_{1,1} + e_{1,1}^3 - e_{1,i}e_{i,1}e_{1,1} \bmod G_1 \\ &\equiv -\left(e_{1,1}e_{1,r}e_{r,1} - e_{1,1}\right) + e_{1,1}^3 - \left(e_{1,1}e_{i,1}e_{1,i} + e_{1,1}\right) \bmod G, \end{aligned}$$

since by definition $i, r \neq 1$. Hence, the (monic) normal form of $S_6^{(r,i,1,1)}$ in this case is

$$\mathcal{G}''^{(i,r)} = e_{1,1}e_{i,1}e_{1,i} + e_{1,1}e_{1,r}e_{r,1} - e_{1,1}^3 \quad (i, r \in \Omega \setminus \{1\}). \tag{6.4}$$

We note that the set $\mathcal{N} = \left\{ \mathcal{G}'^{(t,i)}, \mathcal{G}''^{(i,r)} \mid i, t, r \in \Omega \setminus \{1\} \right\}$ of the normal forms (6.3) and (6.4) is not self-reduced. So we eliminate from $\mathcal{G}'^{(i,i)}$ the leading monomial of $\mathcal{G}''^{(i,i)}$ and obtain

$$\mathcal{G}'^{(i,i)} = -2e_{1,1}e_{1,i}e_{i,1} + e_{1,1}^3 + e_{1,1},$$

whose monic form coincides with $\mathcal{G}_{13}^{(i)}$. We now eliminate from $\mathcal{G}''^{(i,i)}$ the leading monomial of $\mathcal{G}_{13}^{(i)}$ and obtain

$$\mathcal{G}''^{(i,i)} = e_{1,1}e_{i,1}e_{1,i} + \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} - e_{1,1}^3,$$

which coincides with $\mathcal{G}_{14}^{(i)}$. This completes the proof. □

Lemma 6.2.9. *The set of all normal forms modulo* $G \cup G_1 \cup G_2 \cup G_3$ *of nontrivial compositions among elements of* $G \cup G_1 \cup G_2 \cup G_3$ *includes the set* G_4 *which consists of the elements:*

$$\mathcal{G}_{17}^{(i)} = e_{1,1}^3 e_{1,i} - e_{1,1}e_{1,i}, \quad \mathcal{G}_{18}^{(i)} = e_{1,1}^3 e_{i,1} + e_{1,1}e_{i,1} \qquad (i \in \Omega \setminus \{1\}),$$
$$\mathcal{G}_{19} = e_{1,1}^5 - e_{1,1}.$$

Proof. For all $i \in \Omega \setminus \{1\}$ we consider the set of the following compositions:

$$S_1^{(i)} = \mathcal{G}_{14}^{(i)} e_{1,i} - e_{1,1}e_{i,1}\mathcal{G}_3^{(1,i,1,i)}, \quad S_2^{(i)} = \mathcal{G}_{13}^{(i)} e_{i,1} - e_{1,1}e_{1,i}\mathcal{G}_3^{(i,1,i,1)},$$
$$S_3^{(i)} = \mathcal{G}_{13}^{(i)} e_{1,i}e_{i,1} - e_{1,1}e_{1,i}\mathcal{G}_9^{(i)}.$$

We note that $S_1^{(i)} = -\frac{1}{2}e_{1,1}^3 e_{1,i} + \frac{1}{2}e_{1,1}e_{1,i}$ and $S_2^{(i)} = -\frac{1}{2}e_{1,1}^3 e_{i,1} - \frac{1}{2}e_{1,1}e_{i,1}$ are in the normal forms modulo $G \cup G_1 \cup G_2 \cup G_3$ and the monic forms of $S_1^{(i)}$ and $S_2^{(i)}$ coincide with $\mathcal{G}_{17}^{(i)}$ and $\mathcal{G}_{18}^{(i)}$ respectively. For $S_3^{(i)}$, we have

$$S_3^{(i)} = -\tfrac{1}{2}e_{1,1}^3 e_{1,i}e_{i,1} - \tfrac{1}{2}e_{1,1}e_{1,i}e_{i,1} + 2e_{1,1}e_{1,i}e_{1,1}^2 e_{i,1} + e_{1,1}e_{1,i}e_{i,1}$$
$$\equiv -\tfrac{1}{2}e_{1,1}^2\left(\tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1}\right) + \tfrac{1}{2}\left(\tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1}\right) \bmod G_3 = -\tfrac{1}{4}e_{1,1}^5 + \tfrac{1}{4}e_{1,1},$$

whose monic form coincides with $\mathcal{G}_{19}$. □

Lemma 6.2.10. *The self-reduced form* $\mathfrak{G}$ *of the set* $G \cup G_1 \cup G_2 \cup G_3 \cup G_4$ *consists of these elements:*

$$\mathcal{G}_0^{(i,j)} = e_{i,1}e_{1,1}e_{1,j} - e_{1,j}e_{1,1}e_{i,1} - e_{i,j} \quad (i, j \in \Omega \setminus \{1\}),$$
$$\mathcal{G}_1^{(i,j,k)} = e_{i,j}e_{j,k} - e_{i,1}e_{1,k} \quad (i, j, k \in \Omega;\ k \neq i, j \neq 1),$$
$$\mathcal{G}_2^{(i,j,k)} = e_{i,j}e_{k,i} - e_{1,j}e_{k,1} \quad (i, j, k \in \Omega;\ j \neq k,\ i \neq 1),$$
$$\mathcal{G}_3^{(i,j,k,\ell)} = e_{i,j}e_{k,\ell} \quad (i, j, k, \ell \in \Omega;\ i \neq \ell,\ j \neq k),$$
$$\mathcal{G}_4^{(i,j)} = e_{i,j}e_{j,i} - e_{i,1}e_{1,i} - e_{1,j}e_{j,1} + e_{1,1}^2 \quad (i, j \in \Omega \setminus \{1\}),$$
$$\mathcal{G}_5^{(i,j)} = e_{i,1}e_{1,j}e_{j,1} - e_{1,1}^2 e_{i,1} - e_{i,1} \quad (i, j \in \Omega;\ i \neq 1, j \neq i),$$
$$\mathcal{G}_6^{(i,j)} = e_{j,1}e_{1,j}e_{i,1} - e_{1,1}^2 e_{i,1} \quad (i, j \in \Omega \setminus \{1\}; i \neq j),$$
$$\mathcal{G}_7^{(i,j)} = e_{1,i}e_{i,1}e_{1,j} - e_{1,1}^2 e_{1,j} \quad (i, j \in \Omega \setminus \{1\};\ i \neq j),$$
$$\mathcal{G}_8^{(i,j)} = e_{1,i}e_{j,1}e_{1,j} - e_{1,1}^2 e_{1,i} + e_{1,i} \quad (i, j \in \Omega;\ i \neq j, i \neq 1),$$

$$
\begin{aligned}
\mathcal{G}_9^{(i)} &= e_{i,1}e_{1,i}e_{i,1} - 2e_{1,1}^2 e_{i,1} - e_{i,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{10}^{(j)} &= e_{1,j}e_{j,1}e_{1,j} - 2e_{1,1}^2 e_{1,j} + e_{1,j} \quad (j \in \Omega \setminus \{1\}),\\
\mathcal{G}_{11}^{(i,j,k)} &= e_{i,1}e_{1,j}e_{k,1} \quad (k,i,j \in \Omega;\ k,i \neq j),\\
\mathcal{G}_{12}^{(i,j,k)} &= e_{1,i}e_{j,1}e_{1,k} \quad (k,i,j \in \Omega;\ i,k \neq j),\\
\mathcal{G}_{13}^{(i)} &= e_{1,1}e_{1,i}e_{i,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{14}^{(i)} &= e_{1,1}e_{i,1}e_{1,i} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{15}^{(i)} &= e_{1,i}e_{i,1}e_{1,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{16}^{(i)} &= e_{i,1}e_{1,i}e_{1,1} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{17}^{(i)} &= e_{1,1}^3 e_{1,i} - e_{1,1}e_{1,i} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{18}^{(i)} &= e_{1,1}^3 e_{i,1} + e_{1,1}e_{i,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{19} &= e_{1,1}^5 - e_{1,1}.
\end{aligned}
$$

Proof. To obtain the self-reduced set $\mathfrak{G}$, we need to eliminate from $G \cup \bigcup_{i=1}^4 G_i$ all occurrences of any element of $\left\{\mathrm{LM}(u) \mid u \in G \cup \bigcup_{i=1}^4 G_i\right\}$ as a subword of any element of $G \cup \bigcup_{i=1}^4 G_i$. We first note that any element $g \in \bigcup_{i=1}^4 G_i$ is in the normal form modulo $G \cup \bigcup_{i=1}^4 G_i \setminus \{g\}$. So we only consider elements of G (see Definition 6.2.4). For all $k < i$, we have

$$
\begin{aligned}
\mathcal{R}_1^{(i,j,k,t)} &= e_{i,j}e_{j,k}e_{k,t} - e_{k,t}e_{j,k}e_{i,j} - e_{i,t} \equiv e_{i,1}e_{1,k}e_{k,t} - e_{k,t}e_{1,k}e_{i,1} - e_{i,t} \bmod G_1\\
&\equiv \delta_{t,1}\left(e_{i,1}e_{1,k}e_{k,1} - e_{1,1}^2 e_{i,1}\right) + \widehat{\delta}_{t,1}\left(e_{i,1}e_{1,1}e_{1,t} - e_{1,t}e_{1,1}e_{i,1}\right) - e_{i,t} \bmod G_3 \cup G_1.
\end{aligned}
$$

For $t \neq 1$ the last result coincides with $\mathcal{G}_0^{(i,t)}$. For $t = 1$, we combine the result with the set $\left\{\mathcal{G}_5^{(i,k)} \mid 1 < i < k\right\} \subset G_3$ and obtain the set $\left\{\mathcal{G}_5^{(i,k)} \mid k \neq i \neq 1\right\}$. For all $t < j$, we have

$$
\begin{aligned}
\mathcal{R}_2^{(i,j,t)} &= e_{i,j}e_{j,i}e_{i,t} - e_{i,t}e_{j,i}e_{i,j} - e_{i,t} \equiv e_{i,j}e_{j,1}e_{1,t} - e_{1,t}e_{j,1}e_{i,j} - e_{i,t} \bmod G_1\\
&\equiv \delta_{i,1}\left(e_{1,j}e_{j,1}e_{1,t} - e_{1,t}e_{j,1}e_{1,j}\right) + \widehat{\delta}_{i,1}\left(e_{i,1}e_{1,1}e_{1,t} - e_{1,t}e_{1,1}e_{i,1}\right) - e_{i,t} \bmod G_1.
\end{aligned}
$$

For $i \neq 1$ the last result coincides with $\mathcal{G}_0^{(i,t)}$(if $t \neq 1$) and $\mathcal{G}_5^{(i,1)}$ (if $t = 1$). For $i = 1$, we have

$$
\mathcal{R}_2^{(1,j,t)} \equiv \left[\delta_{t,1}\left(e_{1,j}e_{j,1}e_{1,1} - e_{1,1}e_{j,1}e_{1,j}\right) + \widehat{\delta}_{t,1}\left(e_{1,j}e_{j,1}e_{1,t} - e_{1,t}e_{j,1}e_{1,j}\right) - e_{1,t}\right] \bmod G_1
$$

$$\equiv \delta_{t,1}\left(e_{1,j}e_{j,1}e_{1,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1}\right) + \widehat{\delta}_{t,1}\left(e_{1,j}e_{j,1}e_{1,t} - e_{1,1}^2 e_{1,t}\right) \bmod G_3.$$

Clearly, for $t = 1$ the normal form of $\mathcal{R}_2^{(1,j,t)}$ coincides with $\mathcal{G}_{15}^{(i)}$. For $t \neq 1$, we combine the last result with the set $\left\{\mathcal{G}_7^{(j,t)} \mid 1 < j < t\right\} \subset G_3$ and obtain the set $\left\{\mathcal{G}_7^{(j,t)} \mid 1 \neq j \neq t \neq 1\right\}$. For all $t < i$, we have

$$\mathcal{R}_3^{(i,j,k,t)} = e_{i,j}e_{k,i}e_{t,k} - e_{t,k}e_{k,i}e_{i,j} + e_{t,j} \equiv e_{i,j}e_{1,i}e_{t,1} - e_{t,1}e_{1,i}e_{i,j} + e_{t,j} \bmod G_1$$
$$\equiv \delta_{j,1}(e_{i,1}e_{1,i}e_{t,1} - e_{t,1}e_{1,i}e_{i,1}) + \widehat{\delta}_{j,1}(e_{1,j}e_{1,1}e_{t,1} - e_{t,1}e_{1,1}e_{1,j}) + e_{t,j} \bmod G_1.$$

For $j \neq 1$ the monic form of the last result coincides with $\mathcal{G}_0^{(j,t)}$ (if $t \neq 1$) and $\mathcal{G}_8^{(j,1)}$(if $t = 1$). For $j = 1$, we have

$$\mathcal{R}_3^{(i,1,k,t)} \equiv \delta_{t,1}\left(e_{i,1}e_{1,i}e_{1,1} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1}\right) + \widehat{\delta}_{t,1}(e_{i,1}e_{1,i}e_{t,1} - e_{1,1}^2 e_{t,1}) \bmod G_3.$$

Clearly, for $t = 1$ the normal form of $\mathcal{R}_3^{(i,1,k,t)}$ coincides with $\mathcal{G}_{16}^{(i)}$. For $t \neq 1$, we combine the last result with the set $\left\{\mathcal{G}_6^{(i,t)} \mid 1 < i < t\right\} \subset G_3$ and obtain the set $\left\{\mathcal{G}_6^{(i,t)} \mid 1 \neq i \neq t \neq 1\right\}$. For $k < j$, we have

$$\mathcal{R}_4^{(i,j,k)} = e_{i,j}e_{k,i}e_{i,k} - e_{i,k}e_{k,i}e_{i,j} + e_{i,j} \equiv e_{1,j}e_{k,1}e_{i,k} - e_{i,k}e_{k,1}e_{1,j} + e_{i,j} \bmod G_1$$
$$\equiv \delta_{i,1}(e_{1,j}e_{k,1}e_{1,k} - e_{1,k}e_{k,1}e_{1,j}) + \widehat{\delta}_{i,1}\left(e_{1,j}e_{1,1}e_{i,1} - e_{i,1}e_{1,1}e_{1,j}\right) + e_{i,j} \bmod G_1$$
$$\equiv \delta_{i,1}\left(e_{1,j}e_{k,1}e_{1,k} - e_{1,1}^2 e_{1,j} + e_{1,j}\right) + \widehat{\delta}_{i,1}\left(e_{1,j}e_{1,1}e_{i,1} - e_{i,1}e_{1,1}e_{1,j} + e_{i,j}\right) \bmod G_3.$$

For $i \neq 1$ the monic form of the last result coincides with $\mathcal{G}_0^{(j,i)}$. For $i = 1$, we combine the last result with the set $\left\{\mathcal{G}_8^{(j,k)} \mid 1 < j < k\right\} \subset G_3$ and obtain the set $\left\{\mathcal{G}_8^{(j,k)} \mid 1 \neq j \neq k\right\}$. For $r < i$, $j \neq k$ or $t \neq r$, and $s \neq k$ or $t \neq i$, we have

$$\mathcal{R}_5^{(i,j,k,t,r,s)} = e_{i,j}e_{k,t}e_{r,s} - e_{r,s}e_{k,t}e_{i,j} \equiv \delta_{j,k}\left(e_{i,j}e_{j,t}e_{r,s} - e_{r,s}e_{j,t}e_{i,j}\right)$$
$$+ \widehat{\delta}_{j,k}\delta_{i,t}\left(e_{i,j}e_{k,i}e_{r,s} - e_{r,s}e_{k,i}e_{i,j}\right) \bmod G_1 \equiv \delta_{j,k}\Big[\delta_{i,t}\left(e_{i,j}e_{j,i}e_{r,s} - e_{r,s}e_{j,i}e_{i,j}\right)$$
$$+ \widehat{\delta}_{i,t}\left(e_{i,1}e_{1,t}e_{r,s} - e_{r,s}e_{1,t}e_{i,1}\right)\Big] \bmod G_1 \equiv \delta_{j,k}\left[\delta_{s,1}\widehat{\delta}_{i,t}\left(e_{i,1}e_{1,t}e_{r,1} - e_{r,1}e_{1,t}e_{i,1}\right)\right]$$
$$\bmod G_1 \equiv 0 \bmod G_3.$$

Similarly, we can show that $\mathcal{R}_6^{(i,j,k,t,s)} \equiv 0 \bmod G_1 \cup G_3$. □

6.2.2 Gröbner basis and finite dimensionality

The following lemma plays a crucial role in proving that the set $\mathfrak{G}$ of Lemma 6.2.10 is a Gröbner basis for the ideal I.

Lemma 6.2.11. *For the universal enveloping algebra* $\mathfrak{A}$*, either*

$$(i)\ \dim \mathfrak{A} = \infty, \textit{ or } \qquad (ii)\ \dim \mathfrak{A} < \infty \textit{ and } \dim \mathfrak{A} \geq 4n^2 + 1.$$

Proof. Suppose that $\dim(\mathfrak{A}) < \infty$. We show that over a field F containing $\mathrm{I} = \sqrt{-1}$ there exist four inequivalent irreducible representations of degree n of the anti-Jordan triple system $\mathfrak{J}$, in addition to the trivial representation of degree 1. For $k = 1, \ldots, 4$, we define the following maps:

$$\rho_k : \mathfrak{J} \to \operatorname{End} V_k,$$
$$\rho_1(E_{i,j}) = E_{i,j}, \quad \rho_2(E_{i,j}) = -E_{i,j}, \quad \rho_3(E_{i,j}) = \mathrm{I}\, E_{j,i}, \quad \rho_4(E_{i,j}) = -\mathrm{I}\, E_{j,i},$$

where $\mathrm{I} = \sqrt{-1}$. Our first step is to show that the maps ρ_k, $k = 1, \ldots, 4$ are representations of the anti-Jordan triple system $\mathfrak{J}$. Clearly ρ_1 is a representation (the natural representation). For ρ_2, we have

$$\rho_2\left(\langle E_{i,j}, E_{k,\ell}, E_{r,s}\rangle\right) = \rho_2\left(\delta_{j,k}\delta_{\ell,r}E_{i,s} - \delta_{s,k}\delta_{\ell,i}E_{r,j}\right) = -\delta_{j,k}\delta_{\ell,r}E_{i,s} + \delta_{s,k}\delta_{\ell,i}E_{r,j};$$

on the other hand, we have

$$\begin{aligned}\langle \rho_2(E_{i,j}), \rho_2(E_{k,\ell}), \rho_2(E_{r,s})\rangle &= \rho_2(E_{i,j})\rho_2(E_{k,\ell})\rho_2(E_{r,s}) - \rho_2(E_{r,s})\rho_2(E_{k,\ell})\rho_2(E_{i,j}) \\ &= -\delta_{j,k}\delta_{\ell,r}E_{i,s} + \delta_{s,k}\delta_{\ell,i}E_{r,j}.\end{aligned}$$

Thus ρ_2 is a representation. For ρ_3, we have

$$\rho_3\left(\langle E_{i,j}, E_{k,\ell}, E_{r,s}\rangle\right) = \rho_3\left(\delta_{j,k}\delta_{\ell,r}E_{i,s} - \delta_{s,k}\delta_{\ell,i}E_{r,j}\right) = \delta_{j,k}\delta_{\ell,r}\,\mathrm{I}E_{s,i} - \delta_{s,k}\delta_{\ell,i}\,\mathrm{I}E_{j,r};$$

on the other hand, we have

$$\begin{aligned}\langle \rho_3(E_{i,j}), \rho_3(E_{k,\ell}), \rho_3(E_{r,s})\rangle &= \rho_3(E_{i,j})\rho_3(E_{k,\ell})\rho_3(E_{r,s}) - \rho_3(E_{r,s})\rho_3(E_{k,\ell})\rho_3(E_{i,j}) \\ &= -\,\delta_{i,\ell}\delta_{k,s}\,\mathrm{I}E_{j,r} + \delta_{r,\ell}\delta_{k,j}\,\mathrm{I}E_{s,i}.\end{aligned}$$

Similarly, we can show that ρ_4 is a representation. We now show that for all $i, j = 1, \dots, 4$ and $i \neq j$, the representations ρ_i and ρ_j are inequivalent. Indeed, there is no matrix T so that

$$\rho_i(x) = T^{-1}\rho_j(x)\,T, \quad \text{for all } x \in \mathfrak{J},\ i \neq j.$$

This is easily seen by checking the trace on the both sides and using the definitions of the representations: $\mathrm{Tr}(\rho_i(x)) \neq \mathrm{Tr}(\rho_j(x)) = \mathrm{Tr}(T^{-1}\rho_j(x)\,T)$. The representations ρ_i, $i = 1, \dots, 4$ of $\mathfrak{J}$ can be extended to irreducible representations of the universal envelope $\mathfrak{A}$. Hence, $\dim \mathfrak{A} \geq \left(\sum_{k=1}^{4} \dim \operatorname{End} V_k\right) + 1 = 4n^2 + 1$. This completes the proof. □

We now can state the main theorem of this section.

Theorem 6.2.12. *With notation as above. If $\mathfrak{J}$ is the anti-Jordan triple system of all $n \times n$ matrices $(n \geq 2)$ then:*

(*i*) *The set $\mathfrak{G}$ is a Gröbner basis for the ideal I.*

(*ii*) *The universal enveloping algebra $\mathfrak{A}$ of $\mathfrak{J}$ is finite-dimensional with a basis $\mathfrak{B}$ consisting of $4n^2 + 1$ monomials:*

$$\mathfrak{B} = \left\{1,\, e_{i,j},\, e_{i,1}e_{1,j},\, e_{1,1}^2 e_{1,j},\, e_{1,1}^4 \mid i, j \in \Omega\right\} \cup \{e_{1,i}e_{j,1} \mid i, j \in \Omega, (i,j) \neq (1,1)\}$$
$$\cup \{e_{1,i}e_{1,1}e_{j,1} \mid i, j \in \Omega,\ j \neq 1\}.$$

Proof. By Lemma 6.2.10 the set $\mathfrak{G}$ is the self-reduced form of the set $G \cup \bigcup_{i=1}^{4} G_i$, so it remains to show that $\mathfrak{G}$ is closed under any composition. We note first that there are $4n^2 + 1$ monomials of $F\langle X\rangle$ that do not have the leading monomials of $\mathfrak{G}$ as factors, namely,

$$1, \quad e_{i,j}, \quad e_{i,1}e_{1,j}, \quad e_{1,i}e_{j,1},\ (i,j) \neq (1,1), \quad e_{1,i}e_{1,1}e_{j,1},\ j \neq 1, \quad e_{1,1}^2 e_{1,j}, \quad e_{1,1}^4,$$

for all $i, j \in \Omega$. Suppose on the contrary that $\mathfrak{G}$ is not a Gröbner basis for the ideal I. Then $\mathfrak{G}$ is not closed under at least one composition by Theorem 2.4.12, i.e., there exist $f, g \in \mathfrak{G}$ such that $fx - yg \not\equiv 0 \bmod \mathfrak{G}$. We add the normal form of $fx - yg$ to the set $\mathfrak{G}$. Hence, the number of the monomials of $F\langle X\rangle$ that do not have the leading monomials of $\mathfrak{G}$ as factors is less than $4n^2+1$. Hence, $\dim \mathfrak{A} < 4n^2+1$. But Lemma 6.2.11 implies that $\dim \mathfrak{A} \geq 4n^2+1$, which is a contradiction. This shows that $\mathfrak{G}$ is a Gröbner basis for the ideal I. The proof (ii) is obvious by using (i) and Proposition 2.4.4. □

6.3 The structure constants of the universal enveloping algebra

In this section we use Theorem 6.2.12 and the relations of Lemma 6.2.10 to compute the structure constants of the universal enveloping algebra $\mathfrak{A}$.

Lemma 6.3.1. *Define an anti-automorphism $\eta: F\langle X\rangle \to F\langle X\rangle$ of the free associative algebra generated by $X = \{e_{i,j}\}_{i,j\in\Omega}$ by $\eta(e_{i,j}) = e_{j,i}$. Then η induces an anti-automorphism of order 2 on $\mathfrak{A}$ (also denoted η).*

Proof. It suffices to show that the ideal $I = \langle \mathfrak{G}\rangle$ (see Theorem 6.2.12) is invariant under the action of η. We have, for example,

$$\eta\left(\mathcal{G}_0^{(i,j)}\right) = e_{j,1}e_{1,1}e_{1,i} - e_{1,i}e_{1,1}e_{j,1} - e_{j,i} = \mathcal{G}_0^{(j,i)} = 0,$$
$$\eta\left(\mathcal{G}_1^{(i,j,k)}\right) = e_{k,j}e_{j,i} - e_{k,1}e_{1,i} = \mathcal{G}_1^{(k,j,i)} = 0.$$

A similar argument applies to all the other elements of $\mathfrak{G}$. □

The next seven propositions give the explicit structure constants of $\mathfrak{A}$.

Proposition 6.3.2. *Let $i,j,k,\ell \in \Omega$. Then in $\mathfrak{A}$, we have*

$$\begin{aligned} e_{i,j}\cdot e_{k,\ell} = \delta_{j,k}\Big[\delta_{i,\ell}\Big\{ \left(\delta_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\right)e_{1,j}e_{j,1} + \left(\delta_{j,1}\widehat{\delta}_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\right)e_{i,1}e_{1,i} \\ - \widehat{\delta}_{i,1}\widehat{\delta}_{j,1}e_{1,1}^2\Big\} + \widehat{\delta}_{i,\ell}e_{i,1}e_{1,\ell}\Big] + \widehat{\delta}_{j,k}\delta_{i,\ell}e_{1,j}e_{k,1}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} e_{i,j}\cdot e_{k,1}e_{1,\ell} = \delta_{i,1}\Big[\left(\delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j}+\tfrac{1}{2}\widehat{\delta}_{\ell,j}\right)+\tfrac{1}{2}\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{1,1}^3 \\ + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}e_{1,1}^2e_{1,j} + \delta_{j,k}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{\ell,j}\left(\delta_{j,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)e_{1,1}^2e_{1,\ell} \\ + \tfrac{1}{2}\left(\delta_{\ell,1}\delta_{j,k}\widehat{\delta}_{\ell,j}-\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{1,1} - \left(\delta_{j,k}\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\right)e_{1,j}\Big] \\ + \widehat{\delta}_{i,1}\delta_{j,k}(e_{1,\ell}e_{1,1}e_{i,1}+e_{i,\ell}), \end{aligned} \tag{6.6}$$

$$\begin{aligned}
e_{\ell,1}e_{1,k}\cdot e_{j,i} = \delta_{i,1}\Big[&\left(\delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j}+\tfrac{1}{2}\widehat{\delta}_{\ell,j}\right)+\tfrac{1}{2}\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{1,1}^3+\widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}e_{1,1}^2e_{j,1}\\
&+\delta_{j,k}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{\ell,j}\left(\delta_{j,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)\left(e_{1,1}^2e_{\ell,1}+e_{\ell,1}\right)\\
&+\tfrac{1}{2}\Big(\delta_{\ell,1}\delta_{j,k}\widehat{\delta}_{\ell,j}-\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\Big)e_{1,1}-\delta_{j,k}\delta_{\ell,j}\widehat{\delta}_{\ell,1}e_{j,1}\Big]\\
&+\widehat{\delta}_{i,1}\delta_{j,k}\left(\delta_{\ell,1}e_{1,1}^2e_{1,i}+\widehat{\delta}_{\ell,1}\left(e_{1,i}e_{1,1}e_{\ell,1}+e_{\ell,i}\right)\right), \qquad (6.7)\\
e_{i,j}\cdot e_{1,k}e_{\ell,1} = \delta_{j,1}\Big[&\tfrac{1}{2}\left(\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}+\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}\right)e_{1,1}^3\\
&+\widehat{\delta}_{i,k}\delta_{k,\ell}\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\,e_{1,1}^2e_{i,1}+\delta_{i,k}\left(\delta_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\left(2\delta_{i,\ell}+\widehat{\delta}_{i,\ell}\right)\right)e_{1,1}^2e_{\ell,1}\\
&+\tfrac{1}{2}\left(\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}-\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}\right)e_{1,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\left(\delta_{i,k}\delta_{i,\ell}+\widehat{\delta}_{i,k}\delta_{k,\ell}\right)e_{i,1}\Big]\\
&+\widehat{\delta}_{j,1}\delta_{i,k}\left(\delta_{\ell,1}\left(e_{1,1}^2e_{1,j}-e_{1,j}\right)+\widehat{\delta}_{\ell,1}e_{1,j}e_{1,1}e_{\ell,1}\right); \quad (k,\ell)\neq(1,1), \qquad (6.8)\\
e_{1,\ell}e_{k,1}\cdot e_{j,i} = \delta_{j,1}\Big[&\tfrac{1}{2}\left(\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}+\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}\right)e_{1,1}^3+\widehat{\delta}_{i,k}\delta_{k,\ell}\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\,e_{1,1}^2e_{1,i}\\
&+\delta_{i,k}\left(\delta_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\left(2\delta_{i,\ell}+\widehat{\delta}_{i,\ell}\right)\right)\left(e_{1,1}^2e_{1,\ell}-e_{1,\ell}\right)\\
&+\tfrac{1}{2}\left(\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}-\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}\right)e_{1,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\delta_{i,k}\delta_{i,\ell}e_{1,i}\Big]\\
&+\widehat{\delta}_{j,1}\delta_{i,k}\left(\delta_{\ell,1}e_{1,1}^2e_{j,1}+\widehat{\delta}_{\ell,1}e_{1,\ell}e_{1,1}e_{1,j}\right); \quad (k,\ell)\neq(1,1). \qquad (6.9)
\end{aligned}$$

Proof. For (6.5), we use the relations $\mathcal{G}_3^{(i,j,k,\ell)}$, $\mathcal{G}_1^{(i,j,\ell)}$ and $\mathcal{G}_2^{(i,j,k)}$ and get

$$e_{i,j}\cdot e_{k,\ell} = \delta_{j,k}\left(\delta_{i,\ell}e_{i,j}e_{j,i}+\widehat{\delta}_{i,\ell}e_{i,1}e_{1,\ell}\right)+\widehat{\delta}_{j,k}\delta_{i,\ell}e_{1,j}e_{k,1}.$$

Using the relation $\mathcal{G}_4^{(i,j)}$ implies

$$\begin{aligned}
e_{i,j}\cdot e_{k,\ell} = \delta_{j,k}\Big[&\delta_{i,\ell}\left(\delta_{i,1}e_{1,j}e_{j,1}+\delta_{j,1}\widehat{\delta}_{i,1}e_{i,1}e_{1,i}+\widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\left(e_{i,1}e_{1,i}+e_{1,j}e_{j,1}-e_{1,1}^2\right)\right)\\
&+\widehat{\delta}_{i,\ell}e_{i,1}e_{1,\ell}\Big]+\widehat{\delta}_{j,k}\delta_{i,\ell}e_{1,j}e_{k,1}.
\end{aligned}$$

This completes the proof of (6.5). For (6.6), we use (6.5) (of the present proposition) and obtain

$$\begin{aligned}
\left(e_{i,j}e_{k,1}\right)e_{1,\ell} &= \delta_{j,k}\left(\delta_{i,1}e_{1,j}e_{j,1}e_{1,\ell}+\widehat{\delta}_{i,1}e_{i,1}e_{1,1}e_{1,\ell}\right)+\widehat{\delta}_{j,k}\delta_{i,1}e_{1,j}e_{k,1}e_{1,\ell}\\
&= \delta_{i,1}\left(\delta_{j,k}e_{1,j}e_{j,1}e_{1,\ell}+\widehat{\delta}_{j,k}e_{1,j}e_{k,1}e_{1,\ell}\right)+\widehat{\delta}_{i,1}\delta_{j,k}e_{i,1}e_{1,1}e_{1,\ell}.
\end{aligned}$$

We now write

$$A = e_{1,j}e_{j,1}e_{1,\ell}, \qquad B = \widehat{\delta}_{j,k}e_{1,j}e_{k,1}e_{1,\ell},$$

and use the relations $\mathcal{G}_0^{(i,\ell)}$ (if $\ell\neq 1$) and $\mathcal{G}_5^{(i,1)}$ (if $\ell = 1$) for the last term and obtain

$$\left(e_{i,j}e_{k,1}\right)e_{1,\ell} = \delta_{i,1}\left(\delta_{j,k}\,A+B\right)+\widehat{\delta}_{i,1}\delta_{j,k}(e_{1,\ell}e_{1,1}e_{i,1}+e_{i,\ell}). \qquad (6.10)$$

Using the relations $\mathcal{G}_{10}^{(j)}$, $\mathcal{G}_{15}^{(j)}$ and $\mathcal{G}_7^{(j,\ell)}$ gives

$$\begin{aligned}A &= \delta_{\ell,j}\left[\delta_{\ell,1}e_{1,1}^3 + \widehat{\delta}_{\ell,1}\left(2e_{1,1}^2e_{1,j} - e_{1,j}\right)\right] \\ &\quad + \widehat{\delta}_{\ell,j}\left[\delta_{j,1}e_{1,1}^2e_{1,\ell} + \widehat{\delta}_{j,1}\left(\delta_{\ell,1}\tfrac{1}{2}\left(e_{1,1}^3 + e_{1,1}\right) + \widehat{\delta}_{\ell,1}e_{1,1}^2e_{1,\ell}\right)\right] \\ &= \delta_{\ell,1}\left[\left(\delta_{\ell,j} + \tfrac{1}{2}\widehat{\delta}_{\ell,j}\right)e_{1,1}^3 + \tfrac{1}{2}\widehat{\delta}_{\ell,j}e_{1,1}\right] + \left[2\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j}(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1})\right]e_{1,1}^2e_{1,\ell} \\ &\quad - \delta_{\ell,j}\widehat{\delta}_{\ell,1}e_{1,j}.\end{aligned}$$

Using the relations $\mathcal{G}_{12}^{(j,k,\ell)}$, $\mathcal{G}_{14}^{(\ell)}$ and $\mathcal{G}_8^{(j,\ell)}$ gives

$$B = \widehat{\delta}_{j,k}\delta_{k,\ell}e_{1,j}e_{\ell,1}e_{1,\ell} = \widehat{\delta}_{j,k}\delta_{k,\ell}\left(\delta_{j,1}\tfrac{1}{2}\left(e_{1,1}^3 - e_{1,1}\right) + \widehat{\delta}_{j,1}(e_{1,1}^2e_{1,j} - e_{1,j})\right).$$

Using A and B in (6.10), and combining the coefficients completes the proof of (6.6). The proof of (6.7) is obvious by applying the anti-automorphism η (see Lemma 6.3.1) to both sides of (6.6) (of the present Proposition) and using the relations $\mathcal{G}_5^{(j,1)}$, $\mathcal{G}_5^{(\ell,1)}$ and $\mathcal{G}_8^{(i,1)}$. The proofs of (6.8) and (6.9) are similar. □

Proposition 6.3.3. *Let $i,j,k,\ell \in \Omega$. Then in $\mathfrak{A}$, we have*

$$e_{i,j}\cdot e_{1,k}e_{1,1}e_{\ell,1} = \delta_{i,k}\left[-e_{1,j}e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right]; \quad \ell \neq 1, \tag{6.11}$$

$$\begin{aligned}e_{i,j}\cdot e_{1,1}^2e_{1,k} &= \delta_{j,1}\Big[\delta_{i,1}\left(\delta_{k,1}e_{1,1}^4 + \widehat{\delta}_{k,1}e_{1,1}e_{1,k}\right) \\ &\quad + \widehat{\delta}_{i,1}\left(\delta_{i,k}\tfrac{1}{2}(e_{1,1}^4 - e_{1,1}^2) + e_{i,1}e_{1,k}\right)\Big] - \widehat{\delta}_{j,1}\delta_{i,1}\delta_{k,1}e_{1,j}e_{1,1},\end{aligned} \tag{6.12}$$

$$e_{i,j}\cdot e_{1,1}^4 = \delta_{j,1}\left[\delta_{i,1}e_{1,1} + \widehat{\delta}_{i,1}\left(e_{1,1}^2e_{i,1} + e_{i,1}\right)\right] - \widehat{\delta}_{j,1}\delta_{i,1}\left(e_{1,1}^2e_{1,j} - e_{1,j}\right). \tag{6.13}$$

Proof. For (6.11), let $\ell \neq 1$ and consider two cases. Case I. If $k = 1$ then (6.6) of Proposition 6.3.2 implies

$$e_{i,j}e_{1,1}^2 = \delta_{j,1}\left[\delta_{i,1}e_{1,1}^3 + \widehat{\delta}_{i,1}(e_{1,1}^2e_{i,1} + e_{i,1})\right] + \widehat{\delta}_{j,1}\delta_{i,1}\left(e_{1,1}^2e_{1,j} - e_{1,j}\right). \tag{6.14}$$

Multiply (6.14) by $e_{\ell,1}$ and use the relations $\mathcal{G}_{18}^{(\ell)}$, $\mathcal{G}_3^{(i,1,\ell,1)}$, $\mathcal{G}_{11}^{(1,j,\ell)}$ and $\mathcal{G}_{13}^{(j)}$ to obtain

$$\left(e_{i,j}e_{1,1}^2\right)e_{\ell,1} = -\delta_{j,1}\delta_{i,1}e_{1,1}e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{i,1}\left(\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right) - e_{1,j}e_{\ell,1}\right).$$

Case II. If $k \neq 1$ then (6.8) of Proposition 6.3.2 implies

$$(e_{i,j}e_{1,k}e_{1,1})\,e_{\ell,1} = \delta_{j,1}\left[\delta_{i,k}\left(\delta_{i,1}e_{1,1}^3e_{\ell,1} + \widehat{\delta}_{i,1}\tfrac{1}{2}\left(e_{1,1}^3 - e_{1,1}\right)e_{\ell,1}\right)\right] + \widehat{\delta}_{j,1}\delta_{i,k}\left(e_{1,1}^2e_{1,j} - e_{1,j}\right)e_{\ell,1}.$$

Using the relations $\mathcal{G}_{18}^{(\ell)}$, $\mathcal{G}_{11}^{(1,j,\ell)}$ and $\mathcal{G}_{13}^{(j)}$ gives

$$\begin{aligned}(e_{i,j}e_{1,k}e_{1,1})\,e_{\ell,1} &= \delta_{j,1}\left[\delta_{i,k}\left(-\delta_{i,1}e_{1,1}e_{\ell,1} + \widehat{\delta}_{i,1}\tfrac{1}{2}(-e_{1,1} - e_{1,1})e_{\ell,1}\right)\right] \\ &\quad + \widehat{\delta}_{j,1}\delta_{i,k}\left[\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right) - e_{1,j}e_{\ell,1}\right] \\ &= -\delta_{j,1}\delta_{i,k}e_{1,1}e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{i,k}\left[\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right) - e_{1,j}e_{\ell,1}\right].\end{aligned}$$

Combining the results of the two cases completes the proof of (6.11). For (6.12), we multiply (6.14) by $e_{1,k}$ and use the relations $\mathcal{G}_{17}^{(k)}$, $\mathcal{G}_{12}^{(1,i,k)}$, $\mathcal{G}_{14}^{(i)}$, $\mathcal{G}_{11}^{(1,j,1)}$ and $\mathcal{G}_{3}^{(1,j,1,k)}$. The proof of (6.13) is similar. □

Proposition 6.3.4. *Let $i, j, k, \ell \in \Omega$. Then in $\mathfrak{A}$, we have*

$$\begin{aligned}e_{i,1}e_{1,j}\cdot e_{k,1}e_{1,\ell} &= \delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j}\delta_{i,1}e_{1,1}^4 + \tfrac{1}{2}\widehat{\delta}_{\ell,j}\delta_{i,1}\left(e_{1,1}^4 + e_{1,1}^2\right)\right) \\ &\quad + \tfrac{1}{2}\Big[\delta_{j,k}\widehat{\delta}_{i,1}\delta_{i,\ell}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right) \\ &\qquad + \widehat{\delta}_{j,k}\delta_{k,\ell}\left(\widehat{\delta}_{j,1}\widehat{\delta}_{i,1}\delta_{i,j} + \delta_{j,1}\delta_{i,1}\right)\Big]\left(e_{1,1}^4 - e_{1,1}^2\right) \\ &\quad + \delta_{j,k}\left(\delta_{\ell,1}\widehat{\delta}_{i,1} + \delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)e_{i,1}e_{1,\ell}, \qquad (6.15)\\ e_{i,1}e_{1,j}\cdot e_{1,k}e_{\ell,1} &= \delta_{j,1}\widehat{\delta}_{k,1}\delta_{k,\ell}\tfrac{1}{2}\left(\delta_{i,1}\left(e_{1,1}^4 + e_{1,1}^2\right) + 2\widehat{\delta}_{i,1}e_{i,1}e_{1,1}\right) \\ &\quad - \delta_{k,1}\left(\delta_{j,1}\delta_{i,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\delta_{i,j}\right)e_{1,1}e_{\ell,1}; \quad (k,\ell) \neq (1,1), \qquad (6.16)\\ e_{i,1}e_{1,j}\cdot e_{1,k}e_{1,1}e_{\ell,1} &= -\delta_{k,1}\left(\delta_{i,1}\delta_{j,1} + \widehat{\delta}_{j,1}\delta_{i,j}\right)e_{1,1}^2e_{\ell,1}; \quad \ell \neq 1, \qquad (6.17)\\ e_{i,1}e_{1,j}\cdot e_{1,1}^2e_{1,k} &= \delta_{j,1}\Big[\delta_{k,1}\left(\delta_{i,1}e_{1,1} + \widehat{\delta}_{i,1}\left(e_{1,1}^2e_{i,1} + e_{i,1}\right)\right) \\ &\qquad + \widehat{\delta}_{k,1}\left(\delta_{i,1}e_{1,1}^2e_{1,k} + \widehat{\delta}_{i,1}\left(e_{1,k}e_{1,1}e_{i,1} + e_{i,k}\right)\right)\Big] \\ &\quad - \widehat{\delta}_{j,1}\delta_{k,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^3 - e_{1,1}\right), \qquad (6.18)\\ e_{i,1}e_{1,j}\cdot e_{1,1}^4 &= \delta_{j,1}e_{i,1}e_{1,1} + \widehat{\delta}_{j,1}\delta_{j,i}\tfrac{1}{2}\left(e_{1,1}^2 - e_{1,1}^4\right). \qquad (6.19)\end{aligned}$$

Proof. For (6.15), we use (6.6) of Proposition 6.3.2 and get

$$\begin{aligned}e_{i,1}\left(e_{1,j}e_{k,1}e_{1,\ell}\right) &= \left(\delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j} + \tfrac{1}{2}\widehat{\delta}_{\ell,j}\right) + \tfrac{1}{2}\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{i,1}e_{1,1}^3 + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\,e_{i,1}e_{1,1}^2e_{1,j} \\ &\quad + \delta_{j,k}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)e_{i,1}e_{1,1}^2e_{1,\ell} \\ &\quad + \tfrac{1}{2}\left(\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{\ell,j} - \widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{i,1}e_{1,1} \\ &\quad - \left(\delta_{j,k}\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\right)e_{i,1}e_{1,j}. \qquad (6.20)\end{aligned}$$

By (6.12) of Proposition 6.3.3 we have

$$e_{i,1}e_{1,1}^3 = \delta_{i,1}e_{1,1}^4 + \widehat{\delta}_{i,1}e_{i,1}e_{1,1}, \tag{6.21}$$

$$e_{i,1}e_{1,1}^2e_{1,\ell} = e_{i,1}e_{1,\ell} + \tfrac{1}{2}\widehat{\delta}_{i,1}\delta_{i,\ell}\left(e_{1,1}^4 - e_{1,1}^2\right);\ \ \ell \neq 1. \tag{6.22}$$

Using (6.21) and (6.22) in (6.20) gives

$$\begin{aligned} e_{i,1}\left(e_{1,j}e_{k,1}e_{1,\ell}\right) = & \left(\delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j} + \tfrac{1}{2}\widehat{\delta}_{\ell,j}\right) + \tfrac{1}{2}\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)\left(\delta_{i,1}e_{1,1}^4 + \widehat{\delta}_{i,1}e_{i,1}e_{1,1}\right) \\ & + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\ \left(e_{i,1}e_{1,j} + \tfrac{1}{2}\widehat{\delta}_{i,1}\delta_{i,j}\left(e_{1,1}^4 - e_{1,1}^2\right)\right) \\ & + \delta_{j,k}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{\ell,j}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)\left(e_{i,1}e_{1,\ell} + \tfrac{1}{2}\widehat{\delta}_{i,1}\delta_{i,\ell}\left(e_{1,1}^4 - e_{1,1}^2\right)\right) \\ & + \tfrac{1}{2}\left(\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{\ell,j} - \widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{i,1}e_{1,1} - \left(\delta_{j,k}\delta_{\ell,j}\widehat{\delta}_{\ell,1} + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\right)e_{i,1}e_{1,j}. \end{aligned}$$

Combining the coefficients in the last equation completes the proof of (6.15). For (6.16), let $(k,\ell) \neq (1,1)$. Using (6.8) of Proposition 6.3.2 gives

$$\begin{aligned} e_{i,1}\left(e_{1,j}e_{1,k}e_{\ell,1}\right) = & \tfrac{1}{2}\delta_{j,1}\widehat{\delta}_{k,1}\delta_{k,\ell}\,e_{i,1}\left(e_{1,1}^3 + e_{1,1}\right) + \delta_{j,1}\delta_{k,1}\,e_{i,1}e_{1,1}^2e_{\ell,1} \\ & + \widehat{\delta}_{j,1}\delta_{k,1}\widehat{\delta}_{\ell,1}\,e_{i,1}e_{1,j}e_{1,1}e_{\ell,1}. \end{aligned}$$

Using (6.21)(of the present proof) and (6.11) of Proposition 6.3.3 implies

$$\begin{aligned} e_{i,1}\left(e_{1,j}e_{1,k}e_{\ell,1}\right) = & \tfrac{1}{2}\delta_{j,1}\widehat{\delta}_{k,1}\delta_{k,\ell}\ \left(\delta_{i,1}\left(e_{1,1}^4 + e_{1,1}^2\right) + \widehat{\delta}_{i,1}2e_{i,1}e_{1,1}\right) \\ & - \delta_{k,1}\delta_{j,1}\delta_{i,1}e_{1,1}e_{\ell,1} - \widehat{\delta}_{j,1}\delta_{k,1}\widehat{\delta}_{\ell,1}\delta_{i,j}e_{1,1}e_{\ell,1}. \end{aligned}$$

This completes the proof of (6.16). For (6.17), let $\ell \neq 1$. Using (6.11) of Proposition 6.3.3 gives

$$e_{i,1}(e_{1,j}e_{1,k}e_{1,1}e_{\ell,1}) = -\delta_{j,1}\delta_{k,1}e_{i,1}e_{1,1}e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{k,1}e_{i,1}\left(\delta_{j,\ell}\tfrac{1}{2}(e_{1,1}^4 + e_{1,1}^2) - e_{1,j}e_{\ell,1}\right).$$

We write

$$A = \widehat{\delta}_{j,1}\delta_{j,\ell}\tfrac{1}{2}e_{i,1}(e_{1,1}^4 + e_{1,1}^2), \quad B = \widehat{\delta}_{j,1}e_{i,1}e_{1,j}e_{\ell,1},$$

and use the relation $\mathcal{G}_{11}^{(i,1,\ell)}$ for the first term of the last equation to obtain

$$e_{i,1}(e_{1,j}e_{1,k}e_{1,1}e_{\ell,1}) = -\delta_{j,1}\delta_{k,1}\delta_{i,1}e_{1,1}^2e_{\ell,1} + \delta_{k,1}\left(A - B\right). \tag{6.23}$$

By (6.13) of Proposition 6.3.3 and (6.6) of Proposition 6.3.2 we have

$$e_{i,1}e_{1,1}^4 = \delta_{i,1}e_{1,1} + \widehat{\delta}_{i,1}\left(e_{1,1}^2e_{i,1} + e_{i,1}\right), \tag{6.24}$$

$$e_{i,1}e_{1,1}e_{1,k} = \delta_{i,1}e_{1,1}^2 e_{1,k} + \widehat{\delta}_{i,1}\left(e_{1,k}e_{1,1}e_{i,1} + e_{i,k}\right), \tag{6.25}$$

respectively. Using (6.24) and (6.25) (with $k = 1$) implies

$$A = \widehat{\delta}_{j,1}\delta_{j,\ell}\left(\delta_{i,1}\tfrac{1}{2}\left(e_{1,1} + e_{1,1}^3\right) + \widehat{\delta}_{i,1}\left(e_{1,1}^2 e_{i,1} + e_{i,1}\right)\right).$$

Using (6.7) of Proposition 6.3.2 gives

$$B = \widehat{\delta}_{j,1}\delta_{j,\ell}\left[\delta_{i,1}\tfrac{1}{2}\left(e_{1,1}^3 + e_{1,1}\right) + \widehat{\delta}_{i,1}\left(\left(2\delta_{i,\ell} + \widehat{\delta}_{i,\ell}\right)e_{1,1}^2 e_{i,1} + e_{i,1}\right)\right] + \widehat{\delta}_{j,1}\widehat{\delta}_{j,\ell}\delta_{i,j}e_{1,1}^2 e_{\ell,1}.$$

Using A and B in (6.23) gives

$$\begin{aligned} e_{i,1}(e_{1,j}e_{1,k}e_{1,1}e_{\ell,1}) &= -\delta_{j,1}\delta_{k,1}\delta_{i,1}e_{1,1}^2 e_{\ell,1} + \widehat{\delta}_{j,1}\delta_{k,1}\left(-\delta_{j,\ell}\widehat{\delta}_{i,1}\delta_{i,\ell}\, e_{1,1}^2 e_{i,1} - \widehat{\delta}_{j,\ell}\delta_{i,j}\, e_{1,1}^2 e_{\ell,1}\right) \\ &= -\delta_{j,1}\delta_{k,1}\delta_{i,1}e_{1,1}^2 e_{\ell,1} - \widehat{\delta}_{j,1}\delta_{k,1}\delta_{i,j}e_{1,1}^2 e_{\ell,1}. \end{aligned}$$

This completes the proof of (6.17). For (6.18), (6.12) of Proposition (6.3.3) implies

$$e_{i,1}\left(e_{1,j}e_{1,1}^2 e_{1,k}\right) = e_{i,1}\left[\delta_{j,1}\left(\delta_{k,1}e_{1,1}^4 + \widehat{\delta}_{k,1}e_{1,1}e_{1,k}\right) - \widehat{\delta}_{j,1}\delta_{k,1}e_{1,j}e_{1,1}\right]. \tag{6.26}$$

By (6.8) of Proposition 6.3.2 we have

$$\widehat{\delta}_{j,1}e_{i,1}e_{1,j}e_{1,1} = \widehat{\delta}_{j,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^3 - e_{1,1}\right). \tag{6.27}$$

Using (6.24), (6.25) and (6.27) in (6.26) completes the proof of (6.18). For (6.19), we use (6.13) of Proposition 6.3.3 and get

$$e_{i,1}e_{1,j}e_{1,1}^4 = \delta_{j,1}e_{i,1}e_{1,1} - \widehat{\delta}_{j,1}e_{i,1}\left(e_{1,1}^2 e_{1,j} - e_{1,j}\right). \tag{6.28}$$

Using (6.12) of Proposition 6.3.3 gives

$$\widehat{\delta}_{j,1}e_{i,1}e_{1,1}^2 e_{1,j} = \widehat{\delta}_{j,1}\left[\delta_{i,1}e_{1,1}e_{1,j} + \widehat{\delta}_{i,1}\left(\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 - e_{1,1}^2\right) + e_{i,1}e_{1,j}\right)\right]. \tag{6.29}$$

Using (6.29) in (6.28) completes the proof of(6.19). □

Proposition 6.3.5. *Let $i, j, k, \ell \in \Omega$ and $(i, j) \neq (1, 1)$. Then in $\mathfrak{A}$, we have*

$$\begin{aligned} e_{1,i}e_{j,1}\cdot e_{1,k}e_{\ell,1} = \tfrac{1}{2}\Big[&\delta_{i,1}\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1}\left(e_{1,1}^4 - e_{1,1}^2\right) \\ &+ \left\{\delta_{j,k}\widehat{\delta}_{i,1}\delta_{i,\ell}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(2\delta_{j,\ell} + \widehat{\delta}_{j,\ell}\right)\right)\right. \end{aligned}$$

$$+ \widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\left(\widehat{\delta}_{j,1}\widehat{\delta}_{j,\ell}\delta_{i,j} + \delta_{i,1}\delta_{j,1}\right)\Big\}\left(e_{1,1}^4 + e_{1,1}^2\right)\Big]$$

$$- \delta_{j,k}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\left(\delta_{\ell,1}\widehat{\delta}_{i,1} + \widehat{\delta}_{\ell,1}\right)\right)e_{1,i}e_{\ell,1}; \quad (k,\ell) \neq (1,1), \tag{6.30}$$

$$e_{1,i}e_{j,1}\cdot e_{1,1}^2 e_{1,k} = \left(-\delta_{j,1}\widehat{\delta}_{i,1}\delta_{k,1} + \widehat{\delta}_{j,1}\delta_{i,j}\delta_{k,i}\right)e_{1,1}^2 e_{1,i} + \delta_{j,1}\delta_{k,1}\widehat{\delta}_{i,1}e_{1,i}$$

$$+ \delta_{i,j}\widehat{\delta}_{j,1}\left(\tfrac{1}{2}\delta_{k,1}\left(e_{1,1}^3 + e_{1,1}\right) + \widehat{\delta}_{k,i}\widehat{\delta}_{k,1}e_{1,1}^2 e_{1,k}\right), \tag{6.31}$$

$$e_{1,i}e_{j,1}\cdot e_{1,1}^4 = \delta_{j,1}e_{1,i}e_{1,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right), \tag{6.32}$$

$$e_{1,i}e_{j,1}\cdot e_{1,k}e_{1,1}e_{\ell,1} = -\delta_{j,k}e_{1,i}e_{1,1}e_{\ell,1}; \quad \ell \neq 1. \tag{6.33}$$

Proof. Let $(i,j) \neq (1,1)$. For (6.30), let $(k,\ell) \neq (1,1)$. Using (6.8) of Proposition 6.3.2 implies

$$\begin{aligned} e_{1,i}\left(e_{j,1}e_{1,k}e_{\ell,1}\right) = {} & \tfrac{1}{2}\left(\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1} + \widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{j,1}\right)e_{1,i}e_{1,1}^3 + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\, e_{1,i}e_{1,1}^2 e_{j,1} \\ & + \delta_{j,k}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(2\delta_{j,\ell} + \widehat{\delta}_{j,\ell}\right)\right)e_{1,i}e_{1,1}^2 e_{\ell,1} \\ & + \tfrac{1}{2}\left(\widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{j,1} - \delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1}\right)e_{1,i}e_{1,1} \\ & + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(\delta_{j,k}\delta_{j,\ell} + \widehat{\delta}_{j,k}\delta_{k,\ell}\right)e_{1,i}e_{j,1}. \end{aligned} \tag{6.34}$$

By (6.11) and (6.12) of Proposition 6.3.3 we have

$$e_{1,i}e_{1,1}^2 e_{\ell,1} = \left(-e_{1,i}e_{\ell,1} + \widehat{\delta}_{i,1}\delta_{i,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right); \quad \ell \neq 1, \tag{6.35}$$

$$e_{1,i}e_{1,1}^3 = \delta_{i,1}e_{1,1}^4 - \widehat{\delta}_{i,1}e_{1,i}e_{1,1}, \tag{6.36}$$

respectively. Using (6.35) and (6.36) in (6.34) gives

$$\begin{aligned} e_{1,i}\left(e_{j,1}e_{1,k}e_{\ell,1}\right) = {} & \tfrac{1}{2}\left(\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1} + \widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{j,1}\right)\left(\delta_{i,1}e_{1,1}^4 - \widehat{\delta}_{i,1}e_{1,i}e_{1,1}\right) \\ & + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(-e_{1,i}e_{j,1} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right) \\ & + \delta_{j,k}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(2\delta_{j,\ell} + \widehat{\delta}_{j,\ell}\right)\right)\left(-e_{1,i}e_{\ell,1} + \widehat{\delta}_{i,1}\delta_{i,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right) \\ & + \tfrac{1}{2}\left(\widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{j,1} - \delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1}\right)e_{1,i}e_{1,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(\delta_{j,k}\delta_{j,\ell} + \widehat{\delta}_{j,k}\delta_{k,\ell}\right)e_{1,i}e_{j,1}. \end{aligned}$$

Combining the coefficients of the last equation completes the proof of (6.30). For (6.31), we use (6.12) of Proposition 6.3.3 and get

$$e_{1,i}\left(e_{j,1}e_{1,1}^2 e_{1,k}\right) = e_{1,i}\left[\delta_{j,1}\left(\delta_{k,1}e_{1,1}^4 + \widehat{\delta}_{k,1}e_{1,1}e_{1,k}\right) + \widehat{\delta}_{j,1}\left(\delta_{j,k}\tfrac{1}{2}\left(e_{1,1}^4 - e_{1,1}^2\right) + e_{j,1}e_{1,k}\right)\right]. \tag{6.37}$$

By (6.13) of Proposition 6.3.3 and (6.6) of Proposition 6.3.2 we have

$$e_{1,i}e_{1,1}^4 = \delta_{i,1}e_{1,1} - \widehat{\delta}_{i,1}\left(e_{1,1}^2 e_{1,i} - e_{1,i}\right), \quad \widehat{\delta}_{k,1}e_{1,i}e_{1,1}e_{1,k} = \delta_{i,1}\widehat{\delta}_{k,1}e_{1,1}^2 e_{1,k},$$

$$e_{1,i}e_{1,1}^2 = \delta_{i,1}e_{1,1}^3 + \widehat{\delta}_{i,1}\left(e_{1,1}^2 e_{1,i} - e_{1,i}\right),$$

$$\begin{aligned}\widehat{\delta}_{j,1}e_{1,i}e_{j,1}e_{1,k} = \widehat{\delta}_{j,1}\Big[&\tfrac{1}{2}\left(\delta_{i,j}\delta_{k,1} + \widehat{\delta}_{i,j}\delta_{j,k}\delta_{i,1}\right)e_{1,1}^3 + \widehat{\delta}_{i,j}\delta_{j,k}\widehat{\delta}_{i,1}\, e_{1,1}^2 e_{1,i}\\ &+ \delta_{i,j}\left(2\delta_{k,i}\widehat{\delta}_{k,1} + \widehat{\delta}_{k,i}\widehat{\delta}_{i,1}\widehat{\delta}_{k,1}\right)e_{1,1}^2 e_{1,k} + \tfrac{1}{2}\left(\delta_{k,1}\delta_{i,j}\widehat{\delta}_{k,i} - \widehat{\delta}_{i,j}\delta_{j,k}\delta_{i,1}\right)e_{1,1}\\ &- \left(\delta_{i,j}\delta_{k,i}\widehat{\delta}_{k,1} + \widehat{\delta}_{i,j}\delta_{j,k}\widehat{\delta}_{i,1}\right)e_{1,i}\Big].\end{aligned}$$

Using the last four equations in (6.37) (note that $(i,j) \neq (1,1)$ by assumption) gives

$$\begin{aligned}e_{1,i}\left(e_{j,1}e_{1,1}^2 e_{1,k}\right) = &-\delta_{j,1}\delta_{k,1}\widehat{\delta}_{i,1}\left(e_{1,1}^2 e_{1,i} - e_{1,i}\right) + \widehat{\delta}_{j,1}\delta_{j,k}\Big[\delta_{i,1}\tfrac{1}{2}\left(e_{1,1} - e_{1,1}^3\right)\\ &- \widehat{\delta}_{i,1}\left(e_{1,1}^2 e_{1,i} - e_{1,i}\right)\Big] + \widehat{\delta}_{j,1}\Big[\tfrac{1}{2}\left(\delta_{i,j}\delta_{k,1} + \widehat{\delta}_{i,j}\delta_{j,k}\delta_{i,1}\right)e_{1,1}^3 + \widehat{\delta}_{i,j}\delta_{j,k}\widehat{\delta}_{i,1}\, e_{1,1}^2 e_{1,i}\\ &+ \delta_{i,j}\left(2\delta_{k,i}\widehat{\delta}_{k,1} + \widehat{\delta}_{k,i}\widehat{\delta}_{i,1}\widehat{\delta}_{k,1}\right)e_{1,1}^2 e_{1,k} + \tfrac{1}{2}\left(\delta_{i,j}\widehat{\delta}_{k,i}\delta_{k,1} - \widehat{\delta}_{i,j}\delta_{j,k}\delta_{i,1}\right)e_{1,1}\\ &- \left(\delta_{i,j}\delta_{k,i}\widehat{\delta}_{k,1} + \widehat{\delta}_{i,j}\delta_{j,k}\widehat{\delta}_{i,1}\right)e_{1,i}\Big].\end{aligned}$$

Combining the coefficients of the last equation gives

$$\begin{aligned}e_{1,i}\left(e_{j,1}e_{1,1}^2 e_{1,k}\right) = &\left[-\widehat{\delta}_{i,1}\left(\delta_{j,1}\delta_{k,1} + \widehat{\delta}_{j,1}\delta_{j,k}\right) + \widehat{\delta}_{j,1}\left(\widehat{\delta}_{i,j}\delta_{j,k}\widehat{\delta}_{i,1} + 2\delta_{i,j}\delta_{k,i}\widehat{\delta}_{k,1}\right)\right]e_{1,1}^2 e_{1,i}\\ &+ \left[\widehat{\delta}_{i,1}\left(\delta_{j,1}\delta_{k,1} + \widehat{\delta}_{j,1}\delta_{j,k}\right) - \widehat{\delta}_{j,1}\left(\delta_{i,j}\delta_{k,i}\widehat{\delta}_{k,1} + \widehat{\delta}_{i,j}\delta_{j,k}\widehat{\delta}_{i,1}\right)\right]e_{1,i}\\ &+ \tfrac{1}{2}\delta_{k,1}\delta_{i,j}\widehat{\delta}_{j,1}\left(e_{1,1}^3 + e_{1,1}\right) + \delta_{i,j}\widehat{\delta}_{k,i}\widehat{\delta}_{i,1}\widehat{\delta}_{k,1}e_{1,1}^2 e_{1,k},\end{aligned}$$

which proves (6.31). For (6.32), we use (6.13) of Proposition 6.3.3 and get

$$e_{1,i}\left(e_{j,1}e_{1,1}^4\right) = \delta_{j,1}e_{1,i}e_{1,1} + \widehat{\delta}_{j,1}\left(e_{1,i}e_{1,1}^2 e_{j,1} + e_{1,i}e_{j,1}\right).$$

Using (6.35) (of the present proof) with $\ell = j$ gives

$$e_{1,i}\left(e_{j,1}e_{1,1}^4\right) = \delta_{j,1}e_{1,i}e_{1,1} + \widehat{\delta}_{j,1}\left[\left(-e_{1,i}e_{j,1} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right) + e_{1,i}e_{j,1}\right].$$

This completes the proof of (6.32). The proof of (6.33) is obvious; since for $\ell \neq 1$, $e_{j,1}e_{1,k}e_{1,1}e_{\ell,1} = -\delta_{j,k}e_{1,1}e_{\ell,1}$ by (6.11) of Proposition 6.3.3. □

Proposition 6.3.6. *Let $i,j,k \in \Omega$. Then in $\mathfrak{A}$, we have*

$$e_{1,1}^2 e_{1,k}\cdot e_{i,j} = \delta_{k,i}\Big[\delta_{j,1}\Big(\big(\delta_{k,1} + \tfrac{1}{2}\widehat{\delta}_{k,1}\big)e_{1,1}^4 + \tfrac{1}{2}\widehat{\delta}_{k,1}e_{1,1}^2\Big) + \widehat{\delta}_{j,1}e_{1,1}e_{1,j}\Big] - \widehat{\delta}_{k,i}\delta_{j,1}\delta_{k,1}e_{1,1}e_{i,1}, \tag{6.38}$$

$$
\begin{aligned}
e_{1,1}^2 e_{1,k} \cdot e_{i,1} e_{1,j} = &\left[\delta_{k,i}\delta_{j,1}\left(\delta_{k,1} + \tfrac{1}{2}\widehat{\delta}_{k,1}\right) + \tfrac{1}{2}\widehat{\delta}_{k,i}\delta_{k,1}\widehat{\delta}_{i,1}\delta_{i,j}\right] e_{1,1} \\
&+ \tfrac{1}{2}\left(\delta_{k,i}\widehat{\delta}_{k,1}\delta_{j,1} - \widehat{\delta}_{k,i}\delta_{k,1}\widehat{\delta}_{i,1}\delta_{i,j}\right) e_{1,1}^3 \\
&+ \left(\delta_{k,1}\left(\delta_{k,i}\widehat{\delta}_{j,1} - \widehat{\delta}_{k,i}\delta_{i,1}\right) + \widehat{\delta}_{k,1}\widehat{\delta}_{j,1}\right) e_{1,1}^2 e_{1,j}, \qquad (6.39)
\end{aligned}
$$

$$
e_{1,1}^2 e_{1,k} \cdot e_{1,i} e_{j,1} = \delta_{k,1}\left(-\delta_{i,1} e_{1,1}^2 e_{j,1} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^3 + e_{1,1}\right)\right);\ (i,j) \neq (1,1), \qquad (6.40)
$$

$$
e_{1,1}^2 e_{1,k} \cdot e_{1,j} e_{1,1} e_{\ell,1} = \delta_{k,1}\delta_{j,1} e_{1,1} e_{\ell,1}; \quad \ell \neq 1, \qquad (6.41)
$$

$$
e_{1,1}^2 e_{1,k} \cdot e_{1,1}^2 e_{1,j} = \delta_{k,1} e_{1,1} e_{1,j}, \qquad (6.42)
$$

$$
e_{1,1}^2 e_{1,j} \cdot e_{1,1}^4 = \delta_{j,1} e_{1,1}^3. \qquad (6.43)
$$

Proof. For (6.38), we use (6.5) of Proposition 6.3.2 and get

$$
\begin{aligned}
e_{1,1}^2\left(e_{1,k} e_{i,j}\right) &= e_{1,1}^2\left[\delta_{k,i}\left(\delta_{j,1} e_{1,k} e_{k,1} + \widehat{\delta}_{j,1} e_{1,1} e_{1,j}\right) + \widehat{\delta}_{k,i}\delta_{j,1} e_{1,k} e_{i,1}\right] \\
&= \delta_{k,i}\left[\delta_{j,1}\left(\delta_{k,1} e_{1,1}^4 + \widehat{\delta}_{k,1}\tfrac{1}{2}(e_{1,1}^4 + e_{1,1}^2)\right) + \widehat{\delta}_{j,1} e_{1,1} e_{1,j}\right] \\
&\quad - \widehat{\delta}_{k,i}\delta_{j,1}\delta_{k,1} e_{1,1} e_{i,1},
\end{aligned}
$$

using (6.16) of Proposition 6.3.4 and (6.12) of proposition 6.3.3. For (6.39), we use (6.38) (of the present Proposition) and obtain

$$
\begin{aligned}
\left(e_{1,1}^2 e_{1,k} e_{i,1}\right) e_{1,j} &= \delta_{k,i}\left[\delta_{k,1} e_{1,1}^4 + \widehat{\delta}_{k,1}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right] e_{1,j} - \widehat{\delta}_{k,i}\delta_{k,1} e_{1,1} e_{i,1} e_{1,j} \\
&= \delta_{k,i}\left[\delta_{k,1}\left(\delta_{j,1} e_{1,1} + \widehat{\delta}_{j,1} e_{1,1}^2 e_{1,j}\right) + \widehat{\delta}_{k,1}\left(\delta_{j,1}\tfrac{1}{2}\left(e_{1,1} + e_{1,1}^3\right) + \widehat{\delta}_{j,1} e_{1,1}^2 e_{1,j}\right)\right] \\
&\quad - \widehat{\delta}_{k,i}\delta_{k,1}\left(\delta_{i,1} e_{1,1}^2 e_{1,j} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^3 - e_{1,1}\right)\right),
\end{aligned}
$$

by the relation $\mathcal{G}_{19}$, (6.12) of proposition 6.3.3 and (6.6) of Proposition 6.3.2. Combining the coefficients of the last equation completes the proof of (6.39). For (6.40), we suppose that $(i,j) \neq (1,1)$. Using (6.38) (of the present Proposition) gives

$$
e_{1,1}^2 e_{1,k} \dot{e}_{1,i} e_{j,1} = \delta_{k,1}\left(\delta_{i,1} e_{1,1}^4 + \widehat{\delta}_{i,1} e_{1,1} e_{1,i}\right) e_{j,1} = \delta_{k,1}\left(-\delta_{i,1} e_{1,1}^2 e_{j,1} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^3 + e_{1,1}\right)\right),
$$

by (6.11) of proposition 6.3.3 and (6.8) of Proposition 6.3.2. The proof of (6.41) is obvious by (6.38) and the relations $\mathcal{G}_{19}$, $\mathcal{G}_{11}^{(1,j,1)}$. The proofs of (6.42) and (6.43) are similar. □

Proposition 6.3.7. *Let $i,j,k,\ell \in \Omega$ and $\ell \neq 1$. Then in $\mathfrak{A}$, we have*

$$
e_{1,k} e_{1,1} e_{\ell,1} \cdot e_{i,j} = \delta_{j,\ell}\Big[\tfrac{1}{2}\Big\{\left(\delta_{k,1}\widehat{\delta}_{j,1}\delta_{i,1} + \widehat{\delta}_{i,1}\delta_{i,k}\right) e_{1,1}^4 + \left(\widehat{\delta}_{i,1}\delta_{i,k} - \delta_{k,1}\widehat{\delta}_{j,1}\delta_{i,1}\right) e_{1,1}^2\Big\}
$$

$$- \widehat{\delta}_{i,1}\delta_{k,1}e_{1,1}e_{i,1} - \widehat{\delta}_{k,1}e_{1,k}e_{i,1}\Big], \tag{6.44}$$

$$e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,i}e_{j,1} = \delta_{i,\ell}\Big[\delta_{k,1}\left(\delta_{j,1}\tfrac{1}{2}\left(e_{1,1} - e_{1,1}^3\right) - \widehat{\delta}_{j,1}e_{1,1}^2e_{j,1}\right)$$
$$- \widehat{\delta}_{k,1}\left(\widehat{\delta}_{j,1}e_{1,k}e_{1,1}e_{j,1} + \delta_{j,1}\left(e_{1,1}^2e_{1,k} - e_{1,k}\right)\right)\Big]; (i,j) \neq (1,1), \tag{6.45}$$

$$e_{1,i}e_{1,1}e_{j,1} \cdot e_{1,k}e_{1,1}e_{\ell,1} = \delta_{j,k}\left(e_{1,i}e_{\ell,1} - \widehat{\delta}_{i,1}\delta_{i,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right); \quad j \neq 1, \tag{6.46}$$

$$e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{i,1}e_{1,j} = 0, \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,1}^2e_{1,j} = 0, \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,1}^4 = 0. \tag{6.47}$$

Proof. For (6.44), let $\ell \neq 1$. Using the anti-automorphism η of Lemma 6.3.1 gives: $e_{1,k}e_{1,1}e_{\ell,1}e_{i,j} = \eta\left(e_{j,i}e_{1,\ell}e_{1,1}e_{k,1}\right)$. Two cases need to be considered. Case I. If $k = 1$ then we use (6.6) of Proposition 6.3.2 and get

$$e_{1,1}^2e_{\ell,1} \cdot e_{i,j} = \eta\left(e_{j,i}\left(e_{1,1}^2e_{1,\ell} - e_{1,\ell}\right)\right). \tag{6.48}$$

By (6.5) of Proposition 6.3.2 and (6.12) of Proposition 6.3.3 we have

$$e_{j,i}e_{1,\ell} = \delta_{i,1}e_{j,1}e_{1,\ell} + \widehat{\delta}_{i,1}\delta_{\ell,j}e_{1,i}e_{1,1}, e_{j,i}e_{1,1}^2e_{1,\ell}$$
$$= \delta_{i,1}\left(\delta_{j,1}e_{1,1}e_{1.\ell} + \widehat{\delta}_{j,1}\left(\tfrac{1}{2}\delta_{j,\ell}\left(e_{1,1}^4 - e_{1,1}^2\right) + e_{j,1}e_{1,\ell}\right)\right),$$

respectively. Using the last two equations in (6.48) gives

$$e_{1,1}^2e_{\ell,1} \cdot e_{i,j} = \eta\left(\delta_{i,1}\widehat{\delta}_{j,1}\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^4 - e_{1,1}^2\right) - \widehat{\delta}_{i,1}\delta_{\ell,j}e_{1,i}e_{1,1}\right)$$
$$= \delta_{i,1}\widehat{\delta}_{j,1}\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^4 - e_{1,1}^2\right) - \widehat{\delta}_{i,1}\delta_{\ell,j}e_{1,1}e_{i,1}.$$

Case II. If $k \neq 1$ then we use (6.11) of Proposition 6.3.3 and get

$$e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{i,j} = \eta\left(-\delta_{j,\ell}e_{1,i}e_{k,1} + \widehat{\delta}_{i,1}\delta_{\ell,j}\left(\delta_{i,k}\tfrac{1}{2}(e_{1,1}^4 + e_{1,1}^2)\right)\right)$$
$$= -\delta_{j,\ell}e_{1,k}e_{i,1} + \tfrac{1}{2}\widehat{\delta}_{i,1}\delta_{\ell,j}\delta_{i,k}\left(e_{1,1}^4 + e_{1,1}^2\right).$$

Combining the results of the two cases completes the proof of (6.44). For (6.45), we use (6.44)(of the present Proposition) and get

$$\left(e_{1,k}e_{1,1}e_{\ell,1}e_{1,i}\right)e_{j,1} = \delta_{i,\ell}\left[\delta_{k,1}\widehat{\delta}_{i,1}\tfrac{1}{2}\left(e_{1,1}^4 - e_{1,1}^2\right) - \widehat{\delta}_{k,1}e_{1,k}e_{1,1}\right]e_{j,1}. \tag{6.49}$$

By the relations $\mathcal{G}_{19}$, $\mathcal{G}_{18}^{(j)}$ and (6.6), (6.8) of Proposition 6.3.2 we have

$$e_{1,1}^4e_{j,1} = \delta_{j,1}e_{1,1} - \widehat{\delta}_{j,1}e_{1,1}^2e_{j,1}, \tag{6.50}$$

$$\widehat{\delta}_{k,1}e_{1,k}e_{1,1}e_{j,1} = \widehat{\delta}_{k,1}\left(\delta_{j,1}\left(e_{1,1}^2e_{1,k} - e_{1,k}\right) + \widehat{\delta}_{j,1}e_{1,k}e_{1,1}e_{j,1}\right), \tag{6.51}$$

respectively. Using (6.50) and (6.51) in (6.49) completes the proof of (6.45). For (6.46), we suppose that $j, \ell \neq 1$. By (6.16) of Proposition 6.3.4 we have

$$e_{1,i}e_{1,1}\left(e_{j,1}e_{1,k}e_{1,1}e_{\ell,1}\right) = -\delta_{j,k}e_{1,i}e_{1,1}^2e_{\ell,1}.$$

Using (6.35) of the proof of Proposition 6.3.5 completes the proof of (6.46). The proofs of (6.47) are obvious by using the relations $\mathcal{G}_3^{(\ell,1,i,1)}$ and $\mathcal{G}_{12}^{(1,\ell,1)}$. □

Proposition 6.3.8. *Let $i, j \in \Omega$. Then in $\mathfrak{A}$, we have*

$$e_{1,1}^4 \cdot e_{i,j} = \delta_{i,1}\left(\delta_{j,1}e_{1,1} + \widehat{\delta}_{j,1}e_{1,1}^2e_{1,j}\right) - \widehat{\delta}_{i,1}\delta_{j,1}e_{1,1}^2e_{j,1}, \tag{6.52}$$

$$e_{1,1}^4 \cdot e_{1,i}e_{j,1} = \delta_{i,1}e_{1,1}e_{j,1} + \widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right); \quad (i,j) \neq (1,1), \tag{6.53}$$

$$e_{1,1}^4 \cdot e_{i,1}e_{1,j} = \delta_{i,1}e_{1,1}e_{1,j} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^2 - e_{1,1}^4\right), \tag{6.54}$$

$$e_{1,1}^4 \cdot e_{1,1}^2e_{1,j} = e_{1,1}^2e_{1,j}, \tag{6.55}$$

$$e_{1,1}^4 \cdot e_{1,i}e_{1,1}e_{j,1} = \delta_{i,1}e_{1,1}^2e_{j,1}; \quad j \neq 1, \tag{6.56}$$

$$e_{1,1}^4 \cdot e_{1,1}^4 = e_{1,1}^4. \tag{6.57}$$

Proof. For (6.52), using the anti-automorphism η of Lemma 6.3.1 and (6.13) of Proposition 6.3.3 implies

$$\begin{aligned} e_{1,1}^4e_{i,j} &= \eta\left(e_{j,i}e_{1,1}^4\right) = \eta\left(\delta_{i,1}\left(\delta_{j,1}e_{1,1} + \widehat{\delta}_{j,1}\left(e_{1,1}^2e_{j,1} + e_{j,1}\right)\right) - \widehat{\delta}_{i,1}\delta_{j,1}\left(e_{1,1}^2e_{1,i} - e_{1,i}\right)\right) \\ &= \delta_{i,1}\left(\delta_{j,1}e_{1,1} + \widehat{\delta}_{j,1}\left(e_{1,j}e_{1,1}^2 + e_{1,j}\right)\right) - \widehat{\delta}_{i,1}\delta_{j,1}\left(e_{i,1}e_{1,1}^2 - e_{i,1}\right). \end{aligned}$$

By the relations $\mathcal{G}_8^{(j,1)}$ and $\mathcal{G}_5^{(i,1)}$ we have

$$\widehat{\delta}_{j,1}e_{1,j}e_{1,1}^2 = \widehat{\delta}_{j,1}\left(e_{1,1}^2e_{1,j} - e_{1,j}\right), \quad \widehat{\delta}_{i,1}e_{i,1}e_{1,1}^2 = \widehat{\delta}_{i,1}\left(e_{1,1}^2e_{i,1} + e_{i,1}\right).$$

Using this in the last equation completes the proof (6.52). For (6.53), using the anti-automorphism η and (6.32) of Proposition 6.3.5 gives

$$e_{1,1}^4e_{1,i}e_{j,1} = \eta\left(e_{1,j}e_{i,1}e_{1,1}^4\right) = \eta\left(\delta_{i,1}e_{1,j}e_{1,1} + \widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\delta_{j,i}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right).$$

This completes the proof of (6.53). The proof of (6.54) is similar. The proofs of (6.55) and (6.57) (resp. (6.56)) are obvious by the relation $\mathcal{G}_{19}$ (resp. $\mathcal{G}_{12}^{(1,i,1)}$). □

6.4 The center of the universal enveloping algebra

Our next aim is to use the results of Section 6.3 to determine the center of $\mathfrak{A}$:

$$Z(\mathfrak{A}) = \{z \in \mathfrak{A} \mid zu = uz, \text{ for all } u \in \mathfrak{A}\}.$$

Theorem 6.4.1. *The center $Z(\mathfrak{A})$ of the (unital) universal enveloping algebra $\mathfrak{A}$ has dimension 5 with basis:*

$$z_1 = \frac{(n-2)}{n} e_{1,1}^2 - \frac{2}{n}\sum_{i=2}^{n} e_{1,i}e_{i,1} + e_{1,1}^4, \quad z_2 = (2-n)e_{1,1}^2 + \sum_{i=2}^{n} e_{1,i}e_{i,1} + \sum_{i=2}^{n} e_{i,1}e_{1,i},$$
$$z_3 = -\tfrac{1}{2}e_{1,1} + \tfrac{1}{2}e_{1,1}^3 + \sum_{i=2}^{n} e_{1,i}e_{1,1}e_{i,1}, \quad z_4 = \sum_{i=1}^{n} e_{i,i}, \quad z_5 = 1.$$

Proof. To get the center of $\mathfrak{A}$, it is sufficient to determine the elements of $\mathfrak{A}$ which commute with $e_{i,j}$, for all $i, j \in \Omega$. Let

$$x = \sum_{i,j=1}^{n} \zeta_1^{(i,j)} e_{i,j} + \sum_{i=1}^{n}\sum_{j=2}^{n} \zeta_2^{(i,j)} e_{1,i}e_{1,1}e_{j,1} + \sum_{j=1}^{n} \zeta_3^{(j)} e_{1,1}^2 e_{1,j}$$
$$+ \sum_{i,j=1}^{n} \zeta_4^{(i,j)} e_{i,1}e_{1,j} + \sum_{\substack{i,j=1 \\ (i,j)\neq(1,1)}}^{n} \zeta_5^{(i,j)} e_{1,i}e_{j,1} + \zeta e_{1,1}^4,$$

be any element of $Z(\mathfrak{A})$. Then

$$\begin{aligned}
0 &= x\, e_{1,1} - e_{1,1}\, x \\
&= \sum_{i,j=1}^{n} \zeta_1^{(i,j)} \left(e_{i,j}e_{1,1} - e_{1,1}e_{i,j}\right) + \sum_{i=1}^{n}\sum_{j=2}^{n} \zeta_2^{(i,j)} \left(e_{1,i}e_{1,1}e_{j,1}e_{1,1} - e_{1,1}e_{1,i}e_{1,1}e_{j,1}\right) \\
&\quad + \sum_{j=1}^{n} \zeta_3^{(j)} \left(e_{1,1}^2 e_{1,j}e_{1,1} - e_{1,1}^3 e_{1,j}\right) + \sum_{i,j=1}^{n} \zeta_4^{(i,j)} \left(e_{i,1}e_{1,j}e_{1,1} - e_{1,1}e_{i,1}e_{1,j}\right) \\
&\quad + \sum_{\substack{i,j=1 \\ (i,j)\neq(1,1)}}^{n} \zeta_5^{(i,j)} \left(e_{1,i}e_{j,1}e_{1,1} - e_{1,1}e_{1,i}e_{j,1}\right).
\end{aligned} \tag{6.58}$$

Proposition 6.3.2 implies that $e_{i,1}e_{1,j}e_{1,1} = 0 = e_{1,i}e_{j,1}e_{1,1}$ for $i \neq j \neq 1$, $e_{1,1}e_{i,1}e_{1,j} = 0 = e_{1,1}e_{1,i}e_{j,1}$ for $1 \neq i \neq j$, $e_{i,1}e_{1,i}e_{1,1} = \frac{1}{2}\left(e_{1,1}^3 - e_{1,1}\right) = e_{1,1}e_{i,1}e_{1,i}$ and $e_{1,i}e_{i,1}e_{1,1} = \frac{1}{2}\left(e_{1,1}^3 + e_{1,1}\right) = e_{1,1}e_{1,i}e_{i,1}$ for $i \neq 1$. Using this in (6.58) gives

$$0 = \sum_{i,j=1}^{n} \zeta_1^{(i,j)} \left(e_{i,j}e_{1,1} - e_{1,1}e_{i,j}\right) - \sum_{j=2}^{n} \zeta_2^{(1,j)} e_{1,1}^3 e_{j,1} - \sum_{j=2}^{n} \zeta_3^{(j)} e_{1,1}^3 e_{1,j}$$

$$+\sum_{i=2}^{n}\zeta_4^{(i,1)}e_{i,1}e_{1,1}^2-\sum_{j=2}^{n}\zeta_4^{(1,j)}e_{1,1}^2e_{1,j}+\sum_{i=2}^{n}\zeta_5^{(i,1)}e_{1,i}e_{1,1}^2-\sum_{j=2}^{n}\zeta_5^{(1,j)}e_{1,1}^2e_{j,1}.$$

Using (6.5), (6.6) of Proposition 6.3.2 and (6.11), (6.12) of Proposition 6.3.3 implies

$$\begin{aligned}0=&\sum_{j=2}^{n}\zeta_1^{(1,j)}(e_{1,j}e_{1,1}-e_{1,1}e_{1,j})+\sum_{i=2}^{n}\zeta_1^{(i,1)}(e_{i,1}e_{1,1}-e_{1,1}e_{i,1})+\sum_{j=2}^{n}\zeta_2^{(1,j)}e_{1,1}e_{j,1}\\&-\sum_{j=2}^{n}\zeta_3^{(j)}e_{1,1}e_{1,j}+\sum_{i=2}^{n}\zeta_4^{(i,1)}(e_{1,1}^2e_{i,1}+e_{i,1})-\sum_{j=2}^{n}\zeta_4^{(1,j)}e_{1,1}^2e_{1,j}\\&+\sum_{i=2}^{n}\zeta_5^{(i,1)}(e_{1,1}^2e_{1,i}-e_{1,i})-\sum_{j=2}^{n}\zeta_5^{(1,j)}e_{1,1}^2e_{j,1}.\end{aligned}$$

Comparing the coefficients on both sides, we get

$$\zeta_1^{(1,j)}=\zeta_1^{(i,1)}=\zeta_2^{(1,j)}=\zeta_3^{(j)}=\zeta_4^{(i,1)}=\zeta_4^{(1,i)}=\zeta_5^{(i,1)}=\zeta_5^{(1,j)}=0,$$

for all $i,j\in\Omega\smallsetminus\{1\}$. Rewriting x with these values for the coefficients, we obtain

$$\begin{aligned}x=&\zeta_1^{(1,1)}e_{1,1}+\sum_{i,j=2}^{n}\zeta_1^{(i,j)}e_{i,j}+\sum_{i,j=2}^{n}\zeta_2^{(i,j)}e_{1,i}e_{1,1}e_{j,1}+\zeta_3^{(1)}e_{1,1}^3+\zeta_4^{(1,1)}e_{1,1}^2\\&+\sum_{i,j=2}^{n}\zeta_4^{(i,j)}e_{i,1}e_{1,j}+\sum_{i,j=2}^{n}\zeta_5^{(i,j)}e_{1,i}e_{j,1}+\zeta e_{1,1}^4.\end{aligned}$$

Choose $q\neq 1$ and observe that $e_{1,1}e_{q,q}=0=e_{q,q}e_{1,1}$ by (6.5) of Proposition 6.3.2. Hence,

$$\begin{aligned}0&=x\,e_{q,q}-e_{q,q}\,x\\&=\sum_{i,j=2}^{n}\zeta_1^{(i,j)}\left(e_{i,j}e_{q,q}-e_{q,q}e_{i,j}\right)+\sum_{i,j=2}^{n}\zeta_2^{(i,j)}\left(e_{1,i}e_{1,1}e_{j,1}e_{q,q}-e_{q,q}e_{1,i}e_{1,1}e_{j,1}\right)\\&\quad+\sum_{i,j=2}^{n}\zeta_4^{(i,j)}\left(e_{i,1}e_{1,j}e_{q,q}-e_{q,q}e_{i,1}e_{1,j}\right)+\sum_{i,j=2}^{n}\zeta_5^{(i,j)}\left(e_{1,i}e_{j,1}e_{q,q}-e_{q,q}e_{1,i}e_{j,1}\right).\end{aligned}$$

Using Proposition 6.3.2, (6.44) of Proposition 6.3.7 and (6.11) of Proposition 6.3.3 implies

$$\begin{aligned}0=&\sum_{\substack{i=2\\i\neq q}}^{n}\zeta_1^{(i,q)}\left(e_{i,1}e_{1,q}-e_{1,q}e_{i,1}\right)+\sum_{\substack{j=2\\j\neq q}}^{n}\zeta_1^{(q,j)}\left(e_{1,j}e_{q,1}-e_{q,1}e_{1,j}\right)-\sum_{\substack{i=2\\i\neq q}}^{n}\zeta_2^{(i,q)}e_{1,i}e_{q,1}\\&+\sum_{\substack{j=2\\j\neq q}}^{n}\zeta_2^{(q,j)}e_{1,q}e_{j,1}+\sum_{\substack{i=2\\i\neq q}}^{n}\zeta_4^{(i,q)}\left(e_{1,q}e_{1,1}e_{i,1}+e_{i,q}\right)-\sum_{\substack{j=2\\j\neq q}}^{n}\zeta_4^{(q,j)}\left(e_{1,j}e_{1,1}e_{q,1}+e_{q,j}\right)\end{aligned}$$

$$+\sum_{\substack{i=2\\ i\neq q}}^{n} \zeta_5^{(i,q)} e_{1,i}e_{1,1}e_{q,1} - \sum_{\substack{j=2\\ j\neq q}}^{n} \zeta_5^{(q,j)} e_{1,q}e_{1,1}e_{j,1}.$$

Comparing the coefficients on both sides gives

$$\zeta_1^{(i,q)} = \zeta_1^{(q,j)} = \zeta_2^{(i,q)} = \zeta_2^{(q,j)} = \zeta_4^{(i,q)} = \zeta_4^{(q,j)} = \zeta_5^{(i,q)} = \zeta_5^{(q,j)} = 0,$$

for all $i,j,q \in \Omega \setminus \{1\}$ and $i \neq q \neq j$. Rewriting x with these values for the coefficients, we get

$$x = \sum_{i=1}^{n}\left(\zeta_1^{(i,i)} e_{i,i} + \zeta_4^{(i,i)} e_{i,1}e_{1,i}\right) + \sum_{i=2}^{n}\left(\zeta_2^{(i,i)} e_{1,i}e_{1,1}e_{i,1} + \zeta_5^{(i,i)} e_{1,i}e_{i,1}\right) + \zeta_3^{(1)} e_{1,1}^3 + \zeta e_{1,1}^4.$$

We next choose $q,s \in \Omega \setminus \{1\}$ and $q \neq s$ and observe that $e_{1,1}e_{q,s} = 0 = e_{q,s}e_{1,1}$ by (6.5) of Proposition 6.3.2. Hence,

$$\begin{aligned} 0 &= xe_{q,s} - e_{q,s}x \\ &= \sum_{i=1}^{n}\left(\zeta_1^{(i,i)}\left(e_{i,i}e_{q,s} - e_{q,s}e_{i,i}\right) + \zeta_4^{(i,i)}\left(e_{i,1}e_{1,i}e_{q,s} - e_{q,s}e_{i,1}e_{1,i}\right)\right) \\ &\quad + \sum_{i=2}^{n}\left(\zeta_2^{(i,i)}\left(e_{1,i}e_{1,1}e_{i,1}e_{q,s} - e_{q,s}e_{1,i}e_{1,1}e_{i,1}\right) + \zeta_5^{(i,i)}\left(e_{1,i}e_{i,1}e_{q,s} - e_{q,s}e_{1,i}e_{i,1}\right)\right). \end{aligned}$$

Using Proposition 6.3.2, (6.44) of Proposition 6.3.7 and (6.11) of Proposition 6.3.3 gives

$$\begin{aligned} 0 &= \left(\zeta_1^{(q,q)} - \zeta_1^{(s,s)}\right) e_{q,1}e_{1,s} + \left(\zeta_1^{(s,s)} - \zeta_1^{(q,q)}\right) e_{1,s}e_{q,1} + \left(\zeta_4^{(q,q)} - \zeta_4^{(s,s)}\right)\left(e_{1,s}e_{1,1}e_{q,1} + e_{q,s}\right) \\ &\quad - \left(\zeta_2^{(s,s)} - \zeta_2^{(q,q)}\right) e_{1,s}e_{q,1} + \left(\zeta_5^{(s,s)} - \zeta_5^{(q,q)}\right) e_{1,s}e_{1,1}e_{q,1}. \end{aligned}$$

Comparing the coefficients on both sides gives

$$\zeta_1^{(q,q)} = \zeta_1^{(s,s)}, \quad \zeta_4^{(q,q)} = \zeta_4^{(s,s)}, \quad \zeta_2^{(s,s)} = \zeta_2^{(q,q)}, \quad \zeta_5^{(s,s)} = \zeta_5^{(q,q)},$$

for all $q,s \in \Omega \setminus \{1\}$ and $q \neq s$. Hence the values of $\zeta_1^{(q,q)}, \zeta_4^{(q,q)}, \zeta_2^{(q,q)}$ and $\zeta_5^{(q,q)}$ (for all $q \in \Omega \setminus \{1\}$) do not depend on the value of q. We remove the exponents of $\zeta_1^{(q,q)}, \zeta_4^{(q,q)}, \zeta_2^{(q,q)}$ and $\zeta_5^{(q,q)}$ and rewrite x and obtain

$$\begin{aligned} x &= \zeta_1^{(1,1)} e_{1,1} + \zeta_1 \sum_{i=2}^{n} e_{i,i} + \zeta_4^{(1,1)} e_{1,1}^2 + \sum_{i=2}^{n}\left(\zeta_4\, e_{i,1}e_{1,i} + \zeta_2\, e_{1,i}e_{1,1}e_{i,1} + \zeta_5\, e_{1,i}e_{i,1}\right) \\ &\quad + \zeta_3^{(1)} e_{1,1}^3 + \zeta e_{1,1}^4. \end{aligned}$$

We observe that x is invariant under the action of the anti-automorphism η. Therefore, if x commutes with $e_{1,q}$ (resp. $e_{q,1}$)$(q \neq 1)$ then x commutes with $e_{q,1}$(resp. $e_{1,q}$). Choose $q \neq \Omega \setminus \{1\}$ and note that $e_{q,1}e_{i,1} = 0 = e_{i,1}e_{q,1}$ $(i \neq 1)$ by (6.5) of proposition 6.3.2. Hence,

$$\begin{aligned}
0 &= xe_{q,1} - e_{q,1}x \\
&= \zeta_1^{(1,1)}(e_{1,1}e_{q,1} - e_{q,1}e_{1,1}) + \zeta_1 \sum_{i=2}^{n} (e_{i,i}e_{q,1} - e_{q,1}e_{i,i}) + \zeta_4^{(1,1)} \left(e_{1,1}^2 e_{q,1} - e_{q,1}e_{1,1}^2\right) \\
&\quad + \zeta_4 \sum_{i=2}^{n} e_{i,1}e_{1,i}e_{q,1} - \zeta_2 \sum_{i=2}^{n} e_{q,1}e_{1,i}e_{1,1}e_{i,1} - \zeta_5 \sum_{i=2}^{n} e_{q,1}e_{1,i}e_{i,1} + \zeta_3^{(1)} \left(e_{1,1}^3 e_{q,1} - e_{q,1}e_{1,1}^3\right) \\
&\quad + \zeta \left(e_{1,1}^4 e_{q,1} - e_{q,1}e_{1,1}^4\right).
\end{aligned}$$

Using (6.5)-(6.7) of Proposition 6.3.2, (6.38) of Propositions 6.3.6, (6.52) of Proposition 6.3.8 and Proposition 6.3.3 gives

$$\begin{aligned}
0 &= \zeta_1^{(1,1)}(e_{1,1}e_{q,1} - e_{q,1}e_{1,1}) + \zeta_1 \sum_{i=2}^{n} (e_{q,1}e_{1,1} - e_{1,1}e_{q,1}) - \zeta_4^{(1,1)} e_{q,1} \\
&\quad + \zeta_4 \left(ne_{1,1}^2 e_{q,1} + e_{q,1}\right) + \zeta_2\, e_{1,1}e_{q,1} - \zeta_5 \left(ne_{1,1}^2 e_{q,1} + (n-1)e_{q,1}\right) \\
&\quad + \zeta_3^{(1)}(-e_{1,1}e_{q,1} - e_{q,1}e_{1,1}) - \zeta(2e_{1,1}^2 e_{q,1} + e_{q,1}).
\end{aligned}$$

Combining the coefficients gives

$$\begin{aligned}
0 &= \left(\zeta_1^{(1,1)} - \zeta_1 + \zeta_2 - \zeta_3^{(1)}\right) e_{1,1}e_{q,1} + \left(-\zeta_1^{(1,1)} + \zeta_1 - \zeta_3^{(1)}\right) e_{q,1}e_{1,1} \\
&\quad + \left(-\zeta_4^{(1,1)} + \zeta_4 - (n-1)\zeta_5 - \zeta\right) e_{q,1} + (n\zeta_4 - n\zeta_5 - 2\zeta)\, e_{1,1}^2 e_{q,1}.
\end{aligned}$$

Comparing the coefficients on both sides gives

$$\begin{aligned}
&\zeta_1^{(1,1)} - \zeta_1 + \zeta_2 - \zeta_3^{(1)} = 0, \qquad -\zeta_1^{(1,1)} + \zeta_1 - \zeta_3^{(1)} = 0, \\
&-\zeta_4^{(1,1)} + \zeta_4 - (n-1)\zeta_5 - \zeta = 0, \qquad n\zeta_4 - n\zeta_5 - 2\zeta = 0.
\end{aligned}$$

These equations can be reduced to the system

$$\begin{aligned}
&\zeta_1^{(1,1)} - \zeta_1 + \zeta_3^{(1)} = 0, \quad \zeta_2 - 2\zeta_3^{(1)} = 0, \\
&\zeta_4^{(1,1)} + (n-2)\zeta_4 - \left(\frac{n-2}{n}\right)\zeta = 0, \quad \zeta_5 - \zeta_4 + \frac{2}{n}\zeta = 0.
\end{aligned}$$

This is a linear system of four equations in eight variables. Hence, there are four free variables. Setting,

$$(\zeta,\zeta_4,\zeta_2,\zeta_1)=(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1),$$

in the last system gives

$$\left(\zeta_1^{(1,1)},\zeta_3^{(1)},\zeta_4^{(1,1)},\zeta_5\right)=\left(0,0,\tfrac{n-2}{n},\tfrac{-2}{n}\right),\ (0,0,2-n,1),\ \left(-\tfrac{1}{2},\tfrac{1}{2},0,0\right),\ (1,0,0,0),$$

respectively. Using these solutions in x gives z_1, z_2, z_3, z_4 respectively. □

6.5 Explicit decomposition of the universal enveloping algebra

Theorem 6.5.1. *The universal enveloping algebra $\mathfrak{A}$ of the anti-Jordan triple system $\mathfrak{J}$ can be decomposed as follows:*

$$\mathfrak{A}=F\oplus M_{n,n}(F)\oplus M_{n,n}(F)\oplus M_{n,n}(F)\oplus M_{n,n}(F),$$

where $M_{n,n}$ is the ordinary associative algebra of all $n\times n$ matrices.

Proof. We define the first two sets of $n\times n$ matrix units. For all $k\in\{0,1\}$ and $i,j=2,\dots,n$, we set

$$\begin{aligned}
B_{1,1}^{(k)}&=\tfrac{1}{4}\left(e_{1,1}^4+e_{1,1}^2+(-1)^k\left(e_{1,1}^3+e_{1,1}\right)\right),\\
B_{1,i}^{(k)}&=e_{1,1}e_{1,i}+(-1)^k e_{1,1}^2e_{1,i},\\
B_{i,1}^{(k)}&=\tfrac{1}{4}\left(e_{i,1}e_{1,1}+(-1)^k(e_{1,1}^2e_{i,1}+e_{i,1})\right),\\
B_{i,i}^{(k)}&=\tfrac{1}{2}\left(\tfrac{1}{2}\left(e_{1,1}^4-e_{1,1}^2\right)+e_{i,1}e_{1,i}+(-1)^k\left(e_{1,i}e_{1,1}e_{i,1}+e_{i,i}\right)\right),\\
B_{i,j}^{(k)}&=\tfrac{1}{2}\left(e_{i,1}e_{1,j}+(-1)^k(e_{1,j}e_{1,1}e_{i,1}+e_{i,j})\right);\quad i\neq j.
\end{aligned}$$

We wish to show that for each $k\in\{0,1\}$, the elements $B_{i,j}^{(k)}; i,j=1,\dots,n$, satisfy the multiplication table for matrix units and the product of any $B_{i,j}^{(k)}$ by any $B_{t,\ell}^{(s)}$ is 0 for $k\neq s$. We note first that if $i,j\neq 1$ then $B_{1,i}^{(k)}B_{1,j}^{(s)}=0$, since $e_{1,1}e_{1,i}e_{1,1}=0$ by (6.7) of Proposition 6.3.2. Let $i\neq 1$. Then

$$B_{1,1}^{(k)}B_{1,i}^{(s)}=\tfrac{1}{4}\Big[(e_{1,1}^4+e_{1,1}^2)e_{1,1}e_{1,i}+(-1)^s(e_{1,1}^4+e_{1,1}^2)e_{1,1}^2e_{1,i}+(-1)^k\left(e_{1,1}^3+e_{1,1}\right)e_{1,1}e_{1,i}$$

$$+ (-1)^{k+s}\left(e_{1,1}^3 + e_{1,1}\right) e_{1,1}^2 e_{1,i}\Big].$$

Using (6.54), (6.55), (6.52) of Proposition 6.3.8 and (6.38) of Proposition 6.3.6, we get

$$B_{1,1}^{(k)} B_{1,i}^{(s)} = \tfrac{1}{4}\left[2\left(1 + (-1)^{k+s}\right) e_{1,1}e_{1,i} + 2\left((-1)^s + (-1)^k\right) e_{1,1}^2 e_{1,i}\right] = \delta_{k,s} B_{1,i}^{(k)}.$$

Also,

$$\begin{aligned} B_{1,1}^{(k)} B_{i,1}^{(s)} = \tfrac{1}{16}\Big[&\left(e_{1,1}^4 + e_{1,1}^2\right) e_{i,1}e_{1,1} + (-1)^s \left(e_{1,1}^4 + e_{1,1}^2\right)\left(e_{1,1}^2 e_{i,1} + e_{i,1}\right) \\ &+ (-1)^k \left(e_{1,1}^3 + e_{1,1}\right) e_{i,1}e_{1,1} + (-1)^{k+s}\left(e_{1,1}^3 + e_{1,1}\right)\left(e_{1,1}^2 e_{i,1} + e_{i,1}\right)\Big]. \end{aligned}$$

Using (6.52), (6.53), (6.56) of Proposition 6.3.8 and (6.38) of Proposition 6.3.6, and observing that $e_{1,1}e_{i,1}e_{1,1} = 0$ by (6.6) of Proposition 6.3.2, we get

$$\begin{aligned} B_{1,1}^{(k)} B_{i,1}^{(s)} = \tfrac{1}{16}\Big[&(-1)^s \left(e_{1,1}^2 e_{i,1} - e_{1,1}^2 e_{i,1} - e_{1,1}^2 e_{i,1} + e_{1,1}^2 e_{i,1}\right) \\ &+ (-1)^{k+s}\left(e_{1,1}e_{i,1} - e_{1,1}e_{i,1} - e_{1,1}e_{i,1} + e_{1,1}e_{i,1}\right)\Big] = 0. \end{aligned}$$

Next let $i, j \neq 1$. Then

$$\begin{aligned} B_{1,i}^{(k)} B_{j,1}^{(s)} = \tfrac{1}{4}\Big[&e_{1,1}e_{1,i}e_{j,1}e_{1,1} + (-1)^s e_{1,1}e_{1,i}\left(e_{1,1}^2 e_{j,1} + e_{j,1}\right) + (-1)^k e_{1,1}^2 e_{1,i}e_{j,1}e_{1,1} \\ &+ (-1)^{k+s} e_{1,1}^2 e_{1,i}\left(e_{1,1}^2 e_{j,1} + e_{j,1}\right)\Big]. \end{aligned}$$

Using (6.15) of Proposition 6.3.4, (6.8) of Proposition 6.3.2, (6.38) and (6.39) of Proposition 6.3.6 we get

$$\begin{aligned} B_{1,i}^{(k)} B_{j,1}^{(s)} = \tfrac{1}{4}\delta_{j,i}\Big[&\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right) + \tfrac{1}{2}(-1)^s (e_{1,1}^3 + e_{1,1}) + \tfrac{1}{2}(-1)^k \left(e_{1,1} + e_{1,1}^3\right) \\ &+ \tfrac{1}{2}(-1)^{k+s}\left(e_{1,1}^4 + e_{1,1}^2\right)\Big] = \delta_{j,i}\delta_{k,s} B_{1,1}^{(k)}. \end{aligned}$$

Also,

$$\begin{aligned} B_{i,1}^{(k)} B_{1,j}^{(s)} &= \tfrac{1}{4}\Big[e_{i,1}e_{1,1}^2 e_{1,j} + (-1)^s e_{i,1}e_{1,1}^3 e_{1,j} + (-1)^k \left(e_{1,1}^2 e_{i,1}e_{1,1}e_{1,j} + e_{i,1}e_{1,1}e_{1,j}\right) \\ &\quad + (-1)^{k+s}\left(e_{1,1}^2 e_{i,1}e_{1,1}^2 e_{1,j} + e_{i,1}e_{1,1}^2 e_{1,j}\right)\Big] \\ &= \tfrac{1}{4}\left[\left(1 + (-1)^{k+s}\right) e_{i,1}e_{1,1}^2 e_{1,j} + \left((-1)^s + (-1)^k\right) e_{i,1}e_{1,1}e_{1,j}\right], \end{aligned}$$

since $e_{1,1}^3 e_{1,j} = e_{1,1}e_{1,j}$ and $e_{1,1}e_{i,1}e_{1,1} = 0$. Using (6.15) of Proposition 6.3.4 and (6.6) of proposition 6.3.2 implies

$$B_{i,1}^{(k)} B_{1,j}^{(s)} = \tfrac{1}{4}\Big[\big(1 + (-1)^{k+s}\big)\left(\delta_{j,i}\tfrac{1}{2}(e_{1,1}^4 - e_{1,1}^2) + e_{i,1}e_{1,j}\right) \\ + \big((-1)^s + (-1)^k\big)\left(e_{1,j}e_{1,1}e_{i,1} + e_{i,j}\right)\Big] = \delta_{k,s} B_{i,j}^{(k)}.$$

We have shown that

$$B_{1,i}^{(k)} B_{1,j}^{(s)} = 0, \quad B_{1,1}^{(k)} B_{1,i}^{(s)} = \delta_{k,s} B_{1,i}^{(k)}, \quad B_{1,1}^{(k)} B_{i,1}^{(s)} = 0,$$
$$B_{1,i}^{(k)} B_{j,1}^{(s)} = \delta_{k,s}\delta_{j,i} B_{1,1}^{(k)}, \quad B_{i,1}^{(k)} B_{1,j}^{(s)} = \delta_{k,s} B_{i,j}^{(k)},$$

for all $i, j \neq 1$. By applying the anti-automorphism η to both sides of the first three products and observing that $B_{1,i}^{(k)} = 4\eta(B_{i,1}^{(k)})$, we obtain

$$B_{j,1}^{(k)} B_{i,1}^{(s)} = 0, \quad B_{i,1}^{(s)} B_{1,1}^{(k)} = \delta_{k,s} B_{i,1}^{(k)}, \quad B_{1,i}^{(s)} B_{1,1}^{(k)} = 0.$$

Now we use the above products to get all the others. For $k, s \in \{0,1\}$ and $i \neq 1$, we have $B_{1,i}^{(k)} B_{i,1}^{(k)} = B_{1,1}^{(k)}$, hence $B_{1,1}^{(s)} B_{1,i}^{(k)} B_{i,1}^{(k)} = B_{1,1}^{(s)} B_{1,1}^{(k)}$. Thus $B_{1,1}^{(s)} B_{1,1}^{(k)} = \delta_{k,s} B_{1,1}^{(k)}$. We now have $B_{i,q}^{(k)} = B_{i,1}^{(k)} B_{1,q}^{(k)}$ (for all i, q). Hence, $B_{i,q}^{(k)} B_{\ell,t}^{(s)} = B_{i,1}^{(k)} B_{1,q}^{(k)} B_{\ell,1}^{(s)} B_{1,t}^{(s)} = \delta_{k,s}\delta_{q,\ell} B_{i,1}^{(k)} B_{1,1}^{(k)} B_{1,t}^{(k)} = \delta_{k,s}\delta_{q,\ell} B_{i,1}^{(k)} B_{1,t}^{(k)} = \delta_{k,s}\delta_{q,\ell} B_{i,t}^{(k)}$ (for all i, q, ℓ, t). Summarizing

$$B_{i,j}^{(s)} B_{t,\ell}^{(s)} = \delta_{j,t} B_{i,\ell}^{(s)}, \quad B_{i,j}^{(s)} B_{t,\ell}^{(k)} = 0, \tag{6.59}$$

for all $s, k \in \{0,1\}, s \neq k$ and $i, j, t, \ell = 1, \ldots n$.

We define next the two other sets of $n \times n$ matrix units. For $k \in \{0,1\}$ and $i, j = 2, \ldots, n$, we set

$$D_{1,1}^{(k)} = \tfrac{1}{4}\left(e_{1,1}^4 - e_{1,1}^2 + (-1)^k \mathrm{I}\left(e_{1,1} - e_{1,1}^3\right)\right),$$
$$D_{1,i}^{(k)} = -\tfrac{1}{2}\left(e_{1,1}e_{i,1} + (-1)^k \mathrm{I}\, e_{1,1}^2 e_{i,1}\right),$$
$$D_{i,1}^{(k)} = -\tfrac{1}{2}\left(e_{1,i}e_{1,1} + (-1)^k \mathrm{I}\left(e_{1,1}^2 e_{1,i} - e_{1,i}\right)\right),$$
$$D_{i,i}^{(k)} = \tfrac{1}{2}\left(\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right) - e_{1,i}e_{i,1} - (-1)^k \mathrm{I}\, e_{1,i}e_{1,1}e_{i,1}\right),$$
$$D_{i,j}^{(k)} = -\tfrac{1}{2}\left(e_{1,i}e_{j,1} + (-1)^k \mathrm{I}\, e_{1,i}e_{1,1}e_{j,1}\right); \quad i \neq j,$$

where $\mathrm{I} = \sqrt{-1}$. We wish to show that for each $k \in \{0,1\}$, the elements $D_{i,j}^{(k)}; i, j = 1, \ldots, n$, satisfy the multiplication table for matrix units and the product of any

$D^{(k)}_{i,j}$ by any $D^{(s)}_{t,\ell}$ is 0 for $k \neq s$. We note first that if $i, j \neq 1$ then $D^{(k)}_{1,i} D^{(s)}_{1,j} = 0$, since $e_{1,1}e_{i,1}e_{1,1} = 0$. Let $i \neq 1$. Then

$$\begin{aligned} D^{(k)}_{1,1} D^{(s)}_{1,i} = -\tfrac{1}{8}\Big[&(e^4_{1,1} - e^2_{1,1})e_{1,1}e_{i,1} + (-1)^s \mathrm{I}\left(e^4_{1,1} - e^2_{1,1}\right) e^2_{1,1}e_{i,1} + (-1)^k \mathrm{I}\left(e_{1,1} - e^3_{1,1}\right) e_{1,1}e_{i,1} \\ &- (-1)^{k+s}\left(e_{1,1} - e^3_{1,1}\right) e^2_{1,1}e_{i,1}\Big]. \end{aligned}$$

Using (6.53), (6.56), (6.52) of Proposition 6.3.8 and (6.38) of Proposition 6.3.6, we get

$$\begin{aligned} D^{(k)}_{1,1} D^{(s)}_{1,i} &= -\tfrac{1}{8}\left[2e_{1,1}e_{i,1} + 2(-1)^s \mathrm{I}\, e^2_{1,1}e_{i,1} + 2(-1)^k \mathrm{I}\, e^2_{1,1}e_{i,1} + 2(-1)^{k+s} e_{1,1}e_{i,1}\right] \\ &= -\tfrac{1}{4}\left[\left(1 + (-1)^{k+s}\right) e_{1,1}e_{i,1} + \left((-1)^s + (-1)^k\right) \mathrm{I}\, e^2_{1,1}e_{i,1}\right] = \delta_{k,s} D^{(k)}_{1,i}. \end{aligned}$$

Also,

$$\begin{aligned} D^{(k)}_{1,1} D^{(s)}_{i,1} = -\tfrac{1}{8}\Big[&e^4_{1,1}e_{1,i}e_{1,1} + (-1)^s \mathrm{I}\, e^4_{1,1}\left(e^2_{1,1}e_{1,i} - e_{1,i}\right) - e^2_{1,1}e_{1,i}e_{1,1} \\ &- (-1)^s \mathrm{I}\, e^2_{1,1}(e^2_{1,1}e_{1,i} - e_{1,i}) + (-1)^k \mathrm{I}\left(e_{1,1} - e^3_{1,1}\right) e_{1,i}e_{1,1} \\ &- (-1)^{k+s}\left(e_{1,1} - e^3_{1,1}\right)\left(e^2_{1,1}e_{1,i} - e_{1,i}\right)\Big]. \end{aligned}$$

Using (6.52), (6.54), (6.55) of Proposition 6.3.8 and (6.38) of Proposition 6.3.6, and observing that $e_{1,1}e_{1,i}e_{1,1} = 0$, we get

$$\begin{aligned} D^{(k)}_{1,1} D^{(s)}_{i,1} = -\tfrac{1}{8}\Big[&(-1)^s \mathrm{I}\left(e^2_{1,1}e_{1,i} - e^2_{1,1}e_{1,i}\right) - (-1)^s \mathrm{I}\left(e^2_{1,1}e_{1,i} - e^2_{1,1}e_{1,i}\right) \\ &- (-1)^{k+s}\left(e_{1,1}e_{1,i} - e_{1,1}e_{1,i} - e_{1,1}e_{1,i} + e_{1,1}e_{1,i}\right)\Big] = 0. \end{aligned}$$

Next let $i, j \neq 1$. Then

$$\begin{aligned} D^{(k)}_{1,i} D^{(s)}_{j,1} = \tfrac{1}{4}\Big[&e_{1,1}e_{i,1}e_{1,j}e_{1,1} + (-1)^s \mathrm{I}\left(e_{1,1}e_{i,1}e^2_{1,1}e_{1,j} - e_{1,1}e_{i,1}e_{1,j}\right) + (-1)^k \mathrm{I}\, e^2_{1,1}e_{i,1}e_{1,j}e_{1,1} \\ &- (-1)^{k+s} e^2_{1,1}e_{i,1}\left(e^2_{1,1}e_{1,j} - e_{1,j}\right)\Big]. \end{aligned}$$

Using (6.30) of Proposition 6.3.5, (6.6) of Proposition 6.3.2, (6.44) and (6.45) of Proposition 6.3.7, we get

$$\begin{aligned} D^{(k)}_{1,i} D^{(s)}_{j,1} &= \tfrac{1}{4}\Big[\delta_{i,j}\tfrac{1}{2}\left(e^4_{1,1} - e^2_{1,1}\right) - \delta_{i,j}(-1)^s \mathrm{I}\tfrac{1}{2}\left(e^3_{1,1} - e_{1,1}\right) + \delta_{i,j}(-1)^k \tfrac{1}{2}\mathrm{I}(e_{1,1} - e^3_{1,1}) \\ &\quad + \delta_{i,j}(-1)^{k+s}\tfrac{1}{2}(e^4_{1,1} - e^2_{1,1})\Big] \\ &= \tfrac{1}{4}\delta_{i,j}\Big[\tfrac{1}{2}\left(1 + (-1)^{k+s}\right)\left(e^4_{1,1} - e^2_{1,1}\right) + \tfrac{1}{2}\left((-1)^s + (-1)^k\right) \mathrm{I}(e_{1,1} - e^3_{1,1})\Big] \end{aligned}$$

$$= \delta_{s,k}\delta_{i,j}D_{1,1}^{(k)}.$$

Also,

$$D_{i,1}^{(k)}D_{1,\ell}^{(s)} = \tfrac{1}{4}\Big[e_{1,i}e_{1,1}^2e_{\ell,1} + (-1)^s\mathrm{I}\,e_{1,i}e_{1,1}^3e_{\ell,1} + (-1)^k\mathrm{I}\left(e_{1,1}^2e_{1,i}e_{1,1}e_{\ell,1} - e_{1,i}e_{1,1}e_{\ell,1}\right)$$
$$- (-1)^{k+s}\left(e_{1,1}^2e_{1,i}e_{1,1}^2e_{\ell,1} - e_{1,i}e_{1,1}^2e_{\ell,1}\right)\Big].$$

Using (6.11) of Proposition 6.3.3 and (6.33) of Proposition 6.3.5 implies

$$D_{i,1}^{(k)}D_{1,\ell}^{(s)} = \tfrac{1}{4}\Big[(1 + (-1)^{k+s})\left(\delta_{i,\ell}\tfrac{1}{2}(e_{1,1}^4 + e_{1,1}^2) - e_{1,i}e_{\ell,1}\right)$$
$$- \left((-1)^s + (-1)^k\right)\mathrm{I}\,e_{1,i}e_{1,1}e_{\ell,1}\Big] = \delta_{k,s}D_{i,\ell}^{(k)}.$$

The other products can be obtained by using the argument at the end of the proof of the first two sets of $n \times n$ matrix units. Summarizing

$$D_{i,j}^{(s)}D_{k,\ell}^{(s)} = \delta_{j,k}D_{i,\ell}^{(s)}, \quad D_{i,j}^{(s)}D_{k,\ell}^{(t)} = 0, \tag{6.60}$$

for all $s, t \in \{0, 1\}$, $s \neq t$ and $i, j, k, \ell = 1, \dots, n$.

We wish to prove now that the product of any $D_{i,j}^{(k)}$ by any $B_{m,n}^{(s)}$ is 0. Clearly $D_{1,i}^{(k)}B_{\ell,1}^{(s)} = 0$ and $D_{1,i}^{(k)}B_{1,\ell}^{(s)} = 0$ $(i, \ell \neq 1)$, since $e_{i,1}e_{\ell,1} = 0$ and $e_{1,1}e_{i,1}e_{1,1} = 0$. Let $\ell \neq 1$. Then

$$D_{1,1}^{(k)}B_{1,\ell}^{(s)} = \tfrac{1}{4}\Big[\left(e_{1,1}^4 - e_{1,1}^2\right)e_{1,1}e_{1,\ell} + (-1)^s(e_{1,1}^4 - e_{1,1}^2)e_{1,1}^2e_{1,\ell} + (-1)^k\mathrm{I}\left(e_{1,1} - e_{1,1}^3\right)e_{1,1}e_{1,\ell}$$
$$+ (-1)^{k+s}\mathrm{I}\left(e_{1,1} - e_{1,1}^3\right)e_{1,1}^2e_{1,\ell}\Big]$$
$$= \tfrac{1}{4}\Big[\left(e_{1,1}e_{1,\ell} - e_{1,1}e_{1,\ell}\right) + (-1)^s\left(e_{1,1}^2e_{1,\ell} - e_{1,1}^2e_{1,\ell}\right) + (-1)^k\mathrm{I}\left(e_{1,1}^2 - e_{1,1}^2\right)e_{1,\ell}$$
$$+ (-1)^{k+s}\mathrm{I}\left(e_{1,1} - e_{1,1}\right)e_{1,\ell}\Big] = 0,$$

using (6.54), (6.55), (6.52) of Proposition 6.3.8 and (6.38) of Proposition 6.3.6. Also,

$$D_{1,1}^{(k)}B_{\ell,1}^{(s)} = \tfrac{1}{16}\Big[(e_{1,1}^4 - e_{1,1}^2)e_{\ell,1}e_{1,1} + (-1)^s\left(e_{1,1}^4 - e_{1,1}^2\right)\left(e_{1,1}^2e_{\ell,1} + e_{\ell,1}\right)$$
$$+ (-1)^k\mathrm{I}\left(e_{1,1} - e_{1,1}^3\right)e_{\ell,1}e_{1,1} + (-1)^{k+s}\mathrm{I}\left(e_{1,1} - e_{1,1}^3\right)\left(e_{1,1}^2e_{\ell,1} + e_{\ell,1}\right)\Big]$$
$$= \tfrac{1}{16}\Big[(-1)^s\left(e_{1,1}^2e_{\ell,1} - e_{1,1}^2e_{\ell,1} - \left(-e_{1,1}^2e_{\ell,1} + e_{1,1}^2e_{\ell,1}\right)\right)$$
$$+ (-1)^{k+s}\mathrm{I}\left(-e_{1,1}e_{\ell,1} + e_{1,1}e_{\ell,1} - e_{1,1}e_{\ell,1} + e_{1,1}e_{\ell,1}\right)\Big] = 0,$$

using (6.56), (6.52), (6.53) of Proposition 6.3.8 and (6.38) of Proposition 6.3.6. We have shown that

$$D^{(k)}_{1,i}B^{(s)}_{\ell,1} = 0, \quad D^{(k)}_{1,i}B^{(s)}_{1,\ell} = 0, \quad D^{(k)}_{1,1}B^{(s)}_{1,\ell} = 0, \quad D^{(k)}_{1,1}B^{(s)}_{\ell,1} = 0, \tag{6.61}$$

for all $i, \ell \neq 1$ and $k, s \in \{0, 1\}$. Let $\ell \neq 1$, then $D^{(k)}_{1,i}B^{(s)}_{1,1} = D^{(k)}_{1,i}B^{(s)}_{1,\ell}B^{(s)}_{\ell,1} = 0$ (for all i), using (6.59) and (6.61). Combining this result with the first and the last equations of (6.61) gives $D^{(k)}_{1,i}B^{(s)}_{j,1} = 0$ (for all i, j). By (6.59) and (6.60), $D^{(k)}_{i,j}B^{(s)}_{t,\ell} = D^{(k)}_{i,1}D^{(k)}_{1,j}B^{(s)}_{t,1}B^{(s)}_{1,\ell}$ (for all i, j, t, ℓ). Hence, $D^{(k)}_{i,j}B^{(s)}_{t,\ell} = 0$ (for all i, j, t, ℓ). By using the anti-automorphism η, we can show that $B^{(s)}_{t,\ell}D^{(k)}_{i,j} = 0$ (for all i, j, t, ℓ). Summarizing

$$D^{(k)}_{i,j}B^{(s)}_{t,\ell} = 0 = B^{(s)}_{t,\ell}D^{(k)}_{i,j} \quad \text{for all } i, j, t, \ell = 1, \ldots, n, \; s, k \in \{0, 1\}. \tag{6.62}$$

Finally, we define 1×1 matrix unit. We set,

$$A_{1,1} = \sum_{i=2}^{n} e_{1,i}e_{i,1} - \sum_{i=2}^{n} e_{i,1}e_{1,i} - ne^4_{1,1} + 1. \tag{6.63}$$

We wish to show that $A^2_{1,1} = A_{1,1}$ and the products of $A_{1,1}$ by any $B^{(k)}_{i,j}$ and $D^{(k)}_{i,j}$ are 0. We observe that

$$\begin{aligned}
&\sum_{k=0}^{1}\left(B^{(k)}_{1,1} + D^{(k)}_{1,1}\right) + \sum_{i=2}^{n}\sum_{k=0}^{1} B^{(k)}_{i,i} + \sum_{i=2}^{n}\sum_{k=0}^{1} D^{(k)}_{i,i} \\
&= e^4_{1,1} + \sum_{i=2}^{n}\left(\tfrac{1}{2}(e^4_{1,1} - e^2_{1,1}) + e_{i,1}e_{1,i}\right) + \sum_{i=2}^{n}\left(\tfrac{1}{2}(e^4_{1,1} + e^2_{1,1}) - e_{1,i}e_{i,1}\right) \\
&= e^4_{1,1} + \tfrac{1}{2}(n-1)(e^4_{1,1} - e^2_{1,1}) + \sum_{i=2}^{n} e_{i,1}e_{1,i} + \tfrac{1}{2}(n-1)(e^4_{1,1} + e^2_{1,1}) - \sum_{i=2}^{n} e_{1,i}e_{i,1} \\
&= ne^4_{1,1} + \sum_{i=2}^{n} e_{i,1}e_{1,i} - \sum_{i=2}^{n} e_{1,i}e_{i,1}.
\end{aligned} \tag{6.64}$$

Using (6.64) in (6.63) gives

$$A_{1,1} = 1 - \left(\sum_{k=0}^{1}\left(B^{(k)}_{1,1} + D^{(k)}_{1,1}\right) + \sum_{i=2}^{n}\sum_{k=0}^{1} B^{(k)}_{i,i} + \sum_{i=2}^{n}\sum_{k=0}^{1} D^{(k)}_{i,i}\right). \tag{6.65}$$

Multiplying (6.65) by $B^{(k)}_{\ell,m}$ from the right, and using the relations of (6.59), (6.60) and (6.62) (of the present proof), we obtain

$$A_{1,1}B^{(k)}_{\ell,m} = B^{(k)}_{\ell,m} - B^{(k)}_{\ell,m} = 0.$$

Similarly, we can show that $B_{\ell,m}^{(k)} A_{1,1} = 0$ and $A_{1,1} D_{\ell,m}^{(k)} = 0 = D_{\ell,m}^{(k)} A_{1,1}$. To show $A_{1,1}^2 = A_{1,1}$, we multiply (6.65) by $A_{1,1}$ and use the last equations.

Now let $\Phi_n^{(k)}$ (resp. $\Psi_n^{(k)}$ and τ_1) denote the subspace of $\mathfrak{A}$ generated by the $B_{i,j}^{(k)}$ (resp. $D_{i,j}^{(k)}$ and $A_{1,1}$), $k \in \{0,1\}$. Our discussion shows that $\Phi_n^{(k)}$ (resp. $\Psi_n^{(k)}$ and τ_1) is a subalgebra of $\mathfrak{A}$ and isomorphic to $M_{n,n}$ (resp. $M_{n,n}$ and $M_{1,1}$), $\Phi_n^{(k)}\Phi_n^{(s)} = 0 = \Phi_n^{(s)}\Phi_n^{(k)}$, $\Psi_n^{(k)}\Psi_n^{(s)} = 0 = \Psi_n^{(s)}\Psi_n^{(k)}$ $(k \neq s)$, $\Phi_n^{(k)}\Psi_n^{(s)} = 0 = \Psi_n^{(s)}\Phi_n^{(k)}$, $\Phi_n^{(s)}\tau_1 = 0 = \tau_1\Phi_n^{(s)}$ and $\Psi_n^{(s)}\tau_1 = 0 = \tau_1\Psi_n^{(s)}$. By (6.65) and the definitions of $B_{i,j}^{(k)}$ and $D_{i,j}^{(k)}$, we have

$$
\begin{aligned}
1 &= A_{1,1} + \sum_{i=1}^{n} B_{i,i}^{(0)} + \sum_{i=1}^{n} B_{i,i}^{(1)} + \sum_{i=1}^{n} D_{i,i}^{(0)} + \sum_{i=1}^{n} D_{i,i}^{(1)}, \\
e_{1,1} &= B_{1,1}^{(0)} - B_{1,1}^{(1)} - \mathrm{I}\, D_{1,1}^{(0)} + \mathrm{I}\, D_{1,1}^{(1)}, \\
e_{i,j} &= B_{i,j}^{(0)} - B_{i,j}^{(1)} - \mathrm{I}\, D_{j,i}^{(0)} + \mathrm{I}\, D_{j,i}^{(1)}; \quad i,j \neq 1, i \neq j, \\
e_{i,i} &= B_{i,i}^{(0)} - B_{i,i}^{(1)} - \mathrm{I}\, D_{i,i}^{(0)} + \mathrm{I}\, D_{i,i}^{(1)}; \quad i \neq 1, \\
e_{1,i} &= \tfrac{1}{2} B_{1,i}^{(0)} - \tfrac{1}{2} B_{1,i}^{(1)} - \mathrm{I}\, D_{i,1}^{(0)} + \mathrm{I}\, D_{i,1}^{(1)}; \quad i \neq 1, \\
e_{i,1} &= 2B_{i,1}^{(0)} - 2B_{i,1}^{(1)} - \mathrm{I}\, D_{1,i}^{(0)} + \mathrm{I}\, D_{1,i}^{(1)}; \quad i \neq 1.
\end{aligned} \tag{6.66}
$$

Thus all the $1, e_{i,j} \in \tau_1 \oplus \Phi_n^{(0)} \oplus \Phi_n^{(1)} \oplus \Psi_n^{(0)} \oplus \Psi_n^{(1)}$. Hence $\mathfrak{A} = \tau_1 \oplus \Phi_n^{(0)} \oplus \Phi_n^{(1)} \oplus \Psi_n^{(0)} \oplus \Psi_n^{(1)}$. □

Remark 6.5.2. The equations (6.66) (of the last proof) describe all inequivalent irreducible representations of the anti-Jordan triple system $\mathfrak{J}$.

Corollary 6.5.3. *The universal enveloping algebra of the simple anti-Jordan triple system of all $n \times n$ matrices over an algebraically closed field is semisimple.*

6.6 Infinite dimensional envelopes of anti-Jordan triple systems

In this section we provide examples of simple and non-simple finite-dimensional anti-Jordan triple systems with infinite-dimensional envelopes.

6.6.1 Universal associative envelope of a simple polarized anti-Jordan triple system

Let P be the subspace of 3×3 matrices defined by

$$P = \left\{ \begin{pmatrix} 0 & x & y \\ z & 0 & 0 \\ w & 0 & 0 \end{pmatrix} \mid x, y, z, w \in F \right\}.$$

Then P defines 4-dimensional anti-Jordan triple system with the triple product $\langle a, b, c \rangle = abc - cba$. By Bashir's classification [2], P is an example of (simple) polarized (see Definition 2.5.10) anti-Jordan triple systems. Let $\mathcal{B} = \{a = E_{1,2},\ b = E_{1,3},\ c = E_{2,1},\ d = E_{3,1}\}$ be a basis of P of matrix units.

Theorem 6.6.1. *The universal associative envelope $\mathfrak{A}$ of P is infinite dimensional with basis*

$$\{a^i(ca)^j(da)^k(db)^\ell d^m,\ a^i(ca)^j(cb)^k(db)^\ell d^m\ (k \neq 0),$$
$$a^i(ca)^j(cb)^k c,\ (bc)^j(bd)^k b^m \mid j, k, \ell \geq 0,\ 0 \leq i, m \leq 1\}.$$

Proof. The universal associative envelope $\mathfrak{A}$ of P is $\mathfrak{A} = F\langle a, b, c, d \rangle / I$, where I is the two sided ideal generated by 24 elements:

$$ba^2 - a^2b,\ b^2a - ab^2,\ bca - acb + b,\ bda - adb - a,\ ca^2 - a^2c,\ cab - bac,$$
$$cba - abc,\ cb^2 - b^2c,\ c^2a - ac^2,\ c^2b - bc^2,\ cda - adc,\ cdb - bdc,$$
$$da^2 - a^2d,\ dab - bad,\ dac - cad - d,\ dba - abd,\ db^2 - b^2d,\ dbc - cbd + c,$$
$$dca - acd,\ dcb - bcd,\ dc^2 - c^2d,\ d^2a - ad^2,\ d^2b - bd^2,\ d^2c - cd^2.$$

We compute a Gröbner basis for the ideal I. The first iteration of the Gröbner basis algorithm produces 36 compositions with distinct non zero normal forms:

$$ba,\ b^2,\ a^2,\ ab,\ bac,\ a^2c,\ ab^2,\ b^2c,\ abc,\ bdc,\ ac^2,\ cd,\ c^2,\ bad,$$
$$a^2d,\ abd,\ a^2c,\ a^2d,\ bac,\ acd,\ bcd,\ ad^2,\ a^2b,\ b^2d,\ abd,\ ac^2,\ bc^2,$$
$$adc,\ bdc,\ bd^2,\ acd,\ dc,\ d^2,\ cd^2,\ c^2d.$$

Including these elements with the original generators, and self-reducing the resulting set produces the twelve generators

$$dbc - cbd + c,\ dac - cad - d,\ bda - adb - a,\ bca - acb + b,\ d^2,\ dc,\ cd,\ c^2,$$
$$b^2,\ ba,\ ab,\ a^2.$$

It is easy to verify that all compositions of these elements reduce to 0, and so we have a Gröbner basis. A basis for $\mathfrak{A}$ consists of the cosets of the monomials which do not contain the leading monomials: dbc, dac, bda, bca, d^2, dc, cd, c^2, b^2, ba, ab, a^2 as factors. □

6.6.2 Universal associative envelope of a non-simple anti-Jordan triple system

We recall the next example from Chapter 5. Let S be the subspace of 2×2 matrices defined by

$$S = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \mid x, y \in F \right\}.$$

Then S defines 2-dimensional anti-Jordan triple system with the triple product $\langle a, b, c \rangle = abc - cba$. Let $\mathcal{B} = \{E_{1,2}, E_{2,1}\}$ be a basis of S of matrix units. The universal enveloping algebra has two generators $e_{1,2}$, $e_{2,1}$ and relations $e_{2,1}^2 e_{1,2} = e_{1,2} e_{2,1}^2$, $e_{2,1} e_{1,2}^2 = e_{1,2}^2 e_{2,1}$, which is the down-up algebra $A(0, 1, 0)$ (see Definition 5.4.1).

To conclude this chapter, we formulate the following conjecture.

Conjecture 6.6.2. *If the universal enveloping algebra of a simple finite-dimensional anti-Jordan triple system is finite-dimensional, then it is semisimple.*

Appendix A

	$r=0$	2	4	6	8	10	12	14	16	18	20	22
$q=0$	0	0	1	1	2	2	5	4	8	7	12	11
1	17	15	23	21	29	27	37	34	45	42	54	51
2	64	60	75	71	86	82	99	94	112	107	126	121
3	141	135	157	151	173	167	191	184	209	202	228	221
4	248	240	269	261	290	282	313	304	336	327	360	351
5	385	375	411	401	437	427	465	454	493	482	522	511
6	552	540	583	571	614	602	647	634	680	667	714	701
7	749	735	785	771	821	807	859	844	897	882	936	921
8	976	960	1017	1001	1058	1042	1101	1084	1144	1127	1188	1171
9	1233	1215	1279	1261	1325	1307	1373	1354	1421	1402	1470	1451

Table A.1: Multiplicities mult(n) for $n = 24q + r$ with $0 \le q \le 9$ (n even)

$$\begin{array}{rl} 4 & [\ 4,\ 2,\ 0,\ -2\ ] \\ 2 & [\ 4,\ 2,\ 0,\ -4\ ] \\ 0 & [\ 4,\ 2,\ -2,\ -4\ ] \\ -2 & [\ 4,\ 0,\ -2,\ -4\ ] \\ -4 & [\ 2,\ 0,\ -2,\ -4\ ] \end{array} \qquad V(4)\left\{\begin{array}{rl} 4\colon & [1] \\ 2\colon & [4] \\ 0\colon & [6] \\ -2\colon & [4] \\ -4\colon & [1] \end{array}\right.$$

Table A.2: Tensor basis and weight vector basis of $\Lambda^4 V(4)$

$$[4,2,0,-2] = 12 \qquad [4,2,0,-4] = 3 \qquad [4,2,-2,-4] = 2$$
$$[4,0,-2,-4] = 3 \qquad [2,0,-2,-4] = 12$$

Table A.3: Quaternary algebra structure on $V(4)$, integral version

Weight	Quadruples				
12	[6, 4, 2, 0]				
10	[6, 4, 2, -2]				
8	[6, 4, 2, -4]	[6, 4, 0, -2]			
6	[6, 4, 2, -6]	[6, 4, 0, -4]	[6, 2, 0, -2]		
4	[6, 4, 0, -6]	[6, 4, -2, -4]	[6, 2, 0, -4]	[4, 2, 0, -2]	
2	[6, 4, -2, -6]	[6, 2, 0, -6]	[6, 2, -2, -4]	[4, 2, 0, -4]	
0	[6, 4, -4, -6]	[6, 2, -2, -6]	[6, 0, -2, -4]	[4, 2, 0, -6]	[4, 2, -2, -4]
-2	[6, 2, -4, -6]	[6, 0, -2, -6]	[4, 2, -2, -6]	[4, 0, -2, -4]	
-4	[6, 0, -4, -6]	[4, 2, -4, -6]	[4, 0, -2, -6]	[2, 0, -2, -4]	
-6	[6, -2, -4, -6]	[4, 0, -4, -6]	[2, 0, -2, -6]		
-8	[4, -2, -4, -6]	[2, 0, -4, -6]			
-10	[2, -2, -4, -6]				
-12	[0, -2, -4, -6]				

Table A.4: Tensor basis of $\Lambda^4 V(6)$

$$V(12)\left\{\begin{array}{rl} 12\text{:} & [1] \\ 10\text{:} & [4] \\ 8\text{:} & [10,6] \\ 6\text{:} & [20,20,4] \\ 4\text{:} & [45,20,15,1] \\ 2\text{:} & [60,36,20,4] \\ 0\text{:} & [50,64,10,10,6] \\ -2\text{:} & [60,36,20,4] \\ -4\text{:} & [45,20,15,1] \\ -6\text{:} & [20,20,4] \\ -8\text{:} & [10,6] \\ -10\text{:} & [4] \\ -12\text{:} & [1] \end{array}\right. \qquad V(8)\left\{\begin{array}{rl} 8\text{:} & [-2,1] \\ 6\text{:} & [-12,-1,2] \\ 4\text{:} & [-21,-2,4,1] \\ 2\text{:} & [-32,-6,4,3] \\ 0\text{:} & [-40,-16,3,3,4] \\ -2\text{:} & [-32,-6,4,3] \\ -4\text{:} & [-21,-2,4,1] \\ -6\text{:} & [-12,-1,2] \\ -8\text{:} & [-2,1] \end{array}\right.$$

$$V(6)\left\{\begin{array}{rl} 6\text{:} & [20,-5,2] \\ 4\text{:} & [30,-20,0,2] \\ 2\text{:} & [0,30,-20,5] \\ 0\text{:} & [0,0,-20,20,0] \\ -2\text{:} & [0,-30,20,-5] \\ -4\text{:} & [-30,20,0,-2] \\ -6\text{:} & [-20,5,-2] \end{array}\right. \qquad V(4)\left\{\begin{array}{rl} 4\text{:} & [0,5,-3,1] \\ 2\text{:} & [30,-18,-2,2] \\ 0\text{:} & [75,-12,-3,-3,3] \\ -2\text{:} & [30,-18,-2,2] \\ -4\text{:} & [0,5,-3,1] \end{array}\right.$$

$$V(0)\left\{\ 0\text{:}\quad [-15,6,-3,-3,1]\right.$$

Table A.5: Weight vector basis of $\Lambda^4 V(6)$

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Table A.6: The weight vector matrix for $V(6)$.

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	·	1/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	1/22	·	1/11	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	1/220	·	2/55	·	·	·	·	1/22	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	1/165	2/165	·	·	·	1/33	·	·	·	·	·	·	·	·	1/33	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	1/264	·	·	·	1/132	5/264	·	·	·	·	·	·	·	1/33	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	1/924	·	·	·	·	2/231	·	5/462	·	·	·	·	·	5/462	9/308	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/264	·	1/132	·	·	·	·	·	·	5/264	·	1/33	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/165	·	·	·	·	·	·	2/165	·	1/33	·	·	1/33	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/220	·	·	·	·	·	·	·	·	2/55	·	·	1/22	·	·	·
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	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/4	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1
	·	·	3/11	·	-5/11	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	3/88	·	1/44	·	·	·	·	-25/88	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	1/44	3/308	·	·	·	-5/77	·	·	·	·	·	·	·	·	-75/308	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	1/77	·	·	·	5/616	-15/616	·	·	·	·	·	·	·	-45/308	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	2/385	·	·	·	·	1/77	·	-3/154	·	·	·	·	·	-3/154	-9/77	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/77	·	5/616	·	·	·	·	·	·	-15/616	·	-45/308	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/44	·	·	·	·	·	·	3/308	·	-5/77	·	·	-75/308	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3/88	·	·	·	·	·	·	·	·	1/44	·	·	-25/88	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3/11	·	·	-5/11	·	·
Row23 ↘	·	·	·	1/40	·	-1/20	·	·	·	·	1/8	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	1/120	-1/40	·	·	·	·	·	·	·	·	·	·	·	·	1/8	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	1/120	-1/40	·	·	·	·	·	·	·	1/20	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	-1/40	·	·	·	·	·	1/40	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	-1/120	·	·	·	·	·	·	1/40	·	-1/20	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	-1/120	·	·	·	·	·	·	1/40	·	·	·	·	-1/8	·	·	·	·
Row29 ↗	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	-1/40	·	·	·	·	·	·	·	·	1/20	·	·	-1/8	·	·	·
	·	·	·	·	·	·	·	1/21	·	·	·	-2/21	·	·	·	·	·	·	·	·	10/21	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	1/84	·	·	·	-1/42	-1/84	·	·	·	·	·	·	·	2/21	·	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	1/126	·	·	·	·	-1/126	·	-1/63	·	·	·	·	·	-1/63	1/14	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/84	·	-1/42	·	·	·	·	·	·	-1/84	·	2/21	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1/21	·	-2/21	·	·	10/21	·	·	·	·
	·	·	·	·	·	·	·	·	·	1/105	·	·	·	·	-1/42	·	2/21	·	·	·	·	·	2/21	-1/7	·	·	·	·	·	·	·	·	·	·	·

Table A.7: The inverse of the weight vector matrix for $V(6)$.

$[6,4,2,-6]=3$	$[6,4,0,-4]=-6$	$[6,2,0,-2]=15$	$[6,4,0,-6]=1$
$[6,4,-2,-4]=-3$	$[6,2,0,-4]=0$	$[4,2,0,-2]=15$	$[6,4,-2,-6]=0$
$[6,2,0,-6]=1$	$[6,2,-2,-4]=-3$	$[4,2,0,-4]=6$	$[6,4,-4,-6]=0$
$[6,2,-2,-6]=0$	$[6,0,-2,-4]=-3$	$[4,2,0,-6]=3$	$[4,2,-2,-4]=0$
$[6,2,-4,-6]=0$	$[6,0,-2,-6]=-1$	$[4,2,-2,-6]=3$	$[4,0,-2,-4]=-6$
$[6,0,-4,-6]=-1$	$[4,2,-4,-6]=3$	$[4,0,-2,-6]=0$	$[2,0,-2,-4]=-15$
$[6,-2,-4,-6]=-3$	$[4,0,-4,-6]=6$	$[2,0,-2,-6]=-15$	

Table A.8: Quaternary algebra structure on $V(6)$, integral version

Weight	Quadruples					
20	[8, 6, 4, 2]					
18	[8, 6, 4, 0]					
16	[8, 6, 4,-2]	[8, 6, 2, 0]				
14	[8, 6, 4,-4]	[8, 6, 2,-2]	[8, 4, 2, 0]			
12	[8, 6, 4,-6]	[8, 6, 2,-4]	[8, 6, 0,-2]	[8, 4, 2,-2]	[6, 4, 2, 0]	
10	[8, 6, 4,-8]	[8, 6, 2,-6]	[8, 6, 0,-4]	[8, 4, 2,-4]	[8, 4, 0,-2]	[6, 4, 2,-2]
8	[8, 6, 2,-8]	[8, 6, 0,-6]	[8, 6,-2,-4]	[8, 4, 2,-6]	[8, 4, 0,-4]	[8, 2, 0,-2]
	[6, 4, 2,-4]	[6, 4, 0,-2]				
6	[8, 6, 0,-8]	[8, 6,-2,-6]	[8, 4, 2,-8]	[8, 4, 0,-6]	[8, 4,-2,-4]	[8, 2, 0,-4]
	[6, 4, 2,-6]	[6, 4, 0,-4]	[6, 2, 0,-2]			
4	[8, 6,-2,-8]	[8, 6,-4,-6]	[8, 4, 0,-8]	[8, 4,-2,-6]	[8, 2, 0,-6]	[8, 2,-2,-4]
	[6, 4, 2,-8]	[6, 4, 0,-6]	[6, 4,-2,-4]	[6, 2, 0,-4]	[4, 2, 0,-2]	
2	[8, 6,-4,-8]	[8, 4,-2,-8]	[8, 4,-4,-6]	[8, 2, 0,-8]	[8, 2,-2,-6]	[8, 0,-2,-4]
	[6, 4, 0,-8]	[6, 4,-2,-6]	[6, 2, 0,-6]	[6, 2,-2,-4]	[4, 2, 0,-4]	
0	[8, 6,-6,-8]	[8, 4,-4,-8]	[8, 2,-2,-8]	[8, 2,-4,-6]	[8, 0,-2,-6]	[6, 4,-2,-8]
	[6, 4,-4,-6]	[6, 2, 0,-8]	[6, 2,-2,-6]	[6, 0,-2,-4]	[4, 2, 0,-6]	[4, 2,-2,-4]
-2	[8, 4,-6,-8]	[8, 2,-4,-8]	[8, 0,-2,-8]	[8, 0,-4,-6]	[6, 4,-4,-8]	[6, 2,-2,-8]
	[6, 2,-4,-6]	[6, 0,-2,-6]	[4, 2, 0,-8]	[4, 2,-2,-6]	[4, 0,-2,-4]	
-4	[8, 2,-6,-8]	[8, 0,-4,-8]	[8,-2,-4,-6]	[6, 4,-6,-8]	[6, 2,-4,-8]	[6, 0,-2,-8]
	[6, 0,-4,-6]	[4, 2,-2,-8]	[4, 2,-4,-6]	[4, 0,-2,-6]	[2, 0,-2,-4]	
-6	[8, 0,-6,-8]	[8,-2,-4,-8]	[6, 2,-6,-8]	[6, 0,-4,-8]	[6,-2,-4,-6]	[4, 2,-4,-8]
	[4, 0,-2,-8]	[4, 0,-4,-6]	[2, 0,-2,-6]			
-8	[8,-2,-6,-8]	[6, 0,-6,-8]	[6,-2,-4,-8]	[4, 2,-6,-8]	[4, 0,-4,-8]	[4,-2,-4,-6]
	[2, 0,-2,-8]	[2, 0,-4,-6]				
-10	[8,-4,-6,-8]	[6,-2,-6,-8]	[4, 0,-6,-8]	[4,-2,-4,-8]	[2, 0,-4,-8]	[2,-2,-4,-6]
-12	[6,-4,-6,-8]	[4,-2,-6,-8]	[2, 0,-6,-8]	[2,-2,-4,-8]	[0,-2,-4,-6]	
-14	[4,-4,-6,-8]	[2,-2,-6,-8]	[0,-2,-4,-8]			
-16	[2,-4,-6,-8]	[0,-2,-6,-8]				
-18	[0,-4,-6,-8]					
-20	[-2,-4,-6,-8]					

Table A.9: Tensor basis of $\Lambda^4 V(8)$

$V(20)$	20:	[1]
	18:	[4]
	16:	[10, 6]
	14:	[20, 20, 4]
	12:	[35, 45, 20, 15, 1]
	10:	[56, 84, 60, 36, 20, 4]
	8:	[140, 126, 50, 70, 64, 10, 10, 6]
	6:	[224, 140, 120, 140, 60, 36, 20, 20, 4]
	4:	[280, 105, 256, 175, 84, 45, 35, 45, 20, 15, 1]
	2:	[280, 360, 140, 160, 140, 20, 84, 60, 36, 20, 4]
	0:	[196, 384, 300, 126, 70, 126, 50, 70, 64, 10, 10, 6]
	−2:	[280, 360, 160, 84, 140, 140, 60, 36, 20, 20, 4]
	−4:	[280, 256, 35, 105, 175, 84, 45, 45, 20, 15, 1]
	−6:	[224, 120, 140, 140, 20, 60, 36, 20, 4]
	−8:	[140, 126, 70, 50, 64, 10, 10, 6]
	−10:	[56, 84, 60, 36, 20, 4]
	−12:	[35, 45, 20, 15, 1]
	−14:	[20, 20, 4]
	−16:	[10, 6]
	−18:	[4]
	−20:	[1]
$V(16)$	16:	[−3, 2]
	14:	[−18, 1, 4]
	12:	[−63, −24, 2, 11, 2]
	10:	[−168, −119, −28, 6, 16, 7]
	8:	[−364, −168, −35, −49, 16, 12, 12, 11]
	6:	[−560, −217, −224, −84, 2, 24, 7, 26, 9]
	4:	[−756, −217, −448, −140, −14, 21, −28, 21, 22, 26, 3]
	2:	[−896, −696, −182, −208, −49, 12, −56, 17, 33, 31, 10]
	0:	[−784, −928, −440, −105, −14, −105, −10, −14, 48, 17, 17, 14]
	−2:	[−896, −696, −208, −56, −182, −49, 17, 33, 12, 31, 10]
	−4:	[−756, −448, −28, −217, −140, −14, 21, 21, 22, 26, 3]
	−6:	[−560, −224, −217, −84, 7, 2, 24, 26, 9]
	−8:	[−364, −168, −49, −35, 16, 12, 12, 11]
	−10:	[−168, −119, −28, 6, 16, 7]
	−12:	[−63, −24, 2, 11, 2]
	−14:	[−18, 1, 4]
	−16:	[−3, 2]

Table A.10: Weight vector basis of $\Lambda^4 V(8)$: part one

$$
V(14)\left\{\begin{array}{rl}
14\colon & [14,-7,4]\\
12\colon & [98,0,-28,6,4]\\
10\colon & [392,147,-84,18,-16,13]\\
8\colon & [784,0,-140,140,-64,-16,32,12]\\
6\colon & [784,-245,672,28,-150,-72,91,34,5]\\
4\colon & [392,-294,896,-280,-84,-162,280,126,4,12,2]\\
2\colon & [0,504,-378,336,-399,-108,504,63,63,-15,6]\\
0\colon & [0,0,0,-504,-336,504,0,336,0,-24,24,0]\\
-2\colon & [0,-504,-336,-504,378,399,-63,-63,108,15,-6]\\
-4\colon & [-392,-896,-280,294,280,84,-126,162,-4,-12,-2]\\
-6\colon & [-784,-672,245,-28,-91,150,72,-34,-5]\\
-8\colon & [-784,0,-140,140,64,-32,16,-12]\\
-10\colon & [-392,-147,84,-18,16,-13]\\
-12\colon & [-98,0,28,-6,-4]\\
-14\colon & [-14,7,-4]
\end{array}\right.
$$

$$
V(12)'\left\{\begin{array}{rl}
12\colon & [15,-5,3,0,0]\\
10\colon & [120,10,-2,-10,6,0]\\
8\colon & [220,13,-5,-25,-4,9,-5,3]\\
6\colon & [328,10,80,-34,-10,14,-20,-2,6]\\
4\colon & [430,15,176,-55,-1,10,-20,-32,-5,11,3]\\
2\colon & [540,260,-60,104,-20,8,-32,-50,-4,10,8]\\
0\colon & [630,360,190,-50,-4,-50,-60,-4,-20,8,8,10]\\
-2\colon & [540,260,104,-32,-60,-20,-50,-4,8,10,8]\\
-4\colon & [430,176,-20,15,-55,-1,-32,10,-5,11,3]\\
-6\colon & [328,80,10,-34,-20,-10,14,-2,6]\\
-8\colon & [220,13,-25,-5,-4,-5,9,3]\\
-10\colon & [120,10,-2,-10,6,0]\\
-12\colon & [15,-5,3,0,0]
\end{array}\right.
$$

$$
V(12)''\left\{\begin{array}{rl}
12\colon & [-42,14,0,-6,3]\\
10\colon & [-336,-28,56,-8,-24,9]\\
8\colon & [-616,140,140,-56,-32,-36,23,6]\\
6\colon & [-448,560,-560,-56,40,-104,35,32,-6]\\
4\colon & [560,840,-896,280,-224,-100,-70,77,50,-11,-3]\\
2\colon & [2016,-224,672,-896,-196,-80,-112,203,-14,-1,-8]\\
0\colon & [2352,1344,-1120,140,-224,140,315,-224,56,-14,-14,-7]\\
-2\colon & [2016,-224,-896,-112,672,-196,203,-14,-80,-1,-8]\\
-4\colon & [560,-896,-70,840,280,-224,77,-100,50,-11,-3]\\
-6\colon & [-448,-560,560,-56,35,40,-104,32,-6]\\
-8\colon & [-616,140,-56,140,-32,23,-36,6]\\
-10\colon & [-336,-28,56,-8,-24,9]\\
-12\colon & [-42,14,0,-6,3]
\end{array}\right.
$$

Table A.11: Weight vector basis of $\Lambda^4 V(8)$: part two

$$V(10)\begin{cases} 10\text{:} & [-210, 35, -7, -5, 3, 0] \\ 8\text{:} & [-350, 91, -35, 35, -16, 9, -5, 3] \\ 6\text{:} & [-336, 105, -210, 105, -75, 3, 0, -9, 9] \\ 4\text{:} & [-280, 210, -224, 70, 112, -70, -70, 14, -40, 10, 6] \\ 2\text{:} & [0, -280, 210, 56, 70, -70, -98, -35, 56, -35, 14] \\ 0\text{:} & [0, 0, 0, 210, -42, -210, 0, 42, 0, -42, 42, 0] \\ -2\text{:} & [0, 280, -56, 98, -210, -70, 35, -56, 70, 35, -14] \\ -4\text{:} & [280, 224, 70, -210, -70, -112, -14, 70, 40, -10, -6] \\ -6\text{:} & [336, 210, -105, -105, 0, 75, -3, 9, -9] \\ -8\text{:} & [350, -91, -35, 35, 16, 5, -9, -3] \\ -10\text{:} & [210, -35, 7, 5, -3, 0] \end{cases}$$

$$V(8)'\begin{cases} 8\text{:} & [0, -7, 7, 5, -2, 1, 0, 0] \\ 6\text{:} & [-56, 14, 40, -8, 4, 0, 5, -2, 1] \\ 4\text{:} & [-84, 42, -8, 8, -12, 6, 40, -1, -3, 0, 1] \\ 2\text{:} & [-56, -48, 44, -40, 2, 8, 48, -6, -5, -1, 2] \\ 0\text{:} & [-98, -12, -82, 36, 16, 36, 2, 16, -12, 1, 1, 2] \\ -2\text{:} & [-56, -48, -40, 48, 44, 2, -6, -5, 8, -1, 2] \\ -4\text{:} & [-84, -8, 40, 42, 8, -12, -1, 6, -3, 0, 1] \\ -6\text{:} & [-56, 40, 14, -8, 5, 4, 0, -2, 1] \\ -8\text{:} & [0, -7, 5, 7, -2, 0, 1, 0] \end{cases}$$

$$V(8)''\begin{cases} 8\text{:} & [0, -56, 42, 40, -8, 0, -5, 3] \\ 6\text{:} & [-448, 14, 320, -8, 44, -24, 5, -10, 9] \\ 4\text{:} & [-1064, 42, 160, 148, -96, 6, 180, -29, -3, 0, 9] \\ 2\text{:} & [-2016, -48, 324, -96, 2, 8, 216, -6, -61, -1, 18] \\ 0\text{:} & [-3528, -432, -152, 246, 16, 246, 72, 16, -82, 1, 1, 22] \\ -2\text{:} & [-2016, -48, -96, 216, 324, 2, -6, -61, 8, -1, 18] \\ -4\text{:} & [-1064, 160, 180, 42, 148, -96, -29, 6, -3, 0, 9] \\ -6\text{:} & [-448, 320, 14, -8, 5, 44, -24, -10, 9] \\ -8\text{:} & [0, -56, 40, 42, -8, -5, 0, 3] \end{cases}$$

Table A.12: Weight vector basis of $\Lambda^4 V(8)$: part three

$$
V(6)\begin{cases}
6\colon & [224, -70, -160, 8, 20, -8, 15, -6, 3] \\
4\colon & [560, -420, -128, 40, -32, 20, -40, 26, -10, -8, 6] \\
2\colon & [0, 400, -300, -320, 50, 40, -40, 50, -5, -25, 10] \\
0\colon & [0, 0, 0, -200, 160, 200, 0, -160, 0, -20, 20, 0] \\
-2\colon & [0, -400, 320, 40, 300, -50, -50, 5, -40, 25, -10] \\
-4\colon & [-560, 128, 40, 420, -40, 32, -26, -20, 10, 8, -6] \\
-6\colon & [-224, 160, 70, -8, -15, -20, 8, 6, -3]
\end{cases}
$$

$$
V(4)'\begin{cases}
4\colon & [28, -14, -16, 2, 4, -2, 10, -2, 1, 0, 0] \\
2\colon & [56, -8, -16, -16, 12, -8, 8, -1, -2, 1, 0] \\
0\colon & [196, -32, -4, 12, -4, 12, -11, -4, 3, -2, -2, 1] \\
-2\colon & [56, -8, -16, 8, -16, 12, -1, -2, -8, 1, 0] \\
-4\colon & [28, -16, 10, -14, 2, 4, -2, -2, 1, 0, 0]
\end{cases}
$$

$$
V(4)''\begin{cases}
4\colon & [-294, 245, 168, -63, -14, 21, -105, 21, 0, -7, 3] \\
2\colon & [196, -252, 112, 392, -112, 84, -84, 42, 0, -14, 4] \\
0\colon & [686, -112, 154, -168, 70, -168, 182, 70, -42, 14, 14, -4] \\
-2\colon & [196, -252, 392, -84, 112, -112, 42, 0, 84, -14, 4] \\
-4\colon & [-294, 168, -105, 245, -63, -14, 21, 21, 0, -7, 3]
\end{cases}
$$

$$
V(0)\{\ 0\colon \quad [-448, 128, -32, -24, 16, -24, 2, 16, 3, -4, -4, 2]
$$

Table A.13: Weight vector basis of $\Lambda^4 V(8)$: part four

$[8,6,2,-8] = 0$	$[8,6,0,-6] = -168$	$[8,6,-2,-4] = 1904$	$[8,4,2,-6] = 336$
$[8,4,0,-4] = -4172$	$[8,2,0,-2] = 15512$	$[6,4,2,-4] = 14336$	$[6,4,0,-2] = -21504$
$[8,6,0,-8] = -21$	$[8,6,-2,-6] = 392$	$[8,4,2,-8] = 42$	$[8,4,0,-6] = -980$
$[8,4,-2,-4] = -420$	$[8,2,0,-4] = 2688$	$[6,4,2,-6] = 3920$	$[6,4,0,-4] = -3276$
$[6,2,0,-2] = -616$	$[8,6,-2,-8] = 44$	$[8,6,-4,-6] = 168$	$[8,4,0,-8] = -131$
$[8,4,-2,-6] = -288$	$[8,2,0,-6] = -72$	$[8,2,-2,-4] = 1176$	$[6,4,2,-8] = 608$
$[6,4,0,-6] = 744$	$[6,4,-2,-4] = -2352$	$[6,2,0,-4] = 0$	$[4,2,0,-2] = -616$
$[8,6,-4,-8] = 50$	$[8,4,-2,-8] = -84$	$[8,4,-4,-6] = 52$	$[8,2,0,-8] = -143$
$[8,2,-2,-6] = 56$	$[8,0,-2,-4] = 980$	$[6,4,0,-8] = 456$	$[6,4,-2,-6] = -672$
$[6,2,0,-6] = 648$	$[6,2,-2,-4] = -784$	$[4,2,0,-4] = -308$	$[8,6,-6,-8] = 20$
$[8,4,-4,-8] = 30$	$[8,2,-2,-8] = -204$	$[8,2,-4,-6] = 96$	$[8,0,-2,-6] = 448$
$[6,4,-2,-8] = 96$	$[6,4,-4,-6] = -320$	$[6,2,0,-8] = 448$	$[6,2,-2,-6] = -512$
$[6,0,-2,-4] = 784$	$[4,2,0,-6] = 784$	$[4,2,-2,-4] = -1344$	$[8,4,-6,-8] = 50$
$[8,2,-4,-8] = -84$	$[8,0,-2,-8] = -143$	$[8,0,-4,-6] = 456$	$[6,4,-4,-8] = 52$
$[6,2,-2,-8] = 56$	$[6,2,-4,-6] = -672$	$[6,0,-2,-6] = 648$	$[4,2,0,-8] = 980$
$[4,2,-2,-6] = -784$	$[4,0,-2,-4] = -308$	$[8,2,-6,-8] = 44$	$[8,0,-4,-8] = -131$
$[8,-2,-4,-6] = 608$	$[6,4,-6,-8] = 168$	$[6,2,-4,-8] = -288$	$[6,0,-2,-8] = -72$
$[6,0,-4,-6] = 744$	$[4,2,-2,-8] = 1176$	$[4,2,-4,-6] = -2352$	$[4,0,-2,-6] = 0$
$[2,0,-2,-4] = -616$	$[8,0,-6,-8] = -21$	$[8,-2,-4,-8] = 42$	$[6,2,-6,-8] = 392$
$[6,0,-4,-8] = -980$	$[6,-2,-4,-6] = 3920$	$[4,2,-4,-8] = -420$	$[4,0,-2,-8] = 2688$
$[4,0,-4,-6] = -3276$	$[2,0,-2,-6] = -616$	$[8,-2,-6,-8] = 0$	$[6,0,-6,-8] = -168$
$[6,-2,-4,-8] = 336$	$[4,2,-6,-8] = 1904$	$[4,0,-4,-8] = -4172$	$[4,-2,-4,-6] = 14336$
$[2,0,-2,-8] = 15512$	$[2,0,-4,-6] = -21504$		

Table A.14: First quaternary algebra structure f on $V(8)$

$[8, 6, 2, -8] = 0$	$[8, 6, 0, -6] = -112$	$[8, 6, -2, -4] = 56$	$[8, 4, 2, -6] = 224$
$[8, 4, 0, -4] = 252$	$[8, 2, 0, -2] = -1792$	$[6, 4, 2, -4] = -2576$	$[6, 4, 0, -2] = 3864$
$[8, 6, 0, -8] = -14$	$[8, 6, -2, -6] = -42$	$[8, 4, 2, -8] = 28$	$[8, 4, 0, -6] = 105$
$[8, 4, -2, -4] = 175$	$[8, 2, 0, -4] = -483$	$[6, 4, 2, -6] = -420$	$[6, 4, 0, -4] = 91$
$[6, 2, 0, -2] = 1106$	$[8, 6, -2, -8] = -14$	$[8, 6, -4, -6] = -18$	$[8, 4, 0, -8] = 21$
$[8, 4, -2, -6] = 68$	$[8, 2, 0, -6] = -48$	$[8, 2, -2, -4] = -126$	$[6, 4, 2, -8] = -28$
$[6, 4, 0, -6] = -154$	$[6, 4, -2, -4] = 252$	$[6, 2, 0, -4] = 0$	$[4, 2, 0, -2] = 1106$
$[8, 6, -4, -8] = -10$	$[8, 4, -2, -8] = 9$	$[8, 4, -4, -6] = 13$	$[8, 2, 0, -8] = 13$
$[8, 2, -2, -6] = -6$	$[8, 0, -2, -4] = -105$	$[6, 4, 0, -8] = -21$	$[6, 4, -2, -6] = 72$
$[6, 2, 0, -6] = -218$	$[6, 2, -2, -4] = 84$	$[4, 2, 0, -4] = 553$	$[8, 6, -6, -8] = -4$
$[8, 4, -4, -8] = -6$	$[8, 2, -2, -8] = 20$	$[8, 2, -4, -6] = 12$	$[8, 0, -2, -6] = -48$
$[6, 4, -2, -8] = 12$	$[6, 4, -4, -6] = 64$	$[6, 2, 0, -8] = -48$	$[6, 2, -2, -6] = -64$
$[6, 0, -2, -4] = -84$	$[4, 2, 0, -6] = -84$	$[4, 2, -2, -4] = 560$	$[8, 4, -6, -8] = -10$
$[8, 2, -4, -8] = 9$	$[8, 0, -2, -8] = 13$	$[8, 0, -4, -6] = -21$	$[6, 4, -4, -8] = 13$
$[6, 2, -2, -8] = -6$	$[6, 2, -4, -6] = 72$	$[6, 0, -2, -6] = -218$	$[4, 2, 0, -8] = -105$
$[4, 2, -2, -6] = 84$	$[4, 0, -2, -4] = 553$	$[8, 2, -6, -8] = -14$	$[8, 0, -4, -8] = 21$
$[8, -2, -4, -6] = -28$	$[6, 4, -6, -8] = -18$	$[6, 2, -4, -8] = 68$	$[6, 0, -2, -8] = -48$
$[6, 0, -4, -6] = -154$	$[4, 2, -2, -8] = -126$	$[4, 2, -4, -6] = 252$	$[4, 0, -2, -6] = 0$
$[2, 0, -2, -4] = 1106$	$[8, 0, -6, -8] = -14$	$[8, -2, -4, -8] = 28$	$[6, 2, -6, -8] = -42$
$[6, 0, -4, -8] = 105$	$[6, -2, -4, -6] = -420$	$[4, 2, -4, -8] = 175$	$[4, 0, -2, -8] = -483$
$[4, 0, -4, -6] = 91$	$[2, 0, -2, -6] = 1106$	$[8, -2, -6, -8] = 0$	$[6, 0, -6, -8] = -112$
$[6, -2, -4, -8] = 224$	$[4, 2, -6, -8] = 56$	$[4, 0, -4, -8] = 252$	$[4, -2, -4, -6] = -2576$
$[2, 0, -2, -8] = -1792$	$[2, 0, -4, -6] = 3864$		

Table A.15: Second quaternary algebra structure g on $V(8)$

$[10,8,2,-10] = -15$	$[10,8,0,-8] = -540$	$[10,8,0,-10] = -63$	$[10,8,-2,-6] = 1125$
$[10,8,-2,-8] = -45$	$[10,8,-2,-10] = -40$	$[10,8,-4,-6] = 450$	$[10,8,-4,-8] = 80$
$[10,8,-4,-10] = -10$	$[10,8,-6,-8] = 30$	$[10,6,4,-10] = 30$	$[10,6,2,-8] = 1425$
$[10,6,2,-10] = 150$	$[10,6,0,-6] = -945$	$[10,6,0,-8] = 180$	$[10,6,0,-10] = 57$
$[10,6,-2,-4] = -4500$	$[10,6,-2,-6] = -810$	$[10,6,-2,-8] = -125$	$[10,6,-2,-10] = -25$
$[10,6,-4,-6] = 90$	$[10,6,-4,-8] = 50$	$[10,6,-4,-10] = -20$	$[10,6,-6,-8] = 60$
$[10,4,2,-6] = -7200$	$[10,4,2,-8] = -300$	$[10,4,2,-10] = 100$	$[10,4,0,-4] = 15120$
$[10,4,0,-6] = -540$	$[10,4,0,-8] = -160$	$[10,4,0,-10] = 112$	$[10,4,-2,-4] = 3960$
$[10,4,-2,-6] = 300$	$[10,4,-2,-8] = -150$	$[10,4,-2,-10] = 30$	$[10,4,-4,-6] = 240$
$[10,4,-4,-8] = 40$	$[10,4,-6,-8] = 100$	$[10,4,-6,-10] = 20$	$[10,4,-8,-10] = 10$
$[10,2,0,-2] = -32760$	$[10,2,0,-4] = -2520$	$[10,2,0,-6] = -1260$	$[10,2,0,-8] = -455$
$[10,2,0,-10] = 47$	$[10,2,-2,-4] = 1680$	$[10,2,-2,-6] = 105$	$[10,2,-2,-8] = -445$
$[10,2,-4,-6] = 300$	$[10,2,-4,-8] = -150$	$[10,2,-4,-10] = -30$	$[10,2,-6,-8] = 50$
$[10,2,-6,-10] = 25$	$[10,2,-8,-10] = 40$	$[10,0,-2,-4] = 1260$	$[10,0,-2,-6] = 630$
$[10,0,-2,-8] = -235$	$[10,0,-2,-10] = -47$	$[10,0,-4,-6] = 720$	$[10,0,-4,-8] = -80$
$[10,0,-4,-10] = -112$	$[10,0,-6,-8] = 15$	$[10,0,-6,-10] = -57$	$[10,0,-8,-10] = 63$
$[10,-2,-4,-6] = 720$	$[10,-2,-4,-8] = 260$	$[10,-2,-4,-10] = -100$	$[10,-2,-6,-8] = 285$
$[10,-2,-6,-10] = -150$	$[10,-2,-8,-10] = 15$	$[10,-4,-6,-8] = 570$	$[10,-4,-6,-10] = -30$
$[8,6,4,-8] = -6000$	$[8,6,4,-10] = -570$	$[8,6,2,-6] = 13500$	$[8,6,2,-8] = -75$
$[8,6,2,-10] = -285$	$[8,6,0,-4] = -10800$	$[8,6,0,-6] = 3915$	$[8,6,0,-8] = 1020$
$[8,6,0,-10] = -15$	$[8,6,-2,-4] = -9900$	$[8,6,-2,-6] = -2025$	$[8,6,-2,-8] = -25$
$[8,6,-2,-10] = -50$	$[8,6,-4,-6] = -900$	$[8,6,-4,-8] = -200$	$[8,6,-4,-10] = -100$
$[8,6,-6,-10] = -60$	$[8,6,-8,-10] = -30$	$[8,4,2,-4] = -18000$	$[8,4,2,-6] = -1800$
$[8,4,2,-8] = -800$	$[8,4,2,-10] = -260$	$[8,4,0,-2] = 25200$	$[8,4,0,-4] = 5760$
$[8,4,0,-6] = 3600$	$[8,4,0,-8] = 1120$	$[8,4,0,-10] = 80$	$[8,4,-2,-4] = -1200$

Table A.16: First quaternary algebra structure f on $V(10)$: part one

$[8,4,-2,-6]=150$	$[8,4,-2,-8]=600$	$[8,4,-2,-10]=150$	$[8,4,-4,-6]=-600$
$[8,4,-4,-10]=-40$	$[8,4,-6,-8]=200$	$[8,4,-6,-10]=-50$	$[8,4,-8,-10]=-80$
$[8,2,0,-2]=-15120$	$[8,2,0,-4]=-5040$	$[8,2,0,-6]=-315$	$[8,2,0,-8]=380$
$[8,2,0,-10]=235$	$[8,2,-2,-4]=-2100$	$[8,2,-2,-6]=-825$	$[8,2,-2,-10]=445$
$[8,2,-4,-6]=-750$	$[8,2,-4,-8]=-600$	$[8,2,-4,-10]=150$	$[8,2,-6,-8]=25$
$[8,2,-6,-10]=125$	$[8,2,-8,-10]=45$	$[8,0,-2,-6]=225$	$[8,0,-2,-8]=-380$
$[8,0,-2,-10]=455$	$[8,0,-4,-6]=720$	$[8,0,-4,-8]=-1120$	$[8,0,-4,-10]=160$
$[8,0,-6,-8]=-1020$	$[8,0,-6,-10]=-180$	$[8,0,-8,-10]=540$	$[8,-2,-4,-6]=2700$
$[8,-2,-4,-8]=800$	$[8,-2,-4,-10]=300$	$[8,-2,-6,-8]=75$	$[8,-2,-6,-10]=-1425$
$[8,-4,-6,-8]=6000$	$[6,4,2,-4]=-16200$	$[6,4,2,-6]=-7200$	$[6,4,2,-8]=-2700$
$[6,4,2,-10]=-720$	$[6,4,0,-2]=22680$	$[6,4,0,-4]=5040$	$[6,4,0,-6]=540$
$[6,4,0,-8]=-720$	$[6,4,0,-10]=-720$	$[6,4,-2,-4]=1800$	$[6,4,-2,-6]=1350$
$[6,4,-2,-8]=750$	$[6,4,-2,-10]=-300$	$[6,4,-4,-8]=600$	$[6,4,-4,-10]=-240$
$[6,4,-6,-8]=900$	$[6,4,-6,-10]=-90$	$[6,4,-8,-10]=-450$	$[6,2,0,-2]=2520$
$[6,2,0,-6]=135$	$[6,2,0,-8]=-225$	$[6,2,0,-10]=-630$	$[6,2,-2,-4]=-900$
$[6,2,-2,-8]=825$	$[6,2,-2,-10]=-105$	$[6,2,-4,-6]=-1350$	$[6,2,-4,-8]=-150$
$[6,2,-4,-10]=-300$	$[6,2,-6,-8]=2025$	$[6,2,-6,-10]=810$	$[6,2,-8,-10]=-1125$
$[6,0,-2,-4]=-900$	$[6,0,-2,-6]=-135$	$[6,0,-2,-8]=315$	$[6,0,-2,-10]=1260$
$[6,0,-4,-6]=-540$	$[6,0,-4,-8]=-3600$	$[6,0,-4,-10]=540$	$[6,0,-6,-8]=-3915$
$[6,0,-6,-10]=945$	$[6,-2,-4,-6]=7200$	$[6,-2,-4,-8]=1800$	$[6,-2,-4,-10]=7200$
$[6,-2,-6,-8]=-13500$	$[4,2,0,-2]=2520$	$[4,2,0,-4]=1440$	$[4,2,0,-6]=900$
$[4,2,0,-10]=-1260$	$[4,2,-2,-6]=900$	$[4,2,-2,-8]=2100$	$[4,2,-2,-10]=-1680$
$[4,2,-4,-6]=-1800$	$[4,2,-4,-8]=1200$	$[4,2,-4,-10]=-3960$	$[4,2,-6,-8]=9900$
$[4,2,-6,-10]=4500$	$[4,0,-2,-4]=-1440$	$[4,0,-2,-8]=5040$	$[4,0,-2,-10]=2520$
$[4,0,-4,-6]=-5040$	$[4,0,-4,-8]=-5760$	$[4,0,-4,-10]=-15120$	$[4,0,-6,-8]=10800$
$[4,-2,-4,-6]=16200$	$[4,-2,-4,-8]=18000$	$[2,0,-2,-4]=-2520$	$[2,0,-2,-6]=-2520$
$[2,0,-2,-8]=15120$	$[2,0,-2,-10]=32760$	$[2,0,-4,-6]=-22680$	$[2,0,-4,-8]=-25200$

Table A.17: First quaternary algebra structure f on $V(10)$: part two

$[10,8,2,-10] = 165$	$[10,8,0,-8] = 180$	$[10,8,0,-10] = 81$	$[10,8,-2,-6] = -135$
$[10,8,-2,-8] = -117$	$[10,8,-2,-10] = 32$	$[10,8,-4,-6] = -54$	$[10,8,-4,-8] = 64$
$[10,8,-4,-10] = 8$	$[10,8,-6,-8] = -24$	$[10,6,4,-10] = -330$	$[10,6,2,-8] = -375$
$[10,6,2,-10] = -120$	$[10,6,0,-6] = 1215$	$[10,6,0,-8] = -144$	$[10,6,0,-10] = -15$
$[10,6,-2,-4] = 540$	$[10,6,-2,-6] = 648$	$[10,6,-2,-8] = -53$	$[10,6,-2,-10] = 20$
$[10,6,-4,-6] = 234$	$[10,6,-4,-8] = -40$	$[10,6,-4,-10] = 16$	$[10,6,-6,-8] = -48$
$[10,4,2,-6] = 2700$	$[10,4,2,-8] = 240$	$[10,4,2,-10] = -80$	$[10,4,0,-4] = -7200$
$[10,4,0,-6] = 432$	$[10,4,0,-8] = 128$	$[10,4,0,-10] = -59$	$[10,4,-2,-4] = -3168$
$[10,4,-2,-6] = -240$	$[10,4,-2,-8] = -33$	$[10,4,-2,-10] = -24$	$[10,4,-4,-6] = 114$
$[10,4,-4,-8] = -32$	$[10,4,-6,-8] = -80$	$[10,4,-6,-10] = -16$	$[10,4,-8,-10] = -8$
$[10,2,0,-2] = 17640$	$[10,2,0,-4] = 2016$	$[10,2,0,-6] = 1008$	$[10,2,0,-8] = 364$
$[10,2,0,-10] = -7$	$[10,2,-2,-4] = -1344$	$[10,2,-2,-6] = -84$	$[10,2,-2,-8] = 203$
$[10,2,-4,-6] = 66$	$[10,2,-4,-8] = 120$	$[10,2,-4,-10] = 24$	$[10,2,-6,-8] = -40$
$[10,2,-6,-10] = -20$	$[10,2,-8,-10] = -32$	$[10,0,-2,-4] = -1008$	$[10,0,-2,-6] = -504$
$[10,0,-2,-8] = 35$	$[10,0,-2,-10] = 7$	$[10,0,-4,-6] = -270$	$[10,0,-4,-8] = 64$
$[10,0,-4,-10] = 59$	$[10,0,-6,-8] = -12$	$[10,0,-6,-10] = 15$	$[10,0,-8,-10] = -81$
$[10,-2,-4,-6] = -270$	$[10,-2,-4,-8] = -55$	$[10,-2,-4,-10] = 80$	$[10,-2,-6,-8] = -75$
$[10,-2,-6,-10] = 120$	$[10,-2,-8,-10] = -165$	$[10,-4,-6,-8] = -150$	$[10,-4,-6,-10] = 330$
$[8,6,4,-8] = 4800$	$[8,6,4,-10] = 150$	$[8,6,2,-6] = -10800$	$[8,6,2,-8] = 825$
$[8,6,2,-10] = 75$	$[8,6,0,-4] = 8640$	$[8,6,0,-6] = -2673$	$[8,6,0,-8] = -204$
$[8,6,0,-10] = 12$	$[8,6,-2,-4] = 4860$	$[8,6,-2,-6] = 855$	$[8,6,-2,-8] = 20$
$[8,6,-2,-10] = 40$	$[8,6,-4,-6] = 720$	$[8,6,-4,-8] = 160$	$[8,6,-4,-10] = 80$
$[8,6,-6,-10] = 48$	$[8,6,-8,-10] = 24$	$[8,4,2,-4] = 14400$	$[8,4,2,-6] = -1620$
$[8,4,2,-8] = 640$	$[8,4,2,-10] = 55$	$[8,4,0,-2] = -20160$	$[8,4,0,-4] = 288$
$[8,4,0,-6] = -2880$	$[8,4,0,-8] = -284$	$[8,4,0,-10] = -64$	$[8,4,-2,-4] = 960$

Table A.18: Second quaternary algebra structure g on $V(10)$: part one

$[8,4,-2,-6] = -885$	$[8,4,-2,-8] = -480$	$[8,4,-2,-10] = -120$	$[8,4,-4,-6] = 480$
$[8,4,-4,-10] = 32$	$[8,4,-6,-8] = -160$	$[8,4,-6,-10] = 40$	$[8,4,-8,-10] = 64$
$[8,2,0,-2] = 3528$	$[8,2,0,-4] = 4032$	$[8,2,0,-6] = 252$	$[8,2,0,-8] = 308$
$[8,2,0,-10] = -35$	$[8,2,-2,-4] = 1680$	$[8,2,-2,-6] = -105$	$[8,2,-2,-10] = -203$
$[8,2,-4,-6] = 600$	$[8,2,-4,-8] = 480$	$[8,2,-4,-10] = 33$	$[8,2,-6,-8] = -20$
$[8,2,-6,-10] = 53$	$[8,2,-8,-10] = 117$	$[8,0,-2,-6] = -945$	$[8,0,-2,-8] = -308$
$[8,0,-2,-10] = -364$	$[8,0,-4,-6] = -576$	$[8,0,-4,-8] = 284$	$[8,0,-4,-10] = -128$
$[8,0,-6,-8] = 204$	$[8,0,-6,-10] = 144$	$[8,0,-8,-10] = 180$	$[8,-2,-4,-6] = -1395$
$[8,-2,-4,-8] = -640$	$[8,-2,-4,-10] = -240$	$[8,-2,-6,-8] = -825$	$[8,-2,-6,-10] = 375$
$[8,-4,-6,-8] = -4800$	$[6,4,2,-4] = 12960$	$[6,4,2,-6] = 2700$	$[6,4,2,-8] = 1395$
$[6,4,2,-10] = 270$	$[6,4,0,-2] = -18144$	$[6,4,0,-4] = 864$	$[6,4,0,-6] = -891$
$[6,4,0,-8] = 576$	$[6,4,0,-10] = 270$	$[6,4,-2,-4] = 1620$	$[6,4,-2,-6] = -1080$
$[6,4,-2,-8] = -600$	$[6,4,-2,-10] = -66$	$[6,4,-4,-8] = -480$	$[6,4,-4,-10] = -114$
$[6,4,-6,-8] = -720$	$[6,4,-6,-10] = -234$	$[6,4,-8,-10] = 54$	$[6,2,0,-2] = -10584$
$[6,2,0,-6] = -567$	$[6,2,0,-8] = 945$	$[6,2,0,-10] = 504$	$[6,2,-2,-4] = 3780$
$[6,2,-2,-8] = 105$	$[6,2,-2,-10] = 84$	$[6,2,-4,-6] = 1080$	$[6,2,-4,-8] = 885$
$[6,2,-4,-10] = 240$	$[6,2,-6,-8] = -855$	$[6,2,-6,-10] = -648$	$[6,2,-8,-10] = 135$
$[6,0,-2,-4] = 3780$	$[6,0,-2,-6] = 567$	$[6,0,-2,-8] = -252$	$[6,0,-2,-10] = -1008$
$[6,0,-4,-6] = 891$	$[6,0,-4,-8] = 2880$	$[6,0,-4,-10] = -432$	$[6,0,-6,-8] = 2673$
$[6,0,-6,-10] = -1215$	$[6,-2,-4,-6] = -2700$	$[6,-2,-4,-8] = 1620$	$[6,-2,-4,-10] = -2700$
$[6,-2,-6,-8] = 10800$	$[4,2,0,-2] = -10584$	$[4,2,0,-4] = -6048$	$[4,2,0,-6] = -3780$
$[4,2,0,-10] = 1008$	$[4,2,-2,-6] = -3780$	$[4,2,-2,-8] = -1680$	$[4,2,-2,-10] = 1344$
$[4,2,-4,-6] = -1620$	$[4,2,-4,-8] = -960$	$[4,2,-4,-10] = 3168$	$[4,2,-6,-8] = -4860$
$[4,2,-6,-10] = -540$	$[4,0,-2,-4] = 6048$	$[4,0,-2,-8] = -4032$	$[4,0,-2,-10] = -2016$
$[4,0,-4,-6] = -864$	$[4,0,-4,-8] = -288$	$[4,0,-4,-10] = 7200$	$[4,0,-6,-8] = -8640$
$[4,-2,-4,-6] = -12960$	$[4,-2,-4,-8] = -14400$	$[2,0,-2,-4] = 10584$	$[2,0,-2,-6] = 10584$
$[2,0,-2,-8] = -3528$	$[2,0,-2,-10] = -17640$	$[2,0,-4,-6] = 18144$	$[2,0,-4,-8] = 20160$

Table A.19: Second quaternary algebra structure g on $V(10)$: part two

Appendix B

Lemma B.0.3. *For $m \geq 2$, the solution space of the following homogenous linear system has dimension $m-1$:*

$$\begin{aligned}
&\sum_{j=1}^{m} s_{0,j} = 0,\\
&-s_{0,1} + s_{2,0} - s_{1,0} = 0,\\
&-s_{0,j+1} - s_{2,j-1} + s_{2,j} - s_{1,j} = 0, \quad 1 \leq j \leq m-1,\\
&(-1)^r \sum_{j=1}^{m} \binom{j}{r} s_{0,j} + s_{1,r-1} - s_{0,r} = 0, \quad 1 \leq r \leq m,\\
&s_{2,m} = s_{1,m} = s_{2,m-1} = 0.
\end{aligned} \tag{$\mathcal{T}$}$$

Proof. We order the system $(\mathcal{T})$ according to the index of the variables as follows:

$$\begin{aligned}
&(\mathcal{T}_1) : s_{2,m} = 0,\\
&(\mathcal{T}_2) : s_{2,m-1} = 0,\\
&(\mathcal{T}_3) : s_{2,m-1} - s_{2,m-2} - s_{1,m-1} - s_{0,m} = 0,\\
&(\mathcal{T}_4) : s_{2,m-2} - s_{2,m-3} - s_{1,m-2} - s_{0,m-1} = 0,\\
&(\mathcal{T}_5) : s_{2,m-3} - s_{2,m-4} - s_{1,m-3} - s_{0,m-2} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(\mathcal{T}_m) : s_{2,2} - s_{2,1} - s_{1,2} - s_{0,3} = 0,\\
&(\mathcal{T}_{m+1}) : s_{2,1} - s_{2,0} - s_{1,1} - s_{0,2} = 0,\\
&(\mathcal{T}_{m+2}) : s_{2,0} - s_{1,0} - s_{0,1} = 0,\\
&(\mathcal{T}_{m+3}) : s_{1,m} = 0,\\
&(\mathcal{T}_{m+4}) : s_{1,m-1} + ((-1)^m - 1)s_{0,m} = 0,\\
&(\mathcal{T}_{m+5}) : s_{1,m-2} + (-1)^{m-1} m\, s_{0,m} + ((-1)^{m-1} - 1)s_{0,m-1} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(\mathcal{T}_{2m+3}) : s_{1,0} - m s_{0,m} - \cdots - 3s_{0,3} - 2s_{0,2} - 2s_{0,1} = 0,\\
&(\mathcal{T}_{2m+4}) : s_{0,m} + s_{0,m-1} + \cdots + s_{0,1} = 0.
\end{aligned}$$

Applying the operations $-\mathcal{T}_i + \mathcal{T}_{i+1} \to \mathcal{T}_{i+1}$, $-\mathcal{T}_{i+1} \to \mathcal{T}_{i+1}$, $2 \leq i \leq m+1$, we obtain

$$\begin{aligned}
&(\mathcal{T}_1) : s_{2,m} = 0,\\
&(\mathcal{T}_2) : s_{2,m-1} = 0,\\
&(\mathcal{T}_3) : s_{2,m-2} + s_{1,m-1} + s_{0,m} = 0,\\
&(\mathcal{T}_4) : s_{2,m-3} + s_{1,m-1} + s_{1,m-2} + s_{0,m} + s_{0,m-1} = 0,\\
&(\mathcal{T}_5) : s_{2,m-4} + s_{1,m-1} + s_{1,m-2} + s_{1,m-3} + s_{0,m} + s_{0,m-1} + s_{0,m-2} = 0,
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(\mathcal{T}_m): s_{2,1}+s_{1,m-1}+s_{1,m-2}+\cdots+s_{1,2}+s_{0,m}+s_{0,m-1}+\cdots+s_{0,3}=0,$$
$$(\mathcal{T}_{m+1}): s_{2,0}+s_{1,m-1}+\cdots+s_{1,1}+s_{0,m}+s_{0,m-1}+\cdots+s_{0,2}=0,$$
$$(\mathcal{T}_{m+2}): s_{1,m-1}+\cdots+s_{1,1}+s_{1,0}+s_{0,m}+s_{0,m-1}+\cdots+s_{0,1}=0,$$
$$(\mathcal{T}_{m+3}): s_{1,m}=0,$$
$$(\mathcal{T}_{m+4}): s_{1,m-1}+((-1)^m-1)s_{0,m}=0,$$
$$(\mathcal{T}_{m+5}): s_{1,m-2}+(-1)^{m-1}ms_{0,m}+((-1)^{m-1}-1)s_{0,m-1}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(\mathcal{T}_{2m+3}): s_{1,0}-ms_{0,m}-\cdots-3s_{0,3}-2s_{0,2}-2s_{0,1}=0,$$
$$(\mathcal{T}_{2m+4}): s_{0,m}+s_{0,m-1}+\cdots+s_{0,1}=0.$$

Applying the operations $\mathcal{T}_{m+2}-\mathcal{T}_i \to \mathcal{T}_{m+2}$, $m+4 \le i \le 2m+3$, we obtain

$$(\mathcal{T}_1): s_{2,m}=0,$$
$$(\mathcal{T}_2): s_{2,m-1}=0,$$
$$(\mathcal{T}_3): s_{2,m-2}+s_{1,m-1}+s_{0,m}=0,$$
$$(\mathcal{T}_4): s_{2,m-3}+s_{1,m-1}+s_{1,m-2}+s_{0,m}+s_{0,m-1}=0,$$
$$(\mathcal{T}_5): s_{2,m-4}+s_{1,m-1}+s_{1,m-2}+s_{1,m-3}+s_{0,m}+s_{0,m-1}+s_{0,m-2}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(\mathcal{T}_m): s_{2,1}+s_{1,m-1}+s_{1,m-2}+\cdots+s_{1,2}+s_{0,m}+s_{0,m-1}+\cdots+s_{0,3}=0,$$
$$(\mathcal{T}_{m+1}): s_{2,0}+s_{1,m-1}+\cdots+s_{1,1}+s_{0,m}+s_{0,m-1}+\cdots+s_{0,2}=0,$$
$$(\mathcal{T}_{m+2}): 3s_{0,m}+3s_{0,m-1}+\cdots+3s_{0,1}=0,$$
$$(\mathcal{T}_{m+3}): s_{1,m}=0,$$
$$(\mathcal{T}_{m+4}): s_{1,m-1}+((-1)^m-1)s_{0,m}=0,$$
$$(\mathcal{T}_{m+5}): s_{1,m-2}+(-1)^{m-1}ms_{0,m}+((-1)^{m-1}-1)s_{0,m-1}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(\mathcal{T}_{2m+3}): s_{1,0}-ms_{0,m}-\cdots-3s_{0,3}-2s_{0,2}-2s_{0,1}=0,$$
$$(\mathcal{T}_{2m+4}): s_{0,m}+s_{0,m-1}+\cdots+s_{0,1}=0.$$

Using the operation $\mathcal{T}_{2m+4}-\frac{1}{3}\mathcal{T}_{m+2} \to \mathcal{T}_{2m+4}$ reduces $\mathcal{T}_{2m+4}$ to 0. Hence the system reduced to $2m+3$ equations in $3m+2$ variables. □

References

[1] R. Bai, G. Song: The classification of six-dimensional 4-Lie algebras. *J. Phys. A* 42 (2009), no. 3, 035207, 17 pages.

[2] S. Bashir: *Automorphisms of Simple Anti-Jordan Pairs.* Ph.D. thesis, University of Ottawa, 2008.

[3] G. Benkart, T. Roby: Down-up algebras. *J. Algebra* 209 (1998), no. 1, 305–344.

[4] G. M. Bergman: The diamond lemma for ring theory. *Adv. in Math.* 29 (1978), no. 2, 178–218.

[5] M. R. Bremner: Varieties of anticommutative n-ary algebras. *J. Algebra* 191 (1997), no. 1, 76–88.

[6] M. R. Bremner: How to compute the Wedderburn decomposition of a finite-dimensional associative algebra. *Groups Complex. Cryptol.* 3 (2011), no. 1, 47-66.

[7] M. R. Bremner, H. A. Elgendy: Alternating quaternary algebra structures on irreducible representations of sl(2,C). *Linear Algebra Appl.* 433 (2010), no. 8-10, 1686–1705.

[8] M. R. Bremner, I. R. Hentzel: Invariant nonassociative algebra structures on irreducible representations of simple Lie algebras. *Experiment. Math.* 13 (2004), no. 2, 231–256.

[9] M. R. Bremner, I. R. Hentzel: Alternating triple systems with simple Lie algebras of derivations. *Nonassociative Algebra and its Applications*, 55–85, Lect. Notes Pure Appl. Math., 246, Chapman & Hall / CRC, Boca Raton, FL, 2006.

[10] M. R. Bremner, L. A. Peresi: Classification of trilinear operations. *Comm. Algebra* 35 (2007), no. 9, 2932–2959.

[11] J. M. Clifton: A simplification of the computation of the natural representation of the symmetric group S_n. *Proc. Amer. Math. Soc.* 83 (1981), no. 2, 248–250.

[12] N. Cantarini, V G. Kac: Classification of simple linearly compact n-Lie superalgebras. *Comm. Math. Phys.* 298 (2010), no. 3, 833–853.

[13] J. A. De Azcrraga, J. M. Izquierdo: n-Ary algebras: a review with applications. *J. Phys. A: Math. Theor.* 43(2010), 293001.

[14] W. A. de Graaf: *Lie Algebras: Theory and Algorithms*. North-Holland, Amsterdam, 2000.

[15] Y. A. Drozd, V. V. Kirichenko: *Finite-dimensional algebras*. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab. Springer-Verlag, Berlin, 1994.

[16] H. A. Elgendy, M. R. Bremner: Universal associative envelopes of $(n{+}1)$-dimensional n-Lie algebras. *Comm. Algebra* 40 (2012), no. 5, 1827–1842.

[17] H. A. Elgendy: Associative enveloping algebras for non-associative triple systems. (Submitted)

[18] H. A. Elgendy: The universal associative envelope of the anti-Jordan triple system of $n \times n$ matrices. (Submitted)

[19] J. Faulkner and J. Ferrar: Simple anti-Jordan pairs. *Comm. Algebra* 8 (1980), no. 11, 993–1013.

[20] V. T. Filippov: n-Lie algebras. *Sibirsk. Mat. Zh.* 26 (1985), no. 6, 126–140.

[21] K. Friedl, L. Ronyai: Polynomial time solutions of some problems in computational algebra. Proceedings of the 17th Annual ACM symposium on Theory of Computing. Association for Computing Machinery, 1985, pages 153–162.

[22] B. Hall: *Lie Groups, Lie Algebras, and Representations*, An Elementary Introduction, Graduate Texts in Mathematics, 222. Springer-Verlag, New York, 2003.

[23] F. Harary, E. M. Palmer: *Graphical Enumeration*. Academic Press, New York, 1973.

[24] J. E. Humphreys: *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics 9, Springer-Verlag, New York, 1972.

[25] N. Jacobson: *Structure and Representations of Jordan Algebras*. American Mathematical Society Colloquium Publications, Providence, R.I. 1968 x+453 pp.

[26] G. James, A. Kerber: *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, 16. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[27] N. Kamiya, D. Mondoc, S. Okubo: A structure theory of $(-1,-1)$-Freudenthal Kantor triple systems. *Bull. Aust. Math. Soc.* 81 (2010), no. 1, 132–155.

[28] E. Kirkman, I. Musson, D. Passman: Noetherian down-up algebras. *Proc. Amer. Math. Soc.* 127 (1999), no. 11, 3161-3167.

[29] G. R. Krause, T. H. Lenagan: *Growth of Algebras and Gelfand-Kirillov Dimension*. Revised edition. Graduate Studies in Mathematics, 22. American Mathematical Society, Providence, RI, 2000. x+212 pp.

[30] H. Li: *Gröbner bases in ring theory*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. x+284 pp.

[31] W. X. Ling: *On the Structure of n-Lie Algebras*. Ph.D. thesis, University of Siegen, 1993.

[32] W. G. Lister: Ternary rings. *Trans. Amer. Math. Soc.* 154 (1971) 37–55.

[33] O. Loos: Representations of Jordan triples. *Trans. Amer. Math. Soc.* 185(1973), 199-211 (1974).

[34] K. Meyberg: *Lectures on algebras and triple systems*. Notes on a course of lectures given during the academic year 19711972. The University of Virginia, Charlottesville, Va., 1972. v+226 pp.

[35] A. Pozhidaev: Enveloping algebras of Filippov algebras. *Comm. Algebra* 31 (2003), no. 2, 883–900.

[36] J. H. van Lint, R. M. Wilson: *A Course in Combinatorics*. Cambridge University Press, 1992.

[37] X. Wu, C. Chao: An application of Pólya's enumeration theorem to partitions of subsets of positive integers. *Czechoslovak Math. J.* 55(130) (2005), no. 3, 611–623.

[38] K. Zhao: Centers of down-up algebras. *J. Algebra* 214 (1999), no. 1, 103–121.